Triangulated categories of mixed motives

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Abstract. We construct triangulated categories of mixed motives over a noetherian scheme of finite Krull dimension, extending Voevodsky's definition of motives over a field. We prove that motives with rational coefficients satisfy the formalism of the six operations of Grothendieck. This is achieved by studying descent properties of motives, as well as by comparing different presentations of these categories, following insights and constructions of Beilinson, Morel and Voevodsky. Finally, we associate to any mixed Weil cohomology a system of categories of coefficients and well behaved realization functors.

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Partially supported by the ANR (grant No. ANR-07-BLAN-042).

Introduction

The aim of these notes is to define a system of triangulated categories of mixed motives which satisfies Grothendieck's six functors formalism. This is achieved partially with integral coefficients, and rather fully with rational coefficients.

Motivic complexes. We construct, for any noetherian scheme of finite dimension X, a triangulated category of mixed motives $\mathrm{DM}(X)$, from the abelian category of Nisnevich sheaves with transfers over X, extending Voevodsky's theory. This construction has the following functorial properties.

- For any scheme[1] X, $\mathrm{DM}(X)$ is a closed symmetric monoidal category.
- For any morphism of schemes $f : Y \to X$, a pair of adjoint functors
$$f^* : \mathrm{DM}(X) \rightleftarrows \mathrm{DM}(Y) : f_* ,$$
with f^* monoidal.
- For any separated morphism of finite type $f : Y \to X$, a pair of adjoint functors
$$f_! : \mathrm{DM}(Y) \rightleftarrows \mathrm{DM}(X) : f^! ,$$
and a natural transformation
$$f_! \to f_*$$
which is invertible for f proper, while
$$f^! = f^*$$
whenever f is an open immersion.
- For any cartesian square
$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X, \end{array}$$
with f separated of finite type, there is a natural transformation $g^* f_! \to f'_! g'^*$, which is invertible whenever g is smooth separated of finite type.

We also prove the following localization property: given a scheme S and a closed immersion between smooth S-schemes $i : Z \to X$, with complement open immersion $j : U \to X$, we have the six gluing functors:
$$\mathrm{DM}(U) \underset{j_*}{\overset{j_!}{\underset{\longrightarrow}{\overset{\longleftarrow}{\longrightarrow}}}} \mathrm{DM}(X) \underset{i^!}{\overset{i^*}{\underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}}}} \mathrm{DM}(Z) .$$
This means that the functors $j_!$ and i_* are fully faithful, satisfy the identity $i^* j_! \simeq 0$, and that, for any object M of $\mathrm{DM}(X)$, we have a natural distinguished triangle
$$j_! j^!(M) \to M \to i_* i^*(M) \to M[1] .$$
Unfortunately, we don't know if this property remains true in DM without this smoothness assumption. The formalism of the six gluing functors for a general closed immersion is needed if one wants to prove that the exchange maps $g^* f_! \to f'_! g'^*$ are invertible in general.

Beilinson motives. However, if we work with rational coefficients, the situation is much better. We construct, using Morel and Voevodsky's homotopy theory of schemes, for any scheme X, a $\mathbf{Q}$-linear triangulated category $\mathrm{DM}_{\mathrm{B}}(X)$, which we call the triangulated category of Beilinson motives. Essentially by construction, in the case where X is regular, we have a natural identification
$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{B}}(X)}(\mathbf{Q}_X, \mathbf{Q}_X(p)[q]) \simeq Gr^p_\gamma K_{2p-q}(X)_{\mathbf{Q}} ,$$
where the right hand side is the graded part of the algebraic K-theory of X with respect to the γ-filtration.

[1] In this introduction, all the schemes will be assumed to be noetherian of finite Krull dimension.

The six operations of Grothendieck act naturally on DM_{B}: we have all the functors $\otimes$, *Hom*, f^*, f_*, $f_!$, $f^!$ (with the same properties as above). The following result is a direct consequence of its analog in SH.

Theorem 1 (localization)**.** *For any closed immersion $i : Z \to X$ with complement open immersion $j : U \to X$, we have the six gluing functors:*

$$\mathrm{DM}_{\mathrm{B}}(U) \underset{j_*}{\overset{j_!}{\rightleftarrows}} \underset{}{\overset{j^*}{\rightleftarrows}} \mathrm{DM}_{\mathrm{B}}(X) \underset{i^!}{\overset{i^*}{\rightleftarrows}} \underset{}{\overset{i_*}{\rightleftarrows}} \mathrm{DM}_{\mathrm{B}}(Z)\,.$$

The result above allows us to use Ayoub's results on cross functors, which gives the following properties. We recall that, for $f : Y \to X$ proper, we have $f_! \simeq f_*$. Moreover, if f is smooth and quasi-projective of relative dimension d, we have, for any object M of $\mathrm{DM}_{\mathrm{B}}(X)$, a purity isomorphism

$$f^*(M)(d)[2d] \simeq f^!(M)$$

These identifications allow to interpret the following theorem as the proper base change formula and the smooth base change formula in DM_{B}.

Theorem 2 (base change isomorphisms)**.** *For any cartesian square*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X, \end{array}$$

with f separated of finite type, the natural transformation $g^*\, f_! \to f'_!\, g'^*$ as well as its transposed $g'_*\, f'^! \to f^!\, g_*$ are invertible.

We also deduce from Quillen's localization theorem in algebraic K-theory an absolute purity theorem:

Theorem 3 (absolute purity)**.** *For any closed immersion between regular schemes $i : Z \to X$ of codimension c, there is canonical isomorphism in* $\mathrm{DM}_{\mathrm{B}}(Z)$

$$\mathbf{Q}_Z(-c)[-2c] \xrightarrow{\sim} i^!(\mathbf{Q}_X)\,.$$

Using the existence of trace maps in algebraic K-theory, we also prove:

Theorem 4 (proper descent)**.** *Consider a Galois alteration $p : X' \to X$ of group G (i.e. p is an alteration, while G is a finite group acting on X' over X, such that, generically, $X'/G \to X$ is finite surjective and radicial) , as well as a closed subscheme $Z \subset X$, such that $U = X - Z$ is normal, and such that the induced map $p_U : U' = p^{-1}(U) \to U$ is a finite morphism. Then the pullback square*

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & X' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{i} & X \end{array}$$

induces a canonical distinguished triangle

$$M \to i_*\, i^*(M) \oplus p_*\, p^*(M)^G \to i_* q_*\, q^*\, i^*(M)^G \to M[1]$$

for any object M of $\mathrm{DM}_{\mathrm{B}}(X)$.

As a corollary of the previous result, we obtain that DM_{B} satisfies h-descent (in particular, étale descent as well as proper descent).

Given a scheme X, we define $\mathrm{DM}_{\mathrm{B},c}(X)$ as the thick subcategory of $\mathrm{DM}_{\mathrm{B}}(X)$ generated by the objects of shape $M_X(Y)(p) = f_!\, f^!(\mathbf{Q}_X)(p)$, for $f : Y \to X$ separated smooth of finite type, and $p \in \mathbf{Z}$. This category coincides with the full subcategory of compact objects in $\mathrm{DM}_{\mathrm{B}}(X)$.

Theorem 5 (continuity)**.** *Let S be a scheme which is the limit of an essentially affine projective system of schemes $\{S_\alpha\}$. Then there is a canonical equivalence of triangulated categories*

$$2\text{-}\varinjlim_{\alpha} \mathrm{DM}_{\mathrm{B},c}(S_\alpha) \simeq \mathrm{DM}_{\mathrm{B},c}(S)\,.$$

The following theorem uses absolute purity, (a weak form of) proper descent, as well as Gabber's weak uniformisation theorem (i.e. that, locally for the h-topology, any excellent scheme is regular, and any closed immersion between excellent schemes is the embedding of a strict normal crossing divisor into a regular scheme). Its proof relies on a mix of arguments of Gabber and Ayoub.

Theorem 6 (finiteness)**.** *The subcategory $\mathrm{DM}_{\mathrm{B},c}$ is stable under the six operations of Grothendieck (for excellent schemes).*

The absolute purity theorem, proper descent, de Jong's resolution of singularities by Galois alterations, and Ayoub's methods lead to:

Theorem 7 (duality)**.** *Let B be an excelllent scheme of dimension ≤ 2. For any separated B-morphism $f : X \to S$ between B-schemes of finite type, with S regular, $f^!(\mathbf{Q}_S)$ is a dualizing object in $\mathrm{DM}_{\mathrm{B}}(X)$.*

Comparison theorems. Recall the qfh-topology (resp. the h-topology): this is the Grothendieck topology on the category of schemes, generated by étale surjective morphisms and finite (resp. proper) surjective morphisms.

In his first published work on the triangulated categories of mixed motives, Voevodsky studied the $\mathbf{A}^1$-homotopy category of the derived category of qfh-sheaves. We consider a $\mathbf{Q}$-linear and $\mathbf{P}^1$-stable version of it, which we denote by $\underline{\mathrm{DM}}_{\mathrm{qfh},\mathbf{Q}}(S)$. By construction, for any S-scheme of finite type X, there is a qfh-motive $M_S(X)$ in $\underline{\mathrm{DM}}_{\mathrm{qfh},\mathbf{Q}}(S)$. We define $\mathrm{DM}_{\mathrm{qfh},\mathbf{Q}}(S)$ as the smallest triangulated full subcategory of $\underline{\mathrm{DM}}_{\mathrm{qfh},\mathbf{Q}}(S)$ which is stable by direct sums, and which contains the objects $M_S(X)(p)$, for X/S smooth of finite type, and $p \in \mathbf{Z}$. Using Voevodsky's comparison results between rational étale and qfh-cohomologies, we obtain the following:

Theorem 8. *If S is excellent and geometrically unibranch, then there is a canonical equivalence of categories*

$$\mathrm{DM}_{\mathbf{Q}}(S) \simeq \mathrm{DM}_{\mathrm{qfh},\mathbf{Q}}(S)\,,$$

where $\mathrm{DM}_{\mathbf{Q}}(S)$ denotes the $\mathbf{Q}$-linear version of $\mathrm{DM}(S)$.

One constructs $\mathrm{DM}_{\mathrm{h},\mathbf{Q}} \subset \underline{\mathrm{DM}}_{\mathrm{h},\mathbf{Q}}$ in a similar way as we did for $\mathrm{DM}_{\mathrm{qfh},\mathbf{Q}} \subset \underline{\mathrm{DM}}_{\mathrm{qfh},\mathbf{Q}}$, by replacing the qfh-topology by the h-topology. Using h-descent in DM_{B}, we also get the following comparison result.

Theorem 9. *If S is excellent, then we have canonical equivalences of categories*

$$\mathrm{DM}_{\mathrm{B}}(S) \simeq \mathrm{DM}_{\mathrm{qfh},\mathbf{Q}}(S) \simeq \mathrm{DM}_{\mathrm{h},\mathbf{Q}}(S)\,.$$

Denote by KGL_S the algebraic K-theory spectrum in Morel and Voevodsky's stable homotopy category $\mathrm{SH}(S)$. By virtue of a result of Riou, the γ-filtration on K-theory induces a decomposition of $KGL_{S,\mathbf{Q}}$:

$$KGL_{S,\mathbf{Q}} \simeq \bigoplus_{n\in\mathbf{Z}} H_{\mathrm{B},S}(n)[2n]\,.$$

The ring spectrum $H_{\mathrm{B},S}$ represents Beilinson motivic cohomology. By construction, the category $\mathrm{DM}_{\mathrm{B}}(S)$ is the full subcategory of $\mathrm{SH}_{\mathbf{Q}}(S)$ which consists of objects E such that the unit map $E \to H_{\mathrm{B},S} \otimes E$ is an isomorphism. If S is excellent and geometrically unibranch, we have:

$$H_{\mathrm{B},S} \simeq H_{\mathcal{M},S} \otimes \mathbf{Q}\,,$$

where $H_{\mathcal{M},S}$ is Voevodsky's motivic cohomology spectrum in $\mathrm{SH}(S)$. We deduce from the preceding comparison results a version with rational coefficients of a conjecture of Voevodsky [Voe02].

Corollary. *For any morphism $f : T \to S$ of excellent geometrically unibranch schemes, the canonical map*

$$f^* H_{\mathcal{M},S} \otimes \mathbf{Q} \to H_{\mathcal{M},T} \otimes \mathbf{Q}$$

is an isomorphism of ring spectra.

Theorem 10. *Let E be an object of $\mathrm{SH}_{\mathbf{Q}}(S)$. The following conditions are equivalent:*

(i) E is a Beilinson motive (i.e. is in $\mathrm{DM}_{\mathrm{B}}(S)$);
(ii) E is an $H_{\mathrm{B},S}$-module in $\mathrm{SH}_{\mathbf{Q}}(S)$;
(iii) E satisfies qfh*-descent;*
(iv) E satisfies h*-descent.*

The ring spectrum $H_{\mathrm{B},S}$ can be endowed with a structure of commutative monoid in the model category underlying $\mathrm{SH}_{\mathbf{Q}}(S)$. In particular, it makes sense to speak of the homotopy category of $H_{\mathrm{B},S}$-modules.

Theorem 11. *For any scheme S, there is a canonical equivalence of categories*

$$\mathrm{DM}_{\mathrm{B}}(S) \simeq \mathrm{Ho}(H_{\mathrm{B},S}\text{-}\mathrm{mod})\,.$$

According to Morel, the category $\mathrm{SH}_{\mathbf{Q}}(S)$ can be decomposed into two factors, one of them being $\mathrm{SH}_{\mathbf{Q}}(S)_+$, that is the part of $\mathrm{SH}_{\mathbf{Q}}(S)$ on which the map $\epsilon : S^0_{\mathbf{Q}} \to S^0_{\mathbf{Q}}$, induced by the permutation of the factors in $\mathbf{G}_m \wedge \mathbf{G}_m$, acts as -1. Let $S^0_{\mathbf{Q}+}$ be the unit object of $\mathrm{SH}_{\mathbf{Q}}(S)_+$.

Using the previous theorem as well as Morel's computation of the motivic sphere spectrum in terms of Milnor-Witt K-theory, we obtain a proof of:

Theorem 12 (Morel)**.** *For any scheme S, the canonical map $S^0_{\mathbf{Q}+} \to H_{\mathrm{B},S}$ is an isomorphism.*

Corollary. *For any scheme S, there is a canonical equivalence of categories*

$$\mathrm{SH}_{\mathbf{Q}}(S)_+ \simeq \mathrm{DM}_{\mathrm{B}}(S)\,.$$

As a corollary, as DM_{B} satisfies étale descent, and as, locally for the étale topology, -1 is always a square in the residue fields of S, we finally get another comparison result, where $\mathrm{D}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Q})$ is the étale version of $\mathrm{SH}_{\mathbf{Q}}(S)$.

Theorem 13. *For any scheme S, there is a canonical equivalence of categories*

$$\mathrm{D}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Q}) \simeq \mathrm{DM}_{\mathrm{B}}(S)\,.$$

Corollary. *An object of $\mathrm{SH}_{\mathbf{Q}}(S)$ is a Beilinson motive if and only if it satisfies étale descent.*

Realizations.

Given a strict ring spectrum $\mathcal{E}$ in $\mathrm{DM}_{\mathrm{B}}(S)$ (that is a commutative monoid in the underlying model category), one can define, for any S-scheme X, the triangulated category

$$\mathrm{D}(X,\mathcal{E}) = \mathrm{Ho}(\mathcal{E}_X\text{-}\mathrm{mod})\,,$$

where $\mathcal{E}_X = f^*\mathcal{E}$, for $f : X \to S$ the structural map.

We then have realization functors

$$\mathrm{DM}_{\mathrm{B}}(X) \to \mathrm{D}(X,\mathcal{E})\ , \quad M \mapsto \mathcal{E}_X \otimes_X M$$

which commute with the six operations of Grothendieck. Using Ayoub's description of the Betti realization, we obtain:

Theorem 14. *If $S = \mathrm{Spec}\,(k)$ with k a subfield of $\mathbf{C}$, and if $\mathcal{E}_{Betti}$ represents Betti cohomology in $\mathrm{DM}_{\mathrm{B}}(S)$, then, for any k-scheme of finite type, the full subcategory of compact objects of $\mathrm{D}(X,\mathcal{E}_{Betti})$ is canonically equivalent to $\mathrm{D}^b_c(X(\mathbf{C}),\mathbf{Q})$.*

More generally, if S is the spectrum of some field k, given a mixed Weil cohomology $\mathcal{E}$, with coefficient field (of characteristic zero) $\mathbf{K}$, we get realization functors

$$\mathrm{DM}_{\mathrm{B},c}(X) \to \mathrm{D}_c(X,\mathcal{E}) \ , \quad M \mapsto \mathcal{E}_X \otimes_X M$$

(where $\mathrm{D}_c(X,\mathcal{E})$ stands for the category of compact objects of $\mathrm{D}(X,\mathcal{E})$), which commute with the six operations of Grothendieck (which preserve compact objects on both sides). Moreover, the category $\mathrm{D}_c(S,\mathcal{E})$ is then canonically equivalent to the bounded derived category of the abelian category of finite dimensional $\mathbf{K}$-vector spaces. As a byproduct, we get the following concrete finiteness result: for any k-scheme of finite type X, and for any objects M and N in $\mathrm{D}_c(X,\mathcal{E})$, the $\mathbf{K}$-vector space $\mathrm{Hom}_{\mathrm{D}_c(X,\mathcal{E})}(M,N[n])$ is finite dimensional, and it is trivial for all but a finite number of values of n.

If k is of characteristic zero, this abstract construction gives essentially the usual categories of coefficients (as seen above in the case of Betti cohomology), and in a sequel of this work, we shall prove that one recovers in this way the derived categories of constructible ℓ-adic sheaves (of geometric origin) in any characteristic. But something new happens in positive characteristic:

Theorem 15. *Let V be a complete discrete valuation ring of mixed characteristic, with field of functions K, and residue field k. Then rigid cohomology is a K-linear mixed Weil cohomology, and thus defines a ring spectrum $\mathcal{E}_{rig}$ in $\mathrm{DM}_{\mathrm{B}}(k)$. We obtain a system of closed symmetric monoidal triangulated categories $\mathrm{D}_{rig}(X) = \mathrm{D}_c(X,\mathcal{E}_{rig})$, for any k-scheme of finite type X, such that*

$$\mathrm{Hom}_{\mathrm{D}_{rig}(X)}(\mathbb{1}_X, \mathbb{1}_X(p)[q]) \simeq H^q_{rig}(X)(p)\,,$$

as well as realization functors

$$R_{rig} : \mathrm{DM}_{\mathrm{B},c}(X) \to \mathrm{D}_{rig}(X)$$

which preserve the six operations of Grothendieck.

Outline of the work

This work is divided into four parts. The first one contains the foundations, stated in the language of fibred categories. The main notion is that of a *$\mathscr{P}$-fibred category* (definition 1.1.9), eventually equiped with a monoidal structure (definition 1.1.26) and a triangulated structure. We study axioms and constructions for these categories on the following themes: the *six functors formalism of Grothendieck* (section 2), *descent theory* (section 3).

The second part gives a way to construct derived categories which fits into the theory of $\mathscr{P}$-fibred categories.

The third part contains a reformulation of the theory of relative cycles of Suslin and Voevodsky – together with some complements. This is used to construct the example, presented in the first part of this introduction, of motivic complexes as a $\mathscr{P}$-fibred category.

The fourth part is concerned with the rational theory of mixed motives and the fundamental theorems stated above. Besides the main definition of *Beilinson motives* (definition 13.2.1), we also use the theory of *modules over a ring spectrum* (section 4) to construct other interesting $\mathscr{P}$-fibred categories: modules over the algebraic K-theory spectrum (corollary 12.3.3), modules over a mixed Weil cohomology spectrum (section 16).

Notations and conventions

In every section, we will fix a category denoted by $\mathscr{S}$ which will contain our geometric objects. Most of the time, $\mathscr{S}$ will be a category of schemes which are suitable for our needs; the required hypothesis on $\mathscr{S}$ are given at the head of each section. In the text, when no precisions are given, any scheme will be assumed to be an object of $\mathscr{S}$.

When $\mathscr{A}$ is an additive category, we denote by $\mathscr{A}^\natural$ the pseudo-abelian enveloppe of $\mathscr{A}$. We denote by $\mathrm{C}(\mathscr{A})$ the category of complexes of $\mathscr{A}$. We consider $\mathrm{K}(\mathscr{A})$ (resp. $\mathrm{K}^b(\mathscr{A})$) the category of complexes (resp. bounded complexes) of $\mathscr{A}$ modulo the chain homotopy equivalences and when $\mathscr{A}$ is abelian, we let $\mathrm{D}(\mathscr{A})$ be the derived category of $\mathscr{A}$.

If $\mathscr{M}$ is a model category, $\mathrm{Ho}(\mathscr{M})$ will denote its homotopy category.

We will use the notation

$$\alpha : \mathscr{C} \rightleftarrows \mathscr{D} : \beta$$

to mean a pair of functors such that α is left adjoint to β. Similarly, when we speak of an adjoint pair of functors (α, β), α will always be the left adjoint. We will denote by

$$ad(\alpha, \beta) : 1 \to \beta\alpha \text{ (resp. } ad'(\alpha, \beta) : \alpha\beta \to 1)$$

the unit (resp. counit) of the adjunction (α, β). Considering a natural tranformation $\eta : F \to G$ of functors, we usually denote by the same letter η — when the context is clear — the induced natural transformation $AFB \to AGB$ obtained when considering functors A and B composed on the left and right with F and G respectively.

Part 1. Fibred categories and the six functor formalism

1. General definitions and axiomatic

1.0. We assume that $\mathscr{S}$ is an arbitrary category.

We shall say that a class $\mathscr{P}$ of morphisms of $\mathscr{S}$ is *admissible* if it is has the following properties.

(Pa) Any isomorphism is in $\mathscr{P}$.
(Pb) The class $\mathscr{P}$ is stable by composition.
(Pc) The class $\mathscr{P}$ is stable by pullbacks: for any morphism $f : X \to Y$ in $\mathscr{P}$ and any morphism $Y' \to Y$, the pullback $X' = Y' \times_Y X$ is representable in $\mathscr{S}$, and the projection $f' : X' \to Y'$ is in $\mathscr{P}$.

The morphisms which are in $\mathscr{P}$ will be called the *$\mathscr{P}$-morphisms.*[2]

In what follows, we assume that an admissible class of morphisms $\mathscr{P}$ is fixed.

1.1. $\mathscr{P}$-fibred categories.

1.1.a. *Definitions.* Let $\mathscr{C}at$ be the 2-category of categories.

1.1.1. Let $\mathscr{M}$ be a fibred category over $\mathscr{S}$, seen as a 2-functor $\mathscr{M} : \mathscr{S}^{op} \to \mathscr{C}at$; see [Gro03, Exp. VI]

Given a morphism $f : T \to S$ in $\mathscr{S}$, we shall denote by

$$f^* : \mathscr{M}(S) \to \mathscr{M}(T)$$

the corresponding pullback functor between the corresponding fibers. We shall always assume that $(1_S)^* = 1_{\mathscr{M}(S)}$, and for any morphisms $W \xrightarrow{g} T \xrightarrow{f} S$ in $\mathscr{S}$, we have structural isomorphisms

$$g^* f^* \xrightarrow{\sim} (fg)^* \tag{1.1.1.1}$$

which are subject to a cocyle condition with respect to composition of morphisms.

Given a morphism $f : T \to S$ in $\mathscr{S}$, if the corresponding inverse image functor f^* has a left adjoint, we shall denote it by

$$f_\sharp : \mathscr{M}(T) \to \mathscr{M}(S)\,.$$

For any morphisms $W \xrightarrow{g} T \xrightarrow{f} S$ in $\mathscr{S}$ such that f^* and g^* have a left adjoint, we have an isomorphism obtained by transposition from the isomorphism (1.1.1.1):

$$(fg)_\sharp \xrightarrow{\sim} f_\sharp g_\sharp\,. \tag{1.1.1.2}$$

Definition 1.1.2. A *pre-$\mathscr{P}$-fibred category* $\mathscr{M}$ over $\mathscr{S}$ is a fibred category $\mathscr{M}$ over $\mathscr{S}$ such that, for any morphism $p : T \to S$ in $\mathscr{P}$, the pullback functor $p^* : \mathscr{M}(S) \to \mathscr{M}(T)$ has a left adjoint $p_\sharp : \mathscr{M}(T) \to \mathscr{M}(T)$.

[2] In practice, $\mathscr{S}$ will be an adequate subcategory of the category of noetherian schemes and $\mathscr{P}$ will be the class of smooth morphisms (resp. of étale morphisms, resp. of morphisms of finite type) in $\mathscr{S}$.

Convention 1.1.3. Usually, we will consider that (1.1.1.1) and (1.1.1.2) are identities. Similarly, we consider that for any object S of $\mathscr{S}$, $(1_S)^* = 1_{\mathscr{M}(S)}$ and $(1_S)_\sharp = 1_{\mathscr{M}(S)}$.[3]

Example 1.1.4. Let S be an object of $\mathscr{S}$. We let $\mathscr{P}/S$ be the full subcategory of the comma category $\mathscr{S}/S$ made of objects over S whose structural morphism is in $\mathscr{P}$. We will usually call the objects of $\mathscr{P}/S$ the *$\mathscr{P}$-objects over S*.

Given a morphism $f : T \to S$ in $\mathscr{S}$ and a $\mathscr{P}$-morphism $\pi : X \to S$, we put $f^*(\pi) = \pi \times_S T$ using the property (Pc) of $\mathscr{P}$. This defines a functor $f^* : \mathscr{P}/S \to \mathscr{P}/T$.

Given two $\mathscr{P}$-morphisms $f : T \to S$ and $\pi : Y \to T$, we put $f_\sharp(\pi) = f \circ \pi$ using the property (Pb) of $\mathscr{P}$. this defines a functor $f_\sharp : \mathscr{P}/T \to \mathscr{P}/S$. According to the property of pullbacks, $f_\sharp$ is left adjoint to f^*.

We thus get a pre-$\mathscr{P}$-fibred category $\mathscr{P}/? : S \mapsto \mathscr{P}/S$.

Example 1.1.5. Assume $\mathscr{S}$ is the category of noetherian schemes of finite dimension, and $P = Sm$. For a scheme S of $\mathscr{S}$, let $\mathscr{H}_\bullet(S)$ be the pointed homotopy category of schemes over S defined by Morel and Voevodsky in [MV99]. Then according to *op. cit.*, $\mathscr{H}_\bullet$ is a pre-Sm-fibred category over $\mathscr{S}$.

1.1.6. *Exchange structures I.–* Suppose given a weak $\mathscr{P}$-fibred category $\mathscr{M}$.

Consider a commutative square of $\mathscr{S}$

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ {\scriptstyle g}\downarrow & \Delta & \downarrow{\scriptstyle f} \\ T & \xrightarrow[p]{} & S \end{array}$$

such that p and q are $\mathscr{P}$-morphisms, we get using the identification of convention 1.1.3 a canonical natural transformation

$$Ex(\Delta_\sharp^*) : q_\sharp g^* \xrightarrow{ad(p_\sharp, p^*)} q_\sharp g^* p^* p_\sharp = q_\sharp q^* f^* p_\sharp \xrightarrow{ad'(q_\sharp, q^*)} f^* p_\sharp$$

called the *exchange transformation* between $q_\sharp$ and g^*.

Remark 1.1.7. These exchange transformations satisfy a coherence condition with respect to the relations $(fg)^* = g^* f^*$ and $(fg)_\sharp = f_\sharp g_\sharp$. As an example, consider two commutative squares in $\mathscr{S}$:

$$\begin{array}{ccccc} Z & \xrightarrow{q'} & Y & \xrightarrow{q} & X \\ {\scriptstyle h}\downarrow & \Theta & {\scriptstyle g}\downarrow & \Delta & \downarrow{\scriptstyle f} \\ W & \xrightarrow[p']{} & T & \xrightarrow[p]{} & S \end{array}$$

and let $\Delta \circ \Theta$ be the commutative square made by the exterior maps – it is usually called the horizontal composition of the squares. Then, the following diagram of 2-morphisms is commutative:

$$\begin{array}{ccccc} (qq')_\sharp h^* & & \xrightarrow{Ex(\Delta\circ\Theta)_\sharp^*} & & f^*(pp')_\sharp \\ \| & & & & \| \\ q_\sharp q'_\sharp h^* & \xrightarrow{Ex(\Theta_\sharp^*)} & q_\sharp g^* p'_\sharp & \xrightarrow{Ex(\Delta_\sharp^*)} & f^* p_\sharp p'_\sharp \end{array}$$

This follows easily from the equations of the relevant adjunctions (see also [Ayo07a, 1.2.5]). Thus, according to our abuse of notation for natural transformations, Ex behaves as a contravariant functor with respect to the horizontal composition of squares. The same is true for vertical composition of commutative squares.

[3] We can always strictify globally the fibred category structure so that $g^* f^* = (fg)^*$ for any composable morphisms f and g, and so that $(1_S)^* = 1_{\mathscr{M}(S)}$ for any object S of $\mathscr{S}$; moreover, for a morphism h of $\mathscr{S}$ such that a left adjoint of h^* exists, and we can choose the left adjoint functor $h_\sharp$ which we feel as the most convenient for us, depending on the situation we deal with. For instance, if $h = 1_S$, we can choose $h_\sharp$ to be $1_{\mathscr{M}(S)}$, and if $h = fg$, with f^* and g^* having left adjoints, we can choose $h_\sharp$ to be $f_\sharp g_\sharp$ (with the unit and counit naturally induced by composition).

1.1.8. Under the assumptions of 1.1.6, we will consider the following property:

($\mathscr{P}$-BC) *$\mathscr{P}$-base change.*– For any cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ {\scriptstyle g}\downarrow & \Delta & \downarrow{\scriptstyle f} \\ T & \xrightarrow[p]{} & S \end{array}$$

such that p is a $\mathscr{P}$-morphism, the exchange transformation

$$Ex(\Delta_\sharp^*) : q_\sharp g^* \to f^* p_\sharp$$

is an isomorphism.

Definition 1.1.9. A *$\mathscr{P}$-fibred category over $\mathscr{S}$* is a pre-$\mathscr{P}$-fibred category $\mathscr{M}$ over $\mathscr{S}$ which satisfies the property of $\mathscr{P}$-base change.

Example 1.1.10. Consider the notations of example 1.1.4. Then the transitivity property of pull-backs of morphisms in $\mathscr{P}$ amounts to say that $\mathscr{P}/?$ satisfies the $\mathscr{P}$-base change property. Thus, $\mathscr{P}/?$ is in fact a $\mathscr{P}$-fibred category, called *the canonical $\mathscr{P}$-fibred category.*

Definition 1.1.11. A $\mathscr{P}$-fibred category $\mathscr{M}$ over $\mathscr{S}$ is *complete* if, for any morphism $f : T \to S$, the pullback functor $f^* : \mathscr{M}(S) \to \mathscr{M}(T)$ admits a right adjoint $f_* : \mathscr{M}(S) \to \mathscr{M}(T)$.

Remark 1.1.12. In the case where $\mathscr{P}$ is the class of isomorphisms a $\mathscr{P}$-fibred category is what we usually call a bifibred category over $\mathscr{S}$.

Example 1.1.13. The pre-Sm-fibred category $\mathscr{H}_\bullet$ of example 1.1.5 is a complete Sm-fibred category according to [MV99].

1.1.14. *Exchange structures II.*– Let $\mathscr{M}$ be a $\mathscr{P}$-fibred $\mathscr{C}$-category. Consider a commutative square

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ {\scriptstyle g}\downarrow & \Delta & \downarrow{\scriptstyle f} \\ T & \xrightarrow[p]{} & S. \end{array}$$

We obtain an exchange transformation:

$$Ex(\Delta_*^*) : p^* f_* \xrightarrow{ad(g^*, g_*)} g_* g^* p^* f_* = g_* q^* f^* f_* \xrightarrow{ad'(f^*, f_*)} g_* q^*.$$

Assume moreover that p and q are $\mathscr{P}$-morphism. Then we can check that $Ex(\Delta_*^*)$ is the transpose of the exchange $Ex(\Delta_\sharp^*)$. Thus, when Δ is cartesian and p is a $\mathscr{P}$-morphism, $Ex(\Delta_*^*)$ is an isomorphism according to ($\mathscr{P}$-BC).

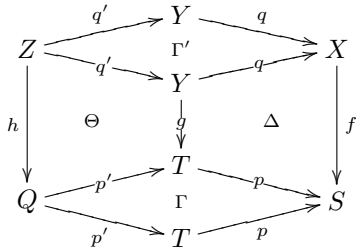
We can also define an exchange transformation:

$$Ex(\Delta_{\sharp *}) : p_\sharp g_* \xrightarrow{ad(f^*, f_*)} f_* f^* p_\sharp g_* \xrightarrow{Ex(\Delta_\sharp^*)^{-1}} f_* q_\sharp g^* g_* \xrightarrow{ad'(g^*, g_*)} f_* q_\sharp.$$

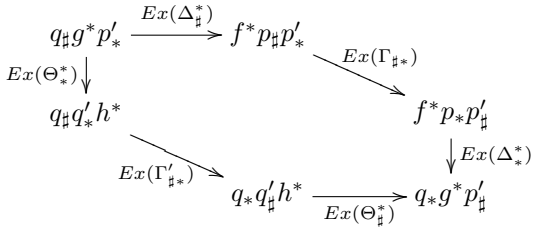
Remark 1.1.15. As in remark 1.1.7, we obtain coherence results for these exchange transformations.

First with respect to the identifications of the kind $f^* g^* = (gf)^*$, $(fg)_* = f_* g_*$, $(fg)_\sharp = f_\sharp g_\sharp$.

Secondly when several exchange transformations of different kind are involved. As an example, we consider the following commutative diagram in $\mathscr{S}$:



Then the following diagram of natural transformations is commutative:



We left the verification to the reader (it is based on the equations of the relevant adjunctions).

Definition 1.1.16. Let $\mathscr{M}$ be a complete $\mathscr{P}$-fibred category. Consider a commutative square in $\mathscr{S}$

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ {\scriptstyle g}\downarrow & \Delta & \downarrow{\scriptstyle f} \\ T & \xrightarrow[p]{} & S. \end{array}$$

We will say that Δ is *$\mathscr{M}$-transversal* if the exchange transformation

$$Ex(\Delta^*_*) : p^* f_* \to g_* q^*$$

of 1.1.14 is an isomorphism.

Given an admissible class of morphisms Q in $\mathscr{S}$, we say that $\mathscr{M}$ has the *transversality* (resp. *cotransversality*) *property with respect to Q-morphisms*, if, for any cartesian square Δ as above such that f is in Q (resp. p is in Q), Δ is $\mathscr{M}$-transversal.

Remark 1.1.17. Assume $\mathscr{S}$ is a sub-category of the category of schemes. When Q is the class of smooth morphisms (resp. proper morphisms),

the cotransversality (resp. transversality) property with respect to Q is usually called the *smooth base change property*

(resp. *proper base change property*).

Note finally that we get the following consequences of the axioms:

Proposition 1.1.18. *Any complete $\mathscr{P}$-fibred category has the cotransversality property with respect to $\mathscr{P}$.*

See paragraph 1.1.14.

Proposition 1.1.19. *If $\mathscr{M}$ is a $\mathscr{P}$-fibred category, then, for any monomorphism $j : U \to S$ in P, the functor $j_\sharp$ is fully faithful. If moreover $\mathscr{M}$ is complete, then the functor j_* is fully faithful as well.*

Proof. Because j is a monomorphism, we get a cartesian square in $\mathscr{S}$:

$$\begin{array}{ccc} U & = & U \\ \| & \Delta & \downarrow{\scriptstyle j} \\ U & \xrightarrow[j]{} & S. \end{array}$$

Remark that $Ex(\Delta^*_\sharp) : 1 \to j^* j_\sharp$ is the unit of the adjunction $(j_\sharp, j^*)$. Thus the $\mathscr{P}$-base change property shows that $j_\sharp$ is fully faithful.

Assume $\mathscr{M}$ is complete. We remark similarly that $Ex(\Delta^*_*) : j^* j_* \to 1$ is the counit of the adjunction (j^*, j_*). Thus, the above proposition shows readily that j_* is fully faithful. □

1.1.b. *Monoidal structures.* Let $\mathscr{C}at^\otimes$ be the sub-2-category of $\mathscr{C}at$ made of symmetric monoidal categories whose 1-morphisms are (strong) symmetric monoidal functors and 2-morphisms are symmetric monoidal transformations.

Definition 1.1.20. A *monoidal pre-$\mathscr{P}$-fibred category over $\mathscr{S}$* is a 2-functor

$$\mathscr{M} : \mathscr{S} \to \mathscr{C}at^{\otimes}$$

such that $\mathscr{M}$ is a pre-$\mathscr{P}$-fibred category.

In other words, $\mathscr{M}$ is a pre-$\mathscr{P}$-fibred category such that each of its fibers $\mathscr{M}(S)$ is endowed with a structure of a monoidal category, and any pullback morphism f^* is monoidal, with the obvious coherent structures. For an object S of $\mathscr{S}$, we will usually denote by $\otimes_S$ (resp. $\mathbb{1}_S$) the tensor product (resp. unit) of $\mathscr{M}(S)$.
In particular, we then have the following natural isomorphisms:

- for a morphism $f : T \to S$ in $\mathscr{S}$, and objects M, N of $\mathscr{M}(S)$,
$$f^*(M) \otimes_T f^*(N) \xrightarrow{\sim} f^*(M \otimes_S N);$$
- for a morphism $f : T \to S$ in $\mathscr{S}$,
$$f^*(\mathbb{1}_S) \xrightarrow{\sim} \mathbb{1}_T\,.$$

Convention 1.1.21. As in convention 1.1.3, we will generally consider that these structural isomorphisms are identities.

Example 1.1.22. Consider the notations of example 1.1.4.

Using the property (Pb) and (Pc) of $\mathscr{P}$, for two S-objects X and Y in $\mathscr{P}/S$, the cartesian product $X \times_S Y$ is an object of $\mathscr{P}/S$. This defines a symmetric monoidal structure on $\mathscr{P}/S$ with unit the trivial S-object S. Moreover, the functor f^* defined in *loc. cit.* is monoidal. Thus, the pre-$\mathscr{P}$-fibred category $\mathscr{P}/?$ is in fact monoidal.

1.1.23. *Monoidal exchange structures I.* Let $\mathscr{M}$ be a monoidal pre-$\mathscr{P}$-fibred category $\mathscr{M}$ over $\mathscr{S}$.

Consider a $\mathscr{P}$-morphism $f : T \to S$, and M (resp. N) an object of $\mathscr{M}(T)$ (resp. $\mathscr{M}(S)$).

We get a morphism in $\mathscr{M}(S)$

$$Ex(f_\sharp^*, \otimes) : f_\sharp(M \otimes_T f^*(N)) \longrightarrow f_\sharp(M) \otimes_S N$$

as the composition

$$f_\sharp(M \otimes_T f^*(N)) \to f_\sharp(f^* f_\sharp(M) \otimes_T f^*(N)) \simeq f_\sharp f^*(f_\sharp(M) \otimes_S N) \to f_\sharp(M) \otimes_S N\,.$$

This map is natural in M and N. It will be called the *exchange transformation* between $f_\sharp$ and $\otimes_T$.

Remark also that the functor $f_\sharp$, as a left adjoint of a symmetric monoidal functor, is colax symmetric monoidal: for any objects M and N of $\mathscr{M}(T)$, there is a canonical morphism

$$f_\sharp(M) \otimes_S f_\sharp(N) \to f_\sharp(M \otimes_T N) \tag{1.1.23.1}$$

natural in M and N, as well as a natural map

$$f_\sharp(\mathbb{1}_T) \to \mathbb{1}_S\,. \tag{1.1.23.2}$$

Remark 1.1.24. As in remark 1.1.7, the preceding exchange transformations satisfy a coherence condition for composable morphisms $W \xrightarrow{g} T \xrightarrow{f} S$. We get in fact a commutative diagram:

$$\begin{array}{ccccc}
(fg)_\sharp\big(M \otimes_S (fg)^*(N)\big) & & \xrightarrow{\quad Ex((fg)_\sharp^*, \otimes)\quad} & & \big((fg)_\sharp(M)\big) \otimes_W N \\
\| & & & & \| \\
f_\sharp g_\sharp\big(M \otimes_S g^* f^*(N)\big) & \xrightarrow{Ex(g_\sharp^*, \otimes)} & f_\sharp\big(g_\sharp(M) \otimes_T f^*(N)\big) & \xrightarrow{Ex(f_\sharp^*, \otimes)} & \big(f_\sharp g_\sharp(M)\big) \otimes_W N
\end{array}$$

As in remark 1.1.15, there is also a coherence relation when different kind of exchange transformations are involved. Consider a commutative square in $\mathscr{S}$

$$\begin{array}{ccc}
Y & \xrightarrow{q} & X \\
{\scriptstyle g}\downarrow & \Delta & \downarrow{\scriptstyle f} \\
T & \xrightarrow[p]{} & S
\end{array}$$

such that p and q are $\mathscr{P}$-morphisms and put $h = f \circ q = p \circ g$. Then the following diagram is commutative:

$$\begin{array}{ccccc} q_\sharp g^*(M \otimes_T p^*N) & \xrightarrow{Ex(\Delta_\sharp^*)} & f^*p_\sharp(M \otimes_T p^*N) & \xrightarrow{Ex(p_\sharp^*,\otimes)} & f^*(p_\sharp M \otimes_S N) \\ \| & & & & \| \\ q_\sharp(g^*M \otimes_Y q^*f^*N) & \xrightarrow{Ex(q_\sharp^*,\otimes)} & (q_\sharp g^*M) \otimes_X f^*N & \xrightarrow{Ex(\Delta_\sharp^*)} & (f^*p_\sharp M) \otimes_X f^*N \end{array}$$

We left the verification to the reader.

1.1.25. Under the assumptions of 1.1.23, we will consider the following property:

($\mathscr{P}$-PF) *$\mathscr{P}$-projection formula.*– For any $\mathscr{P}$-morphism $f : T \to S$ the exchange transformation

$$Ex(f_\sharp, \otimes_T) : f_\sharp(M \otimes_T f^*(N)) \to f_\sharp(M) \otimes_S N$$

is an isomorphism for all M and N.

Definition 1.1.26. A *monoidal* $\mathscr{P}$-fibred category over $\mathscr{S}$ is a monoidal pre-$\mathscr{P}$-fibred category $\mathscr{M} : \mathscr{S}^{op} \to \mathscr{C}at^{\otimes}$ over $\mathscr{S}$ which satisfies the $\mathscr{P}$-projection formula.

Example 1.1.27. Consider the canonical monoidal weak $\mathscr{P}$-fibred category $\mathscr{P}/?$ (see example 1.1.22). The transitivity property of pullbacks implies readily that $\mathscr{P}/?$ satisfies the property ($\mathscr{P}$-PF). Thus, $\mathscr{P}/?$ is in fact a monoidal $\mathscr{P}$-fibred category called *canonical*.

Definition 1.1.28. A monoidal $\mathscr{P}$-fibred category $\mathscr{M}$ over $\mathscr{S}$ is *complete* if it satisfies the following conditions:

(1) $\mathscr{M}$ is complete as a $\mathscr{P}$-fibred category.
(2) For any object S of $\mathscr{S}$, the monoidal category $\mathscr{M}(S)$ is closed (i.e. has an internal Hom).

In this case, we will usually denote by Hom_S the internal Hom in $\mathscr{M}(S)$, so that we have natural bijections

$$\mathrm{Hom}_{\mathscr{M}(S)}(A \otimes_S B, C) \simeq \mathrm{Hom}_{\mathscr{M}(S)}(A, Hom_S(B, C)) .$$

Example 1.1.29. The $\mathscr{P}$-fibred category $\mathscr{H}_\bullet$ of example 1.1.13 is in fact a complete monoidal $\mathscr{P}$-fibred category. The tensor product is given by the smash product (see [MV99]).

1.1.30. *Monoidal exchange structures II.*– Let $\mathscr{M}$ be a complete monoidal $\mathscr{P}$-fibred category.

Consider a morphism $f : T \to S$ in $\mathscr{S}$. Then we obtain an exchange transformation:

$$\begin{aligned} Ex(f_*^*, \otimes_S) : (f_*M) \otimes_S N &\xrightarrow{ad(f^*,f_*)} f_*f^*\big((f_*M) \otimes_S N\big) \\ &= f_*\big((f^*f_*M) \otimes_T f^*N\big) \xrightarrow{ad'(f^*,f_*)} f_*(M \otimes_T f^*N). \end{aligned}$$

Remark 1.1.31. As in remark 1.1.24, these exchange transformations are compatible with the identifications $(fg)_* = f_*g_*$ and $(fg)^* = g^*f^*$.
Moreover, there is a coherence relation when composing the exchange transformations of the kind $Ex(f_*^*, \otimes)$ with exchange transformations of the kind $Ex(\Delta_*^*)$ as in *loc. cit.*
Finally, note another kind of coherence relations involving $Ex(f_*^*, \otimes)$, $Ex(\Delta_\sharp^*)$ (resp. $Ex(f_\sharp^*, \otimes)$) and $Ex(\Delta_{\sharp *})$.
We left the formulation of these coherence relations to the reader, on the model of the preceding ones.

1.1.32. *Monoidal exchange structures III.*– Let $\mathscr{M}$ be a complete monoidal $\mathscr{P}$-fibred category and $f : T \to S$ be a morphism in $\mathscr{S}$.

Because f^* is monoidal, we get by adjunction a canonical isomorphism

$$Hom_S(M, f_*N) \to f_*Hom_T(f^*M, N).$$

Assume that f is a $\mathscr{P}$-morphism. Then from the $\mathscr{P}$-projection formula, we get by adjunction two canonical isomorphisms:

$$f^*Hom_S(M, N) \to Hom_T(f^*M, f^*N),$$
$$Hom_S(f_\sharp M, N) \to f_*Hom_T(M, f^*N)$$

1.1.c. *Geometric sections.*

1.1.33. Consider a weak $\mathscr{P}$-fibred category $\mathscr{M}$.

Let S be a scheme. For any $\mathscr{P}$-morphism $p : X \to S$, we put $M_S(X) := p_\sharp(\mathbb{1}_X)$. According to our conventions, this object is identified with $p_\sharp p^*(\mathbb{1}_S)$. In particular, it defines a covariant functor $M_S : P_S \to \mathscr{M}(S)$.

Consider a cartesian square in $\mathscr{S}$

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ {\scriptstyle q}\downarrow & \Delta & \downarrow{\scriptstyle p} \\ T & \xrightarrow[f]{} & S \end{array}$$

such that p is a $\mathscr{P}$-morphism. With the notations of example 1.1.4, $Y = f^*(X)$. Then we get a natural exchange transformation

$$Ex(M_T, f^*) : M_T(f^*(X)) = q_\sharp(\mathbb{1}_Y) = q_\sharp g^*(\mathbb{1}_X) \xrightarrow{Ex(\Delta^*_\sharp)} f^* p_\sharp(\mathbb{1}_X) = f^* M_S(X).$$

In other words, M defines a lax natural transformation $\mathscr{P}/? \to \mathscr{M}$.

Consider $\mathscr{P}$-morphisms $p : X \to S$, $q : Y \to S$. Let $Z = X \times_S Y$ be the cartesian product and consider the cartesian square:

$$\begin{array}{ccc} Z & \xrightarrow{p'} & Y \\ {\scriptstyle q'}\downarrow & \Theta & \downarrow{\scriptstyle q} \\ X & \xrightarrow{p} & S. \end{array}$$

Using the exchange transformations of the preceding paragraph, we get a canonical morphism

$$Ex(M_S, \otimes_S) : M_S(X \times_S Y) \longrightarrow M_S(X) \otimes_S M_S(Y)$$

as the composition

$$M_S(X \times_S Y) = p_\sharp q'_\sharp p'^*(\mathbb{1}_Y) \xrightarrow{Ex(\Theta^*_\sharp)} p_\sharp p^* q_\sharp(\mathbb{1}_Y) \quad p_\sharp(\mathbb{1}_X \otimes_X p^* q_\sharp(\mathbb{1}_Y))$$
$$\xrightarrow{Ex(p_\sharp, \otimes_X)} p_\sharp(\mathbb{1}_X) \otimes_S q_\sharp(\mathbb{1}_Y) = M_S(X) \otimes_S M_S(Y).$$

In other words, the functor M_S is symmetric colax monoidal.

Remark finally that for any $\mathscr{P}$-morphism $p : T \to S$, and any $\mathscr{P}$-object Y over T, we obtain according to convention an identification $p_\sharp M_T(Y) = M_S(X)$.

Definition 1.1.34. Given a monoidal pre-$\mathscr{P}$-fibred category $\mathscr{M}$ over $\mathscr{S}$, the lax natural transformation $M : \mathscr{P}/? \to \mathscr{M}$ constructed above will be called the *geometric sections of* $\mathscr{M}$.

The following lemma is obvious from the definitions above:

Lemma 1.1.35. *let $\mathscr{M}$ be a monoidal $\mathscr{P}$-fibred category. Let $M : \mathscr{P}/? \to \mathscr{M}$ be the geometric sections of $\mathscr{M}$. Then:*

(i) For any morphism $f : T \to S$ in $\mathscr{S}$, the exchange $Ex(M_T, f^*)$ defined above is an isomorphism.

(ii) For any scheme S, the exchange $Ex(M_S, \otimes_S)$ defined above is an isomorphism.

In other words, M is a cartesian functor and M_S is a (strong) *symmetric monoidal functor.*

1.1.36. In the situation of the lemma we thus obtain the following identifications:

- $f^* M_S(X) \simeq M_T(X \times_S T)$,
- $p_\sharp M_T(Y) \simeq M_S(X)$,
- $M_S(X \times_S Y) \simeq M_S(X) \otimes_S M_S(Y)$,

whenever it makes sense.

1.1.d. *Twists.*

1.1.37. Let $\mathscr{M}$ be a pre-$\mathscr{P}$-fibred category of $\mathscr{S}$. Recall that a cartesian section of $\mathscr{M}$ (*i.e.* a cartesian functor $A : \mathscr{S} \to \mathscr{M}$) is the data of an object A_S of $\mathscr{M}(S)$ for each object S of $\mathscr{S}$ and of isomorphisms

$$f^*(A_S) \xrightarrow{\sim} A_T$$

for each morphism $f : T \to S$, subject to coherence identities; see [Gro03, Exp. VI].

If $\mathscr{M}$ is monoidal, the tensor product of two cartesian sections is defined termwise.

Definition 1.1.38. let $\mathscr{M}$ be a monoidal pre-$\mathscr{P}$-fibred category. A set of *twists* τ for $\mathscr{M}$ is a set of cartesian sections of $\mathscr{M}$ stable by tensor product. For short, we say also that $\mathscr{M}$ is τ*-twisted*.

1.1.39. Let $\mathscr{M}$ be a monoidal pre-$\mathscr{P}$-fibred category endowed with a set of twists τ.

The tensor product on τ induces a monoid structure that we will denote by $+$ (the unit object of τ will be written 0).

Consider an object $i \in \tau$. For any object S of $\mathscr{S}$, we thus obtain an object i_S in $\mathscr{M}(S)$ associated to i. Given any object M of $\mathscr{M}(S)$, we simply put:

$$M\{i\} = M \otimes_S i_S$$

and call this object the twist of M by i. We have, by definition: $M\{0\} = M$.

For any $i, j \in \tau$, and any object M of $\mathscr{M}(S)$, we obtain $M\{i+j\} = (M\{i\})\{j\}$ – using the structural associativity isomorphism of the monoidal structure. Given a morphism $f : T \to S$, an object M of $\mathscr{M}(S)$ and a twist $i \in \tau$, we also obtain $f^*(M\{i\}) = (f^*M)\{i\}$. If f is a $\mathscr{P}$-morphism, for any object M of $\mathscr{M}(T)$, the exchange transformation $Ex(f_\sharp^*, \otimes_T)$ of paragraph 1.1.6 induces a canonical morphism

$$Ex(f_\sharp, \{i\}) : f_\sharp(M\{i\}) \to (f_\sharp M)\{i\}.$$

We will say that $f_\sharp$ commutes with τ-twists (or simply twists when τ is clear) if for any $i \in I$, the natural transformation $Ex(f_\sharp, \{i\})$ is an isomorphism.

Definition 1.1.40. Let $\mathscr{M}$ be a monoidal pre-$\mathscr{P}$-fibred category with a set of twists τ and $M : \mathscr{P}/? \to \mathscr{M}$ be the geometric sections of $\mathscr{M}$.

We say $\mathscr{M}$ is τ*-generated* if for any object S of $\mathscr{S}$, the family of functors

$$\mathrm{Hom}_{\mathscr{M}(S)}(M_S(X)\{i\}, -) : \mathscr{M}(S) \to \mathscr{S}et$$

indexed by a $\mathscr{P}$-object X/S and an element $i \in \tau$ is conservative.

Of course, we do not exclude the case where τ is trivial , but then, we shall simply say that $\mathscr{M}$ is *geometrically generated*.

We shall frequently use the following proposition to characterize complete monoidal $\mathscr{P}$-fibred categories over $\mathscr{S}$:

Proposition 1.1.41. *Let $\mathscr{M} : \mathscr{S} \to \mathscr{C}at^\otimes$ be a 2-functor such that:*

(1) For any $\mathscr{P}$-morphism $f : T \to S$, the pullback functor $f^* : \mathscr{M}(S) \to \mathscr{M}(T)$ is monoidal and admits a left adjoint $f_\sharp$ in $\mathscr{C}$.
(2) For any morphism $f : T \to S$, the pullback functor $f^* : \mathscr{M}(S) \to \mathscr{M}(T)$ admits a right adjoint f_* in $\mathscr{C}$.

We consider $\mathscr{M}$ as a monoidal weak $\mathscr{P}$-fibred category and denote by $M : \mathscr{P}/? \to \mathscr{M}$ its associated geometric sections. Suppose given a set of twists τ such that $\mathscr{M}$ is τ-generated. Then, the following assertions are equivalent:

(3) $\mathscr{M}$ satisfies properties ($\mathscr{P}$-BC) and ($\mathscr{P}$-PF)
*(*i.e. *$\mathscr{M}$ is a complete monoidal $\mathscr{P}$-fibred category.)*
(3') (a) M is a cartesian functor.
(b) For any object S of $\mathscr{S}$, M_S is (strongly) monoidal.
(c) For any $\mathscr{P}$-morphism f, $f_\sharp$ commutes with τ-twists.

Proof. $(i) \Rightarrow (ii)$: This is obvious (see lemma 1.1.35).

$(ii) \Rightarrow (i)$: We use the following easy lemma:

Lemma 1.1.42. *Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be categories, $F, G : \mathscr{C}_1 \to \mathscr{C}_2$ be two left adjoint functors and $\eta : F \to G$ be a natural transformation. Let $\mathcal{G}$ be a class of objects of $\mathscr{C}_1$ which is generating in the sense that the family of functors $\mathrm{Hom}_{\mathscr{C}_1}(X, -)$ for X in $\mathcal{G}$ is conservative.*

Then the following conditions are equivalent:

(1) η is an isomorphism.

(2) For all X in $\mathcal{G}$, η_X is an isomorphism.

Given this lemma, to prove ($\mathscr{P}$-BC), we are reduced to check that the exchange transformation $Ex(\Delta_\sharp^*)$ is an isomorphism when evaluated on an object $M_T(U)\{i\}$ for an object U of $\mathscr{P}/T$ and a twist $i \in \tau$. Then it follows from (ii), 1.1.39 and example 1.1.10.[4]
To prove ($\mathscr{P}$-PF), we proceed in two steps first proving the case $M = M_T(U)\{i\}$ and N any object of $\mathscr{M}(S)$ using the same argument as above with the help of 1.1.27. Then, we can prove the general case by another application of the same argument. □

Suppose given a complete monoidal $\mathscr{P}$-fibered category $\mathscr{M}$ with a set of twists τ. Let $f : T \to S$ be a morphism of $\mathscr{S}$. Then the exchange transformation 1.1.30 induces for any $i \in \tau$ an exchange transformation

$$Ex(f_*, \{i\}) : (f_* M)\{i\} \to f_*(M\{i\})\,.$$

Definition 1.1.43. In the situation above, we say that f_* *commutes with τ-twists* (or simply with twists when τ is clear) if for any $i \in \tau$, the exchange transformation $Ex(f_*, \{i\})$ is an isomorphism.

1.2. Morphisms of $\mathscr{P}$-fibred categories.

1.2.a. *General case.*

1.2.1. Consider two $\mathscr{P}$-fibred categories $\mathscr{M}$ and $\mathscr{M}'$ over $\mathscr{S}$, as well as a cartesian functor $\varphi^* : \mathscr{M} \to \mathscr{M}'$ between the underlying fibred categories: for any object S of $\mathscr{S}$, we have a functor

$$\varphi_S^* : \mathscr{M}(S) \to \mathscr{M}'(S)\,,$$

and for any map $f : T \to S$ in $\mathscr{S}$, we have an isomorphism of functors c_f

$$\begin{array}{ccc} \mathscr{M}(S) & \xrightarrow{\varphi_S^*} & \mathscr{M}'(S) \\ {\scriptstyle f^*}\downarrow & \not\Downarrow c_f & \downarrow{\scriptstyle f^*} \\ \mathscr{M}(T) & \xrightarrow[\varphi_T^*]{} & \mathscr{M}'(T) \end{array} \qquad c_f : f^*\,\varphi_S^* \xrightarrow{\sim} \varphi_T^*\,f^* \tag{1.2.1.1}$$

satisfying some cocycle condition with respect to composition in $\mathscr{S}$.

For any $\mathscr{P}$-morphism $p : T \to S$, we construct an exchange morphism

$$Ex(p_\sharp, \varphi^*) : p_\sharp\,\varphi_T^* \longrightarrow \varphi_S^*\,p_\sharp$$

as the composition

$$p_\sharp\varphi_T^* \xrightarrow{ad(p_\sharp, p^*)} p_\sharp\varphi_T^* p^* p_\sharp \xrightarrow{c_p^{-1}} p_\sharp p^* \varphi_S^* p_\sharp \xrightarrow{ad'(p_\sharp, p^*)} \varphi_S^* p_\sharp.$$

Definition 1.2.2. Consider the situation above. We say that the cartesian functor

$$\varphi^* : \mathscr{M} \to \mathscr{M}'$$

is a *morphism of $\mathscr{P}$-fibred categories* if, for any $\mathscr{P}$-morphism p, the exchange transformation $Ex(p_\sharp, \varphi^*)$ is an isomorphism.

Example 1.2.3. If $\mathscr{M}$ is a monoidal $\mathscr{P}$-fibred category, then the geometric sections $M : \mathscr{P}/? \to \mathscr{M}$ is a morphism of $\mathscr{P}$-fibred categories (1.1.35).

---

[4] The cautious reader will use remark 1.1.7 to check that the corresponding map

$$M_X(U \times_T Y)\{i\} \to M_X(U \times_T Y)\{i\}$$

is the identity.

Definition 1.2.4. Let $\mathscr{M}$ and $\mathscr{M}'$ be two complete $\mathscr{P}$-fibred categories. A *morphism of complete $\mathscr{P}$-fibred categories* is a morphism of $\mathscr{P}$-fibred categories

$$\varphi^* : \mathscr{M} \to \mathscr{M}'$$

such that, for any object S of $\mathscr{S}$, the functor $\varphi_S^* : \mathscr{M}(S) \to \mathscr{M}'(S)$ has a right adjoint

$$\varphi_{*,S} : \mathscr{M}'(S) \to \mathscr{M}(S)\,.$$

When we want to indicate a notation for the right adjoint of a morphism as above, we use the notation

$$\varphi^* : \mathscr{M} \rightleftarrows \mathscr{N} : \varphi_*$$

the left adjoint being in the left hand side.

1.2.5. *Exchange structures III.* Consider a morphism $\varphi^* : \mathscr{M} \to \mathscr{M}'$ of complete $\mathscr{P}$-fibred categories.

Then for any morphism $f : T \to S$ in $\mathscr{S}$, we define exchange transformations

$$Ex(\varphi^*, f_*) : \varphi_S^* f_* \longrightarrow f_* \varphi_T^*, \tag{1.2.5.1}$$

$$Ex(f^*, \varphi_*) : f^* \varphi_{*,S} \longrightarrow \varphi_{*,T} f^*, \tag{1.2.5.2}$$

as the respective compositions

$$\varphi_S^* f_* \xrightarrow{ad(f^*,f_*)} f_* f^* \varphi_S^* f_* \simeq f_* \varphi_T^* f^* f_* \xrightarrow{ad'(f^*,f_*)} f_* \varphi_T^*,$$

$$f^* \varphi_{*,S} \xrightarrow{ad(f^*,f_*)} f^* \varphi_{*,S} f_* f^* \simeq f^* f_* \varphi_{*,T} f^* \xrightarrow{ad'(f^*,f_*)} \varphi_{*,T} f^*.$$

Remark 1.2.6. We warn the reader that $\varphi_* : \mathscr{M}' \to \mathscr{M}$ is not a cartesian functor in general, meaning that the exchange transformation $Ex(f^*, \varphi_*)$ is not necessarily an isomorphism, even when f is a $\mathscr{P}$-morphism.

1.2.b. *Monoidal case.*

Definition 1.2.7. Let $\mathscr{M}$ and $\mathscr{M}'$ be monoidal $\mathscr{P}$-fibred categories.

A *morphisms of monoidal $\mathscr{P}$-fibred categories* is a morphism $\varphi^* : \mathscr{M} \to \mathscr{M}'$ of $\mathscr{P}$-fibred categories such that for any object S of $\mathscr{S}$, the functor $\varphi_S^* : \mathscr{M}(X) \to \mathscr{N}(S)$ has the structure of a (strong) symmetric monoidal functor, and such that the structural isomorphisms (1.2.1.1) are isomorphisms of symmetric monoidal functors.

In the case where $\mathscr{M}$ and $\mathscr{M}'$ are complete monoidal $\mathscr{P}$-fibred categories, we shall say that such a morphism φ^* is a *morphism of complete monoidal $\mathscr{P}$-fibred categories* if φ^* is also a morphism of complete $\mathscr{P}$-fibred categories.

Remark 1.2.8. If we denote by $M(-, \mathscr{M})$ and $M(-, \mathscr{M}')$ the geometric sections of $\mathscr{M}$ and $\mathscr{M}'$ respectively, we have a natural identification:

$$\varphi_S^*(M_S(X, \mathscr{M})) \simeq M_S(X, \mathscr{M}')\,.$$

1.2.9. *Monoidal exchange structures IV.* Consider a a morphism $\varphi^* : \mathscr{M} \to \mathscr{M}'$ of complete monoidal $\mathscr{P}$-fibred categories. For objects M (resp. N) of $\mathscr{M}(S)$ (resp. $\mathscr{M}'(S)$), we define an exchange transformation

$$Ex(\varphi_*, \otimes, \varphi^*) : (\varphi_{*,S} M) \otimes_S N \to \varphi_{*,S}(M \otimes_T \varphi_S^* N),$$

natural in M and N, as the following composite

$$(\varphi_{*,S} M) \otimes_S N \xrightarrow{ad(\varphi^*,\varphi_*)} \varphi_{*,S} \varphi_S^* ((\varphi_{*,S} M) \otimes_S N)$$

$$= \varphi_{*,S}((\varphi_S^* \varphi_{*,S} M) \otimes_T \varphi_S^* N) \xrightarrow{ad'(\varphi^*,\varphi_*)} \varphi_{*,S}(M \otimes_T \varphi_S^* N).$$

As in remark 1.1.31, we get coherence relations between the various exchange transformations associated with a morphism of monoidal $\mathscr{P}$-fibred categories. We left the formulation to the reader.

Note also that, because φ^* is monoidal, we get by adjunction a canonical isomorphism:

$$Hom_{\mathscr{M}(S)}(M, \varphi_{*,S} M') \xrightarrow{\sim} \varphi_{*,S} Hom_{\mathscr{M}'(S)}(\varphi_S^* M, M')\,.$$

1.2.10. Consider two monoidal $\mathscr{P}$-fibred categories $\mathscr{M}$, $\mathscr{M}'$ and a cartesian functor $\varphi^* : \mathscr{M} \to \mathscr{M}'$ such that, for any scheme S, $\varphi_S^* : \mathscr{M}(S) \to \mathscr{M}'(S)$ is monoidal.

Given a cartesian section $K = (K_S)_{S\in\mathscr{S}}$ of $\mathscr{M}$, we obtain for any morphism $f : T \to S$ in $\mathscr{S}$ a canonical map

$$f^*\varphi_S^*(K_S) = \varphi_T^*(f^*(K_S)) \to \varphi_T^*(K_T)$$

which defines a cartesian section of $\mathscr{M}'$, which we denote by $\varphi^*(K)$.

Definition 1.2.11. Let $(\mathscr{M}, \tau)$ and $(\mathscr{M}', \tau')$ be twisted monoidal $\mathscr{P}$-fibred categories. Let $\varphi^* : \mathscr{M} \to \mathscr{M}'$ be a cartesian functor as above (resp. a morphism of monoidal $\mathscr{P}$-fibred categories).

We say that $\varphi^* : (\mathscr{M}, \tau) \to (\mathscr{M}', \tau')$ commutes with twists (resp. is a morphism of twisted $\mathscr{P}$-fibred categories) if for any $i \in \tau$, the cartesian section $\varphi^*(i)$ is in τ' (up to isomorphism in $\mathscr{M}'$).

In particular, φ^* induces a morphism of monoids $\tau \to \tau'$ (if we consider the isomorphism classes of objects). Moreover, for any object K of $\mathscr{M}(S)$ and any twist $i \in \tau$, we get an identification:

$$\varphi_S^*(K\{i\}) \simeq (\varphi_S^*K)\{\varphi^*(i)\}.$$

Moreover, the exchange transformation $Ex(\varphi_*, \otimes)$ induces an exchange:

$$Ex(\varphi_*, \{i\}) : \varphi_{*,S}(K)\{i\} \to \varphi_{*,S}\big(K\{\varphi^*(i)\}\big).$$

When this transformation is an isomorphism for any twist $i \in \tau$, we say that φ_* commutes with twists.

Remark 1.2.12. In every examples, the morphism $\tau \to \tau'$ will be an explicit injection and we will cancel it in the notations of twists.

Note finally that lemma 1.1.42 allows to prove, as for proposition 1.1.41, the following useful lemma:

Lemma 1.2.13. *Consider two complete monoidal $\mathscr{P}$-fibred categories $\mathscr{M}$, $\mathscr{M}'$ and denote by $M(-, \mathscr{M})$ and $M(-, \mathscr{M}')$ their respective geometric sections. Let $\varphi^* : \mathscr{M} \to \mathscr{M}'$ be a cartesian functor such that*

(1) For any scheme S, $\varphi_S^* : \mathscr{M}(S) \to \mathscr{M}'(S)$ is monoidal.
(2) For any scheme S, φ_S^* admits a right adjoint $\varphi_{*,S}$.

Assume $\mathscr{M}$ (resp. $\mathscr{M}'$) is τ-generated (resp. τ'-twisted) and φ^* commutes with twists. Then the following conditions are equivalent:

(3) φ^* is a morphism of complete monoidal $\mathscr{P}$-fibred categories.
(3') For any object X of $\mathscr{P}/S$, the exchange transformation (cf. 1.2.1)

$$\varphi^* M_S(X, \mathscr{M}) \to M_S(X, \mathscr{M}')$$

is an isomorphism.

1.3. **Structures on $\mathscr{P}$-fibred categories.**

1.3.a. *Abstract definition.*

1.3.1. We fix a sub-2-category $\mathscr{C}$ of $\mathscr{C}at$ with the following properties[5]:

(1) the 2-functor

$$\mathscr{C}at \to \mathscr{C}at' , \quad A \mapsto A^{op}$$

sends $\mathscr{C}$ to $\mathscr{C}'$, where $\mathscr{C}'$ denotes the 2-category whose objects and maps are those of $\mathscr{C}$ and whose 2-morphisms are the 2-morphisms of $\mathscr{C}$, put in the reverse direction.

(2) $\mathscr{C}$ is closed under adjunction: for any functor $u : A \to B$ in $\mathscr{C}$, if a functor $v : B \to A$ is a right adjoint or a left adjoint to u, then v is in $\mathscr{C}$.

[5]See the following sections for examples.

(3) the 2-morphisms of $\mathscr{C}$ are closed by transposition: if

$$u : A \rightleftarrows B : v \text{ and } u' : A \rightleftarrows B : v'$$

are two adjunctions in $\mathscr{C}$ (with the left adjoints on the left hand side), a natural transformation $u \to u'$ is in $\mathscr{C}$ if and only if the corresponding natural transformation $v' \to v$ is in $\mathscr{C}$.

We can then define and manipulate $\mathscr{C}$-structured $\mathscr{P}$-fibred categories as follows.

Definition 1.3.2. A *$\mathscr{C}$-structured $\mathscr{P}$-fibred category* (resp. *$\mathscr{C}$-structured complete $\mathscr{P}$-fibred category*) $\mathscr{M}$ over $\mathscr{S}$ is simply a $\mathscr{P}$-fibred category (resp. a complete $\mathscr{P}$-fibred category) whose underlying 2-functor $\mathscr{M} : \mathscr{S}^{op} \to \mathscr{C}at$ factors through $\mathscr{C}$.

If $\mathscr{M}$ and $\mathscr{M}'$ are $\mathscr{C}$-structured fibred categories over $\mathscr{S}$, a cartesian functor $\mathscr{M} \to \mathscr{M}'$ is *$\mathscr{C}$-structured* if the functors $\mathscr{M}(S) \to \mathscr{M}'(S)$ are in $\mathscr{C}$ for any object S of $\mathscr{S}$, and if all the structural 2-morphisms (1.2.1.1) are in $\mathscr{C}$ as well.

Definition 1.3.3. A morphism of $\mathscr{C}$-structured $\mathscr{P}$-fibred categories (resp. $\mathscr{C}$-structured complete $\mathscr{P}$-fibred categories) is a morphism of $\mathscr{P}$-fibred categories (resp. of complete $\mathscr{P}$-fibred categories) which is $\mathscr{C}$-structured as a cartesian functor.

1.3.4. Consider a 2-category $\mathscr{C}$ as in the paragraph 1.3.1. In order to deal with the monoidal case, we will consider also a sub-2-category $\mathscr{C}^{\otimes}$ of $\mathscr{C}$ such that:

(1) The objects of $\mathscr{C}^{\otimes}$ are objects of $\mathscr{C}$ equipped with a symmetric monoidal structure;
(2) the 1-morphisms of $\mathscr{C}^{\otimes}$ are exactly the 1-morphisms of $\mathscr{C}$ which are symmetric monoidal as functors;
(3) the 2-morphisms of $\mathscr{C}^{\otimes}$ are exactly the 2-morphisms of $\mathscr{C}$ which are symmetric monoidal as natural transformations.

Note that $\mathscr{C}^{\otimes}$ satisfies condition (1) of 1.3.1, but it does not satisfies conditions (2) and (3) in general. Instead, we get the following properties:

(2′) If $u : A \to B$ is a functor in $\mathscr{C}^{\otimes}$, a right (resp. left) adjoint v is a lax[6] (resp. colax) monoidal functor in $\mathscr{C}$.
(3′) Consider adjunctions

$$u : A \rightleftarrows B : v \text{ and } u' : A \rightleftarrows B : v'$$

in $\mathscr{C}$ (with the left adjoints on the left hand side). If $u \to u'$ (resp. $v \to v'$) is a 2-morphism in $\mathscr{C}^{\otimes}$ then $v \to v'$ (resp. $u \to u'$) is a 2-morphism in $\mathscr{C}$ which is a symmetric monoidal transformation of lax (resp. colax) monoidal functors.

We thus adopt the following definition:

Definition 1.3.5. A *$(\mathscr{C}, \mathscr{C}^{\otimes})$-structured monoidal $\mathscr{P}$-fibred category* (resp. a *$(\mathscr{C}, \mathscr{C}^{\otimes})$-structured complete monoidal $\mathscr{P}$-fibred category*) is simply a monoidal $\mathscr{P}$-fibred category (resp. a complete monoidal $\mathscr{P}$-fibred category) whose underlying 2-functor $\mathscr{M} : \mathscr{S}^{op} \to \mathscr{C}at^{\otimes}$ factors through $\mathscr{C}^{\otimes}$. Morphisms of such objects are defined in the same way.

Note that, with the hypothesis made on $\mathscr{C}$, all the exchange natural transformations defined in the preceding paragraphs lie in $\mathscr{C}$ and satisfy the appropriate coherence property with respect to the monoidal structure.

1.3.b. *The abelian case.*

1.3.6. Let $\mathscr{A}b$ be the sub-2-category of $\mathscr{C}at$ made of the abelian categories, with the additive functors as 1-morphisms, and the natural transformations as 2-morphisms. Obviously, it satisfies properties of 1.3.1. When we will apply one of the definitions 1.3.2, 1.3.3 to the case $\mathscr{C} = \mathscr{A}b$, we will use the simple adjective *abelian* for $\mathscr{A}b$-structured.

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[6]For any object a, a' in A, F is lax if there exists a structural map $F(a) \otimes F(a') \xrightarrow{(1)} F(a \otimes a')$ satisfying coherence relations (see [Mac98, XI. 2]). Colax is defined by reversing the arrow (1).

Let $\mathscr{A}b^{\otimes}$ be the sub-2-category of $\mathscr{A}b$ made of the abelian monoidal categories, with 1-morphisms the symmetric monoidal additive functors and 2-morphisms the symmetric monoidal natural transformations. It satisfies the hypothesis of paragraph 1.3.4. When we will apply definition 1.3.5 to the case of $(\mathscr{A}b, \mathscr{A}b^{\otimes})$, we will use the simple expression *monoidal abelian* for $(\mathscr{A}b, \mathscr{A}b^{\otimes})$-structured monoidal.

Lemma 1.3.7. *Consider an abelian $\mathscr{P}$-fibred category $\mathscr{A}$ such that for any object S of $\mathscr{S}$, $\mathscr{A}(S)$ is a Grothendieck abelian category. Then the following conditions are equivalent:*

(i) $\mathscr{A}$ is complete.
(ii) For any morphism $f : T \to S$ in $\mathscr{S}$, f^* commutes with sums.

If in addition, $\mathscr{A}$ is monoidal, the following conditions are equivalent:

(i′) $\mathscr{A}$ is monoidal complete.
(ii′) (a) For any morphism $f : T \to S$ in $\mathscr{S}$, f^* is right exact.
(b) For any object S of $\mathscr{S}$, the bifunctor $\otimes_S$ is right exact.

In view of this lemma, we adopt the following definition:

Definition 1.3.8. A Grothendieck abelian (resp. Grothendieck abelian monoidal) $\mathscr{P}$-fibred category $\mathscr{A}$ over $\mathscr{S}$ is an abelian $\mathscr{P}$-fibred category which is complete (resp. complete monoidal) and such that for any scheme S, $\mathscr{A}(S)$ is a Grothendieck abelian category.

Remark 1.3.9. Let $\mathscr{A}$ be a Grothendieck abelian monoidal $\mathscr{P}$-fibred category. Conventionally, we will denote by $M_S(-, \mathscr{A})$ its geometric sections. Note that if $\mathscr{A}$ is τ-twisted, then any object of $\mathscr{A}$ is a quotient of a direct sum of objects of shape $M_S(X, \mathscr{A})\{i\}$ for a $\mathscr{P}$-object X/S and a twist $i \in \tau$.

1.3.c. *The triangulated case.*

1.3.10. Let $\mathscr{T}ri$ be the sub-2-category of $\mathscr{C}at$ made of the triangulated categories, with the triangulated functors as 1-morphisms, and the triangulated natural transformations as 2-morphisms. Then $\mathscr{T}ri$ satisfies the properties of 1.3.1 (property (2) can be found for instance in [Ayo07a, Lemma 2.1.23], and we leave property (3) as an exercise for the reader). When we will apply one of the definitions 1.3.2, 1.3.3 to the case $\mathscr{C} = \mathscr{T}ri$, we will use the simple adjective *triangulated* for $\mathscr{T}ri$-structured.

Let $\mathscr{T}ri^{\otimes}$ be the sub-2-category of $\mathscr{T}ri$ made of the triangulated monoidal categories, with 1-morphisms the symmetric monoidal triangulated functors and 2-morphisms the symmetric monoidal natural transformations. It satisfies the hypothesis of paragraph 1.3.4. When we will apply definition 1.3.5 to the case of $(\mathscr{T}ri, \mathscr{T}ri^{\otimes})$, we will use the expression *monoidal triangulated* for $(\mathscr{T}ri, \mathscr{T}ri^{\otimes})$-structured monoidal.

Convention 1.3.11. The set of twists of a triangulated monoidal $\mathscr{P}$-fibred category $\mathscr{T}$ will always be of the form $\mathbf{Z} \times \tau$ where the first factor corresponds to the cartesian sections defined by suspensions $\mathbb{1}[n]$, $n \in \mathbf{Z}$. In the notation, we shall often make the abuse of only indicating τ. In particular, the expression $\mathscr{T}$ is τ-generated will mean conventionally that $\mathscr{T}$ is $(\mathbf{Z} \times \tau)$-generated in the sense of definition 1.1.40.

1.3.12. Consider a triangulated category $\mathscr{T}$ which admits small sums. Recall the following definitions:
An object X of $\mathscr{T}$ is called *compact* if the functor $\mathrm{Hom}_{\mathscr{T}}(X, -)$ commutes with small sums. A class $\mathcal{G}$ of objects of $\mathscr{T}$ is called generating if the family of functor $\mathrm{Hom}_{\mathscr{T}}(X[n], -)$, $X \in \mathcal{G}$, $n \in \mathbf{Z}$, is conservative.
The triangulated category $\mathscr{T}$ is called *compactly generated* if there exists a generating set $\mathcal{G}$ of compact objects of $\mathscr{T}$. This property of being compact has been generalized by A. Neeman to the property of being α-*small* for some cardinal α (*cf.* [Nee01, 4.1.1]) – recall compact=$\aleph_0$-small. Then the property of being compactly generated has been generalized by Neeman to the property of being *well generated*; see [Kra01] for a convenient characterization of well generated triangulated categories.

Definition 1.3.13. Let $\mathscr{T}$ be a triangulated $\mathscr{P}$-fibred category over $\mathscr{S}$. We say that $\mathscr{T}$ is *compactly generated* (resp. *well generated*) if for any object S of $\mathscr{S}$, $\mathscr{T}(S)$ admits small sums and is compactly generated (resp. well generated).

Remark 1.3.14. Remember that a monoidal triangulated $\mathscr{P}$-fibred category is compactly τ-generated in the sense of definition 1.1.40 if it is compactly generated in the sense of the previous definition and for any $\mathscr{P}$-object X/S, any twist $i \in \tau$, $M_S(X)\{i\}$ is compact.

1.3.15. For a triangulated category $\mathscr{T}$ which has small sums, given a family $\mathcal{G}$ of objects of $\mathscr{T}$, we denote by $\langle \mathcal{G} \rangle$ the localizing subcategory of $\mathscr{T}$ generated by $\mathcal{G}$, i.e. $\langle \mathcal{G} \rangle$ is the smallest triangulated full subcategory of $\mathscr{T}$ which is stable by small sums and which contains all the objects in $\mathcal{G}$. Recall that, in the case $\mathscr{T}$ is well generated (e.g. if $\mathscr{T}$ compactly generated), then the family $\mathcal{G}$ generates $\mathscr{T}$ (in the sense that the family of functors $\{\mathrm{Hom}_{\mathscr{T}}(X, -)\}_{X \in \mathcal{G}}$ is conservative) if and only if $\mathscr{T} = \langle \mathcal{G} \rangle$. The following lemma is a consequence of [Nee01]:

Lemma 1.3.16. *Let $\mathscr{T}$ be a triangulated monoidal $\mathscr{P}$-fibred category over $\mathscr{S}$ with geometric sections M. Assume $\mathscr{T}$ is τ-generated.*

If $\mathscr{T}$ is well generated, then for any object S of $\mathscr{S}$,

$$\mathscr{T}(S) = \langle M_S(X)\{i\}; X/S \text{ a } \mathscr{P}\text{-object}, i \in \tau \rangle$$

Moreover, there exists a regular cardinal α such that all the objects of shape $M_S(X)\{i\}$ are α-compact.

Note finally that the Brown representability theorem of Neeman (*cf.* [Nee01]) gives the following lemma (analog of 1.3.7):

Lemma 1.3.17. *Consider a well generated triangulated $\mathscr{P}$-fibred category $\mathscr{T}$. Then the following conditions are equivalent:*

(i) $\mathscr{T}$ is complete.
(ii) For any morphism $f : T \to S$ in $\mathscr{S}$, f^* commutes with sums.

If in addition, $\mathscr{T}$ is monoidal, the following conditions are equivalent:

(i′) $\mathscr{T}$ is monoidal complete.
(ii′) (a) For any morphism $f : T \to S$ in $\mathscr{S}$, f^* is right exact.
(b) For any object S of $\mathscr{S}$, the bifunctor $\otimes_S$ is right exact.

We finish this section with a proposition which will constitute a useful trick:

Proposition 1.3.18. *Consider an adjunction of triangulated categories*

$$a : \mathscr{T} \rightleftarrows \mathscr{T}' : b.$$

Assume that $\mathscr{T}$ admits a set of compact generators $\mathcal{G}$ such that any object in $a(\mathcal{G})$ in compact in $\mathscr{T}'$. Then b commutes with direct sums. If in addition $\mathscr{T}'$ is well generated then b admits a right adjoint.

Proof. The second assertion follows from the first one according to a corollary of the Brown representability theorem of Neeman (*cf.* [Nee01, 8.4.4]).

For the first one, we consider a family $(X_i)_{i \in I}$ of objects of $\mathscr{T}'$ and prove that the canonical morphism

$$\oplus_{i \in I} b(X_i) \to b\,(\oplus_{i \in I} X_i)$$

is an isomorphism in $\mathscr{T}$. To prove this, it is sufficient to apply the functor $\mathrm{Hom}_{\mathscr{T}}(G, -)$ for any object G of $\mathcal{G}$. Then the result is obvious from the assumptions. □

We shall use often the following standard argument to produce equivalences of triangulated categories.

Corollary 1.3.19. *Let $a : \mathscr{T} \to \mathscr{T}'$ be a triangulated functor between triangulated categories. Assume that the functor a preserves small sums, and that $\mathscr{T}$ admits a small set of compact*

generators $\mathcal{G}$, such that $a(\mathcal{G})$ form a family of compact objects in $\mathscr{T}'$. Then a is fully faithful if and only if, for any couple of objects G and G' in $\mathcal{G}$, the map

$$\mathrm{Hom}_{\mathscr{T}}(G, G'[n]) \to \mathrm{Hom}_{\mathscr{T}'}(a(G), a(G')[n])$$

is bijective for any integer n. If a is fully faithful, then a is an equivalence of categories if and only if $a(\mathcal{G})$ is a generating family in $\mathscr{T}'$.

Proof. Let us prove that this is a sufficient condition. As $\mathscr{T}$ is in particular well generated, by the Brown representability theorem, the functor b admits a right adjoint $b : \mathscr{T}' \to \mathscr{T}$. By virtue of the preceding proposition, the functor b preserves small sums. Let us prove that a is fully faithful. We have to check that, for any object M of $\mathscr{T}$, the map $M \to b(a(M))$ is invertible. As a and b are triangulated and preserve small sums, it is sufficient to check this when M runs over a generating family of objects of $\mathscr{T}$ (e.g. $\mathcal{G}$). As $\mathcal{G}$ is generating, it is sufficient to prove that the map

$$\mathrm{Hom}_{\mathscr{T}}(G, M[n]) \to \mathrm{Hom}_{\mathscr{T}'}(a(G), a(M)[n]) = \mathrm{Hom}_{\mathscr{T}'}(a(G), b(a(M))[n])$$

is bijective for any integer n, which hold then by assumption. The functor a thus identifies $\mathscr{T}$ with the localizing subcategory of $\mathscr{T}'$ generated by $a(\mathcal{G})$; if moreover $a(\mathcal{G})$ is a generating family in $\mathscr{T}'$, then $\mathscr{T}' = \langle a(\mathcal{G})\rangle$, which also proves the last assertion. □

1.3.d. *The model category case.*

1.3.20. We shall use Hovey's book [Hov99] for a general reference to the theory of model categories. Note that, following *loc. cit.*, all the model categories we shall consider will have small limits and small colimits.

Let $\mathscr{M}$ be the sub-2-category of $\mathscr{C}at$ made of the model categories, with 1-morphisms the left Quillen functors and 2-morphisms the natural transformations. When we will apply definition 1.3.2 (resp. 1.3.3) to $\mathscr{C} = \mathscr{M}$, we will speak of a *$\mathscr{P}$-fibred model category* for a $\mathscr{M}$-structured $\mathscr{P}$-fibred category $\mathscr{M}$ (resp. morphism of $\mathscr{P}$-fibred model categories). Note that according to the definition of left Quillen functors, $\mathscr{M}$ is then automatically complete.

Given a property (P) of model categories (like being cofibrantly generated, left and/or right proper, combinatorial, stable, etc), we will say that a P-fibred model category $\mathscr{M}$ over $\mathscr{S}$ has the property (P) if, for any object S of $\mathscr{S}$, the model category $\mathscr{M}(S)$ has the property (P).

For the monoidal case, we let $\mathscr{M}^{\otimes}$ be the sub-2-categories of $\mathscr{M}$ made of the symmetric monoidal model categories (see [Hov99, Definition 4.2.6]), with 1-morphisms the symmetric monoidal left Quillen functors and 2-morphisms the symmetric monoidal natural transformations, following the conditions of 1.3.4. When we will apply definition 1.3.5 to the case of $(\mathscr{M}, \mathscr{M}^{\otimes})$, we will speak simply of a *monoidal $\mathscr{P}$-fibred model category* for a $(\mathscr{M}, \mathscr{M}^{\otimes})$-structured monoidal $\mathscr{P}$-fibred category $\mathscr{M}$. Again, $\mathscr{M}$ is then monoidal complete.

Remark 1.3.21. Let $\mathscr{M}$ be a $\mathscr{P}$-fibred model category over $\mathscr{S}$. Then for any $\mathscr{P}$-morphism $p : X \to Y$, the inverse image functor $p^* : \mathscr{M}(Y) \to \mathscr{M}(X)$ has very strong exactness properties: it preserves small limits and colimits (having both a left and a right adjoint), and it preserves weak equivalences, cofibrations, and fibrations. The only non (completely) trivial assertion here is about the preservation of weak equivalences. For this, one notices first that it preserves trivial cofibrations and trivial fibrations (being both a left Quillen functor and a right Quillen functor). In particular, by virtue of Ken Brown Lemma [Hov99, Lemma 1.1.12], it preserves weak equivalences between cofibrant (resp. fibrant) objects. Given a weak equivalence $u : M \to N$ in $\mathscr{M}(Y)$, we can find a commutative square

$$\begin{array}{ccc} M' & \xrightarrow{u'} & N' \\ \downarrow & & \downarrow \\ M & \xrightarrow[u]{} & N \end{array}$$

in which the two vertical maps are trivial fibrations, and where u' is a weak equivalence between cofibrant objects, from which we deduce easily that $p^*(u)$ is a weak equivalence in $\mathscr{M}(X)$.

1.3.22. Consider a $\mathscr{P}$-fibred model category $\mathscr{M}$ over $\mathscr{S}$. By assumption, we get the following pairs of adjoint functors:

(a) For any morphism $f : X \to S$ of $\mathscr{S}$,

$$\mathbf{L}f^* : \mathrm{Ho}(\mathscr{M}(S)) \rightleftarrows \mathrm{Ho}(\mathscr{M}(X)) : \mathbf{R}f_*$$

(b) For any P-morphism $p : T \to S$, the pullback functor

$$\mathbf{L}p_\sharp : \mathrm{Ho}(\mathscr{M}(S)) \rightleftarrows \mathrm{Ho}(\mathscr{M}(T)) : \mathbf{L}p^* = p^* = \mathbf{R}p^*$$

Moreover, the canonical isomorphism of shape $(fg)^* \simeq g^*f^*$ induces a canonical isomorphism $\mathbf{R}(fg)^* \simeq \mathbf{R}g^*\mathbf{R}f^*$. In the situation of the $\mathscr{P}$-base change formula 1.1.8, we obtain also that the base change map

$$\mathbf{L}q_\sharp\mathbf{L}g^* \to \mathbf{L}f^*\mathbf{L}p_\sharp$$

is an isomorphism from the equivalent property of $\mathscr{M}$. Thus, we have defined a complete $\mathscr{P}$-fibred category whose fiber over S is $\mathrm{Ho}(\mathscr{M}(S))$.

Definition 1.3.23. Given a $\mathscr{P}$-fibred model category $\mathscr{M}$ as above, the complete $\mathscr{P}$-fibred category defined above will be denoted by $\mathrm{Ho}(\mathscr{M})$ and called the *homotopy $\mathscr{P}$-fibred category* associated with $\mathscr{M}$.

1.3.24. Assume that $\mathscr{M}$ is a monoidal $\mathscr{P}$-fibred model category over $\mathscr{S}$. Then, for any object S of $\mathscr{S}$, $\mathrm{Ho}(\mathscr{M})(S)$ has the structure of a symmetric closed monoidal category; see [Hov99, Theorem 4.3.2]. The (derived) tensor product of $\mathrm{Ho}(\mathscr{M})(S)$ will be denoted by $M \otimes^{\mathbf{L}}_S N$, and the (derived) internal Hom will be written $\mathbf{R}Hom_S(M, N)$, while the unit object will be written $\mathbb{1}_S$.

For any morphism $f : T \to S$ in $\mathscr{S}$, the derived functor $\mathbf{L}f^*$ is symmetric monoidal as follows from the equivalent property of its homologue f^*.

Moreover, for any P-morphism $p : T \to S$ and for any object M in $\mathrm{Ho}(\mathscr{M})(T)$ and any object N in $\mathrm{Ho}(\mathscr{M})(S)$, the exchange map of 1.1.23

$$\mathbf{L}p_\sharp(M \otimes^{\mathbf{L}} p^*(N)) \to \mathbf{L}p_\sharp(M) \otimes^{\mathbf{L}} N$$

is an isomorphism.

Definition 1.3.25. Given a monoidal $\mathscr{P}$-fibred model category $\mathscr{M}$ as above, the complete monoidal $\mathscr{P}$-fibred category defined above will be denoted by $\mathrm{Ho}(\mathscr{M})$ and called the *monoidal homotopy $\mathscr{P}$-fibred category* associated with $\mathscr{M}$.

1.4. **Premotivic categories.** In the present article, we will focus on a particular type of $\mathscr{P}$-fibred category.

1.4.1. Let $\mathcal{S}$ be a scheme. Assume $\mathscr{S}$ is a full subcategory of the category of $\mathcal{S}$-schemes. We let $\mathscr{S}^{ft}$ be the class of morphisms of finite type in $\mathscr{S}$ and Sm be the class of smooth morphisms of finite type in $\mathscr{S}$. In practice, the classes Sm and $\mathscr{S}^{ft}$ are admissible in $\mathscr{S}$ in the sense of 1.0 (this is automatic, for instance, if $\mathscr{S}$ is stable by pullbacks).

Definition 1.4.2. Let $\mathscr{P}$ be an admissible class of morphisms in $\mathscr{S}$.

A *$\mathscr{P}$-premotivic category over $\mathscr{S}$* – or simply *$\mathscr{P}$-premotivic category* when $\mathscr{S}$ is clear – is a complete monoidal $\mathscr{P}$-fibred category over $\mathscr{S}$ endowed with a small set of twists τ such that $\mathscr{M}$ is τ-generated.

We will also say: *premotivic* for Sm-premotivic and *generalized premotivic* for $\mathscr{S}^{ft}$-premotivic.

The sections of a $\mathscr{P}$-premotivic category will be called *premotives*.

Given a 2-category $\mathscr{C}$ as in 1.3.1, we define similarly the notion of a $\mathscr{C}$-structured $\mathscr{P}$-premotivic (resp. premotivic, generalized premotivic) category. This will be particularly applied in the abelian and triangulated cases (*cf.* respectively 1.3.6 and 1.3.10). In partcilular, we shall consider *compactly τ-generated* triangulated premotivic categories: these are the triangulated premotivic categories $\mathscr{T}$, such that, for any scheme S in $\mathscr{S}$, the objects $M_S(X)\{n\}$, for X/S smooth of finite type, and $n \in \tau$, form a generating family of compact objects (in particular, such a $\mathscr{T}$ is compactly generated as a triangulated $\mathscr{P}$-fibred category; see 1.3.12).

Example 1.4.3. Let $\mathscr{S}$ be the category of noetherian schemes of finite dimension.

For such a scheme S, recall $\mathscr{H}_\bullet(S)$ is the pointed homotopy category of Morel and Voevodsky; *cf.* examples 1.1.5, 1.1.13, 1.1.29. Then, according to the fact recalled in these examples the 2-functor $\mathscr{H}_\bullet$ is an $\mathbf{N}$-generated premotivic category.

For such a scheme S, consider the stable homotopy category $\mathrm{SH}(S)$ of Morel and Voevodsky (see [Jar00, Ayo07b]). According to [Ayo07b], it defines a triangulated premotivic category denoted by SH. Moreover, it is compactly $(\mathbf{Z}\times\mathbf{Z})$-generated in the sense of definition 1.1.40 where the first factor refers to the suspension and the second one refers to the Tate twist (*i.e.* as a triangulated premotivic category, it is compactly generated by the Tate twists).

Definition 1.4.4. Let $\mathscr{M}$ and $\mathscr{M}'$ be $\mathscr{P}$-premotivic categories.

A *premotivic morphism* from $\mathscr{M}$ to $\mathscr{M}'$ is a morphism $\varphi^* : \mathscr{M} \to \mathscr{M}'$ of twisted complete monoidal $\mathscr{P}$-fibred categories. We shall also say that

$$\varphi^* : \mathscr{M} \rightleftarrows \mathscr{M}' : \varphi_*$$

is a premotivic adjunction.

Given a 2-category $\mathscr{C}$ as in 1.3.1, we define similarly the notion of a morphism (resp. adjunction) of $\mathscr{C}$-structured $\mathscr{P}$-premotivic (resp. premotivic, generalized premotivic) category.

Example 1.4.5. With the hypothesis and notations of 1.4.3, we get a premotivic adjunction

$$\Sigma^\infty : \mathscr{H}_\bullet \rightleftarrows \mathrm{SH} : \Omega^\infty$$

induced by the infinite suspension functor according to [Jar00].

1.4.6. Let $\mathscr{T}$ (resp. $\mathscr{A}$) be an triangulated $\mathscr{P}$-premotivic category with geometric sections M and set of twists τ. For any scheme S, we let $\mathscr{T}_{\tau-gm}(S)$ be the smallest triangulated thick subcategory of $\mathscr{T}(S)$ which contains premotives of shape $M_S(S)\{i\}$ (resp. $M_S(X,\mathscr{A})\{i\}$) for a $\mathscr{P}$-scheme X/S and a twist $i\in\tau$. This subcategory is stable by the operations f^*, $p_\sharp$ and $\otimes$. In particular, $\mathscr{T}_c$ defines a *not necessarily complete* triangulated (resp. abelian) $\mathscr{P}$-fibred category over $\mathscr{S}$. We also obtain a morphism of triangulated (resp. abelian) monoidal $\mathscr{P}$-fibred categories, fully faithful as a functor,

$$\iota : \mathscr{T}_c \to \mathscr{T}$$

Definition 1.4.7. Consider the notations introduced above. We will call $\mathscr{T}_c$ the *τ-constructible part of $\mathscr{T}$*. For any scheme S, the objects of $\mathscr{T}_c(S)$ will be called *τ-constructible*, or simply, *constructible*.

When τ is clear from the context, we will not indicate it in the notation or terminology of this definition. Note also that if $\mathscr{T}$ is compactly τ-generated, then τ-geometric premotives over S coincide with the compact objects of $\mathscr{T}(S)$. Thus, in this case, the $\mathscr{P}$-fibred sub-category of τ-constructible premotives does not depend on τ so that the preceding abuse of notation is fully legitimated (this will be the case in practice).

Definition 1.4.8. Consider a τ-generated premotivic category $\mathscr{M}$.

An enlargement of $\mathscr{M}$ is the data of a τ'-twisted generalized premotivic category $\underline{\mathscr{M}}$ together with a premotivic adjunction

$$\rho_\sharp : \mathscr{M} \longrightarrow \underline{\mathscr{M}} : \rho^*$$

(where $\underline{\mathscr{M}}$ is considered as a premotivic category in the obvious way), satisfying the following properties:

(a) For any scheme S in $\mathscr{S}$, the functor $\rho_{\sharp,S} : \mathscr{M}(S) \to \underline{\mathscr{M}}(S)$ is fully faithful and its right adjoint $\rho_S^* : \underline{\mathscr{M}}(S) \to \mathscr{M}(S)$ commutes with sums.

(b) $\rho_\sharp$ induces an equivalence $\tau \simeq \tau'$.

Again, this notion is defined similarly for a $\mathscr{C}$-structured $\mathscr{P}$-premotivic category.

Note that for any smooth S-scheme X, we get in the context of an enlargement as above the following identifications:

$$\rho_{\sharp,S}(M_S(X)) \simeq \underline{M}_S(X),$$
$$\rho_S^*(\underline{M}_S(X)) \simeq M_S(X)$$

where M (resp. $\underline{M}$) denote the geometric sections of $\mathscr{M}$ (resp. $\underline{\mathscr{M}}$).

Remember also that for any morphism of schemes f and any smooth morphism p, $\rho_\sharp$ commutes with f^* and $p_\sharp$, while ρ^* commutes with f_* and p^*.

2. Triangulated $\mathscr{P}$-fibred categories in algebraic geometry

2.0. In this entire section, we fix a base scheme $\mathcal{S}$ assumed to be noetherian and a full subcategory $\mathscr{S}$ of the category of noetherian $\mathcal{S}$-schemes satisfying the following properties:

(a) $\mathscr{S}$ is closed under finite sums and pullback along morphisms of finite type.
(b) For any scheme S in $\mathscr{S}$, any quasi-projective S-scheme belongs to $\mathscr{S}$.

In sections 2.2 and 2.4, we will add the following assumption on $\mathscr{S}$:

(c) Any separated morphism $f : Y \to X$ in $\mathscr{S}$, admits a compactification in $\mathscr{S}$ in the sense of [AGV73, 3.2.5], i.e. admits a factorization of the form

$$Y \xrightarrow{j} \bar{Y} \xrightarrow{p} X$$

where j is an open immersion, p is proper, and $\bar{Y}$ belongs to $\mathscr{S}$. Furthermore, if f is quasi-projective, then p can be chosen to be projective.

(d) Chow's lemma holds in $\mathscr{S}$ (i.e., for any proper morphism $Y \to X$ in $\mathscr{S}$, there exists a projective birational morphism $p : Y_0 \to Y$ in $\mathscr{S}$ such that fp is projective as well).

A category $\mathscr{S}$ satisfying all these properties will be called *adequate* for future references.[7]

We also fix an admissible class $\mathscr{P}$ of morphisms in $\mathscr{S}$ and a complete triangulated $\mathscr{P}$-fibred category $\mathscr{T}$. We will add the following assumptions:

(d) In section 2.2 and 2.3, $\mathscr{P}$ contains the open immersions.
(e) In section 2.4, $\mathscr{P}$ contains the smooth morphisms of $\mathscr{S}$.

In the case $\mathscr{T}$ is monoidal, we denote by

$$M : \mathscr{P}/? \to \mathscr{T}$$

its geometric sections.

According to the convention of 1.4.2, we will speak of the *premotivic case* when $\mathscr{P}$ is the class of smooth morphisms in $\mathscr{S}$ and $\mathscr{T}$ is a premotivic triangulated category.

2.1. **Elementary properties.**

Definition 2.1.1. We say that $\mathscr{T}$ is additive, if for any finite family $(S_i)_{i\in\mathscr{I}}$ of schemes in $\mathscr{S}$, the canonical map

$$\mathscr{T}\left(\coprod_i S_i\right) \to \prod_i \mathscr{T}(S_i)$$

is an equivalence.

Recall this property implies in particular that $\mathscr{T}(\varnothing) = 0$.

Lemma 2.1.2. *Let S be a scheme, $p : \mathbf{A}^1_S \to S$ be the canonical projection. The following conditions are equivalent:*

(i) The functor $p^* : \mathscr{T}(S) \to \mathscr{T}(\mathbf{A}^1_S)$ is fully faithful.
(ii) The counit adjunction morphism $1 \to p_*p^*$ is an isomorphism.

In the premotivic case, these conditions are equivalent to the following ones:

(iii) The unit adjunction morphism $p_\sharp p^* \to 1$ is an isomorphism.
(iv) The morphism $M_S(\mathbf{A}^1_S) \xrightarrow{p_*} \mathbb{1}_S$ induced by p is an isomorphism.
(iv') For any smooth S-scheme X, the morphism $M_S(\mathbf{A}^1_X) \xrightarrow{(1_X\times p)_*} M_S(X)$ is an isomorphism.

[7] For instance, the scheme $\mathcal{S}$ can be the spectrum of a prime field or of a Dedekind domain. The category $\mathscr{S}$ might be the category of all noetherian $\mathcal{S}$-schemes of finite dimension or simply the category of quasi-projective $\mathcal{S}$-schemes. In all these cases, property (c) is ensured by Nagata's theorem (see [Con07]) and property (d) by Chow's lemma (see [GD61, 5.6.1]).

The only thing to recall is that in the premotivic case, $p_\sharp p^*(M) = M_S(\mathbf{A}^1_S)\otimes M$ and $p_*p^*(M) = Hom_S(M_S(\mathbf{A}^1_S), M)$.

Definition 2.1.3. The equivalent conditions of the previous lemma will be called the *homotopy property* for $\mathscr{T}$, denoted by (Htp).

2.1.4. Consider a scheme S, $p : \mathbf{P}^1_S \to S$ the canonical projection and $s : S \to \mathbf{P}^1_S$ the ∞-section. The composite natural transformation

$$1 \xrightarrow{ad(p^*,p_*)} p_*p^* \xrightarrow{ad(s^*,s_*)} p_*s_*s^*p = 1$$

is an isomorphism. In particular, for any premotive M over S, the map $M \to p_*p^*M$ admits a cokernel in the additive category $\mathscr{T}(S)$. The cokernel of this map thus defines a functor which we denote by $\mathrm{coKer}(1 \to p_*p^*)$. The following lemma is easy:

Lemma 2.1.5. *Consider the notations above and the assumption:*

(i) The functor $\mathrm{coKer}(1 \to p_*p^*)$ *is an equivalence of categories.*

Assume $\mathscr{T}$ *is premotivic and put* $\mathbb{1}_S(1) = \mathrm{Ker}\big(M_S(\mathbf{P}^1_S) \xrightarrow{p_*} \mathbb{1}_S\big)[-2]$. *Then the condition (i) above is equivalent to the following ones:*

(ii) The counit map $p_\sharp p^* \to 1$ *is a split epimorphism and its kernel* $\mathrm{Ker}(p_\sharp p^* \to 1)$ *is an equivalence of categories.*

(iii) The premotive $\mathbb{1}_S(1)$ *is* $\otimes$*-invertible in* $\mathscr{T}(S)$.

Definition 2.1.6. In any case, we call the condition (i) above the *stability property* for $\mathscr{T}$. We denote it by (Stab).
In the premotivic case, we call $\mathbb{1}_S(1)$ the Tate premotive. Under (Stab), its inverse will be denoted by $\mathbb{1}_S(-1)$.

Remark 2.1.7. In the premotivic case and under the property (Stab), the Tate inverse premotive $\mathbb{1}(-1)$ is a cartesian section and it generates a set of twists (most of the time isomorphic to the monoid $\mathbf{N}$). Then, a natural assumption on the triangulated premotivic category $\mathscr{T}$ is that it is $\mathbb{1}(-1)$-generated (equivalently, *generated by (negative) Tate twists*).

2.1.8. Recall that a *sieve* R of a scheme X is a class of morphisms in $\mathscr{S}/X$ which is stable by composition on the right by any morphism of schemes (see [AGV73, I.4]).

Given such a sieve R, we will say that $\mathscr{T}$ *is R-separated* if the class of functors f^* for $f \in R$ is conservative. Given two sieves R, R' of X, the following properties are immediate:

(a) If $R' \subset R$ then $\mathscr{T}$ is R-separated implies $\mathscr{T}$ is R'-separated.
(b) If $\mathscr{T}$ is R-separated and is R'-separated then $\mathscr{T}$ is $(R \cup R')$-separated.

A family of morphisms $(f_i : X_i \to X)_{i\in I}$ of schemes defines a sieve $R = \langle f_i, i \in I\rangle$ such that f is in R if and only if there exists $i \in I$ such that f can be factored through f_i. Obviously,

(c) $\mathscr{T}$ is R-separated if and only if the family of functors $(f_i^*)_{i\in I}$ is conservative.

Recall that a topology on $\mathscr{S}$ is the data for any scheme X of a set of sieves of X satisfying certain stability conditions (*cf.* [AGV73, II, 1.1]), called t-covering sieves. A pre-topology t_0 on $\mathscr{S}$ is the data for any scheme X of a set of families of morphisms of shape $(f_i : X_i \to X)_{i\in I}$ satisfying certain stability conditions (*cf.* [AGV73, II, 1.3]), called t_0-covers. A pre-topology t_0 generated a unique topology t.

Definition 2.1.9. Let t be a Grothendieck topology on $\mathscr{S}$. We say that $\mathscr{T}$ is *t-separated* if the following property holds:

(t-sep) For any t-covering sieve R, $\mathscr{T}$ is R-separated in the sense defined above.

Obviously, given two topologies t and t' on $\mathscr{S}$ such that t is finer than t', if $\mathscr{T}$ is t-separated then it is t'-separated.

If the topology t on $\mathscr{S}$ is generated by a pre-topology t_0 then $\mathscr{T}$ is t-separated if and only if for any t_0-covers $(f_i)_{i\in I}$, the family of functors $(f_i^*)_{i\in I}$ is conservative – use [AGV73, 1.4] and 2.1.8(a)+(c).

2.1.10. Recall that a morphism of schemes $f : T \to S$ is radicial if it is injective and for any point t of T, the residual extension induced by f at t is radicial (*cf.* [GD60, 3.5.4, 3.5.8])[8].

Definition 2.1.11. We simply say that $\mathscr{T}$ is *separated* (resp. *semi-separated*) if $\mathscr{T}$ is separated for the topology generated by surjective families of morphisms of finite type (resp. finite radicial morphisms) in $\mathscr{S}$. We also denote by (Sep) (resp. (sSep)) this property.

Remark 2.1.12. If $\mathscr{T}$ is additive, property (Sep) (resp. (sSep)) is equivalent to ask that for any surjective morphism of finite type (resp. finite surjective radicial morphism) $f : T \to S$ in $\mathscr{S}$, the functor f^* is conservative.

Note the following interesting result:

Proposition 2.1.13. *Assume $\mathscr{T}$ is semi-separated and satisfies the transversality property with respect to finite surjective radicial morphisms.*

Then for any finite surjective radicial morphism $f : Y \to X$, the functor

$$f^* : \mathscr{T}(X) \to \mathscr{T}(Y)$$

is an equivalence of categories.

Proof. We first consider the case when $f = i$ is in addition a closed immersion. In this case, we can consider the pullback square bellow.

$$\begin{array}{ccc} Y & = & Y \\ \| & & \downarrow{\scriptstyle i} \\ Y & \xrightarrow[i]{} & Z \end{array}$$

Using the transversality property with respect to i, we see that the counit $i^*i_* \to 1$ is an isomorphism. It thus remains to prove that the unit map $1 \to i_*i^*$ is an isomorphism. As i^* is conservative by semi-separability, it is sufficient to check that

$$i^* \to i^*i_*i^*(M)$$

is an isomorphism. But this is a section of the map $i^*\, i_*i^*(M) \to i^*(M)$, which is already known to be an isomorphism.

Consider now the general case of a finite radicial extension f. We introduce the pullback square

$$\begin{array}{ccc} Y \times_X Y & \xrightarrow{p} & Y \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[f]{} & X \end{array}$$

Consider the diagonal immersion $i : Y \to Y \times_X Y$. Because Y is noetherian and p is separable, i is finite (*cf.* [GD61, 6.1.5]) thus a closed immersion. As p is a universal homeomorphism, the same is true for its section i. The preceding case thus implies that i^* is an equivalence of categories. Moreover, as $pi = qi = 1_Y$, we see that p^* and q^* are both quasi-inverses to i^*, which implies that they are isomorphic equivalences of categories. More precisely, we get canonical isomorphisms of functors

$$i^* \simeq p_* \simeq q_* \quad \text{and} \quad i_* \simeq p^* \simeq q^*.$$

We check that the unit map $1 \to f_*f^*$ is an isomorphism. Indeed, by semi-separability, it is sufficient to prove this after applying the functor f^*, and we get, using the transversality property for f:

$$f^* \simeq i^*p^*f^* \simeq q_*p^*f^* \simeq f^*f_*f^*.$$

We then check that the counit map $f^*f_* \to 1$ is an isomorphism as well. In fact, using again the transversality property for f, we have isomorphisms

$$f^*f_*(M) \simeq q_*p^*(M) \simeq i^*i_*(M) \simeq M.$$

[8]It is equivalent to ask that f is universally injective. When f is surjective, this is equivalent to ask that f is a universal homeomorphism.

□

2.1.14. Recall from [Voe00a] that a cd-structure on $\mathscr{S}$ is a collection P of commutative squares of schemes

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & Q & \downarrow f \\ A & \xrightarrow[e]{} & X \end{array}$$

which is closed under isomorphisms. We will say that a square Q in P is P-distinguished.

Voevodsky associates to P a topology t_P, the smallest topology such that:

- for any distinguished square Q as above, the sieve generated by $\{f : A \to X, e : Y \to X\}$ is t_P-covering on X.
- the empty sieve covers the empty scheme.

Example 2.1.15. A *Nisnevich distinguished square* is a square Q as above such that Q is cartesian, f is étale, e is an open embedding with reduced complement Z and the induced map $p^{-1}(Z) \to Z$ is an isomorphism. The corresponding cd-structure is called the *upper cd-structure* (see section 2 of [Voe00b]). Because we work with noetherian schemes, the corresponding topology is the *Nisnevich topology* (see proposition 2.16 of *loc.cit.*).

A *proper* cdh-*distinguished square* is a square Q as above such that Q is cartesian, f is proper, e is a closed embedding with open complement U and the induced map $p^{-1}(U) \to U$ is an isomorphism. The corresponding cd-structure is called the *lower cd-structure*. The topology associated to the lower cd-structure is called the *proper cdh-topology*.

The topology generated by the lower and upper cd-structures is by definition (according to the preceding remark on Nisnevich topology) the *cdh-topology*.

All these three examples are complete cd-structures in the sense of [Voe00a, 2.3].

Lemma 2.1.16. *Let P be a complete cd-structure (see* [Voe00a, def 2.3]*) on $\mathscr{S}$ and t_P be the associated topology. The following conditions are equivalent:*

(i) $\mathscr{T}$ is t_P-separated.

(ii) For any distinguished square Q for P of the above form, the pair of functors (e^*, f^*) is conservative.

Proof. This follows from the definition of a complete cd-structure and 2.1.8(a). □

Remark 2.1.17. If we assume that $\mathscr{S}$ is stable by arbitrary pullback then any cd-structure P on $\mathscr{S}$ such that P-distinguished squares are stable by pullback is complete (see [Voe00a, 2.4]).

2.2. **Exceptional functors, following Deligne.**

2.2.1. Consider an open immersion $j : U \to S$. Applying 1.1.14 to the cartesian square

$$\begin{array}{ccc} U & = & U \\ \| & & \downarrow j \\ U & \xrightarrow[j]{} & S \end{array}$$

we get a canonical natural transformation

$$\gamma_j : j_\sharp = j_\sharp 1_* \xrightarrow{Ex(\Delta_{\sharp *})} j_* 1_\sharp = j_*.$$

Recall that the functors $j_\sharp$ and j_* are fully faithful (1.1.19).

Note that according to remark 1.1.7, this natural transformation is compatible with the identifications of the kind $(jk)_\sharp = j_\sharp k_\sharp$ and $(jk)_* = j_* k_*$.

Lemma 2.2.2. *Let S be a scheme, U and V be subschemes such that $S = U \sqcup V$. We let $h : U \to S$ (resp. $k : V \to S$) be the canonical open immersions.*

Assume that the functor $(h^*, k^*) : \mathscr{T}(S) \to \mathscr{T}(U) \times \mathscr{T}(V)$ is conservative and that $\mathscr{T}(\varnothing) = 0$. Then the natural transformation γ_h (resp. γ_k) is an isomorphism. Moreover, the functor (h^*, k^*) is then an equivalence of categories.

Proof. As $h_\sharp$ and h_* are fully faithful, we have $h^*h_\sharp \simeq h^*h_*$. By $\mathscr{P}$-base change, we also get $k^*h_\sharp \simeq k^*h_* \simeq 0$. It remains to prove the last assertion. The functor $R = (h^*, k^*)$ has a left adjoint L defined by L=$h_! \oplus k_!$:

$$L(M,N) = h_!(M) \oplus k_!(N)\,.$$

The natural transformation $LR \to 1$ is an isomorphism: to see this, is it sufficient to evaluate at h^* and k^*, which gives an isomorphism in $\mathscr{T}(U)$ and $\mathscr{T}(V)$ respectively. The natural transformation $1 \to RL$ is also an isomorphism because $h_\sharp$ and $k_\sharp$ are fully faithful. □

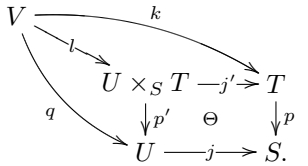
Remark 2.2.3. Assume $\mathscr{T}$ is Zariski separated (definition 2.1.9). Then, as a corollary of this lemma, $\mathscr{T}$ is additive (definition 2.1.1) if and only if $\mathscr{T}(\varnothing) = 0$.

2.2.4. *Exchange structures V.–* Assume $\mathscr{T}$ is additive. We consider a commutative square of schemes

$$\begin{array}{ccc} V & \xrightarrow{k} & T \\ {\scriptstyle q}\downarrow & \Delta & \downarrow{\scriptstyle p} \\ U & \xrightarrow[j]{} & S \end{array} \tag{2.2.4.1}$$

such that j, k are an open immersions and p, q are a proper morphisms.

This diagram can be factored into the following commutative diagram:



Then l is an open and closed immersion so that the previous lemma implies the canonical morphism $\gamma_l : l_\sharp \to l_*$ is an isomorphism. As a consequence, we get a natural exchange transformation

$$Ex(\Delta_{\sharp *}) : j_\sharp q_* = j_\sharp p'_* l_* \xrightarrow{Ex(\Theta_{\sharp *})} p_* j'_\sharp l_* \xrightarrow{\gamma_l^{-1}} p_* j'_\sharp l_\sharp = p_* k_\sharp$$

using the exchange of 1.1.14. Note that, with the notations introduced in 2.2.1, the following diagram is commutative.

$$\begin{array}{ccccc} j_\sharp q_* & & \xrightarrow{\qquad Ex(\Delta_{\sharp *})\qquad} & & p_* k_\sharp \\ {\scriptstyle \gamma_j q_*}\downarrow & & & & \downarrow{\scriptstyle p_* \gamma_k} \\ j_* q_* & \xrightarrow{\sim} & (jq)_* = (pk)_* & \xleftarrow{\sim} & p_* k_\sharp \end{array} \tag{2.2.4.2}$$

Indeed one sees first that it is sufficient to treat the case where Δ is cartesian. Then, as $j_\sharp$ is a fully faithful left adjoint to j^* it is sufficient to check that (2.2.4.2) commutes after having applied j^*. Using the cotransversality property with respect to open immersions, one sees then that this consists to verify the commutativity of (2.2.4.2) when j is the identity, in which case it is trivial.

Definition 2.2.5. We say that the triangulated $\mathscr{P}$-fibred category $\mathscr{T}$ satisfies the *support property*, denoted by (Supp), if it is additive and for any commutative square of shape (2.2.4.1) the exchange transformation $Ex(\Delta_{\sharp *}) : j_\sharp q_* \to p_* k_\sharp$ defined above is an isomorphism.

By definition, it is sufficient to check the last property of property (Supp) in the case where Δ is cartesian.

2.2.6. We denote by $\mathscr{S}^{sep}$ (resp. $\mathscr{S}^{open}$, $\mathscr{S}^{prop}$) the sub-category of the category $\mathscr{S}$ with the same objects but morphisms are separated morphisms of finite type (resp. open immersions, proper morphisms). We denote by

$$\mathscr{T}_* : \mathscr{S} \to \mathscr{T}ri^\otimes$$
$$\text{resp. } \mathscr{T}_\sharp : \mathscr{S}^{open} \to \mathscr{T}ri^\otimes$$

the 2-functor defined respectively by morphisms of type f_* and $j_\sharp$ (f any morphism of schemes). The proposition below is essentially based on a result of Deligne [AGV73, XVII, 3.3.2]:

Proposition 2.2.7. *Assume the triangulated premotivic category $\mathscr{T}$ satisfies (Supp).*

Then there exists a unique 2-functor

$$\mathscr{T}_! : \mathscr{S}^{sep} \to \mathscr{T}ri^{\otimes}$$

with the property that

$$\mathscr{T}_!|_{\mathscr{S}^{prop}} = \mathscr{T}_*|_{\mathscr{S}^{prop}}, \quad \mathscr{T}_!|_{\mathscr{S}^{open}} = \mathscr{T}_\sharp$$

and for any commutative square Δ of shape (2.2.4.1) *the composition of the structural isomorphisms*

$$j_\sharp q_* = j_! q_! \simeq (jq)_! = (pk)_! \simeq p_! k_! = p_* k_\sharp$$

is equal to the exchange transformation $Ex(\Delta_{\sharp *})$.

2.2.8. Under the assumptions of the proposition, for any separated morphism of finite type $f : Y \to X$, we will denote by $f_! : \mathscr{T}(Y) \to \mathscr{T}(X)$ the functor $\mathscr{T}_!(f)$. The functor $f_!$ is called the *direct image functor with compact support* (associated with f).

Proof. We recall the principle of the proof of Deligne. Let $f : Y \to X$ be a separated morphism of finite type in $\mathscr{S}$.

Let $\mathscr{C}_f$ be the category of compactifications of f in $\mathscr{S}$, i.e. of factorizations of f of the form

$$Y \xrightarrow{j} \bar{Y} \xrightarrow{p} X \tag{2.2.8.1}$$

where j is an open immersion, p is proper, and $\bar{Y}$ belongs to $\mathscr{S}$. Morphisms of $\mathscr{C}_f$ are given by commutative diagrams of the form

$$\begin{array}{ccccc} & & \bar{Y}' & & \\ & \overset{j'}{\nearrow} & \downarrow \pi & \overset{p'}{\searrow} & \\ Y & & & & X. \\ & \underset{j}{\searrow} & \bar{Y} & \underset{p}{\nearrow} & \end{array} \tag{2.2.8.2}$$

in $\mathscr{S}$. To any compactification of f of shape (2.2.8.1), we associate the functor $p_* j_\sharp$.
To any morphism of compactifications (2.2.8.2), we associate a natural isomorphism

$$p'_* j'_\sharp = p_* \pi_* j'_\sharp \xrightarrow{Ex(\Delta_{\sharp *})^{-1}} p_* j_\sharp 1_* = p_* j_\sharp.$$

where Δ stands for the commutative square made by removing π in the diagram (2.2.8.2), and $Ex(\Delta_{\sharp *})$ is the corresponding natural transformation (see 2.2.4). The compatibility of $Ex(\Delta_{\sharp *})$ with composition of morphisms of schemes shows that we have defined a functor

$$\Gamma_f : \mathscr{C}_f^{op} \to Hom(\mathscr{T}(Y), \mathscr{T}(X))$$

which sends all the maps of $\mathscr{C}_f$ to isomorphisms (by the support property).

The category $\mathscr{C}_f$ is non-empty by the assumption 2.0(c) on $\mathscr{S}$, and it is in fact left filtering; see [AGV73, XVII, 3.2.6(ii)]. This defines a canonical functor $f_! : \mathscr{T}(Y) \to \mathscr{T}(X)$, independent of any choice compactification of f, defined in the category of functors $Hom(\mathscr{T}(Y), \mathscr{T}(X))$ by the formula

$$f_! = \varinjlim_{\mathscr{C}_f^{op}} \Gamma_f \,.$$

If $f = p$ is proper, then the compactification

$$Y \xrightarrow{=} Y \xrightarrow{p} X$$

is an initial object of $\mathscr{C}_f$, which gives a canonical identification $p_! = p_*$. Similarly, if $f = j$ is an open immersion, then the compactification

$$Y \xrightarrow{j} X \xrightarrow{=} X$$

is a terminal object of $\mathscr{C}_j$, so that we get a canonical identification $j_! = j_\sharp$.

This construction is compatible with composition of morphisms. Let $g : Z \to Y$ and $f : Y \to X$ be two separated morphisms of finite type in $\mathscr{S}$. For any a couple of compactifications

$$Z \xrightarrow{k} \bar{Z} \xrightarrow{q} Y \text{ and } Y \xrightarrow{j} \bar{Y} \xrightarrow{p} X$$

of f and g respectively, we can choose a compactification

$$\bar{Z} \xrightarrow{h} T \xrightarrow{r} Y$$

of jq, and we get a canonical isomorphisms

$$f_! \, g_! \simeq p_* \, j_! \, q_* \, k_! \simeq p_* \, r_* \, h_! \, k_! \simeq (pr)_* \, (hk)_\sharp \simeq (fg)_! \, .$$

The independence of these isomorphic with respect to the choices of compactification follows from [AGV73, XVII, 3.2.6(iii)]. The cocycle conditions (i.e. the associativity) also follows formally from [AGV73, XVII, 3.2.6]. The uniqueness statement is obvious. □

2.2.9. This construction is functorial in the following sense.

Define a 2-*functor with support on* $\mathscr{T}$ to be a triple $(\mathscr{D}, a, b)$, where:

(i) $\mathscr{D} : \mathscr{S}^{sep} \to \mathscr{T}ri$ is a 2-functor (we shall write the structural coherence isomorphisms as $c_{g,f} : \mathscr{D}(gf) \xrightarrow{\sim} \mathscr{D}(g)\mathscr{D}(f)$ for composable arrows f and g in $\mathscr{S}^{sep}$;

(ii) $a : \mathscr{T}_*|_{\mathscr{S}^{prop}} \to \mathscr{D}|_{\mathscr{S}^{prop}}$ and $b : \mathscr{T}_\sharp \to \mathscr{D}$ are morphisms of 2-functors which agree on objects, i.e. such that for any scheme S in $\mathscr{S}$, we have

$$\psi_S = a_S = b_S : \mathscr{T}(S) \to \mathscr{D}(S) \, ;$$

(iii) for any commutative square of shape (2.2.4.1) in which j and k are open immersions, while p and q are proper morphisms, the diagram below commutes.

$$\begin{array}{ccccc}
\psi_S \, j_\sharp q_* & & \xrightarrow{\psi_S \, Ex(\Delta_{\sharp *})} & & \psi_S \, p_* k_\sharp \\
{\scriptstyle b\,q_*}\downarrow & & & & \downarrow{\scriptstyle a k_\sharp} \\
\mathscr{D}(j)\psi_U q_* & & & & \mathscr{D}(p)\psi_T k_\sharp \\
{\scriptstyle \mathscr{D}(j)a}\downarrow & & & & \downarrow{\scriptstyle \mathscr{D}(p)b} \\
\mathscr{D}(j)\mathscr{D}(q)\psi_V & \xrightarrow{c^{-1}_{j,q}} & \mathscr{D}(jq) = \mathscr{D}(pk)\psi_V & \xleftarrow{c^{-1}_{p,k}} & \mathscr{D}(p)\mathscr{D}(k)\psi_V
\end{array}$$

Morphisms of 2-functors with support on $\mathscr{T}$

$$(\mathscr{D}, a, b) \to (\mathscr{D}', a', b')$$

are defined in the obvious way: these are morphisms of 2-functors $\mathscr{D} \to \mathscr{D}'$ which preserve all the structure on the nose.

Using the arguments of the proof of 2.2.7, one checks easily that the category of 2-functors with support has an initial object, which is nothing but the 2-functor $\mathscr{T}_!$ together with the identities of $\mathscr{T}_*|_{\mathscr{S}^{prop}}$ and of $\mathscr{T}_\sharp$ respectively. In particular, for any 2-functor $\mathscr{D} : \mathscr{S}^{sep} \to \mathscr{T}ri$, a morphism of 2-functors $\mathscr{T}_! \to \mathscr{D}$ is completely determined by its restrictions to $\mathscr{S}^{prop}$ and $\mathscr{S}^{open}$, and by its compatibility with the exchange isomorphisms of type $Ex(\Delta_{\sharp *})$ in the sense described in condition (iii) above.

Proposition 2.2.10. *Assume that $\mathscr{T}$ satisfies the support property. For any separated morphism of finite type f in $\mathscr{S}$, there exists a canonical natural transformation*

$$\alpha_f : f_! \to f_* \, .$$

The collection of maps α_f defines a morphism of 2*-functors*

$$\alpha : \mathscr{T}_! \to \mathscr{T}_*|_{\mathscr{S}^{sep}} \ , \quad f \mapsto (\alpha_f : f_! \to f_*)$$

whose restrictions to $\mathscr{S}^{prop}$ and $\mathscr{S}^{open}$ are respectively the identity and the morphism of 2*-functors $\gamma : \mathscr{T}_\sharp \to \mathscr{T}_*|_{\mathscr{S}^{open}}$ defined in 2.2.1.*

Proof. The identities $f_* = f_*$ for f proper (resp. projective) and the exchange natural transformations of type $Ex(\Delta_{\sharp *})$ turns $\mathscr{T}_*|_{\mathscr{S}^{sep}}$ into a 2-functor with support (resp. restricted support) on $\mathscr{T}$ (property (iii) of 2.2.9 is expressed by the commutative square (2.2.4.2)). □

Proposition 2.2.11. *Let $\mathscr{T}'$ be another triangulated complete $\mathscr{P}$-fibred category over $\mathscr{S}$. Assume that $\mathscr{T}$ and $\mathscr{T}'$ both have the support property, and consider given a triangulated morphism of $\mathscr{P}$-fibred categories $\varphi^* : \mathscr{T} \to \mathscr{T}'$ (recall definition 1.2.2).*

Then, there is a canonical family of natural transformations

$$Ex(\varphi^*, f_!) : \varphi_X^*\, f_! \to f_!\, \varphi_Y^*$$

for each separated morphism of finite type $f : Y \to X$ in $\mathscr{S}$, which is functorial with respect to composition in $\mathscr{S}$ (i.e. defines a morphism of 2-functors) and such that, the following conditions are verified:

(a) if f is proper, then, under the identification $f_! = f_*$, the map $Ex(\varphi^*, f_!)$ is the exchange transformation $Ex(\varphi^*, f_*) : \varphi_X^*\, f_* \to f_*\, \varphi_Y^*$ defined in 1.2.5;

(b) if f is an open immersion, then, under the identification $f_! = f_\sharp$, the map $Ex(\varphi^*, f_!)$ is the inverse of the exchange isomorphism $Ex(f_\sharp, \varphi^*) : f_\sharp\, \varphi_Y^* \to \varphi_X^*\, f_\sharp$ defined in 1.2.1.

Proof. The exchange maps of type $Ex(\varphi^*, f_*)$ define a morphism of 2-functors

$$a : \mathscr{T}_*|_{\mathscr{S}prop} \to \mathscr{T}'_*|_{\mathscr{S}prop} = \mathscr{T}'_!|_{\mathscr{S}prop}$$

while the inverse of the exchange isomorphisms of type $Ex(f_\sharp, \varphi^*)$ define a morphism of 2-functors

$$b : \mathscr{T}_\sharp \to \mathscr{T}'_\sharp = \mathscr{T}'_!|_{\mathscr{S}^{open}},$$

in such a way that the triple $(\mathscr{T}'_!, a, b)$ is a 2-functor with support on $\mathscr{T}$. □

Corollary 2.2.12. *Suppose $\mathscr{T}$ satisfies the support property.*

(i) For any cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & \Delta & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X, \end{array}$$

such that f is separated of finite type, there exists a natural transformation

$$Ex(\Delta_!^*) : g^* f_! \to f'_! g'^*$$

which is natural with respect to the horizontal composition of such squares, and such that, in each of the following cases, we have the following identifications in $\mathscr{T}(X')$

(a) f proper:

$$\begin{array}{ccc} g^* f_! & \xrightarrow{Ex(\Delta_!^*)} & f'_! g'^* \\ \| & & \| \\ g^* f_* & \xrightarrow{Ex(\Delta_*^*)} & f'_* g'^*, \end{array}$$

(b) f open immersion:

$$\begin{array}{ccc} g^* f_! & \xrightarrow{Ex(\Delta_!^*)} & f_! g'^* \\ \| & & \| \\ g^* f_\sharp & \xrightarrow{Ex(\Delta_\sharp^*)^{-1}} & f'_\sharp g'^*. \end{array}$$

Moreover, when g is a $\mathscr{P}$-morphism, $Ex(\Delta_!^*)$ is an isomorphism.

(ii) If, furthermore, $\mathscr{T}$ is a monoidal triangulated $\mathscr{P}$-fibred category over $\mathscr{S}$, then, for any separated morphism of finite type $f : Y \to X$, there is a natural transformation

$$Ex(f_!^*, \otimes_Y) : (f_! K) \otimes_X L \to f_!(K \otimes_Y f^* L)$$

which is natural with respect to composition in $\mathscr{S}$, and such that, in each of the following cases, we have the following identifications:

(a) f proper:

$$\begin{array}{ccc} (f_! K) \otimes L & \xrightarrow{Ex(f_!^*, \otimes)} & f_!(K \otimes f^* L) \\ \| & & \| \\ (f_* K) \otimes L & \xrightarrow{Ex(f_*^*, \otimes)} & f_*(K \otimes_Y f^* L), \end{array}$$

(b) f open immersion:

$$\begin{array}{ccc} (f_! K) \otimes L & \xrightarrow{Ex(f_!^*, \otimes)} & f_!(K \otimes_f^* L) \\ \| & & \| \\ (f_\sharp K) \otimes_L & \xrightarrow{Ex(f_\sharp^*, \otimes)^{-1}} & f_\sharp(K \otimes_Y f^* L). \end{array}$$

As in the previous analogous cases, the natural transformations $Ex(\Delta_!^*)$ and $Ex(f_!^*, \otimes_Y)$ will be called *exchanged transformations*.

Proof. To prove (i), consider a fixed map $g : X' \to X$ in $\mathscr{S}$. We consider the triangulated P/X-fibred categories $\mathscr{T}'$ and $\mathscr{T}''$ over $\mathscr{S}/X$ defined by $\mathscr{T}'(Y) = \mathscr{T}(Y)$ and $\mathscr{T}''(Y) = \mathscr{T}(Y')$ for any X-scheme Y (in $\mathscr{S}$), with $g' : Y' = Y \times_X X' \to Y$ the map obtained from $Y \to X$ by pullback along g. The collection of functors

$$g'^* : \mathscr{T}(Y) \to \mathscr{T}(Y')$$

define an exact morphism of triangulated P/X-fibred categories over $\mathscr{S}/X$ (by the $\mathscr{P}$-base change formula):

$$\varphi^* : \mathscr{T}' \to \mathscr{T}'' .$$

Applying the preceding proposition to the latter gives (i). The fact that we get an isomorphism whenever g is a $\mathscr{P}$-morphism follows from the $\mathscr{P}$-base change formula and from 1.1.18.

The proof of (ii) is similar: fix a scheme X in $\mathscr{S}$, as well as an object L in $\mathscr{T}(X)$. We can consider L as a cartesian section of $\mathscr{T}|_{\mathscr{S}/X}$, and by the $\mathscr{P}$-projection formula, we then have an exact morphism of triangulated P/X-fibred categories over $\mathscr{S}/X$:

$$L \otimes (-) : \mathscr{T}|_{\mathscr{S}/X} \to \mathscr{T}|_{\mathscr{S}/X} .$$

Here again, we can apply the preceding proposition and conclude. □

Recall from definition 1.1.16 that we say $\mathscr{T}$ satisfies the transversality property with respect to proper morphisms (in $\mathscr{S}$) if the exchange transformation $Ex(\Delta_*^*) : p^* f_* \to g_* q^*$ defined in 1.1.14 is an isomorphism for any cartesian square Δ as soon as f is a proper morphism (in $\mathscr{S}$).

Proposition 2.2.13. *Assume that $\mathscr{T}$ satisfies the support property. Then the following conditions are equivalent:*

(a) $\mathscr{T}$ satisfies the transversality property with respect to proper morphisms in $\mathscr{S}$.
(b) For any cartesian square of $\mathscr{S}$

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & \Delta & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X, \end{array}$$

in which f is separated and of finite type, the exchange transformation

$$Ex(\Delta_!^*) : g^* f_! \to f'_! g'^*$$

is an isomorphism.

Proof. Property (*b*) is always verified in the case where f is an open immersion by assumption ($\mathscr{P}$-base change formula). As any separated morphism factors as an open immersion followed by a proper morphism, this implies the equivalence between (a) and (b). □

As a conclusion of this part, we get the following:

Theorem 2.2.14. *Consider the following assumptions:*

(a) $\mathscr{T}$ satisfies (Supp).
(b) For any proper morphism $f : T \to S$ in $\mathscr{S}$, f_* admits a right adjoint $f^!$.
(c) $\mathscr{T}$ satisfies the proper transversality property.
(d) $\mathscr{T}$ is monoidal.

Under assumptions (a) and (b), for any separated morphism of finite type $f : Y \to X$ in $\mathscr{S}$, there exists an essentially unique pair of adjoint functors

$$f_! : \mathscr{T}(Y) \rightleftarrows \mathscr{T}(X) : f^!$$

such that:

(1) There exists a natural transformation $\alpha_f : f_! \to f_*$ which is an isomorphism when f is proper.
(2) For any open immersion j, $j_! = j_\sharp$ and $j^! = j^*$.

(3) For any cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & \Delta & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X, \end{array}$$

in which f is separated and of finite type, there exists natural transformations

$$Ex(\Delta_!^*) : g^* f_! \to f'_! g'^*,$$

$$Ex(\Delta_*^!) : g'_* f'^! \to f^! g_*$$

which are isomorphisms whenever f is an open immersion or g is a $\mathscr{P}$-morphism.

If we assume (a), (b), (c) then the following additional property holds:

(4c) For any cartesian square Δ in $\mathscr{S}$, the natural transformations $Ex(\Delta_!^*)$ and $Ex(\Delta_*^!)$ are isomorphisms.

If we assume (a), (b), (d) then the following additional property holds:

(4d) For any separated morphism of finite type $f : Y \to X$ in $\mathscr{S}$, there exists a natural transformation

$$Ex(f_!^*, \otimes) : (f_! K) \otimes_X L \to f_!(K \otimes_Y f^* L).$$

2.2.15. Recall that an exact functor between well generated triangulated categories admits a right adjoint if and only if it commutes with small sums: this is an immediate consequence of the *Brown representability theorem* proved by Neeman (*cf.* [Nee01, 8.4.4]). We deduce the following useful result.

Proposition 2.2.16. *Assume that $\mathscr{T}$ is a compactly τ-generated triangulated premotivic category over $\mathscr{S}$.*

Then, for any morphism of schemes $f : T \to S$, the functor $f_* : \mathscr{T}(T) \to \mathscr{T}(S)$ admits a right adjoint.

Proof. This follows directly from proposition 1.3.18. □

2.3. **The localization property.**

2.3.a. *Definition.*

2.3.1. Consider a closed immersion $i : Z \to S$ in $\mathscr{S}$. Let $U = S - Z$ be the complement open subscheme of S and $j : U \to S$ the canonical immersion. We will use the following consequence of the triangulated $\mathscr{P}$-fibred structure on $\mathscr{T}$:

(a) The unit $1 \to j^* j_\sharp$ is an isomorphism.
(b) The counit $j^* j_* \to 1$ is an isomorphism.
(c) $i^* j_\sharp = 0$.
(d) $j^* i_* = 0$.
(e) The composite map $j_\sharp j^* \xrightarrow{ad'(j_\sharp, j^*)} 1 \xrightarrow{ad(i^*, i_*)} i_* i^*$ is zero.

In fact, the first four relations all follow from the base change property ($\mathscr{P}$-BC). Relation (e) is a consequence of (d) once we have noticed that the following square is commutative

$$\begin{array}{ccc} j_\sharp j^* & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ j_\sharp j^* i_* i^* & \to & i_* i^*. \end{array}$$

For the closed immersion i and the triangulated premotivic category $\mathscr{T}$, we introduce the property (Loc_i) made of the following assumptions:

(a) The pair of functors (j^*, i^*) is conservative.
(b) The counit $i^* i_* \xrightarrow{ad'(i^*, i_*)} 1$ is an isomorphism.

Definition 2.3.2. We say that $\mathscr{T}$ satisfies the *localization property*, denoted by (Loc), if:

(1) $\mathscr{T}(\varnothing) = 0$.
(2) For any closed immersion i in $\mathscr{S}$, (Loc_i) is satisfied.

The main consequence of the localization axiom is that it leads to the situation of the six gluing functor (*cf.* [BBD82, prop. 1.4.5]):

Proposition 2.3.3. *Let $i : Z \to S$ be a closed immersion such that (Loc_i) is satisfied.*

(1) The functor i_* admits a right adjoint $i^!$.
(2) For any K in $\mathscr{T}(S)$, there exists a unique map $\partial_{i,K} : i_*i^*K \to j_\sharp j^*K[1]$ such that the triangle

$$j_\sharp j^*K \xrightarrow{ad'(j_\sharp, j^*)} K \xrightarrow{ad(i^*, i_*)} i_*i^*K \xrightarrow{\partial_{i,K}} j_\sharp j^*K[1]$$

is distinguished. The map $\partial_{i,K}$ is functorial in K.
(3) For any K in $\mathscr{T}(S)$, there exists a unique map $\partial'_{i,K} : j_*j^*K \to i_*i^!K[1]$ such that the triangle

$$i_*i^!K \xrightarrow{ad'(i_*, i^!)} K \xrightarrow{ad(j^*, j_*)} j_*j^*K \xrightarrow{\partial'_{i,K}} i_*i^!K[1]$$

is distinguished. The map $\partial'_{i,K}$ is functorial in K.

Under the property (Loc_i), the canonical triangles appearing in 5) and (3) above are called the *localization triangles* associated with i.

Proof. We first consider point (2). For the existence, we consider a distinguished triangle

$$j_\sharp j^*K \xrightarrow{ad'(j_\sharp, j^*)} K \xrightarrow{\pi} C \xrightarrow{+1}$$

Applying 2.3.1(e), we obtain a factorization

$$\begin{array}{ccc} K & \xrightarrow{ad(i^*, i_*)} & i_*i^*K \\ & \searrow^{\pi} \quad \nearrow_{w} & \\ & C & \end{array}$$

We prove w is an isomorphism. According to the above triangle, $j^*w = 0$. From 2.3.1(d), $j^*i_*i^*K = 0$ so that j^*w is an isomorphism. Applying i^* to the above distinguished triangle, we obtain from 2.3.1(c) that $i^*\pi$ is an isomorphism. Thus, applying i^* to the above commutative diagram together with (Loc_i) (a), we obtain that i^*w is an isomorphism which concludes.

Consider a map $K \xrightarrow{u} L$ in $\mathscr{T}(S)$ and suppose we have chosen maps a and b in the diagram:

$$\begin{array}{ccccccc} j_\sharp j^*K & \xrightarrow{ad'(j_\sharp, j^*)} & K & \xrightarrow{ad(i^*, i_*)} & i_*i^*K & \xrightarrow{a} & j_\sharp j^*K[1] \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u} & & & & \downarrow{\scriptstyle u} \\ j_\sharp j^*L & \xrightarrow{ad'(j_\sharp, j^*)} & L & \xrightarrow{ad(i^*, i_*)} & i_*i^*L & \xrightarrow{b} & j_\sharp j^*L[1] \end{array}$$

such that the horizontal lines are distinguished triangles. Then we can find a map $h : i_*i^*K \to i_*i^*L$ completing the previous diagram into a morphism of triangles. Then the map $w = h - u$ satisfy the relation $w \circ a_i = 0$. Thus it can be lifted to a map in $\mathrm{Hom}(j_\sharp j^*K[1], i_*i^*L)$. But this is zero by adjunction and the relation 2.3.1(d). This proves both the naturality of $\partial_{i,K}$ and its uniqueness.

For point (1) and (3), for any object K of $\mathscr{T}(S)$, we consider a distinguished triangle

$$D \to K \xrightarrow{ad(j^*, j_*)} j_*j^*K \xrightarrow{+1}$$

According to 2.3.1(b), $j^*D = 0$. Thus according to the triangle of point (2) applied to D, we obtain $D = i_*i^*D$. Arguing as for point (2), we thus obtain that D is unique and depends functorialy on K so that, if we put $i^!K = i^*D$, point (1) and (3) follows. □

Remark 2.3.4. Consider the hypothesis and notations of the previous proposition.

(1) By transposition from 2.3.1(d), we deduce that $i^!j_* = 0$.

(2) Assume that i is a $\mathscr{P}$-morphism. Then the $\mathscr{P}$-base change formula implies that $i^*j_* = 0$. Dually, we get that $i^!j_\sharp = 0$. By adjunction, we thus obtain $\partial_{i,K} = 0$ and $\partial'_{i,K} = 0$ for any object K so that both localization triangles are split. In that case, we get that $\mathscr{T}(S) = \mathscr{T}(Z) \times \mathscr{T}(U)$.[9]

The preceding lemma admits the following reciprocal statement:

Lemma 2.3.5. *Consider a closed immersion $i : Z \to S$ in $\mathscr{S}$ with complementary open immersion $j : U \to S$. Then the following properties are equivalent:*

(i) $\mathscr{T}$ satisfies (Loc$_i$).

(ii) (a) The functor i_* is conservative.

(b) For any object K of $\mathscr{T}(S)$, there exists a map $i_*i^*(K) \to j_\sharp j^*(K)[1]$ which fits into a distinguished triangle

$$j_\sharp j^*(K) \xrightarrow{ad'(j_\sharp, j^*)} K \xrightarrow{ad(i^*, i_*)} i_*i^*(K) \to j_\sharp j^*(K)[1]$$

Proof. The fact (i) implies (ii) follows from proposition 2.3.3. Conversely, (ii)(b) implies that the pair (i^*, j^*) is conservative and it remains to prove (Loc$_i$) (b). Let K be an object of $\mathscr{T}(S)$. Consider the distinguished triangle given by (ii)(b):

$$j_\sharp j^*(K) \xrightarrow{ad'(j_\sharp, j^*)} K \xrightarrow{ad(i^*, i_*)} i_*i^*(K) \to j_\sharp j^*(K)[1].$$

If we apply i_* on the left to this triangle, we get using 2.3.1(d) that the morphism

$$i_*(K) \xrightarrow{ad(i^*, i_*).i_*} i_*i^*i_*(K)$$

is an isomorphism. Hence, by the zig-zag equation, the morphism

$$i_*i^*i_*(K) \xrightarrow{i_*.ad'(i^*, i_*)} i_*(K)$$

is an isomorphism. Property (ii)(a) thus implies that $i^*i_*(K) \simeq K$. □

2.3.b. *First consequences of localization.* The following statement is straightforward.

Proposition 2.3.6. *Assume $\mathscr{T}$ satisfies the localization property and consider a scheme S in $\mathscr{S}$.*

(1) Let S_{red} be the reduced scheme associated with S. The canonical immersion $S_{red} \xrightarrow{\nu} S$ induces an equivalence of categories:

$$\nu^* : \mathscr{T}(S) \to \mathscr{T}(S_{red}).$$

(2) For any any partition $(S_i \xrightarrow{\nu_i} S)_{i \in I}$ of S by locally closed subset, the family of functors $(\nu_i^*)_{i \in I}$ is conservative (S_i is considered with its canonical structure of a reduced subscheme of S).

Lemma 2.3.7. *If $\mathscr{T}$ satisfies the localization property (Loc) then it is additive.*

Proof. Note that, by assumption, $\mathscr{T}(\varnothing) = 0$. Then the assertion follows directly from lemma 2.2.2. □

Proposition 2.3.8. *If $\mathscr{T}$ satisfies the localization property then it satisfies the property of separation with respect to the* cdh*-topology.*

Proof. Consider a cartesian square of schemes

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & Q & \downarrow p \\ A & \xrightarrow[e]{} & X. \end{array}$$

According to lemma 2.1.16, we have only to check that the pair of functors (e^*, p^*) is conservative when Q is a Nisnevich (or respectively a proper cdh) distinguished square. Let $\nu : A' \to X$ be the

[9] This remark explains why the localization property is too strong for generalized premotivic categories.

complementary closed (resp. open) immersion to e, where A' have the induced reduced subscheme (resp. induced subscheme) structure. Consider the cartesian square

$$\begin{array}{ccc} Y & \longleftarrow & B' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xleftarrow[\nu]{} & A \end{array}$$

By assumption on Q, q is an isomorphism. According to (Loc) (ii), (e^*, ν^*) is conservative. This concludes. □

The following proposition can be found in a slightly less precise and general form in [Ayo07a, 2.1.162].[10]

Proposition 2.3.9. *Assume $\mathscr{T}$ satisfies the localization property.*
Then the following conditions are equivalent:

(i) $\mathscr{T}$ is separated.
(ii) For a morphism $f : T \to S$ in $\mathscr{S}$, $f^* : \mathscr{T}(S) \to \mathscr{T}(T)$ is conservative whenever f is:
  (a) a finite étale cover;
  (b) finite, faithfully flat and radicial.

Proof. Only $(ii) \Rightarrow (i)$ requires a proof. Consider a surjective morphism of finite type $f : T \to S$ in $\mathscr{S}$. According to [GD67, 17.16.4], there exists a partition $(S_i)_{i\in I}$ of S by (affine) subschemes and a family of maps of the form

$$S_i'' \xrightarrow{g_i} S_i' \xrightarrow{h_i} S_i$$

such that g_i (resp. h_i) satisfies assumption (a) (resp. (b)) above and such that for any $i \in I$, $f \times_S S_i''$ admits a section. Thus, proposition 2.3.6 concludes. □

2.3.c. *Localization and the support property.* The following lemma is obvious from relations 2.3.1.

Lemma 2.3.10. *Consider a closed immersion $i : Z \to S$ such that $\mathscr{T}$ satisfies the property (Loc_i).*
Then $\mathscr{T}$ satisfies the transversality property with respect to i.

Proposition 2.3.11. *Assume that $\mathscr{T}$ satisfies the localization property.*
Then the following conditions are equivalent:

(i) For any scheme S and any integer $n > 0$, $\mathscr{T}$ satisfies the transversality property with respect to the projection $\mathbf{P}^n_S \to S$.
(ii) $\mathscr{T}$ satisfies the proper transversality property.

Proof. We prove that (i) implies (ii). We have to prove that for any cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & \Delta & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X, \end{array}$$

such that f is proper, the exchange transformation $Ex(\Delta_*^*) : g^* f_* \to f'_* g'^*$ is an isomorphism.

We first treat the case where f is projective. Recall that, from proposition 1.1.18, we know that $Ex(\Delta_*^*)$ is an isomorphism as soon as g is an open immersion. In particular, given any open immersion $j : U \to X$, denoting by Δ_U the pullback of Δ along j, we can check that $j^* Ex(\Delta_*^*)$ can be identified with $Ex((\Delta_U)_*^*)$ through exchange isomorphisms. Thus, according to the Zariski separation property (*cf.* proposition 2.3.8), we can assume that X is affine. Then f can be factored into a closed immersion $Y \to \mathbf{P}^n_X$ and the projection $p : \mathbf{P}^n_X \to X$ so that the preceding lemma and assumption (i) concludes.

[10] A warning: the proof in *loc. cit.* seems to require that the schemes are excellent.

To treat the general case, we argue by noetherian induction on Y, assuming that for any proper closed subscheme T of Y, the result is known for the composite square $(\Delta\Theta)$ of the two cartesian squares:

(2.3.11.1)
$$\begin{array}{ccccc} T' & \longrightarrow & Y' & \longrightarrow & X' \\ \downarrow & \Theta & \downarrow & \Delta & \downarrow g \\ T & \xrightarrow[i]{} & Y & \xrightarrow[f]{} & X. \end{array}$$

In fact, the case $Z = \varnothing$ is obvious because $\mathscr{T}(\varnothing) = 0$.

According to Chow's lemma [GD61, 5.6.2], there exists a morphism $p : Y_0 \to Y$ such that:

(a) p and $f \circ p$ are projective morphisms.

(b) There exists an open immersion $j : V \to Y$ such that $q : p^{-1}(V) \to V$ is an isomorphism.

If j is the identity, then we are in the case treated above. Thus we can assume that the complement T of V in Y is a proper subscheme. Let $i : T \to Y$ be the closed immersion where T is seen with its reduced structure.

Consider the following cartesian square

$$\begin{array}{ccc} Y'_0 & \xrightarrow{p'} & Y' \\ {\scriptstyle g_0}\downarrow & \Omega & \downarrow{\scriptstyle g'} \\ Y_0 & \xrightarrow[p]{} & Y. \end{array}$$

Then the following diagram of exchange transformations

(2.3.11.2)
$$\begin{array}{ccccc} g^* f_* p_* & \xrightarrow{Ex(\Delta^*_*).p_*} & f'_* g'^* p_* & \xrightarrow{f'_*.Ex(\Omega^*_*)} & f'_* p'_* g_0^* \\ \| & & & & \| \\ g^*(fp)_* & & \xrightarrow{\qquad Ex((\Delta\Omega)^*_*) \qquad} & & (f'p')_* g_0^* \end{array}$$

is commutative so that (a) and the preceding case implies that $Ex(\Delta^*_*).p_*$ is an isomorphism. But point (b) now implies that $Ex(\Delta^*_*).j_*$ is an isomorphism. Consider the diagram (2.3.11.1) for the immersion i introduced above. The case of a closed immersion treated above implies that $Ex(\Theta^*_*)$ is an isomorphism. By the inductive assumption, $Ex((\Delta\Theta)^*_*)$ is an isomorphism. Thus, considering the obvious analog of the commutative diagram (2.3.11.2), we conclude that $Ex(\Delta^*_*).i_*$ is an isomorphism. Thus, the point (3) of proposition 2.3.3 allows to conclude. □

Lemma 2.3.12. *Consider a closed immersion $i : Z \to S$ such that $\mathscr{T}$ satisfies the property (Loc$_i$) and a cartesian square in $\mathscr{S}$*

$$\begin{array}{ccc} V & \xrightarrow{k} & T \\ {\scriptstyle q}\downarrow & \Delta & \downarrow{\scriptstyle p} \\ U & \xrightarrow[j]{} & S \end{array}$$

where j is the open immersion complementary to i and p satisfies the $\mathscr{T}$-transversality property.

Then the exchange transformation (see 2.2.4)

$$Ex(\Delta_{\sharp *}) : j_\sharp p_* \to q_* k_\sharp$$

is an isomorphism.

Proof. Using property (Loc$_i$), we have only to prove that $j^* Ex(\Delta_{\sharp *})$ and $i^* Ex(\Delta_{\sharp *})$ are invertible. This follows from the $\mathscr{P}$-base change formula and from the $\mathscr{T}$-transversality assumption on p by 2.3.1(a) and 2.3.1(c). □

Corollary 2.3.13. *Assume $\mathscr{T}$ satisfies the localization property and the the transversality property with respect to the projection $\mathbf{P}^n_S \to S$ for any scheme S and any integer $n > 0$.*

Then $\mathscr{T}$ satisfies the support property.

Proof. Lemma 2.3.7 implies $\mathscr{T}$ is additive and the preceding lemma concludes. □

2.3.d. *Localization and monoidal structures.*

2.3.14. Assume $\mathscr{T}$ is monoidal and let M denote its geometric sections. Fix a closed immersion $i : Z \to S$ in $\mathscr{S}$ with complementary open immersion $j : U \to S$. We fix an object $M_S(S/S-Z)$ of $\mathscr{T}(S)$ and a distinguished triangle

$$M_S(S-Z) \xrightarrow{j_*} \mathbb{1}_S \xrightarrow{p_i} M_S(S/S-Z) \xrightarrow{d_i} M_S(S-Z)[1]. \tag{2.3.14.1}$$

Remark that according to 2.3.1(c), the map $i^*(p_i) : \mathbb{1}_Z \to i^*M_S(S/S-Z)$ is an isomorphism. Given any object K in $\mathscr{T}(S)$, we thus obtain an isomorphism

$$i^*(M_S(S/S-Z) \otimes_S K) = i^*(M_S(S/S-Z)) \otimes_Z i^*(K) \xrightarrow{(i^*p_i)^{-1}} \mathbb{1}_Z \otimes_Z i^*(K) = i^*(K)$$

which is natural in K. It induces by adjunction a map

$$\psi_{i,K} : M_S(S/S-Z) \otimes_S K \to i_*i^*(K) \tag{2.3.14.2}$$

which is natural in K.
For any $\mathscr{P}$-scheme X/S, we put $M_S(X/X-X_Z) = M_S(S/S-Z) \otimes_S M_S(X)$ so that we get from (2.3.14.1) a canonical distinguished triangle:

$$M_S(X-X_Z) \xrightarrow{j_{X*}} M_S(X) \to M_S(X/X-X_Z) \to M_S(X-X_Z)[1].$$

The map (2.3.14.2) for $K = M_S(X)$ gives a canonical map

$$\psi_{i,X} : M_S(X/X-X_Z) \to i_*(M_Z(X_Z)). \tag{2.3.14.3}$$

Proposition 2.3.15. *Consider the previous hypothesis and notations. Then the following conditions are equivalent:*

(i) $\mathscr{T}$ *satisfies the property (Loc$_i$).*

(ii)
- *(a) The functor i_* is conservative.*
- *(b) The morphism $\psi_{i,S} : M_S(S/S-Z) \to i_*(\mathbb{1}_Z)$ is an isomorphism.*
- *(c) For any object K of $\mathscr{T}(S)$, the exchange transformation*

$$Ex(i_*^*, \otimes) : (i_*\mathbb{1}_Z) \otimes_S K \to i_*i^*K$$

is an isomorphism.

(iii)
- *(a) The functor i_* is conservative.*
- *(b) The morphism $\psi_{i,S} : M_S(S/S-Z) \to i_*(\mathbb{1}_Z)$ is an isomorphism.*
- *(c) For any objects K and L of $\mathscr{T}(S)$, the exchange transformation*

$$Ex(i_*^*, \otimes) : (i_*K) \otimes_S L \to i_*(K \otimes_Z i^*L)$$

is an isomorphism.

Assume in addition that $\mathscr{T}$ is well generated and τ-twisted as a triangulated $\mathscr{P}$-fibred category. Then the above conditions are equivalent to the following one:

(iv)
- *(a) The functor i_* is conservative, commutes with direct sums and with τ-twists.*
- *(b) The morphism $\psi_{i,X} : M_S(X/X-X_Z) \to i_*(M_Z(X_Z))$ is an isomorphism for any $\mathscr{P}$-scheme X/S.*

In particular, (Loc$_i$) implies that for any object K of $\mathscr{T}(S)$, the localization triangle of 2.3.3

$$j_\sharp j^*(K) \to K \to i_*i^*(K) \xrightarrow{\partial_K} j_\sharp j^*(K)[1]$$

is canonically isomorphic (through exchange transformations) to the triangle (2.3.14.1) tensored with K.

Proof. $(i) \Rightarrow (iii)$: According to (Loc$_i$) (a), we need only to check that the maps in (iii)(b) and (iii)(c) are isomorphisms after applying i^* and j^*. This follows easily from (Loc$_i$) (b).
$(iii) \Rightarrow (ii)$: Obvious
$(ii) \Rightarrow (i)$: According to (ii)(b), the distinguished triangle (2.3.14.1) is isomorphic to a triangle of the form

$$j_\sharp j^*(\mathbb{1}_S) \xrightarrow{ad'(j_\sharp, j^*)} \mathbb{1}_S \xrightarrow{ad(i^*, i_*)} i_*i^*(\mathbb{1}_Z) \to j_\sharp j^*(\mathbb{1}_S).$$

According to (ii)(c), this latter triangle tensored with K is isomorphic through exchange transformations to a triangle of the form

$$j_\sharp j^*(K) \xrightarrow{ad'(j_\sharp, j^*)} K \xrightarrow{ad(i^*, i_*)} i_* i^*(K) \to j_\sharp j^*(K).$$

Thus lemma 2.3.5 allows to conclude.

To end the proof, we remark by using the equations for the adjunction (i^*, i_*) that for any object M of $\mathscr{T}(S)$, the following diagram is commutative:

$$\begin{array}{ccccc} & & i_* i^*(\mathbb{1}_S) \otimes K & = & i_*(\mathbb{1}_Z) \otimes K \\ & \overset{\psi_i \otimes 1_K}{\nearrow} & & & \downarrow {\scriptstyle Ex(i_*^*, \otimes)} \\ M_S(S/S-Z) \otimes K & & & & \\ & \underset{\psi_{i,K}}{\searrow} & & & \\ & & i_* i^*(K) & = & i_*(\mathbb{1}_Z \otimes i^* i^*(K)). \end{array}$$

Note that (i) implies that i_* is conservative and commutes with direct sums (see 2.3.3) and (ii)(c) implies it commutes with twists. According to the above diagram, (ii)(b) implies (iv)(b).
We prove that reciprocally that (iv) implies (ii). Because (ii)(b) (resp. (ii)(a)) is a particular case of (iv)(b) (resp. (iv)(a)), we have only to prove (ii)(b). In view of the previous diagram, we are reduced to prove that for any object K of $\mathscr{T}(S)$, the map $\psi_{i,K}$ is an isomorphism. Consider the full subcategory $\mathscr{U}$ of $\mathscr{T}(S)$ made of the objects K such that $\psi_{i,K}$ is an isomorphism. Then $\mathscr{U}$ is triangulated. Using (iv)(a), $\mathscr{U}$ is stable by small sums and τ-twists. By assumption, it contains the objects of the form $M_S(X)$ for a $\mathscr{P}$-scheme X/S. Thus, because $\mathscr{T}$ is well generated by assumption, lemma 1.3.16 concludes. □

Lemma 2.3.16. *Consider a closed immersion $i : Z \to S$. We assume the following assumption (in addition to that of 2.0):*

- *$\mathscr{T}$ is well generated, τ-twisted and satisfies the Zariski separation property.*
- *For any $\mathscr{P}$-scheme X_0/Z and any point x_0 of X_0, there exists an open neighbourhood U_0 of x_0 in X_0 and a $\mathscr{P}$-scheme U/S such that $U_0 = U \times_S Z$.*[11]

Then the functor i_* is conservative.

Proof. Consider an object K of $\mathscr{T}(Z)$ such that $i_*(K) = 0$. We prove that $K = 0$.
Because $\mathscr{T}$ is τ-generated, it is sufficient to prove that for a $\mathscr{P}$-morphism $p_0 : X_0 \to Z$ and a twist $(n, i) \in \mathbf{Z} \times \tau$,

$$\mathrm{Hom}_{\mathscr{T}(Z)}(M_Z(X_0)\{i\}[n], K) = 0.$$

Because $M_Z(X_0) = p_{0\sharp}(\mathbb{1}_{X_0})$, this equivalent to prove that

$$\mathrm{Hom}_{\mathscr{T}(X_0)}(\mathbb{1}_{X_0}\{i\}[n], p_0^*(K)) = 0.$$

Using the Zariski separation property on $\mathscr{T}$, this latter assumption is local in X_0. Thus, according to the assumption on the class $\mathscr{P}$, we can assume there exists a $\mathscr{P}$-scheme X/S such that $X_0 = X \times_S Z$. Thus $M_Z(X_0)\{i\}[n] = i^*(M_S(X)\{i\}[n])$ and the initial assumption on K allows to conclude. □

Note for future applications the following interesting corollaries:

Corollary 2.3.17. *Assume $\mathscr{T}$ is a premotivic triangulated category which is compactly τ-generated for a group of twists τ (*i.e. *any twists in τ admits a tensor inverse) and which satisfies the Zariski separation property.*

Then, for any closed immersion i, the functor i_* is conservative, commutes with sums and with twists.

This is a consequence of lemmas 2.3.16 and 2.2.16. In fact, under these conditions, i_* commutes with arbitrary τ-twists because it is true for its (left) adjoint i^*.

[11] This property is trivial when $\mathscr{P}$ is the class of open immersions or the class of morphisms of finite type in $\mathscr{S}$. It is also true when $\mathscr{P}$ is the class of étale morphism or $\mathscr{P} = Sm$ (*cf.* [GD67, 18.1.1]).

Corollary 2.3.18. *Assume $\mathscr{T}$ satisfies the assumptions of the preceding corollary. Then the following conditions on a closed immersion i are equivalent:*

(i) $\mathscr{T}$ satisfies the property (Loc$_i$).
(ii) For any scheme S in $\mathscr{S}$ and any smooth S-scheme X, the map (2.3.14.3)

$$\psi_{i,X} : M_S(X/X - X_Z) \to i_* M_Z(X_Z)$$

is an isomorphism.

2.3.e. *Localization and morphisms.*

Lemma 2.3.19. *Let $\mathscr{T}'$ be another complete triangulated $\mathscr{P}$-fibred category over $\mathscr{S}$, and consider a morphism of triangulated $\mathscr{P}$-fibred categories over $\mathscr{S}$*

$$\varphi^* : \mathscr{T} \rightleftarrows \mathscr{T}' : \varphi_* .$$

Let $i : Z \to S$ be a closed immersion with complementary open immersion $j : U \to S$. If $\mathscr{T}$ and $\mathscr{T}'$ satisfies the property (Loc$_i$), then the following exchange transformation is an isomorphism

$$Ex(\varphi^*, i_*) : \varphi^* i_* \longrightarrow i_* \varphi.$$

Proof. We use the facts that (i^*, j^*) is conservative and φ^* commutes with i^*, j^*. Then relations 2.3.1(d) and (Loc$_i$) (b) concludes. □

2.3.20. Recall that to any morphism

$$\varphi^* : \mathscr{T} \to \mathscr{T}'$$

of complete triangulated $\mathscr{P}$-fibred categories satisfying the support property, and to any separable morphism of finite type $f : T \to S$ in $\mathscr{S}$, we have associated in 2.2.11 a canonical exchange transformation

$$Ex(\varphi^*, f_!) : \varphi^* f_! \to f_! \varphi^*.$$

Proposition 2.3.21. *Let $\mathscr{T}'$ be another complete triangulated $\mathscr{P}$-fibred category over $\mathscr{S}$, and consider a morphism of triangulated $\mathscr{P}$-fibred categories over $\mathscr{S}$*

$$\varphi^* : \mathscr{T} \rightleftarrows \mathscr{T}' : \varphi_* .$$

Assume $\mathscr{T}$ and $\mathscr{T}'$ satisfy the localization and the support properties. Then the following conditions are equivalent:

(i) For any scheme S and any integer $n > 0$, given the projection $p : \mathbf{P}^n_S \to S$, the exchange transformation

$$Ex(\varphi^*, p_*) : \varphi^* p_* \to p_* \varphi^*$$

is an isomorphism.
(ii) For any separated morphism of finite type $f : T \to S$, the exchange transformation

$$Ex(\varphi^*, f_!) : \varphi^* f_! \to f_! \varphi^*$$

is an isomorphism.

Proof. The only thing to prove is that, assuming (i), condition (ii) holds. We consider a morphism of finite type $f : T \to S$ and we prove that $Ex(\varphi^*, f_!)$ is an isomorphism.

This holds when f is an immersion according to lemma 2.3.19.

Suppose that f is affine. Then there exists an integer $n \geq 0$ such that it admits a factorization into an immersion and the projection $\mathbf{P}^n_S \to S$ so that condition (i) allows to conclude.

To finish the proof, we use a noetherian induction on T. Indeed, as $\mathscr{T}(\varnothing) = 0$ and $\mathscr{T}'(\varnothing) = 0$, the proposition holds trivially when $T = \varnothing$ (or is a consequence of the case of an immersion). Thus, by noetherian induction, we can assume that for any closed immersion $i : Z \to T$ of a proper closed subscheme Z in T, the exchange transformation $Ex(\varphi^*, (fi)_!)$ is an isomorphism.
Consider an open immersion $j : U \to T$ of some non empty affine scheme. Then, according to the case treated above, $Ex(\varphi^*, (fj)_!)$ is an isomorphism. Let $i : Z \to T$ be the complementary open

immersion to j. By induction, $Ex(\varphi^*,(fi)_!)$ is invertible. Let k be one of the immersions i, j. Then the following diagram is commutative:

$$\begin{array}{ccccc} \varphi^* f_! k_! & \xrightarrow{Ex(\varphi^*,f_!).k_!} & f_!\varphi^* k_! & \xrightarrow{f_!.Ex(\varphi^*,k_!)} & f_! k_! \varphi^* \\ \| & & & & \| \\ \varphi^*(fk)_! & & \xrightarrow{Ex(\varphi^*,(fk)_!)} & & (fk)_!\varphi^*. \end{array}$$

Thus, according to the case treated above, $Ex(\varphi^*, f_!).k_!$ is an isomorphism and we can conclude using the localization triangle for i (see 2.3.3(2)). □

Remark 2.3.22. When the equivalent conditions of this proposition will be satisfied, we will simply say that φ^* commutes with $f_!$.

We finish this section with the following useful result:

Proposition 2.3.23. *Assume $\mathscr{T}$ is τ-twisted and consider a τ'-twisted triangulated $\mathscr{P}$-fibred category $\mathscr{T}'$ and a morphism*

$$\varphi^* : (\mathscr{T},\tau) \rightleftarrows (\mathscr{T}',\tau') : \varphi_*$$

compatible with twists. We assume the following assumption:

(a) The map $\tau \to \tau'$ induced by φ^* is (essentially) surjective.
(b) $\mathscr{T}'$ is well generated.

We consider a closed immersion $i : Z \to S$ and assume the following:

(c) $\mathscr{T}$ satisfies the property (Loc_i).
(d) The exchange transformation $Ex(\varphi^*, i_*) : \varphi^* i_* \to i_* \varphi^*$ is an isomorphism.
(e) The functor $i_* : \mathscr{T}'(Z) \to \mathscr{T}'(S)$ commutes with τ'-twists.[12]

Then $\mathscr{T}'$ satisfies the property (Loc_i).

Proof. Note that, under the above assumptions, φ_* is conservative (in fact, for any $\mathscr{P}$-scheme X/S and any twists $i \in \tau'$, $M'_S(X)\{i\}$ is in the essential image of φ^*). Thus, if $i_* : \mathscr{T}(Z) \to \mathscr{T}(S)$ is conservative (resp. commute with sums), then $i_* : \mathscr{T}'(S) \to \mathscr{T}'(S)$ is conservative (resp. commute with sums) using the isomorphism $\varphi_* i_* \simeq i_* \varphi_*$.
Let M (resp. M') be the geometric sections of $\mathscr{T}$ (resp. $\mathscr{T}'$). As in 2.3.14, we fix a distinguished triangle

$$M_S(S-Z) \xrightarrow{j_*} \mathbb{1}_S \xrightarrow{p_i} M_S(S/S-Z) \xrightarrow{d_i} M_S(S-Z)[1].$$

and we put $M'_S(S/S-Z) = \varphi^* M_S(S/S-Z)$. According to *loc. cit.*, we thus get for any $\mathscr{P}$-scheme X/S canonical maps

$$\psi_{i,X} : M_S(X/X-X_Z) \to i_* M_Z(X_Z),$$
$$\psi'_{i,X} : M'_S(X/X-X_Z) \to i_* M'_Z(X_Z).$$

By construction, the following diagram is commutative:

$$\begin{array}{ccccc} \varphi^* M_S(X/X-X_Z) & \xrightarrow{\varphi^*\psi_{i,X}} & \varphi^* i_* M_Z(X_Z) & \xrightarrow{Ex(\varphi^*,i_*)} & i_*\varphi^* M_Z(X_Z) \\ \| & & & & \| \\ M'_S(X/X-X_Z) & & \xrightarrow{\psi'_{i,X}} & & M'_Z(X_Z) \end{array}$$

Thus, proposition 2.3.15 allows to conclude. □

[12]This will be satisfied if any τ'-twists is invertible because the left adjoint of i_* commutes with τ'-twists.

2.4. **The theorem of Ayoub.**

2.4.1. Recall that in this part, according to assumption 2.0, $\mathscr{T}$ is a complete triangulated Sm-fibred category. In the next definition, we give a redundant terminology for some crucial properties of $\mathscr{T}$.

Definition 2.4.2. A *pregeometric category over* $\mathscr{S}$ is a triangulated complete Sm-fibred category over $\mathscr{S}$ which satisfies the homotopy, stability and localization properties.

A *geometric category over* $\mathscr{S}$ is a pregeometric category $\mathscr{T}$ over $\mathscr{S}$ such that, for any proper morphism f in $\mathscr{S}$, the functor f_* has a right adjoint.

A *motivic category over* $\mathscr{S}$ is a premotivic triangulated category over $\mathscr{S}$ which is also geometric.

Remark 2.4.3. Assume $\mathscr{T}$ is a premotivic triangulated category such that:

(1) $\mathscr{T}$ is compactly τ-generated.
(2) $\mathscr{T}$ satisfies the homotopy and stability properties.
(3) $\mathscr{T}$ satisfies the localization property.

Then $\mathscr{T}$ is a motivic triangulated category in the above sense (see lemma 2.2.16).[13] More generally, we will see below that, if $\mathscr{T}$ is pregeometric and well generated, then it is geometric; see 2.4.18.

Example 2.4.4. The premotivic category SH of example 1.4.3 is a motivic category (applying the above remark).

Following [Ayo07a, section 1.5], we define:

Definition 2.4.5. Let $p : E \to S$ be a vector bundle over a scheme in $\mathscr{S}$, and s its zero-section. We define the Thom transformation associated with E/X as the composite

$$\mathcal{T}h(E) = p_\sharp s_* : \mathscr{T}(S) \to \mathscr{T}(S).$$

2.4.6. Let us recall an important construction about Thom transformations from [Ayo07a, p. 97]. Consider an exact sequence of vector bundles over a scheme X in $\mathscr{S}$:

$$0 \to E' \xrightarrow{\nu} E \xrightarrow{\pi} E'' \to 0. \tag{σ}$$

Note that π is smooth and ν is a closed immersion. Let (p, s), (p', s'), (p'', s'') be the pair of natural projection/section of the respective vector bundles. We can derive from this exact sequence a commutative diagram in $\mathscr{S}$:

$$\begin{array}{ccccc} X & & \xrightarrow{s} & & \\ {\scriptstyle s'}\downarrow & & & & \\ E' & \xrightarrow{\nu} & E & & \\ {\scriptstyle p'}\downarrow & \Delta & \downarrow{\scriptstyle \pi} & \searrow{\scriptstyle p} & \\ X & \xrightarrow[s'']{} & E'' & \xrightarrow[p'']{} & X \end{array}$$

where Δ is cartesian. Then we can define an exchange transformation:

$$Ex(\sigma) : \mathcal{T}h(E) = p_\sharp s_* = p''_\sharp \pi_\sharp \nu_* s'_* \xrightarrow{Ex(\Delta_{\sharp *})} p''_\sharp s''_* p'_\sharp s'_* = \mathcal{T}h(E'') \circ \mathcal{T}h(E').$$

Let S be a scheme. We denote by p and s (resp. q and t) the projection and zero section of $\mathbf{P}^1_S/S$ (resp. $\mathbf{A}^1_S/S$). The unit map $1 \to p_* p^*$ is a split monomorphism. Thus, by adjunction, the counit map $p_\sharp p^* \xrightarrow{a_p} 1$ is a split epimorphism.

Proposition 2.4.7. *Assume $\mathscr{T}$ satisfies properties (Loc) and (Htp).*

Then using the notations above, there exists a canonical isomorphism

$$\mathrm{Ker}(p_\sharp p^* \xrightarrow{a_p} 1) \simeq \mathcal{T}h(\mathbf{A}^1_S).$$

[13] In our examples, (1) will always be satisfied, (2) will be obtained by construction and (3) will be the hard point.

Proof. Let j (resp. k) be the complementary open immersion to s (resp. t). We obtain a cartesian square

$$\begin{array}{ccc} \mathbf{G}_m & \xrightarrow{k} & \mathbf{A}^1_S \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u} \\ \mathbf{A}^1_S & \xrightarrow{j} & \mathbf{P}^1_S \end{array}$$

where u is the complementary immersion to the ∞-section. Let $h = u \circ k$. According to the $\mathscr{P}$-base change formula, we get an exchange isomorphism $h_\sharp h^* \simeq j_\sharp j^* u_\sharp u^*$ so that we obtain a commutative diagram made of counit morphisms:

$$\begin{array}{ccccc} j_\sharp j^* u_\sharp u^* & = & h_\sharp h^* & \longrightarrow & u_\sharp u^* \\ \downarrow & & (1) & & \downarrow \\ j_\sharp j^* & & \longrightarrow & & 1 \end{array}$$

From the localization property, we thus obtain the following morphism of split (because s and t are split monomorphisms) distinguished squares:

$$(\star) \qquad \begin{array}{ccccccc} h_\sharp h^* & \longrightarrow & u_\sharp u^* & \xrightarrow{a_t} & u_\sharp t_* t^* u^* & \longrightarrow & h_\sharp h^*[1] \\ \downarrow & (1) & \downarrow & & \downarrow{\scriptstyle \epsilon} & & \downarrow \\ j_\sharp j^* & \longrightarrow & 1 & \xrightarrow{a_s} & s_* s^* & \longrightarrow & j_\sharp j^*[1]. \end{array}$$

But ϵ is an isomorphism: because (j, u) defines an open cover and $\mathscr{T}$ is Zariski separated according to 2.3.8, it is sufficient to check that $j^*\epsilon$ and $u^*\epsilon$ are isomorphisms and this follows from the previous diagram and relation 2.3.1(a).

We thus obtain a canonical isomorphism

$$q_\sharp t_* = p_\sharp u_\sharp t_* t^* u^* p^* \xrightarrow{\epsilon} p_\sharp s_* s^* p^* = p_\sharp s_*.$$

From the bottom line of diagram $(\star)$, we get a distinguished triangle

$$q_\sharp q^* \xrightarrow{a'_j} p_\sharp p^* \xrightarrow{a_s} p_\sharp t_* \to q_\sharp q^*[1]$$

so that $q_\sharp t^*$ is the cokernel of the split monomorphism a'_j. But the following triangle, made of counit morphisms, is commutative

$$\begin{array}{ccc} q_\sharp q^* & \xrightarrow{a'_j} & p_\sharp p^* \\ \downarrow & \swarrow{\scriptstyle a_p} & \\ 1 & & \end{array}$$

so that property (Htp) allows to conclude. □

2.4.8. We introduce a useful terminology in order to simplify the next statements:
A morphism of schemes $f : Y \to X$ is *strictly quasi-projective (resp. strictly projective)* if it can be factored over X through an immersion (resp. a closed immersion) $Y \xrightarrow{i} \mathbf{P}^n_X$ for an integer $n \geq 0$. We let $\mathscr{S}_0$ be the subcategory of $\mathscr{S}$ made of strictly quasi-projective $\mathcal{S}$-schemes which are in $\mathscr{S}$. As a corollary of the preceding lemma, we get the following comparison of our axioms with that of J. Ayoub (see [Ayo07a]).

Corollary 2.4.9. *The following conditions on a 2-functor $\mathscr{T}_0 : {\mathscr{S}_0}^{op} \to \mathscr{T}ri$ are equivalent:*

(i) $\mathscr{T}_0$ is a pregeometric Sm-fibred category over $\mathscr{S}_0$.
(ii) $\mathscr{T}_0$ is a stable homotopy 2-functor in the sense of [Ayo07a, 1.4.1].

Proof. Indeed, all the axioms of *loc. cit.* are clear, except axiom 6 which follows from the preceding proposition. □

Theorem 2.4.10 (Ayoub)**.** *If $\mathscr{T}$ is pregeometric, then, for any scheme S in $\mathscr{S}$ and any integer $n \geq 1$, if $p : \mathbf{P}^n_S \to S$ denotes the structural map, there is a canonical isomorphism of functors $p_\sharp(-n)[-2n] \simeq p_*$ in $\mathscr{T}$.*

Proof. Thanks to corollary 2.4.9, we can refer to [Ayo07a, thm. 1.7.9]. □

Corollary 2.4.11. *Consider a morphism of triangulated Sm-fibred categories $\varphi^* : \mathscr{T} \to \mathscr{T}'$ over $\mathscr{S}$. Assume that $\mathscr{T}$ and $\mathscr{T}'$ are pregeometric. Let S be a scheme in $\mathscr{S}$, $n \geq 1$ an integer, and $p : \mathbf{P}^n_S \to S$ be the canonical projection. Then the exchange transformation $\varphi^* p_* \to p_* \varphi^*$ is an isomorphism.*

Proof. By virtue of the preceding theorem, as φ^* commutes with $p_\sharp$ (by definition of morphisms of Sm-fibred categories), it is now sufficient to prove that φ^* commutes with Thom equivalences $\mathcal{T}h(-E)$ for any vector bundle E over some scheme X in $\mathscr{S}$, or, equivalently, that φ^* commutes with $\mathcal{T}h(E)$, the quasi-inverse of $\mathcal{T}h(-E)$. By definition, we have $\mathcal{T}h(E) = q_\sharp s_*$, where $q : E \to X$ is the projection, and $s : X \to E$ is the zero section. As φ^* commutes with $q_\sharp$, it is sufficient to prove that φ^* commutes with i_* for any closed immersion i. This latter property is given by lemma 2.3.19. □

Theorem 2.4.12. *Assume $\mathscr{T}$ is pregeometric. Then the following properties hold:*

(a) $\mathscr{T}$ satisfies the proper transversality property;
(b) $\mathscr{T}$ satisfies the support property.

Proof. By virtue of proposition 2.3.11 and corollary 2.3.13 it is sufficient to prove that $\mathscr{T}$ has the transversality property with respect to the projection $p : \mathbf{P}^n_S \to S$ for any scheme S in $\mathscr{S}$ and any integer $n > 0$. But this is a particular case of the preceding corollary (see the proof of 2.2.12 (i)). □

Remark 2.4.13. This holds in particular for the motivic category SH of example 2.4.4, by Ayoub's results [Ayo07b].

2.4.14. When $\mathscr{T}$ is pregeometric, according to the previous theorem, we can apply the construction of 2.2.7, so that, for any separated morphism of finite type $f : T \to S$, we get a well defined functor $f_! : \mathscr{T}(T) \to \mathscr{T}(S)$. Recall also that the localization property shows that for any closed immersion i, the functor i_* admits a right adjoint which we denote by $i^!$.
We obtain from [Ayo07a] the following result:

Theorem 2.4.15 (Ayoub)**.** *Assume $\mathscr{T}$ is pregeometric and consider the above notations. Then the following conditions hold.*

(i) For any vector bundle E over a scheme S in $\mathscr{S}$, the Thom transformation $\mathcal{T}h(E) : \mathscr{T}(S) \to \mathscr{T}(S)$ is an equivalence of category with quasi-inverse $\mathcal{T}h(-E) = s^! p^*$, where p is the projection of E onto X, while s denotes the zero section of E).

(ii) For any short exact sequence of vector bundles

$$0 \to E' \xrightarrow{\nu} E \xrightarrow{\pi} E'' \to 0$$

over a scheme X in $\mathscr{S}$, the map

$$\mathcal{T}h(E) \to \mathcal{T}h(E'') \circ \mathcal{T}h(E')$$

is an isomorphism.

(iii) For any smooth strictly quasi-projective morphism $f : Y \to X$ with tangent bundle T_f, there exists a canonical isomorphism

$$f_! \xrightarrow{\sim} f_\sharp \mathcal{T}h(-T_f)\,.$$

Proof. In fact, point (i) is easy from the homotopy and stability properties (see [Ayo07a, 1.5.7]) Point (ii) is proved in [Ayo07a, 1.5.18]. Point (iii) is a consequence of the theorem 1.4.2 of *op. cit.* once we have noticed that the functor $f_!$ constructed by Ayoub agrees with that of proposition 2.2.7 (which we can apply, for any scheme Y in $\mathscr{S}$, over the category of strictly quasi-projective Y-schemes by 2.4.9), according to the uniqueness statement of this proposition. □

Remark 2.4.16. The isomorphism of point (iii) is usually called the *purity isomorphism* associated to f. Note that this isomorphism is compatible with composition in a suitable sense.[14] Using

[14]The interested reader is referred to [Ayo07a, p. 105]. We will state this compatibility in the monoidal case below.

descent theory we will show that one can construct this isomorphism without the assumption that f is strictly quasi-projective.

Proposition 2.4.17. *Consider a morphism of triangulated Sm-fibred categories over $\mathscr{S}$*

$$\varphi^* : \mathscr{T} \to \mathscr{T}' .$$

Assume that $\mathscr{T}$ and $\mathscr{T}'$ are pregeometric. For any separated morphism of finite type $f : T \to S$ in $\mathscr{S}$, the exchange transformation

$$\varphi^* f_! \to f_! \varphi^*$$

is an isomorphism.

Proof. This is a direct consequence of proposition 2.3.21, lemma 2.4.11, and theorem 2.4.12. □

Corollary 2.4.18. *If $\mathscr{T}$ is pregeometric and well generated, then it is geometric.*

Proof. By the Brown representability theorem [Nee01, 8.4.4], it is sufficient to prove that, for any separated morphism of finite type $f : T \to S$, the functor $f_!$ preserves small sums. Let I be small set. For a scheme X in $\mathscr{S}$, we write $\mathscr{T}^I(X)$ for the category of families of objects of $\mathscr{T}(X)$ indexed by I. This defines a pregeometric category over $\mathscr{S}$ (the adjunctions in $\mathscr{T}^I$ are just computed termwise in $\mathscr{T}$). For any smooth morphism of finite type f in $\mathscr{S}$, the functor $f_\sharp$ preserves small sums, and, for any morphism f in $\mathscr{S}$, the functor f^* preserves smalls sums. In other words, we have a morphism of (complete) triangulated *Sm*-fibred categories

$$\mathscr{T}^I \to \mathscr{T} \ , \quad (M)_{i\in I} \mapsto \bigoplus_{i\in I} M_i \, .$$

We conclude by applying proposition 2.4.17 to the latter. □

Corollary 2.4.19. *Assume that $\mathscr{T}$ is geometric. Then, for any separated morphism of finite type $f : Y \to X$, we have an adjunction*

$$f_! : \mathscr{T}(Y) \rightleftarrows \mathscr{T}(X) : f^! \, ,$$

and a natural tranformation $f_! \to f_*$, which is an isomorphism whenever f is proper. Moreover, for any cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & \Delta & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X, \end{array}$$

in which f is separated and of finite type, the exchange maps

$$g^* f_! \to f'_! g'^* \quad \textit{and} \quad g'_* f'^! \to f^! g_*$$

are isomorphisms.

Proof. The existence of $f_!$ and $f^!$ follows from theorems 2.2.14 and 2.4.12. Consider a pullback square Δ as above. The isomorphism $g^* f_! \simeq f'_! g'^*$ follows from proposition 2.4.17, and the isomorphism $g'_* f'^! \simeq f^! g_*$ is obtained from the preceding one by transposition. □

2.4.20. We assume now that $\mathscr{T}$ is a geometric and monoidal (i.e. symmetric monoidal as a complete triangulated *Sm*-fibred category over $\mathscr{S}$), with geometric section M.

Let $p : E \to S$ be a vector bundle with zero section $s : S \to E$. We define the *Thom motive* associated with E/S as the object $MTh(E) = p_\sharp s_*(\mathbb{1}_X)$ in $\mathscr{T}(S)$. Note that the localization axiom implies that this motive is defined uniquely (*cf.* 2.3.3(2)) by the distinguished triangle

$$M_S(E - S) \xrightarrow{j_*} M_S(E) \to MTh(E) \to M_S(E - S)[1].$$

Also, by proposition 2.4.7, we have $MTh(\mathbf{A}^1_S) \simeq \mathbb{1}_S(1)[2]$.
Moreover, according to 2.3.15, for any object K of $\mathscr{T}(X)$, we can compute the Thom transformation:

$$\mathcal{T}h(E)(K) \simeq K \otimes MTh(E). \tag{2.4.20.1}$$

Thus the premotive $MTh(E)$ is $\otimes$-invertible. We denote by $MTh(-E) = s^!(\mathbb{1}_E)$ its $\otimes$-inverse. Finally, the construction of Ayoub recalled in 2.4.6 allows to associate to any short exact sequence σ as in *loc. cit.* an isomorphism in $\mathscr{T}(S)$:

$$\epsilon_\sigma : MTh(E) \xrightarrow{\sim} MTh(E') \otimes MTh(E'').$$

Theorem 1.4.2 of [Ayo07a] then implies:

Theorem 2.4.21. *Under the assumptions of 2.4.20, for any separated morphism of finite type $f : Y \to X$ in $\mathscr{S}$, there exists a pair of adjoint functors*

$$f_! : \mathscr{T}(Y) \rightleftarrows \mathscr{T}(X) : f^!$$

such that:

(i) There exists a structure of a covariant (resp. contravariant) 2-functor on $f \mapsto f_!$ (resp. $f \mapsto f^!$).

(ii) There exists a natural transformation $\alpha_f : f_! \to f_*$ which is an isomorphism when f is proper. Moreover, α is a morphism of 2-functors.

(iii) For any smooth strictly projective morphism f in $\mathscr{S}$ with cotangent bundle Ω_f, there are canonical natural isomorphisms

$$\mathfrak{p}_f : f^! \xrightarrow{\sim} MTh(\Omega_f) \otimes f^* \quad \text{and} \quad \mathfrak{q}_f : f_\sharp \otimes MTh(-\Omega_f) \xrightarrow{\sim} f_!$$

which are dual to each other. Moreover, for any smooth strictly quasi-projective morphisms $Z \xrightarrow{g} Y \xrightarrow{f} X$ in $\mathscr{S}$, considering the exact sequence of cotangent bundles

$$(\sigma) \qquad 0 \to g^*\Omega_f \to \Omega_{gf} \to \Omega_g \to 0,$$

the following diagram is commutative:

$$\begin{array}{ccc}
(fg)^! & = & g^!f^! \\
 & & \downarrow{\scriptstyle \mathfrak{p}_g.\mathfrak{p}_f} \\
 & & MTh(\Omega_g) \otimes MTh(g^{-1}\Omega_f) \otimes g^*f^* \\
{\scriptstyle \mathfrak{p}_{gf}}\downarrow & & \downarrow{\scriptstyle (*)} \\
 & & MTh(g^*\Omega_f) \otimes MTh(\Omega_g) \otimes g^*f^* \\
 & & \downarrow{\scriptstyle \epsilon_\sigma} \\
MTh(\Omega_{gf}) \otimes (fg)^* & = & MTh(\Omega_{gf}) \otimes g^*f^*
\end{array}$$

where $(*)$ denotes the natural symmetry isomorphism.

(iv) For any cartesian square:

$$\begin{array}{ccc}
Y' & \xrightarrow{f'} & X' \\
{\scriptstyle g'}\downarrow & \Delta & \downarrow{\scriptstyle g} \\
Y & \xrightarrow[f]{} & X,
\end{array}$$

such that f is separated of finite type, there exist natural isomorphisms

$$g^*f_! \xrightarrow{\sim} f'_!g'^*,$$
$$g'_*f'^! \xrightarrow{\sim} f^!g_*.$$

(v) For any separated morphism of finite type $f : Y \to X$ in $\mathscr{S}$, there exist natural isomorphisms

$$(f_!K) \otimes_X L \xrightarrow{\sim} f_!(K \otimes_X f^*L),$$
$$Hom_X(L, f_!(K)) \xrightarrow{\sim} f_*Hom_Y(f^!(L), K),$$
$$f^!Hom_X(L, M) \xrightarrow{\sim} Hom_Y(f^!(L), f^*(M)).$$

Proof. Given theorem 2.4.12, everything except point (iii) are consequences of theorem 2.2.14 (the first isomorphism of property (v) is a special instance of proposition 2.4.17, while the other ones are obtained from the first by transposition). Point (iii) follows from 2.4.15 applied to $\mathscr{T}|_{\mathscr{S}_0}$. □

Remark 2.4.22. In fact, the main step in the proof of [Ayo07a, 1.4.1] is to construct the morphism $f^!$ by choosing a factorization:

$$Y \xrightarrow{i} P \xrightarrow{p} X \tag{2.4.22.1}$$

where i is a closed immersion and p is a smooth morphism and defining:

$$f^! = (MTh(-N_i) \otimes i^!)(MTh(T_f) \otimes f^*).$$

By virtue of a result of Ayoub [Ayo07a, 1.3.1], this construction is independent of the factorization. Note also that the two possible definitions of the pair of functors $(f_!, f^!)$ was already evoked by Deligne in [AGV73, XVII] (the other one being credited to Grothendieck).

3. Descent in $\mathscr{P}$-fibred model categories

3.0. In this section, $\mathscr{S}$ is an abstract category and $\mathscr{P}$ an admissible class of morphisms in $\mathscr{S}$.

In section 3.3 however, we will consider as in 2.0 a noetherian base scheme $\mathcal{S}$ and we will assume that $\mathscr{S}$ is an adequate category of $\mathcal{S}$-schemes satisfying the following condition on $\mathscr{S}$:

(a) Any scheme in $\mathscr{S}$ is finite dimensional.

Moreover, in sections 3.3.c and 3.3.d, we will even assume:

(a′) Any scheme in $\mathscr{S}$ is quasi-excellent and finite dimensional.

We fix an admissible class $\mathscr{P}$ of morphisms in $\mathscr{S}$ which contains the class of étale morphisms in $\mathscr{S}$ and a stable combinatorial P-fibred model category $\mathscr{M}$ over $\mathscr{S}$.
In section 3.3.d, we will assume furthermore that:

(b) The stable model $\mathscr{P}$-fibred category $\mathscr{M}$ is $\mathbf{Q}$-linear (see 3.2.14).

3.1. Extension of $\mathscr{P}$-fibred categories to diagrams.

3.1.a. *The general case.*

3.1.1. Assume given a $\mathscr{P}$-fibered category $\mathscr{M}$ over $\mathscr{S}$. Then $\mathscr{M}$ can be extended to $\mathscr{S}$-diagrams (i.e. functors from a small category to $\mathscr{S}$) as follows. Let I be a small category, and $\mathscr{X}$ a functor from I to $\mathscr{S}$. For an object i of I, we will denote by $\mathscr{X}_i$ the fiber of $\mathscr{X}$ at i (i.e. the evaluation of $\mathscr{X}$ at i), and, for a map $u : i \to j$ in I, we will still denote by $u : \mathscr{X}_i \to \mathscr{X}_j$ the morphism induced by u. We define the category $\mathscr{M}(\mathscr{X}, I)$ as follows.

An object of $\mathscr{M}(\mathscr{X}, I)$ is a couple (M, a), where M is the data of an object M_i in $\mathscr{M}(\mathscr{X}_i)$ for any object i of I, and a is the data of a morphism $a_u : u^*(M_j) \to M_i$ for any morphism $u : i \to j$ in I, such that, for any object i of I, the map a_{1_i} is the identity of M_i (we will always assume that 1_i^* is the identity functor), and, for any composable morphisms $u : i \to j$ and $v : j \to k$ in I, the following diagram commutes.

$$\begin{array}{ccc} u^*v^*(M_k) & \xrightarrow{\simeq} & (vu)^*(M_k) \\ {\scriptstyle u^*(a_v)}\downarrow & & \downarrow{\scriptstyle a_{vu}} \\ u^*(M_j) & \xrightarrow[a_u]{} & M_i \end{array}$$

A morphism $p : (M, a) \to (N, b)$ is a collection of morphisms

$$p_i : M_i \to N_i$$

in $\mathscr{M}(\mathscr{X}_i)$, for each object i in I, such that, for any morphism $u : i \to j$ in I, the following diagram commutes.

$$\begin{array}{ccc} u^*(M_j) & \xrightarrow{u^*(p_j)} & u^*(N_j) \\ {\scriptstyle a_u}\downarrow & & \downarrow{\scriptstyle b_u} \\ M_i & \xrightarrow[p_i]{} & N_i \end{array}$$

In the case where $\mathscr{M}$ is a monoidal $\mathscr{P}$-fibred category, the category $\mathscr{M}(\mathscr{X}, I)$ is naturally endowed with a symmetric monoidal structure. Given two objects (M, a) and (N, b) of $\mathscr{M}(\mathscr{X}, I)$, their tensor product

$$(M, a) \otimes (N, b) = (M \otimes N, a \otimes b)$$

is defined as follows. For any object i of I,

$$(M \otimes N)_i = M_i \otimes N_i \,,$$

and for any map $u : i \to j$ in I, the map $(a \otimes b)_u$ is the composition of the isomorphism $u^*(M_j \otimes N_j) \simeq u^*(M_j) \otimes u^*(N_j)$ with the morphism

$$a_u \otimes b_u : u^*(M_j) \otimes u^*(N_j) \to M_i \otimes N_i \,.$$

Note finally that if $\mathscr{M}$ is complete monoidal $\mathscr{P}$-fibred category, then $\mathscr{M}(\mathscr{X}, I)$ admits an internal Hom.

3.1.2. *Evaluation functors.* Assume now that for any S, $\mathscr{M}(S)$ admits small sums.

For each object i of I, we have a functor

$$\begin{aligned} i^* : \mathscr{M}(\mathscr{X}, I) &\to \mathscr{M}(\mathscr{X}_i) \\ (M, a) &\longmapsto M_i \end{aligned} \tag{3.1.2.1}$$

This functor i^* has a left adjoint

$$i_\sharp : \mathscr{M}(\mathscr{X}_i) \to \mathscr{M}(\mathscr{X}, I) \tag{3.1.2.2}$$

defined as follows. If M is an object of $\mathscr{M}(\mathscr{X}_i)$, then $i_\sharp(M)$ is the data (M', a') such that for any object j of I,

$$(i_\sharp(M))_j = M'_j = \coprod_{u \in \mathrm{Hom}_I(j,i)} u^*(M) \,, \tag{3.1.2.3}$$

and, for any morphism $v : k \to j$ in I, the map a'_v is the canonical map induced by the collection of maps

$$v^*u^*(M) \simeq (uv)^*(M) \to \coprod_{w \in \mathrm{Hom}_I(k,i)} w^*(M) \tag{3.1.2.4}$$

for $u \in \mathrm{Hom}_I(j, i)$.

If we assume that $\mathscr{M}$ is a complete $\mathscr{P}$-fibred category and that $\mathscr{M}(S)$ admits small products for any S, then i^* has a right adjoint

$$i_* : \mathscr{M}(\mathscr{X}_i) \to \mathscr{M}(\mathscr{X}, I) \tag{3.1.2.5}$$

given, for any object M of $\mathscr{M}(\mathscr{X}_i)$ by the formula

$$(i_*(M))_j = \prod_{u \in \mathrm{Hom}_I(i,j)} u_*(M), \tag{3.1.2.6}$$

with transition map given by the dual formula of 3.1.2.4.

3.1.3. *Functoriality.* Assume that $\mathscr{M}$ if a $\mathscr{P}$-fibred category suth that for any object S of $\mathscr{S}$, $\mathscr{M}(S)$ has small colimits.

Remember that, if $\mathscr{X}$ and $\mathscr{Y}$ are $\mathscr{S}$-diagrams, indexed respectively by small categories I and J, a morphism $\varphi : (\mathscr{X}, I) \to (\mathscr{Y}, J)$ is a couple $\varphi = (\alpha, f)$, where $f : I \to J$ is a functor, and

$\alpha : \mathscr{X} \to f^*(\mathscr{Y})$ is a natural transformation (where $f^*(\mathscr{Y}) = \mathscr{Y} \circ f$). In particular, for any object i of I, we have a morphism

$$\alpha_i : \mathscr{X}_i \to \mathscr{Y}_{f(i)}$$

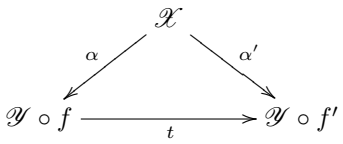
in $\mathscr{S}$. This turns $\mathscr{S}$-diagrams into a strict 2-category: the identity of $(\mathscr{X}, I)$ is the couple $(1_{\mathscr{X}}, 1_I)$, and, if $\varphi = (\alpha, f) : (\mathscr{X}, I) \to (\mathscr{Y}, J)$ and $\psi = (\beta, g) : (\mathscr{Y}, J) \to (\mathscr{Z}, K)$ are two composable morphisms, the morphism $\psi \circ \varphi : (\mathscr{X}, I) \to (\mathscr{Z}, K)$ is the couple (gf, γ), where for each object i of I, the map

$$\gamma_i : \mathscr{X}_i \to \mathscr{Z}_{g(f(i))}$$

is the composition

$$\mathscr{X}_i \xrightarrow{\alpha_i} \mathscr{Y}_{f(i)} \xrightarrow{\beta_{f(i)}} \mathscr{Z}_{g(f(i))}\,.$$

There is also a notion of natural transformation between morphisms of $\mathscr{S}$-diagrams: if $\varphi = (\alpha, f)$ and $\varphi' = (\alpha', f')$ are two morphisms from $(\mathscr{X}, I)$ to $(\mathscr{Y}, J)$, a natural transformation t from φ to φ' is a natural transformation $t : f \to f'$ such that the following diagram of functors commutes.



This makes the category of $\mathscr{S}$-diagrams a (strict) 2-category.

To a morphism of diagrams $\varphi = (\alpha, f) : (\mathscr{X}, I) \to (\mathscr{Y}, J)$, we associate a functor

$$\varphi^* : \mathscr{M}(\mathscr{Y}, J) \to \mathscr{M}(\mathscr{X}, I)$$

as follows. For an object (M, a) of $\mathscr{M}(\mathscr{Y})$, $\varphi^*(M, a) = (\varphi^*(M), \varphi^*(a))$ is the object of $\mathscr{M}(\mathscr{X})$ defined by $\varphi^*(M)_i = M_{f(i)}$ for i in I, and by the formula $\varphi^*(a)_u = a_{f(u)} : f(u)^*(M_{f(j)}) \to M_{f(i)}$ for $u : i \to j$ in I.

We will say that a morphism $\varphi : (\mathscr{X}, I) \to (\mathscr{Y}, J)$ is a *$\mathscr{P}$-morphism* if, for any object i in I, the morphism $\alpha_i : \mathscr{X}_i \to \mathscr{Y}_{f(i)}$ is a $\mathscr{P}$-morphism. For such a morphism φ, the functor φ^* has a left adjoint which we denote by

$$\varphi_\sharp : \mathscr{M}(\mathscr{X}, I) \to \mathscr{M}(\mathscr{Y}, J)\,.$$

For instance, given a $\mathscr{S}$-diagram $\mathscr{X}$ indexed by a small category I, each object i of I defines a $\mathscr{P}$-morphism of diagrams $i : \mathscr{X}_i \to (\mathscr{X}, I)$ (where $\mathscr{X}_i$ is indexed by the terminal category), so that the corresponding the functor $i_\sharp$ corresponds precisely to (3.1.2.2).

Assume that $\mathscr{M}$ is a complete $\mathscr{P}$-fibred category such that $\mathscr{M}(S)$ has small limits for any object X of $\mathscr{S}$. Then the functor φ^* has a right adjoint which we denote by

$$\varphi_* : \mathscr{M}(\mathscr{X}, I) \to \mathscr{M}(\mathscr{Y}, J)\,.$$

In the case where φ is the morphism $i : \mathscr{X}_i \to (\mathscr{X}, I)$ defined by an object i of I, i_* corresponds precisely to (3.1.2.5).

Remark 3.1.4. This construction can be applied in particular to any Grothendieck abelian (monoidal) $\mathscr{P}$-fibred category (*cf.* definition 1.3.8). The triangulated case cannot be treated in general without assuming a thorough structure – this is the purpose of the next section.

3.1.b. *The model category case.*

3.1.5. Let $\mathscr{M}$ be a $\mathscr{P}$-fibred model category over $\mathscr{S}$ (*cf.* 1.3.20). Given a $\mathscr{S}$-diagram $\mathscr{X}$ indexed by a small category I, we will say that a morphism of $\mathscr{M}(\mathscr{X}, I)$ is a *termwise weak equivalence* (resp. a *termwise fibration*, resp. a *termwise cofibration*) if, for any object i of I, its image by the functor i^* is a weak equivalence (resp. a fibration, resp. a cofibration) in $\mathscr{M}(\mathscr{X}_i)$.

Proposition 3.1.6. *If $\mathscr{M}$ is a cofibrantly generated $\mathscr{P}$-fibred model category over $\mathscr{S}$, then, for any $\mathscr{S}$-diagram $\mathscr{X}$ indexed by a small category I, the category $\mathscr{M}(\mathscr{X}, I)$ is a cofibrantly generated model category whose weak equivalences (resp. fibrations) are the termwise weak equivalences (resp. the termwise fibrations). This model category structure on $\mathscr{M}(\mathscr{X}, I)$ will be called the* projective model structure.

Moreover, any cofibration of $\mathscr{M}(\mathscr{X},I)$ is a termwise cofibration, and the family of functors

$$i^* : \mathrm{Ho}(\mathscr{M})(\mathscr{X},I) \to \mathrm{Ho}(\mathscr{M})(\mathscr{X}_i)\ , \quad i \in \mathrm{Ob}(I)\,,$$

is conservative.

If $\mathscr{M}$ is left proper (resp. right proper, resp. combinatorial, resp. stable), then so is the projective model category structure on $\mathscr{M}(\mathscr{X})$.

Proof. Let $\mathscr{X}^\delta$ be the $\mathscr{S}$-diagram indexed by the set of objects of I (seen as a discrete category), whose fiber at i is $\mathscr{X}_i$. Let $\varphi : (\mathscr{X}^\delta, Ob\, I) \to (\mathscr{X}, I)$ be the inclusion (i.e. the map which is the identity on objects and which is the identity on each fiber). As φ is clearly a $\mathscr{P}$-morphism, we have an adjunction

$$\varphi_\sharp : \mathscr{M}(\mathscr{X}^\delta, Ob\, I) \simeq \prod_i \mathscr{M}(\mathscr{X}_i) \rightleftarrows \mathscr{M}(\mathscr{X}, I) : \varphi^*\,.$$

The functor $\varphi_\sharp$ can be made explicit: it sends a family of objects $(M_i)_i$ (with M_i in $\mathscr{M}(\mathscr{X}_i)$) to the sum of the $i_\sharp(M_i)$'s indexed by the set of objects of I. Note also that this proposition is trivially verified whenever $\mathscr{X}^\delta = \mathscr{X}$. Using the explicit formula for $i_\sharp$ given in 3.1.2, it is then straightforward to check that the adjunction $(\varphi_\sharp, \varphi^*)$ satisfies the assumptions of [Cra95, Theorem 3.3], which proves the existence of the projective model structure on $\mathscr{M}(\mathscr{X},I)$. Furthermore, the generating cofibrations (resp. trivial cofibrations of $\mathscr{M}(\mathscr{X},I)$) can be described as follows. For each object i of I, let A_i (resp. B_i) be a generating set of cofibrations (resp. of trivial cofibrations in $\mathscr{M}(\mathscr{X}_i)$. The class of termwise trivial fibrations (resp. of termwise fibrations) of $\mathscr{M}(\mathscr{X},I)$ is the class of maps which have the right lifting property with respect to the set $A = \cup_{i\in I} i_\sharp(A_i)$ (resp. to the set $B = \cup_{i\in I} i_\sharp(B_i)$). Hence, the set A (resp. B) generates the class of cofibrations (resp. of trivial cofibrations). In particular, as any element of A is a termwise cofibration (which follows immediately from the explicit formula for $i_\sharp$ given in 3.1.2), and as termwise cofibrations are stable by pushouts, transfinite compositions and retracts, any cofibration is a termwise cofibration (by the small object argument).

As any fibration (resp. cofibration) of $\mathscr{M}(\mathscr{X},I)$ is a termwise fibration (resp. a termwise cofibration), it is clear that, whenever the model categories $\mathscr{M}(\mathscr{X}_i)$ are right (resp. left) proper, the model category $\mathscr{M}(\mathscr{X},I)$ has the same property.

The functor φ^* preserves fibrations and cofibrations, while it also preserves and detects weak equivalences (by definition). This implies that the induced functor

$$\varphi^* : \mathrm{Ho}(\mathscr{M})(\mathscr{X},I) \to \mathrm{Ho}(\mathscr{M})(\mathscr{X}^\delta, Ob\, I) \simeq \prod_i \mathrm{Ho}(\mathscr{M})(\mathscr{X}_i)$$

is conservative (using the facts that the set of maps from a cofibrant object to a fibrant object in the homotopy category of a model category is the set of homotopy classes of maps, and that a morphism of a model category is a weak equivalence if and only if it induces an isomorphism in the homotopy category). As φ^* commutes to limits and colimits, this implies that it commutes to homotopy limits and to homotopy colimits (up to weak equivalences). Using the conservativity property, this implies that a commutative square of $\mathscr{M}(\mathscr{X},I)$ is a homotopy pushout (resp. a homotopy pullback) if and only if it is so in $\mathscr{M}(\mathscr{X}^\delta, Ob\, I)$. Remember that stable model categories are characterized as those in which a commutative square is a homotopy pullback square if and only if it is a homotopy pushout square. As a consequence, if all the model categories $\mathscr{M}(\mathscr{X}_i)$ are stable, as $\mathscr{M}(\mathscr{X}^\delta, Ob\, I)$ is then obviously stable as well, the model category $\mathscr{M}(\mathscr{X},I)$ has the same property.

It remains to prove that, if $\mathscr{M}(X,I)$ is a combinatorial model category for any object X of $\mathscr{S}$, then $\mathscr{M}(\mathscr{X},I)$ is combinatorial as well. For each object i in I, let G_i be a set of accessible generators of $\mathscr{M}(\mathscr{X}_i)$. Note that, for any object i of I, the functor $i_\sharp$ has a left adjoint i^* which commutes to colimits (having itself a right adjoint i_*). It is then easy to check that the set of objects of shape $i_\sharp(M)$, for M in G_i and i in I, is a small set of accessible generators of $\mathscr{M}(\mathscr{X},I)$. This implies that $\mathscr{M}(\mathscr{X},I)$ is accessible and ends the proof. □

Proposition 3.1.7. *Let $\mathscr{M}$ be a combinatorial $\mathscr{P}$-fibred model category over $\mathscr{S}$. Then, for any $\mathscr{S}$-diagram $\mathscr{X}$ indexed by a small category I, the category $\mathscr{M}(\mathscr{X},I)$ is a combinatorial model*

category whose weak equivalences (resp. cofibrations) are the termwise weak equivalences (resp. the termwise cofibrations). This model category structure on $\mathscr{M}(\mathscr{X}, I)$ will be called the injective model structure[15]*. Moreover, any fibration of the injective model structure on $\mathscr{M}(\mathscr{X}, I)$ is a termwise fibration.*

If $\mathscr{M}$ is left proper (resp. right proper, resp. stable), then so is the injective model category structure on $\mathscr{M}(\mathscr{X}, I)$.

Proof. See [Bar09, Theorem 2.28] for the existence of such a model structure (if, for any object X in $\mathscr{S}$, all the cofibrations of $\mathscr{M}(X)$ are monomorphisms, this can also be done following mutatis mutandis the proof of [Ayo07a, Proposition 4.5.9]). Any trivial cofibration of the projective model structure being a termwise trivial cofibration, any fibration of the injective model structure is a fibration of the projective model structure, hence a termwise fibration.

The assertions about properness follows from their analogs for the projective model structure and from [Cis06, Corollary 1.5.21] (or can be proved directly; see [Bar09, Proposition 2.31]). Similarly, the assertion on stability follows from their analogs for the projective model structure. □

3.1.8. From now on, we assume that a combinatorial $\mathscr{P}$-fibred model category $\mathscr{M}$ over $\mathscr{S}$ is given. Then, for any $\mathscr{S}$-diagram $(\mathscr{X}, I)$, we have two model category structures on $\mathscr{M}(\mathscr{X}, I)$, and the identity defines a left Quillen equivalence from the projective model structure to the injective model structure. These will be used for the understanding of the functorialities coming from morphisms of diagrams of S-schemes.

3.1.9. The category of $\mathscr{S}$-diagrams admits small sums. If $\{(\mathscr{Y}_j, I_j)\}_{j\in J}$ is a small family of $\mathscr{S}$-diagrams, then their sum is the $\mathscr{S}$-diagram $(\mathscr{X}, I)$, where

$$I = \coprod_{j\in J} I_j\,,$$

and $\mathscr{X}$ is the functor from I to $\mathscr{S}$ defined by

$$\mathscr{X}_i = \mathscr{Y}_{j,i} \quad \text{whenever } i \in I_j.$$

Proposition 3.1.10. *For any small family of $\mathscr{S}$-diagrams $\{(\mathscr{Y}_j, I_j)\}_{j\in J}$, the canonical functor*

$$\mathrm{Ho}(\mathscr{M})\Big(\coprod_{j\in J} \mathscr{Y}_j\Big) \to \prod_{j\in J} \mathrm{Ho}(\mathscr{M})(\mathscr{Y}_j)$$

is an equivalence of categories.

Proof. The functor

$$\mathscr{M}\Big(\coprod_{j\in J} \mathscr{Y}_j\Big) \to \prod_{j\in J} \mathscr{M}(\mathscr{Y}_j)$$

is an equivalence of categories. It thus remains an equivalence after localization. To conclude, it is sufficient to see that the homotopy category of a product of model categories is the product of their homotopy categories, which follows rather easily from the explicit description of the homotopy category of a model category; see e.g. [Hov99, Theorem 1.2.10]. □

Proposition 3.1.11. *Let $\varphi = (\alpha, f) : (\mathscr{X}, I) \to (\mathscr{Y}, J)$ be a morphism of $\mathscr{S}$-diagrams.*

(i) The adjunction $\varphi^* : \mathscr{M}(\mathscr{Y}, J) \rightleftarrows \mathscr{M}(\mathscr{X}, I) : \varphi_*$ is a Quillen adjunction with respect to the injective model structures. In particular, it induces a derived adjunction

$$\mathbf{L}\varphi^* : \mathrm{Ho}(\mathscr{M})(\mathscr{Y}, J) \rightleftarrows \mathrm{Ho}(\mathscr{M})(\mathscr{X}, I) : \mathbf{R}\varphi_*\,.$$

(ii) If φ is a $\mathscr{P}$-morphism, then the adjunction $\varphi_\sharp : \mathscr{M}(\mathscr{X}, I) \rightleftarrows \mathscr{M}(\mathscr{Y}, J) : \varphi^*$ is a Quillen adjunction with respect to the projective model structures, and the functor φ^* preserves weak equivalences. In particular, we get a derived adjunction

$$\mathbf{L}\varphi_\sharp : \mathrm{Ho}(\mathscr{M})(\mathscr{X}, I) \rightleftarrows \mathrm{Ho}(\mathscr{M})(\mathscr{Y}, J) : \mathbf{L}\varphi^* = \varphi^* = \mathbf{R}\varphi^*\,.$$

[15] Quite unfortunately, this corresponds to the 'semi-projective' model structure introduced in [Ayo07a, Def. 4.5.8].

Proof. The functor φ^* obviously preserves termwise cofibrations and termwise trivial cofibrations (we reduce to the case of a morphism of $\mathscr{S}$ using the explicit description of φ^* given in 3.1.3), which proves the first assertion. Similarly, the second assertion follows from the fact that, under the assumption that φ is a $\mathscr{P}$-morphism, the functor φ^* preserves termwise weak equivalences (see Remark 1.3.21), as well as termwise fibrations. □

3.1.12. The computation of the (derived) functors $\mathbf{R}\varphi_*$ (and $\mathbf{L}\varphi_\sharp$ whenever it makes sense) given by Proposition 3.1.11 has to do with homotopy limits (and colimits). It is easier to understand this first in the non derived version as follows.

Consider first the trivial case of a constant $\mathscr{S}$-diagram: let X be an object of $\mathscr{S}$, and I a small category. Then, seeing X as the constant functor $I \to \mathscr{S}$ with value X, we have a projection map $p_I : (X, I) \to X$. ¿From the very definition, the category $\mathscr{M}(X, I)$ is simply the category of presheaves on I with values in $\mathscr{M}(X)$, so that the inverse image functor

$$p_I^* : \mathscr{M}(X) \to \mathscr{M}(X, I) = \mathscr{M}(X)^{I^{op}} \tag{3.1.12.1}$$

is the 'constant diagram functor', while its right adjoint

$$\varprojlim_{I^{op}} = p_{I,*} : \mathscr{M}(X, I) \to \mathscr{M}(X) \tag{3.1.12.2}$$

is the limit functor, and its left adjoint,

$$\varinjlim_{I^{op}} = p_{I,\sharp} : \mathscr{M}(X, I) \to \mathscr{M}(X) \tag{3.1.12.3}$$

is the colimit functor.

Let S be an object of $\mathscr{S}$. A *$\mathscr{S}$-diagram over S* is the data of a $\mathscr{S}$-diagram $(\mathscr{X}, I)$, together with a morphism of $\mathscr{S}$-diagrams $p : (\mathscr{X}, I) \to S$ (i.e. its a $\mathscr{S}/S$-diagram). Such a map p factors as

$$(\mathscr{X}, I) \xrightarrow{\pi} (S, I) \xrightarrow{p_I} S\,, \tag{3.1.12.4}$$

where $\pi = (p, 1_I)$. Then one checks easily that, for any object M of $\mathscr{M}(\mathscr{X}, I)$, and for any object i of I, one has

$$\pi_*(M)_i \simeq p_{i,*}(M_i)\,, \tag{3.1.12.5}$$

where $p_i : \mathscr{X}_i \to S$ is the structural map, from which we deduce the formula

$$p_*(M) \simeq \varprojlim_{i \in I^{op}} \pi_*(M)_i \simeq \varprojlim_{i \in I^{op}} p_{i,*}(M_i)\,, \tag{3.1.12.6}$$

Remark that, if I is a small category with a terminal object ω, then any $\mathscr{S}$-diagram $\mathscr{X}$ indexed by I is a $\mathscr{S}$-diagram over $\mathscr{X}_\omega$, and we deduce from the computations above that, if $p : (\mathscr{X}, I) \to \mathscr{X}_\omega$ denotes the canonical map, then, for any object M of $\mathscr{M}(\mathscr{X}, I)$,

$$p_*(M) \simeq M_\omega\,. \tag{3.1.12.7}$$

Consider now a morphism of $\mathscr{S}$-diagrams $\varphi = (\alpha, f) : (\mathscr{X}, I) \to (\mathscr{Y}, J)$. For each object j, we can form the following pullback square of categories.

$$\begin{array}{ccc} I/j & \xrightarrow{u_j} & I \\ {\scriptstyle f/j}\downarrow & & \downarrow{\scriptstyle f} \\ J/j & \xrightarrow[v_j]{} & J \end{array} \tag{3.1.12.8}$$

in which J/j is the category of objects of J over j (which has a terminal object, namely $(j, 1_j)$, and v_j is the canonical projection; the category I/j is thus the category of pairs (i, a), where i is

an object of I, and $a : f(i) \to j$ a morphism in J. From this, we can form the following pullback of $\mathscr{S}$-diagrams

$$\begin{array}{ccc} (\mathscr{X}/j, I/j) & \xrightarrow{\mu_j} & (\mathscr{X}, I) \\ {\scriptstyle \varphi/j}\downarrow & & \downarrow{\scriptstyle \varphi} \\ (\mathscr{Y}/j, J/j) & \xrightarrow[\nu_j]{} & (\mathscr{Y}, J) \end{array} \tag{3.1.12.9}$$

in which $\mathscr{X}/j = \mathscr{X} \circ u_j$, $\mathscr{Y}/j = \mathscr{Y} \circ v_j$, and the maps μ_j and ν_j are the one induced by u_j and v_j respectively. For an object M of $\mathscr{M}(\mathscr{X}, I)$ (resp. an object N of $\mathscr{M}(\mathscr{Y}, J)$), we define M/j (resp. N/j) as the object of $\mathscr{M}(\mathscr{X}/j, I/j)$ (resp. of $\mathscr{M}(\mathscr{Y}/j, J/j)$) obtained as $M/j = \mu_j^*(M)$ (resp. $N/j = \nu_j^*(N)$). With these conventions, for any object M of $\mathscr{M}(\mathscr{X}, I)$ and any object j of the indexing category J, one gets the formula

$$\varphi_*(M)_j \simeq (\varphi/j)_*(M/j)_{(j,1_j))} \simeq \varprojlim_{(i,a)\in I/j^{op}} \alpha_{i,*}(M_i)\,. \tag{3.1.12.10}$$

This implies that the natural map

$$\varphi_*(M)/j = \nu_j^*\,\varphi_*(M) \to (\varphi/j)_*\,\mu_j^*(M) = (\varphi/j)_*(M/j) \tag{3.1.12.11}$$

is an isomorphism: to prove this, it is sufficient to obtain an isomorphism from (3.1.12.11) after evaluating by any object $(j', a : j' \to j)$ of J/j, which follows readily from (3.1.12.10) and from the obvious fact that $(I/j)/(j', a)$ is canonically isomorphic to I/j'.

In order to deduce from the computations above their derived versions, we need two lemmata.

Lemma 3.1.13. *Let $\mathscr{X}$ be a $\mathscr{S}$-diagram indexed by a small category I, and i an object of I. Then the evaluation functor*

$$i^* : \mathscr{M}(\mathscr{X}, I) \to \mathscr{M}(\mathscr{X}_i)$$

is a right Quillen functor with respect to the injective model structure, and it preserves weak equivalences.

Proof. Proving that the functor i^* is a right Quillen functor is equivalent to proving that its left adjoint (3.1.2.2) is a left Quillen functor with respect to the injective model structure, which follows immediately from its computation (3.1.2.3), as, in any model category, cofibrations and trivial cofibrations are stable by small sums. The last assertion is obvious from the very definition of the weak equivalences in $\mathscr{M}(\mathscr{X}, I)$. □

Lemma 3.1.14. *For any pullback square of $\mathscr{S}$-diagrams of shape* (3.1.12.8)*, the functors*

$$\mu_j^* : \mathscr{M}(\mathscr{X}, I) \to \mathscr{M}(\mathscr{X}/j, I/j)\,, \quad M \mapsto M/j$$
$$\nu_j^* : \mathscr{M}(\mathscr{Y}, I) \to \mathscr{M}(\mathscr{Y}/j, J/j)\,, \quad N \mapsto N/j$$

are right Quillen functors with respect to the injective model structure, and they preserve weak equivalences.

Proof. It is sufficient to prove this for the functor μ_j^* (as ν_j^* is simply the special case where $I = J$ and f is the identity). The fact that μ_j^* preserves weak equivalences is obvious, so that it remains to prove that it is a right Quillen functor. We thus have to prove that left adjoint of μ_j^*,

$$\mu_{j,\sharp} : \mathscr{M}(\mathscr{X}/j, I/j) \to \mathscr{M}(\mathscr{X}, I)\,,$$

is a left Quillen functor. In other words, we have to prove that, for any object i of I, the functor

$$i^*\mu_{j,\sharp} : \mathscr{M}(\mathscr{X}, I) \to \mathscr{M}(\mathscr{X})$$

is a left Quillen functor. For any object M of $\mathscr{M}(\mathscr{X}, I)$, we have a natural isomorphism

$$i^*\mu_{j,\sharp}(M) \simeq \coprod_{a\in \mathrm{Hom}_J(f(i),j)} (i,a)_\sharp(M_i)\,.$$

But we know that the functors $(i,a)_\sharp$ are left Quillen functors, so that the stability of cofibrations and trivial cofibrations by small sums and this description of the functor $i^*\mu_{j,\sharp}$ achieves the proof. □

Proposition 3.1.15. *Let S be an object of $\mathscr{S}$, and $p : (\mathscr{X}, I) \to S$ a $\mathscr{S}$-diagram over S, and consider the canonical factorization* (3.1.12.4)*. For any object M of* $\mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)$*, there are canonical isomorphisms and* $\mathrm{Ho}(\mathscr{M})(S)$*:*

$$\mathbf{R}\pi_*(M)_i \simeq \mathbf{R}p_{i,*}(M_i) \quad \textit{and} \quad \mathbf{R}p_*(M) \simeq \mathbf{R}\varprojlim_{i \in I^{op}} \mathbf{R}p_{i,*}(M_i)\,.$$

In particular, if furthermore the category I has a terminal object ω, then

$$\mathbf{R}p_*(M) \simeq \mathbf{R}p_{\omega,*}(M_\omega)\,.$$

Proof. This follows immediately from (3.1.12.5) and (3.1.12.6) and from the fact that deriving (right) Quillen functors is compatible with composition. □

Proposition 3.1.16. *We consider the pullback square of $\mathscr{S}$-diagrams* (3.1.12.8) *(as well as the notations thereof). For any object M of* $\mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)$*, and any object j of J, we have natural isomorphisms*

$$\mathbf{R}\varphi_*(M)_j \simeq \mathbf{R}\varprojlim_{(i,a)\in I/j^{op}} \mathbf{R}\alpha_{i,*}(M_i) \quad \textit{and} \quad \mathbf{R}\varphi_*(M)/j \simeq \mathbf{R}(\varphi/j)_*(M/j)$$

in $\mathrm{Ho}(\mathscr{M})(\mathscr{Y}_j)$ *and in* $\mathrm{Ho}(\mathscr{M})(\mathscr{Y}/j, J/j)$ *respectively.*

Proof. Using again the fact that deriving right Quillen functors is compatible with composition, by virtue of Lemma 3.1.13 and Lemma 3.1.14, this is a direct translation of (3.1.12.10) and (3.1.12.11). □

Proposition 3.1.17. *Let $u : T \to S$ be a $\mathscr{P}$-morphism of $\mathscr{S}$, and $p : (\mathscr{X}, I) \to S$ a $\mathscr{S}$-diagram over S. Consider the pullback square of $\mathscr{S}$-diagrams*

$$\begin{array}{ccc} (\mathscr{Y}, I) & \xrightarrow{\varphi} & (\mathscr{X}, I) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ T & \xrightarrow[u]{} & S \end{array}$$

(i.e. $\mathscr{Y}_i = T \times_S \mathscr{X}_i$ for any object i of I). Then, for any object M of $\mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)$*, the canonical map*

$$\mathbf{L}u^*\,\mathbf{R}p_*(M) \to \mathbf{R}q_*\,\mathbf{L}v^*(M)$$

is an isomorphism in $\mathrm{Ho}(\mathscr{M})(T)$*.*

Proof. By Remark 1.3.21, the functor ν^* is both a left and a right Quillen functor which preserves weak equivalences, so that the functor $\mathbf{L}\nu^* = \nu^* = \mathbf{R}\nu^*$ preserves homotopy limits. Hence, by Proposition 3.1.15, one reduces to the case where I is the terminal category, i.e. to the transposition of the isomorphism given by the $\mathscr{P}$-base change formula ($\mathscr{P}$-BC) for the homotopy $\mathscr{P}$-fibred category $\mathrm{Ho}(\mathscr{M})$ (see 1.1.18). □

3.1.18. A morphism of $\mathscr{S}$-diagrams $\nu = (\alpha, f) : (\mathscr{Y}', J') \to (\mathscr{Y}, J)$, is *cartesian* if, for any arrow $i \to j$ in J', the induced commutative square

$$\begin{array}{ccc} \mathscr{Y}'_i & \longrightarrow & \mathscr{Y}'_j \\ {\scriptstyle \alpha_i}\downarrow & & \downarrow{\scriptstyle \alpha_j} \\ \mathscr{Y}_{f(i)} & \longrightarrow & \mathscr{Y}_{f(j)} \end{array}$$

is cartesian.

A morphism of $\mathscr{S}$-diagrams $\nu = (\alpha, f) : (\mathscr{Y}', J') \to (\mathscr{Y}, J)$ is *reduced* if $J = J'$ and $f = 1_J$.

Proposition 3.1.19. *Let $\nu : (\mathscr{Y}', J) \to (\mathscr{Y}, J)$ be a reduced cartesian $\mathscr{P}$-morphism of $\mathscr{S}$-diagrams, and $\varphi = (\alpha, f) : (\mathscr{X}, I) \to (\mathscr{Y}, J)$ a morphism of $\mathscr{S}$-diagrams. Consider the pullback square of $\mathscr{S}$-diagrams*

$$\begin{array}{ccc} (\mathscr{X}', I) & \xrightarrow{\mu} & (\mathscr{X}, I) \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ (\mathscr{Y}', J) & \xrightarrow[\nu]{} & (\mathscr{Y}, J) \end{array}$$

(i.e. $\mathscr{X}'_i = \mathscr{Y}'_{f(i)} \times_{\mathscr{Y}_{f(i)}} \mathscr{X}_i$ for any object i of I). Then, for any object M of $\mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)$, the canonical map

$$\mathbf{L}\nu^* \mathbf{R}\varphi_*(M) \to \mathbf{R}\psi_* \mathbf{L}\mu^*(M)$$

is an isomorphism in $\mathrm{Ho}(\mathscr{M})(\mathscr{Y}', J)$.

Proof. By virtue of Proposition 3.1.6, it is sufficient to prove that the map

$$j^*\mathbf{L}\nu^* \mathbf{R}\varphi_*(M) \to j^*\mathbf{R}\psi_* \mathbf{L}\mu^*(M)$$

is an isomorphism for any object j of J. Let $p : (\mathscr{X}/j, I/j) \to \mathscr{Y}_j$ and $q : (\mathscr{X}'/j, J, j) \to \mathscr{Y}'_j$ be the canonical maps. As ν is cartesian, we have a pullback square of $\mathscr{S}$-diagrams

$$\begin{array}{ccc} (\mathscr{X}'/j, I/j) & \xrightarrow{\mu/j} & (\mathscr{X}/j, I/j) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ \mathscr{Y}'_j & \xrightarrow[\nu_j]{} & \mathscr{Y}_j \end{array}$$

But ν_j being a $\mathscr{P}$-morphism, by virtue of Proposition 3.1.17, we thus have an isomorphism

$$\mathbf{L}\nu_j^* \mathbf{R}p_*(M/j) \simeq \mathbf{R}q_* \mathbf{L}(\mu/j)^*(M/j) = \mathbf{R}q_*(\mathbf{L}\mu^*(M)/j)\,.$$

Applying Proposition 3.1.16 and the last assertion of Proposition 3.1.15 twice, we also have canonical isomorphisms

$$j^*\mathbf{R}\varphi_*(M) \simeq \mathbf{R}p_*(M/j) \quad \text{and} \quad j^*\mathbf{R}\psi_* \mathbf{L}\mu^*(M) \simeq \mathbf{R}q_*(\mathbf{L}\mu^*(M)/j)\,.$$

The obvious identity $j^*\mathbf{L}\nu^* = \mathbf{L}\nu_j^* j^*$ achieves the proof. □

Corollary 3.1.20. *Under the assumptions of Proposition 3.1.19, for any object N of $\mathrm{Ho}(\mathscr{M})(\mathscr{Y}', j)$, the canonical map*

$$\mathbf{L}\mu_\sharp \mathbf{L}\psi^*(N) \to \mathbf{L}\varphi^* \mathbf{L}\nu_\sharp(N)$$

is an isomorphism in $\mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)$.

Remark 3.1.21. The class of cartesian $\mathscr{P}$-morphisms form an admissible class of morphisms in the category of $\mathscr{S}$-diagrams, which we denote by $\mathscr{P}_{cart}$. Proposition 3.1.11 and the preceding corollary thus asserts that $\mathrm{Ho}(\mathscr{M})$ is a $\mathscr{P}_{cart}$-fibred category over the category of $\mathscr{S}$-diagrams.

3.1.22. We shall deal sometimes with diagrams of $\mathscr{S}$-diagrams. Let I be a small category, and $\mathscr{F}$ a functor from I to the category of $\mathscr{S}$-diagrams. For each object i of I, we have a $\mathscr{S}$-diagram $(\mathscr{F}(i), J_i)$, and, for each map $u : i \to i'$, we have a functor $f_u : J_i \to J_{i'}$ as well as a natural transformation $\alpha_u : \mathscr{F}(i) \to \mathscr{F}(i') \circ f_u$, subject to coherence identities. In particular, the correspondance $i \mapsto J_i$ defines a functor from I to the category of small categories. Let $I_{\mathscr{F}}$ be the cofibred category over I associated to it; see [Gro03, Exp. VI]. Explicitely, $I_{\mathscr{F}}$ is described as follows. The objects are the couples (i, x), where i is an object of I, and x is an object of J_i. A morphism $(i, x) \to (i', x')$ is a couple (u, v), where $u : i \to i'$ is a morphism of I, and $v : f_u(x) \to x'$ is a morphism of $J_{i'}$. The identity of (i, x) is the couple $(1_i, 1_x)$, and, for two morphisms $(u, v) : (i, x) \to (i', x')$ and $(u', v') : (i', x') \to (i'', x'')$, their composition $(u'', v'') : (i, x) \to (i'', x'')$ is defined by $u'' = u' \circ u$, while v'' is the composition of the map

$$f_{u''}(x) = f_{u'}(f_u(x)) \xrightarrow{f_{u'}(v)} f_{u'}(x') \xrightarrow{v'} x''\,.$$

The functor $p : I_{\mathscr{F}} \to I$ is simply the projection $(i,x) \mapsto i$. For each object i of I, we get a canonical pullback square of categories

$$\begin{array}{ccc} J_i & \xrightarrow{\ell_i} & I_{\mathscr{F}} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ e & \xrightarrow[i]{} & I \end{array} \tag{3.1.22.1}$$

in which i is the functor from the terminal category e which corresponds to the object i, and ℓ_i is the functor defined by $\ell_i(x) = (i,x)$.

The functor $\mathscr{F}$ defines a $\mathscr{S}$-diagram $(\int \mathscr{F}, I_{\mathscr{F}})$: for an object (i,x) of $I_{\mathscr{F}}$, $(\int \mathscr{F})_{(i,x)} = \mathscr{F}(i)_x$, and for a morphism $(u,v) : (i,x) \to (i',x')$, the map

$$(u,v) : (\textstyle\int \mathscr{F})_{(i,x)} = \mathscr{F}(i)_x \to (\int \mathscr{F})_{(i',x')} = \mathscr{F}(i')_{x'}$$

is simply the morphism induced by α_u and v. For each object i of I, there is a natural morphism of $\mathscr{S}$-diagrams

$$\lambda_i : (\mathscr{F}(i), J_i) \to (\textstyle\int \mathscr{F}, I_{\mathscr{F}})\,, \tag{3.1.22.2}$$

given by $\lambda_i = (1_{\mathscr{F}(i)}, \ell_i)$

Proposition 3.1.23. *Let X be an object of $\mathscr{S}$, and $f : \mathscr{F} \to X$ a morphism of functors (where X is considered as the constant functor from I to $\mathscr{S}$-diagrams with value the functor from e to $\mathscr{S}$ defined by X). Then, for each object i of I, we have a canonical pullback square of $\mathscr{S}$-diagrams*

$$\begin{array}{ccc} (\mathscr{F}(i), J_i) & \xrightarrow{\lambda_i} & (\textstyle\int \mathscr{F}, I_{\mathscr{F}}) \\ {\scriptstyle \varphi_i}\downarrow & & \downarrow{\scriptstyle \varphi} \\ X & \xrightarrow[i]{} & (X, I) \end{array}$$

in which φ and φ_i are the obvious morphisms induced by f (where, this time, (X,I) is seen as the constant functor from I to $\mathscr{S}$ with value X).

Moreover, for any object M of $\mathrm{Ho}(\mathscr{M})(\int \mathscr{F}, I_{\mathscr{F}})$, the natural map

$$i^* \mathbf{R}\varphi_*(M) = \mathbf{R}\varphi_*(M)_i \to \mathbf{R}\varphi_{i,*}\, \lambda_i^*(M)$$

is an isomorphism. In particular, if we also write by abuse f for the induced map of $\mathscr{S}$-diagrams from $(\int \mathscr{F}, I_{\mathscr{F}})$ to X, we have a natural isomorphism

$$\mathbf{R}f_*(M) \simeq \mathbf{R}\varprojlim_{i \in I^{op}} \mathbf{R}\varphi_{i,*}\, \lambda_i^*(M)\,.$$

Proof. This pullback square is the one induced by (3.1.22.1). We shall prove first that the map

$$i^* \mathbf{R}\varphi_*(M) = \mathbf{R}\varphi_*(M)_i \to \mathbf{R}\varphi_{i,*}\, \lambda_i^*(M)$$

is an isomorphism in the particular case where I has a terminal object ω and $i = \omega$. By virtue of Propositions 3.1.15 and 3.1.16, we have isomorphisms

$$\omega^* \mathbf{R}\varphi_*(M) \simeq \mathbf{R}\varprojlim_{i \in I^{op}} \mathbf{R}\varphi_*(M)_i \simeq \quad \mathbf{R}\varprojlim_{(i,x) \in I_{\mathscr{F}}^{op}} \mathbf{R}\varphi_{i,x,*}\,(M_{(i,x)})\,, \tag{3.1.23.1}$$

where $\varphi_{i,x} : \mathscr{F}(i)_x \to X$ denotes the map induced by f. We are thus reduced to prove that the canonical map

$$\mathbf{R}\varprojlim_{(i,x) \in I_{\mathscr{F}}^{op}} \mathbf{R}\varphi_{i,x,*}\,(M_{(i,x)}) \to \mathbf{R}\varprojlim_{x \in J_\omega^{op}} \mathbf{R}\varphi_{\omega,x,*}\,(M_{(\omega,x)}) \simeq \mathbf{R}\varphi_{\omega,*}\, \lambda_\omega^*(M) \tag{3.1.23.2}$$

is an isomorphim. As $I_{\mathscr{F}}$ is cofibred over I, and as ω is a terminal object of I, the inclusion functor $\ell_\omega : J_\omega \to I_{\mathscr{F}}$ has a left adjoint, whence is coaspherical in any weak basic localizer (i.e. is homotopy cofinal); see [Mal05, 1.1.9, 1.1.16 and 1.1.25]. As any model category defines a Grothendieck derivator ([Cis03, Thm. 6.11]), it follows from [Cis03, Cor. 1.15] that the map (3.1.23.2) is an isomorphism.

To prove the general case, we proceed as follows. Let $\mathscr{F}/i$ be the functor obtained by composing $\mathscr{F}$ with the canonical functor $v_i : I/i \to I$. Then, keeping track of the conventions adopted in 3.1.12, we check easily that $(I/i)_{\mathscr{F}/i} = (I_{\mathscr{F}})/i$ and that $\int(\mathscr{F}/i) = (\int\mathscr{F})/i$. Moreover, the pullback square (3.1.22.1) is the composition of the following pullback squares of categories.

$$\begin{array}{ccccc} J_i & \xrightarrow{a_i} & I_{\mathscr{F}}/i & \xrightarrow{u_i} & I_{\mathscr{F}} \\ {\scriptstyle q}\downarrow & & {\scriptstyle p/i}\downarrow & & \downarrow{\scriptstyle p} \\ e & \xrightarrow[(i,1_i)]{} & I/i & \xrightarrow[v_i]{} & I \end{array}$$

The pullback square of the proposition is thus the composition of the following pullback squares.

$$\begin{array}{ccccc} (\mathscr{F}(i), J_i) & \xrightarrow{\alpha_i} & (\int\mathscr{F}/i, I_{\mathscr{F}}/i) & \xrightarrow{\mu_i} & (\int\mathscr{F}, I_{\mathscr{F}}) \\ {\scriptstyle \varphi_i}\downarrow & & {\scriptstyle \varphi/i}\downarrow & & \downarrow{\scriptstyle \varphi} \\ X & \xrightarrow[(i,1_i)]{} & (X, I/i) & \xrightarrow[v_i]{} & (X, I) \end{array}$$

The natural transformations

$$(i,1_i)^*\,\mathbf{R}(\varphi/i)_* \to \mathbf{R}\varphi_{i,*}\,\alpha_i^* \quad \text{and} \quad v_i^*\,\mathbf{R}\varphi_* \to \mathbf{R}(\varphi/i)_*\,\mu_i^*$$

are both isomorphisms: the first one comes from the fact that $(i,1_i)$ is a terminal object of I/i, and the second one from Proposition 3.1.16. We thus get:

$$\begin{aligned} i^*\,\mathbf{R}\varphi_*(M) &\simeq (i,1_i)^*\,v_i^*\,\mathbf{R}\varphi_*(M) \\ &\simeq (i,1_i)^*\,\mathbf{R}(\varphi/i)_*\,\mu_i^*(M) \\ &\simeq \mathbf{R}\varphi_{i,*}\,\alpha_i^*\,\mu_i^*(M) \\ &\simeq \mathbf{R}\varphi_{i,*}\,\lambda_i^*(M)\,. \end{aligned}$$

The last assertion of the proposition is then a straightforward application of Proposition 3.1.15. □

Proposition 3.1.24. *If $\mathscr{M}$ is a monoidal $\mathscr{P}$-fibred combinatorial model category over $\mathscr{S}$, then, for any $\mathscr{S}$-diagram $\mathscr{X}$ indexed by a small category I, the injective model structure turns $\mathscr{M}(\mathscr{X}, I)$ into a symmetric monoidal model category. In particular, the categories* $\mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)$ *are canonically endowed with a closed symmetric monoidal structure, in such a way that, for any morphism of $\mathscr{S}$-diagrams $\varphi : (\mathscr{X}, I) \to (\mathscr{Y}, J)$, the functor* $\mathbf{L}\varphi^* : \mathrm{Ho}(\mathscr{M})(\mathscr{Y}, J) \to \mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)$ *is symmetric monoidal.*

Proof. This is obvious from the definition of a symmetric monoidal model category, as the tensor product of $\mathscr{M}(\mathscr{X}, I)$ is defined termwise, as well as the cofibrations and the trivial cofibrations. □

Proposition 3.1.25. *Assume that $\mathscr{M}$ is a monoidal $\mathscr{P}$-fibred combinatorial model category over $\mathscr{S}$, and consider a reduced cartesian $\mathscr{P}$-morphism $\varphi = (\alpha, f) : (\mathscr{X}, I) \to (\mathscr{Y}, I)$. Then, for any object M in* $\mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)$ *and any object N in* $\mathrm{Ho}(\mathscr{M})(\mathscr{Y}, I)$*, the canonical map*

$$\mathbf{L}\varphi_\sharp(M \otimes^{\mathbf{L}} \varphi^*(N)) \to \mathbf{L}\varphi_\sharp(M) \otimes^{\mathbf{L}} N$$

is an isomorphism.

Proof. Let i be an object of I. It is sufficient to prove that the map

$$i^*\mathbf{L}\varphi_\sharp(M \otimes^{\mathbf{L}} \varphi^*(N)) \to i^*\mathbf{L}\varphi_\sharp(M) \otimes^{\mathbf{L}} N$$

is an isomorphism in $\mathrm{Ho}(\mathscr{M})(\mathscr{X}_i)$. Using Corollary 3.1.20, we see that this map can be identified with the map

$$\mathbf{L}\varphi_{i,\sharp}(M_i \otimes^{\mathbf{L}} \varphi_i^*(N_i)) \to \mathbf{L}\varphi_{i,\sharp}(M_i) \otimes^{\mathbf{L}} N_i\,,$$

which is an isomorphism according to the $\mathscr{P}$-projection formula for the homotopy $\mathscr{P}$-fibred category $\mathrm{Ho}(\mathscr{M})$. □

3.1.26. Let $(\mathscr{X}, I)$ be a $\mathscr{S}$-diagram. An object M of $\mathscr{M}(\mathscr{X}, I)$ is *homotopy cartesian* if, for any map $u : i \to j$ in I, the structural map $u^*(M_j) \to M_i$ induces an isomorphism

$$\mathbf{L}u^*(M_i) \simeq M_j$$

in $\mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)$ (i.e. if there exists a weak equivalence $M'_j \to M_j$ with M'_j cofibrant in $\mathscr{M}(\mathscr{X}_j)$ such that the map $u^*(M'_j) \to M_i$ is a weak equivalence in $\mathscr{M}(\mathscr{X}_i)$).

We denote by $\mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)_{hcart}$ the full subcategory of $\mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)$ spanned by homotopy cartesian sections.

An *homotopy cartesian resolution* of an object M of $\mathscr{M}(\mathscr{X}, I)$ is a weak equivalence $M' \to M$ in $\mathscr{M}(\mathscr{X}, I)$ with M' cofibrant (for the injective model structure) and homotopy cartesian.

Definition 3.1.27. A cofibrantly generated model category $\mathscr{V}$ is *tractable* is there exist sets I and J of cofibrations between cofibrant objects which generate the class of cofibrations and the class of trivial cofibrations respectively.

Remark 3.1.28. If $\mathscr{M}$ is a combinatorial and tractable $\mathscr{P}$-fibred model category over $\mathscr{S}$, then so are the projective and the injective model structures on $\mathscr{M}(\mathscr{X}, I)$; see [Bar09, Thm. 2.28 and 2.30].

Proposition 3.1.29. *If $\mathscr{M}$ is tractable, then any object of $\mathscr{M}(\mathscr{X}, I)$ admits an homotopy cartesian resolution (and there even exists a functorial homotopy cartesian resolution in $\mathscr{M}(\mathscr{X}, I)$). In particular, the inclusion functor*

$$\mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)_{hcart} \to \mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)$$

admits a left adjoint.

Proof. This follows from the fact that the cofibrant homotopy cartesian sections are the cofibrant object of a right Bousfield localization of the injective model structure on $\mathscr{M}(\mathscr{X}, I)$; see [Bar09, Theorem 5.25]. □

Definition 3.1.30. Let $\mathscr{M}$ and $\mathscr{M}'$ two $\mathscr{P}$-fibred model categories over $\mathscr{S}$. A *Quillen morphism* γ from $\mathscr{M}$ to $\mathscr{M}'$ is a morphism of $\mathscr{P}$-fibred categories $\gamma : \mathscr{M} \to \mathscr{M}'$ such that $\gamma^* : \mathscr{M}(X) \to \mathscr{M}'(X)$ is a left Quillen functor for any object X of $\mathscr{S}$.

Remark 3.1.31. If $\gamma : \mathscr{M} \to \mathscr{M}'$ is a Quillen morphism between $\mathscr{P}$-fibred combinatorial model categories, then, for any $\mathscr{S}$-diagram $(\mathscr{X}, I)$, we get a Quillen adjunction

$$\gamma^* : \mathscr{M}(\mathscr{X}, I) \rightleftarrows \mathscr{M}'(\mathscr{X}, I) : \gamma_*$$

(with the injective model structures as well as with the projective model structures).

Proposition 3.1.32. *For any Quillen morphism $\gamma : \mathscr{M} \to \mathscr{M}'$, the derived adjunctions*

$$\mathbf{L}\gamma^* : \mathrm{Ho}(\mathscr{M})(X) \rightleftarrows \mathrm{Ho}(\mathscr{M}')(X) : \mathbf{R}\gamma_*$$

define a morphism of $\mathscr{P}$-fibred categories $\mathrm{Ho}(\mathscr{M}) \to \mathrm{Ho}(\mathscr{M}')$ over $\mathscr{S}$. If moreover $\mathscr{M}$ and $\mathscr{M}'$ are combinatorial, then the morphism $\mathrm{Ho}(\mathscr{M}) \to \mathrm{Ho}(\mathscr{M}')$ extends to a morphism of $\mathscr{P}_{cart}$-fibred categories over the category of $\mathscr{S}$-diagrams.

Proof. This follows immediately from [Hov99, Theorem 1.4.3]. □

3.2. **Hypercoverings, descent, and derived global sections.**

3.2.1. Let $\mathscr{S}$ be an essentially small category, and $\mathscr{P}$ an admissible class of morphisms in $\mathscr{S}$. We assume that a Grothendieck topology t on $\mathscr{S}$ is given. We shall write $\mathscr{S}^{\amalg}$ for the full subcategory of the category of $\mathscr{S}$-diagrams whose objects are the small families $X = \{X_i\}_{i \in I}$ of objects of $\mathscr{S}$ (seen as functors from a discrete category to $\mathscr{S}$). The category $\mathscr{S}^{\amalg}$ is equivalent to the full subcategory of the category of presheaves of sets on $\mathscr{S}$ spanned by sums of representable presheaves. In particular, small sums are representable in $\mathscr{S}^{\amalg}$ (but note that the functor from $\mathscr{S}$ to $\mathscr{S}^{\amalg}$ does not preserve sums). Finally, we remark that the topology t extends naturally to a Grothendieck topology on $\mathscr{S}^{\amalg}$ such that the topology t on $\mathscr{S}$ is the topology induced from the inclusion $\mathscr{S} \subset \mathscr{S}^{\amalg}$.The covering maps for this topology on $\mathscr{S}^{\amalg}$ will be called *t-coverings*

(note that the inclusion $\mathscr{S} \subset \mathscr{S}^{\amalg}$ is continuous and induces an equivalence between the topos of t-sheaves on $\mathscr{S}$ and the topos of t-sheaves on $\mathscr{S}^{\amalg}$).

Let Δ be the category of non-empty finite ordinals. Remember that a simplicial object of $\mathscr{S}^{\amalg}$ is a presheaf on Δ with values in $\mathscr{S}^{\amalg}$. For a simplicial set K and an object X of $\mathscr{S}^{\amalg}$, we denote by $K \times X$ the simplicial object of $\mathscr{S}^{\amalg}$ defined by

$$(K \times X)_n = \coprod_{x \in K_n} X \quad , \qquad n \geq 0 \, .$$

We write Δ^n for the standard combinatorial simplex of dimension n, and $i_n : \partial\Delta^n \to \Delta^n$ for its boundary inclusion.

A morphism $p : \mathscr{X} \to \mathscr{Y}$ between simplicial objects of $\mathscr{S}^{\amalg}$ is a *t-hypercovering* if, locally for the t-topology, it has the right lifting property with respect to boundary inclusions of standard simplices, which, in a more precise way, means that, for any integer $n \geq 0$, any object U of $\mathscr{S}^{\amalg}$, and any commutative square

$$\begin{array}{ccc} \partial\Delta^n \times U & \xrightarrow{x} & \mathscr{X} \\ {\scriptstyle i_n \times 1}\downarrow & & \downarrow{\scriptstyle p} \\ \Delta^n \times U & \xrightarrow[y]{} & \mathscr{Y} \end{array} \quad ,$$

there exists a t-covering $q : V \to U$, and a morphism of simplicial objects $z : \Delta^n \times V \to \mathscr{X}$, such that the diagram bellow commutes.

$$\begin{array}{ccc} \partial\Delta^n \times V & \xrightarrow{x(1\times q)} & \mathscr{X} \\ {\scriptstyle i_n \times 1}\downarrow & \overset{z}{\nearrow} & \downarrow{\scriptstyle p} \\ \Delta^n \times V & \xrightarrow[y(1\times q)]{} & \mathscr{Y} \end{array}$$

A *t-hypercovering* of an object X of $\mathscr{S}^{\amalg}$ is a a t-hypercovering $p : \mathscr{X} \to X$ (where X is considered as a constant simplicial object).

Remark 3.2.2. This definition of t-hypercovering is equivalent to the one given in [AGV73, Exp. V, 7.3.1.4].

3.2.3. Let $\mathscr{X}$ be a simplicial object of $\mathscr{S}^{\amalg}$. It is in particular a functor from the category Δ^{op} to the category of $\mathscr{S}$-diagrams, so that the constructions and considerations of 3.1.22 apply to $\mathscr{X}$. In particular, there is a $\mathscr{S}$-diagram $\tilde{\mathscr{X}}$ associated to $\mathscr{X}$, namely $\tilde{\mathscr{X}} = (\int \mathscr{X}, (\Delta^{op})_{\mathscr{X}})$. More explicitely, for each integer $n \geq 0$, there is a family $\{\mathscr{X}_{n,x}\}_{x \in K_n}$ of objects of $\mathscr{S}$, such that

$$\mathscr{X}_n = \coprod_{x \in K_n} \mathscr{X}_{n,x} \, . \tag{3.2.3.1}$$

In fact, the sets K_n form a simplicial set K, and the category $(\Delta^{op})_{\mathscr{X}})$ can be identified over Δ^{op} to the category $(\Delta/K)^{op}$, where Δ/K is the fibred category over Δ whose fiber over n is the set K_n (seen as a discrete category), i.e. the category of simplices of K. We shall call K the *underlying simplicial set of $\mathscr{X}$*, while the decomposition (3.2.3.1) will be called the *local presentation of $\mathscr{X}$*. The construction $\mathscr{X} \mapsto \tilde{\mathscr{X}}$ is functorial. If $p : \mathscr{X} \to \mathscr{Y}$ is a morphism of simplicial objects of $\mathscr{S}^{\amalg}$, we shall denote by $\tilde{p} : \tilde{\mathscr{X}} \to \tilde{\mathscr{Y}}$ the induced morphism of $\mathscr{S}$-diagrams. However, for a morphism of $p : \mathscr{X} \to X$, where X is an object of $\mathscr{S}^{\amalg}$, we shall still denote by $p : \tilde{\mathscr{X}} \to X$ the corresponding morphism of $\mathscr{S}$-diagrams.

Let $\mathscr{M}$ be a $\mathscr{P}$-fibred combinatorial model category over $\mathscr{S}$. Given a simplicial object $\mathscr{X}$ of $\mathscr{S}^{\amalg}$, we define the category $\mathrm{Ho}(\mathscr{M})(\mathscr{X})$ by the formula:

$$\mathrm{Ho}(\mathscr{M})(\mathscr{X}) = \mathrm{Ho}(\mathscr{M})(\textstyle\int \mathscr{X}, (\Delta^{op})_{\mathscr{X}}) \, . \tag{3.2.3.2}$$

Given an object X of $\mathscr{S}^{\amalg}$ and a morphism $p : \mathscr{X} \to X$, we have a derived adjunction

$$\mathbf{L}p^* : \mathrm{Ho}(\mathscr{M})(X) \rightleftarrows \mathrm{Ho}(\mathscr{M})(\mathscr{X}) : \mathbf{R}p_* \, . \tag{3.2.3.3}$$

Proposition 3.2.4. *Consider an object X of $\mathscr{S}$, a simplicial object $\mathscr{X}$ of $\mathscr{S}^{\mathrm{II}}$, as well as a morphism $p : \mathscr{X} \to X$. Denote by K the underlying simplicial set of $\mathscr{X}$, and for each integer $n \geq 0$ and each simplex $x \in K_n$, write $p_{n,x} : \mathscr{X}_{n,x} \to X$ for the morphism of $\mathscr{S}^{\mathrm{II}}$ induced by the local presentation of $\mathscr{X}$ (3.2.3.1). Then, for any object M of $\mathrm{Ho}(\mathscr{M})(X)$, there are canonical isomorphisms*

$$\mathbf{R}p_*\mathbf{R}p^*(M) \simeq \mathbf{R}\varprojlim_{n\in\Delta} \mathbf{R}p_{n,*}\mathbf{L}p_n^*(M) \simeq \mathbf{R}\varprojlim_{n\in\Delta}\Big(\prod_{x\in K_n} \mathbf{R}p_{n,x,*}\mathbf{L}p_{n,x}^*(M)\Big).$$

Proof. The first isomorphism is a direct application of the last assertion of Proposition 3.1.23 for $\mathscr{F} = \mathscr{X}$, while the second follows from the first by Proposition 3.1.10. □

Definition 3.2.5. Given an object Y of $\mathscr{S}^{\mathrm{II}}$, an object M of $\mathrm{Ho}(\mathscr{M})(Y)$ will be said to satisfy *t-descent* if it has the following property: for any morphism $f : X \to Y$ and any t-hypercovering $p : \mathscr{X} \to X$, the map

$$\mathbf{R}f_*\,\mathbf{L}f^*(M) \to \mathbf{R}f_*\,\mathbf{R}p_*\,\mathbf{L}p^*\,\mathbf{L}f^*(M)$$

is an isomorphism in $\mathrm{Ho}(\mathscr{M})(Y)$.

We shall say that $\mathscr{M}$ (or by abuse, that $\mathrm{Ho}(\mathscr{M})$) satisfies *t-descent* if, for any object Y of $\mathscr{S}^{\mathrm{II}}$, any object of $\mathrm{Ho}(\mathscr{M})(Y)$ satisfies t-descent.

Proposition 3.2.6. *If $Y = \{Y_i\}_{i\in I}$ is a small family of objects of $\mathscr{S}$ (seen as an object of $\mathscr{S}^{\mathrm{II}}$), then an object M of $\mathrm{Ho}(\mathscr{M})(Y)$ satisfies t-descent if and only if, for any $i \in I$, any morphism $f : X \to Y_i$ of $\mathscr{S}$, and any t-hypercovering $p : \mathscr{X} \to X$, the map*

$$\mathbf{R}f_*\,\mathbf{L}f^*(M_i) \to \mathbf{R}f_*\,\mathbf{R}p_*\,\mathbf{L}p^*\,\mathbf{L}f^*(M_i)$$

is an isomorphism in $\mathrm{Ho}(\mathscr{M})(Y_i)$.

Proof. This follows from the definition and from Proposition 3.1.10. □

Corollary 3.2.7. *The $\mathscr{P}$-fibred model category $\mathscr{M}$ satisfies t-descent if and only if, for any object X of $\mathscr{S}$, and any t-hypercovering $p : \mathscr{X} \to X$, the functor*

$$\mathbf{L}p^* : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(\mathscr{X})$$

is fully faithful.

Proposition 3.2.8. *If $\mathscr{M}$ satisfies t-descent, then, for any t-covering $f : Y \to X$, the functor*

$$\mathbf{L}f^* : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(Y)$$

is conservative.

Proof. Let $f : Y \to X$ be a t-covering, and $u : M \to M'$ a morphism of $\mathrm{Ho}(\mathscr{M})(X)$ whose image by $\mathbf{L}f^*$ is an isomorphism. We can consider the Čech t-hypercovering associated to f, that is the simplicial object $\mathscr{Y}$ over X defined by

$$\mathscr{Y}_n = \underbrace{Y \times_X Y \times_X \cdots \times_X Y}_{n+1 \text{ times}}.$$

Let $p : \mathscr{Y} \to X$ be the canonical map. For each $n \geq 0$, the map $p_n : \mathscr{Y}_n \to X$ factor through f, from wich we deduce that the functor

$$\mathbf{L}p_n^* : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(\mathscr{Y}_n)$$

sends u to an isomorphism. This implies that the functor

$$\mathbf{L}p^* : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(\mathscr{Y})$$

sends u to an isomorphism as well. But, as $\mathscr{Y}$ is a t-hypercovering of X, the functor $\mathbf{L}p^*$ is fully faithful, from which we deduce that u is an isomorphism by the Yoneda Lemma. □

3.2.9. Let $\mathscr{V}$ be a complete and cocomplete category. For an object X of $\mathscr{S}$, define $Pr(\mathscr{S}/X, \mathscr{V})$ as the category of presheaves on $\mathscr{S}/X$ with values in $\mathscr{V}$. Then $Pr(C/-, \mathscr{V})$ is a $\mathscr{P}$-fibred category (where, by abuse of notations, $\mathscr{S}$ denotes also the class of all maps in $\mathscr{S}$): this is a special case of the constructions explained in 3.1.2 applied to $\mathscr{V}$, seen as a fibred category over the terminal category. To be more explicit, for each object X of $\mathscr{S}^{\mathrm{II}}$, we have a $\mathscr{V}$-enriched Yoneda embedding

$$\mathscr{S}^{\mathrm{II}}/X \times \mathscr{V} \to Pr(\mathscr{S}/X, \mathscr{V}) \quad , \qquad (U, M) \mapsto U \otimes M \,, \tag{3.2.9.1}$$

where, if $U = \{U_i\}_{i\in I}$ is a small family of objects of $\mathscr{S}$, $U \otimes M$ is the presheaf

$$V \mapsto \coprod_{i\in I} \coprod_{a\in \mathrm{Hom}_{\mathscr{S}/S}(V,U_i)} M \,. \tag{3.2.9.2}$$

For a morphism $f : X \to Y$ in $\mathscr{S}$, the functor

$$f^* : Pr(\mathscr{S}/Y, \mathscr{V}) \to Pr(\mathscr{S}/X, \mathscr{V})$$

is the functor defined by composition with the corresponding functor $\mathscr{S}/X \to \mathscr{S}/Y$. The functor f^* has always a left adjoint

$$f_\sharp : Pr(\mathscr{S}/X, \mathscr{V}) \to Pr(\mathscr{S}/Y, \mathscr{V}) \,,$$

which is the unique colimit preserving functor defined by

$$f_\sharp(U \otimes M) = U \otimes M \,,$$

where, on the left hand side U is considered as an object over X, while, on the right hand side, U is considered as an object over Y by composition with f. Similary, if all the pullbacks by f are representable in $\mathscr{S}$ (e.g. if f is a $\mathscr{P}$-morphism), the functor f^* can be described as the colimit preserving functor defined by the formula

$$f^*(U \otimes M) = (X \times_Y U) \otimes M \,.$$

If $\mathscr{V}$ is a cofibrantly generated model category, then, for each object X of $\mathscr{S}$, the category $Pr(\mathscr{S}/X, \mathscr{V})$ is naturally endowed with the *projective model category structure*, i.e. with the cofibrantly generated model category structure whose weak equivalences and fibrations are defined termwise (this is Proposition 3.1.6 applied to $\mathscr{V}$, seen as a fibred category over the terminal category). The cofibrations of the projective model category structure on $Pr(\mathscr{S}/X, \mathscr{V})$ will be called the *projective cofibrations.* If moreover $\mathscr{V}$ is combinatorial (resp. left proper, resp. right proper, resp. stable), so is $Pr(\mathscr{S}/X, \mathscr{V})$. In particular, if $\mathscr{V}$ is a combinatorial model category, then $Pr(\mathscr{S}/-, \mathscr{V})$ is a $\mathscr{P}$-fibred combinatorial model category over $\mathscr{S}$.

According to Definition 3.2.5, it thus makes sense to speak of t-descent in $Pr(\mathscr{S}/-, \mathscr{V})$.

If $U = \{U_i\}_{i\in I}$ is a small family of objects of $\mathscr{S}$ over X, and if F is a presheaf over $\mathscr{S}/X$, we define

$$F(U) = \prod_{i\in I} F(U_i) \,. \tag{3.2.9.3}$$

the functor $F \mapsto F(U)$ is a right adjoint to the functor $E \mapsto U \otimes E$.

We remark that a termwise fibrant presheaf F on $\mathscr{S}/X$ satisfies t-descent if and only if, for any object Y of $\mathscr{S}^{\mathrm{II}}$, and any t-hypercovering $\mathscr{Y} \to Y$ over X, the map

$$F(Y) \to \mathbf{R}\varprojlim_{n\in\Delta} F(\mathscr{Y}_n)$$

is an isomorphism in $\mathrm{Ho}(\mathscr{V})$.

Proposition 3.2.10. *If $\mathscr{V}$ is combinatorial and left proper, then the category of presheaves $Pr(\mathscr{S}/X, \mathscr{V})$ admits a combinatorial model category structure whose cofibrations are the projective cofibrations, and whose fibrant objects are the termwise fibrant objects which satisfy t-descent. This model category structure will be called the* t-local model category structure*, and the corresponding homotopy category will be denoted by* $\mathrm{Ho}_t(Pr(\mathscr{S}/X, \mathscr{V}))$.

Moreover, any termwise weak equivalence is a weak equivalence for the t-local model structure, and the induced functor

$$a^* : \mathrm{Ho}(Pr(\mathscr{S}/X, \mathscr{V})) \to \mathrm{Ho}_t(Pr(\mathscr{S}/X, \mathscr{V}))$$

admits a fully faithful right adjoint

$$a_* : \mathrm{Ho}_t(Pr(\mathscr{S}/X, \mathscr{V})) \to \mathrm{Ho}(Pr(\mathscr{S}/X, \mathscr{V}))$$

whose essential image consists precisely of the full subcategory of $\mathrm{Ho}(Pr(\mathscr{S}/X, \mathscr{V}))$ *spanned by the presheaves which satisfy t-descent.*

Proof. Let H be the class of maps of shape

$$\operatorname*{hocolim}_{n\in\Delta^{op}} \mathscr{Y}_n \otimes E \to Y \otimes E\,, \tag{3.2.10.1}$$

where Y is an object of $\mathscr{S}^{\mathrm{II}}$ over X, $\mathscr{Y} \to Y$ is a t-hypercovering, and E is a cofibrant replacement of an object which is either a source or a target of a generating cofibration of $\mathscr{V}$. Define the t-local model category structure as the left Bousfield localization of $Pr(\mathscr{S}/X, \mathscr{V})$ by H; see [Bar09, Theorem 4.7]. We shall call *t-local weak equivalences* the weak equivalences of the t-local model category structure. For each object Y over X, the functor $Y \otimes (-)$ is a left Quillen functor from $\mathscr{V}$ to $Pr(\mathscr{S}/X, \mathscr{V})$. We thus get a total left derived functor

$$Y \otimes^{\mathbf{L}} (-) : \mathrm{Ho}(\mathscr{V}) \to \mathrm{Ho}_t(Pr(\mathscr{S}/X, \mathscr{V}))$$

whose right adjoint is the evaluation at Y. For any object E of $\mathscr{V}$ and any t-local fibrant presheaf F on $\mathscr{S}/X$ with values in $\mathscr{V}$, we thus have natural bijections

$$\mathrm{Hom}(E, F(Y)) \simeq \mathrm{Hom}(Y \otimes^{\mathbf{L}} E, F)\,, \tag{3.2.10.2}$$

and, for any simplicial object $\mathscr{Y}$ of $\mathscr{S}/X$, identifications

$$\mathrm{Hom}(E, \mathbf{R}\varprojlim_{n\in\Delta} F(\mathscr{Y}_n)) \simeq \mathrm{Hom}(\mathbf{L}\varinjlim_{n\in\Delta} \mathscr{Y}_n \otimes^{\mathbf{L}} E, F)\,, \tag{3.2.10.3}$$

One sees easily that, for any t-hypercovering $\mathscr{Y} \to Y$ and any cofibrant object E of $\mathscr{V}$, the map

$$\mathbf{L}\varinjlim_{n\in\Delta} \mathscr{Y}_n \otimes^{\mathbf{L}} E \to Y \otimes^{\mathbf{L}} E \tag{3.2.10.4}$$

is an isomorphism in the t-local homotopy category $\mathrm{Ho}_t(Pr(\mathscr{S}/X, \mathscr{V}))$: by the small object argument, the smallest full subcategory of $\mathrm{Ho}(Pr(\mathscr{S}/X, \mathscr{V}))$ which is stable by homotopy colimits and which contains the source and the targets of the generating cofibrations is $\mathrm{Ho}_t(Pr(\mathscr{S}/X, \mathscr{V}))$ itself, and the class of objects E of $\mathscr{V}$ such that the map (3.2.10.4) is an isomorphism in $\mathrm{Ho}(\mathscr{V})$ is sable by homotopy colimits. Similarly, we see that, for any object E, the functor $(-) \otimes^{\mathbf{L}} E$ preserves sums. As a consequence, we get from (3.2.10.2) and (3.2.10.3) that the fibrant objects of the t-local model category structure are precisely the termwise fibrant objects F of the projective model structure which satisfy t-descent. The last part of the proposition follows from the general yoga of left Bousfield localizations. □

3.2.11. Let $\mathscr{M}$ be a $\mathscr{P}$-fibred combinatorial model category over $\mathscr{S}$, and S an object of $\mathscr{S}$. Denote by

$$\mathscr{S} : \mathscr{S}/S \to \mathscr{S}$$

the canonical forgetful functor. Then there is a canonical morphism of $\mathscr{S}$-diagrams

$$\sigma : (\mathscr{S}, \mathscr{S}/S) \to (S, \mathscr{S}/S) \tag{3.2.11.1}$$

(where $(S, \mathscr{S}/S)$ stands for the constant diagram with value S). This defines a functor

$$\mathbf{R}\sigma_* : \mathrm{Ho}(\mathscr{M})(\mathscr{S}, \mathscr{S}/S) \to \mathrm{Ho}(\mathscr{M})(S, \mathscr{S}/S) = \mathrm{Ho}(Pr(\mathscr{S}/S, \mathscr{M}(S)))\,. \tag{3.2.11.2}$$

For an object M of $\mathrm{Ho}(\mathscr{M})(S)$, one defines the presheaf of *geometric derived global sections of M over S* by the formula

$$\mathbf{R}\Gamma_{geom}(-, M) = \mathbf{R}\sigma_* \mathbf{L}\sigma^*(M)\,. \tag{3.2.11.3}$$

This is a presheaf on $\mathscr{S}/S$ with values in $\mathscr{M}(S)$ whose evaluation on a morphism $f : X \to S$ is, by virtue of Propositions 3.1.15 and 3.1.16,

$$\mathbf{R}\Gamma_{geom}(X, M) \simeq \mathbf{R}f_* \, \mathbf{L}f^*(M)\,. \tag{3.2.11.4}$$

Proposition 3.2.12. *For an object M of $\mathrm{Ho}(\mathscr{M})(S)$, the following conditions are equivalent.*

(a) The object M satisfies t-descent.

(b) The presheaf $\mathbf{R}\Gamma_{geom}(-, M)$ satisfies t-descent.

Proof. For any morphism $f : X \to S$ and any t-hypercovering $p : \mathscr{X} \to X$ over S, we have, by Proposition 3.2.4 and formula (3.2.11.4), an isomorphism

$$\mathbf{R}f_*\mathbf{R}p_* \, \mathbf{L}p^* \, \mathbf{L}f^*(M) \simeq \mathbf{R}\varprojlim_{n\in\Delta} \mathbf{R}\Gamma_{geom}(\mathscr{X}_n, M)\,.$$

From there, we see easily that conditions (a) and (b) are equivalent. □

3.2.13. The preceding proposition allows to reduce descent problems in a fibred model category to descent problems in a category of presheaves with values in a model category. On can even go further and reduce the problem to category of presheaves with values in an 'elementary model category' as follows.

Consider a model category $\mathscr{V}$. Then one can associate to $\mathscr{V}$ its corresponding *prederivator* $\mathbf{Ho}(\mathscr{V})$, that is the strict 2-functor from the 2-category of small categories to the 2-category of categories, defined by

$$\mathbf{Ho}(\mathscr{V})(I) = \mathrm{Ho}(\mathscr{V}^{I^{op}}) = \mathrm{Ho}(Pr(I, \mathscr{V})) \tag{3.2.13.1}$$

for any small category I. More explicitly: for any functor $u : I \to J$, one gets a functor

$$u^* : \mathbf{Ho}(\mathscr{V})(J) \to \mathbf{Ho}(\mathscr{V})(I)$$

(induced by the composition with u), and for any morphism of functors

$$I \;\underset{v}{\overset{u}{\Downarrow \alpha}}\; J \quad ,$$

one has a morphism of functors

$$\mathbf{Ho}(\mathscr{V})(I) \;\underset{v^*}{\overset{u^*}{\Uparrow \alpha^*}}\; \mathbf{Ho}(\mathscr{V})(J) \quad .$$

Moreover, the prederivator $\mathbf{Ho}(\mathscr{V})$ is then a Grothendieck derivator; see [Cis03, Thm. 6.11]. This means in particular that, for any functor between small categories $u : I \to J$, the functor u^* has a left adjoint

$$\mathbf{L}u_\sharp : \mathbf{Ho}(\mathscr{V})(I) \to \mathbf{Ho}(\mathscr{V})(J) \tag{3.2.13.2}$$

as well as a right adjoint

$$\mathbf{R}u_* : \mathbf{Ho}(\mathscr{V})(I) \to \mathbf{Ho}(\mathscr{V})(J) \tag{3.2.13.3}$$

(in the case where $J = e$ is the terminal category, then $\mathbf{L}u_\sharp$ is the homotopy colimit functor, while $\mathbf{R}u_*$ is the homotopy limit functor).

If $\mathscr{V}$ and $\mathscr{V}'$ are two model categories, a *morphism of derivators*

$$\Phi : \mathbf{Ho}(\mathscr{V}) \to \mathbf{Ho}(\mathscr{V}')$$

is simply a morphism of 2-functors, that is the data of functors

$$\Phi_I : \mathbf{Ho}(\mathscr{V})(I) \to \mathbf{Ho}(\mathscr{V}')(I)$$

together with coherent isomorphisms

$$u^*(\Phi_J(F)) \simeq \Phi_I(u^*(F))$$

for any functor $u : I \to J$ and any presheaf F on J with values in $\mathscr{V}$ (see [Cis03, p. 210] for a precise definition).

Such a morphism Φ is said to be *continuous* if, for any functor $u : I \to J$, and any object F of $\mathbf{Ho}(\mathscr{V})(I)$, the canonical map

$$\Phi_J \, \mathbf{R}u_*(F) \to \mathbf{R}u_* \, \Phi_I(F) \tag{3.2.13.4}$$

is an isomorphism. One can check that a morphism of derivators Φ is continuous if and only if it commutes with homotopy limits (i.e. if and only if the maps (3.2.13.4) are isomorphisms in the case where $J = e$ is the terminal category); see [Cis08, Prop. 2.6]. For instance, the total right derived functor of any right Quillen functor defines a continuous morphism of derivators; see [Cis03, Prop. 6.12].

Dually a morphism Φ of derivators is *cocontinuous* if, for any functor $u : I \to J$, and any object F of $\mathbf{Ho}(\mathscr{V})(I)$, the canonical map

$$\mathbf{L}u_! \, \Phi_I(F) \to \Phi_J \, \mathbf{L}u_!(F) \tag{3.2.13.5}$$

is an isomorphism.

3.2.14. We shall say that a stable model category $\mathscr{V}$ is $\mathbf{Q}$-*linear* if all the objects of the triangulated category $\mathrm{Ho}(\mathscr{V})$ are uniquely divisible.

Theorem 3.2.15. *Let $\mathscr{V}$ be a model category (resp. a stable model category, resp. a $\mathbf{Q}$-linear stable model category), and denote by $\mathcal{S}$ the model category of simplicial sets (resp. the stable model category of S^1-spectra, resp. the $\mathbf{Q}$-linear stable model category of complexes of $\mathbf{Q}$-vector spaces). Denote by $\mathbb{1}$ the unit object of the closed symmetric monoidal category* $\mathrm{Ho}(\mathcal{S})$.

Then, for each object E of $\mathrm{Ho}(\mathscr{V})$*, there exists a unique continuous morphism of derivators*

$$\mathbf{R}\mathrm{Hom}(E, -) : \mathbf{Ho}(\mathscr{V}) \to \mathbf{Ho}(\mathcal{S})$$

such that, for any object F of $\mathrm{Ho}(\mathscr{V})$*, there is a functorial bijection*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{S})}(\mathbb{1}, \mathbf{R}\mathrm{Hom}(E, F)) \simeq \mathrm{Hom}_{\mathrm{Ho}(\mathscr{V})}(E, F)) \, .$$

Proof. Note that the stable $\mathbf{Q}$-linear case follows from the stable case and from the fact that the derivator of complexes of $\mathbf{Q}$-vector spaces is (equivalent to) the full subderivator of the derivator of S^1-spectra spanned by uniquely divisible objects.

It thus remains to prove the theorem in the case where $\mathscr{V}$ be a model category (resp. a stable model category) and $\mathcal{S}$ is the model category of simplicial sets (resp. the stable model category of S^1-spectra). The existence of $\mathbf{R}\mathrm{Hom}(E, -)$ follows then from [Cis03, Prop. 6.13] (resp. [CT09, Lemma A.6]).

For the unicity, as we don't really need it here, we shall only sketch the proof (the case of simplicial sets is done in [Cis03, Rem. 6.14]). One uses the universal property of the derivator $\mathbf{Ho}(\mathcal{S})$: by virtue of [Cis08, Cor. 3.26] (resp. of [CT09, Thm. A.5]), for any model category (resp. stable model category) $\mathscr{V}'$ there is a canonical equivalence of categories between the category of cocontinous morphisms from $\mathbf{Ho}(\mathcal{S})$ to $\mathbf{Ho}(\mathscr{V}')$ and the homotopy category $\mathrm{Ho}(\mathscr{V})$. As a consequence, the derivator $\mathbf{Ho}(\mathcal{S})$ admits a unique closed symmetric monoidal structure, and any derivator (resp. triangulated derivator) is naturally and uniquely enriched in $\mathbf{Ho}(\mathcal{S})$; see [Cis08, Thm. 5.22]. More concretely, this universal property gives, for any object E in $\mathrm{Ho}(\mathscr{V}')$, a unique cocontinuous morphism of derivators

$$\mathbf{Ho}(\mathcal{S}) \to \mathbf{Ho}(\mathscr{V}') \quad , \qquad K \mapsto K \otimes E$$

such that $\mathbb{1} \otimes E = E$. For a fixed K in $\mathbf{Ho}(\mathcal{S})(I)$, this defines a cocontinuous morphism of derivators

$$\mathbf{Ho}(\mathscr{V}') \to \mathbf{Ho}(\mathscr{V}'^{I^{op}}) \quad , \qquad E \mapsto K \otimes E$$

which has a right adjoint

$$\mathbf{Ho}(\mathscr{V}'^{I^{op}}) \to \mathbf{Ho}(\mathscr{V}') \quad , \qquad F \mapsto F^K \, .$$

Let

$$\mathbf{R}\mathrm{Hom}(E, -) : \mathbf{Ho}(\mathscr{V}) \to \mathbf{Ho}(\mathcal{S})$$

be a continuous morphism such that, for any object F of $\mathscr{V}$, there is a functorial bijection

$$i_F : \mathrm{Hom}_{\mathrm{Ho}(\mathcal{S})}(\mathbb{1}, \mathbf{R}\mathrm{Hom}(E,F)) \simeq \mathrm{Hom}_{\mathrm{Ho}(\mathscr{V})}(E,F)) .$$

Then, for any object K of $\mathbf{Ho}(\mathcal{S})(I)$, and any object F of $\mathbf{Ho}(\mathscr{V})(I)$ a canonical isomorphism

$$\mathbf{R}\mathrm{Hom}(E, F^K) \simeq \mathbf{R}\mathrm{Hom}(E,F)^K$$

which is completely determined by being the identity for $K = \mathbb{1}$ (this requires the full universal property of $\mathbf{Ho}(\mathcal{S})$ given by by [Cis08, Thm. 3.24] (resp. by the dual version of [CT09, Thm. A.5])). We thus get from the functorial bijections i_F the natural bijections:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Ho}(\mathcal{S})(I)}(K, \mathbf{R}\mathrm{Hom}(E,F)) \simeq & \mathrm{Hom}_{\mathrm{Ho}(\mathcal{S})}(\mathbb{1}, \mathbf{R}\mathrm{Hom}(E,F)^K) \\ \simeq & \mathrm{Hom}_{\mathrm{Ho}(\mathcal{S})}(\mathbb{1}, \mathbf{R}\mathrm{Hom}(E,F^K)) \\ \simeq & \mathrm{Hom}_{\mathrm{Ho}(\mathscr{V})}(E, F^K) \\ \simeq & \mathrm{Hom}_{\mathbf{Ho}(\mathscr{V})(I)}(K \otimes E, F) .\end{aligned}$$

In other words, $\mathbf{R}\mathrm{Hom}(E,-)$ has to be a right adjoint to $(-) \otimes E$. □

Remark 3.2.16. The preceding theorem mostly holds for abstract derivators. The only problem is for the existence of the morphism $\mathbf{R}\mathrm{Hom}(E,-)$ (the unicity is always clear). However, this problem disapears for derivators which have a Quillen model (as we have seen above), as well as for triangulated derivators (see [CT09, Lemma A.6]). Hence Theorem 3.2.15 holds in fact for any triangulated Grothendieck derivator.

In the case when $\mathscr{V}$ is a combinatorial model category (which, in practice, will essentially always be the case), the enrichment over simplicial sets (resp, in the stable case, over spectra) can be constructed via Quillen functors by Dugger's presentation theorems [Dug01] (resp. [Dug06]).

Corollary 3.2.17. *Let $\mathscr{M}$ be a $\mathscr{P}$-fibred combinatorial model category (resp. a stable $\mathscr{P}$-fibred combinatorial model category, resp. a $\mathbf{Q}$-linear stable $\mathscr{P}$-fibred combinatorial model category) over $\mathscr{S}$, and $\mathcal{S}$ the model category of simplicial sets (resp. the stable model category of S^1-spectra, resp. the $\mathbf{Q}$-linear stable model category of complexes of $\mathbf{Q}$-vector spaces).*

Consider an object S of $\mathscr{S}$, a morphism $f : X \to S$, and a morphism of $\mathscr{S}$-diagrams $p : (\mathscr{X}, I) \to X$ over S. Then, for an object M of $\mathrm{Ho}(\mathscr{M})(S)$, the following conditions are equivalent.

(a) The map

$$\mathbf{R}f_* \mathbf{L}f^*(M) \to \mathbf{R}f_* \mathbf{R}p_* \mathbf{L}p^* \mathbf{L}f^*(M)$$

is an isomorphism in $\mathrm{Ho}(\mathscr{M})(S)$.

(b) The map

$$\mathbf{R}\Gamma_{geom}(X,M) \to \mathbf{R}\varprojlim_{i \in I^{op}} \mathbf{R}\Gamma_{geom}(\mathscr{X}_i, M)$$

is an isomorphism in $\mathrm{Ho}(\mathscr{M})(S)$.

(c) For any object E of $\mathrm{Ho}(\mathscr{M})(S)$, the map

$$\mathbf{R}\mathrm{Hom}(E, \mathbf{R}\Gamma_{geom}(X,M)) \to \mathbf{R}\varprojlim_{i \in I^{op}} \mathbf{R}\mathrm{Hom}(E, \mathbf{R}\Gamma_{geom}(\mathscr{X}_i, M))$$

is an isomorphism in $\mathrm{Ho}(\mathcal{S})$.

Proof. The equivalence between (a) and (b) follows from Propositions 3.1.15 and 3.1.16, which give the formula

$$\mathbf{R}f_* \mathbf{R}p_* \mathbf{L}p^* \mathbf{L}f^*(M) \simeq \mathbf{R}\varprojlim_{i \in I^{op}} \mathbf{R}\Gamma_{geom}(\mathscr{X}_i, M) .$$

The identification

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{S})}(\mathbb{1}, \mathbf{R}\mathrm{Hom}(E,F)) \simeq \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(S)}(E,F)$$

and the Yoneda Lemma show that a map in $\mathrm{Ho}(\mathscr{M})(S)$ is an isomorphism if and only its image by $\mathbf{R}\mathrm{Hom}(E,-)$ is an isomorphism for any object E of $\mathrm{Ho}(\mathscr{M})(S)$. Moreover, as $\mathbf{R}\mathrm{Hom}(E,-)$

is continuous, for any small category I and any presheaf F on I with values in $\mathscr{M}(S)$, there is a canonical isomorphism

$$\mathbf{R}\mathrm{Hom}(E,\ \mathbf{R}\varprojlim_{i\in I^{op}} F_i)) \simeq\ \mathbf{R}\varprojlim_{i\in I^{op}} \mathbf{R}\mathrm{Hom}(E, F_i))\,.$$

This proves the equivalence between contitions (b) and (c). □

Corollary 3.2.18. *Under the assumptions of Corollary 3.2.17, given an object S of $\mathscr{S}$, an object M of $\mathrm{Ho}(\mathscr{M})(S)$ satisfies t-descent if and only if, for any object E of $\mathrm{Ho}(\mathscr{M})(S)$ the presheaf of simplicial sets (resp. of S^1-spectra, resp. of complexes of $\mathbf{Q}$-vector spaces) $\mathbf{R}\mathrm{Hom}(E, \mathbf{R}\Gamma_{geom}(-, M))$ satisfies t-descent over $\mathscr{S}/S$.*

Proof. This follows from the preceding corollary, using the formula given by Proposition 3.2.4. □

Remark 3.2.19. We need the category $\mathscr{S}$ to be small in some sense to apply the two preceding corollaries because we need to make sense of the model projective category structure of Proposition 3.2.10. However, we can use these corollaries even if the site $\mathscr{S}$ is not small as well: we can either use the theory of universes, or apply these corollaries to all the adequate small subsites of $\mathscr{S}$. As a consequence, we shall feel free to use Corollaries 3.2.17 and 3.2.18 for non necessarily small sites $\mathscr{S}$, leaving to the reader the task to avoid set-theoretic difficulties according to her/his taste.

Definition 3.2.20. For an S^1-spectrum E and an integer n, we define its nth cohomology group $H^n(E)$ by the formula

$$H^n(E) = \pi_{-n}(E)\,,$$

where π_i stands for the ith stable homotopy group functor.

Let $\mathscr{M}$ be a monoidal $\mathscr{P}$-fibred stable combinatorial model category over $\mathscr{S}$. Given an object S of $\mathscr{S}$ as well as an object M of $\mathrm{Ho}(\mathscr{M})(S)$, we define the presheaf of *absolute derived global sections of M over S* by the formula

$$\mathbf{R}\Gamma(-, M) = \mathbf{R}\mathrm{Hom}(\mathbb{1}_S, \mathbf{R}\Gamma_{geom}(-, M))\,.$$

For a map $X \to S$ of $\mathscr{S}$, we thus have the *absolute cohomology of X with coefficients in M*, $\mathbf{R}\Gamma(X, M)$, as well as the *cohomology groups of X with coefficients in M*:

$$H^n(X, M) = H^n(\mathbf{R}\Gamma(X, M))\,.$$

We have canonical isomorphisms of abelian groups

$$\begin{aligned} H^n(X, M) &\simeq \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(S)}(\mathbb{1}_S, \mathbf{R}f_*\,\mathbf{L}f^*(M)) \\ &\simeq \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(X)}(\mathbb{1}_X, \mathbf{L}f^*(M))\,. \end{aligned}$$

Note that, if moreover $\mathscr{M}$ is $\mathbf{Q}$-linear, the presheaf $\mathbf{R}\Gamma(-, M)$ can be considered as a presheaf of complexes of $\mathbf{Q}$-vector spaces on $\mathscr{S}/S$.

3.3. **Descent over schemes.** The aim of this section is to give natural sufficient conditions for $\mathscr{M}$ to satisfy descent with respect to various Grothendieck topologies[16].

3.3.a. *localization and Nisnevich descent.*

3.3.1. Recall from example 2.1.15 that a *Nisnevich distinguished square* is a pullback square of schemes

$$\begin{array}{ccc} V & \xrightarrow{l} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow[j]{} & X \end{array} \qquad (3.3.1.1)$$

[16] In fact, using remark 3.2.16, all of this section (results and proofs) holds for an abstract algebraic prederivator in the sense of Ayoub [Ayo07a, Def. 2.4.13] without any changes (note that the results of 3.1.b are in fact a proof that (stable) combinatorial fibred model categories over $\mathscr{S}$ give rise to algebraic prederivators). The only interest of considering a fibred model category over $\mathscr{S}$ is that it allows to formulate things in a little more naive way.

in which f is étale, j is an open immersion with reduced complement Z and the induced morphism $f^{-1}(Z) \to Z$ is an isomorphism.

For any scheme X in $\mathscr{S}$, we denote by X_{Nis} the small Nisnevich site of X.

Theorem 3.3.2 (Morel-Voevodsky)**.** *Let $\mathscr{V}$ be a (combinatorial) model category and T a scheme in $\mathscr{S}$. For a presheaf F on T_{Nis} with values in $\mathscr{V}$, the following conditions are equivalent.*

(i) $F(\varnothing)$ is a terminal object in $\mathrm{Ho}(\mathscr{V})$, and for any Nisnevich distinguished square (3.3.1.1) *in T_{Nis}, the square*

$$\begin{array}{ccc} F(X) & \longrightarrow & F(Y) \\ \downarrow & & \downarrow \\ F(U) & \longrightarrow & F(V) \end{array}$$

is a homotopy pullback square in $\mathscr{V}$.

(ii) The presheaf F satisfies Nisnevich descent on T_{Nis}.

Proof. By virtue of corollaries 3.2.17 and 3.2.18, it is sufficient to prove this in the case where $\mathscr{V}$ is the usual model category of simplicial sets, in which case this is precisely Morel and Voevodsky's theorem; see [MV99, Voe00a, Voe00b]. □

3.3.3. Consider a Nisnevich distinguished square (3.3.1.1) and put $a = jg = fl$. According to our general assumption 3.0, the maps a, j and f are $\mathscr{P}$-morphisms. For any object M of $\mathscr{M}(X)$, we obtain a commutative square in $\mathscr{M}$ (which is well defined as an object in the homotopy of commutative squares in $\mathscr{M}(X)$):

$$\begin{array}{ccc} \mathbf{L}a_\sharp a^* M & \longrightarrow & \mathbf{L}f_\sharp f^*(M) \\ \downarrow & & \downarrow \\ \mathbf{L}j_\sharp j^*(M) & \longrightarrow & M. \end{array} \tag{3.3.3.1}$$

We also obtain another commutative square in $\mathscr{M}$ by appyling the functor $\mathbf{R}Hom_X(-, \mathbb{1}_X)$:

$$\begin{array}{ccc} M & \longrightarrow & \mathbf{R}f_*\, f^*(M) \\ \downarrow & & \downarrow \\ \mathbf{R}j_*\, j^*(M) & \longrightarrow & \mathbf{R}a_*\, a^*(M). \end{array} \tag{3.3.3.2}$$

Proposition 3.3.4. *If the category $\mathrm{Ho}(\mathscr{M})$ has the localization property, then for any Nisnevich distinguished square* (3.3.1.1) *and any object M of $\mathrm{Ho}(\mathscr{M})(X)$, the squares* (3.3.3.1) *and* (3.3.3.2) *are homotopy cartesians.*

Proof. Let $i : Z \to X$ be the complement of the open immersion j (Z being endowed with the reduced structure) and $p : f^{-1}(Z) \to Z$ the map induced by f.

We have only to prove that one of the squares (3.3.3.1), (3.3.3.2) are cartesian. We choose the square (3.3.3.1).

Because the pair of functor $(\mathbf{L}i^*, j^*)$ is conservative on $\mathrm{Ho}(\mathscr{M})(X)$, we have only to check that the pullback of (3.3.3.1) along j^* or $\mathbf{L}i^*$ is homotopy cartesian.

But, using the $\mathscr{P}$-base change property, we see that the image of (3.3.3.1) by j^* is (canonically isomorphic to) the commutative square

$$\begin{array}{ccc} \mathbf{L}g_\sharp a^*(M) & = & \mathbf{L}g_\sharp a^*(M) \\ \downarrow & & \downarrow \\ j^*(M) & = & j^*(M) \end{array}$$

which is obviously homotopy cartesian.

Using again the $\mathscr{P}$-base change property, we obtain that the image of (3.3.3.1) by $\mathbf{L}i^*$ is isomorphic in $\mathrm{Ho}(\mathscr{M})$ to the square

$$\begin{array}{ccc} 0 & \longrightarrow & p_\sharp p^* \mathbf{L}i^*(M) \\ \| & & \downarrow \\ 0 & \longrightarrow & \mathbf{L}i^*(M) \end{array}$$

which is again obviously homotopy cartesian because p is an isomorphism (note for this last reason, $p_\sharp = \mathbf{L}p_\sharp$). □

Corollary 3.3.5. *If* $\mathrm{Ho}(\mathscr{M})$ *has the localization property, then it satisfies Nisnevich descent.*

Proof. This corollary thus follows immediately from corollary 3.2.17, theorem 3.3.2 and proposition 3.3.4. □

3.3.b. *Proper base change isomorphism and descent by blow-ups.*

3.3.6. Recall from example 2.1.15 that a cdh-*distinguished square* is a pullback square of schemes

$$\begin{array}{ccc} T & \xrightarrow{k} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[i]{} & X \end{array} \tag{3.3.6.1}$$

in which f is proper surjective, i a closed immersion and the induced map $f^{-1}(X - Z) \to X - Z$ is an isomorphism.

The cdh-*topology* is the Grothendieck topology on the category of schemes generated by Nisnevich coverings and by coverings of shape $\{Z \to X, Y \to X\}$ for any cdh-distinguished square (3.3.6.1).

Theorem 3.3.7 (Voevodsky)**.** *Let* $\mathscr{V}$ *be a (combinatorial) model category. For a presheaf* F *on* $\mathscr{S}$ *with values in* $\mathscr{V}$*, the following conditions are equivalent.*

(i) The presheaf F *satisfies* cdh-*descent on* $\mathscr{S}$*.*

(ii) The presheaf F *satisfies Nisnevich descent and, for any* cdh-*distinguished square* (3.3.6.1) *of* $\mathscr{S}$*, the square*

$$\begin{array}{ccc} F(X) & \longrightarrow & F(Y) \\ \downarrow & & \downarrow \\ F(Z) & \longrightarrow & F(T) \end{array}$$

is a homotopy pullback square in $\mathscr{V}$*.*

Proof. It is sufficient to prove this in the case where $\mathscr{V}$ is the usual model category of simplicial sets; see corollaries 3.2.17 and 3.2.18. As the distinguished cdh-squares define a bounded regular and reduced *cd*-structure on $\mathscr{S}$, the equivalence between (i) and (ii) follows from Voevodsky's theorems on descent with respect to topologies defined by *cd*-structures [Voe00a, Voe00b]. □

3.3.8. Consider a cdh-distinguished square (3.3.6.1) and put $a = ig = fk$. For any object M of $\mathscr{M}(X)$, we obtain a commutative square in $\mathscr{M}$ (which is well defined as an object in the homotopy of commutative squares in $\mathscr{M}(X)$):

$$\begin{array}{ccc} M & \longrightarrow & \mathbf{R}f_* \mathbf{L}f^*(M) \\ \downarrow & & \downarrow \\ \mathbf{R}i_* \mathbf{L}i^*(M) & \longrightarrow & \mathbf{R}a_* \mathbf{L}a^*(M) \end{array} \tag{3.3.8.1}$$

Proposition 3.3.9. *Assume* $\mathrm{Ho}(\mathscr{M})$ *satisfies the localization property and the transversality property with respect to proper morphisms.*[17] *Then the following conditions hold:*

(i) For any cdh*-distinguished square* (3.3.6.1)*, and any object* M *of* $\mathrm{Ho}(\mathscr{M})(X)$ *the commutative square* (3.3.8.1) *is homotopy cartesian.*
(ii) The $\mathscr{P}$*-fibred model category* $\mathrm{Ho}(\mathscr{M})$ *satisfies* cdh*-descent.*

Proof. We first prove (i). Consider a cdh-distinguished square (3.3.6.1) and let $j : U \to X$ be the complement open immersion of i. As the pair of functor $(\mathbf{L}i^*, j^*)$ is conservative on $\mathrm{Ho}(\mathscr{M})(X)$, we have only to check that the image of (3.3.8.1) under $\mathbf{L}i^*$ and j^* is homotopy cartesian.

Using projective transversality, we see that the image of (3.3.8.1) by the functor $\mathbf{L}i^*$ is (isomorphic to) the homotopy pullback square

$$\begin{array}{ccc} \mathbf{L}i^*(M) & \longrightarrow & \mathbf{R}g_*\,\mathbf{L}g^*\,\mathbf{L}i^*(M) \\ \| & & \| \\ \mathbf{L}i^*(M) & \longrightarrow & \mathbf{R}g_*\,\mathbf{L}g^*\,\mathbf{L}i^*(M) \end{array} .$$

Let $h : f^{-1}(U) \to U$ be the pullback of f over U. As j is an open immersion, it is by assumption a $\mathscr{P}$-morphism and the $\mathscr{P}$-base change formula implies that the image of (3.3.8.1) by j^* is (isomorphic to) the commutative square

$$\begin{array}{ccc} \mathbf{L}j^*(M) & \longrightarrow & \mathbf{R}h_*\mathbf{L}h^*\mathbf{L}j^*(M) \\ \downarrow & & \downarrow \\ 0 & = & 0 \end{array}$$

which is obviously homotopy cartesian because h is an isomorphism.

We then prove (ii). We already know that $\mathscr{M}$ satisfies Nisnevich descent (corollary 3.3.5). Thus, by virtue of the equivalence between conditions (i) and (ii) of theorem 3.3.7, the computation above, together with corollaries 3.2.17 and 3.2.18 imply that $\mathscr{M}$ satisfies cdh-descent. □

3.3.10. To any cdh-distinguished square (3.3.6.1), one associates a diagram of schemes $\mathscr{Y}$ over X as follows. Let $\ulcorner$ be the category freely generated by the oriented graph

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \\ c & & \end{array} \tag{3.3.10.1}$$

Then $\mathscr{Y}$ is the functor from $\ulcorner$ to $\mathscr{S}/X$ defined by the following diagram.

$$\begin{array}{ccc} T & \xrightarrow{h} & Y \\ {\scriptstyle g}\downarrow & & \\ Z & & \end{array} \tag{3.3.10.2}$$

We then have a canonical map $\varphi : \mathscr{Y} \to X$, and the second assertion of theorem 3.3.9 can be reformulated by saying that the adjunction map

$$M \to \mathbf{R}\varphi_*\,\mathbf{L}\varphi^*(M)$$

[17] Recall from proposition 2.3.11 it is sufficient that $\mathrm{Ho}(\mathscr{M})$ satisfies transversality with respect to the projections $\mathbf{P}^n_S \to S$.

is an isomorphism for any object M of $\mathrm{Ho}(\mathscr{M})(X)$: indeed, by virtue of proposition 3.1.15, $\mathbf{R}\varphi_*\,\mathbf{L}\varphi^*(M)$ is the homotopy limit of the diagram

$$\begin{array}{ccc} & & \mathbf{R}f_*\,\mathbf{L}f^*(M) \\ & & \downarrow \\ \mathbf{R}i_*\,\mathbf{L}i^*(M) & \longrightarrow & \mathbf{R}a_*\,\mathbf{L}a^*(M) \end{array}$$

in $\mathrm{Ho}(\mathscr{M})(X)$. In other words, if $\mathscr{M}$ has the properties of localization and of projective transversality, then the functor

$$\mathbf{L}\varphi^* : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(\mathscr{Y}, \sqcap)$$

is fully faithful.

3.3.c. *Proper descent with rational coefficients I: Galois excision.* From now on, we assume that any scheme in $\mathscr{S}$ is quasi-excellent (we shall use several times the fact that the normalization of a quasi-excellent schemes gives rise to a finite surjective morphism). We fix a scheme S in $\mathscr{S}$, and we shall work with S-schemes in $\mathscr{S}$ (assuming these form an essentially small category).

3.3.11. The h-*topology* (resp. the qfh-*topology)* is the Grothendieck topology on the category of schemes associated to the pretopology whose coverings are the universal topological epimorphisms (resp. the quasi-finite universal topological epimorphisms). This topology has been introduced and studied by Voevodsky in [Voe96].

The h-topology is finer than the cdh-topology and, of course, finer than the qfh-topology. The qfh-topology is in turn finer than the étale topology. An interesting feature of the h-topology (resp. of the qfh-topology) is that any proper (resp. finite) surjective map is an h-covering. In fact, the h-topology (resp. the qfh-topology) can be described as the topology generated by the Nisnevich coverings and by the proper (resp. finite) surjective maps; see lemma 3.3.27 (resp. lemma 3.3.26) below for a precise statement.

3.3.12. Consider a morphism of schemes $f : Y \to X$. Consider the group of automorphisms $G = \mathrm{Aut}_Y(X)$ of the X-scheme Y.

Assuming X is connected, we say according to [Gro03, exp. V] that f is a *Galois cover* if it is finite étale (thus surjective) and G operates transitively and faithfully on any (or simply one) of the geometric fibers of Y/X. Then G is called the *Galois group* of Y/X.[18]

When X is not connected, we will still say that f is a *Galois cover* if it is so over any connected component of X. Then G will be called the *Galois group* of X. If $(X_i)_{i\in I}$ is the family connected components of X, then G is the product of the Galois groups G_i of $f \times_X X_i$ for each $i \in I$. The group G_i is equal to the Galois group of any residual extension over a generic point of X_i.

The following definition is an extension of the definition 5.5 of [SV00b]:

Definition 3.3.13. A *pseudo-Galois cover* is a finite surjective morphism of schemes $f : Y \to X$ which can be factored as

$$Y \xrightarrow{f'} X' \xrightarrow{p} X$$

where f' is a Galois cover and p is radicial[19] (such a p is automatically finite and surjective).

Note that the group G defined by the Galois cover f' is independent of the choice of the factorization. In fact, if $\bar{X}$ denotes the semi-localization of X at its generic points, considering the cartesian squares

$$\begin{array}{ccccc} \bar{Y} & \longrightarrow & \bar{X}' & \longrightarrow & \bar{X} \\ \downarrow & & \downarrow & & \downarrow \\ Y & \xrightarrow{f'} & X' & \xrightarrow{p} & X \end{array}$$

then $G = \mathrm{Aut}_{\bar{X}}(\bar{Y})$ – for any point $y \in \bar{Y}$, $x' = f'(y)$, $x = f(y)$, $\kappa_{x'}/\kappa_x$ is the maximal radicial sub-extension of the normal extension κ_y/κ_x. It will be called the *Galois group* of Y/X.

[18] The map f induces a one to one correspondence between the generic points of Y and that of X. For any generic point $y \in Y$, $x = f(y)$, the residual extension κ_y/κ_x is a Galois extension with Galois group G.

[19] See 2.1.10 for a reminder on radicial morphisms.

Remark also that Y is a G-torsor over X locally for the qfh-topology (i.e. it is a Galois object of group G in the qfh-topos of X): this comes from the fact that finite radicial epimorphisms are isomorphisms locally for the qfh-topology (any universal homeomorphism has this property by [Voe96, prop. 3.2.5]).

Let $f : Y \to X$ be a finite morphism, and G a finite group acting on Y over X. Note that, as Y is affine on X, the scheme theoretic quotient Y/G exists; see [Gro03, Exp. V, Cor. 1.8]. Such scheme-theoretic quotients are stable by flat pullbacks; see [Gro03, Exp. V, Prop. 1.9].

Definition 3.3.14. Let G be finite group. A qfh-*distinguished square of group* G is a pullback square of S-schemes of shape

$$\begin{array}{ccc} T & \xrightarrow{h} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[i]{} & X \end{array} \tag{3.3.14.1}$$

in which Y is endowed with an action of G over X, and satisfying the following three conditions.

(a) The morphism f is finite and surjective.
(b) The induced morphism $f^{-1}(X - Z) \to f^{-1}(X - Z)/G$ is flat.
(c) The morphism $f^{-1}(X - Z)/G \to X - Z$ is radicial.

Immediate examples of qfh-distinguished squares of trivial group are the following. The scheme Y might be the normalization of X, and Z is a nowhere dense closed subscheme out of which f is an isomorphism; or Y is dense open subscheme of X which is the disjoint union of its irreducible components; or Y is a closed subscheme of X inducing an isomorphism $Y_{red} \simeq X_{red}$.

A qfh-distinguished square of group G (3.3.14.1) will be said to be *pseudo-Galois* if Z is nowhere dense in X and if the map $f^{-1}(X - Z) \to X - Z$ is a pseudo-Galois cover of group G.

The main examples of pseudo-Galois qfh-distinguished squares will come from the following situation.

Proposition 3.3.15. *Consider an irreducible normal scheme X, and a finite extension L of its field of functions $k(X)$. Let K be the inseparable closure of $k(X)$ in L, and assume that L/K is a Galois extension of group G. Denote by Y the normalization of X in L. Then the action of G on $k(Y) = L$ extends naturally to an action on Y over X. Furthermore, there exists a closed subscheme Z of X, such that the pullback square*

$$\begin{array}{ccc} T & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[i]{} & X \end{array}$$

is a pseudo-Galois qfh-*distinguished square of group G.*

Proof. The action of G on L extends naturally to an action on Y over X by functoriality. Furthermore, Y/G is the normalization of X in K, so that $Y/G \to X$ is finite radicial and surjective (see [Voe96, Lemma 3.1.7] or [Bou98, V, §2, n° 3, lem. 4]). By construction, Y is generically a Galois cover over Y/G, which implies the result (see [GD67, Cor. 18.2.4]). □

3.3.16. For a given S-scheme T, we shall denote by $L(T)$ the corresponding representable qfh-sheaf of sets (remember that the qfh-topology is not subcanonical, so that $L(T)$ has to be distinguished from T itself). Beware that, in general, there is no reason that, given a finite group G acting on T, the scheme-theoretic quotient $L(T/G)$ (whenever defined) and the qfh-sheaf-theoretic quotient $L(T)/G$ would coincide.

Lemma 3.3.17. *Let $f : Y \to X$ be a separated morphism, G a finite group acting on Y over X, and Z a closed subscheme of X such that f is finite and surjective over $X - Z$, and such that the quotient map $f^{-1}(X - Z) \to f^{-1}(X - Z)/G$ is flat, while the map $f^{-1}(X - Z)/G \to X - Z$ is radicial. For $g \in G$, write $g : Y \to Y$ for the corresponding automorphism of Y, and define Y_g as*

the image of the diagonal $Y \to Y \times_X Y$ *composed with the automorphism* $1_Y \times_X g : Y \times_X Y \to Y \times_X Y$*. Then, if* $T = Z \times_X Y$*, we get a* qfh*-covering of* $Y \times_X Y$ *by closed subschemes:*

$$Y \times_X Y = (T \times_Z T) \cup \bigcup_{g \in G} Y_g \,.$$

Proof. Note that, as f is separated, the diagonal $Y \to Y \times_X Y$ is a closed embedding, so that the Y_g's are closed subschemes of $Y \times_X Y$. As the map $Y \times_{Y/G} Y \to Y \times_X Y$ is a universal homeomorphism, we may assume that $Y/G = X$. It is sufficient to prove that, if y and y' are two geometric points of Y whose images coincide in X and do not belong to Z, there exists an element g of G such that $y' = gy$ (which means that the pair (y, y') belongs to Y_g). For this purpose, we may assume, without loss of generality, that $Z = \varnothing$. Then, by assumption, Y is flat over X, from which we get the identification $(Y \times_X Y)/G \simeq Y \times_X (Y/G) \simeq Y$ (where the action of G on $Y \times_X Y$ is trivial on the first factor and is induced by the action on Y on the second factor). This achieves the proof. □

Proposition 3.3.18. *For any* qfh*-distinguished square of group* G (3.3.14.1)*, the commutative square*

$$\begin{array}{ccc} L(T)/G & \longrightarrow & L(Y)/G \\ \downarrow & & \downarrow \\ L(Z) & \longrightarrow & L(X) \end{array}$$

is a pullback and a pushout in the category of qfh*-sheaves. Moreover, if* X *is normal and if* Z *is nowhere dense in* X*, then the canonical map* $L(Y)/G \to L(Y/G) \simeq L(X)$ *is an isomorphism of* qfh*-sheaves (which implies that* $L(T)/G \to L(Z)$ *is an isomorphism as well).*

Proof. Note that this commutative square is a pullback because it was so before taking the quotients by G (as colimits are universal in any topos). As f is an qfh-covering, it is sufficient to prove that

$$\begin{array}{ccc} L(T) \times_{L(Z)} L(T)/G & \longrightarrow & L(Y) \times_{L(X)} L(Y)/G \\ \downarrow & & \downarrow \\ L(T) & \longrightarrow & L(Y) \end{array}$$

is a pushout square. This latter square fits into the following commutative diagram

$$\begin{array}{ccc} L(T) & \longrightarrow & L(Y) \\ \downarrow & & \downarrow \\ L(T) \times_{L(Z)} L(T)/G & \longrightarrow & L(Y) \times_{L(X)} L(Y)/G \\ \downarrow & & \downarrow \\ L(T) & \longrightarrow & L(Y) \end{array}$$

in which the two vertical composed maps are identities (the vertical maps of the upper commutative square are obtained from the diagonals by taking the quotients under the natural action of G on the right component). It is thus sufficient to prove that the upper square is a pushout. As the lower square is a pullback, the upper one shares the same property; moreover, all the maps in the upper commutative square are monomorphisms of qfh-sheaves, so that it is sufficient to prove that the map $(L(T) \times_{L(Z)} L(T)/G) \amalg L(Y) \to L(Y) \times_{L(X)} L(Y)/G$ is an epimorphism of qfh-sheaves.

According to lemma 3.3.17, this follows from the commutativity of the diagram

$$\begin{array}{ccc} L(T\times_Z T)\amalg\left(\coprod_{g\in G}L(Y_g)\right) & \longrightarrow & L(Y\times_X Y) \\ \downarrow & & \downarrow \\ (L(T)\times_{L(Z)}L(T)/G)\amalg L(Y) & \longrightarrow & L(Y)\times_{L(X)}L(Y)/G \end{array}$$

in which the vertical maps are obviously epimorphic.

Assume now that X is normal and that Z is nowhere dense in X, and let us prove that the canonical map $L(Y)/G \to L(X)$ is an isomorphism of qfh-sheaves. This is equivalent to prove that, for any qfh-sheaf of sets F, the map $f^* : F(X) \to F(Y)$ induces a bijection

$$F(X) \simeq F(Y)^G\,.$$

Let F be a qfh-sheaf. The map $f^* : F(X) \to F(Y)$ is injective because f is a qfh-covering, and it is clear that the image of f^* lies in $F(Y)^G$.

Let a be a section of F over Y which is invariant under the action of G. Denote by $pr_1, pr_2 : Y\times_X Y \to Y$ the two canonical projections. With the notations introduced in lemma 3.3.17, we have

$$pr_1^*(a)|_{Y_g} = a = a.g = pr_2^*(a)|_{Y_g}$$

for every element g in G. As Z does not contain any generic point of X, the scheme $T\times_Z T$ does not contain any generic point of $Y\times_X Y$ neither: as any irreducible component of Y dominates an irreducible component of X, and, as X is normal, the finite map $Y \to X$ is universally open; in particular, the projection $pr_1 : Y\times_X Y \to Y$ is universally open, which implies that any generic point of $Y\times_X Y$ lies over a generic point of Y. By virtue of [Voe96, prop. 3.1.4], lemma 3.3.17 thus gives a qfh-covering of $Y\times_X Y$ by closed subschemes of shape

$$Y\times_X Y = \bigcup_{g\in G} Y_g\,.$$

This implies that

$$pr_1^*(a) = pr_2^*(a)\,.$$

The morphism $Y \to X$ being a qfh-covering and F a qfh-sheaf, we deduce that the section a lies in the image of f^*. □

Corollary 3.3.19. *For any* qfh-*distinguished square of group G* (3.3.14.1)*, we get a bicartesian square of* qfh-*sheaves of abelian groups*

$$\begin{array}{ccc} \mathbf{Z}_{\mathrm{qfh}}(T)_G & \longrightarrow & \mathbf{Z}_{\mathrm{qfh}}(Y)_G \\ \downarrow & & \downarrow \\ \mathbf{Z}_{\mathrm{qfh}}(Z) & \longrightarrow & \mathbf{Z}_{\mathrm{qfh}}(X) \end{array}$$

(where the subscript G stands for the coinvariants under the action of G). In other words, there is a canonical short exact sequence of sheaves of abelian groups

$$0 \to \mathbf{Z}_{\mathrm{qfh}}(T)_G \to \mathbf{Z}_{\mathrm{qfh}}(Z)\oplus\mathbf{Z}_{\mathrm{qfh}}(Y)_G \to \mathbf{Z}_{\mathrm{qfh}}(X) \to 0\,.$$

Proof. As the abelianization functor preserves colimits and monomorphisms, the preceding proposition implies formally that we have a short exact sequence of shape

$$\mathbf{Z}_{\mathrm{qfh}}(T)_G \to \mathbf{Z}_{\mathrm{qfh}}(Z)\oplus\mathbf{Z}_{\mathrm{qfh}}(Y)_G \to \mathbf{Z}_{\mathrm{qfh}}(X) \to 0\,,$$

while the left exactness follows from the fact that $Z \to X$ being a monomorphism, the map obtained by pullback, $L(T)/G \to L(Y)/G$, is a monomorphism as well. □

3.3.20. Let $\mathscr{V}$ be a $\mathbf{Q}$-linear stable model category (see 3.2.14).

Consider a finite group G, and an object E of $\mathscr{V}$, endowed with an action of G. By viewing G as a category with one object we can see E as functor from G to $\mathscr{V}$ and take its homotopy limit in $\mathrm{Ho}(\mathscr{V})$, which we denote by E^{hG} (in the literature, E^{hG} is called the object of homotopy fixed points under the action of G on E). One the other hand, the category $\mathrm{Ho}(\mathscr{V})$ is, by assumption, a $\mathbf{Q}$-linear triangulated category with small sums, and, in particular, a $\mathbf{Q}$-linear pseudo-abelian category so that we can define E^G as the object of $\mathrm{Ho}(\mathscr{V})$ defined by

$$E^G = \mathrm{Im}\, p\,, \tag{3.3.20.1}$$

where $p : E \to E$ is the projector defined in $\mathrm{Ho}(\mathscr{V})$ by the formula

$$p(x) = \frac{1}{\#G} \sum_{g \in G} g.x\,. \tag{3.3.20.2}$$

The inclusion $E^G \to E$ induces a canonical isomorphism

$$E^G \xrightarrow{\sim} E^{hG} \tag{3.3.20.3}$$

in $\mathrm{Ho}(\mathscr{V})$: to see this, by virtue of theorem 3.2.15, we can assume that $\mathscr{V}$ is the model category of complexes of $\mathbf{Q}$-vector spaces, in which case it is obvious.

Corollary 3.3.21. *Let C be a presheaf of complexes of $\mathbf{Q}$-vector spaces on the category of S-schemes. Then, for any* qfh*-distinguished square of group G* (3.3.14.1)*, the commutative square*

$$\begin{array}{ccc} \mathbf{R}\Gamma_{\mathrm{qfh}}(X, C_{\mathrm{qfh}}) & \longrightarrow & \mathbf{R}\Gamma_{\mathrm{qfh}}(Y, C_{\mathrm{qfh}})^G \\ \downarrow & & \downarrow \\ \mathbf{R}\Gamma_{\mathrm{qfh}}(Z, C_{\mathrm{qfh}}) & \longrightarrow & \mathbf{R}\Gamma_{\mathrm{qfh}}(T, C_{\mathrm{qfh}})^G \end{array}$$

is a homotopy pullback square in the derived category of $\mathbf{Q}$-vector spaces. In particular, we get a long exact sequence of shape

$$H^n_{\mathrm{qfh}}(X, C_{\mathrm{qfh}}) \to H^n_{\mathrm{qfh}}(Z, C_{\mathrm{qfh}}) \oplus H^n_{\mathrm{qfh}}(Y, C_{\mathrm{qfh}})^G \to H^n_{\mathrm{qfh}}(T, C_{\mathrm{qfh}})^G \to H^{n+1}_{\mathrm{qfh}}(X, C_{\mathrm{qfh}})$$

If furthermore X is normal and Z is nowhere dense in X, then the maps

$$H^n_{\mathrm{qfh}}(X, C_{\mathrm{qfh}}) \to H^n_{\mathrm{qfh}}(Y, C_{\mathrm{qfh}})^G \quad \textit{and} \quad H^n_{\mathrm{qfh}}(Z, C_{\mathrm{qfh}}) \to H^n_{\mathrm{qfh}}(T, C_{\mathrm{qfh}})^G$$

are isomorphisms for any integer n.

Proof. Let $C_{\mathrm{qfh}} \to C'$ be a fibrant resolution in the qfh-local injective model category structure on the category of qfh-sheaves of complexes of $\mathbf{Q}$-vector spaces; see for instance [Ayo07a, Cor. 4.4.42]. Then for $U = Y, T$, we have a natural isomorphism of complexes

$$\mathrm{Hom}(\mathbf{Q}_{\mathrm{qfh}}(U)_G, C') = C'(U)^G$$

which gives an isomorphism

$$\mathbf{R}\mathrm{Hom}(\mathbf{Q}_{\mathrm{qfh}}(U)_G, C_{\mathrm{qfh}}) \simeq \mathbf{R}\Gamma_{\mathrm{qfh}}(U, C_{\mathrm{qfh}})^G$$

in the derived category of the abelian category of $\mathbf{Q}$-vector spaces. This corollary thus follows formally from corollary 3.3.19 by evaluating at the derived functor $\mathbf{R}\mathrm{Hom}(-, C_{\mathrm{qfh}})$.

If furthermore X is normal, then one deduces the isomorphism $H^n_{\mathrm{qfh}}(X, C_{\mathrm{qfh}}) \simeq H^n_{\mathrm{qfh}}(Y, C_{\mathrm{qfh}})^G$ from the fact that $L(Y)/G \simeq L(Y/G) \simeq X$ (proposition 3.3.18), which implies that $\mathbf{Z}_{\mathrm{qfh}}(Y)_G \simeq \mathbf{Z}_{\mathrm{qfh}}(X)$. The isomorphism $H^n_{\mathrm{qfh}}(Z, C_{\mathrm{qfh}}) \simeq H^n_{\mathrm{qfh}}(T, C_{\mathrm{qfh}})^G$ then comes as a byproduct of the long exact sequence above. □

Theorem 3.3.22. *Let X be a scheme, and C be a presheaf of complexes of $\mathbf{Q}$-vector spaces on the small étale site of X. Then C satisfies étale descent if and only if it has the following properties.*

(a) The complex C satisfies Nisnevich descent.

(b) For any étale X-scheme U and any Galois cover $V \to U$ of group G, the map $C(U) \to C(V)^G$ is a quasi-isomorphism.

Proof. These are certainly necessary conditions. To prove that they are sufficient, note that the Nisnevich cohomological dimension and the rational étale cohomological dimension of a noetherian scheme are bounded by the Krull dimension; see [MV99, proposition 1.8, page 98] and [Voe96, Lemma 3.4.7]. By virtue of [SV00a, Theorem 0.3], for $\tau = \mathrm{Nis}, \text{ét}$, we have strongly convergent spectral sequences

$$E_2^{p,q} = H^p_\tau(U, H^q(C)_\tau) \Rightarrow H^{p+q}_\tau(U, C_\tau)\,.$$

Condition (a) gives isomorphisms $H^{p+q}(C(U)) \simeq H^{p+q}_{\mathrm{Nis}}(U, C_{\mathrm{Nis}})$, so that it is sufficient to prove that, for each of the cohomology presheaves $F = H^q(C)$, we have

$$H^p_{\mathrm{Nis}}(U, F_{\mathrm{Nis}}) \simeq H^p_{\text{ét}}(U, F_{\text{ét}})\,.$$

As the rational étale cohomology of any henselian scheme is trivial in non-zero degrees, it is sufficient to prove that, for any local henselian scheme U (obtained as the henselisation of an étale X-scheme at some point), $F_{\mathrm{Nis}}(U) \simeq F_{\text{ét}}(U)$. Let G be the absolute Galois group of the closed point of U. Then we have

$$F_{\mathrm{Nis}}(U) = F(U) \quad \text{and} \quad F_{\text{ét}}(U) = \varinjlim_\alpha F(U_\alpha)^{G_\alpha}\,,$$

where the U_α's run over all the Galois covers of U corresponding to the finite quotients $G \to G_\alpha$. But it follows from (b) that $F(U) \simeq F(U_\alpha)^{G_\alpha}$ for any α, so that $F_{\mathrm{Nis}}(U) \simeq F_{\text{ét}}(U)$. □

Lemma 3.3.23. *Any* qfh*-covering admits a refinement of the form $Z \to Y \to X$, where $Z \to Y$ is a finite surjective morphism, and $Y \to X$ is an étale covering.*

Proof. This property being clearly local on X with respect to the étale topology, we can assume that X is strictly henselian, in which case this follows from [Voe96, Lemma 3.4.2]. □

Theorem 3.3.24. *A presheaf of complexes of $\mathbf{Q}$-vector spaces C on the category of S-schemes satisfies* qfh*-descent if and only if it has the following two properties:*

(a) the complex C satisfies Nisnevich descent;

(b) for any pseudo-Galois qfh*-distinguished square of group G* (3.3.14.1)*, the commutative square*

$$\begin{array}{ccc} C(X) & \longrightarrow & C(Y)^G \\ \downarrow & & \downarrow \\ C(Z) & \longrightarrow & C(T)^G \end{array}$$

is a homotopy pullback square in the derived category of $\mathbf{Q}$-vector spaces.

Proof. Any complex of presheaves of $\mathbf{Q}$-vector spaces satisfying qfh-descent satisfies properties (a) and (b): property (a) follows from the fact that the qfh-topology is finer than the étale topology; property (b) is corollary 3.3.21.

Assume now that C satisfies these two properties. Let $\varphi : C \to C'$ be a morphism of presheaves of complexes of $\mathbf{Q}$-vector spaces which is a quasi-isomorphism locally for the qfh-topology, and such that C' satisfies qfh-descent (such a morphism exists thanks to the qfh-local model category structure on the category of presheaves of complexes of $\mathbf{Q}$-vector spaces; see proposition 3.2.10). Then the cone of φ also satisfies conditions (a) and (b). Hence it is sufficient to prove the theorem in the case where C is acyclic locally for the qfh-topology.

Assume from now on that C_{qfh} is an acyclic complex of qfh-sheaves, and denote by $H^n(C)$ the nth cohomology presheaf associated to C. We know that the associated qfh-sheaves vanish, and we want to deduce that $H^n(C) = 0$.

We shall prove by induction on d that, for any S-scheme X of dimension d and for any integer n, the group $H^n(C)(X) = H^n(C(X))$ vanishes. The case where $d < 0$ follows from the fact, that by (a), the presheaves $H^n(C)$ send finite sums to finite direct sums, so that, in particular, $H^n(C)(\varnothing) = 0$. Before going further, notice that condition (b) implies $H^n(C)(X_{red}) = H^n(C)(X)$ for any S-scheme X (consider the case where, in the diagram (3.3.14.1), $Z = Y = T = X_{red}$), so that it is always harmless to replace X by its reduction. Assume now that $d \geq 0$, and that the

vanishing of $H^n(C)(X)$ is known whenever X is of Krull dimension $< d$ and for any integer n. Under this inductive assumption, we have the following reduction principle.

Consider a pseudo-Galois qfh-distinguished square of group G (3.3.14.1). If Z and T are of dimension $< d$, then by condition (b), the map $H^n(C)(X) \to H^n(C)(Y)^G$ is an isomorphism: indeed, we have an exact sequence of shape

$$H^{n-1}(C)(T)^G \to H^n(C)(X) \to H^n(C)(Z) \oplus H^n(C)(Y)^G \to H^n(C)(T)^G\,,$$

which implies our assertion by induction on d.

We shall prove now the vanishing of $H^n(C)(T)$ for normal S-schemes T of dimension d. Let a be a section of $H^n(C)$ over such a T. As $H^n(C)_{\mathrm{qfh}}(T) = 0$, there exists a qfh-covering $g : Y \to T$ such that $g^*(a) = 0$. But, by virtue of lemma 3.3.23, we can assume g is the composition of a finite surjective morphism $f : Y \to X$ and of an étale covering $e : X \to T$. We claim that $e^*(a) = 0$. To prove it, as, by (a), the presheaf $H^n(C)$ sends finite sums to finite direct sums, we can assume that X is normal and connected. Refining f further, we can assume that Y is the normalization of X in a finite extension of $k(X)$, and that $k(Y)$ is a Galois extension of group G over the inseparable closure of $k(X)$ in $k(Y)$. By virtue of proposition 3.3.15, we get by the reduction principle the identification $H^n(C)(X) = H^n(C)(Y)^G$, whence $e^*(a) = 0$. As a consequence, the restriction of the presheaf of complexes C to the category of normal S-schemes of dimension $\leq d$ is acyclic locally for the étale topology (note that this is quite meaningful, as any étale scheme over a normal scheme is normal; see [GD67, Prop. 18.10.7]). But C satisfies étale descent (by virtue of theorem 3.3.22 this follows formally from property (a) and from property (b) for $Z = \varnothing$), so that $H^n(C)(T) = H^n_{\text{ét}}(T, C_{\text{ét}}) = 0$ for any normal S-scheme T of dimension $\leq d$ and any integer n.

Consider now a reduced S-scheme X of dimension $\leq d$. Let $p : T \to X$ be the normalization of X. As p is birational (see [GD61, Cor. 6.3.8]) and finite surjective (because X is quasi-excellent), we can apply the reduction principle and see that the pullback map $p^* : H^n(C)(X) \to H^n(C)(T) = 0$ is an isomorphism for any integer n, which achieves the induction and the proof. □

Lemma 3.3.25. *Étale coverings are finite étale coverings locally for the Nisnevich topology: any étale covering admits a refinement of the form $Z \to Y \to X$, where $Z \to Y$ is a finite étale covering and $Y \to X$ is a Nisnevich covering.*

Proof. This property being local on X for the Nisnevich topology, it is sufficient to prove this in the case where X is local henselian. Then, by virtue of [GD67, Cor. 18.5.12 and Prop. 18.5.15], we can even assume that X is the spectrum of field, in which case this is obvious. □

Lemma 3.3.26. *Any* qfh*-covering admits a refinement of the form $Z \to Y \to X$, where $Z \to Y$ is a finite surjective morphism, and $Y \to X$ is a Nisnevich covering.*

Proof. As finite surjective morphisms are stable by pullback and composition, this follows immediately from lemmata 3.3.23 and 3.3.25. □

Lemma 3.3.27. *Any* h*-covering of an integral scheme X admits a refinement of the form*

$$U \to Z \to Y \to X\,,$$

where $U \to Z$ is a finite surjective morphism, $Z \to Y$ is a Nisnevich covering, $Y \to X$ is a proper surjective birational map, and Y is normal.

Proof. By virtue of [Voe96, Theorem 3.1.9], any h-covering admits a refinement of shape

$$W \to V \to X\,,$$

where $W \to V$ is a qfh-covering, and $V \to X$ is a proper surjective birational map. By replacing V by its normalization Y, we get a refinement of shape

$$W \times_V Y \to Y \to X$$

where $W \times_V Y \to Y$ is a qfh-covering, and $Y \to X$ is proper surjective birational map. We conclude by lemma 3.3.26. □

Lemma 3.3.28. *Let C be a presheaf of complexes of $\mathbf{Q}$-vector spaces on the category of S-schemes satisfying* qfh*-descent. Then, for any finite surjective morphism $f : Y \to X$ with X normal, the map $f^* : H^n(C)(X) \to H^n(C)(Y)$ is a monomorphism.*

Proof. It is clearly sufficient to prove this when X is connected. Then, up to refinement, we can assume that f is a map as in proposition 3.3.15. In this case, by virtue of corollary 3.3.21, the $\mathbf{Q}$-vector space $H^n(C)(X) \simeq H^n(C)(Y)^G$ is a direct factor of $H^n(C)(Y)$. □

Theorem 3.3.29. *A presheaf of complexes of $\mathbf{Q}$-vector spaces on the category of S-schemes satisfies* h*-descent if and only if it satisfies* qfh*-descent and* cdh*-descent.*

Proof. This is certainly a necessary condition, as the h-topology is finer than the qfh-topology and the cdh-topology. For the converse, as in the proof of theorem 3.3.24, it is sufficient to prove that any presheaf of complexes of $\mathbf{Q}$-vector spaces C on the category of S-schemes satisfying qfh-descent and cdh-descent, and which is acyclic locally for the h-topology, is acyclic. We shall prove by noetherian induction that, given such a complex C, for any integer n, and any S-scheme X, for any section a of $H^n(C)$ over X, there exists a cdh-covering $X' \to X$ on which a vanishes. In other words, we shall get that C is acyclic locally for the cdh-topology, and, as C satisfies cdh-descent, this will imply that $H^n(C)(X) = H^n_{\mathrm{cdh}}(X, C_{\mathrm{cdh}}) = 0$ for any integer n and any S-scheme X. Note that the presheaves $H^n(C)$ send finite sums to finite direct sums (which follows, for instance, from the fact that C satisfies Nisnevich descent). In particular, $H^n(C)(\varnothing) = 0$ for any integer n.

Let X be an S-scheme, and $a \in H^n(C)(X)$. We have a cdh-covering of X of shape $X' \amalg X'' \to X$, where X' is the sum of the irreducible components of X_{red} and X'' is a nowhere dense closed subscheme of X, so that we can assume X is integral. Let a be a section of the presheaf $H^n(C)$ over X. As $H^n(C)_h = 0$, by virtue of lemma 3.3.27, there exists a proper surjective birationnal map $p : Y \to X$ with Y normal, a Nisnevich covering $q : Z \to Y$, and a surjective finite morphism $r : U \to Z$ such that $r^*(q^*(p^*(a))) = 0$ in $H^n(C)(U)$. But then, Z is normal as well (see [GD67, Prop. 18.10.7]), so that, by lemma 3.3.28, we have $q^*(p^*(a)) = 0$ in $H^n(C)(Z)$. Let T be a nowhere dense closed subscheme of X such that p is an isomorphism over $X - T$. By noetherian induction, there exists a cdh-covering $T' \to T$ such that $a|_{T'}$ vanishes. Hence the section a vanishes on the cdh-covering $T' \amalg Z \to X$. □

3.3.d. *Proper descent with rational coefficients II: separation.* From now on, we assume that $\mathrm{Ho}(\mathscr{M})$ is $\mathbf{Q}$-linear.

Proposition 3.3.30. *Let $f : Y \to X$ be a morphism of schemes in $\mathscr{S}$, and G a finite group acting on Y over X. Denote by $\mathscr{Y}$ the scheme Y considered a functor from G to the category of S-schemes, and denote by $\varphi : (\mathscr{Y}, G) \to X$ the morphism induced by f. Then, for any object M of $\mathrm{Ho}(\mathscr{M})(X)$, there are canonical isomorphisms*

$$(\mathbf{R}f_* \mathbf{L}f^*(M))^G \simeq (\mathbf{R}f_* \mathbf{L}f^*(M))^{hG} \simeq \mathbf{R}\varphi_* \mathbf{L}\varphi^*(M)\,.$$

Proof. The second isomorphism comes from proposition 3.1.15, and the first, from (3.3.20.3). □

Theorem 3.3.31. *If $\mathrm{Ho}(\mathscr{M})$ satisfies Nisnevich descent, the following conditions are equivalent:*

(i) $\mathrm{Ho}(\mathscr{M})$ *satisfies étale descent.*

(ii) for any finite étale cover $f : Y \to X$, the functor

$$\mathbf{L}f^* : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(Y)$$

is conservative;

(iii) for any finite Galois cover $f : Y \to X$ of group G, and for any object M of $\mathrm{Ho}(\mathscr{M})(X)$, the canonical map

$$M \to (\mathbf{R}f_* \mathbf{L}f^*(M))^G$$

is an isomorphism.

Proof. The equivalence between (i) and (iii) follows from theorem 3.3.22 by corollaries 3.2.17 and 3.2.18, and proposition 3.2.8 shows that (i) implies (ii). It is thus sufficient to prove that (ii) implies (iii). Let $f : Y \to X$ be a finite Galois cover of group G. As the functor $f^* = \mathbf{L}f^*$ is

conservative by assumption, it is sufficient to check that the map $M \to (\mathbf{R}f_* \mathbf{L}f^*(M))^G$ becomes an isomorphism after applying f^*. By virtue of proposition 3.1.17, this just means that it is sufficient to prove (iii) when f has a section, i.e. when Y is isomorphic to the trivial G-torsor over X. In this case, we have the (equivariant) identification $\bigoplus_{g\in G} M \simeq \mathbf{R}f_* \mathbf{L}f^*(M)$, where G acts on the left term by permuting the factors. Hence $M \simeq (\mathbf{R}f_* \mathbf{L}f^*(M))^G$. □

Proposition 3.3.32. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *has the localization property. The following conditions are equivalent:*

(i) $\mathrm{Ho}(\mathscr{M})$ *is separated.*
(ii) $\mathrm{Ho}(\mathscr{M})$ *is semi-separated and satisfies étale descent.*

Proof. This follows from proposition 2.3.9 and theorem 3.3.31. □

Corollary 3.3.33. *Assume that all the residue fields of* $\mathcal{S}$ *are of characteristic zero, and that* $\mathscr{M}$ *has the property of localization. Then the following conditions are equivalent:*

(i) $\mathrm{Ho}(\mathscr{M})$ *is separated.*
(ii) $\mathrm{Ho}(\mathscr{M})$ *satisfies étale descent.*

Proof. In this case, a finite surjective morphism $f : Y \to X$ is radicial if and only if it induces an isomorphism after reduction $Y_{red} \simeq X_{red}$. But it is clear that, by the localization property, such a morphism f induces an equivalence of categories $\mathbf{L}f^*$, so that $\mathrm{Ho}(\mathscr{M})$ is automatically semi-separated. We conclude by proposition 3.3.32. □

Proposition 3.3.34. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is separated, satisfies the localization property the proper transversality property. Then, for any pseudo-Galois cover* $f : Y \to X$ *of group* G*, and for any object* M *of* $\mathrm{Ho}(\mathscr{M})(X)$*, the canonical map*

$$M \to (\mathbf{R}f_* \mathbf{L}f^*(M))^G$$

is an isomorphism.

Proof. By proposition 3.3.32, this is an easy consequence of proposition 2.1.13 and of condition (iii) of theorem 3.3.31. □

3.3.35. From now on, we assume furthermore that any scheme in $\mathscr{S}$ is quasi-excellent.

Theorem 3.3.36. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *satisfies the localization and proper transversality properties. Then the following conditions are equivalent:*

(i) $\mathrm{Ho}(\mathscr{M})$ *is separated;*
(ii) $\mathrm{Ho}(\mathscr{M})$ *satisfies* h*-descent;*
(iii) $\mathrm{Ho}(\mathscr{M})$ *satisfies* qfh*-descent;*
(iv) for any qfh*-distinguished square* (3.3.14.1) *of group* G*, if we write* $a = fh = ig : T \to X$ *for the composed map, then, for any object* M *of* $\mathrm{Ho}(\mathscr{M})(X)$*, the commutative square*

$$\begin{array}{ccc} M & \longrightarrow & (\mathbf{R}f_* \mathbf{L}f^*(M))^G \\ \downarrow & & \downarrow \\ \mathbf{R}i_* \mathbf{L}i^*(M) & \longrightarrow & (\mathbf{R}a_* \mathbf{L}a^*(M))^G \end{array} \tag{3.3.36.1}$$

is homotopy cartesian;
(v) the same as condition (iv), but only for pseudo-Galois qfh*-distinguished squares.*

Proof. As $\mathscr{M}$ satisfies cdh-descent (theorem 3.3.9), the equivalence between conditions (ii) and (iii) follows from theorem 3.3.29 by corollary 3.2.18. Similarly, theorem 3.3.24 and corollaries 3.3.21, 3.2.17 and 3.2.18 show that conditions (iii), (iv) and (v) are equivalent. As étale surjective morphisms as well as finite radicial epimorphisms are qfh-coverings, it follows from proposition 3.2.8, theorem 3.3.31 and proposition 3.3.32, that condition (iii) implies condition (i). It thus remains to prove that condition (i) implies condition (v). So let us consider a pseudo-Galois qfh-distinguished square (3.3.14.1) of group G, and prove that (3.3.36.1) is homotopy cartesian. Using

proper transversality, we see that the image of (3.3.36.1) by the functor $\mathbf{L}i^*$ is (isomorphic to) the homotopy pullback square

$$\begin{array}{ccc} \mathbf{L}i^*(M) & \longrightarrow & (\mathbf{R}g_*\,\mathbf{L}g^*\,\mathbf{L}i^*(M))^G \\ \| & & \| \\ \mathbf{L}i^*(M) & \longrightarrow & (\mathbf{R}g_*\,\mathbf{L}g^*\,\mathbf{L}i^*(M))^G \end{array} .$$

Write $j : U \to X$ for the complement open immersion of i, and $b : f^{-1}(U) \to U$ for the map induced by f. As j is étale, we see, using proposition 3.1.17, that the image of (3.3.8.1) by $j^* = \mathbf{L}j^*$ is (isomorphic to) the square

$$\begin{array}{ccc} j^*(M) & \longrightarrow & (\mathbf{R}b_*\,\mathbf{L}b^*\,j^*(M))^G \\ \downarrow & & \downarrow \\ 0 & = & 0 \end{array} .$$

in which the upper horizontal map is an isomorphism by proposition 3.3.34. Hence it is a homotopy pullback square. Thus, because the pair of functors $(\mathbf{L}i^*, j^*)$ is conservative on $\mathrm{Ho}(\mathscr{M})(X)$, the square (3.3.36.1) is homotopy cartesian. □

Corollary 3.3.37. *Assume that all the residue fields of $\mathcal{S}$ are of characteristic zero, and that* $\mathrm{Ho}(\mathscr{M})$ *has the localization and proper transversality properties. Then* $\mathrm{Ho}(\mathscr{M})$ *satisfies* h*-descent if and only if it satisfies étale descent.*

Proof. This follows from corollary 3.3.33 and theorem 3.3.36. □

Corollary 3.3.38. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is separated and has the localization and proper transversality properties. Let $f : Y \to X$ be a finite surjective morphism, with X normal, and G a group acting on Y over X, such that the map $Y/G \to X$ is generically radicial (i.e. radicial over a dense open subscheme of X). Consider at last a pullback square of the following shape.*

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \longrightarrow & X \end{array}$$

Then, for any object M of $\mathrm{Ho}(\mathscr{M})(X')$*, the natural map*

$$M \to (\mathbf{R}f'_*\,\mathbf{L}f'^*(M))^G$$

is an isomorphism.

Proof. For any presheaf C of complexes of $\mathbf{Q}$-vector spaces on $\mathscr{S}/X$, one has an isomorphism

$$\mathbf{R}\Gamma_{\mathrm{qfh}}(X', C_{\mathrm{qfh}}) \simeq \mathbf{R}\Gamma_{\mathrm{qfh}}(Y', C_{\mathrm{qfh}})^G .$$

This follows from the fact that we have an isomorphism of qfh-sheaves of sets $L(Y)/G \simeq L(X)$ (the map $Y \to Y/G$ being generically flat, this is proposition 3.3.18), which implies that the map $L(Y')/G \to L(X')$ is an isomorphism of qfh-sheaves (by the universality of colimits in topoi), and implies this assertion (as in the proof of 3.3.21).

By virtue of theorem 3.3.36, $\mathrm{Ho}(\mathscr{M})$ satisfies qfh-descent, so that the preceding computations imply the result by corollaries 3.2.17 and 3.2.18. □

Corollary 3.3.39. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is separated and has the localization and proper transversality properties. Then for any finite surjective morphism $f : Y \to X$ with X normal, the morphism*

$$M \to \mathbf{R}f_*\,\mathbf{L}f^*(M)$$

is a monomorphism and admits a functorial splitting in $\mathrm{Ho}(\mathscr{M})(X)$*. Furthermore, this remains true after base change by any map $X' \to X$.*

Proof. It is sufficient to treat the case where X is connected. We may replace Y by a normalization of X in a suitable finite extension of its field of function, and assume that a finite group G acts on Y over X, so that the properties described in the preceding corollary are fulfilled (see 3.3.15). □

Remark 3.3.40. The condition (iv) of theorem 3.3.36 can be reformulated in a more global way as follows (this won't be used in these notes, but this might be useful for the reader who might want to formulate all this in terms of (pre-)algebraic derivators [Ayo07a, Def. 2.4.13]). Given a qfh-distinguished square (3.3.14.1) of group G, we can form a functor $\mathscr{F}$ from category $I = \ulcorner$ (3.3.10.1) to the category of diagrams of S-schemes corresponding to the diagram of diagrams of S-schemes

$$\begin{array}{ccc} (\mathscr{T}, G) & \xrightarrow{(h,1_G)} & (\mathscr{Y}, G) \\ {\scriptstyle g}\downarrow & & \\ Z & & \end{array}$$

in which $\mathscr{T}$ and $\mathscr{Y}$ correspond to T anf Y respectively, seen as functor from G to $\mathscr{S}/X$. The construction of 3.1.22 gives a diagram of X-schemes $(\int \mathscr{F}, I_{\mathscr{F}})$ which can be described explicitely as follows. The category $I_{\mathscr{F}}$ is the cofibred category over $\ulcorner$ associated to the functor from $\ulcorner$ to the category of small categories defined by the diagram

$$\begin{array}{ccc} G & \xrightarrow{1_G} & G \\ \downarrow & & \\ e & & \end{array}$$

in which e stands for the terminal category, and G for the category with one object associated to G. It has thus three objects a, b, c (see (3.3.10.1)), and the morphisms are determined by

$$\mathrm{Hom}_{I_{\mathscr{F}}}(x, y) = \begin{cases} * & \text{if } y = c; \\ \varnothing & \text{if } x \neq y \text{ and } x = b, c; \\ \mathrm{G} & \text{otherwise.} \end{cases}$$

The functor $\mathscr{F}$ sends a, b, c to T, Y, Z respectively, and simply encodes the fact that the diagram

$$\begin{array}{ccc} T & \xrightarrow{h} & Y \\ {\scriptstyle g}\downarrow & & \\ Z & & \end{array}$$

is G-equivariant, the action on Z being trivial. Now, by propositions 3.1.23 and 3.3.30, if $\varphi : (\mathscr{F}, I_{\mathscr{F}}) \to (X, \ulcorner)$ denotes the canonical map, for any object M of $\mathrm{Ho}(\mathscr{M})(X)$, the object $\mathbf{R}\varphi_* \mathbf{L}\varphi^*(M)$ is the functor from $\lrcorner = \ulcorner^{op}$ to $\mathscr{M}(X)$ corresponding to the diagram below (of course, this is well defined only in the homotopy category of the category of functors from $\lrcorner$ to $\mathscr{M}(X)$).

$$\begin{array}{ccc} & & (\mathbf{R}f_* \mathbf{L}f^*(M))^G \\ & & \downarrow \\ \mathbf{R}i_* \mathbf{L}i^*(M) & \longrightarrow & (\mathbf{R}a_* \mathbf{L}a^*(M))^G \end{array}$$

As a consequence, if $\psi : (\int \mathscr{F}, I_{\mathscr{F}}) \to X$ denotes the structural map, the object $\mathbf{R}\psi_* \mathbf{L}\psi^*(M)$ is simply the homotopy homotopy limit of the diagram of $\mathscr{M}(X)$ above, so that condition (iv) of theorem 3.3.36 can now be reformulated by saying that the map

$$M \to \mathbf{R}\psi_* \mathbf{L}\psi^*(M)$$

is an isomorphism, i.e. that the functor

$$\mathbf{L}\psi^* : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(\int \mathscr{F}, I_{\mathscr{F}})$$

is fully faithful.

4. Basic homotopy commutative algebra

4.1. **Rings.**

Definition 4.1.1. A symmetric monoidal model category $\mathscr{V}$ satisfies the *monoid axiom* if, for any trivial cofibration $A \to B$ and any object X, the smallest class of maps of $\mathscr{V}$ which contains the map $X \otimes A \to X \otimes B$ and is stable by pushouts and transfinite compositions is contained in the class of weak equivalences.

4.1.2. Let $\mathscr{V}$ be a symmetric monoidal category. We denote by $Mon(\mathscr{V})$ the category of monoids in $\mathscr{V}$. If $\mathscr{V}$ has small colimits, the forgetful functor

$$U : Mon(\mathscr{V}) \to \mathscr{V}$$

has a left adjoint

$$F : \mathscr{V} \to Mon(\mathscr{V})\,.$$

Theorem 4.1.3. *Let $\mathscr{V}$ a symmetric monoidal combinatorial model category which satisfies the monoid axiom. The category of monoids $Mon(\mathscr{V})$ is endowed with the structure of a combinatorial model category whose weak equivalences (resp. fibrations) are the morphisms of commutative monoids which are weak equivalences (resp. fibrations) in $\mathscr{V}$. In particular, the forgetful functor $U : Mon(\mathscr{V}) \to \mathscr{V}$ is a right Quillen functor. Moreover, if the unit object of $\mathscr{V}$ is cofibrant, then any cofibrant object of $Mon(\mathscr{V})$ is cofibrant as an object of $\mathscr{V}$.*

Proof. This is very a particular case of the third assertion of [SS00, Theorem 4.1] (the fact that $Mon(\mathscr{V})$ is combinatorial whenever $\mathscr{V}$ is so comes for instance from [Bek00, Proposition 2.3]). □

Definition 4.1.4. A symmetric monoidal model category $\mathscr{V}$ is *strongly* $\mathbf{Q}$*-linear* if the underlying category of $\mathscr{V}$ is additive and $\mathbf{Q}$-linear (i.e. all the objects of $\mathscr{V}$ are uniquely divisible).

Remark 4.1.5. If $\mathscr{V}$ is a strongly $\mathbf{Q}$-linear stable model category, then it is $\mathbf{Q}$-linear in the sense of 3.2.14.

Lemma 4.1.6. *Let $\mathscr{V}$ be a strongly $\mathbf{Q}$-linear model category, G a finite group, and $u : E \to F$ an equivariant morphism of representations of G in $\mathscr{V}$. Then, if u is a cofibration in $\mathscr{V}$, so is the induced map $E_G \to F_G$ (where the subscript G denotes the coinvariants under the action of the group G).*

Proof. The map $E_G \to F_G$ is easily seen to be a direct factor (retract) of the cofibration $E \to F$. □

4.1.7. If $\mathscr{V}$ is a symmetric monoidal category, we denote by $Comm(\mathscr{V})$ the category of commutative monoids in $\mathscr{V}$. If $\mathscr{V}$ has small colimits, the forgetful functor

$$U : Comm(\mathscr{V}) \to \mathscr{V}$$

has a left adjoint

$$F : \mathscr{V} \to Comm(\mathscr{V})\,.$$

Theorem 4.1.8. *Let $\mathscr{V}$ a symmetric monoidal combinatorial model category. Assume that $\mathscr{V}$ is left proper and tractable, satisfies the monoid axiom, and is strongly $\mathbf{Q}$-linear. Then the category of commutative monoids $Comm(\mathscr{V})$ is endowed with the structure of a combinatorial model category whose weak equivalences (resp. fibrations) are the morphisms of commutative monoids which are weak equivalences (resp. fibrations) in $\mathscr{V}$. In particular, the forgetful functor $U : Comm(\mathscr{V}) \to \mathscr{V}$ is a right Quillen functor.*

If moreover the unit object of $\mathscr{V}$ is cofibrant, then any cofibrant object of $Comm(\mathscr{V})$ is cofibrant as an object of $\mathscr{V}$.

Proof. The preceding lemma implies immediately that $\mathscr{V}$ is freely powered in the sense of [Lur09, Definition 4.3.17], so that the existence of this model category structure follows from a general result of Lurie [Lur09, Proposition 4.3.21]. The second assertion is then true by definition. The last assertion is proved by a careful analysis of pushouts by free maps in $Comm(\mathscr{V})$ as follows. For two cofibrations $u : A \to B$ and $v : C \to D$ in $\mathscr{V}$, write $u \wedge v$ for the map

$$u \wedge v : A \otimes D \amalg_{A \otimes C} B \otimes C \to B \otimes D$$

(which is a cofibration by definition of monoidal model categories). By iterating this construction, we get, for a cofibration $u : A \to B$ in $\mathscr{V}$, a cofibration

$$\wedge^n(u) = \underbrace{u \wedge \cdots \wedge u}_{n \text{ times}} : \square^n(u) \to B^{\otimes n} .$$

Note that the symmetric group $\mathfrak{S}_n$ acts naturally on $B^{\otimes n}$ and $\square^n(u)$. We define

$$Sym^n(B) = (B^{\otimes n})_{\mathfrak{S}_n} \quad \text{and} \quad Sym^n(B, A) = \square^n(u)_{\mathfrak{S}_n} .$$

By virtue of Lemma 4.1.6, we get a cofibration of $\mathscr{V}$:

$$\sigma^n(u) : Sym^n(B, A) \to Sym^n(B) .$$

Consider now the free map $F(u) : F(A) \to F(B)$ can be filtered by $F(A)$-modules as follows. Define $D_0 = F(A)$. As $A = Sym^1(B, A)$, we have a natural morphism $F(A) \otimes Sym^1(B, A) \to F(A)$. The objects D_n are then defined by induction with the pushouts below.

$$\begin{array}{ccc} F(A) \otimes Sym^n(B, A) & \xrightarrow{1_{F(A) \otimes \sigma^n(u)}} & F(A) \otimes Sym^n(B) \\ \downarrow & & \downarrow \\ D_{n-1} & \longrightarrow & D_n \end{array}$$

We get natural maps $D_n \to F(B)$ which induce an isomorphism

$$\varinjlim_{n \geq 0} D_n \simeq F(B)$$

in such a way that the morphism $F(u)$ correspond to the canonical map

$$F(A) = D_0 \to \varinjlim_{n \geq 0} D_n .$$

Hence, if $F(A)$ is cofibrant, all the maps $D_{n-1} \to D_n$ are cofibrations, so that the map $F(A) \to F(B)$ is a cofibration in $\mathscr{V}$. In the particular case where A is the initial object of $\mathscr{V}$, we see that for any cofibrant object B of $\mathscr{V}$, the free commutative monoid $F(B)$ is cofibrant as an object of $\mathscr{V}$ (because the initial object of $Comm(\mathscr{V})$ is the unit object of $\mathscr{V}$). This also implies that, if u is a cofibration between cofibrant objects, the map $F(u)$ is a cofibration in $\mathscr{V}$.

This description of $F(u)$ also allows to compute the pushouts of $F(u)$ in $Comm(\mathscr{V})$ in $\mathscr{V}$ as follows. Consider a pushout

$$\begin{array}{ccc} F(A) & \xrightarrow{F(u)} & F(B) \\ \downarrow & & \downarrow \\ R & \xrightarrow[v]{} & S \end{array}$$

in $Comm(\mathscr{V})$. For $n \geq 0$, define R_n by the pushouts of $\mathscr{V}$:

$$\begin{array}{ccc} F(A) & \longrightarrow & D_n \\ \downarrow & & \downarrow \\ R & \longrightarrow & R_n \end{array}$$

We then have an isomorphism

$$\varinjlim_{n \geq 0} R_n \simeq S .$$

In particular, if u is a cofibration between cofibrant objects, the morphism of commutative monoids $v : R \to S$ is then a cofibration in $\mathscr{V}$. As the forgetful functor U preserves filtered colimits, conclude easily from there (with the small object argument [Hov99, Theorem 2.1.14]) that any cofibration of $Comm(\mathscr{V})$ is a cofibration of $\mathscr{V}$. Using again that the unit object of $\mathscr{V}$ is cofibrant in $\mathscr{V}$ (i.e. that the initial object of $Comm(\mathscr{V})$ is cofibrant in $\mathscr{V}$) this proves the last assertion of the theorem. □

Corollary 4.1.9. *Let $\mathscr{V}$ a symmetric monoidal combinatorial model category. Assume that $\mathscr{V}$ is left proper and tractable, satisfies the monoid axiom, and is strongly* $\mathbf{Q}$*-linear. Consider a small set H of maps of $\mathscr{V}$, and denote by $L_H\mathscr{V}$ the left Bousfield localization of $\mathscr{V}$ by H; see* [Bar09, Theorem 4.7]*. Define the class of H-equivalences in* $\mathrm{Ho}(\mathscr{V})$ *to be the class of maps which become invertible in* $\mathrm{Ho}(L_H\mathscr{V})$*. If H-equivalences are stable by (derived) tensor product in* $\mathrm{Ho}(\mathscr{V})$*, then $L_H\mathscr{V}$ is a symmetric monoidal combinatorial model category (which is again left proper and tractable, satisfies the monoid axiom, and is strongly* $\mathbf{Q}$*-linear).*

In particular, under these assumtions, there exists a morphism of commutative monoids $\mathbb{1} \to R$ in $\mathscr{V}$ which is a weak equivalence of $L_H\mathscr{V}$, with R a cofibrant and fibrant object of $L_H\mathscr{V}$.

Proof. The first assertion is a triviality. The last assertion follows immediately: the map $\mathbb{1} \to R$ is simply obtained as a fibrant replacement of $\mathbb{1}$ in the model category $Comm(L_H\mathscr{V})$ obtained from Theorem 4.1.8 applied to $L_H\mathscr{V}$. □

4.1.10. Consider now a category $\mathscr{S}$, as well as a closed symmetric monoidal bifibred category $\mathscr{M}$ over $\mathscr{S}$. We shall also assume that the fibers of $\mathscr{M}$ admit limits and colimits.

Then the categories $Mon(\mathscr{M}(X))$ (resp. $Comm(\mathscr{M}(X))$) define a bifibred category over $\mathscr{S}$ as follows. Given a morphism $f : X \to Y$, the functor

$$f^* : \mathscr{M}(Y) \to \mathscr{M}(X)$$

is symmetric monoidal, so that it preserves monoids (resp. commutative monoids) as well as morphisms between them. It thus induces a functor

$$\begin{aligned} &f^* : Mon(\mathscr{M}(Y)) \to Mon(\mathscr{M}(X)) \\ &(\text{resp. } f^* : Comm(\mathscr{M}(Y)) \to Comm(\mathscr{M}(X))\,). \end{aligned} \tag{4.1.10.1}$$

As $f^* : \mathscr{M}(Y) \to \mathscr{M}(X)$ is symmetric monoidal, its right adjoint f_* is lax monoidal: there is a natural morphism

$$\mathbb{1}_Y \to f_*(\mathbb{1}_X) = f_*\, f^*(\mathbb{1}_Y)\,, \tag{4.1.10.2}$$

and, for any objects A and B of $\mathscr{M}(X)$, there is a natural morphism

$$f_*(A) \otimes_Y f_*(B) \to f_*(A \otimes_X B) \tag{4.1.10.3}$$

which corresponds by adjunction to the map

$$f^*(f_*(A) \otimes_Y f_*(B)) \simeq f^*\, f_*(A) \otimes f^*\, f_*(B) \to A \otimes B\,.$$

Hence the functor f_* preserves also monoids (resp. commutative monoids) as well as morphisms between them, so that we get a functor

$$\begin{aligned} &f_* : Mon(\mathscr{M}(X)) \to Mon(\mathscr{M}(Y)) \\ &(\text{resp. } f_* : Comm(\mathscr{M}(X)) \to Comm(\mathscr{M}(Y))\,). \end{aligned} \tag{4.1.10.4}$$

By construction, the functor f^* of (4.1.10.1) is a left adjoint ot the functor f_* of (4.1.10.4). These constructions extend to morphisms of $\mathscr{S}$-diagrams in a similar way.

Proposition 4.1.11. *Let $\mathscr{M}$ be a symmetric monoidal combinatorial fibred model category over $\mathscr{S}$. Assume that, for any object X of $\mathscr{S}$, the model category $\mathscr{M}(X)$ satisfies the monoid axiom (resp. is left proper and tractable, satisfies the monoid axiom, and is strongly* $\mathbf{Q}$*-linear).*

(a) For any object X of $\mathscr{S}$, the category $Mon(\mathscr{M})(X)$ (resp. $Comm(\mathscr{M})(X)$) of monoids (resp. of commutative monoids) in $\mathscr{M}(X)$ is a combinatorial model category structure whose weak equivalences (resp. fibrations) are the morphisms of commutative monoids which are weak equivalences (resp. fibrations) in $\mathscr{M}(X)$. This turns $Mon(\mathscr{M})$ (resp. $Comm(\mathscr{M})$) into a combinatorial fibred model category over $\mathscr{S}$.

(b) For any morphism of $\mathscr{S}$-diagrams $\varphi : (\mathscr{X}, I) \to (Y, J)$, the adjunction

$$\varphi^* : Mon(\mathscr{M})(\mathscr{Y}, J) \rightleftarrows Mon(\mathscr{M})(\mathscr{X}, I) : \varphi_*$$

$$(resp.\ \varphi^* : Comm(\mathscr{M})(\mathscr{Y}, J) \rightleftarrows Comm(\mathscr{M})(\mathscr{X}, I) : \varphi_*)$$

is a Quillen adjunction (where the categories of monoids $Mon(\mathscr{M})(\mathscr{X}, I)$ (resp. of commutative monoids $Comm(\mathscr{M})(\mathscr{X}, I)$) are endowed with the injective model category structure obtained from Proposition 3.1.7 applied to $Mon(\mathscr{M})$ (resp. to $Comm(\mathscr{M})$).

(d) If moreover, for any object X of $\mathscr{S}$, the unit $\mathbb{1}_X$ is cofibrant in $\mathscr{M}(X)$, then, for morphism of $\mathscr{S}$-diagrams $\varphi : (\mathscr{X}, I) \to (Y, J)$, the square

$$\begin{array}{ccc} \mathrm{Ho}(Mon(\mathscr{M}))(\mathscr{Y}, J) & \xrightarrow{\mathbf{L}\varphi^*} & \mathrm{Ho}(Mon(\mathscr{M}))(\mathscr{X}, I) \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathrm{Ho}(\mathscr{M})(\mathscr{Y}, J) & \xrightarrow{\mathbf{L}\varphi^*} & \mathrm{Ho}(\mathscr{M})(\mathscr{X}, I) \end{array} \tag{4.1.11.1}$$

is essentially commutative. Similarly, in the respective case, the square

$$\begin{array}{ccc} \mathrm{Ho}(Comm(\mathscr{M}))(\mathscr{Y}, J) & \xrightarrow{\mathbf{L}\varphi^*} & \mathrm{Ho}(Comm(\mathscr{M}))(\mathscr{X}, I) \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathrm{Ho}(\mathscr{M})(\mathscr{Y}, J) & \xrightarrow{\mathbf{L}\varphi^*} & \mathrm{Ho}(\mathscr{M})(\mathscr{X}, I) \end{array} \tag{4.1.11.2}$$

is essentially commutative.

Proof. Assertion (a) is an immediate consequence of Theorem 4.1.3 (resp. of Theorem 4.1.8), and assertion (b) is a particular case of Proposition 3.1.11 (beware that the injective model category structure on $Comm(\mathscr{M})(\mathscr{X}, I)$ does not necessarily coincide with the model category structure given by Theorem 4.1.3 (resp. of Theorem 4.1.8) applied to the injective model structure on $\mathscr{M}(\mathscr{X}, I)$). For assertion (d), we see by the second assertion of Proposition 3.1.6 that it is sufficient to prove it when $\varphi : X \to Y$ is simply a morphism of $\mathscr{S}$. In this case, by construction of the total left derived functor of a left Quillen functor, this follows from the fact that φ^* commutes with the forgetful functor and from the fact that, by virtue of the last assertion of Theorem 4.1.3 (resp. of Theorem 4.1.8), the forgetful functor U preserves weak equivalences and cofibrant objects. □

Remark 4.1.12. The main application of the preceding corollary will come from assertion (d): it says that, given a monoid (resp. a commutative monoid) R in $\mathscr{M}(Y)$ and a morphism $f : X \to Y$, the image of R by the functor

$$\mathbf{L}f^* : \mathrm{Ho}(\mathscr{M})(Y) \to \mathrm{Ho}(\mathscr{M})(X)$$

is canonically endowed with a structure of monoid (resp. of commutative monoid) in the strongest sense possible. Under the assumptions of assertion (c) of Proposition 4.1.11, we shall often make the abuse of saying that $\mathbf{L}f^*(R)$ is a monoid (resp. a commutative monoid) in $\mathscr{M}(X)$ without refereeing explicitely to the model category structure on $Mon(\mathscr{M})(X)$ (resp. on $Comm(\mathscr{M})(X)$). Similarly, for any monoid (resp. commutative monoid) R in $\mathscr{M}(X)$, $\mathbf{R}f_*(R)$ will be canonically endowed with a structure of a monoid (resp. a commutative monoid) in $\mathscr{M}(Y)$. In particular, for any monoid (resp. commutative monoid) R in $\mathscr{M}(Y)$, the adjunction map

$$R \to \mathbf{R}f_* \mathbf{L}f^*(R)$$

is a morphism of monoids (i.e. is a map in the homotopy category $\mathrm{Ho}(Mon(\mathscr{M}))(X)$ (resp. $\mathrm{Ho}(Comm(\mathscr{M}))(X)$)), and, for any monoid (resp. commutative monoid) R in $\mathscr{M}(X)$, the adjunction map

$$\mathbf{L}f^* \mathbf{R}f_*(R) \to R$$

is a morphism of monoids (i.e. is a map in the homotopy category $\mathrm{Ho}(Mon(\mathscr{M}))(Y)$ (resp. $\mathrm{Ho}(Comm(\mathscr{M}))(Y)$)).

Remark 4.1.13. In order to get a good homotopy theory of commutative monoids wihout the strongly **Q**-linear assumption, we should replace commutative monoids by E_∞-algebras (i.e. objects endowed with a structure of commutative monoid up to a bunch of coherent homotopies). More generally, we should prove the analog of Theorem 4.1.3 and of Theorem 4.1.8 by replacing $Mon(\mathscr{V})$ by the category of algebras of some 'well behaved' operad, and then get as a consequence the analog of Proposition 4.1.11. All this should be a consequence of the general constructions and results of [Spi01, BM03, BM08].

However, in the case we are interested in the homotopy theory of commutative monoids in some category of spectra $\mathscr{V}$, it seems that some version of Shipley's *positive stable model structure* (*cf.* [Shi04, Proposition 3.1]) would provide a good model category for commutative monoids, which, by Lurie's strictification theorem [Lur09, Theorem 4.3.22], would be equivalent to the homotopy theory of E_∞-algebras in $\mathscr{V}$. This kind of technics should certainly be available in the context of stable homotopy theory of schemes, which would provide the good setting to speak of motivic commutative ring spectra: with these positive stable model structures, Theorem 4.1.8 and Proposition 4.1.11 should be true for genuine commutative monoids wihout any **Q**-linearity assumption (in general, given a nice enough combinatorial symmetric monoidal model category $\mathscr{V}$ with unit $\mathbb{1}$, the 'positive stable model category of symmetric $\mathbb{1}$-spectra' should provide a combinatorial symmetric model category which should be Quillen equivalent to $\mathscr{V}$, and in which commutative monoids are models for E_∞-algebras in $\mathscr{V}$; this would provide an alternative to the point of view of $\mathbb{S}$-modules of Elmendorf, Kriz, Mandell, May, and Spitzweck [EKMM97, Spi01]).

4.2. **Modules.**

4.2.1. Given a monoid R in a symmetric monoidal category $\mathscr{V}$, we shall write $R\text{-}\operatorname{mod}(\mathscr{V})$ for the category of (left) R-modules. The forgetful functor

$$U : R\text{-}\operatorname{mod}(\mathscr{V}) \to \mathscr{V}$$

is a left adjoint to the free R-module functor

$$R \otimes (-) : \mathscr{V} \to R\text{-}\operatorname{mod}(\mathscr{V}) .$$

If $\mathscr{V}$ has enough small colimits, and if R is a commutative monoid, the category $R\text{-}\operatorname{mod}(\mathscr{V})$ is endowed with a unique symmetric monoidal structure such that the functor $R \otimes (-)$ is naturally symmetric monoidal. We shall denote by $\otimes_R$ the tensor product of $R\text{-}\operatorname{mod}(\mathscr{V})$.

Theorem 4.2.2. *Let $\mathscr{V}$ be a combinatorial symmetric model category which satisfies the monoid axiom.*

(i) For any monoid R in $\mathscr{V}$, the category of right (resp. left) R-modules is a combinatorial model category with weak equivalences (resp. fibrations) the morphisms of R-modules which are weak equivalences (resp. fibrations) in $\mathscr{V}$.

(ii) For any commutative monoid R in $\mathscr{V}$, the model category of R-modules given by (i) is a combinatorial symmetric monoidal model category which satisfies the monoid axiom.

Proof. Assertions (i) and (ii) are particular cases of the first two assertions of [SS00, Theorem 4.1]. □

Definition 4.2.3. A symmetric monoidal model category $\mathscr{V}$ is *perfect* if it has the following properties.

(a) $\mathscr{V}$ is combinatorial and tractable (3.1.27);
(b) $\mathscr{V}$ satisfies the monoid axiom;
(c) For any weak equivalence of monoids $R \to S$, the functor $M \mapsto S \otimes_R M$ is a left Quillen equivalence from the category of left R-modules to the category of left S-modules.
(d) weak equivalences are stable by small sums in $\mathscr{V}$.

Remark 4.2.4. If $\mathscr{V}$ is a perfect symmetric monoidal model category, then, for any commutative monoid R, the symmetric monoidal model category of R-modules in $\mathscr{V}$ given by Theorem 4.2.2 (ii) is also perfect: condition (c) is quite obvious, and condition (d) comes from the fact that the forgetful functor $U : R\text{-}\operatorname{mod} \to \mathscr{V}$ commutes with small sums, while it preserves and detects weak

equivalences. Note that condition (d) implies that the functor $U : \mathrm{Ho}(R\text{-}\mathrm{mod}) \to \mathrm{Ho}(\mathscr{V})$ preserves small sums.

Remark 4.2.5. If $\mathscr{V}$ is a stable symmetric monoidal model category which satisfies the monoid axiom, then for any monoid R of $\mathscr{V}$, the model category of (left) R-modules given by Theorem 4.2.2 is stable as well: the suspension functor of $\mathrm{Ho}(R\text{-}\mathrm{mod})$ is given by the derived tensor product by the R-bimodule $R[1]$, which is clearly invertible with inverse $R[-1]$.

Proposition 4.2.6. *Let $\mathscr{V}$ be a stable perfect symmetric monoidal model category. Assume furthermore that $\mathrm{Ho}(\mathscr{V})$ admits a small family $\mathcal{G}$ of compact generators (as a triangulated category). For any monoid R in $\mathscr{V}$, the triangulated category $\mathrm{Ho}(R\text{-}\mathrm{mod}(\mathscr{V}))$ admits the set $\{R \otimes^{\mathbf{L}} E \mid E \in \mathcal{G}\}$ as a family of compact generators.*

Proof. We have a derived adjunction

$$R \otimes^{\mathbf{L}} (-) : \mathrm{Ho}(\mathscr{V}) \rightleftarrows \mathrm{Ho}(R\text{-}\mathrm{mod}(\mathscr{V})) : U\,.$$

As the functor U preserves small sums the functor $R \otimes^{\mathbf{L}} (-)$ preserves compact objects. But U is also conservative, so that $\{R \otimes^{\mathbf{L}} E \mid E \in \mathcal{G}\}$ is a family of compact generators of $\mathrm{Ho}(R\text{-}\mathrm{mod}(\mathscr{V}))$. □

Remark 4.2.7. If $\mathscr{V}$ is a combinatorial symmetric model category which satisfies the monoid axiom, then there are two ways to derive the tensor product. The first one consists to derive the left Quillen bifunctor $(-) \otimes (-)$, which gives the usual derived tensor product

$$(-) \otimes^{\mathbf{L}} (-) : \mathrm{Ho}(\mathscr{V}) \times \mathrm{Ho}(\mathscr{V}) \to \mathrm{Ho}(\mathscr{V})\,.$$

Remember that, by construction, $A \otimes^{\mathbf{L}} B = A' \otimes B'$, where A' and B' are cofibrant replacements of A and B respectively. On the other hand, the monoid axiom gives that, for any object A of $\mathscr{V}$, the functor $A \otimes (-)$ preserves weak equivalences between cofibrant objects, which implies that it has also a total left derived functor

$$A \otimes^{\mathbf{L}} (-) : \mathrm{Ho}(\mathscr{V}) \to \mathrm{Ho}(\mathscr{V})\,.$$

Despite the fact we have adopted very similar (not to say identical) notations for these two derived functor, there is no reason they would coincide in general: by construction, the second one is defined by $A \otimes^{\mathbf{L}} B = A \otimes B'$, where B' is some cofibrant replacement of B. However, they coincide quite often in practice (e.g. for simplicial sets, for the good reason that all of them are cofibrant, or for symmetric S^1-spectra, or for complexes of quasi-coherent $\mathcal{O}_X$-modules over a quasi-compact and quasi-separated scheme X).

Proposition 4.2.8. *Let $\mathscr{V}$ be a stable combinatorial symmetric monoidal model category which satisfies the monoid axiom. Assume furthermore that, for any cofibrant object A of $\mathscr{V}$, the functor $A \otimes (-)$ preserve weak equivalences (in other words, that the two ways to derive the tensor product explained in Remark 4.2.7 coincide), and that weak equivalences are stable by small sums in $\mathscr{V}$. Then the symmetric monoidal model category $\mathscr{V}$ is perfect.*

Proof. We just have to check condition (c) of Definition 4.2.3. Consider a weak equivalence of monoids $R \to S$. We then get a derived adjunction

$$S \otimes^{\mathbf{L}}_{R} (-) : \mathrm{Ho}(R\text{-}\mathrm{mod}(\mathscr{V})) \rightleftarrows \mathrm{Ho}(S\text{-}\mathrm{mod}(\mathscr{V})) : U\,,$$

where $S \otimes^{\mathbf{L}}_{R} (-)$ is the left derived functor of the functor $M \mapsto S \otimes_R M$. We have to prove that, for any left R-module M, the map

$$M \to S \otimes^{\mathbf{L}}_{R} M$$

is an isomorphism in $\mathrm{Ho}(\mathscr{V})$. As this is a morphism of triangulated functors which commutes with sums, and as $\mathrm{Ho}(R\text{-}\mathrm{mod}(\mathscr{V}))$ is well generated in the sense of Neeman [Nee01] (as the localization of a stable combinatorial model category), it is sufficient to check this when M runs over a small family of generators of $\mathrm{Ho}(R\text{-}\mathrm{mod}(\mathscr{V}))$. Let us chose is a small family of generators $\mathcal{G}$ of $\mathrm{Ho}(\mathscr{V})$. As the forgetful functor from $\mathrm{Ho}(R\text{-}\mathrm{mod}(\mathscr{V}))$ to $\mathrm{Ho}(\mathscr{V})$ is conservative, we see that $\{R \otimes^{\mathbf{L}} E \mid E \in \mathcal{G}\}$ is a small generating family of $\mathrm{Ho}(R\text{-}\mathrm{mod}(\mathscr{V}))$. We are thus reduced to prove that the map

$$R \otimes^{\mathbf{L}} E \to S \otimes^{\mathbf{L}}_{R} (R \otimes^{\mathbf{L}} E) \simeq S \otimes^{\mathbf{L}} E$$

is an isomorphism for any object E in $\mathcal{G}$. For this, we can assume that E is cofibrant, and this follows then from the fact that the functor $(-)\otimes E$ preserves weak equivalences by assumption. □

4.2.9. Let $\mathscr{S}$ be a category endowed with an admissible class of morphisms $\mathscr{P}$, and $\mathscr{M}$ a symmetric monoidal $\mathscr{P}$-fibred category. Consider a monoid R in the symmetric monoidal category $\mathscr{M}(1_{\mathscr{S}},\mathscr{S})$ (i.e. a section of the fibred category $Mon(\mathscr{M})$ over $\mathscr{S}$). In other words, R consists of the data of a monoid R_X for each object X of $\mathscr{S}$, and of a morphism of monoids $a_f : f^*(R_Y) \to R_X$ for each map $f : X \to Y$ in $\mathscr{S}$, subject to coherence relations; see 3.1.2.

For an object X of $\mathscr{S}$, we shall write $R\text{-}\operatorname{mod}(X)$ for the category of (left) R_X-modules in $\mathscr{M}(X)$, i.e.

$$R\text{-}\operatorname{mod}(X) = R_X\text{-}\operatorname{mod}(\mathscr{M}(X))\,.$$

This defines a fibred category $R\text{-}\operatorname{mod}$ over $\mathscr{S}$ as follows.

For a morphism $f : X \to Y$, the inverse image functor

$$f^* : R\text{-}\operatorname{mod}(Y) \to R\text{-}\operatorname{mod}(X) \tag{4.2.9.1}$$

is defined by

$$M \mapsto R_X \otimes_{f^*(R_Y)} f^*(M) \tag{4.2.9.2}$$

(where, on the right hand side, f^* stands for the inverse image functor in $\mathscr{M}$). The functor (4.2.9.1) has a right adjoint

$$f_* : R\text{-}\operatorname{mod}(X) \to R\text{-}\operatorname{mod}(Y) \tag{4.2.9.3}$$

which is simply the functor induced by $f_* : \mathscr{M}(X) \to \mathscr{M}(Y)$ (as the latter sends R_X-modules to $f_*(R_X)$-modules, which are themselves R_Y-modules via the map a_f).

If the map f is a $\mathscr{P}$-morphism, then, for any R_X-module M, the object $f_\sharp(M)$ has a natural structure of R_Y-module: using the map a_f, M has a natural structure of $f^*(R_Y)$-module

$$f^*(R_Y) \otimes_X M \to M\,,$$

and applying $f_\sharp$, we get by the $\mathscr{P}$-projection formula (1.1.25) a morphism

$$R_Y \otimes f_\sharp(M) \simeq f_\sharp(f^*(R_Y) \otimes M) \to f_\sharp(M)$$

which defines a natural R_Y-module structure on $f_\sharp(M)$. For a $\mathscr{P}$-morphism $f : X \to Y$, we define a functor

$$f_\sharp : R\text{-}\operatorname{mod}(X) \to R\text{-}\operatorname{mod}(Y) \tag{4.2.9.4}$$

as the functor induced by $f_\sharp : \mathscr{M}(X) \to \mathscr{M}(Y)$. Note that the functor (4.2.9.4) is a left adjoint to the functor (4.2.9.1) whenever the map $a_f : f^*(R_Y) \to R_X$ is an isomorphism in $\mathscr{M}(X)$.

We shall say that R is a *cartesian monoid in $\mathscr{M}$ over $\mathscr{S}$* if R is a monoid of $\mathscr{M}(1_{\mathscr{C}},\mathscr{C})$ such that all the structural maps $f^*(R_Y) \to R_X$ are isomorphisms (i.e. if R is a cartesian section of the fibred category $Mon(\mathscr{M})$ over $\mathscr{S}$)

If R is a cartesian monoid in $\mathscr{M}$ over $\mathscr{S}$, then $R\text{-}\operatorname{mod}$ is a $\mathscr{P}$-fibred category over $\mathscr{S}$: to see this, it remains to prove that, for any pullback square of $\mathscr{S}$

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[h]{} & Y \end{array}$$

in which f is a $\mathscr{P}$-morphism, and for any R_X-module M, the base change map

$$f'_\sharp\, g^*(M) \to h^*\, f_\sharp(M)$$

is an isomorphism, which follows immediately from the analogous formula for $\mathscr{M}$.

Similarly, we see that whenever R is a commutative monoid of $\mathscr{M}(1_{\mathscr{C}},\mathscr{C})$ (i.e. R_X is a commutative monoid in $\mathscr{M}(X)$ for all X in $\mathscr{S}$), then $R\text{-}\operatorname{mod}$ is a symmetric monoidal $\mathscr{P}$-fibred category.

Proposition 4.2.10. *Let $\mathscr{M}$ be a combinatorial symmetric monoidal $\mathscr{P}$-fibred model category over $\mathscr{S}$ which satisfies the monoid axiom, and R a monoid in $\mathscr{M}(1_{\mathscr{S}},\mathscr{S})$ (resp. a cartesian monoid in $\mathscr{M}$ over $\mathscr{S}$). Then 4.2.2 (i) applied termwise turns R-mod into combinatorial fibred model category (resp. a combinatorial $\mathscr{P}$-fibred model category).*

If moreover R is commutative, then R-mod is a combinatorial symmetric monoidal fibred model category (resp. a combinatorial symmetric monoidal $\mathscr{P}$-fibred model category).

Proof. Choose, for each object X of $\mathscr{S}$, two small sets of maps I_X and J_X which generate the class of cofibrations and the class of trivial cofibrations in $\mathscr{M}(X)$ respectively. Then $R_X \otimes_X I_X$ and $R_X \otimes_X J_X$ generate the class of cofibrations and the class of trivial cofibrations in $R\text{-mod}(X)$ respectively. For a map $f : X \to Y$ in $\mathscr{S}$, we see from formula (4.2.9.2) that the functor (4.2.9.1) sends these generating cofibrations and trivial cofibrations to cofibrations and trivial cofibrations respectively, from which we deduce that the functor (4.2.9.1) is a left Quillen functor. In the respective case, if f is a $\mathscr{P}$-morphism, then we deduce similarly from the projection formula (1.1.25) in $\mathscr{M}$ that the functor (4.2.9.4) sends generating cofibrations and trivial cofibrations to cofibrations and trivial cofibrations respectively. The last assertion follows easily by applying 4.2.2 (ii) termwise. □

Proposition 4.2.11. *Let $\mathscr{M}$ be a perfect symmetric monoidal $\mathscr{P}$-fibred model category over $\mathscr{S}$, and consider a homotopy cartesian monoid R in $\mathscr{M}$ over $\mathscr{S}$ (i.e. a homotopy cartesian section of $Mon(\mathscr{M})$ (3.1.26)). Then* $\mathrm{Ho}(R\text{-mod})$ *is a $\mathscr{P}$-fibred category over $\mathscr{S}$, and*

$$R \otimes^{\mathbf{L}} (-) : \mathrm{Ho}(\mathscr{M}) \to \mathrm{Ho}(R\text{-mod})$$

is a morphism of $\mathscr{P}$-fibred categories. In the case where R is commutative, $\mathrm{Ho}(R\text{-mod})$ *is even a symmetric monoidal $\mathscr{P}$-fibred category.*

Moreover, for any weak equivalence between homotopy cartesian monoids $R \to S$ over $\mathscr{S}$, the Quillen morphism

$$S \otimes_R (-) : R\text{-mod} \to S\text{-mod}$$

induces an equivalence of $\mathscr{P}$-fibred categories over $\mathscr{S}$

$$S \otimes^{\mathbf{L}}_R (-) : \mathrm{Ho}(R\text{-mod}) \to \mathrm{Ho}(S\text{-mod})\,.$$

Proof. It is sufficient to prove these asertions by restricting everything over $\mathscr{S}/S$, where S runs over all the objects of $\mathscr{S}$. In particular, we may (and shall) assume that $\mathscr{S}$ has a terminal object S. As $\mathscr{M}$ is perfect, it follows from condition (c) of Definition 4.2.3 that we can replace R by any of its homotopy cartesian resolution (see Proposition 3.1.29). In particular, we may assume that R_S is a cofibrant object of $Mon(\mathscr{M})(S)$. We can thus define a termwise cofibrant cartesian monoid R' as the family of monoids $f^*(R_S)$, where $f : X \to S$ runs over all the objects of $\mathscr{S} \simeq \mathscr{S}/S$. There is a canonical morphism of homotopy cartesian monoids $R' \to R$ which is a termwise weak equivalence. We thus get, by condition (c) of Definition 4.2.3, an equivalence of fibred categories

$$R \otimes^{\mathbf{L}}_{R'} (-) : \mathrm{Ho}(R'\text{-mod}) \to \mathrm{Ho}(R\text{-mod})\,.$$

We can thus replace R by R', which just means that we can assume that R is cartesian and termwise cofibrant. The first assertion follows then easily from Proposition 4.2.10. In the case where R is commutative, we prove that $\mathrm{Ho}(R\text{-mod})$ is a $\mathscr{P}$-fibred symmetric monoidal category as follows. Let $f : X \to Y$ a morphism of $\mathscr{S}$. We would like to prove that, for any object M in $\mathrm{Ho}(R\text{-mod})(X)$ and any object N in $\mathrm{Ho}(R\text{-mod})(Y)$, the canonical map

$$\mathbf{L}f_\sharp(M \otimes^{\mathbf{L}}_R f^*(N)) \to \mathbf{L}f_\sharp(M) \otimes^{\mathbf{L}}_R N \tag{4.2.11.1}$$

is an isomorphism. By adjunction, this is equivalent to prove that, for any objects N and E in $\mathrm{Ho}(R\text{-mod})(Y)$, the map

$$f^*\mathbf{R}Hom_R(N,E) \to \mathbf{R}Hom_R(f^*(N), f^*(E)) \tag{4.2.11.2}$$

is an isomorphism in $\mathrm{Ho}(R\text{-mod})(X)$ (where $\mathbf{R}Hom_R$ stands for the internal Hom of $\mathrm{Ho}(R\text{-mod})$). But the forgetful functors

$$U : \mathrm{Ho}(R\text{-mod})(X) \to \mathrm{Ho}(\mathscr{M})(X)$$

are conservative, commute with f^* for any $\mathscr{P}$-morphism f, and commute with internal Hom: by adjunction, this follows immediately from the fact that the functors

$$R \otimes^{\mathbf{L}} (-) : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(R\text{-}\mathrm{mod})(X) \simeq \mathrm{Ho}(R'\text{-}\mathrm{mod})(X)$$

are symmetric monoidal and define a morphism of $\mathscr{P}$-fibred categories (and thus, in particular, commute with $f_\sharp$ for any $\mathscr{P}$-morphism f). Hence, to prove that (4.2.11.2) is an isomorphism, it is sufficient to prove that its analog in $\mathrm{Ho}(\mathscr{M})$ is so, which follows immediately from the fact that the analog of (4.2.11.1) is an isomorphism in $\mathrm{Ho}(\mathscr{M})$ by assumption.

For the last assertion, we are also reduced to the case where R and S are cartesian and termwise cofibrant, in which case this follows easily again from condition (c) of Definition 4.2.3. □

Proposition 4.2.12. *Let $\mathscr{M}$ be a combinatorial symmetric monoidal model category over $\mathscr{S}$ which satisfies the monoid axiom. Then, for any cartesian monoid R in $\mathscr{M}$ over $\mathscr{S}$ we have a Quillen morphism*

$$R \otimes (-) : \mathscr{M} \to R\text{-}\mathrm{mod}\ .$$

If, for any object X of $\mathscr{S}$, the unit object $\mathbb{1}_X$ is cofibrant in $\mathscr{M}(X)$ and the monoid R_X is cofibrant in $Mon(\mathscr{M})(X)$, then the forgetful functors also define a Quillen morphism

$$U : R\text{-}\mathrm{mod} \to \mathscr{M}\ .$$

Proof. The first assertion is obvious. For the second one, note that, for any object X of $\mathscr{S}$, the monoid R_X is also cofibrant as an object of $\mathscr{M}(X)$; see Theorem 4.1.3. This implies that the forgetful functor

$$U : R_X\text{-}\mathrm{mod} \to \mathscr{M}(X)$$

is a left Quillen functor: by the small object argument and by by definition of the model category structure of Theorem 4.2.2 (i), this follows from the trivial fact that the endofunctor

$$R_X \otimes (-) : \mathscr{M}(X) \to \mathscr{M}(X)$$

is a left Quillen functor itself whenever R_X is cofibrant in $\mathscr{M}(X)$. □

Remark 4.2.13. The results of the preceding proposition (as well as their proofs) are also true in terms of P_{cart}-fibred categories (3.1.21) over the category of $\mathscr{S}/S$-diagrams for any object S of $\mathscr{S}$ (whence over all $\mathscr{S}$-diagrams whenever $\mathscr{S}$ has a terminal object).

4.2.14. Consider now a noetherian scheme S of finite Krull dimension. We choose a full subcategory of the category of separated noetherian S-schemes of finite Krull dimension which is stable by finite limits, contains separated S-schemes of finite type, and such that, for any étale S-morphism $Y \to X$, if X is in $\mathscr{S}/S$, so is Y. We denote by $\mathscr{S}/S$ this chosen category of S-schemes.

We also fix an admissible class $\mathscr{P}$ of morphisms of $\mathscr{S}/S$ which contains the class of étale morphisms.

Definition 4.2.15. A property P of $\mathrm{Ho}(\mathscr{M})$, for $\mathscr{M}$ a stable combinatorial $\mathscr{P}$-fibred model category over $\mathscr{S}/S$, is *homotopy linear* if the following implications are true.

(a) If $\gamma : \mathscr{M} \to \mathscr{M}'$ is a Quillen equivalence (i.e. a Quillen morphism which is termwise a Quillen equivalence) between stable combinatorial $\mathscr{P}$-fibred model category over $\mathscr{S}/S$, then $\mathscr{M}$ has property P is and only if $\mathscr{M}'$ has property P.

(b) If $\mathscr{M}$ is a stable combinatorial symmetric monoidal $\mathscr{P}$-model category which satisfies the monoid axiom, and such that the unit $\mathbb{1}_X$ of $\mathscr{M}(X)$ is cofibrant, then, for any cartesian and termwise cofibrant monoid R in $\mathscr{M}$ over $\mathscr{S}/S$, R-mod has property P.

Proposition 4.2.16. *The following properties are homotopy linear: $\mathbf{A}^1$-homotopy invariance, $\mathbf{P}^1$-stability, the localization property, the property of proper transversality, separability, semi-separability, t-descent (for a given Grothendieck topology t on $\mathscr{S}/S$).*

Proof. Property (a) of the definition above is obvious. Property (b) comes from the fact that the forgetful functors

$$U : \mathrm{Ho}(R\text{-}\mathrm{mod}) \to \mathrm{Ho}(\mathscr{M})$$

are conservative and commute with all the operations: $\mathbf{L}f^*$ and $\mathbf{R}f_*$ for any morphism f, as well as $\mathbf{L}f_\sharp$ for any $\mathscr{P}$-morphism (by Proposition 4.2.12). Hence any property formulated in terms of equations involving only these operations is homotopy linear. □

Part 2. Construction of fibred categories

5. Fibred derived categories

5.0. In this entire section, we fix a full subcategory $\mathscr{S}$ of the category of noetherian $\mathcal{S}$-schemes satisfying the following properties:

(a) $\mathscr{S}$ is closed under finite sums and pullback along morphisms of finite type.
(b) For any scheme S in $\mathscr{S}$, any quasi-projective S-scheme belongs to $\mathscr{S}$.

We fix an admissible class of morphisms $\mathscr{P}$ of $\mathscr{S}$. All our $\mathscr{P}$-premotivic categories (*cf.* definition 1.4.2) are defined over $\mathscr{S}$. Moreover, for any abelian $\mathscr{P}$-premotivic category $\mathscr{A}$ in this section, we assume the following:

(c) $\mathscr{A}$ is a *Grothendieck* abelian $\mathscr{P}$-premotivic category (see definition 1.3.8 and the recall below).
(d) $\mathscr{A}$ is given with a generating set of twists τ. We sometimes refer to it as *the twists of* $\mathscr{A}$.
(e) We will denote by $M_S(X, \mathscr{A})$, or simply by $M_S(X)$, the geometric section over a $\mathscr{P}$-scheme X/S.

Without precision, any scheme will be assumed to be an object of $\mathscr{S}$.

In section 5.2, except possibly for 5.2.a, we assume further:

(f) $\mathscr{P}$ contains the class of smooth finite type morphisms.

In section 5.3, we assume (f) and instead of (d) above.

5.0.17. We will refer sometimes to the canonical dg-structure of the category of complexes $\mathrm{C}(\mathscr{A})$ over an abelian category $\mathscr{A}$. Recall that to any complexes K and L over $\mathscr{A}$, we associate a complex of abelian groups $\mathrm{Hom}^\bullet_{\mathscr{A}}(K, L)$ whose component in degree $n \in \mathbf{Z}$ is

$$\prod_{p \in \mathbf{Z}} \mathrm{Hom}_{\mathscr{A}}(K^p, L^{p+n})$$

and whose differential in degree $n \in \mathbf{Z}$ is defined by the formula:

$$(f_p)_{p\in\mathbf{Z}} \mapsto \big(d_L \circ f_p - (-1)^n . f_{p+1} \circ d_K)\big)_{p\in\mathbf{Z}}.$$

In other words, this is the image of the bicomplex $\mathrm{Hom}_{\mathscr{A}}(K, L)$ by the Tot-product functor which we denote by Tot^π. Of course, the associated homotopy category is the category $\mathrm{K}(\mathscr{A})$ of complexes up to chain homotopy equivalence.

5.1. From abelian premotives to triangulated premotives.

5.1.a. *Abelian premotives: recall and examples.* Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$. According to the convention of 5.0, for any scheme S, $\mathscr{A}_S$ is a Grothendieck abelian closed symmetric monoidal category. Moreover, if τ denotes the twists of $\mathscr{A}$, the essentially small family

$$\big(M_S(X)\{i\}\big)_{X\in\mathscr{P}/S, i\in\tau}$$

is a family of generators of $\mathscr{A}_S$ in the sense of [Gro57].

Example 5.1.1. Consider a fixed ring Λ. Let $\mathrm{PSh}(\mathscr{P}/S, \Lambda)$ be the category of Λ-presheaves (i.e. presheaves of Λ-modules) on $\mathscr{P}/S$. For any $\mathscr{P}$-scheme X/S, we let $\Lambda_S(X)$ be the free Λ-presheaf on $\mathscr{P}/S$ represented by X. Then $\mathrm{PSh}(\mathscr{P}/S, \Lambda)$ is a Grothendieck abelian category generated by the essentially small family $\big(\Lambda_S(X)\big)_{X\in\mathscr{P}/S}$.

There is a unique symmetric closed monoidal structure on $\mathrm{PSh}(\mathscr{P}/S, \Lambda)$ such that

$$\Lambda_S(X) \otimes_S \Lambda_S(Y) = \Lambda_S(X \times_S Y).$$

Finally the existence of functors f^*, f_* and, in the case when f is a $\mathscr{P}$-morphism, of $f_\sharp$, follows from general sheaf theory (*cf.* [AGV73]).

Thus, $\mathrm{PSh}(\mathscr{P}, \Lambda)$ defines an abelian $\mathscr{P}$-premotivic category.

5.1.2. Consider an abstract abelian $\mathscr{P}$-premotivic category $\mathscr{A}$. To any premotive M of $\mathscr{A}_S$, we can associate a presheaf of abelian groups

$$X \mapsto \mathrm{Hom}_{\mathscr{A}_S}(M_S(X), M)$$

which we denote by $\gamma_*(M)$.
This defines a functor $\gamma_* : \mathscr{A}_S \to \mathrm{PSh}(\mathscr{P}/S, \mathbf{Z})$. It admits the following left adjoint:

$$\gamma^* : \mathrm{PSh}(\mathscr{P}/S, \mathbf{Z}) \to \mathscr{A}_S \ , \quad F \mapsto \varinjlim_{X/F} M_S(X, \mathscr{A})$$

where the colimit runs over the category of representable presheaves over F.

It is now easy to check we have defined a morphism of (complete) abelian $\mathscr{P}$-premotivic categories:

$$\gamma^* : \mathrm{PSh}(\mathscr{P}, \mathbf{Z}) \rightleftarrows \mathscr{A} : \gamma_*. \tag{5.1.2.1}$$

Moreover $\mathrm{PSh}(\mathscr{P}, \mathbf{Z})$ appears as the initial abelian $\mathscr{P}$-premotivic category.

Remark that the functor $\gamma_* : \mathscr{A}_S \to \mathrm{PSh}(S, \mathbf{Z})$ is conservative if the set of twists τ of $\mathscr{A}$ is trivial.

Definition 5.1.3. A $\mathscr{P}$-admissible topology t is a Grothendieck pretopology t on the category $\mathscr{S}$, such that any t-covering family consists of $\mathscr{P}$-morphisms.

Note that, for any scheme S in $\mathscr{S}$, such a topology t induces a pretopology on $\mathscr{P}/S$ (which we denote by the same letter). For any morphism (resp. $\mathscr{P}$-morphism) $f : T \to S$, the functor f^* (resp. $f_\sharp$) preserves t-covering families.

As $\mathscr{P}$ is fixed in all this section, we will simply say *admissible* for $\mathscr{P}$-admissible.

Example 5.1.4. Let t be an admissible topology. We denote by $\mathrm{Sh}_t(\mathscr{P}/S, \Lambda)$ the category of t-sheaves of Λ-modules on $\mathscr{P}/S$. Given a $\mathscr{P}$-scheme X/S, we let $\Lambda^t_S(X)$ be the free Λ-linear t_S-sheaf represented by X. Then, $\mathrm{Sh}_t(\mathscr{P}/S, \Lambda)$ is an abelian Grothendieck category with generators $(\Lambda^t_S(X))_{X \in Sm_S}$.

As in the preceding example, the category $\mathrm{Sh}_t(\mathscr{P}/S, \Lambda)$ admits a unique closed symmetric monoidal structure such that $\Lambda^t_S(X) \otimes_S \Lambda^t_S(Y) = \Lambda^t_S(X \times_S Y)$. Finally, for any morphism $f : T \to S$ of schemes, the existence of functors f^*, f_* (resp. $f_\sharp$ when f is a $\mathscr{P}$-morphism) follows from the general theory of sheaves (see again [AGV73]: according to our assumption on t and [AGV73, III, 1.6], the functors $f^* : \mathscr{P}/S \to \mathscr{P}/T$ and $f_\sharp : \mathscr{P}/T \to \mathscr{P}/S$ (for f in $\mathscr{P}$) are continuous).

Thus, $\mathrm{Sh}_t(\mathscr{P}, \Lambda)$ defines an abelian $\mathscr{P}$-premotivic category (with trivial set of twists).

The associated t-sheaf functor induces a morphism

$$a_t^* : \mathrm{PSh}(\mathscr{P}, \Lambda) \rightleftarrows \mathrm{Sh}_t(\mathscr{P}, \Lambda) : a_{t,*}. \tag{5.1.4.1}$$

Remark 5.1.5. Recall the abelian category $\mathrm{Sh}_t(\mathscr{P}/S, \mathbf{Z})$ is a localization of the category $\mathrm{PSh}(S, \mathbf{Z})$ in the sense of Gabriel-Zisman. In particular, given an abstract abelian $\mathscr{P}$-premotivic category $\mathscr{A}$, the canonical morphism

$$\gamma^* : \mathrm{PSh}(\mathscr{P}/S, \mathbf{Z}) \rightleftarrows \mathscr{A}_S : \gamma_*$$

induces a unique morphism

$$\mathrm{Sh}_t(\mathscr{P}/S, \mathbf{Z}) \rightleftarrows \mathscr{A}_S$$

if and only if for any presheaf of abelian groups F on $\mathscr{P}/S$ such that $a_t(F) = F_t = 0$, one has $\gamma^*(F) = 0$.

We leave to the reader the exercise which consists to formulate the universal property of the abelian $\mathscr{P}$-premotivic category $\mathrm{Sh}_t(\mathscr{P}, \mathbf{Z})$.[20]

[20] We will formulate a derived version in the paragraph on descent properties for derived premotives (*cf.* 5.2.9).

5.1.b. *The t-descent model category structure.*

5.1.6. Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$ with set of twists τ.

We let $\mathrm{C}(\mathscr{A})$ be the $\mathscr{P}$-fibered abelian category over $\mathscr{S}$ whose fibers over a scheme S is the category $\mathrm{C}(\mathscr{A}_S)$ of (unbounded) complexes in $\mathscr{A}_S$. For any scheme S, we let $\iota_S : \mathscr{A}_S \to \mathrm{C}(\mathscr{A}_S)$ the embedding which sends an object of $\mathscr{A}_S$ to the corresponding complex concentrated in degree zero.

If $\mathscr{A}$ is τ-twisted, then the category $\mathrm{C}(\mathscr{A}_S)$ is obviously $(\mathbf{Z} \times \tau)$-twisted. The following lemma is straightforward :

Lemma 5.1.7. *With the notations above, there is a unique structure of abelian $\mathscr{P}$-premotivic category on $\mathrm{C}(\mathscr{A})$ such that the functor $\iota : \mathscr{A} \to \mathrm{C}(\mathscr{A})$ is a morphism of abelian $\mathscr{P}$-premotivic categories.*

5.1.8. For a scheme S, let $\mathscr{P}/S^{\amalg}$ be the category introduced in 3.2.1. The functor $M_S(-)$ can be extended to $\mathscr{P}/S^{\amalg}$ by associating to a family $(X_i)_{i\in I}$ of $\mathscr{P}$-schemes over S the premotive

$$\bigoplus_{i\in I} M_S(X_i).$$

If $\mathcal{X}$ is a simplicial object of $\mathscr{P}/S^{\amalg}$, we denote by $M_S(\mathcal{X})$ the complex associated with the simplicial object of $\mathscr{A}_S$ obtained by applying degreewise the above extension of $M_S(-)$.

Definition 5.1.9. Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category and t be an admissible topology. Let S be a scheme and C be an object of $\mathrm{C}(\mathscr{A}_S)$:

(1) The complex C is said to be *local* (with respect to the geometric section) if, for any $\mathscr{P}$-scheme X/S and any pair $(n, i) \in \mathbf{Z} \times \tau$, the canonical morphism

$$\mathrm{Hom}_{\mathrm{K}(\mathscr{A}_S)}(M_S(X)\{i\}[n], C) \to \mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(M_S(X)\{i\}[n], C)$$

is an isomorphism.

(2) The complex C is said to be *t-flasque* if for any t-hypercover $\mathcal{X} \to X$ in $\mathscr{P}/S$, for any $(n, i) \in \mathbf{Z} \times \tau$, the canonical morphism

$$\mathrm{Hom}_{\mathrm{K}(\mathscr{A}_S)}(M_S(X)\{i\}[n], C) \to \mathrm{Hom}_{\mathrm{K}(\mathscr{A}_S)}(M_S(\mathcal{X})\{i\}[n], C)$$

is an isomorphism.

We say the abelian $\mathscr{P}$-premotivic category $\mathscr{A}$ satifies *cohomological t-descent* if for any t-hypercover $\mathcal{X} \to X$ of a $\mathscr{P}$-scheme X/S, and for any $i \in \tau$, the map

$$M_S(\mathcal{X})\{i\} \to M_S(X)\{i\}$$

is a quasi-isomorphism (or equivalently, if any local complex is t-flasque).

We say that $\mathscr{A}$ is *compatible with* t if $\mathscr{A}$ satisfies cohomological t-descent, and if, for any scheme S, any t-flasque complex of $\mathscr{A}_S$ is local.

Example 5.1.10. Consider the notations of 5.1.4.

Consider the canonical dg-structure on $\mathrm{C}(\mathrm{Sh}_t(\mathscr{P}/S, \Lambda))$ (see 5.1.1). By definition, for any complexes D and C of sheaves, we get an equality:

$$\mathrm{Hom}_{\mathrm{K}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda))}(D, C) = H^0(\mathrm{Hom}^{\bullet}_{\mathscr{A}}(D, C)) = H^0(\mathrm{Tot}^{\pi}\,\mathrm{Hom}_{\mathscr{A}}(D, C)).$$

In the case where $D = \Lambda^t_S(X)$ (resp. $D = \Lambda^t_S(\mathcal{X})$) for a $\mathscr{P}$-scheme X/S (resp. a simplicial $\mathscr{P}$-scheme over S) we obtain the following identification:

$$\mathrm{Hom}_{\mathrm{K}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda))}(\Lambda^t_S(X), C) = H^0(C(X)).$$
$$(\text{resp. } \mathrm{Hom}_{\mathrm{K}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda))}(\Lambda^t_S(\mathcal{X}), C) = H^0(\mathrm{Tot}^{\pi}\, C(\mathcal{X}))\,).$$

Thus, we get the following equivalences:

$$C \text{ is local} \Leftrightarrow \text{for any } \mathscr{P}\text{-scheme } X/S,\ H^n_t(X, C) \simeq H^n(C(X)).$$
$$C \text{ is } t\text{-flasque} \Leftrightarrow \text{for any } t\text{-hypercover } \mathcal{X} \to X,\ H^n(C(X)) \simeq H^n(\mathrm{Tot}^{\pi}\, C(\mathcal{X})).$$

According to the computation of cohomology with hypercovers (*cf.* [Bro74]), if the complex C is t-flasque, it is local. In other words, we have the expected property that the abelian $\mathscr{P}$-premotivic category $\mathrm{Sh}_t(\mathscr{P}, \Lambda)$ is compatible with t.

5.1.11. Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$ and an admissible topology t.

Fix a base scheme S. A morphism $p : C \to D$ of complexes on $\mathscr{A}_S$ is called a *t-fibration* if its kernel is a t-flasque complex and if for any $\mathscr{P}$-scheme X/S, any $i \in \tau$ and any integer $n \in \mathbf{Z}$, the map of abelian groups

$$\mathrm{Hom}_{\mathscr{A}_S}(M_S(X)\{i\}, C^n) \to \mathrm{Hom}_{\mathscr{A}_S}(M_S(X)\{i\}, D^n)$$

is surjective.

For any object A of $\mathscr{A}_S$, we let $S^n A$ (resp. $D^n A$) be the complex with only one non trivial term (resp. two non trivial terms) equal to A in degree n (resp. in degree n and $n+1$, with the identity as only non trivial differential). We define the class of *cofibrations* as the smallest class of morphisms of $\mathrm{C}(\mathscr{A}_S)$ which :

(1) contains the map $S^{n+1}M_S(X)\{i\} \to D^n M_S(X)\{i\}$ for any $\mathscr{P}$-scheme X/S, any $i \in \tau$, and any integer n;
(2) is stable by pushout, transfinite composition and retract.

A complex C is said to be *cofibrant* if the canonical map $0 \to C$ is a cofibration. For instance, for any $\mathscr{P}$-scheme X/S and any $i \in \tau$, the complex $M_S(X)\{i\}[n]$ is cofibrant.

Let $\mathcal{G}_S$ be the essentially small family made of premotives $M_S(X)\{i\}$ for a $\mathscr{P}$-scheme X/S and a twist $i \in \tau$, and $\mathcal{H}_S$ be the family of complexes of the form $\mathrm{Cone}(M_S(\mathcal{X})\{i\} \to M_S(X)\{i\})$ for any t-hypercover $\mathcal{X} \to X$ and any twist $i \in \tau$. By the very definition, as $\mathscr{A}$ is compatible with t (definition 5.1.9), $(\mathcal{G}_S, \mathcal{H}_S)$ is a descent structure on $\mathscr{A}_S$ in the sense of [CD09, def. 2.2]. Moreover, it is weakly flat in the sense of [CD09, par. 3.1]. Thus the following proposition is a particular case of [CD09, theorem 2.5, proposition 3.2, and corollary 5.5] :

Proposition 5.1.12. *Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category, which we assume to be compatible with an admissible topology t. Then for any scheme S, the category $\mathrm{C}(\mathscr{A}_S)$ with the preceding definition of fibrations and cofibrations, with quasi-isomorphisms as weak equivalences is a proper symmetric monoidal model category.*

5.1.13. We will call this model structure on $\mathrm{C}(\mathscr{A}_S)$ the *t-descent model category structure* (over S). Note that, for any $\mathscr{P}$-scheme X/S and any twist $i \in \tau$, the complex $M_S(X)\{i\}$ concentrated in degree 0 is cofibrant by definition, as well as any of its suspensions and twists. They form a family of generators for the triangulated category $\mathrm{D}(A_S)$.
Observe also that the fibrant objects for the t-descent model category structure are exactly the t-flasque complexes in $\mathscr{A}_S$. Moreover, essentially by definition, a complex of $\mathscr{A}_S$ is local if and only if it is t-flasque (see [CD09, 2.5]).

5.1.14. Consider again the notations and hypothesis of 5.1.11.

Consider a morphism of schemes $f : T \to S$. Then the functor

$$f^* : \mathrm{C}(\mathscr{A}_S) \to \mathrm{C}(\mathscr{A}_T)$$

sends $\mathcal{G}_S$ in $\mathcal{G}_T$, and $\mathcal{H}_S$ in $\mathcal{H}_T$ because the topology t is admissible. This means it satisfies descent according to the definition of [CD09, 2.4]. Applying theorem 2.14 of *op. cit.*, the functor f^* preserves cofibrations and trivial cofibrations, i.e. the pair of functors (f^*, f_*) is a Quillen adjunction with respect to the t-descent model category structures.

Assume that f is a $\mathscr{P}$-morphism. Then, similarly, the functor

$$f_\sharp : \mathrm{C}(\mathscr{A}_T) \to \mathrm{C}(\mathscr{A}_S)$$

sends $\mathcal{G}_S$ (resp. $\mathcal{H}_S$) in $\mathcal{G}_T$ (resp. $\mathcal{H}_T$) so that it $f_\sharp$ also satisfies descent in the sense of *op. cit.* Therefore, it preserves cofibrations and trivial cofibrations, and the pair of adjoint functors $(f_\sharp, f^*)$ is a Quillen adjunction for the t-descent model category structures.

In other words, we have obtained the following result.

Corollary 5.1.15. *Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category compatible with an admissible topology t. The $\mathscr{P}$-fibred category $\mathrm{C}(\mathscr{A})$ with the t-descent model category structure defined in 5.1.12 is a symmetric monoidal $\mathscr{P}$-fibred model category. Moreover, it is stable, proper and combinatorial.*

5.1.16. Recall the following consequences of this corollary (see also 1.3.22 for the general theory). Consider a morphism $f : T \to S$ of schemes. Then the pair of adjoint functors (f^*, f_*) admits total left/right derived functors

$$\mathbf{L}f^* : \mathrm{D}(\mathscr{A}_S) \rightleftarrows \mathrm{D}(\mathscr{A}_T) : \mathbf{R}f_*.$$

More precisely, f_* (resp. f^*) preserves t-local (resp. cofibrant) complexes. For any complex K on $\mathscr{A}_S$, $\mathbf{R}f_*(K) = f_*(K')$ (resp. $\mathbf{L}f^*(K) = f^*(K'')$) where $K' \to K$ (resp. $K \to K''$) is a t-local (resp. cofibrant) resolution of K.[21]

When f is a $\mathscr{P}$-morphism, the functor f^* is even exact and thus preserves quasi-isomorphisms. This implies that $\mathbf{L}f^* = f^*$. The functor $f_\sharp$ admits a total left derived functor

$$\mathbf{L}f_\sharp : \mathrm{D}(\mathscr{A}_T) \rightleftarrows \mathrm{D}(\mathscr{A}_S) : \mathbf{R}f_*$$

defined by the formula $\mathbf{L}f_\sharp(K) = f_\sharp(K'')$ for a complex K on $\mathscr{A}_T$ and a cofibrant resolution $K'' \to K$.

Note also that the tensor product (resp. internal Hom) of $\mathrm{C}(\mathscr{A}_S)$ admits a total left derived functor (resp. total right derived functor). For any complexes K and L on $\mathscr{A}_S$, this derived functors are defined by the formula:

$$K \otimes_S^{\mathbf{L}} L = K'' \otimes_S L''$$
$$\mathbf{R}\mathit{Hom}_S(K, L) = \mathit{Hom}_S(K'', L')$$

where $K \to K''$ and $L \to L''$ are cofibrant resolutions and $L' \to L$ is a t-local resolution.

It is now easy to check that these functors define a triangulated $\mathscr{P}$-premotivic category $\mathrm{D}(\mathscr{A})$, which is τ-generated according to 5.1.13.

Definition 5.1.17. Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category compatible with an admissible topology t.

The triangulated $\mathscr{P}$-premotivic category $\mathrm{D}(\mathscr{A})$ defined above is called the *derived $\mathscr{P}$-premotivic category* associated with $\mathscr{A}$.[22]

The geometric section of a $\mathscr{P}$-scheme X/S in the category $\mathrm{D}(\mathscr{A})$ is the complex concentrated in degree 0 equal to the object $M_S(X)$. The triangulated $\mathscr{P}$-fibred category is τ-generated and well generated in the sense of 1.3.13. This means that $\mathrm{D}(\mathscr{A}_S)$ is equal to the following localizing[23] subcategory generated by the family

$$\{M_S(X)\{i\}; X/S \ \mathscr{P}\text{-scheme}, i \in \tau\} \tag{5.1.17.1}$$

Example 5.1.18. Given any admissible topology t, the abelian $\mathscr{P}$-premotivic category $\mathrm{Sh}_t(\mathscr{P}, \Lambda)$ introduced in example 5.1.4 is compatible with t (*cf.* 5.1.10) and defines the derived $\mathscr{P}$-premotivic category $\mathrm{D}(\mathrm{Sh}_t(\mathscr{P}, \Lambda))$.
Remark also that the abelian $\mathscr{P}$-premotivic category $\mathrm{PSh}(\mathscr{P}, \Lambda)$ introduced in example 5.1.1 is compatible with the coarse topology and gives the derived $\mathscr{P}$-premotivic category $\mathrm{D}(\mathrm{PSh}(\mathscr{P}, \Lambda))$.

Remark 5.1.19. Recall from 5.0.17 the canonical dg-structure on $\mathrm{C}(\mathscr{A}_S)$. Then we can define a derived dg-structure by defining for any complexes K and L of $\mathscr{A}_S$, the complex of morphisms:

$$\mathbf{R}\mathrm{Hom}_{\mathscr{A}_S}(K, L) = \mathrm{Hom}^\bullet_{\mathscr{A}_S}(Q(K), R(L))$$

where R and Q are respectively some fibrant and cofibrant (functorial) resolutions for the t-descent model structure. The homotopy category associated with this new dg-structure on $\mathrm{C}(\mathscr{A}_S)$ is the derived category $\mathrm{D}(\mathscr{A}_S)$. Moreover, for any morphism (resp. $\mathscr{P}$-morphism) of schemes f, the

[21] Recall also that fibrant/cofibrant resolutions can be made functorially, because our model categories are cofibrantely generated, so that the left or right derived functors are in fact defined at the level of complexes.

[22] Remark indeed that $\mathrm{D}(\mathscr{A})$ does not depend on the topology t.

[23] *i.e.* triangulated and stable by sums.

pair $(\mathbf{L}f^*, \mathbf{R}f_*)$ (resp. $(\mathbf{L}f_\sharp, f^*)$) is a dg-adjunction. The same is true for the pair of bifunctors $(\otimes_S^{\mathbf{L}}, \mathbf{R}Hom_S)$.

5.1.20. Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$ compatible with a topology t. According to section 3.1.b, the 2-functor $\mathrm{D}(\mathscr{A})$ can be extended to the category of $\mathscr{S}$-diagrams: to any diagram of schemes $\mathscr{X} : I \to \mathscr{S}$ indexed by a small category I, we can associate a symmetric monoidal closed triangulated category $\mathrm{D}(\mathscr{A})(\mathscr{X}, I)$ which coincides with $\mathrm{D}(\mathscr{A})(X)$ when $I = e$, $\mathscr{X} = X$ for a scheme X.

Let us be more specific. The fibred category $\mathscr{A}$ admits an extension to $\mathscr{S}$-diagrams: a section of $\mathscr{A}$ over a diagram of schemes $\mathscr{X} : I \to \mathscr{S}$, indexed by a small category I, is the following data:

(1) A family $(A_i)_{i\in I}$ such that A_i is an object of $\mathscr{A}_{X_i}$.
(2) A family $(a_u)_{u\in Fl(I)}$ such that for any arrow $u : i \to j$ in I, $a_u : u^*(A_j) \to A_i$ is a morphism in $\mathscr{A}_{X_i}$ and this family of morphisms satisfies a cocyle condition (see paragraph 3.1.1).

Then, $\mathrm{D}(\mathscr{A})(\mathscr{X}, I)$ is the derived category of the abelian category $\mathscr{A}(\mathscr{X}, I)$. In particular, objects of $\mathrm{D}(\mathscr{A})(\mathscr{X}, I)$ are complexes of sections of $\mathscr{A}$ over $(\mathscr{X}, I)$ (or, what amount to the same thing, families of complexes $(K_i)_{i\in I}$ with transition maps (a_u) as above, relative to the fibred category $\mathrm{C}(\mathscr{A})$).

Recall that a morphism of $\mathscr{S}$-diagrams $\varphi : (\mathscr{X}, I) \to (\mathscr{Y}, J)$ is given by a functor $f : I \to J$ and a natural transformation $\varphi : \mathscr{X} \to \mathscr{Y} \circ f$. We say that φ is a $\mathscr{P}$-morphism if for any $i \in I$, $\varphi_i : \mathscr{X}_i \to \mathscr{Y}_{f(i)}$ is a $\mathscr{P}$-morphism. For any morphism (resp. $\mathscr{P}$-morphism) φ, we have defined in 3.1.3 adjunctions of (abelian) categories:

$$\varphi^* : \mathscr{A}(\mathscr{Y}, J) \rightleftarrows \mathscr{A}(\mathscr{X}, I) : \varphi_*$$
$$\text{resp. } \varphi_\sharp : \mathscr{A}(\mathscr{X}, I) \rightleftarrows \mathscr{A}(\mathscr{Y}, J) : \varphi^*$$

which extends the adjunctions we had on trivial diagrams.

According to proposition 3.1.11, these respective adjunctions admits left/right derived functors as follows:

$$\mathbf{L}\varphi^* : \mathrm{D}(\mathscr{A})(\mathscr{Y}, J) \rightleftarrows \mathrm{D}(\mathscr{A})(\mathscr{X}, I) : \mathbf{R}\varphi_* \tag{5.1.20.1}$$
$$\text{resp. } \mathbf{L}\varphi_\sharp : \mathrm{D}(\mathscr{A})(\mathscr{X}, I) \rightleftarrows \mathrm{D}(\mathscr{A})(\mathscr{Y}, J) : \mathbf{L}\varphi^* = \varphi^* \tag{5.1.20.2}$$

Again, these adjunctions coincide on trivial diagrams with the map we already had.

Note also that the symmetric closed monoidal structure on $\mathrm{C}(\mathscr{A}(\mathscr{X}, I))$ can be derived and induces a symmetric monoidal structure on $\mathrm{D}(\mathscr{A})(\mathscr{X}, I)$ (see proposition 3.1.24).[24]

Recall from 3.2.5 and 3.2.7 that, given a topology t' (not necessarily admissible) over $\mathscr{S}$, we say that $\mathrm{D}(\mathscr{A})$ satisfies t'-descent if for any t'-hypercover $p : \mathscr{X} \to X$ (here $\mathscr{X}$ is considered as a $\mathscr{S}$-diagram), the functor

$$\mathbf{L}p^* : \mathrm{D}(\mathscr{A})(X) \to \mathrm{D}(\mathscr{A})(\mathscr{X}) \tag{5.1.20.3}$$

is fully faithful (see corollary 3.2.7).

Proposition 5.1.21. *Consider the notations and hypothesis introduced above. Let t' be an admissible topology on $\mathscr{S}$. Then the following conditions are equivalent:*

(i) $\mathrm{D}(\mathscr{A})$ *satisfies t'-descent.*
(ii) $\mathscr{A}$ *satisfies cohomological t'-descent.*

Proof. We prove (i) implies (ii). Consider a t'-hypercover $p : \mathscr{X} \to X$ in $\mathscr{P}_S$. This is a $\mathscr{P}$-morphism. Thus, by the fully faithfulness of (5.1.20.3), the counit map $\mathbf{L}p_\sharp p^* \to 1$ is an isomorphism. By applying the latter to the unit object $\mathbb{1}_X$ of $\mathrm{D}(\mathscr{A}_X)$, we thus obtain that

$$M_X(\mathscr{X}) \to \mathbb{1}_X$$

is an isomorphism in $\mathrm{D}(\mathscr{A}_X)$. If $\pi : X \to S$ is the structural $\mathscr{P}$-morphism, by applying the functor $\mathbf{L}\pi_\sharp$ to this isomorphism, we obtain that

$$M_S(\mathscr{X}) \to M_S(X)$$

[24] In fact, $\mathrm{D}(\mathscr{A})$ is then a monoidal $\mathscr{P}_{cart}$-fibred category over the category of $\mathscr{S}$-diagrams (remark 3.1.21).

is an isomorphism in $\mathrm{D}(\mathscr{A}_S)$ and this concludes.

Reciprocally, to prove (i), we can restrict to t'-hypercovers $p : \mathscr{X} \to X$ which are $\mathscr{P}$-morphisms because t' is admissible. Because $\mathbf{R}p^* = p^*$ admits a left adjoint $\mathbf{L}p_\sharp$, we have to prove that the counit

$$\mathbf{L}p_\sharp p^* \to 1$$

is an isomorphism. This is a natural transformation between triangulated functors which commutes with small sums. Thus, according to (5.1.17.1), we have only to check this is an isomorphism when evaluated at a complex of the form $M_X(Y)\{i\}$ for a $\mathscr{P}$-scheme Y/X and a twist $i \in \tau$. But the resulting morphism is then $M_X(\mathscr{X} \times_X Y)\{i\} \to M_X(Y)\{i\}$ and we can conclude because $\mathscr{X} \times_X Y \to Y$ is a t'-hypercover in $\mathscr{P}/S$ (again because t' is admissible). □

5.1.22. Consider the situation of 5.1.20 Let S be a scheme. An interesting particular case is given for constant $\mathscr{S}$-diagrams over S; for a small category I, we let I_S be the constant $\mathscr{S}$-diagram $I \to \mathscr{S}, i \mapsto S, u \mapsto 1_S$. Then the adjunctions (5.1.20.1) for this kind of diagrams define a *Grothendieck derivator*

$$I \mapsto \mathrm{D}(\mathscr{A})(I_S).$$

Recall that, if $f : I \to e$ is the canonical functor to the terminal category and $\varphi = f_X : I_X \to X$ the corresponding morphism of $\mathscr{S}$-diagrams, for any I-diagram $K_\bullet = (K_i)_{i\in I}$ of complexes over $\mathscr{A}_S$, we get right derived limits and left derived colimits:

$$\mathbf{R}\varphi_*(K_\bullet) = \mathbf{R}\varprojlim_{i\in I} K_i.$$
$$\mathbf{L}\varphi_\sharp(K_\bullet) = \mathbf{L}\varinjlim_{i\in I} K_i.$$

5.1.23. The associated derived $\mathscr{P}$-premotivic category is functorial in the following sense.

Consider an adjunction

$$\varphi : \mathscr{A} \rightleftarrows \mathscr{B} : \psi$$

of abelian $\mathscr{P}$-premotivic categories. Let τ (resp. τ') be the set of twists of $\mathscr{A}$ (resp. $\mathscr{B}$), and recall that φ induces a morphisms of monoid $\tau \to \tau'$ still denoted by φ. Consider two topologies t and t' such that t' is finer than t. Suppose $\mathscr{A}$ (resp. $\mathscr{B}$) is compatible with t (resp. t') and let $(\mathcal{G}_S^{\mathscr{A}}, \mathcal{H}_S^{\mathscr{A}})$ (resp. $(\mathcal{G}_S^{\mathscr{B}}, \mathcal{H}_S^{\mathscr{B}})$) be the descent structure on $\mathscr{A}_S$ (resp. $\mathscr{B}_S$) defined in 5.1.11.

For any scheme S, consider the evident extensions

$$\varphi_S : \mathrm{C}(\mathscr{A}_S) \rightleftarrows \mathrm{C}(\mathscr{B}_S) : \psi_S$$

of the above adjoint functors to complexes. Recall that for any $\mathscr{P}$-scheme X/S and any twist $i \in \tau$, $\varphi_S(M_S(X, \mathscr{A})\{i\}) = M_S(X, \mathscr{B})\{\varphi(i)\}$ by definition. Thus, φ_S sends $\mathcal{G}_S^{\mathscr{A}}$ to $\mathcal{G}_S^{\mathscr{A}}$. Because t' is finer than t, it sends also $\mathcal{H}_S^{\mathscr{A}}$ to $\mathcal{H}_S^{\mathscr{B}}$. In other words, it satifies descent in the sense of [CD09, par. 2.4] so that the pair (φ_S, ψ_S) is a Quillen adjunction with respect to the respective t-descent and t'-descent model structure on $\mathrm{C}(\mathscr{A}_S)$ and $\mathrm{C}(\mathscr{B}_S)$.

Considering the derived functors, it is now easy to check we have obtained a $\mathscr{P}$-premotivic adjunction[25]

$$\mathbf{L}\varphi : \mathrm{D}(\mathscr{A}) \rightleftarrows \mathrm{D}(\mathscr{B}) : \mathbf{R}\psi.$$

Example 5.1.24. Let t be an admissible topology. Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$ compatible with t. Then the morphism of abelian $\mathscr{P}$-premotivic categories (5.1.2.1) induces a morphism of triangulated $\mathscr{P}$-premotivic categories:

$$\mathbf{L}\gamma^* : \mathrm{D}(\mathrm{PSh}(\mathscr{P}, \mathbf{Z})) \rightleftarrows \mathrm{D}(\mathscr{A}) : \mathbf{R}\gamma_* \tag{5.1.24.1}$$

[25]Remark also that this adjunction extends on $\mathscr{S}$-diagrams considering the situation described in 5.1.20: for any diagram $\mathcal{X} : I \to \mathscr{S}$, we get an adjunction

$$\mathbf{L}\varphi_{\mathcal{X}} : \mathrm{D}(\mathscr{A})(\mathcal{X}) \rightleftarrows \mathrm{D}(\mathscr{B})(\mathcal{X}) : \mathbf{R}\psi_{\mathcal{X}}$$

and this defines a morphism of triangulated monoidal $\mathscr{P}_{cart}$-fibred categories over the $\mathscr{S}$-diagrams (*cf.* proposition 3.1.32).

Similarly, the morphism (5.1.4.1) induces a morphism of triangulated $\mathscr{P}$-premotivic categories

$$a_t^* : \mathrm{D}(\mathrm{PSh}(\mathscr{P}, \Lambda)) \rightleftarrows \mathrm{D}(\mathrm{Sh}_t(\mathscr{P}, \Lambda)) : \mathbf{R}a_{t,*}. \tag{5.1.24.2}$$

Note that $a_t^* = \mathbf{L}a_t^*$ on objects, because the functor a_t^* is exact.

The following result can be used to check the compatibility to a given admissible topology:

Proposition 5.1.25. *Let t be an admissible topology. Consider a morphism of abelian $\mathscr{P}$-premotivic categories*

$$\varphi : \mathscr{A} \rightleftarrows \mathscr{B} : \psi$$

such that:

(a) For any scheme S, ψ_S is exact.

(b) The morphism φ induces an isomorphism of the underlying set of twists of $\mathscr{A}$ and $\mathscr{B}$.

According to the last property, we identify the set of twists of $\mathscr{A}$ and $\mathscr{B}$ to a monoid τ in such a way that φ acts on τ by the identity.

Assume that $\mathscr{A}$ is compatible with t. Then the following conditions are equivalent:

(i) $\mathscr{B}$ is compatible with t.

(ii) $\mathscr{B}$ satisfies cohomological t-descent,

Proof. The fact (i) implies (ii) is clear from the definition and we prove the converse using the following lemma :

Lemma 5.1.26. *Consider a morphism of $\mathscr{P}$-premotivic abelian categories*

$$\varphi : \mathscr{A} \rightleftarrows \mathscr{B} : \psi$$

satisfying conditions (a) and (b) of the above proposition and a base scheme S.

Given a simplicial $\mathscr{P}$-scheme $\mathcal{X}$ over S, a twist $i \in \tau$ and a complex C over $\mathscr{B}_S$, we denote by

$$\epsilon_{\mathcal{X},i,C} : \mathrm{Hom}_{\mathrm{C}(\mathscr{B}_S)}\big(M_S(\mathcal{X}, \mathscr{B})\{i\}, C\big) \to \mathrm{Hom}_{\mathrm{C}(\mathscr{A}_S)}\big(M_S(\mathcal{X}, \mathscr{A})\{i\}, \psi_S(C)\big)$$

the adjunction isomorphism obtained for the adjoint pair (φ_S, ψ_S).
Then there exists a unique isomorphism $\epsilon'_{\mathcal{X},i,C}$ making the following diagram commutative:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{C}(\mathscr{B}_S)}\big(M_S(\mathcal{X}, \mathscr{B})\{i\}, C\big) & \xrightarrow{\epsilon_{\mathcal{X},i,C}} & \mathrm{Hom}_{\mathrm{C}(\mathscr{A}_S)}\big(M_S(\mathcal{X}, \mathscr{A})\{i\}, \psi_S(C)\big) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{K}(\mathscr{B}_S)}\big(M_S(\mathcal{X}, \mathscr{B})\{i\}, C\big) & \xrightarrow{\epsilon'_{\mathcal{X},i,C}} & \mathrm{Hom}_{\mathrm{K}(\mathscr{A}_S)}\big(M_S(\mathcal{X}, \mathscr{A})\{i\}, \psi_S(C)\big). \end{array}$$

Assume moreover that $\mathscr{B}$ satisfies cohomological t-descent.
Then there exists an isomorphism $\epsilon''_{\mathcal{X},i,C}$ making the following diagram commutative:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{K}(\mathscr{B}_S)}\big(M_S(\mathcal{X}, \mathscr{B})\{i\}, C\big) & \xrightarrow{\epsilon'_{\mathcal{X},i,C}} & \mathrm{Hom}_{\mathrm{K}(\mathscr{A}_S)}\big(M_S(\mathcal{X}, \mathscr{A})\{i\}, \psi_S(C)\big) \\ \downarrow{\scriptstyle \pi^{\mathscr{B}}_{\mathcal{X},i,C}} & & \downarrow{\scriptstyle \pi^{\mathscr{A}}_{\mathcal{X},i,C}} \\ \mathrm{Hom}_{\mathrm{D}(\mathscr{B}_S)}\big(M_S(\mathcal{X}, \mathscr{B})\{i\}, C\big) & \xrightarrow{\epsilon''_{\mathcal{X},i,C}} & \mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}\big(M_S(\mathcal{X}, \mathscr{A})\{i\}, \psi_S(C)\big), \end{array} \tag{5.1.26.1}$$

where $\pi^{\mathscr{A}}_{\mathcal{X},i,C}$ and $\pi^{\mathscr{B}}_{\mathcal{X},i,C}$ are induced by the obvious localization functors.

The existence and unicity of isomorphism $\epsilon'_{\mathcal{X},i,C}$ follows from the fact that the functors φ_S and ψ_S are additive. Indeed, this implies that the isomorphism $\epsilon_{\mathcal{X},i,C}$ is compatible with chain homotopies.

Consider the injective model structure on $\mathrm{C}(\mathscr{A}_S)$ and $\mathrm{C}(\mathscr{B}_S)$ (see for example [CD09, 1.2] for the definition). We first treat the case when C is fibrant for this model structure on $\mathrm{C}(\mathscr{B}_S)$. Because the premotive $M_S(\mathcal{X}, \mathscr{B})\{i\}$ is cofibrant for the injective model structure, we obtain that the canonical map $\pi^{\mathscr{B}}_{\mathcal{X},i,C}$ is an isomorphism. This implies there exists a unique map $\epsilon''_{\mathcal{X},i,C}$ making diagram (5.1.26.1) commutative. On the other hand, the isomorphism $\epsilon'_{\mathcal{X},i,C}$ obtained previously

is obviously functorial in $\mathcal{X}$. Thus, because $\mathscr{B}$ satisfies t-descent, we obtain that $\psi_S(C)$ is t-flasque. Because $\mathscr{A}$ is compatible with t, this implies $\psi_S(C)$ is t-local, and because $M_S(\mathcal{X},\mathscr{B})\{i\}$ is cofibrant for the t-descent model structure on $\mathrm{C}(\mathscr{A}_S)$, this implies $\pi^{\mathscr{B}}_{\mathcal{X},i,C}$ is an isomorphism. Thus finally, $\epsilon''_{\mathcal{X},i,C}$ is an isomorphism as required.

To treat the general case, we consider a fibrant resolution $C \to D$ for the injective model structure on $\mathrm{C}(\mathscr{B}_S)$. Because ψ_S is exact, it preserves isomorphisms. Using the previous case, We define $\epsilon''_{\mathcal{X},i,C}$ by the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{D}(\mathscr{B}_S)}\big(M_S(\mathcal{X},\mathscr{B})\{i\},C\big) & \xrightarrow{\epsilon''_{\mathcal{X},i,C}} & \mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}\big(M_S(\mathcal{X},\mathscr{A})\{i\},\psi_S(C)\big) \\ \sim\downarrow & & \downarrow\sim \\ \mathrm{Hom}_{\mathrm{D}(\mathscr{B}_S)}\big(M_S(\mathcal{X},\mathscr{B})\{i\},D\big) & \xrightarrow{\epsilon''_{\mathcal{X},i,D}} & \mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}\big(M_S(\mathcal{X},\mathscr{A})\{i\},\psi_S(D)\big). \end{array}$$

The required property for $\epsilon''_{\mathcal{X},i,C}$ then follows easily and the lemma is proved.

To finish the proof that (ii) implies (i), we note the lemma immediately implies, under (ii), that the following two conditions are equivalent :

- C is t-flasque (resp. t-local) in $\mathrm{C}(\mathscr{B}_S)$;
- $\psi_S(C)$ is t-flasque (resp. t-local) in $\mathrm{C}(\mathscr{A}_S)$.

This concludes. □

5.1.c. *Compacity and geometric triangulated premotives.*

Definition 5.1.27. Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category compatible with an admissible topology t. We will say that t is *bounded in* $\mathscr{A}$ if for any scheme S, there exists an essentially small family $\mathcal{N}^t_S$ of bounded complexes which are direct factors of finite sums of objects of type $M_S(X,\mathscr{A})\{i\}$ in each degree, such that, for any complex C of $\mathscr{A}_S$, the following conditions are equivalent.

(i) C is t-local.
(ii) For any H in $\mathcal{N}^t_S$, the abelian group $\mathrm{Hom}_{K(\mathscr{A}_S)}(H,C)$ vanishes.

In this case, we say the family $\mathcal{N}^t_S$ is a *bounded generating family for t-hypercoverings in $\mathscr{A}_S$.*

Example 5.1.28. (1) Assume $\mathscr{P}$ contains the open immersions so that the Zariski topology is admissible. Let MV_S to be the family of complexes of the form

$$\Lambda_S(U\cap V) \xrightarrow{l_*-k_*} \Lambda_S(U)\oplus\Lambda_S(V) \xrightarrow{i_*+j_*} \Lambda_S(X)$$

for any open cover $X = U \cup V$, where i,j,k,l denotes the obvious open immersions. It follows then from [BG73] that MV_S is a bounded generating family of Zariski hypercovers in $\mathrm{Sh}_{\mathrm{Zar}}(\mathscr{P}/S,\Lambda)$.

(2) Assume $\mathscr{P}$ contains the étale morphisms so that the Nisnevich topology is admissible. We let BG_S be the family of complexes of the form

$$\Lambda_S(W) \xrightarrow{g_*-l_*} \Lambda_S(U)\oplus\Lambda_S(V) \xrightarrow{j_*+f_*} \Lambda_S(X)$$

for a Nisnevich distinguished square in $\mathscr{S}$ (*cf.* 2.1.15)

$$\begin{array}{ccc} W & \xrightarrow{l} & V \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & X. \end{array}$$

Then, by applying 3.3.2, we see that BG_S is a bounded generating family for Nisnevich hypercovers in $\mathrm{Sh}_{\mathrm{Nis}}(\mathscr{P}/S,\Lambda)$.

(3) Assume that $\mathscr{P} = \mathscr{S}^{ft}$ is the class of morphisms of finite type in $\mathscr{S}$. We let $PCDH_S$ be the family of complexes of the form

$$\Lambda_S(T) \xrightarrow{g_*-k_*} \Lambda_S(Z)\oplus\Lambda_S(Y) \xrightarrow{i_*+f_*} \Lambda_S(X)$$

for a cdh-distinguished square in $\mathscr{S}$ (*cf.* 2.1.15)

$$\begin{array}{ccc} T & \xrightarrow{k} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X. \end{array}$$

Then, by virtue of 3.3.7, $CDH_S = BG_S \cup PCDH_S$ is a bounded generating family for cdh-hypercovers in $\mathrm{Sh}_{\mathrm{cdh}}\big(\mathscr{S}^{ft}/S, \Lambda\big)$.

(4) The étale topology is not bounded in $\mathrm{Sh}_{\text{ét}}(Sm, \Lambda)$ for an arbitray ring Λ. However, if $\Lambda = \mathbf{Q}$, it is bounded: by virtue of theorem 3.3.22, a bounded generating family for étale hypercoverings in $\mathrm{Sh}_{\text{ét}}(Sm, \mathbf{Q})_S$ is just BG_S (to be more precise, we should add the complexes of the form $\mathbf{Q}_S(Y)_G \to \mathbf{Q}_S(X)$ for any Galois cover $Y \to X$ of group G, but these are contractible (because $Y/G \simeq X$), so that we can just drop them).

(5) As in the case of étale topology, the qfh-topology is not bounded in general, but it is with rational coefficients. Let $PQFH_S$ be the family of complexes of the form

$$\mathbf{Q}_S(T)_G \xrightarrow{g_* - k_*} \mathbf{Q}_S(Z) \oplus \mathbf{Q}_S(Y)_G \xrightarrow{i_* + f_*} \mathbf{Q}_S(X)$$

for a qfh-distinguished square of group G in $\mathscr{S}$ (*cf.* 3.3.14)

$$\begin{array}{ccc} T & \xrightarrow{k} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X. \end{array}$$

Then, by virtue of theorem 3.3.24, $QFH_S = PQFH_S \cup BG_S$ is a bounded generating family for qfh-hypercoverings in $\mathrm{Sh}_{\mathrm{qfh}}\big(\mathscr{S}^{ft}/S, \mathbf{Q}\big)$.

(6) Similarly, by theorem 3.3.29, $H_S = CDH_S \cup QFH_S$ is a bounded generating family for h-hypercoverings in $\mathrm{Sh}_{\mathrm{h}}\big(\mathscr{S}^{ft}/S, \mathbf{Q}\big)$.

Proposition 5.1.29. *Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category compatible with an admissible topology t. We make the following assumptions:*

(a) t is bounded in $\mathscr{A}$;

(b) for any $\mathscr{P}$-morphism $X \to S$ and any $n \in \tau$, the functor $\mathrm{Hom}_{\mathscr{A}_S}(M_S(X)\{n\}, -)$ preserves filtered colimits.

Then t-local complexes are stable by filtering colimits.

Proof. Let $\mathcal{N}_S^t$ is a bounded generating family for t-hypercovers in $\mathscr{A}_S$. Then a complex C of $\mathscr{A}_S$ is t-flasque if and only if for any $H \in \mathcal{N}_S^t$, the abelian group $\mathrm{Hom}_{K(\mathscr{A}_S)}(H, C)$ is trivial. Hence it is sufficient to prove that the functor

$$C \mapsto \mathrm{Hom}_{K(\mathscr{A}_S)}(H, C))$$

preserves filtering colimits of complexes. This will follow from the fact that the functor

$$C \mapsto \mathrm{Hom}_{\mathrm{C}(\mathscr{A}_S)}(H, C))$$

preserves filtering colimits. As H a is bounded complex that is degreewise compact, this latter property is obvious. □

5.1.30. Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$ compatible with an admissible topology t, with generating set of twists τ. Assume that t is bounded in $\mathscr{A}$ and consider a bounded generating family $\mathscr{N}_S^t$ for t-hypercovers in $\mathscr{A}_S$.

Let $M(\mathscr{P}/S, \mathscr{A})$ be the full subcategory of $\mathscr{A}_S$ spanned by direct factors of finite sums of premotives of shape $M_S(X)\{i\}$ for a $\mathscr{P}$-scheme X/S and a twist $i \in \tau$. This category is additive and we can associate to it its category of complexes up to chain homotopy. We get an obvious triangulated functor

$$K^b\big(M(\mathscr{P}/S, \mathscr{A})\big) \to \mathrm{D}(\mathscr{A}_S). \tag{5.1.30.1}$$

Then the previous functor induces a triangulated functor

$$K^b\big(M(\mathscr{P}/S, \mathscr{A})\big)/\mathcal{N}_S^t \to \mathrm{D}(\mathscr{A}_S)$$

where the left hand side stands for the Verdier quotient of $K^b\big(M(\mathscr{P}/S,\mathscr{A})\big)$ by the thick subcategory generated by $\mathcal{N}_S^t$.

The category $K^b\big(M(\mathscr{P}/S,\mathscr{A})\big)/\mathcal{N}_S^t$ may not be pseudo-abelian while the aim of the previous functor is. Thus we can consider its pseudo-abelian envelope and the induced functor

$$\Big(K^b\big(M(\mathscr{P}/S,\mathscr{A})\big)/\mathcal{N}_S^t\Big)^\natural \to \mathrm{D}(\mathscr{A}_S). \tag{5.1.30.2}$$

Following definitions, the image of this functor is made of geometric premotives of the triangulated $\mathscr{P}$-premotivic category $\mathrm{D}(\mathscr{A}_S)$.

Then the following proposition is a corollary of [CD09, theorem 6.2] :

Proposition 5.1.31. *Consider thef hypothesis and notations above.*

If $\mathscr{A}$ is compactly τ-generated then $\mathrm{D}(\mathscr{A})$ *is compactly τ-generated. Moreover, the functor* (5.1.30.2) *is fully faithful.*

Recall from definition 1.4.7 that the category $\mathrm{D}_c(\mathscr{A})$ of τ-constructible premotives in $\mathrm{D}(\mathscr{A}_S)$ is the thick triangulated subcategory generated by premotives of shape $M_S(X)\{i\}$.

Corollary 5.1.32. *Consider the situation of 5.1.30, and assume that $\mathscr{A}$ is compactly τ-generated. For any premotive $\mathcal{M}$ in* $\mathrm{D}(\mathscr{A}_S)$*, the following conditions are equivalent:*

(i) $\mathcal{M}$ is compact.
(ii) $\mathcal{M}$ is τ-geometric.

Moreover, the functor (5.1.30.2) *induces an equivalence of categories:*

$$\Big(K^b\big(M(\mathscr{P}/S,\mathscr{A})\big)/\mathcal{N}_S^t\Big)^\natural \to \mathrm{D}_c(\mathscr{A}_S).$$

Example 5.1.33. According to example 5.1.28, we get the following examples:

(1) Let $\Lambda(Sm/S) = M(Sm/S,\mathscr{A})$ for $\mathscr{A} = \mathrm{Sh}_{\mathrm{Nis}}(Sm/S,\Lambda)$. We obtain a fully faithful functor
$$\Big(\mathrm{K}^b\left(\Lambda(Sm/S)\right)/BG_S\Big)^\natural \to \mathrm{D}\left(\mathrm{Sh}_{\mathrm{Nis}}(Sm/S,\Lambda)\right).$$
which is essentially surjective on compact objects.

(2) Let $\Lambda(\mathscr{S}_S^{ft}) = M(Sm/S,\mathscr{A})$ for $\mathscr{A} = \mathrm{Sh}_{\mathrm{cdh}}\big(\mathscr{S}^{ft}/S,\Lambda\big)$. We obtain a fully faithful functor
$$\Big(\mathrm{K}^b\left(\Lambda(\mathscr{S}^{ft}/S)\right)/BG_S \cup CDH_S\Big)^\natural \to \mathrm{D}\Big(\mathrm{Sh}_{\mathrm{cdh}}(\mathscr{S}^{ft}/S,\Lambda)\Big).$$
which is essentially surjective on compact objects.

(3) Let $\mathbf{Q}_{\text{ét}}(Sm/S) = M(Sm/S,\mathscr{A})$ for $\mathscr{A} = \mathrm{Sh}_{\text{ét}}(Sm/S,\mathbf{Q})$. We obtain a fully faithful functor
$$\Big(\mathrm{K}^b\left(\mathbf{Q}_{\text{ét}}(Sm/S)\right)/BG_S\Big)^\natural \to \mathrm{D}\left(\mathrm{Sh}_{\text{ét}}(Sm/S,\mathbf{Q})\right).$$
which is essentially surjective on compact objects.

5.2. The $\mathbf{A}^1$-derived premotivic category.

5.2.a. *Localization of triangulated premotivic categories.*

5.2.1. Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category compatible with an admissible topology t and $\mathrm{D}(\mathscr{A})$ be the associated derived $\mathscr{P}$-premotivic category.

Suppose given an essentially small family of morphisms $\mathscr{W}$ in $\mathrm{C}(\mathscr{A})$ which is stable by the operations f^*, $f_\sharp$ (in other words, $\mathscr{W}$ is a sub-$\mathscr{P}$-fibred category of $\mathrm{C}(\mathscr{A})$). Remark that the localizing subcategory $\mathcal{T}$ of $\mathrm{D}(\mathscr{A})$ generated by the cones of arrows in $\mathscr{W}$ is again stable by these operations. Moreover, as for any $\mathscr{P}$-morphism $f : X \to S$ we have $f_\sharp f^* = M_S(X) \otimes_S (-)$, the category $\mathcal{T}$ is stable by tensor product with a geometric section.

We will say that a complex K over $\mathscr{A}_S$ is *$\mathscr{W}$-local* if for any object T of $\mathcal{T}$ and any integer $n \in \mathbf{Z}$, $\mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(T, K[n]) = 0$. A morphism of complexes $p : C \to D$ over $\mathscr{A}_S$ is a *$\mathscr{W}$-equivalence* if for any $\mathscr{W}$-local complex K over $\mathscr{A}_S$, the induced map

$$\mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(D,K) \to \mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(C,K)$$

is bijective.

A morphism of complexes over $\mathscr{A}_S$ is called a *$\mathscr{W}$-fibration* if it is a t-fibration with a $\mathscr{W}$-local kernel. A complex over $\mathscr{A}_S$ will be called *$\mathscr{W}$-fibrant* if it is t-local and $\mathscr{W}$-local.

As consequence of [CD09, 4.3, 4.11 and 5.6], we obtain :

Proposition 5.2.2. *Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category compatible with an admissible topology t and $\mathscr{W}$ be an essentially small family of morphisms in $\mathrm{C}(\mathscr{A})$ stable by f^* and $f_\sharp$.*

Then the category $\mathrm{C}(\mathscr{A}_S)$ is a proper closed symmetric monoidal category with the $\mathscr{W}$-fibrations as fibrations, the cofibrations as defined in 5.1.11, and the $\mathscr{W}$-equivalences as weak equivalences.

The homotopy category associated with this model category will be denoted by $\mathrm{D}(\mathscr{A}_S)[\mathscr{W}_S^{-1}]$. Ti can be described as the Verdier quotient $\mathrm{D}(\mathscr{A}_S)/\mathcal{T}_S$.

In fact, the $\mathscr{W}$-local model category on $\mathrm{C}(\mathscr{A}_S)$ is nothing else than the left Bousfield localization of the t-local model category structure. As a consequence, we obtain an adjunction of triangulated categories:

$$\pi_S : \mathrm{D}(\mathscr{A}_S) \rightleftarrows \mathrm{D}(\mathscr{A}_S)[\mathscr{W}_S^{-1}] : \mathcal{O}_S \tag{5.2.2.1}$$

such that $\mathcal{O}_S$ is fully faithful with essential image the $\mathscr{W}$-local complexes. In fact, the model structure gives a functorial $\mathscr{W}$-fibrant resolution $1 \to R_{\mathscr{W}}$

$$R_{\mathscr{W}} : \mathrm{C}(\mathscr{A}_S) \to \mathrm{C}(\mathscr{A}_S)\,,$$

which induces $\mathcal{O}_S$.

Note that the triangulated category $\mathrm{D}(\mathscr{A}_S)[\mathscr{W}_S^{-1}]$ is generated by the complexes concentrated in degree 0 of the form $M_S(X)\{i\}$ – or, equivalently, the $\mathscr{W}$-local complexes $R_{\mathscr{W}}(M_S(X)\{i\})$ – for a $\mathscr{P}$-scheme X and a twist $i \in \tau$.

Remark 5.2.3. Another very useful property is that $\mathscr{W}$-equivalences are stable by filtering colimits; see [CD09, prop. 3.8].

5.2.4. Recall from 5.1.14 that for any morphism (resp. $\mathscr{P}$-morphism) $f : T \to S$, the functor f^* (resp. $f_\sharp$) satisfies descent; as it also preserves $\mathscr{W}$, it follows from [CD09, 4.9] that the adjunction

$$f^* : \mathrm{C}(\mathscr{A}_S) \to \mathrm{C}(\mathscr{A}_T) : f_*$$
$$(\text{resp. } f_\sharp : \mathrm{C}(\mathscr{A}_S) \to \mathrm{C}(\mathscr{A}_T) : f^*)$$

is a Quillen adjunction with respect to the $\mathscr{W}$-local model structures. This gives the following corollary.

Corollary 5.2.5. *The $\mathscr{P}$-fibred category $\mathrm{C}(\mathscr{A})$ with the $\mathscr{W}$-local model structure on its fibers defined above is a monoidal $\mathscr{P}$-fibred model category, which is moreover stable, proper and combinatorial.*

We will denote by $\mathrm{D}(\mathscr{A})[\mathscr{W}^{-1}]$ the triangulated $\mathscr{P}$-premotivic category whose fiber over a scheme S is the homotopy category of the $\mathscr{W}_S$-local model category $\mathrm{C}(\mathscr{A}_S)$. The adjunction (5.2.2.1) readily defines an adjuntion of triangulated $\mathscr{P}$-premotivic categories

$$\pi : \mathrm{D}(\mathscr{A}) \rightleftarrows \mathrm{D}(\mathscr{A})[\mathscr{W}^{-1}] : \mathcal{O}. \tag{5.2.5.1}$$

The $\mathscr{P}$-fibred categories $\mathrm{D}(\mathscr{A})$ and $\mathrm{D}(\mathscr{A})[\mathscr{W}^{-1}]$ are both τ-generated (and this adjunction is compatible with τ-twists in a strong sense).

Remark 5.2.6. For any scheme S, the category $\mathrm{D}(\mathscr{A}_S)[\mathscr{W}_S^{-1}]$ is well generated and has a canonical dg-structure (see also 5.1.19).

5.2.7. With the notations above, let us put $\mathscr{T} = \mathrm{D}(\mathscr{A})[\mathscr{W}^{-1}]$ to clarify the following notations. As in 5.1.20, the fibred category $\mathscr{T}$ has a canonical extension to diagram of schemes $\mathscr{X} : I \to \mathscr{S}$.

If we define $\mathscr{W}_{\mathscr{X}}$ as the class of morphisms $(f_i)_{i\in I}$ in $\mathrm{C}(\mathscr{A}(\mathscr{X}, I))$ such that for any object i, f_i is a $\mathscr{W}$-equivalence, then $\mathscr{T}(X)$ is the triangulated category $\mathrm{D}(\mathscr{A}(\mathscr{X}, I))[\mathscr{W}_{\mathscr{X}}]$.

Again, this triangulated category is symmetric monoidal closed and for any morphism (resp. $\mathscr{P}$-morphism) $\varphi : (\mathscr{X}, I) \to (\mathscr{Y}, J)$, we get (derived) adjunctions as in 5.1.20:

$$\mathbf{L}\varphi^* : \mathscr{T}(\mathscr{Y}, J) \rightleftarrows \mathscr{T}(\mathscr{X}, I) : \mathbf{R}\varphi_* \tag{5.2.7.1}$$

$$(\text{resp. } \mathbf{L}\varphi_\sharp : \mathscr{T}(\mathscr{X}, I) \rightleftarrows \mathscr{T}(\mathscr{Y}, J) : \mathbf{L}\varphi^* = \varphi^*) \tag{5.2.7.2}$$

In fact, $\mathscr{T}$ is then a complete monoidal $\mathscr{P}_{cart}$-fibred category over the category of diagrams of schemes and the adjunction (5.2.5.1) extends to an adjunction of complete monoidal $\mathscr{P}_{cart}$-fibred categories.

Example 5.2.8. Suppose we are under the hypothesis of example 5.1.24.2.

Let $\mathscr{W}_{t,S}$ denote the family of maps which are of the form $\Lambda_S(\mathcal{X}) \to \Lambda_S(X)$ for a t-hypercovering $\mathcal{X} \to X$ in $\mathscr{P}/S$. Then $\mathscr{W}_t$ is obviously stable by f^* and $f_\sharp$.

Recall now that a complex of t-sheaves on $\mathscr{P}/S$ is local if and only if its t-hypercohomology and its hypercohomology computed in the coarse topology agree (*cf.* 5.1.10).

This readily implies the adjunction considered in example 5.1.24.2

$$a_t^* : \mathrm{D}(\mathrm{PSh}(\mathscr{P}, \Lambda)) \rightleftarrows \mathrm{D}(\mathrm{Sh}_t(\mathscr{P}, \Lambda)) : \mathbf{R}a_{t,*}.$$

induces an equivalence of triangulated $\mathscr{P}$-premotivic categories

$$\mathrm{D}(\mathrm{PSh}(\mathscr{P}, \Lambda))[\mathscr{W}_t^{-1}] \rightleftarrows \mathrm{D}(\mathrm{Sh}_t(\mathscr{P}, \Lambda)).$$

Recall $\mathbf{R}a_{t,*}$ is fully faithful and identifies $\mathrm{D}(\mathrm{Sh}_t(S, \Lambda))$ with the full subcategory of $\mathrm{D}(\mathrm{PSh}(S, \Lambda))$ made by t-local complexes.

5.2.9. A triangulated $\mathscr{P}$-premotivic category $(\mathscr{T}, M)$ such that there exists:

(1) an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$ compatible with an admissible topology t_0 on Sm.
(2) an essentially small family $\mathscr{W}$ of morphisms in $\mathrm{C}(\mathscr{A})$ stable by f^* and $f_\sharp$
(3) an adjunction of triangulated $\mathscr{P}$-premotivic categories $\mathrm{D}(\mathscr{A})[\mathscr{W}^{-1}] \simeq \mathscr{T}$

will be called for short a *derived $\mathscr{P}$-premotivic category.* According to convention 5.0(d) and from the above construction, $\mathscr{T}$ is τ-generated for some set of twists τ. [26]

Let us denote simply by $M_S(X)$ the geometric sections of $\mathscr{T}$. In this case, using the morphisms (5.1.24.1) and (5.2.5.1), we get a canonical morphism of triangulated $\mathscr{P}$-premotivic categories:

$$\varphi^* : \mathrm{D}(\mathrm{PSh}(\mathscr{P}, \mathbf{Z})) \rightleftarrows \mathscr{T} : \varphi_*. \tag{5.2.9.1}$$

By definition, for any premotive $\mathcal{M}$, any scheme X and any integer $n \in \mathbf{Z}$, we get a canonical identification:

$$\mathrm{Hom}_{\mathscr{T}(S)}(M_S(X), \mathcal{M}[n]) = H^n\Gamma(X, \varphi_*(\mathcal{M})). \tag{5.2.9.2}$$

Given any simplicial scheme $\mathcal{X}$, we put $M_S(\mathcal{X}) = \varphi^*\big(\mathbf{Z}_S(\mathcal{X})\big)$, so that we also obtain:

$$\mathrm{Hom}_{\mathscr{T}(S)}(M_S(\mathcal{X}), \mathcal{M}[n]) = H^n\big(\,\mathrm{Tot}^\pi\,\Gamma(\mathcal{X}, \mathbf{R}\gamma_*(\mathcal{M}))\big). \tag{5.2.9.3}$$

Proposition 5.2.10. *Consider the above notations and t an admissible topology. The following conditions are equivalent.*

(i) For any t-hypercovering $\mathcal{X} \to X$ in $\mathscr{P}/S$, the induced map $M_S(\mathcal{X}) \to M_S(X)$ is an isomorphism in $\mathscr{T}(S)$.
(i′) For any t-hypercovering $p : \mathcal{X} \to X$ in $\mathscr{P}/S$, the induced functor $\mathbf{L}p^* : \mathscr{T}(X) \to \mathscr{T}(\mathcal{X})$ is fully faithful.
(i″) $\mathscr{T}$ satisfies t-descent.

[26] We will formulate in some remarks below universal properties of some derived $\mathscr{P}$-premotivic categories. When doing so, we will restrict to morphisms of derived $\mathscr{P}$-premotivic categories which can be written as

$$\mathbf{L}\varphi : \mathrm{D}(\mathscr{A}_1)[\mathscr{W}_1^{-1}] \to \mathrm{D}(\mathscr{A}_2)[\mathscr{W}_2^{-1}]$$

for a morphism $\varphi : \mathscr{A}_1 \to \mathscr{A}_2$ of abelian $\mathscr{P}$-premotivic categories compatible with suitable topologies. More natural universal properties could be obtained if one considers the framework of dg-categories or triangulated derivator.

(ii) There exists an essentially unique map $\varphi_t^* : \mathrm{D}(\mathrm{Sh}_t(\mathscr{P}/S, \mathbf{Z})) \to \mathscr{T}(S)$ *making the following diagram essentially commutative:*

$$\begin{array}{ccc} \mathrm{D}(\mathrm{PSh}(\mathscr{P}/S, \mathbf{Z})) & \xrightarrow{\varphi^*} & \mathscr{T}(S) \\ {\scriptstyle a_t}\downarrow & \nearrow_{\varphi_t^*} & \\ \mathrm{D}(\mathrm{Sh}_t(\mathscr{P}/S, \mathbf{Z})) & & \end{array}$$

(ii′) For any complex $C \in \mathrm{C}(\mathrm{PSh}(\mathscr{P}/S, \mathbf{Z}))$ *such that* $a_t(C) = 0$, $\varphi^*(C) = 0$.

(ii″) For any map $f : C \to D$ *in* $\mathrm{C}(\mathrm{PSh}(\mathscr{P}/S, \mathbf{Z}))$ *such that* $a_t(f)$ *is an isomorphism,* $\varphi^*(f)$ *is an isomorphism.*

(iii) There exists an essentially unique map $\varphi_{t*} : \mathscr{T}(S) \to \mathrm{D}(\mathrm{Sh}_t(\mathscr{P}/S, \mathbf{Z}))$ *making the following diagram essentially commutative:*

$$\begin{array}{ccc} \mathrm{D}(\mathrm{PSh}(\mathscr{P}/S, \mathbf{Z})) & \xleftarrow{\varphi_*} & \mathscr{T}(S) \\ {\scriptstyle \mathbf{R}\mathcal{O}_t}\uparrow & \swarrow_{\varphi_{t*}} & \\ \mathrm{D}(\mathrm{Sh}_t(\mathscr{P}/S, \mathbf{Z})) & & \end{array}$$

(iii′) For any premotive $\mathcal{M}$ *in* $\mathscr{T}(S)$*, the complex* $\varphi_*(\mathcal{M})$ *is local.*

(iii″) For any premotive $\mathcal{M}$ *in* $\mathscr{T}(S)$*, any* $\mathscr{P}$*-scheme* X/S *and any integer* $n \in \mathbf{Z}$*,*

$$\mathrm{Hom}_{\mathscr{T}(S)}(M_S(X), \mathcal{M}[n]) = H_t^n(X, \varphi_*(\mathcal{M})).$$

When these conditions are fulfilled for any scheme S*, the functors appearing in (ii) and (iii) induce a morphism of triangulated* $\mathscr{P}$*-premotivic categories:*

$$\varphi_t^* : \mathrm{D}(\mathrm{Sh}_t(\mathscr{P}, \mathbf{Z})) \rightleftarrows \mathscr{T} : \varphi_{t*}.$$

Proof. The equivalence between conditions (i), (i') and (i'') is clear (we proceed as in the proof of 5.1.21). The equivalences $(ii) \Leftrightarrow (ii') \Leftrightarrow (ii'')$ and $(iii) \Leftrightarrow (iii')$ follows from example 5.2.8 and the definition of a localization. The equivalence $(i) \Leftrightarrow (ii'')$ follows again from *loc. cit.* The equivalences $(i) \Leftrightarrow (iii') \Leftrightarrow (iii'')$ follows finally from (5.2.9.2), (5.2.9.3), and the characterisation of a local complex of sheaves (*cf.* 5.1.10). □

Remark 5.2.11. The preceding proposition express the fact that the category $\mathrm{D}(\mathrm{Sh}_t(\mathscr{P}, \mathbf{Z}))$ is the universal derived $\mathscr{P}$-premotivic category satisfying t-descent.

5.2.12. We end this section by making explicit two particular cases of the descent property for derived $\mathscr{P}$-premotivic categories.

Consider a derived $\mathscr{P}$-premotivic category $\mathscr{T}$ with geometric sections M. Considering any diagram $\mathcal{X} : I \to \mathscr{P}/S$ of $\mathscr{P}$-schemes over S, with projection $p : \mathcal{X} \to S$, we can associate a premotive in $\mathscr{T}$:

$$M_S(\mathcal{X}) = \mathbf{L}p_\sharp(\mathbb{1}_S) = \mathbf{L}\varinjlim_{i \in I} M_S(\mathcal{X}_i).$$

In particular, when I is the category $\bullet \to \bullet$, we associate to every S-morphism $f : Y \to X$ of $\mathscr{P}$-schemes over S a canonical[27] *bivariant premotive*

$$M_S(X \xrightarrow{f} Y).$$

When f is an immersion, we will also write $M_S(Y/X)$ for this premotive. Note that in any case, there is a canonical distinguished triangle in $\mathscr{T}(S)$:

$$M_S(X) \xrightarrow{f_*} M_S(Y) \xrightarrow{\pi_f} M_S(X \xrightarrow{f} Y) \xrightarrow{\partial_f} M_S(X)[1].$$

[27] In fact, if $\mathscr{T} = \mathrm{D}(\mathscr{A})[\mathscr{W}^{-1}]$ for an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$, then we can define $M_S(X \to Y)$ as the cone of the morphism of complexes (concentrated in degree 0) $M_S(X) \xrightarrow{f_*} M_S(Y)$.

This triangle is functorial in the arrow f – with respect to *commutative* squares.
Given a commutative square of $\mathscr{P}$-schemes over S

$$\begin{array}{ccc} B & \xrightarrow{e'} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{e} & X \end{array} \tag{5.2.12.1}$$

we will say that the image square in $\mathscr{T}(S)$

$$\begin{array}{ccc} M_S(B) & \xrightarrow{e'_*} & M_S(Y) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ M_S(A) & \xrightarrow{e_*} & M_S(X) \end{array}$$

is *homotopy cartesian*[28] if the premotive associated with diagram 5.2.12.1 is zero.

Proposition 5.2.13. *Consider a derived $\mathscr{P}$-premotivic category $\mathscr{T}$. We assume that $\mathscr{P}$ contains the étale morphisms (resp. $\mathscr{P} = \mathscr{S}^{ft}$). Then, with the above definitions, the following conditions are equivalent:*

(i) $\mathscr{T}$ satisfies Nisnevich (resp. proper cdh *descent.)*
(ii) For any scheme S and any Nisnevich (resp. proper cdh*) distinguished square Q of S-schemes, the square $M_S(Q)$ is homotopy cartesian in $\mathscr{T}(S)$.*
(iii) For any Nisnevich (resp. proper cdh*) distinguished square of shape* (5.2.12.1)*, the canonical map $M_S(Y/B) \xrightarrow{(f/g)_*} M_S(X/A)$ is an isomorphism.*

Moreover, under these conditions, to any Nisnevich (resp. proper cdh*) distinguished square Q of shape* (5.2.12.1)*, we associate a map*

$$\partial_Q : M_S(X) \xrightarrow{\pi_e} M_S(X/A) \xrightarrow{(f/g)_*^{-1}} M_S(Y/B) \xrightarrow{\partial_{e'}} M_S(Y)[1]$$

which defines a distinguished triangle in $\mathscr{T}(S)$:

$$M_S(B) \xrightarrow{\binom{e'_*}{-g_*}} M_Z(Y) \oplus M_S(A) \xrightarrow{(f_*,e_*)} M_S(X) \xrightarrow{\partial_Q} M_S(Y)[1].$$

Proof. The equivalence of (i) and (ii) follows from the theorem of Morel-Voevodsky 3.3.2 (resp. the theorem of Voevodsky 3.3.7). To prove the equivalence of (ii) and (iii), we assume $\mathscr{T} = \mathrm{C}(\mathscr{A})[\mathscr{W}^{-1}]$. Then, the homotopy colimit of a square of shape 5.2.12.1 is given by the complex

$$\mathrm{Cone}\big(\mathrm{Cone}(M_S(B) \to M_S(Y)) \to \mathrm{Cone}(M_S(A) \to M_S(X))\big).$$

This readily proves the needed equivalence, together with the remaining assertion. □

Remark 5.2.14. In the first of the respective cases of the proposition, condition (ii) is what we usually called the *Brown-Gersten* property (BG) for $\mathscr{T}$, whereas condition (iii) can be called the *excision property*. In the second respective case, condition (ii) will be called the *proper* cdh property for the *generalized premotivic category* $\mathscr{T}$. We say also that $\mathscr{T}$ satisfies the (cdh) property if it satisfies condition (ii) with respect to any cdh distinguished square Q.

5.2.b. *The homotopy relation.*

5.2.15. Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category compatible with an admissible topology t.

We consider $\mathscr{W}_{\mathbf{A}^1}$ to be the family of morphisms $M_S(\mathbf{A}^1_X)\{i\} \to M_S(X)\{i\}$ for a $\mathscr{P}$-scheme X/S and an element i of I. The family $\mathscr{W}_{\mathbf{A}^1}$ is obviously stable by f^* and $f_\sharp$.

Definition 5.2.16. Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category compatible with an admissible topology t. With the notation above, we define $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}) = \mathrm{D}(\mathscr{A})[\mathscr{W}_{\mathbf{A}^1}^{-1}]$ and refer to it as the (effective) *$\mathscr{P}$-premotivic $\mathbf{A}^1$-derived category* with coefficients in $\mathscr{A}$.

[28] If $\mathscr{T} = \mathrm{D}(\mathscr{A})[\mathscr{W}^{-1}]$, this amount to say that the diagram obtained of complexes by applying the functor $M_S(-)$ if homotopy cartesian in the $\mathscr{W}$-local model category $\mathrm{C}(\mathscr{A})$.

By definition, the category $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ satisfies the homotopy property (Htp) (see 2.1.3).

Example 5.2.17. We can divide our examples into two types:
1) Assume $\mathscr{P} = Sm$:

Consider the admissible topology $t = \mathrm{Nis}$. Following F. Morel, we define the (effective) $\mathbf{A}^1$-derived category over S to be $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(S,\Lambda) = \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sh}_{\mathrm{Nis}}(Sm_S,\Lambda))$. We indeed get a triangulated premotivic category (see also the construction of [Ayo07b]):

$$\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,\Lambda} := \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sh}_{\mathrm{Nis}}(Sm,\Lambda))\,. \tag{5.2.17.1}$$

We shall also write its fibres

$$\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(S,\Lambda) := \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,\Lambda}(S) \tag{5.2.17.2}$$

for a scheme S. For $\Lambda = \mathbf{Z}$, we shall often write simply

$$\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1} := \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sh}_{\mathrm{Nis}}(Sm,\mathbf{Z}))\,. \tag{5.2.17.3}$$

Another interesting case is when $t = \text{ét}$; we get a triangulated premotivic category of *effective étale premotives*:

$$\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sh}_{\text{ét}}(Sm,\Lambda))\,.$$

In each of these cases, we denote by $\Lambda^t_S(X)$ the premotive associated with a smooth S-scheme X.
2) Assume $\mathscr{P} = \mathscr{S}^{ft}$:

Consider the admissible topology $t = \mathrm{qfh}$ (resp. $t = \mathrm{h}$). Following V. Voeovdsky [Voe96], we define the category of *effective t-motives* over S as:

$$\underline{DM}^{\mathit{eff}}_t(S,\Lambda) = \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}\big(\mathrm{Sh}_t\big(\mathscr{S}^{ft},\Lambda\big)/S\big)\,. \tag{5.2.17.4}$$

We will denote simply by $\underline{\Lambda}^t_S(X)$ the corresponding premotive associated with X in $\underline{DM}^{\mathit{eff}}_t(S,\Lambda)$.

Another interesting case is obtained when $t = \mathrm{cdh}$. We get an $\mathbf{A}^1$-derived generalized premotivic category $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}\big(\mathrm{Sh}_{\mathrm{cdh}}\big(\mathscr{S}^{ft},\Lambda\big)\big)$ whose premotives are simply denoted by $\underline{\Lambda}^{\mathrm{cdh}}_S(X)$ for any finite type S-scheme X.

5.2.18. Let C be a complex with coefficients in $\mathscr{A}_S$. According to the general case, we say that C is $\mathbf{A}^1$-*local* if for any $\mathscr{P}$-scheme X/S and any $(i,n) \in I \times \mathbf{Z}$, the map induced by the canonical projection

$$\mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(M_S(X)\{i\}[n], C) \to \mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(M_S(\mathbf{A}^1_X)\{i\}[n], C)$$

is an isomorphism. The adjunction (5.2.2.1) defines a morphism of triangulated $\mathscr{P}$-premotivic categories

$$\mathrm{D}(\mathscr{A}) \rightleftarrows \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$$

such that for any scheme S, $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$ is identified with the full subcategory of $\mathrm{D}(\mathscr{A}_S)$ made of $\mathbf{A}^1$-local complexes.

Fibrant objects for the model category structure on $\mathrm{C}(\mathscr{A}_S)$ appearing in proposition 5.2.2 relatively to $\mathscr{W}_{\mathbf{A}^1}$, simply called $\mathbf{A}^1$-*fibrant* objects, are the t-flasque and $\mathbf{A}^1$-local complexes.

We say a morphism $f : C \to D$ of complexes of $\mathscr{A}_S$ is an $\mathbf{A}^1$-*equivalence* if it becomes an isomorphism in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$. Considering moreover two morphisms $f, g : C \to D$ of complexes of $\mathscr{A}_S$, we say they are $\mathbf{A}^1$-*homotopic* if there exists a morphism of complexes

$$H : M_S(\mathbf{A}^1_S) \otimes_S C \to D$$

such that $H \circ (s_0 \otimes 1_C) = f$ and $H \circ (s_1 \otimes 1_C) = g$, where s_0 and s_1 are respectively induced by the zero and the unit section of $\mathbf{A}^1_S/S$. When f and g are $\mathbf{A}^1$-homotopic, they are equal as morphisms of $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$. We say the morphism $p : C \to D$ is a *strong* $\mathbf{A}^1$-*equivalence* if there exists a morphism $q : D \to C$ such that the morphisms $p \circ q$ and $q \circ p$ are $\mathbf{A}^1$-homotopic to the identity. A complex C is $\mathbf{A}^1$-*contractible* if the map $C \to 0$ is a strong $\mathbf{A}^1$-equivalence.

As an example, for any integer $n \in \mathbf{N}$, and any $\mathscr{P}$-scheme X/SS, the map

$$p_* : M_S(\mathbf{A}^n_X) \to M_S(X)$$

induced by the canonical projection is a strong $\mathbf{A}^1$-equivalence with inverse the zero section $s_{0,*} : M_S(X) \to M_S(\mathbf{A}^n_X)$.

5.2.19. The category $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ is functorial in $\mathscr{A}$.

Let $\varphi : \mathscr{A} \rightleftarrows \mathscr{B} : \psi$ be an adjunction of abelian $\mathscr{P}$-premotivic categories. Consider two topologies t and t' such that t' is finer than t. Suppose $\mathscr{A}$ (resp. $\mathscr{B}$) is compatible with t (resp. t').

For any scheme S, consider the evident extensions $\varphi_S : \mathrm{C}(\mathscr{A}_S) \rightleftarrows \mathrm{C}(\mathscr{B}_S) : \psi_S$ of the above adjoint functors to complexes. We easily check that the functor ψ_S preserves $\mathbf{A}^1$-local complexes. Thus, applying 5.1.23, the pair (φ_S, ψ_S) is a Quillen adjunction for the respective $\mathbf{A}^1$-localized model structure on $\mathrm{C}(\mathscr{A}_S)$ and $\mathrm{C}(\mathscr{B}_S)$; see [CD09, 3.11]. Considering the derived functors, it is now easy to check we have obtained an adjunction

$$\mathbf{L}\varphi : \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}) \rightleftarrows \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{B}) : \mathbf{R}\psi$$

of triangulated $\mathscr{P}$-premotivic categories.

Example 5.2.20. Consider the notations of 5.2.17. In the case where $\mathscr{P} = Sm$, we get from the adjunction of (5.1.24.2) the following adjunction of triangulated premotivic categories

$$a^*_{\text{ét}} : \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,\Lambda} \rightleftarrows \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sh}_{\text{ét}}(Sm, \Lambda)) : \mathbf{R}a_{\text{ét},*}.$$

Example 5.2.21. Let $\mathscr{T}$ be a derived $\mathscr{P}$-premotivic category as in 5.2.9. If $\mathscr{T}$ satisfies the property (Htp), then the canonical morphism (5.2.9.1) induces a morphism

$$\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{PSh}(\mathscr{P}, \mathbf{Z})) \rightleftarrows \mathscr{T}.$$

If moreover $\mathscr{T}$ satisfies t-descent for an admissible topology t, we further obtain as in 5.2.10 a morphism

$$\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sh}_t(\mathscr{P}, \mathbf{Z})) \rightleftarrows \mathscr{T}.$$

Particularly interesting cases are given by $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}$ (resp. $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}\big(\mathrm{Sh}_{\mathrm{cdh}}\big(\mathscr{S}^{ft}, \mathbf{Z}\big)\big)$) which is the universal derived premotivic category (resp. generalized premotivic category), *i.e.* initial premotivic category satisfying Nisnevich descent (resp. cdh descent) and the homotopy property.

We remark the following useful property.

Proposition 5.2.22. *Consider a morphism*

$$\varphi^* : \mathscr{A} \rightleftarrows \mathscr{B} : \varphi_*$$

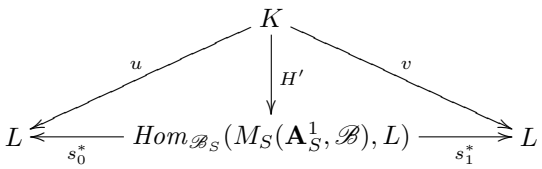
of abelian $\mathscr{P}$-premotivic categories such that $\mathscr{A}$ (resp. $\mathscr{B}$) is compatible with an admissible topology t (resp. t'). Assume t' is finer than t.

Let S be a base scheme. Assume that $\varphi_{*,S}$ commutes with colimits[29]*. Then φ_* respects $\mathbf{A}^1$-equivalences.*

In other words, the right derived functor $\mathbf{R}\varphi_* : \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{B}_S) \to \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$ satisfies the relation $\mathbf{R}\varphi_* = \varphi_*$.

Proof. In this proof, we write φ_* for $\varphi_{*,S}$. We first prove that φ_* preserves strong $\mathbf{A}^1$-equivalences (see 5.2.18).

Consider two maps $u, v : K \to L$ in $\mathrm{C}(\mathscr{B}_S)$. To give an $\mathbf{A}^1$-homotopy $H : M_S(\mathbf{A}^1_S, \mathscr{B}) \otimes_S K \to L$ between u and v is equivalent by adjunction to give a map $H' : K \to Hom_{\mathscr{B}_S}(M_S(\mathbf{A}^1_S, \mathscr{B}), L)$ which fits into the following commutative diagram:



[29]This amounts to ask that $\varphi_{*,S}$ is exact and commutes with direct sums.

where s_0 and s_1 are the respective zero and unit section of $\mathbf{A}^1_S/S$.

Because $M_S(\mathbf{A}^1_S, \mathscr{B}) = \varphi_S^*(M_S(\mathbf{A}^1_S, \mathscr{A}))$, we get a canonical isomorphism (see paragraph 1.2.9)

$$\varphi_*(Hom_{\mathscr{B}_S}(M_S(\mathbf{A}^1_S, \mathscr{B}), L)) \simeq Hom_{\mathscr{B}_S}(M_S(\mathbf{A}^1_S, \mathscr{A}), \varphi_*(L)).$$

Thus, applying φ_* to the previous commutative diagram and using this identification, we obtain that $\varphi_*(u)$ is $\mathbf{A}^1$-homotopic to $\varphi_*(v)$.

As a consequence, for any $\mathscr{P}$-scheme X over S, and any $\mathscr{B}$-twist i, the map

$$\varphi_*(M_S(\mathbf{A}^1_X, \mathscr{B})\{i\}) \to \varphi_*(M_S(X, \mathscr{B})\{i\})$$

induced by the canonical projection is a strong $\mathbf{A}^1$-equivalence, thus an $\mathbf{A}^1$-equivalence.

The functor $\varphi_* : \mathscr{B}_S \to \mathscr{A}_S$ commutes with colimits. Thus it admits a right adjoint that we will denote by $\varphi^!$. Consider the injective model structure on $\mathrm{C}(A_S)$ and $\mathrm{C}(\mathscr{B}_S)$ (see [CD09, 2.1]). Because φ_* is exact, it is a left Quillen functor for these model structures. Thus, the right derived functor $\mathbf{R}\varphi^!$ is well defined. From the result we just get, we see that $\mathbf{R}\varphi^!$ preserves $\mathbf{A}^1$-local objects, and this readily implies $\mathbf{L}\varphi_* = \varphi_*$ preserves $\mathbf{A}^1$-equivalences. □

5.2.23. To relate the category $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(S)$ with the homotopy category of schemes of Morel and Voevodsky [MV99], we have to consider the category of simplicial Nisnevich sheaves of sets denoted by $\Delta^{op}\,\mathrm{Sh}(Sm_S)$. Considering the free abelian sheaf functor, we obtain an adjunction of categories

$$\Delta^{op}\,\mathrm{Sh}(Sm_S) \rightleftarrows \mathrm{C}(\mathrm{Sh}(Sm_S, \mathbf{Z})).$$

If we consider Blander's projective $\mathbf{A}^1$-model structure [Bla03] on the category $\Delta^{op}\,\mathrm{Sh}(Sm_S)$, we can easily see that this is a Quillen pair, so that we obtain a $\mathscr{P}$-premotivic adjunction of simple $\mathscr{P}$-premotivic categories

$$N : \mathscr{H} \rightleftarrows \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1} : K.$$

Note that the functor N sends cofiber sequences in $\mathscr{H}(S)$ to distinguished triangles in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(S)$.

5.2.c. *Explicit $\mathbf{A}^1$-resolution.*

5.2.24. Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$ compatible with an admissible topology t.

Consider the canonically split exact sequence

$$0 \to \mathbb{1}_S \xrightarrow{s_0} M_S(\mathbf{A}^1_X) \to U \to 0$$

where the map $s_0 : \mathbb{1}_S \to M_S(\mathbf{A}^1_X)$ is induced by the zero section of $\mathbf{A}^1$. The section corresponding to 1 in $\mathbf{A}^1$ defines another map

$$s_1 : \mathbb{1}_S \to M_S(\mathbf{A}^1_X)$$

which does not factor through s_0, so that we get canonically a non trivial map $u : \mathbb{1}_S \to U$. This defines for any complex C of $\mathscr{A}_S$ a map, called the *evaluation at* 1,

$$Hom(U, C) = \mathbb{1}_S \otimes_S Hom(U, C) \xrightarrow{u\otimes 1} U \otimes_S Hom(U, C) \xrightarrow{ev} C.$$

We define the complex $R^{(1)}_{\mathbf{A}^1}(C)$ to be

$$R^{(1)}_{\mathbf{A}^1}(C) = \mathrm{Cone}\big(Hom(U, C) \to C\big).$$

We have by construction a map

$$r_C : C \to R^{(1)}_{\mathbf{A}^1}(C).$$

This defines a morphism of functors from the identity functor to $R^{(1)}_{\mathbf{A}^1}$. For an integer $n \geq 1$, we define by induction a complex

$$R^{(n+1)}_{\mathbf{A}^1}(C) = R^{(1)}_{\mathbf{A}^1}(R^{(n)}_{\mathbf{A}^1}(C)),$$

and a map

$$r_{R^{(n)}_{\mathbf{A}^1}(C)} : R^{(n)}_{\mathbf{A}^1}(C) \to R^{(n+1)}_{\mathbf{A}^1}.$$

We finaly define a complex $R_{\mathbf{A}^1}(C)$ by the formula

$$R_{\mathbf{A}^1}(C) = \varinjlim_n R^{(n)}_{\mathbf{A}^1}(C).$$

We have a functorial map

$$C \to R_{\mathbf{A}^1}(C).$$

Lemma 5.2.25. *With the above hypothesis and notations, the map $C \to R_{\mathbf{A}^1}(C)$ is an $\mathbf{A}^1$-equivalence.*

Proof. For any closed symmetric monoidal category $\mathscr{C}$ and any objects A, B, C and I in $\mathscr{C}$, we have

$$\begin{aligned}\mathrm{Hom}(I \otimes \mathit{Hom}(B,C), \mathit{Hom}(A,C)) &= \mathrm{Hom}(\mathit{Hom}(B,C), \mathit{Hom}(I, \mathit{Hom}(A,C)))\\ &= \mathrm{Hom}(\mathit{Hom}(B,C), \mathit{Hom}(I \otimes A, C)).\end{aligned}$$

Hence any map $I \otimes A \to B$ induces a map $I \otimes \mathit{Hom}(B,C) \to \mathit{Hom}(A,B)$ for any object C. If we apply this to $\mathscr{C} = \mathrm{C}(\mathscr{A}_S)$ and $I = M_S(\mathbf{A}^1)$, we see immediately that the functor $\mathit{Hom}(-,C)$ preserves strong $\mathbf{A}^1$-homotopy equivalences. In particular, for any complex C, the map $C \to \mathit{Hom}(M_S(\mathbf{A}^1_X), C)$ is a strong $\mathbf{A}^1$-homotopy equivalence. This implies that $\mathit{Hom}(U,C) \to 0$ is an $\mathbf{A}^1$-equivalence, so that the map r_C is an $\mathbf{A}^1$-equivalence as well. As $\mathbf{A}^1$-equivalences are stable by filtering colimits, this implies our result. □

Proposition 5.2.26. *Consider the above notations and hypothesis, and assume that t is bounded in $\mathscr{A}$.*

For any t-flasque complex C of $\mathscr{A}_S$, the complex $R_{\mathbf{A}^1}(C)$ is t-flasque and $\mathbf{A}^1$-local. Moreover, the morphism $C \to R_{\mathbf{A}^1}(C)$ is an $\mathbf{A}^1$-equivalence. If furthermore C is t-flasque, so is $R_{\mathbf{A}^1}(C)$.

Proof. The last assertion is a particular case of Lemma 5.2.25. The functor $R^{(1)}_{\mathbf{A}^1}$ preserves t-flasque complexes. By virtue of 5.1.29, the functor $R_{\mathbf{A}^1}$ has the same gentle property. It thus remains to prove that the functor $R_{\mathbf{A}^1}$ sends t-flasque complexes on $\mathbf{A}^1$-local ones. We shall use that the derived category $\mathrm{D}(\mathscr{A}_S)$ is compactly generated; see 5.1.29.

Let C be a t-flasque complex of $\mathscr{A}_S$. To prove $R_{\mathbf{A}^1}(C)$ is $\mathbf{A}^1$-local, we are reduced to prove that the map

$$R_{\mathbf{A}^1}(C) \to \mathit{Hom}(M_S(\mathbf{A}^1_X), R_{\mathbf{A}^1}(C))$$

is a quasi-isomorphism, or, equivalently, that the complex $\mathit{Hom}(U, R_{\mathbf{A}^1}(C))$ is acyclic. As U is a direct factor of $M_S(\mathbf{A}^1_X, \mathscr{A})$, for any $\mathscr{P}$-scheme X over S and any i in I, the object $\mathbf{Z}_S(X;\mathscr{A})\{i\} \otimes_S U$ is compact. This implies that the canonical map

$$\varinjlim_n \mathit{Hom}(U, R^{(n)}_{\mathbf{A}^1}(C)) \to \mathit{Hom}(U, R_{\mathbf{A}^1}(C))$$

is an isomorphism of complexes. As filtering colimits preserve quasi-isomorphisms, the complex $\mathit{Hom}(U, R_{\mathbf{A}^1}(C))$ (resp. $R_{\mathbf{A}^1}(C)$) can be considered as the homotopy colimit of the complexes $\mathit{Hom}(U, R^{(n)}_{\mathbf{A}^1}(C))$ (resp. $R^{(n)}_{\mathbf{A}^1}(C)$). In particular, for any compact object K of $\mathrm{D}(\mathscr{A}_S)$, the canonical morphisms

$$\varinjlim_n \mathrm{Hom}(K, \mathit{Hom}(U, R^{(n)}_{\mathbf{A}^1}(C))) \to \mathrm{Hom}(K, \mathit{Hom}(U, R_{\mathbf{A}^1}(C)))$$

$$\varinjlim_n \mathrm{Hom}(K, R^{(n)}_{\mathbf{A}^1}(C)) \to \mathrm{Hom}(K, R_{\mathbf{A}^1}(C))$$

are bijective.

By construction, we have distinguished triangles

$$\mathit{Hom}(U, R^{(n)}_{\mathbf{A}^1}(C)) \to R^{(n)}_{\mathbf{A}^1}(C) \to R^{(n+1)}_{\mathbf{A}^1}(C) \to \mathit{Hom}(U, R^{(n)}_{\mathbf{A}^1}(C))[1].$$

This implies that the evaluation at 1 morphism

$$ev_1 : \mathit{Hom}(U, R_{\mathbf{A}^1}(C)) \to R_{\mathbf{A}^1}(C)$$

induces the zero map

$$\mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(K, \mathit{Hom}(U, R_{\mathbf{A}^1}(C))) \to \mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(K, R_{\mathbf{A}^1}(C))$$

for any compact object K of $\mathrm{D}(\mathscr{A}_S)$. Hence the induced map

$$a = \mathit{Hom}(U, ev_1) : \mathit{Hom}(U, \mathit{Hom}(U, R_{\mathbf{A}^1}(C))) \to \mathit{Hom}(U, R_{\mathbf{A}^1}(C))$$

has the same property: for any compact object K, the map

$$\mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(K, Hom(U, Hom(U, R_{\mathbf{A}^1}(C)))) \to \mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(K, Hom(U, R_{\mathbf{A}^1}(C)))$$

is zero.

The multiplication map $\mathbf{A}^1 \times \mathbf{A}^1 \to \mathbf{A}^1$ induces a map

$$\mu : U \otimes_S U \to U$$

such that the composition of

$$\mu^* : Hom(U, R_{\mathbf{A}^1}(C)) \to Hom(U \otimes_S U, R_{\mathbf{A}^1}(C)) = Hom(U, Hom(U, R_{\mathbf{A}^1}(C)))$$

with a is the identity of $Hom(U, R_{\mathbf{A}^1}(C))$. As $\mathrm{D}(\mathscr{A}_S)$ is compactly generated, this implies that $Hom(U, R_{\mathbf{A}^1}(C)) = 0$ in the derived category $\mathrm{D}(\mathscr{A}_S)$. □

Remark 5.2.27. Consider a t-flasque resolution functor (*i.e.* a fibrant resolution for the t-local model structure) $R_t : \mathrm{C}(\mathscr{A}_S) \to \mathrm{C}(\mathscr{A}_S)$, $1 \to R_t$. As a corollary of the proposition, the composite functor $R_{\mathbf{A}^1} \circ R_t$ is a resolution functor by t-local and $\mathbf{A}^1$-local complexes.

Example 5.2.28. Consider an admissible topology t and the $\mathscr{P}$-premotivic $\mathbf{A}^1$-derived category $D = \mathrm{D}^{eff}_{\mathbf{A}^1}(\mathrm{Sh}_t(\mathscr{P}, \Lambda))$. Suppose that t is bounded for abelian t-sheaves (for example, this is the case for the Zariski and the Nisnevich topologies, see 5.1.28).

Let C be a complex of abelian t-sheaves on $\mathscr{P}/S$. If C is $\mathbf{A}^1$-local, then

$$\mathrm{Hom}_{D(S)}(\Lambda^t_S(X), C) = \mathrm{H}^n_t(X; C)$$

(this is true without any condition on t).
Consider a t-local resolution C_t of C in $\mathrm{C}\big(\mathrm{Sh}_t(\mathscr{P}/S, \Lambda)\big)$. Then we get the following formula:

$$\mathrm{Hom}_{D(S)}\big(\Lambda^t_S(X), C[n]\big) = \mathrm{H}^n\big(\Gamma\big(X, R_{\mathbf{A}^1}(C_t)\big)\big).$$

Corollary 5.2.29. *Consider a morphism of abelian $\mathscr{P}$-premotivic categories*

$$\varphi : \mathscr{A} \rightleftarrows \mathscr{B} : \psi$$

Suppose there are admissible topologies t and t', with t' finer than t, such that the following conditions are verified.

(i) $\mathscr{A}$ is compatible with t and $\mathscr{B}$ is compatible with t'.
(ii) $\mathscr{B}$ and $\mathrm{D}(\mathscr{B})$ are compactly τ-generated.
(iii) For any scheme S, the functor $\psi_S : \mathscr{B}_S \to \mathscr{A}_S$ preserves filtering colimits.

Then, $\psi_S : \mathrm{C}(\mathscr{B}_S) \to \mathrm{C}(\mathscr{A}_S)$ preserves $\mathbf{A}^1$-equivalences between t'-flasque objects. If moreover ψ_S is exact, the functor ψ_S preserves $\mathbf{A}^1$-equivalences.

Proof. We already know that ψ_S is a right Quillen functor, so that it preserves local objects and $\mathbf{A}^1$-fibrant objects. This implies also that ψ_S preserves $\mathbf{A}^1$-equivalences between $\mathbf{A}^1$-fibrant objects (this is Ken Brown's lemma [Hov99, 1.1.12]). Let D be a t'-flasque complex of $\mathscr{B}_S$. Then $\psi_S(D)$ is a t-flasque complex of $\mathscr{A}_S$. It follows from proposition 5.2.26 that $R_{\mathbf{A}^1}(D)$ is $\mathbf{A}^1$-local and that $D \to R_{\mathbf{A}^1}(D)$ is an $\mathbf{A}^1$-equivalence. Lemma 5.2.25 implies the map

$$\psi_S(D) \to R_{\mathbf{A}^1}(\psi_S(D)) = \psi_S(R_{\mathbf{A}^1}(D))$$

is a an $\mathbf{A}^1$-equivalence. This implies the first assertion.

The last assertion is a direct consequence of the first one. □

5.2.30. Consider the usual cosimplicial scheme $\Delta^\bullet$ defined by

$$\Delta^n = \mathrm{Spec}\,(\mathbf{Z}[t_0, \ldots, t_n]/(t_1 + \cdots + t_n - 1)) \simeq \mathbf{A}^n$$

(see [MV99]). For any scheme S, we get a cosimplicial object of $\mathscr{A}_S$, namely $M_S(\Delta^\bullet_S)$. For a complex C of $\mathscr{A}_S$, we put

$$\underline{C}^*(C) = \mathrm{Tot}^\oplus Hom(M_S(\Delta^\bullet_S), C),$$

where $Hom(M_S(\Delta^\bullet_S), C)$ is considered as a bicomplex by the Dold-Kan correspondence. The canonical map $M_S(\Delta^\bullet_S) \to \mathbb{1}_S$ induces a map

$$C \to \underline{C}^*(C).$$

Lemma 5.2.31. *For any complex C of $\mathscr{A}_S$, the map*

$$\underline{C}^*(C) \to Hom(M_S(\mathbf{A}^1_X), \underline{C}^*(C) = \underline{C}^*(Hom(M_S(\mathbf{A}^1_X), C))$$

is a chain homotopy equivalence.

Proof. The composite morphism

$$s_0 p : M_S(\mathbf{A}^1 \times \Delta^\bullet_S) \to M_S(\mathbf{A}^1 \times \Delta^\bullet_S),$$

where s_0 is the map induced by the zero section, and p is the map induced by the obvious projection of $\mathbf{A}^1$ on its base, is chain homotopic to the identity. Indeed, the homotopy relation is given by the formula

$$s_n = \sum_{i=0}^{n} (-1)^i . (1 \otimes_S \psi_i) : M_S(\mathbf{A}^1 \times \Delta^{n+1}_S) \to M_S(\mathbf{A}^1 \times \Delta^n_S)$$

where 1 is the identity of $M_S(\mathbf{A}^1_X)$, and ψ_i is induced by the map $\Delta^{n+1}_S \to \mathbf{A}^1 \times \Delta^n_S$ which sends the j-th vertex $v_{j,n+1}$ to either $0 \times v_{j,n}$, if $j \leq i$, or to $1 \times v_{j-1,n}$ otherwise. This implies the lemma. □

Lemma 5.2.32. *For any t-flasque complex C of $\mathscr{A}_S$, we have a canonical isomorphism*

$$\underline{C}^*(C) \simeq \mathbf{L} \varinjlim_n \mathbf{R}Hom(M_S(\Delta^n_S), C)$$

in $\mathrm{D}(\mathscr{A}_S)$.

This is a variation on the Dold-Kan correspondence. As a direct consequence, we get:

Lemma 5.2.33. *For any complex C of $\mathscr{A}_S$, the map $C \to \underline{C}^*(C)$ is an $\mathbf{A}^1$-equivalence.*

Proposition 5.2.34. *If t is bounded in $\mathscr{A}$, then, for any t-flasque complex C of $\mathscr{A}_S$, $\underline{C}^*(C)$ is $\mathbf{A}^1$-local.*

Proof. Using the first premotivic adjunction of example 5.2.21 and the fact that $\mathrm{D}(\mathscr{A})$ is compactly generated (5.1.29), we can reduce the proposition to the case where $\mathscr{A}_S$ is the category of presheaves of abelian groups over $\mathscr{P}/S$, in which case this is well known. □

5.2.d. *Geometric $\mathbf{A}^1$-local premotives.*

5.2.35. Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$ compatible with an admissible topology t, with generating set of twists τ. Assume that t is bounded in $\mathscr{A}$ and consider a bounded generating family $\mathscr{N}^t_S$ for t-hypercovers in $\mathscr{A}_S$.

Let $\mathcal{T}_{\mathbf{A}^1_S}$ be the family of complexes of $\mathrm{C}(\mathscr{A}_S)$ of shape

$$M_S(\mathbf{A}^1_X)\{i\} \to M_S(X)\{i\}$$

for a $\mathscr{P}$-scheme X over S and a twist $i \in I$. Then the functor (5.1.30.1) obviously induces the following functor

$$\Big(K^b\big(M(\mathscr{P}/S, \mathscr{A})\big)/\mathcal{N}^t_S \cup \mathcal{T}_{\mathbf{A}^1_S}\Big)^\natural \to \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S), \tag{5.2.35.1}$$

where the category on the left is the pseudo-abelian category associated to the Verdier quotient of $K^b\big(M(\mathscr{P}/S, \mathscr{A})\big)$ by the thick subcategory generated by $\mathcal{N}^t_S \cup \mathcal{T}_{\mathbf{A}^1_S}$. Applying Thomason's localization theorem [Nee01], we get from proposition 5.1.31 the following result:

Proposition 5.2.36. *Consider the previous hypothesis and notations. Assume furthermore that, for any $\mathscr{P}$-morphism $X \to S$, and any $n \in \tau$, the object $M_S(X)\{n\}$ is of finite presentation in $\mathscr{A}_S$ (by which we mean that the functor $\mathrm{Hom}_{\mathscr{A}_S}(M_S(X)\{n\}, -)$ preserves filtered colimits). Then $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ is compactly τ-generated. Moreover, the functor* (5.2.35.1) *is fully faithful.*

Corollary 5.2.37. *Under the assumptions of 5.2.36, for any premotive $\mathcal{M}$ in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$, the following conditions are equivalent:*

(i) $\mathcal{M}$ is compact;
(ii) $\mathcal{M}$ is τ-constructible.

Moreover, the functor (5.2.35.1) *induces an equivalence of categories:*

$$\Big(K^b\big(M(\mathscr{P}/S,\mathscr{A})\big)/\mathcal{N}_S^t \cup \mathcal{T}_{\mathbf{A}^1_S}\Big)^\natural \to \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,c}(\mathscr{A}_S).$$

Example 5.2.38. With the notations of 5.1.33, we get the following equivalences of categories:

$$\Big(\mathrm{K}^b\left(\Lambda(Sm_S)\right)/(BG_S \cup \mathcal{T}_{\mathbf{A}^1_S})\Big)^\natural \to \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,c}(S,\Lambda).$$

$$\Big(\mathrm{K}^b\left(\Lambda(\mathscr{S}_S^{ft})\right)/CDH_S \cup \mathcal{T}_{\mathbf{A}^1_S}\Big)^\natural \to \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,c}\left(\mathrm{Sh}_{\mathrm{cdh}}(\mathscr{S}_S^{ft},\Lambda)\right).$$

This statement is the analog of the embedding theorem [VSF00, chap. 5, 3.2.6].

5.3. **The stable $\mathbf{A}^1$-derived premotivic category.**

5.3.a. *Modules.* Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category with generating set of twists τ.

A *cartesian commutative monoid* R of $\mathscr{A}$ is a cartesian section of the fibred category $\mathscr{A}$ over $\mathscr{S}$ such that for any scheme S, R_S has a commutative monoid structure in $\mathscr{A}(S)$ and for any morphism of schemes $f : T \to S$, the structural transition maps $\phi_f : f^*(R_S) \to R_T$ are isomorphisms of monoids.

Let us fix a cartesian commutative monoid R of $\mathscr{A}$.

Consider a base scheme S. We denote by R_S- mod the category of modules in the monoidal category $\mathscr{A}_S$ over the monoid R_S. For any $\mathscr{P}$-scheme X/S and any twist $i \in \tau$, we put

$$R_S(X)\{i\} = R_S \otimes_S M_S(X)\{i\}$$

endowed with its canonical R_S-module structure. The category R_S- mod is a Grothendieck abelian category such that the forgetful functor $U_S : R_S\text{- mod} \to \mathscr{A}_S$ is exact and conservative. A family of generators for R_S- mod is given by the modules $R_S(X)\{i\}$ for a $\mathscr{P}$-scheme X/S and a twist $i \in \tau$. As A_S is commutative, R_S- mod has a unique symmetric monoidal structure such that the free R_S-module functor is symmetric monoidal. We denote by $\otimes_R$ this tensor product. Note that $R_S(X) \otimes_R R_S(Y) = R_S(X \times_S Y)$. Finally the categories of modules R_S- mod form a symmetric monoidal $\mathscr{P}$-fibred category, such that the following proposition holds (see 4.2.9).

Proposition 5.3.1. *Let $\mathscr{A}$ be a τ-generated abelian $\mathscr{P}$-premotivic category and R be a cartesian commutative monoid of $\mathscr{A}$.*

*Then the category R-*mod *equipped with the structures introduced above is a τ-generated abelian $\mathscr{P}$-premotivic category.*

Moreover, we have an adjunction of abelian $\mathscr{P}$-premotivic categories:

$$R_S \otimes (-) : \mathscr{A} \rightleftarrows R\text{-mod} : U\,. \tag{5.3.1.1}$$

Remark 5.3.2. With the hypothesis of the preceding proposition, for any morphism of schemes $f : T \to S$, the exchange transformation $f^*U_S \to U_T f^*$ is an isomorphism by construction of R- mod (4.2.9).

Proposition 5.3.3. *Let $\mathscr{A}$ be a τ-generated abelian $\mathscr{P}$-premotivic category compatible with an admissible topology t. Consider a cartesian commutative monoid R of $\mathscr{A}$ such that for any scheme S, tensoring quasi-isomorphisms between cofibrant complexes by R_S gives quasi-isomorphisms (e.g. R_S might be cofibrant (as a complex concentrated in degree zero), or flat). Then the abelian $\mathscr{P}$-premotivic category R-*mod *is compatible with t.*

Proof. In view of proposition 5.1.25, we have only to show that R- mod satisfies cohomological t-descent. Consider a t-hypercover $p : \mathcal{X} \to X$ in $\mathscr{P}/S$. We prove that the map $p_* : R_S(\mathcal{X}) \to R_S(X)$ is a quasi-isomorphism in $\mathrm{C}(R_S\text{- mod})$. The functor U_S is conservative, and $U_S(p_*)$ is equal to the map:

$$R_S \otimes_S M_S(\mathcal{X}) \to R_S \otimes_S M_S(X).$$

But this is a quasi-isomorphism in $\mathrm{C}(\mathscr{A}_S)$ by assumption on R_S. □

Remark 5.3.4. According to lemma 5.1.26, for any simplicial $\mathscr{P}$-scheme $\mathcal{X}$ over S, any twist $i \in \tau$ and any R_S-module C, we get canonical isomorphisms:

$$\mathrm{Hom}_{\mathrm{K}(R_S\text{-}\mathrm{mod})}\big(R_S(\mathcal{X})\{i\}, C\big) \simeq \mathrm{Hom}_{\mathrm{K}(\mathscr{A}_S)}(M_S(\mathcal{X})\{i\}, C) \tag{5.3.4.1}$$

$$\mathrm{Hom}_{\mathrm{D}(R_S\text{-}\mathrm{mod})}(R_S(\mathcal{X})\{i\}, C) \simeq \mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(M_S(\mathcal{X})\{i\}, C). \tag{5.3.4.2}$$

5.3.b. *Symmetric sequences.* Let $\mathscr{A}$ be an abelian category.

Let G be a group. An action of G on an object $A \in \mathscr{A}_S$ is a morphism of groups $G \to \mathrm{Aut}_{\mathscr{A}}(A), g \mapsto \gamma_g^A$. We say that A is a G-object of $\mathscr{A}$. A G-equivariant morphism $A \xrightarrow{f} B$ of G-objects of $\mathscr{A}$ is a morphism f in $\mathscr{A}$ such that $\gamma_g^B \circ f = f \circ \gamma_g^A$.

If E is any object of $\mathscr{A}$, we put $G \times E = \bigoplus_{g\in G} E$ considered as a G-object via the permutation isomorphisms of the summands.

If H is a subgroup of G, and E is an H-object, $G \times E$ has two actions of H : the first one, say γ, is obtained via the inclusion $H \subset G$, and the second one denoted by γ', is obtained using the structural action of H on E. We define $G \times_H E$ as the equalizer of the family of morphisms $(\gamma_\sigma - \gamma'_\sigma)_{\sigma\in H}$, and consider it equipped with its induced action of G.

Definition 5.3.5. Let $\mathscr{A}$ be an abelian category.

A symmetric sequence of $\mathscr{A}$ is a sequence $(A_n)_{n\in\mathbf{N}}$ such that for each $n \in \mathbf{N}$, E_n is a $\mathfrak{S}_n$-object of $\mathscr{A}$. A morphism of symmetric sequences of $\mathscr{A}$ is a collection of $\mathfrak{S}_n$-equivariant morphism $(f_n : A_n \to B_n)_{n\in\mathbf{N}}$.

We let $\mathscr{A}^{\mathfrak{S}}$ be the category of symmetric sequences of $\mathscr{A}$.

It is straightforward to check $\mathscr{A}^{\mathfrak{S}}$ is abelian. For any integer $n \in \mathbf{N}$, we define the *n-th evaluation functor* as follows:

$$ev_n : \mathscr{A}^{\mathfrak{S}} \to \mathscr{A}, A_* \mapsto A_n.$$

Any object A of $\mathscr{A}$ can be considered as the trivial symmetric sequence $(A, 0, \ldots)$. The functor $i_0 : A \mapsto (A, 0, \ldots)$ is obviously left adjoint to ev_0 and we obtain an adjunction

$$i_0 : \mathscr{A} \rightleftarrows \mathscr{A}^{\mathfrak{S}} : ev_0. \tag{5.3.5.1}$$

Remark i_0 is also right adjoint to ev_0. Thus i_0 preserves every limits and colimits.

For any integer $i \in \mathbf{N}$ and any symmetric sequence A_* of $\mathscr{A}$, we put

$$A_*\{-n\} = m \mapsto \begin{cases} \mathfrak{S}_m \times_{\mathfrak{S}_{m-n}} A_{m-n} & \text{if } m \geq n \\ 0 & \text{otherwise.} \end{cases} \tag{5.3.5.2}$$

This define an endofunctor on $\mathscr{A}^{\mathfrak{S}}$, and we have $A_*\{-n\}\{-m\} = A_*\{-n-m\}$ (through a canonical isomorphism). Remark finally that for any integer $n \in \mathbf{N}$, the functor

$$i_n : \mathscr{A} \to \mathscr{A}^{\mathfrak{S}}, A \mapsto (i_0(A))\{-n\}$$

is left adjoint to ev_n.

Remark 5.3.6. Let $\mathfrak{S}$ be the category of finite sets with bijective maps as morphisms. Then the category of symmetric sequences is canonically equivalent to the category of functors $\mathfrak{S} \to \mathscr{A}$. This presentation is useful to define a tensor product on $\mathscr{A}^{\mathfrak{S}}$.

Definition 5.3.7. Let $\mathscr{A}$ be a symmetric closed monoidal abelian category.

Given two functors $A_*, B_* : \mathfrak{S} \to \mathscr{A}$, we put:

$$\begin{array}{rcl} E \otimes^{\mathfrak{S}} F : \mathfrak{S} & \mapsto & \mathscr{A} \\ N & \mapsto & \bigoplus_{A=P\sqcup Q} E(P) \otimes F(Q). \end{array}$$

If $\mathbb{1}_{\mathscr{A}}$ is the unit object of the monoidal category $\mathscr{A}$, the category $\mathscr{A}^{\mathfrak{S}}$ is then a symmetric closed monoidal category with unit object $i_0(\mathbb{1}_{\mathscr{A}})$.

5.3.8. Let A be an object of $\mathscr{A}$. Then the n-th tensor power $A^{\otimes n}$ of A is endowed with a canonical action of the group $\mathfrak{S}_n$ through the structural permutation isomorphism of the symmetric structure on $\mathscr{A}$. Thus the sequence $Sym(A) = (A^{\otimes n})_{n\in\mathbf{N}}$ is a symmetric sequence.

Moreover, the isomorphism $A^{\otimes n} \otimes_{\mathscr{A}} A^{\otimes m} \to A^{\otimes n+m}$ is $\mathfrak{S}_n \times \mathfrak{S}_m$-equivariant. Thus it induces a morphism $\mu : Sym(A) \otimes^{\mathfrak{S}}_{\mathscr{A}} Sym(A) \to Sym(A)$ of symmetric sequences. We also consider the

obvious morphism $\eta : i_0(\mathbb{1}_{\mathscr{A}}) = i_0(A^{\otimes 0}) \to Sym(A)$. One can check easily that $Sym(A)$ equipped with the multiplication μ and the unit η is a commutative monoid in the monoidal category $\mathscr{A}^{\mathfrak{S}}$.

Definition 5.3.9. Let $\mathscr{A}$ be an abelian symmetric monoidal category. The commutative monoid $Sym(A)$ of $\mathscr{A}^{\mathfrak{S}}$ defined above will be called the symmetric monoid generated by A.

Remark 5.3.10. One can describe $Sym(A)$ by a universal property: given a commutative monoid R in $\mathscr{A}^{\mathfrak{S}}$, to give a morphism of commutative monoids $Sym(A) \to R$ is equivalent to give a morphism $A \to R_1$ in $\mathscr{A}$.

5.3.11. Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$.

Consider a base scheme S. According to the previous paragraph, the category $\mathscr{A}_S^{\mathfrak{S}}$ is an abelian category, endowed with a symmetric tensor product $\otimes_S^{\mathfrak{S}}$. For any $\mathscr{P}$-scheme X/S and any integer $n \in \mathbf{N}$, using (5.3.5.2), we put

$$M_S(X, \mathscr{A}^{\mathfrak{S}})\{-n\} = i_0(M_S(X, \mathscr{A}))\{-n\}.$$

It is immediate that the class of symmetric sequences of the form $M_S(X, \mathscr{A}^{\mathfrak{S}})\{-n\}$ for a smooth S-scheme X and an integer $n \geq 0$ is a generating family for the abelian category $\mathscr{A}_S^{\mathfrak{S}}$ which is therefore a Grothendieck abelian category. It is clear that for any $\mathscr{P}$-scheme X and Y over S,

$$M_S(X, \mathscr{A}^{\mathfrak{S}})\{-n\} \otimes_S^{\mathfrak{S}} M_S(Y, \mathscr{A}^{\mathfrak{S}})\{-n\} = M_S(X \times_S Y, \mathscr{A}^{\mathfrak{S}})\{-n\}.$$

Given a morphism (resp. $\mathscr{P}$-morphism) of schemes $f : T \to S$ and a symmetric sequence A_* of $\mathscr{A}_S$, we put $f_{\mathfrak{S}}^*(A_*) = (f^* A_n)_{n \in \mathbf{N}}$ (resp. $f_\sharp^{\mathfrak{S}}(A_*) = (f_\sharp A_n)_{n \in \mathbf{N}}$). This defines a functor $f_{\mathfrak{S}}^* : \mathscr{A}_S^{\mathfrak{S}} \to \mathscr{A}_T^{\mathfrak{S}}$ (resp. $f_\sharp^{\mathfrak{S}} : A_T^{\mathfrak{S}} \to \mathscr{A}_S^{\mathfrak{S}}$) which is obviously right exact. Thus the functor $f_{\mathfrak{S}}^*$ admits a right adjoint which we denote by $f_*^{\mathfrak{S}}$. When f is in $\mathscr{P}$, we check easily the functor $f_\sharp^{\mathfrak{S}}$ is left adjoint to $f_{\mathfrak{S}}^*$.

From criterion 1.1.41 and lemma 1.2.13, we check easily the following proposition:

Proposition 5.3.12. *Consider the previous hypothesis and notations.*

The association $S \mapsto \mathscr{A}_S^{\mathfrak{S}}$ together with the structures introduced above defines an $\mathbf{N} \times \tau$-generated abelian $\mathscr{P}$-premotivic category.

Moreover, the different adjunctions of the form (5.3.5.1) *other each fibers over a scheme S define an adjunction of $\mathscr{P}$-premotivic categories:*

$$i_0 : \mathscr{A} \rightleftarrows \mathscr{A}^{\mathfrak{S}} : \mathsf{ev}_0 \tag{5.3.12.1}$$

Indeed, i_0 is trivially compatible with twists.

Proposition 5.3.13. *Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category, and t be an admissible topology. If $\mathscr{A}$ is compatible with t then $\mathscr{A}^{\mathfrak{S}}$ is compatible with t.*

Proof. This is based on the following lemma (see [CD09, 7.5, 7.6]):

Lemma 5.3.14. *Assume that $\mathscr{A}$ is compatible with t. Then for any complex C of $\mathscr{A}_S$, any complex E of $\mathscr{A}_S^{\mathfrak{S}}$ and any integer $n \geq 0$, there are canonical isomorphisms:*

$$\mathrm{Hom}_{\mathrm{K}(\mathscr{A}_S^{\mathfrak{S}})}(i_0(C)\{-n\}, E) \simeq \mathrm{Hom}_{\mathrm{K}(\mathscr{A}_S)}(C, E_n) \tag{5.3.14.1}$$

$$\mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S^{\mathfrak{S}})}(i_0(C)\{-n\}, E) \simeq \mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(C, E_n) \tag{5.3.14.2}$$

The lemma implies that E is local (resp. t-flasque) if and only if for any $n \geq 0$, E_n is local (resp. t-flasque). This concludes. □

5.3.c. *Symmetric Tate spectra.*

5.3.15. Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$.

For any scheme S, the unit point of $\mathbf{G}_{m,S}$ defines a split monomorphism of $\mathscr{A}$-premotives $\mathbb{1}_S \to M_S(\mathbf{G}_{m,S})$. We denote by $\mathbb{1}_S\{1\}$ the cokernel of this monomorphism and call it the *suspended Tate S-premotive* with coefficients in $\mathscr{A}$. The collection of these objects for any scheme S is a cartesian section of $\mathscr{A}$ denoted by $\mathbb{1}\{1\}$. For any integer $n \geq 0$, we denote by $\mathbb{1}\{n\}$ its n-the tensor power.

With the notations of 5.3.9, we define the *symmetric Tate spectrum* over S as the symmetric sequence $\mathbb{1}_S\{*\} = Sym(\mathbb{1}_S\{1\})$ in $\mathscr{A}_S^{\mathfrak{S}}$. The corresponding collection defines a cartesian commutative monoid of the fibred category $\mathscr{A}^{\mathfrak{S}}$, called the *absolute Tate spectrum*.

Definition 5.3.16. Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$.

We denote by $\mathrm{Sp}(\mathscr{A})$ the abelian $\mathscr{P}$-premotivic category of modules over $\mathbb{1}\{*\}$ in the category $\mathscr{A}^{\mathfrak{S}}$. The objects of $\mathrm{Sp}(\mathscr{A})$ are called the abelian (symmetric) Tate spectra.[30]

The category $\mathrm{Sp}(\mathscr{A})$ is $\mathbf{N} \times \tau$-generated. Composing the adjunctions (5.3.1.1) and (5.3.12.1), we get an adjuntion

$$\Sigma^\infty : \mathscr{A} \rightleftarrows \mathrm{Sp}(\mathscr{A}) : \Omega^\infty \tag{5.3.16.1}$$

of abelian $\mathscr{P}$-premotivic categories.

Let us explicit the definitions. An abelian Tate spectrum (E, σ) is the data of :

(1) for any $n \in \mathbf{N}$, an object E_n of $\mathscr{A}_S$ endowed with an action of $\mathfrak{S}_n$
(2) for any $n \in \mathbf{N}$, a morphism $\sigma_n : E_n\{1\} \to E_{n+1}$ in $\mathscr{A}_S$

such that the composite map

$$E_m\{n\} \xrightarrow{\sigma_m\{n-1\}} E_{m+1}\{n-1\} \to ... \xrightarrow{\sigma_{m+n-1}} E_{m+n}$$

is $\mathfrak{S}_n \times \mathfrak{S}_m$-equivariant with respect to the canonical action of $\mathfrak{S}_n$ on $\mathbb{1}_S\{n\}$ and the structural action of $\mathfrak{S}_m$ on E_m. By definition, $\omega(E) = E_0$. Recall that ω is exact.

Given an object A of $\mathscr{A}_S$, the abelian Tate spectrum $\Sigma^\infty A$ is defined such that $(\Sigma^\infty A)_n = A\{n\}$ with the action of $\mathfrak{S}_n$ given by its action on $\mathbb{1}_S\{n\}$ by permutations of the factors.

Be careful we consider the category $\mathrm{Sp}(\mathscr{A}_S)$ as $\mathbf{N}$-twisted by negative twists. For any abelian Tate spectrum E_*, $(E_*\{-n\})_m = \mathfrak{S}_n \times_{\mathfrak{S}_{m-n}} E_{m-n}$ for $n \geq m$.

5.3.17. Consider a morphism

$$\varphi : \mathscr{A} \to \mathscr{B}$$

of abelian $\mathscr{P}$-premotivic categories. Then as $\varphi(\mathbf{Z}^{\mathscr{A}}\{1\}) = \mathbf{Z}^{\mathscr{B}}\{1\}$, φ can be extended to abelian Tate spectra in such a way that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\varphi} & \mathscr{B} \\ {\scriptstyle \Sigma^\infty_{\mathscr{A}}}\downarrow & & \downarrow{\scriptstyle \Sigma^\infty_{\mathscr{B}}} \\ \mathrm{Sp}(\mathscr{A}) & \xrightarrow{\mathrm{Sp}(\varphi)} & \mathrm{Sp}(\mathscr{B}). \end{array}$$

(Of course the obvious diagram for the corresponding right adjoints also commutes.)

Definition 5.3.18. For any scheme S, a complex of abelian Tate spectra over S will be called simply a *Tate spectrum* over S.

A Tate spectrum E is a bigraded object. In the notation E_n^m, the index m corresponds to the (cochain) complex structure and the index n to the symmetric sequence structure.

From propositions 5.3.3 and 5.3.13, we get the following:

Proposition 5.3.19. *Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category compatible with an admissible topology t. Then $\mathrm{Sp}(\mathscr{A})$ is compatible with t.*

Note also that remark 5.3.4 and lemma 5.3.14 implies that for any simplicial $\mathscr{P}$-scheme $\mathcal{X}$ over S, any integer $n \in \mathbf{N}$, and any Tate spectrum E, we have canonical isomorphisms:

$$\mathrm{Hom}_{\mathrm{K}(\mathrm{Sp}(\mathscr{A}_S))}(\Sigma^\infty M_S(\mathcal{X}, \mathscr{A})\{-n\}, E) \simeq \mathrm{Hom}_{\mathrm{K}(\mathscr{A}_S)}(\Sigma^\infty M_S(\mathcal{X}, \mathscr{A}), E_n) \tag{5.3.19.1}$$

$$\mathrm{Hom}_{\mathrm{D}(\mathrm{Sp}(\mathscr{A}_S))}(\Sigma^\infty M_S(\mathcal{X}, \mathscr{A})\{-n\}, E) \simeq \mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(\Sigma^\infty M_S(\mathcal{X}, \mathscr{A}), E_n) \tag{5.3.19.2}$$

According to the proposition, the category $\mathrm{C}(\mathrm{Sp}(\mathscr{A}_S))$ of Tate spectra over S has a t-descent model structure. The previous isomorphisms allow to describe this structure as follows:

[30] As we will almost never consider non symmetric spectra, we will cancel the word "symmetric" in our terminology.

(1) For any simplicial $\mathscr{P}$-scheme $\mathcal{X}$ over S, and any integer $n \geq 0$, the Tate spectrum $\Sigma^\infty M_S(\mathcal{X}, \mathscr{A})\{-n\}$ is cofibrant.
(2) A Tate spectrum E over S is fibrant if and only if for any integer $n \geq 0$, the complex E_n over $\mathscr{A}_S$ is local (*i.e.* t-flasque).
(3) Let $f : E \to F$ be a morphism of Tate spectra over S. Then f is a fibration (resp. quasi-isomorphism) if and only if for any integer $n \geq 0$, the morphism $f_n : E_n \to F_n$ of complexes over $\mathscr{A}_S$ is a fibration (resp. quasi-isomorphism).

Note that properties (2) and (3) follows from (5.3.4.1) and (5.3.14.1).

5.3.20. We can also introduce the $\mathbf{A}^1$-localization of this model structure. The corresponding homotopy category is the $\mathbf{A}^1$-derived $\mathscr{P}$-premotivic category $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}))$ introduced in 5.2.16. The isomorphism (5.3.19.2) gives the following assertion: From the above, a Tate spectrum E is $\mathbf{A}^1$-local if and only if for any integer $n \geq 0$, E_n is $\mathbf{A}^1$-local.

(1) A Tate spectrum E over S is $\mathbf{A}^1$-local if and only if for any integer $n \geq 0$, the complex E_n over $\mathscr{A}_S$ is $\mathbf{A}^1$-local.
(2) Let $f : E \to F$ be a morphism of Tate spectra over S. Then f is a $\mathbf{A}^1$-local fibration (resp. weak $\mathbf{A}^1$-equivalence) if and only if for any integer $n \geq 0$, the morphism $f_n : E_n \to F_n$ of complexes over $\mathscr{A}_S$ is a $\mathbf{A}^1$-local fibration (resp. weak $\mathbf{A}^1$-equivalence).

As a consequence, the isomorphism (5.3.19.2) induces an isomorphism

$$\mathrm{Hom}_{\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}_S))}(\Sigma^\infty M_S(\mathcal{X}, \mathscr{A})\{-n\}, E) \simeq \mathrm{Hom}_{\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)}(\Sigma^\infty M_S(\mathcal{X}, \mathscr{A}), E_n). \tag{5.3.20.1}$$

Similarly, the adjunction (5.3.16.1) induces an adjunction of triangulated $\mathscr{P}$-premotivic categories

$$\mathbf{L}\Sigma^\infty : \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}) \rightleftarrows \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A})) : \mathbf{R}\Omega^\infty. \tag{5.3.20.2}$$

5.3.d. *Symmetric Tate Ω-spectra.*

5.3.21. The final step is to localize further the category $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}))$. The aim is to relate the positive twists on $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ obtained by tensoring with $\mathbb{1}_S\{1\}$ and the negative twists on $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}))$ induced by the consideration of symmetric sequences.

Let X be a $\mathscr{P}$-scheme over S. From the definition of Σ^∞, there is a canonical morphism of abelian Tate spectra:

$$\big[\Sigma^\infty\big(\mathbb{1}_S\{1\}\big)\big]\{-1\} \to \Sigma^\infty \mathbb{1}_S.$$

Tensoring this map by $\Sigma^\infty M_S(X, \mathscr{A})\{-n\}$ for any $\mathscr{P}$-scheme X over S and any integer $n \in \mathbf{N}$, we obtain a family of morphisms of Tate spectra concentrated in cohomological degree 0:

$$\big[\Sigma^\infty\big(M_S(X, \mathscr{A})\{1\}\big)\big]\{-n-1\} \to \Sigma^\infty M_S(X, \mathscr{A})\{-n\}.$$

We denote by $\mathscr{W}_\Omega$ this family and put $\mathscr{W}_{\Omega,\mathbf{A}^1} = \mathscr{W}_\Omega \cup \mathscr{W}_{\mathbf{A}^1}$. Obviously, $\mathscr{W}_{\Omega,\mathbf{A}^1}$ is stable by the operations f^* and $f_\sharp$.

Definition 5.3.22. Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category compatible with an admissible topology t. With the notations introduced above, we define the *stable $\mathbf{A}^1$-derived $\mathscr{P}$-premotivic category with coefficients in $\mathscr{A}$* as the derived $\mathscr{P}$-premotivic category

$$\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}) = \mathrm{D}(\mathrm{Sp}(\mathscr{A}))[\mathscr{W}^{-1}_{\mathfrak{S},\mathbf{A}^1}]$$

defined in corollary 5.2.5.

Note that Obviously $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}) = \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}))[\mathscr{W}_\Omega^{-1}]$. Using the left Bousfield localization of the $\mathbf{A}^1$-local model structure on $\mathrm{C}(\mathrm{Sp}(\mathscr{A}))$, we thus obtain a canonical adjunction of triangulated $\mathscr{P}$-fibred premotivic categories

$$\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A})) \rightleftarrows \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}))[\mathscr{W}_\Omega^{-1}]$$

which allows to describe $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S)$ as the full subcategory of $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}_S))$ made of Tate spectra which are $\mathscr{W}_\Omega$-local in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}_S))$. Recall a Tate spectrum E is a sequence of complexes $(E_n)_{n\in\mathbf{N}}$ over $\mathscr{A}_S$ together with suspension maps in $\mathrm{C}(\mathscr{A}_S)$

$$\sigma_n : \mathbb{1}_S\{1\} \otimes E_n \to E_{n+1}.$$

From this, we deduce a canonical morphism $\mathbb{1}_S\{1\} \otimes^{\mathbf{L}} E_n \to E_{n+1}$ in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ whose adjoint morphism we denote by

$$u_n : E_n \to \mathbf{R}\mathit{Hom}_{\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)}(\mathbb{1}_S\{1\}, E_{n+1}) \tag{5.3.22.1}$$

According to (5.3.20.1), the condition that E is $\mathscr{W}_\Omega$-local in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}))$ is equivalent to ask that for any integer $n \geq 0$, the map (5.3.22.1) is an isomorphism in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}))$.

Considering the adjunction (5.3.20.2), we obtain finally an adjunction of triangulated $\mathscr{P}$-fibred categories:

$$\Sigma^\infty : \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}) \rightleftarrows \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A})) \rightleftarrows \mathrm{D}_{\mathbf{A}^1}(\mathscr{A}) : \Omega^\infty.$$

Note that tautologically, the Tate spectrum $\Sigma^\infty(\mathbb{1}_S\{1\})$ has a tensor inverse $(\Sigma^\infty \mathbb{1}_S)\{-1\}$ in $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S)$. This means that the triangulated $\mathscr{P}$-premotivic category $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$ satisfies the stability property (we have $M\{1\} = M(1)[1]$ for any object M of $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$).

Definition 5.3.23. Consider the assumptions of definition 5.3.22.

For any scheme S, we say that a Tate spectrum E over S is a Tate Ω-spectrum if the following conditions are fulfilled:

(a) For any integer $n \geq 0$, E_n is t-flasque and $\mathbf{A}^1$-local.
(b) For any integer $n \geq 0$, the adjoint of the structural suspension map

$$E_n \to \mathit{Hom}_{\mathrm{C}(\mathscr{A}_S)}(\mathbb{1}_S\{1\}, E_{n+1})$$

is a quasi-isomorphism.

In particular, a Tate Ω-spectrum is $\mathscr{W}_\Omega$-local in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}_S))$. In fact, it is also $\mathscr{W}_{\Omega,\mathbf{A}^1}$-local in the category $\mathrm{D}(\mathrm{Sp}(\mathscr{A}_S))$ so that the category $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$ is also equivalent to the full subcategory of $\mathrm{D}(\mathrm{Sp}(\mathscr{A}_S))$ spanned by Tate Ω-spectra.

Fibrant objects of the $\mathscr{W}_{\Omega,\mathbf{A}^1}$-local model category on $\mathrm{C}(\mathrm{Sp}(\mathscr{A}))$ obtained in definition 5.3.22 are exactly the Tate Ω-spectra.

Essentially by definition, we have:

Proposition 5.3.24. *Let $\mathscr{A}$ be as definition 5.3.22. Then $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$ is a $\mathscr{P}$-premotivic category which satisifies t-descent, the homotopy property, and the stability property.*

Proposition 5.3.25. *Consider the above notations. Let S be a base scheme.*

(1) If the endofunctor

$$\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S) \to \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S), C \mapsto \mathbf{R}\mathit{Hom}_{\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)}(\mathbb{1}_S\{1\}, C)$$

is conservative, then the functor Ω^∞_S is conservative.

(2) If the Tate twist $E \mapsto E(1)$ is fully faithful in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$, then Σ^∞_S is fully faithful.

(3) If the Tate twist $E \mapsto E(1)$ induces an auto-equivalence of $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$, then $(\Sigma^\infty_S, \Omega^\infty_S)$ are adjoint equivalences of categories.

Remark 5.3.26. Similar statements can be obtained for the derived categories rather than the $\mathbf{A}^1$-derived categories. We left their formulation and proof to the reader.

Proof. Consider point (1). We have to prove that for any $\mathscr{W}_\Omega$-local Tate spectrum E in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}_S))$, if $\mathbf{R}\Omega^\infty(E) = 0$, then $E = 0$. But $\mathbf{R}\Omega^\infty(E) = \Omega^\infty(E) = E_0$ (see 5.3.20). Because for any integer $n \geq 0$, the map (5.3.22.1) is an $\mathbf{A}^1$-equivalence, we deduce that for any integer $n \in \mathbf{Z}$, the complex E_n is (weakly) $\mathbf{A}^1$-acyclic. According to (5.3.20.1), this implies $E = 0$ – because $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S)$ is $\mathbf{N}$-generated.

Consider point (2). We want to prove that for any complex C over $\mathscr{A}_S$, the counit map $C \to \mathbf{R}\Omega^\infty\mathbf{L}\Sigma^\infty(C)$ is an isomorphism. It is enough to treat the case where C is cofibrant.

Considering the left adjoint $\mathbf{L}\Sigma^\infty$ of (5.3.20.2), we first prove that $\mathbf{L}\Sigma^\infty(C)$ is $\mathscr{W}_\Omega$-local. Because C is cofibrant, this Tate spectrum is equal in degree n to the complex $C\{n\}$ (with its natural action of $\mathfrak{S}_n$). Moreover, the suspension map is given by the isomorphism (in the monoidal category $\mathrm{C}(\mathscr{A}_S)$)

$$\sigma_n : \mathbb{1}_S\{1\} \otimes_S C\{n\} \to C\{n+1\}.$$

In particular, the corresponding map in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$

$$\sigma'_n : \mathbb{1}_S\{1\} \otimes^L_S C\{n\} \to C\{n+1\}.$$

is canonically isomorphic to

$$\mathbb{1}_S\{1\} \otimes^L_S C\{n\} \xrightarrow{1\otimes 1} \mathbb{1}_S\{1\} \otimes^L_S C\{n\}.$$

Thus, because the Tate twist is fully faithful in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$, the adjoint map to σ'_n is an $\mathbf{A}^1$-equivalence. In other words, $\mathbf{L}\Sigma^\infty(C)$ is $\mathscr{W}_\Omega$-local. But then, as C is cofibrant, $C = \Omega^\infty\Sigma^\infty(C) = \mathbf{R}\Omega^\infty\mathbf{L}\Sigma^\infty(C)$, and this concludes.

Point (3) is then a consequence of (1) and (2). □

Remark 5.3.27. (1) The construction of the triangulated category $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$ can also be obtained using the more general construction of [CD09, §7] – see also [Hov01, 7.11] and [Ayo07b, chap. 4] for even more general accounts. Here, we exploit the simplification arising from the fact that we invert a complex concentrated in degree 0: this allowed us to describe $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$ simply as a Verdier quotient of the derived category of an abelian category. However, we can also consider the category of symmetric spectra in $\mathrm{C}(\mathscr{A}_S)$ with respect to one of the complexes $\mathbb{1}_S(1)[2]$ or $\mathbb{1}_S(1)$ and this leads to the equivalent categories; see [Hov01, 8.3].

(2) Point (3) of proposition 5.3.25 is a particular case of [Hov01, 8.1].

5.3.28. Consider a morphism of abelian $\mathscr{P}$-premotivic categories

$$\varphi : \mathscr{A} \rightleftarrows \mathscr{B} : \psi$$

such that $\mathscr{A}$ (resp. $\mathscr{B}$) is compatible with a system of topology t (resp. t'). Suppose t' is finer than t. According to 5.3.17, we obtain an adjunction of abelian $\mathscr{P}$-premotivic categories

$$\varphi : \mathrm{C}(\mathrm{Sp}(\mathscr{A})) \rightleftarrows \mathrm{C}(\mathrm{Sp}(\mathscr{B})) : \psi.$$

The pair (φ_S, ψ_S) is a Quillen adjunction for the stable model structures (apply again [CD09, prop. 3.11]). Thus we obtain a morphism of triangulated $\mathscr{P}$-premotivic categories:

$$\mathbf{L}\varphi : \mathrm{D}_{\mathbf{A}^1}(\mathscr{A}) \rightleftarrows \mathrm{D}_{\mathbf{A}^1}(\mathscr{B}) : \mathbf{R}\psi.$$

Remark 5.3.29. Under the light of Proposition 5.3.25, the category $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$ might be considered as the universal derived $\mathscr{P}$-premotivic category $\mathscr{T}$ with a morphism $\mathrm{D}(\mathscr{A}) \to \mathscr{T}$, and such that $\mathscr{T}$ satisfies the homotopy and the stability property. This can be made precise in the setting of algebraic derivators or of dg-categories (or any other kind of stable ∞-categories).

Proposition 5.3.30. *Assume that $\mathscr{P}$ contains the class of smooth morphisms of finite type. Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category. Assume that $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ is compactly generated by its geometric sections $M_S(X)$, and satisfies Nisnevich descent. Then, for any scheme S, for any compact object C of $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$ and for any Tate spectrum E in $\mathscr{A}_S$, we have a canonical isomorphisms*

$$\varinjlim_{r>>0} \mathrm{Hom}_{\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)}(C\{a+r\}), E_r[i]) \simeq \mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S)}(\mathbf{L}\Sigma^\infty(C)\{a\}, E[i])$$

for any integers a and i.

Proof. The permutation $\sigma = (123)$ acts as the identity on $\mathbb{1}_S\{1\}^{\otimes 3} = \mathbb{1}_S\{3\}$ in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$: using example 5.2.21, it is sufficient to prove this in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Z}}(S)$, which is well known; see [Ayo07b, 4.5.65]. This proposition is then a direct consequence of [Ayo07b, theorems 4.3.61 and 4.3.79]. □

Corollary 5.3.31. *Under the assumptions of the preceding proposition, the triangulated category $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S)$ is compactly generated. More precisely, if $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)_c$ denotes the category of compact objects in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$, then the category of compact objects in $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S)$ is canonically equivalent to the pseudo-abelian completion of the category obtained as the 2-colimit of the following diagram.*

$$\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)_c \xrightarrow{\otimes \mathbb{1}_S\{1\}} \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)_c \xrightarrow{\otimes \mathbb{1}_S\{1\}} \cdots \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)_c \xrightarrow{\otimes \mathbb{1}_S\{1\}} \cdots$$

Proposition 5.3.32. *Let t and t' be two admissible topologies, with t' finer than t. Then $\mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_{t'}(\mathscr{P},\Lambda))$ is canonically equivalent to the the full subcategory of $\mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_t(\mathscr{P},\Lambda))$ spanned by the objects which satisfy t'-descent.*

Proof. It is sufficient to prove this proposition in the case where t is the coarse topology. We deduce from [Ayo07b, 4.4.42] that, for any scheme S in $\mathscr{S}$, we have

$$\mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_{t'}(\mathscr{P}/S,\Lambda)) = \mathrm{D}\left(\mathrm{PSh}(\mathscr{P}/S,\Lambda)\right)[\mathscr{W}^{-1}]\,,$$

with $\mathscr{W} = \mathscr{W}_{t'} \cup \mathscr{W}_{\mathbf{A}^1} \cup \mathscr{W}_{\Omega}$, where $\mathscr{W}_{t'}$ is the set of maps of shape

$$\Sigma^{\infty} M_S(\mathcal{X})\{n\}[i] \to \Sigma^{\infty} M_S(X)\{n\}[i]\,,$$

for any t'-hypercovering $\mathcal{X} \to X$ and any integers $n \leq 0$ and i. The assertion is then a particular case of the description of the homotopy category of a left Bousfield localization. □

Proposition 5.3.33. *Let t be an admissible topology. Then, for any scheme S in $\mathscr{S}$, the symmetric monoidal model structure on $\mathrm{C}(\mathrm{Sp}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda)))$ underlying $\mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda))$ is perfect (see 4.2.3).*

Proof. The generating family of $\mathrm{Sh}_t(\mathscr{P}_S,\Lambda)$ is flat in the sense of [CD09, 3.1], so that, by virtue of [CD09, prop. 7.22 and cor. 7.24], the assumptions of proposition 4.2.8 are fulfilled. □

Example 5.3.34. We have the stable versions of the $\mathscr{P}$-premotivic categories introduced in example 5.2.17:
1) Consider the admissible topology $t = \mathrm{Nis}$. Following F. Morel, we define the stable $\mathbf{A}^1$-derived premotivic category as (see also the construction of [Ayo07b]):

$$\mathrm{D}_{\mathbf{A}^1,\Lambda} := \mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_{\mathrm{Nis}}(Sm,\Lambda)) \quad \text{and} \quad \underline{\mathrm{D}}_{\mathbf{A}^1,\Lambda} := \mathrm{D}_{\mathbf{A}^1}\left(\mathrm{Sh}_{\text{ét}}\left(\mathscr{S}^{ft},\Lambda\right)\right). \tag{5.3.34.1}$$

We shall also write sometimes

$$\mathrm{D}_{\mathbf{A}^1}(S,\Lambda) := \mathrm{D}_{\mathbf{A}^1,\Lambda}(S) \quad \text{and} \quad \underline{\mathrm{D}}_{\mathbf{A}^1}(S,\Lambda) := \underline{\mathrm{D}}_{\mathbf{A}^1,\Lambda}(S) \tag{5.3.34.2}$$

for a scheme S.

In the case when $t = \text{ét}$, we get the triangulated premotivic categories of *étale premotives*:

$$\mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_{\text{ét}}(Sm,\Lambda)) \quad \text{and} \quad \mathrm{D}_{\mathbf{A}^1}\left(\mathrm{Sh}_{\text{ét}}\left(\mathscr{S}^{ft},\Lambda\right)\right).$$

In each of these cases, we denote by $\Sigma^{\infty}\Lambda^t_S(X)$ the premotive associated with a smooth S-scheme X.

From the adjunction (5.1.24.2), we get an adjunction of triangulated premotivic categories:

$$a_{\text{ét}} : \mathrm{D}_{\mathbf{A}^1,\Lambda} \rightleftarrows \mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_{\text{ét}}(Sm,\Lambda)) : \mathbf{R}O_{\text{ét}}.$$

2) Assume $\mathscr{P} = \mathscr{S}^{ft}$:

Consider the admissible topology $t = \mathrm{qfh}$ (resp. $t = \mathrm{h}$). Extending the construction of Voevodsky ([Voe96]), we define the category of (non effective) qfh-*motives* (resp. h-*motives*) over S as:

$$\underline{\mathrm{DM}}_t(S,\Lambda) = \mathrm{D}_{\mathbf{A}^1}\left(\mathrm{Sh}_t\left(\mathscr{S}^{ft},\Lambda\right)\right). \tag{5.3.34.3}$$

For an S-scheme of finite type X, we will denote simply by $\Sigma^{\infty}\Lambda^{\mathrm{qfh}}_S(X)$ (resp $\Sigma^{\infty}\Lambda^{\mathrm{h}}_S(X)$) the corresponding premotive associated with X in $\underline{\mathrm{DM}}_t(S,\Lambda)$. We shall write sometimes $\underline{\mathrm{DM}}_{t,\Lambda}(S)$ instead of $\underline{\mathrm{DM}}_t(S,\Lambda)$. The h-sheafification functor induces a premotivic adjunction

$$\underline{\mathrm{DM}}_{\mathrm{qfh},\Lambda} \rightleftarrows \underline{\mathrm{DM}}_{\mathrm{h},\Lambda}\,. \tag{5.3.34.4}$$

We define $\mathrm{DM}_t(S,\Lambda)$ as the localizing subcategory of $\underline{\mathrm{DM}}_t(S,\Lambda)$ generated by objects of shape $\Sigma^{\infty}\Lambda^t_S(X)(p)[q]$ for any smooth S-scheme of finite type X and any integers p and q. These define an Sm-premotivic triangulated category, and the inclusions

$$\mathrm{DM}_t(S,\Lambda) \to \underline{\mathrm{DM}}_t(S,\Lambda) \tag{5.3.34.5}$$

form a premotivic morphism (the existence of right adjoints is ensured by the Brown representability theorem). As before, we shall write sometimes $\mathrm{DM}_{t,\Lambda}(S)$ instead of $\mathrm{DM}_t(S,\Lambda)$.

Proposition 5.3.35. *For $t = \mathrm{qfh}, \mathrm{h}$, the premotivic category $\mathrm{DM}_{t,\Lambda}$ satisfies t-descent.*

Proof. This is true for $\underline{\mathrm{DM}}_{t,\Lambda}$ by construction, which implies formally the assertion for $\mathrm{DM}_{t,\Lambda}$. □

Remark 5.3.36. According to proposition 5.2.10 and remark 5.3.29, for any admissible topology t, $\mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_t(\mathscr{P}, \Lambda))$ is the universal derived $\mathscr{P}$-premotivic category satisfying t-descent as well as the homotopy and stability properties.

A crucial example for us: the stable $\mathbf{A}^1$-derived premotivic category $\mathrm{D}_{\mathbf{A}^1}$ is the universal derived premotivic category satisfying the properties of homotopy, of stability and of Nisnevich descent.

5.3.37. We assume $\mathscr{P} = Sm$.
Let $\mathrm{Sh}_\bullet(Sm)$ be the category of pointed Nisnevich sheaves of sets. Consider the pointed version of the adjunction of $\mathscr{P}$-premotivic categories

$$N : \Delta^{op}\,\mathrm{Sh}_\bullet(Sm) \rightleftarrows \mathrm{C}(\mathrm{Sh}_{\mathrm{Nis}}(Sm, \Lambda)) : K$$

constructed in 5.2.23.

If we consider on the left hand side the $\mathbf{A}^1$-model category defined by Blander [Bla03], (N_S, K_S) is a Quillen adjunction for any scheme S.

We consider $(\mathbf{G}_m, 1)$ as a constant pointed simplicial sheaf. The construction of symmetric $\mathbf{G}_m$-spectra respectively to the model category $\Delta^{op}\,\mathrm{Sh}_\bullet(Sm)$ can now be carried out following [Jar00] or [Ayo07b] and yields a symmetric monoidal model category whose homotopy category is the stable homotopy category of Morel and Voevodsky $\mathrm{SH}(S)$.

Using the functoriality statements [Hov01, th. 8.3 and 8.4], we finally obtain a $\mathscr{P}$-premotivic adjunction

$$N : \mathrm{SH} \rightleftarrows \mathrm{D}_{\mathbf{A}^1} : K.$$

The functor K is the analog of the Eilenberg-Mac Lane functor in algebraic topology; in fact, this adjunction is actually induced by the Eilenberg-MacLane functor (see [Ayo07b, chap. 4]). In particular, as the rational model category of topological (symmetric) S^1-spectra is Quillen equivalent to the model category of complexes of $\mathbf{Q}$-vector spaces, we have a natural equivalence of premotivic categories

$$\mathrm{SH}_{\mathbf{Q}} \rightleftarrows \mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}\,,$$

(where $\mathrm{SH}_{\mathbf{Q}}(S)$ denotes the Verdier quotient of $\mathrm{SH}(S)$ by the localizing subcategory generated by compact torsion objects).

6. Localization and the universal derived example

6.0. In this section, $\mathscr{S}$ is an adequate category of $\mathcal{S}$-schemes as in 2.0. In sections 6.2 and 6.3, we assume in addition that the schemes in $\mathscr{S}$ are finite dimensional.

We will apply the definitions of the preceding section to the admissible class made of morphisms of finite type (resp. smooth morphisms of finite type) in $\mathscr{S}$, denoted by $\mathscr{S}^{ft}$ (resp. Sm).

Recall the general convention of section 1.4:

- *premotivic* means Sm-premotivic.
- *generalized premotivic* means $\mathscr{S}^{ft}$-premotivic.

6.1. **Generalized derived premotivic categories.**

Example 6.1.1. Let t be a system of topologies on $\mathscr{S}^{ft}$. For a scheme S, we denote by $\mathrm{Sh}_t\left(\mathscr{S}_S^{ft}, \Lambda\right)$ the category of sheaves of abelian groups on $\mathscr{S}_S^{ft}$ for the topology t_S. For an S-scheme of finite type X, we let $\underline{\Lambda}_S^t(X)$ be the free t-sheaf of Λ-modules represented by X. Recall $\mathrm{Sh}_t\left(\mathscr{S}^{ft}, \Lambda\right)$ is a generalized abelian premotivic category (see 5.1.4).
Let $\rho : Sm_S \to \mathscr{S}^{ft}/S$ be the obvious inclusion functor. Then it induces (*cf.* [AGV73, IV, 4.10]) a sequence of adjoint functors

$$\mathrm{Sh}_t(Sm/S, \Lambda) \underset{\rho_*}{\overset{\rho_\sharp}{\rightleftarrows}} \mathrm{Sh}_t\left(\mathscr{S}^{ft}/S, \Lambda\right) \quad (\rho^*)$$

and we checked easily that this induces an enlargement of abelian premotivic categories:

$$\rho_\sharp : \mathrm{Sh}_t(Sm, \Lambda) \rightleftarrows \mathrm{Sh}_t\big(\mathscr{S}^{ft}, \Lambda\big) : \rho^*. \tag{6.1.1.1}$$

Remark 6.1.2. Note that for any scheme S, the abelian category $\mathrm{Sh}_t(Sm/S, \Lambda)$ can be described as the Gabriel quotient of the abelian category $\mathrm{Sh}_t\big(\mathscr{S}^{ft}/S, \Lambda\big)$ with respect to the sheaves $\underline{F}$ over $\mathscr{S}^{ft}/S$ such that $\rho^*(\underline{F}) = 0$.

An example of such a sheaf in the case where $t = \mathrm{Nis}$ and $\dim(S) > 0$ is the Nisnevich sheaf $\underline{\Lambda}_S(Z)$ on $\mathscr{S}_S^{ft}$ represented by a nowhere dense closed subscheme Z of S is zero when restricted to Sm_S.

6.1.3. Consider an abelian premotivic category $\mathscr{A}$ compatible with an admissible topology t on Sm and a generalized abelian premotivic category $\underline{\mathscr{A}}$ compatible with an admissible topology t' on $\mathscr{S}$. We denote by M (resp. $\underline{M}$) the geometric sections of $\mathscr{A}$ (resp. $\underline{\mathscr{A}}$). We assume that t' is finer that t, and consider an adjunction of abelian premotivic categories:

$$\rho_\sharp : \mathscr{A} \rightleftarrows \underline{\mathscr{A}} : \rho^*.$$

Let S be a scheme in $\mathscr{S}$. The functors $\rho_\sharp$ and ρ^* induce a derived adjunction (see 5.2.19):

$$\mathbf{L}\rho_\sharp : \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S) \rightleftarrows \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\underline{\mathscr{A}}_S) : \mathbf{R}\rho^*$$

(where $\underline{\mathscr{A}}$ is considered as an Sm-fibred category).

Proposition 6.1.4. *Consider the previous hypothesis, and fix a scheme S. Assume furthermore that we have the following properties.*

(i) The functor $\rho_\sharp : \mathscr{A}_S \to \underline{\mathscr{A}}_S$ is fully faithful.
(ii) The functor $\rho^* : \underline{\mathscr{A}}_S \to \mathscr{A}_S$ commutes with small colimits.

Then, the following conditions hold :

(a) The induced functor

$$\rho^* : \mathrm{C}(\underline{\mathscr{A}}_S) \to \mathrm{C}(\mathscr{A}_S)$$

preserves $\mathbf{A}^1$-equivalences.

(b) The $\mathbf{A}^1$-derived functor $\mathbf{L}\rho_\sharp : \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S) \to \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\underline{\mathscr{A}}_S)$ is fully faithful.

Proof. Point (a) follows from proposition 5.2.22. To prove (b), we have to prove that the unit map

$$M \to \rho^*\mathbf{L}\rho_\sharp(M)$$

is an isomorphism for any object M of $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$. For this purpose, we may assume that M is cofibrant, so that we have

$$M \simeq \rho^*\rho_\sharp(M) \simeq \rho^*\mathbf{L}\rho_\sharp(M)$$

(where the first isomorphism holds already in $\mathrm{C}(\mathscr{A}_S)$). □

Corollary 6.1.5. *Consider the hypothesis of the previous proposition. Then the family of adjunctions $\mathbf{L}\rho_\sharp : \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S) \to \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\underline{\mathscr{A}}_S) : \mathbf{R}\rho^*$ indexed by a scheme S induces an enlargement of triangulated premotivic categories*

$$\mathbf{L}\rho_\sharp : \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}) \to \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\underline{\mathscr{A}}) : \mathbf{R}\rho^*.$$

Example 6.1.6. Considering the situation of 6.1.1, we will be particularly interested in the case of the Nisnevich topology. We denote by $\underline{\mathrm{D}}^{\mathit{eff}}_{\mathbf{A}^1,\Lambda}$ the generalized $\mathbf{A}^1$-derived premotivic category associated with $\mathrm{Sh}\big(\mathscr{S}^{ft}, \Lambda\big)$. The preceding corollary gives a canonical enlargement:

$$\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,\Lambda} \rightleftarrows \underline{\mathrm{D}}^{\mathit{eff}}_{\mathbf{A}^1,\Lambda} \tag{6.1.6.1}$$

6.1.7. Consider again the hypothesis of 6.1.3. We denote simply by M (resp. $\underline{M}$) the geometric sections of the premotivic triangulated category $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$ (resp. $\mathrm{D}_{\mathbf{A}^1}(\underline{\mathscr{A}})$).

Recall from 5.3.15 that we have defined $\mathbb{1}_S\{1\}$ (resp. $\underline{\mathbb{1}}_S\{1\}$) as the cokernel of the canonical map $\mathbb{1}_S \to M_S(\mathbf{G}_{m,S})$ (resp. $\underline{\mathbb{1}}_S \to \underline{M}_S(\mathbf{G}_{m,S})$. Thus, it is obvious that we get a canonical

identification $\rho_\sharp(\mathbb{1}_S\{1\}) = \underline{\mathbb{1}}_S\{1\}$. Therefore, the enlargement $\rho_\sharp$ can be extended canonically to an enlargement

$$\rho_\sharp : \mathrm{Sp}(\mathscr{A}) \rightleftarrows \mathrm{Sp}(\underline{\mathscr{A}}) : \rho^*$$

of abelian premotivic categories in such a way that for any scheme S, the following diagram commutes:

$$\begin{array}{ccc} \mathscr{A}_S & \xrightarrow{\rho_\sharp} & \underline{\mathscr{A}}_S \\ {\scriptstyle \Sigma^\infty_{\mathscr{A}}}\downarrow & & \downarrow{\scriptstyle \Sigma^\infty_{\underline{\mathscr{A}}}} \\ \mathrm{Sp}(\mathscr{A}_S) & \xrightarrow{\rho_\sharp} & \mathrm{Sp}(\underline{\mathscr{A}}_S). \end{array}$$

According to proposition 5.3.13, $\mathrm{Sp}(\mathscr{A})$ (resp. $\mathrm{Sp}(\underline{\mathscr{A}})$) is compatible with t (resp. t'), and we obtain an adjoint pair of functors (5.3.28):

$$\mathbf{L}\rho_\sharp : \mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S) \rightleftarrows \mathrm{D}_{\mathbf{A}^1}(\underline{\mathscr{A}}_S) : \mathbf{R}\rho^*.$$

From the preceding commutative square, we get the identification:

$$\mathbf{L}\rho_\sharp \circ \Sigma^\infty_{\mathscr{A}} = \Sigma^\infty_{\underline{\mathscr{A}}} \circ \mathbf{L}\rho_\sharp \tag{6.1.7.1}$$

As in the non effective case, we get the following result:

Proposition 6.1.8. *Keep the assumptions of proposition 6.1.4, and suppose furthermore that both* $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ *and* $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\underline{\mathscr{A}})$ *are compactly generated by their geometric sections. Then the derived functor* $\mathbf{L}\rho_\sharp : \mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S) \to \mathrm{D}_{\mathbf{A}^1}(\underline{\mathscr{A}}_S)$ *is fully faithful.*

Proof. We have to prove that for any Tate spectrum E of $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S)$, the adjunction morphism

$$E \to \mathbf{L}\rho^*\mathbf{R}\rho_\sharp(E)$$

is an isomorphism. According to proposition 1.3.18, the functor $\mathbf{R}\tilde{\rho}^*$ admits a right adjoint. Thus, applying lemma 1.1.42, it is sufficient to consider the case where $E = M_S(X)\{i\}[n]$ for a smooth S-scheme X, and a couple $(n, i) \in \mathbf{Z} \times \mathbf{Z}$.

Moreover, it is sufficient to prove that for another smooth S-scheme Y and an integer $j \in \mathbf{Z}$, the induced morphism

$$\mathrm{Hom}(\Sigma^\infty M_S(Y)\{j\}, \Sigma^\infty M_S(X)\{i\}[n]) \to \mathrm{Hom}(\Sigma^\infty \underline{M}_S(Y)j\}, \Sigma^\infty \underline{M}_S(X)i\}[n])$$

is an isomorphism. Using the identification (6.1.7.1), propositions 5.3.30 and 6.1.4 allows to conclude. □

Corollary 6.1.9. *If the assumptions of proposition 6.1.8 hold for any scheme* S *in* $\mathscr{S}$*, then we obtain an enlargement of triangulated premotivic categories*

$$\mathbf{L}\rho_\sharp : \mathrm{D}_{\mathbf{A}^1}(\mathscr{A}) \rightleftarrows \mathrm{D}_{\mathbf{A}^1}(\underline{\mathscr{A}}) : \mathbf{R}\rho^*.$$

Example 6.1.10. Considering again the situation of 6.1.1, in the case of the Nisnevich topology. We denote by $\underline{\mathrm{D}}_{\mathbf{A}^1,\Lambda}$ the generalized stable $\mathbf{A}^1$-derived premotivic category associated with $\mathrm{Sh}\big(\mathscr{S}^{ft}, \Lambda\big)$. The preceding corollary gives a canonical enlargement:

$$\mathbf{L}\rho_\sharp : \mathrm{D}_{\mathbf{A}^1,\Lambda} \rightleftarrows \underline{\mathrm{D}}_{\mathbf{A}^1,\Lambda} : \mathbf{R}\rho^* \tag{6.1.10.1}$$

which is compatible with the enlargement (6.1.6.1) in the sense that the following diagram is essentially commutative:

$$\begin{array}{ccc} \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,\Lambda} & \longrightarrow & \underline{\mathrm{D}}^{\mathit{eff}}_{\mathbf{A}^1,\Lambda} \\ {\scriptstyle \Sigma^\infty}\downarrow & & \downarrow{\scriptstyle \underline{\Sigma}^\infty} \\ \mathrm{D}_{\mathbf{A}^1,\Lambda} & \longrightarrow & \underline{\mathrm{D}}_{\mathbf{A}^1,\Lambda} \end{array}$$

Proposition 6.1.11. *Consider a Grothendieck topology* t *on our category of schemes* $\mathscr{S}$*. Let* S *be a scheme in* $\mathscr{S}$*, and* M *an object of* $\mathrm{D}_{\mathbf{A}^1,\Lambda}(S)$*. Then* M *satisfies* t*-descent in* $\mathrm{D}_{\mathbf{A}^1,\Lambda}(S)$ *if and only if* $\mathbf{L}\rho_\sharp(M)$ *satisfies* t*-descent in* $\underline{\mathrm{D}}_{\mathbf{A}^1,\Lambda}(S)$*.*

Proof. Let $f : \mathscr{X} \to S$ be a diagram of S-schemes of finite type. Define

$$H^q(\mathscr{X}, M(p)) = \mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1,\Lambda}(S)}(\Lambda_{\mathscr{X}}, \mathbf{L}f^*(M)(p)[q])$$

$$\underline{H}^q(\mathscr{X}, M(p)) = \mathrm{Hom}_{\underline{\mathrm{D}}_{\mathbf{A}^1,\Lambda}(S)}(\underline{\Lambda}_{\mathscr{X}}, \mathbf{L}f^* \mathbf{L}\rho_\sharp(M)(p)[q])$$

for any integers p and q. The fully faithfulness of $\mathbf{L}\rho_\sharp$ ensures that the comparison map

$$H^q(\mathscr{X}, M(p)) \to \underline{H}^q(\mathscr{X}, M(p))$$

is always bijective. This proposition follows then from the fact that M (resp. $\mathbf{L}\rho_\sharp(M)$) satisfies t-descent if and only if, for any integers p and q, for any S-scheme of finite type X, and any t-hypercovering $\mathscr{X} \to X$, the induced map

$$H^q(X, M(p)) \to H^q(\mathscr{X}, M(p)) \text{ (resp. } \underline{H}^q(X, M(p)) \to \underline{H}^q(\mathscr{X}, M(p))\text{)}$$

is bijective. □

6.2. **The fundamental example.** Recall from [Ayo07b] the following theorem of Morel and Voevodsky:

Theorem 6.2.1. *The triangulated premotivic categories* $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,\Lambda}$ *and* $\mathrm{D}_{\mathbf{A}^1,\Lambda}$ *satisfy the localization property.*

Corollary 6.2.2. *(1) The premotivic category* $\mathrm{D}_{\mathbf{A}^1,\Lambda}$ *is a motivic category.*

(2) Suppose that $\mathscr{T}$ *is a derived premotivic category (see 5.2.9) which is a motivic category. Then there exists a canonical morphism of derived premotivic categories:*

$$\mathrm{D}_{\mathbf{A}^1,\mathbf{Z}} \to \mathscr{T}.$$

Proof. The first assertion follows from the previous theorem and remark 2.4.3. The second assertion follows from proposition 3.3.5 and example 5.3.36. □

Remark 6.2.3. Thus, theorem 2.4.21 can be applied to $\mathrm{D}_{\mathbf{A}^1,\Lambda}$. In particular, for any separated morphism of finite type $f : T \to S$, there exists a pair of adjoint functors

$$f_! : \mathrm{D}_{\mathbf{A}^1,\Lambda}(T) \rightleftarrows \mathrm{D}_{\mathbf{A}^1,\Lambda}(S) : f^!$$

as in the theorem *loc. cit.* so that we have removed the quasi-projective assumption in [Ayo07a].

6.2.4. Because the cdh topology is finer than the Nisnevich topology, we get a morphism of generalized premotivic categories:

$$a^*_{\mathrm{cdh}} : \underline{\mathrm{D}}_{\mathbf{A}^1,\Lambda} \rightleftarrows \mathrm{D}_{\mathbf{A}^1}\left(\mathrm{Sh}_{\mathrm{cdh}}\left(\mathscr{S}^{ft}, \Lambda\right)\right) : \mathbf{R}a_{\mathrm{cdh},*}.$$

Corollary 6.2.5. *For any scheme* S*, the composite functor*

$$\mathrm{D}_{\mathbf{A}^1}(S,\Lambda) \to \underline{\mathrm{D}}_{\mathbf{A}^1}(S,\Lambda) \xrightarrow{a_{\mathrm{cdh}}} \mathrm{D}_{\mathbf{A}^1}\left(\mathrm{Sh}_{\mathrm{cdh}}\left(\mathscr{S}^{ft}_S, \Lambda\right)\right)$$

is fully faithful.

Moreover, it induces an enlargement of premotivic categories:

$$\mathrm{D}_{\mathbf{A}^1,\Lambda} \rightleftarrows \mathrm{D}_{\mathbf{A}^1}\left(\mathrm{Sh}_{\mathrm{cdh}}\left(\mathscr{S}^{ft}, \Lambda\right)\right) \tag{6.2.5.1}$$

Remark 6.2.6. This corollary is a generalisation in our derived setting of the main theorem of [Voe00b]. Note that if $\dim(S) > 0$, there is no hope that the above composite functor is essentially surjective because as soon as Z is a nowhere dense closed subscheme of S, the premotive $\underline{M}^{\mathrm{cdh}}_S(Z, \Lambda)$ does not belong to its image (*cf.* remark 6.1.2).

Proof. According to corollary 6.2.2 and proposition 3.3.9, any Tate spectrum E of $\mathrm{D}_{\mathbf{A}^1}(S, \Lambda)$ satisfies cdh-descent in the derived premotivic category $\mathrm{D}_{\mathbf{A}^1,\Lambda}$, and this implies the first assertion by 5.3.32 and 6.1.11. The second one then follows from the fact the forgetful functor

$$\mathrm{D}_{\mathbf{A}^1}\left(\mathrm{Sh}_{\mathrm{cdh}}\left(\mathscr{S}^{ft}_S, \Lambda\right)\right) \to \underline{\mathrm{D}}_{\mathbf{A}^1}(S, \Lambda).$$

commutes with direct sums (its left adjoint preserves compact objects). □

6.3. **Nearly Nisnevich sheaves.**

6.0. In all this section, we fix an abelian premotivic category $\mathscr{A}$ and we consider the canonical premotivic adjunction (5.1.2.1) associated with $\mathscr{A}$.

We assume $\mathscr{A}$ satisfies the following properties.

(i) $\mathscr{A}$ is compatible with Nisnevich topology, so that we have from (5.1.2.1) a premotivic adjunction:

$$\gamma^* : \mathrm{Sh}_{\mathrm{Nis}}(Sm, \mathbf{Z}) \rightleftarrows \mathscr{A} : \gamma_*. \tag{6.3.0.1}$$

(ii) $\mathscr{A}$ is geometrically generated and the geometric sections are of finite presentation (*i.e.* the functors $\mathrm{Hom}_{\mathscr{A}_S}(M_S(X), -)$ preserve filtered colimits and form a conservative family).

(iii) For any scheme S, and for any open immersion $U \to X$ of smooth S-schemes, the map $M_S(U) \to M_S(X)$ is a monomorphism.

(iv) For any scheme S, the functor $\gamma_* : \mathscr{A}_S \to \mathrm{Sh}_{\mathrm{Nis}}(Sm/S, \mathbf{Z})$ is exact.

Note that the functor $\gamma_* : \mathscr{A}_S \to \mathrm{Sh}_{\mathrm{Nis}}(Sm/S, \mathbf{Z})$ is exact and conservative. As it also preserves filtered colimits, this functor preserves in facts small colimits.

Observe also that, according to these assumption, the abelian premotivic category of Tate spectra $\mathrm{Sp}(\mathscr{A})$ is compatible with Nisnevich topology, $\mathbf{N}$-generated. Moreover, we get a canonical premotivic adjunction

$$\gamma^* : \mathrm{Sp}(\mathrm{Sh}_{\mathrm{Nis}}(Sm, \mathbf{Z})) \rightleftarrows \mathrm{Sp}(\mathscr{A}) : \gamma_* \tag{6.3.0.2}$$

such that γ_* is conservative and preserves small colimits.

In the following, we show how one can deduce properties of the premotivic triangulated categories $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ and $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$ from the good properties of $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,\mathbf{Z}}$ and $\mathrm{D}_{\mathbf{A}^1,\mathbf{Z}}$.

6.3.a. *Support property (effective case).*

Proposition 6.3.1. *For any scheme S, the functor $\gamma_* : \mathrm{C}(\mathscr{A}_S) \to \mathrm{C}(\mathrm{Sh}_{\mathrm{Nis}}(Sm_S, \mathbf{Z}))$ preserves and detects $\mathbf{A}^1$-equivalences.*

Proof. It follows immediately from corollary 5.2.29 that γ_* preserves $\mathbf{A}^1$-equivalences. The fact it detects them can be rephrased by saying that the induced functor

$$\gamma_* : \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S) \to \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,\mathbf{Z}}(S)$$

is conservative. This is obviously true once we noticed that its left adjoint is essentially surjective on generators. □

Corollary 6.3.2. *The right derived functor*

$$\mathbf{R}\gamma_* = \gamma_* : \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S) \to \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,\mathbf{Z}}(S)$$

is conservative.

Proposition 6.3.3. *Let $f : S' \to S$ be a finite morphism of schemes. Then the induced functor*

$$f_* : \mathrm{C}(\mathscr{A}_{S'}) \to \mathrm{C}(\mathscr{A}_S)$$

preserves colimits and $\mathbf{A}^1$-equivalences.

Proof. We first prove f_* preserves colimits. We know the functors γ_* preserve colimits and are conservative. As we have the identification $\gamma_* f_* = f_* \gamma_*$, it is sufficient to prove the property for $\mathscr{A} = \mathrm{Sh}_{\mathrm{Nis}}(Sm, \mathbf{Z})$. Let X be a smooth S-scheme. It is sufficient to prove that, for any point x of X, if X^h_x denotes the henselianization of X at x, the functor

$$\mathrm{Sh}_{\mathrm{Nis}}(Sm_{S'}, \mathbf{Z}) \to \mathscr{A}b \quad , \qquad F \mapsto f_*(F)(X^h_x) = F(S' \times_S X^h_x)$$

commutes to colimits. Moreover the scheme $S' \times_S X^h_x$ is finite over X^h_x, so that we have $S' \times_S X^h_x = \amalg_i Y_i$, where the Y_i's are a finite family of henselian local schemes over $S' \times_S X^h_x$. Hence we have to check that the functor $F \mapsto \bigoplus_i F(Y_i)$ preserves colimits. As colimits commute to sums, it is thus sufficient to prove that the functors $F \mapsto F(Y_i)$ commute to colimits. This follows from the

fact that the local henselian schemes Y_i are points of the topos of sheaves over the small Nisnevich site of X.

We are left to prove that the functor $f_* : \mathrm{C}(\mathscr{A}_{S'}) \to \mathrm{C}(\mathscr{A}_S)$ respects $\mathbf{A}^1$-equivalences. For this, we shall study the behaviour of f_* with respect to the $\mathbf{A}^1$-resolution functor constructed in 5.2.24. Note that f_* commutes to limits because it has a left adjoint. In particular, we know that f_* is exact. Moreover, one checks easily that $f_* R^{(n)}_{\mathbf{A}^1} = f_* R^{(n)}_{\mathbf{A}^1}$. As f_* commutes to colimits, this gives the formula $f_* R_{\mathbf{A}^1} = R_{\mathbf{A}^1} f_*$. Let C be a complex of Nisnevich sheaves of abelian groups on Sm/S'. Choose a quasi-isomorphism $C \to C'$ with C' a Nis-flasque complex. Applying proposition 5.2.26, we know that $R_{\mathbf{A}^1}(C')$ is $\mathbf{A}^1$-fibrant and that we get a canonical $\mathbf{A}^1$-equivalence

$$f_*(C) \to f_*(C') \to f_*(R_{\mathbf{A}^1}(C')) = R_{\mathbf{A}^1}(f_*(C')).$$

Hence we are reduced to prove that f_* preserves $\mathbf{A}^1$-equivalences between $\mathbf{A}^1$-fibrant objects. But such $\mathbf{A}^1$-equivalences are quasi-isomorphisms, so that we can conclude using the exactness of f_*. □

Proposition 6.3.4. *For any open immersion of schemes $j : U \to S$, the exchange natural transformation $j_\sharp \gamma_* \to \gamma_* j_\sharp$ is an isomorphism of functors.*

Proof. Let X be a scheme, and F a Nisnevich sheaf of abelian groups on Sm_X. Define the category $\mathscr{C}_F$ as follows. The objects are the couples (Y, s), where Y is a smooth scheme over X, and s is a section of F over Y. The arrows $(Y, s) \to (Y', s')$ are the morphisms $f \in \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Nis}}(Sm/X, \mathbf{Z})}(\mathbf{Z}_X(Y), \mathbf{Z}_X(Y'))$ such that $f^*(s') = s$. We have a canonical functor

$$\varphi_F : \mathscr{C}_F \to \mathrm{Sh}_{\mathrm{Nis}}(Sm/X, \mathbf{Z})$$

defined by $\varphi_F(Y, s) = \mathbf{Z}_X(Y)$, and one checks easily that the canonical map

$$\varinjlim_{\mathscr{C}_F} \varphi_F = \varinjlim_{(Y,s) \in \mathscr{C}_F} \mathbf{Z}_X(Y) \to F$$

is an isomorphism in $\mathrm{Sh}_{\mathrm{Nis}}(Sm/X, \mathbf{Z})$ (this is essentially a reformulation of the Yoneda lemma).

Consider now an object F in the category $\mathscr{A}_U$. We get two categories $\mathscr{C}_{\gamma_*(F)}$ and $\mathscr{C}_{\gamma_*(j_\sharp(F))}$. There is a functor

$$i : \mathscr{C}_{\gamma_*(F)} \to \mathscr{C}_{\gamma_*(j_\sharp(F))}$$

which is defined by the formula $i(Y, s) = (Y, j_\sharp(s))$. To explain our notations, let us say that we see s as a morphism from $M_S(U, \mathscr{A})$ to F, so that $j_\sharp(s)$ is a morphism from $M_S(Y, \mathscr{A}) = j_\sharp M_S(U, \mathscr{A})$ to $j_\sharp(F)$. This functor i has right adjoint

$$i' : \mathscr{C}_{\gamma_*(j_\sharp(F))} \to \mathscr{C}_{\gamma_*(F)}$$

defined by $i'(Y, s) = (Y_U, s_U)$, where $Y_U = Y \times_S U$, and s_U is the section of $\gamma_*(F)$ over Y_U that corresponds to the section $j^*(s)$ of $j^* j_\sharp \gamma_*(F)$ over Y_U under the canonical isomorphism $\gamma_*(F) \simeq j^* j_\sharp \gamma_*(F)$ (here, we use strongly the fact the functor $j_\sharp$ is fully faithful). The existence of a right adjoint implies i is cofinal. This latter property is sufficient for the canonical morphism

$$\varinjlim_{C_{\gamma_*(F)}} \varphi_{\gamma_*(j_\sharp(F))} \circ i \to \varinjlim_{\mathscr{C}_{\gamma_*(j_\sharp(F))}} \varphi_{\gamma_*(j_\sharp(F))} = \gamma_*(j_\sharp(F))$$

to be an isomorphism. But the functor $\varphi_{\gamma_*(j_\sharp(F))} \circ i$ is exactly the composition of the functor $j_\sharp$ with $\varphi_{\gamma_*(F)}$. As the functor $j_\sharp$ commutes to colimits, we have

$$\varinjlim_{C_{\gamma_*(F)}} \varphi_{\gamma_*(j_\sharp(F))} \circ i = \varinjlim_{C_{\gamma_*(F)}} j_\sharp \varphi_{\gamma_*(F)} \simeq j_\sharp \varinjlim_{C_{\gamma_*(F)}} \varphi_{\gamma_*(F)} \simeq j_\sharp(\gamma_*(F)).$$

Hence we obtain a canonical isomorphism $j_\sharp(\gamma_*(F)) \simeq \gamma_*(j_\sharp(F))$. It is easily seen that the corresponding map $\gamma_*(F) \to j^*(\gamma_*(j_\sharp(F))) = \gamma_*(j^* j_\sharp(F))$ is the image by γ_* of the unit map $F \to j^* j_\sharp(F)$. This shows the isomorphism we have constructed is the exchange morphism. □

Corollary 6.3.5. *For any open immersion of schemes $j : U \to S$, the functor $j_\sharp : \mathscr{A}_U \to \mathscr{A}_S$ is exact. Moreover, the induced functor*

$$j_\sharp : \mathrm{C}(\mathscr{A}_U) \to \mathrm{C}(\mathscr{A}_S)$$

preserves $\mathbf{A}^1$*-equivalences.*

Proof. Using the fact γ_* is exact and conservative, and propositions 6.3.1 and 6.3.4, it is sufficient to prove this corollary when $\mathscr{A} = \mathrm{Sh}_{\mathrm{Nis}}(Sm, \mathbf{Z})$. It is straightforward to prove exactness using Nisnevich points. The fact $j_\sharp$ preserves $\mathbf{A}^1$-equivalences follows from the exactness property and from the obvious fact it preserves strong $\mathbf{A}^1$-equivalences. □

Corollary 6.3.6. *Let $j : U \to S$ be an open immersion of schemes. For any object M of $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_U)$ the exchange morphism*

$$\mathbf{L}j_\sharp(\mathbf{R}\gamma_*(M)) \to \mathbf{R}\gamma_*(\mathbf{L}j_\sharp(M)) \tag{6.3.6.1}$$

is an isomorphism in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(S, \mathbf{Z})$.

Corollary 6.3.7. *The triangulated premotivic category $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ satisfies the support property.*

Proof. According to corollary 6.3.2, the functor $\mathbf{R}\gamma_*$ is conservative. Thus, by virtue of the preceding corollary, to prove the support property in the case of $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ it is sufficient to prove it in the case where $\mathscr{A} = \mathrm{Sh}_{\mathrm{Nis}}(Sm, \mathbf{Z})$. This follows from theorems 6.2.1 and 2.4.12. □

6.3.b. *Support property (stable case).*

6.3.8. Recall from 5.3.17 that the premotivic adjunction (γ^*, γ_*) induces a canonical adjunction of abelian premotivic categories that we denote by:

$$\tilde{\gamma}^* : \mathrm{Sp}(\mathrm{Sh}_{\mathrm{Nis}}(Sm, \mathbf{Z})) \rightleftarrows \mathrm{Sp}(\mathscr{A}_S) : \tilde{\gamma}_*$$

Proposition 6.3.9. *For any scheme S, the functor induced functor*

$$\tilde{\gamma}_* : \mathrm{C}\left(\mathrm{Sp}(\mathscr{A}_S)\right) \rightleftarrows \mathrm{C}\left(\mathrm{Sp}(\mathrm{Sh}_{\mathrm{Nis}}(Sm_S, \mathbf{Z}))\right)$$

preserves and detects stable $\mathbf{A}^1$-equivalences.

Proof. Using the equivalence between symmetric Tate spectra and non symmetric Tate spectra, we are reduced to prove this for complexes of non symmetric Tate spectra. Consider a non symmetric Tate spectrum $(E_n)_{n\in\mathbf{N}}$ with suspension maps $\sigma_n : E_n\{1\} \to E_{n+1}$. The non symmetric Tate spectrum $\tilde{\gamma}_*(E)$ is equal to $\gamma_*(E_n)$ in degree $n \in \mathbf{Z}$, and the suspension map is given by the composite:

$$\mathbb{1}_S\{1\} \otimes_S \gamma_*(E_n) \to \gamma_*(\gamma^*(\mathbb{1}_S\{1\}) \otimes_S E_n) = \gamma_*(E_n\{1\}) \xrightarrow{\gamma_*(\sigma_n)} E_{n+1}.$$

Thus, propositions 6.3.1 and 5.3.31 allows to conclude. □

Corollary 6.3.10. *The right derived functor*

$$\mathbf{R}\gamma_* = \gamma_* : \mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S) \to \mathrm{D}_{\mathbf{A}^1,\mathbf{Z}}(S)$$

is conservative.

Proposition 6.3.11. *Let $j : U \to X$ be an open immersion of schemes. For any object M of $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_U)$, the exchange morphism*

$$\mathbf{L}j_\sharp(\mathbf{R}\gamma_*(M)) \to \mathbf{R}\gamma_*(\mathbf{L}j_\sharp(M))$$

is an isomorphism in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Z}}(X)$.

Proof. From corollary 6.3.5 and the $\mathscr{P}$-base change formula for the open immersion j, one deduces easily that $j_\sharp$ preserves stable $\mathbf{A}^1$-equivalences of (non symmetric) Tate spectra. Moreover, proposition 6.3.4 shows that $j_\sharp\gamma_* = \gamma_* j_\sharp$ at the level of Tate spectra. This concludes. □

One deduces the following corollary as in the effective case (see 6.3.7).

Corollary 6.3.12. *The triangulated premotivic category $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$ satisfies the support property.*

6.3.c. *Localization for smooth schemes.*

Lemma 6.3.13. *Let $i : Z \to S$ be a closed immersion which admits a smooth retraction $p : S \to Z$. Then the exchange transformation*

$$\mathbf{L}\gamma^*\mathbf{R}i_* \to \mathbf{R}i_*\mathbf{L}\gamma^*$$

is an isomorphism in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$ (resp. $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S)$).

Proof. We first remark that for any object C of $\mathrm{C}(\mathscr{A}_Z)$ (resp. $\mathrm{C}(\mathrm{Sp}(\mathscr{A}_Z))$) the canonical sequence

$$j_\sharp(pj)^*(C) \to p^*(C) \to i_*(C)$$

is a cofiber sequence in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$ (resp. $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})_S)$). Indeed, we can check this after applying the exact conservative functor γ_*. The sequence we obtain is canonically isomorphic through exchange transformations to

$$j_\sharp j^*p^*(\gamma_*C) \to p^*(\gamma_*C) \to i_*i^*p^*(\gamma_*C)$$

using corollary 6.3.6, the commutation of γ_* with j^*, p^* and i_* (recall it is the right adjoint of a premotivic adjunction) and the relation $pi = 1$. But this last sequence is a cofiber sequence in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,\mathbf{Z}}(S)$ (resp. $\mathrm{D}_{\mathbf{A}^1,\mathbf{Z}}(S)$) because it satisfies the localization property (see 6.2.1).

Using exchange transformations, we obtain a morphism of distinguished triangles in $\mathrm{DM}^{\mathit{eff}}_{\mathbf{Z}}(S)$

$$\begin{array}{ccccccc}
\gamma^*j_\sharp j^*p^*(C) & \longrightarrow & \gamma^*p^*(C) & \longrightarrow & \gamma^*i_*(C) & \longrightarrow & \gamma^*j_\sharp j^*p^*(C)[1] \\
\| & & \| & & \downarrow{\scriptstyle Ex(\gamma^*,i_*)} & & \| \\
j_\sharp j^*p^*(\gamma^*C) & \longrightarrow & p^*(\gamma^*C) & \longrightarrow & i_*(\gamma^*C) & \longrightarrow & j_\sharp j^*p^*(\gamma^*C)[1]
\end{array}$$

The first two vertical arrows are isomorphisms as γ^* is the left adjoint of a premotivic adjunction; thus the morphism $Ex(\gamma^*, i_*)$ is also an isomorphism. □

Proposition 6.3.14. *Let $i : Z \to S$ be a closed immersion. If i admits a smooth retraction, then $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ satisfies localization with respect to i.*

Proof. This follows from proposition 2.3.23 and the preceding lemma. □

Corollary 6.3.15. *Let S be a scheme. Then the premotivic category $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ (resp. $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$) satisfies localization with respect to any closed immersion between smooth S-schemes.*

Proof. Let $i : Z \to X$ be closed immersion between smooth S-schemes. We want to prove that $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ (resp. $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$) satisfies localization with respect to i. According to 2.3.18, it is sufficient to prove that for any smooth S-scheme S, the canonical map

$$M_S(X/X - X_Z) \to i_*M_Z(X_Z)$$

is an isomorphism where we use the notation of *loc. cit.* and $M(., \mathscr{A})$ denotes the geometric sections of $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ (resp. $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$). But the premotivic triangulated category category $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$ (resp. $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$) satisfies the Nisnevich separation property and the *Sm*-base change property. Thus, we can argue locally in S for the Nisnevich topology. Thus, the statement is reduced to the preceding proposition as i admits locally for the Nisnevich topology a smooth retraction (see for example [Dég07, 4.5.11]). □

Part 3. Motivic complexes and relative cycles

7. Relative cycles

7.0. In this entire section, $\mathscr{S}$ is the category of noetherian schemes; any scheme is assumed to be noetherian. We fix a subring $\Lambda \subset \mathbf{Q}$ which will be the ring of coefficients of the algebraic cycles considered in the following section.

7.1. **Definitions.**

7.1.a. *Category of cycles.*

7.1.1. Let X be a scheme. As usual, an element of the underlying set of X will be called a *point* and a morphism $\mathrm{Spec}\,(k) \to X$ where k is a field will be called a *geometric point*. We often identify a point $x \in X$ with the corresponding geometric point $\mathrm{Spec}\,(\kappa_x) \to X$. However, the explicit expression "the point $\mathrm{Spec}\,(k) \to X$" always refers to a geometric point.
As our schemes are assumed to be noetherian, any immersion $f : X \to Y$ is quasi-compact. Thus, according to [GD60, 9.5.10], the schematic closure $\bar{X}$ of X in Y exists which gives a unique factorization of f into

$$X \xrightarrow{j} \bar{X} \xrightarrow{i} Y$$

such that i is a closed immersion and j is an open immersion with dense image[31]. Note that when Y is reduced, $\bar{Y}$ coincide with the topological closure of Y in X with its induced reduced subscheme structure. In this case, we simply call $\bar{Y}$ the closure of Y in X.

Definition 7.1.2. A Λ-cycle is a couple (X, α) such that X is a scheme and α is a Λ-linear combination of points of X. A generic point of (X, α) is a point which appears in the Λ-linear combination α with a non zero coefficient. The support $\mathrm{Supp}(\alpha)$ of α is the closure of the generic points of α.

A morphism of Λ-cycles $(Y, \beta) \to (X, \alpha)$ is a morphism of scheme $f : Y \to X$ such that $f(\mathrm{Supp}(\beta)) \subset \mathrm{Supp}(\alpha)$.

When considering such a pair (X, α), we will denote it simply by α and refer to X as the domain of α. We also use the notation $\alpha \subset X$ to mean the domain of the cycle α is the scheme X.

The category of Λ-cycle is functorial in Λ with respect to morphisms of integral rings. In what follows, cycles are assumed to have coefficients in Λ unless explicitly stated.

7.1.3. Given a property $(\mathcal{P})$ of morphisms of schemes, we will say that a morphism $f : \beta \to \alpha$ of cycles satisfies property $(\mathcal{P})$ if the induced morphism $f|^{\mathrm{Supp}(\alpha)}_{\mathrm{Supp}(\beta)}$ satisfies property $(\mathcal{P})$. A morphism $f : Y \to X$ will be said to be *pseudo-dominant* if any irreducible component of Y dominates an irreducible component of X. Thus a morphism of cycles $\sum_{j\in J} m_j.y_j \xrightarrow{f} \sum_{i\in I} n_i.x_i$ is pseudo-dominant if and only if for any $j \in J$ there exist $i \in I$ such that $f(y_j) = x_i$.

Definition 7.1.4. Let X be a scheme. We denote by $X^{(0)}$ the set of generic points of X. We define as usual the cycle associated with X as the cycle with domain X :

$$\langle X \rangle = \sum_{x\in X^{(0)}} \mathrm{lg}(\mathcal{O}_{X,x}).x.$$

The integer $\mathrm{lg}(\mathcal{O}_{X,x})$, length of an artinian local ring, is called the geometric multiplicity of x in X.

When no confusion is possible, we usually omit the delimiters in the notation $\langle X \rangle$. As an example, we say that α *is a cycle over* X to mean the existence of a structural morphism of cycles $\alpha \to \langle X \rangle$.

7.1.5. When Z is a closed subscheme of a scheme X, we denote by $\langle Z \rangle_X$ the cycle $\langle Z \rangle$ considered as a cycle with domain X.
Consider a cycle α with domain X. Let $(Z_i)_{i\in I}$ be the irreducible components of $\mathrm{Supp}(\alpha)$. Then we can write α uniquely as $\alpha = \sum_{i\in I} n_i.\langle Z_i \rangle_X$. We call this writing the *standard form* of α for short.

Definition 7.1.6. Let $\alpha = \sum_{i\in I} n_i.x_i$ be a cycle with domain X and $f : X \to Y$ be any morphism.

For any $i \in I$, put $y_i = f(x_i)$. Then f induces an extension field $\kappa(x_i)/\kappa(y_i)$ between the residue fields. We let d_i be the degree of this extension field in case it is finite and 0 otherwise.

[31] Recall the scheme $\bar{X}$ is characterized by the property of being the smallest sub-scheme of Y with the existence of such a factorization.

We define the pushout of α by f as the cycle with domain Y

$$f_*(\alpha) = \sum_{i\in I} n_i d_i.f(x_i).$$

Thus, when f is an immersion, $f_*(\alpha)$ is the same cycle as α but seen as a cycle with domain X. Remark also that we obtain the following equality

$$f_*\big(\langle X\rangle\big) = \big\langle \bar{X}\big\rangle_Y \tag{7.1.6.1}$$

where $\bar{X}$ is the schematic closure of X in Y (indeed X is a dense open subscheme in $\bar{X}$). When f is clear, we sometimes abusively put: $\langle X\rangle_Y := f_*(\langle X\rangle)$.

We always have a canonical morphism $\alpha \to f_*(\alpha)$. In case we have a morphism of cycles $\alpha \to \gamma$ with domain the morphism $p : X \to S$, any commutative diagram of schemes

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S & \end{array}$$

induces a unique commutative diagram of cycles

$$\begin{array}{ccc} \alpha & \longrightarrow & f_*(\alpha). \\ & \searrow \quad \swarrow & \\ & \gamma & \end{array}$$

By transitivity of degrees, we obviously have $f_*g_* = (fg)_*$ for a composable pair of morphisms (f, g).

Definition 7.1.7. Let $\alpha = \sum_{i\in I} n_i.x_i$ be a cycle over a scheme S with domain $f : X \to S$ and $U \subset S$ be an open subscheme. Let $I' = \{i \in I \mid f(x_i) \in U\}$. We define the restriction of α over U as the cycle $\alpha|_U = \sum_{i\in I'} n_i.x_i$ with domain $X \times_S U$ considered as a cycle over U.

If $\alpha = \sum_{i\in I} n_i.\langle Z_i\rangle_X$, then obviously $\alpha|_U = \sum_{i\in I} n_i.\langle Z_i \times_S U\rangle_{X_U}$. We state the following obvious lemma for convenience :

Lemma 7.1.8. *Let S be a scheme, $U \subset S$ an open subscheme and X be an S-scheme. Let $j : X_U \to X$ be the obvious open immersion.*

(i) For any cycle $\alpha' \subset X_U$, $\big(j_*(\alpha')\big)|_U = \alpha'$.
(ii) Assume $\bar{U} = S$. For any cycle $\alpha \subset X$, $j_*(\alpha\,|_U) = \alpha$.

7.1.b. *Hilbert cycles.*

7.1.9. Recall that a finite dimensional scheme X is equidimensional – we will also say *absolutely equidimensional* – if its irreducible components have all the same dimension.

Recall a flat morphism $f : X \to S$ is equidimensional if it is of finite type and for any generic points η, η' of the same connected component of S, the fibers X_η, $X_{\eta'}$ are absolutely equidimensional of the same dimension.

Definition 7.1.10. Let S be a scheme.

Let α be a cycle over S with domain X. We say that α is a Hilbert cycle over S if there exists a finite family $(Z_i)_{i\in I}$ of closed subschemes of X which are flat equidimensional over S and a finite family $(n_i)_{i\in I} \in \Lambda^I$ such that

$$\alpha = \sum_{i\in I} n_i.\langle Z_i\rangle_X.$$

Example 7.1.11. Any cycle over a field k is a Hilbert cycle over $\mathrm{Spec}\,(k)$. Let S be the spectrum of a discrete valuation ring. A cycle $\alpha = \sum_{i\in I} n_i.x_i$ over S is a Hilbert cycle if and only if each point x_i lies over the generic point of S. Indeed, an integral S-scheme is flat if and only if it is dominant.

Consider a Hilbert S-cycle $\alpha \subset X$ and a morphism of schemes $f : S' \to S$. Put $X' = X \times_S S'$. We choose finite families $(Z_i)_{i\in I}$ of flat S-schemes and $(n_i)_{i\in I} \in \Lambda^I$ such that $\alpha = \sum_{i\in I} n_i.\langle Z_i\rangle_X$. Then proposition 3.2.2 of [SV00b] says exactly the cycle

$$\sum_{i\in I} n_i.\langle Z_i \times_S S'\rangle_{X'}$$

depends only on α and not on the chosen families.

Definition 7.1.12. Adopting the previous notations and hypothesis, we define the pullback cycle of α along the morphism $f : S' \to S$ as the cycle with domain X'

$$\alpha \otimes^{\flat}_S S' = \sum_{i\in I} n_i.\langle Z_i \times_S S'\rangle_{X'}.$$

In this setting the following lemma is obvious :

Lemma 7.1.13. *Let α be a Hilbert cycle over S, and $S'' \to S' \to S$ be morphisms of schemes. Then $(\alpha \otimes^{\flat}_S S') \otimes^{\flat}_{S'} S'' = \alpha \otimes^{\flat}_S S''$.*

We will use another important computation from [SV00b] (it is a particular case of *loc. cit.*, 3.6.1).

Proposition 7.1.14. *Let R be a discrete valuation ring with residue field k.*
Let $\alpha \subset X$ be a Hilbert cycle over $\mathrm{Spec}\,(R)$ *and $f : X \to Y$ a morphism over* $\mathrm{Spec}\,(R)$*. We denote by $f' : X' \to Y'$ the pullback of f over* $\mathrm{Spec}\,(k)$*.*
Suppose that the support of α is proper with respect to f.
Then $f_*(\alpha)$ is a Hilbert cycle over R and the following equality of cycles holds in X':

$$f'_*(\alpha \otimes^{\flat}_S k) = f_*(\alpha) \otimes^{\flat}_S k.$$

Let us introduce the following classical definition:

Definition 7.1.15. Let $p : \tilde{S} \to S$ be a birational morphism. Let C be the minimal closed subset of S such that p induces an isomorphism $(\tilde{S} - \tilde{S} \times_S C) \to (S - C)$.

Consider $\alpha = \sum_{i\in I} n_i.\langle Z_i\rangle_X$ a cycle over S written in standard form.

We define the strict transform $\tilde{Z}_i$ of the closed subscheme Z_i in X along p is the schematic closure of $(Z_i - Z_i \times_S C) \times_S \tilde{S}$ in $X \times_S \tilde{S}$. We define the strict transform of α along p as the cycle over $\tilde{S}$

$$\tilde{\alpha} = \sum_{i\in I} n_i.\langle \tilde{Z}_i\rangle_{X\times_S \tilde{S}}.$$

As in [SV00b], we remark that a corollary of the platification theorem of Gruson-Raynaud is the following :

Lemma 7.1.16. *Let S be a reduced scheme and α be a pseudo-dominant cycle over S.*
Then there exists a dominant blow-up $p : \tilde{S} \to S$ such that the strict transform $\tilde{\alpha}$ of α along p is a Hilbert cycle over $\tilde{S}$.

We conclude this part by recalling an elementary lemma about cycles and Galois descent which will be used extensively in the next sections :

Lemma 7.1.17. *Let L/K be an extension of fields and X be a K-scheme. We put $X_L = X \times_K$* $\mathrm{Spec}\,(L)$ *and consider the faithfully flat morphism $f : X_L \to X$.*
Denote by $\mathrm{Cycl}(X)$ *(resp.* $\mathrm{Cycl}(X_L)$*) the cycles with domain X (resp. X_L).*

1. *The morphism $f^* : \mathrm{Cycl}(X) \to \mathrm{Cycl}(X_L), \beta \mapsto \beta \otimes^{\flat}_K L$ is a monomorphism.*
2. *Suppose L/K is finite. For any K-cycle $\beta \in \mathrm{Cycl}(X)$, $f_*(\beta \otimes^{\flat}_K L) = [L : K].\beta$.*
3. *Suppose L/K is finite normal with Galois group G.*
   The image of f^* consists of cycles invariant under the action of G. For any cycle $\beta \in \mathrm{Cycl}(X_L)^G$, there exists a unique cycle $\beta_K \in \mathrm{Cycl}(X)$ such that

$$\beta_K \otimes^{\flat}_K L = [L : K]_i.\beta$$

where $[L : K]_i$ is the inseparable degree *of L/K.*

7.1.c. *Specialization.* The aim of this section is to give conditions on cycles so that one can define a *relative tensor product* on them.

Definition 7.1.18. Consider two cycles $\alpha = \sum_{i\in I} n_i.s_i$ and $\beta = \sum_{j\in J} m_i.x_j$. Let S be the support of α.

A morphism $\beta \xrightarrow{f} \alpha$ of cycles is said to be *pre-special* if it is of finite type and for any $j \in J$, there exists $i \in I$ such that $f(x_j) = s_i$ and $n_i|m_j$ in Λ. We define the reduction of β/α as the cycle over S

$$\beta_0 = \sum_{j\in J, f(x_j)=s_i} \frac{m_j}{n_i}.x_j.$$

Example 7.1.19. Let S be a scheme and α a Hilbert S-cycle. Then the canonical morphism of cycles $\alpha \to \langle S\rangle$ is pre-special. If S is the spectrum of a discrete valuation ring, an S-cycle α is pre-special if and only if it is a Hilbert S-cycle.

Definition 7.1.20. Let α be a cycle.

A *point* (resp. *dash*) of α will be a morphism $\operatorname{Spec}(k) \xrightarrow{x} \alpha$ (resp. $\operatorname{Spec}(R) \xrightarrow{\tau} \alpha$) such that k is a field (resp. R is a discrete valuation ring). We simply say that x (resp. τ) is *dominant* if the image of the generic point in the domain of α is a generic point of α.
Let $x : \operatorname{Spec}(k_0) \to \alpha$ be a point. An extension of x will be a point y on α of the form $\operatorname{Spec}(k) \to \operatorname{Spec}(k_0) \xrightarrow{x} \alpha$.

A *fat point* of α will be morphisms

$$\operatorname{Spec}(k) \xrightarrow{s} \operatorname{Spec}(R) \xrightarrow{\tau} \alpha$$

such that τ is a dominant dash and the image of s is the closed point of $\operatorname{Spec}(R)$.
Given a point $x : \operatorname{Spec}(k) \to \alpha$, a fat point over x is a factorization of x through a dominant dash as above.

In the situation of the last definition, we denote simply by (R, k) a fat point over x, without indicating in the notation the morphisms s and τ.

Remark 7.1.21. With our choice of terminology, a point of α is in general an extension of a specialization of a generic point of α. As a further example, a dominant point of α is an extension of a generic point of α.

Lemma 7.1.22. *For any cycle α and any non dominant point $x : \operatorname{Spec}(k_0) \to \alpha$, there exists an extension $y : \operatorname{Spec}(k) \to \alpha$ of x and a fat point (R, k) over y.*

Proof. Replacing α by its support S, we can assume $\alpha = \langle S\rangle$. Let s be the image of x in S, κ its residue field. We can assume S is reduced, irreducible by taking one irreducible component containing s, and local with closed point s. Let $S = \operatorname{Spec}(A)$, $K = \operatorname{Frac}(A)$. According to [GD61, 7.1.7], there exists a discrete valuation ring R such that $A \subset R \subset K$, and R/A is an extension of local rings. Then any composite extension k/κ of k_0 and the residue field of R over κ gives the desired fat point (R, k). □

Definition 7.1.23. Let $\beta \to \alpha$ be a pre-special morphism of cycles. Consider S the support of α and X the domain of β. Let $\beta_0 = \sum_{j\in J} m_j.\langle Z_j\rangle_X$ be the reduction of β/α written in standard form.

(1) Let $\operatorname{Spec}(K) \to \alpha$ be a dominant point. We define the following cycle over $\operatorname{Spec}(K)$ with domain $X_K = X \times_S \operatorname{Spec}(K)$:

$$\beta_K = \sum_{j\in J} m_j.\langle Z_j \times_S \operatorname{Spec}(K)\rangle_{X_K}.$$

(2) Let $\operatorname{Spec}(R) \xrightarrow{\tau} S$ be a dominant dash, K be the fraction field of R and $j : X_K \to X_R$ be the canonical open immersion. We define the following cycle over R with domain X_R :

$$\beta_R = j_*(\beta_K).$$

According to example 7.1.11, β_R is a Hilbert cycle over R.

(3) Let $x : \mathrm{Spec}\,(k) \to \alpha$ be a point on α and (R, k) be a fat point over x.
We define the specialization of β along the fat point (R, k) as the cycle

$$\beta_{R,k} = \beta_R \otimes_R^{\flat} k$$

using the above notation and definition 7.1.12. It is a cycle over $\mathrm{Spec}\,(k)$ with domain $X_k = X \times_S \mathrm{Spec}\,(k)$.

Remark 7.1.24. Let $\beta \subset X$ be an S-cycle, $x : \mathrm{Spec}\,(K) \to S$ be a dominant point and U be an open neighbourhood of x in S.
Then if β is pre-special over S, $\beta|_U$ is pre-special over U and $\beta_K = (\beta|_U)_K$.
If $\tau : \mathrm{Spec}\,(R) \to S$ (resp. (R, k)) is a dash (resp. fat point) with generic point x, we also get $\beta_R = (\beta|_U)_R$ (resp. $\beta_{R,k} = (\beta|_U)_{R,k}$).

7.1.25. Let S be a reduced scheme, and $\beta = \sum_{i\in I} n_i.x_i$ be an S-cycle with domain X. For any index $i \in I$, let κ_i be the residue field of x_i.

Consider a dominant point $x : \mathrm{Spec}\,(K) \to S$. Let s be its image in S and F be the residue field of s. We put $I' = \{i \in I \mid f(x_i) = s\}$ where $f : X \to S$ is the structural morphism. With these notations, we get

$$\beta_K = \sum_{i\in I'} n_i.\langle \mathrm{Spec}\,(\kappa_i \otimes_F K)\rangle_{X_K},$$

and for a dominant dash $\mathrm{Spec}\,(R) \to S$ with generic point x,

$$\beta_R = \sum_{i\in I'} n_i.\langle \mathrm{Spec}\,(\kappa_i \otimes_F K)\rangle_{X_R}, \tag{7.1.25.1}$$

where $\mathrm{Spec}\,(\kappa_i \otimes_F K)$ is seen as a subscheme of X_K (resp. X_R).

Consider a fat point (R, k) with generic point x and write $\beta = \sum_{i\in I} n_i.\langle Z_i\rangle_X$ in standard form (*i.e.* Z_i is the closure of $\{x_i\}$ in X). Then according to (7.1.6.1), we obtain[32]

$$\beta_{R,k} = \sum_{i\in I'} n_i.\left\langle \overline{Z_{i,K}} \times_R \mathrm{Spec}\,(k)\right\rangle_{X_k}$$

where $Z_{i,K} = Z_i \times_S \mathrm{Spec}\,(K)$ is considered as a subscheme of X_K and the schematic closure is taken in X_R.

Considering the description of the schematic closure for the generic fiber of an R-scheme (*cf.* [GD67, 2.8.5]), we obtain the following way to compute $\beta_{R,k}$. By definition, R is an F-algebra. For $i \in I'$, let A_i be the image of the canonical morphism

$$\kappa_i \otimes_F R \to \kappa_i \otimes_F K.$$

It is an R-algebra without R-torsion. Moreover, the factorization

$$\mathrm{Spec}\,(\kappa_i \otimes_F K) \to \mathrm{Spec}\,(A_i) \to \mathrm{Spec}\,(\kappa_i \otimes_F R)$$

defines $\mathrm{Spec}\,(A_i)$ as the schematic closure of the left hand side in the right hand side (*cf.* [GD67, 2.8.5]). In particular, we get an immersion $\mathrm{Spec}\,(A_i \otimes_R k) \to X_k$ and the nice formula :

$$\beta_{R,k} = \sum_{i\in I'} n_i.\left\langle \mathrm{Spec}\,(A_i \otimes_R k)\right\rangle_{X_k}.$$

Definition 7.1.26. Consider a morphism of cycles $f : \beta \to \alpha$ and a point $x : \mathrm{Spec}\,(k_0) \to \alpha$.
We say that f is special at x if it is pre-special and for any extension $y : \mathrm{Spec}\,(k) \to \alpha$ of x, for any fat points (R, k) and (R', k) over y, the equality $\beta_{R,k} = \beta_{R',k}$ holds in X_k. Equivalently, we say that β/α is special at s.
We say that f is special (or that β is special over α) if it is special at every point of α.

Remark 7.1.27. (1) Trivially, f is special at every dominant point of α.

[32] This shows that our definition coincide with the one given in [SV00b] (p. 23, paragraph preceding 3.1.3) in the case where $\alpha = \langle S\rangle$, S reduced.

(2) Given an extension y of x, it is equivalent for f to be special at x or at y (use lemma 7.1.17(1)). Thus, in the case where $\alpha = \langle S \rangle$, we can restrict our attention to the points $s \in S$.
(3) According to 7.1.24, the property that β/S is special at $s \in S$ depends only on an open neighbourhood U of s in S. More precisely, the following conditions are equivalent :
  (i) β is special at s over S.
  (ii) $\beta|_U$ is special at s over U.

Example 7.1.28. Let S be a scheme and β be a Hilbert cycle over S. We have already seen that $\beta \to \langle S \rangle$ is pre-special. The next lemma shows this morphism is in fact special.

Lemma 7.1.29. *Let S be a scheme and β be a Hilbert cycle over S. Consider a point $x : \operatorname{Spec}(k) \to S$ and a fat point (R, k) over x.*

Then $\beta_{R,k} = \beta \otimes^{\flat}_S k$.

Proof. According to the preceding definition and lemma 7.1.13 it is sufficient to prove $\beta_R = \beta \otimes^{\flat}_S R$. As the two sides of this equation are unchanged when replacing β by the reduction β_0 of β/S, we can assume that S is reduced. By additivity, we are reduced to the case where $\beta = \langle X \rangle$ is the fundamental cycle associated with a flat S-scheme X. According to 7.1.6.1, $\beta_R = \langle \overline{X_K} \rangle_{X_R}$. Applying now [GD67, 2.8.5], $\overline{X_K}$ is the unique closed subscheme Z of X_R such that Z is flat over $\operatorname{Spec}(R)$ and $Z \times_R \operatorname{Spec}(K) = X_K$. Thus, as X_R is flat over $\operatorname{Spec}(R)$, we get $\overline{X_K} = X_R$ and this concludes. □

Lemma 7.1.30. *Let $p : \tilde{S} \to S$ be a birational morphism and consider a commutative diagram*

$$\begin{array}{ccc} \operatorname{Spec}(k) & \longrightarrow & \operatorname{Spec}(R) \\ & \searrow & \downarrow \\ & & S \end{array} \qquad \begin{array}{c} \nearrow \tilde{S} \\ \downarrow p \\ \end{array}$$

such that (R, k) is a fat point of $\tilde{S}$ and S.

Consider a pre-special cycle β over S and $\tilde{\beta}$ its strict transform along p. Then, $\tilde{\beta}$ is pre-special and $\tilde{\beta}_{R,k} = \beta_{R,k}$.

Proof. Using 7.1.24, we reduce to the case where p is an isomorphism which is trivial. □

Lemma 7.1.31. *Let S be a reduced scheme, $x : \operatorname{Spec}(k_0) \to S$ be a point and α be a pre-special cycle over S. Let $p : \tilde{S} \to S$ be a dominant blow-up such that the strict transform $\tilde{\alpha}$ of α along p is a Hilbert cycle over $\tilde{S}$. Then the following conditions are equivalent :*

(i) α is special at x.
(ii) for every points $x_1, x_2 : \operatorname{Spec}(k) \to \tilde{S}$ such that $p \circ x_1 = p \circ x_2$ and $p \circ x_1$ is an extension of x, $\tilde{\alpha} \otimes^{\flat}_{\tilde{S}} x_1 = \tilde{\alpha} \otimes^{\flat}_{\tilde{S}} x_2$.

Proof. The case where x is a dominant point follows from the definitions and the fact p is an isomorphism at the generic point. We thus assume x is non dominant.
$(i) \Rightarrow (ii)$: Applying lemma 7.1.22 to x_i, $i = 1, 2$, we can find an extension $x'_i : \operatorname{Spec}(k_i) \to \tilde{S}$ of x_i and a fat point (R_i, k_i) over x'_i. Taking a composite extension L of k_1 and k_2 over k, we can further assume $L = k_1 = k_2$ and $p \circ x'_1 = p \circ x'_2$. Then for $i = 1, 2$, we get

$$\left(\tilde{\alpha} \otimes^{\flat}_{\tilde{S}} x_i\right) \otimes^{\flat}_k L \overset{7.1.13}{=\!=} \tilde{\alpha} \otimes^{\flat}_{\tilde{S}} x'_i \overset{7.1.29}{=\!=} \tilde{\alpha}_{R_i,L} \overset{7.1.30}{=\!=} \alpha_{R_i,L},$$

and this concludes according to 7.1.17(1).
$(ii) \Rightarrow (i)$: Consider an extension $y : \operatorname{Spec}(k) \to \alpha$ over x and two fat point (R_1, k), (R_2, k) over y. Fix $i \in \{1, 2\}$. As p is proper birational, the dash $\operatorname{Spec}(R_i)$ on S can be extended (uniquely) to $\tilde{S}$. Let $x_i : \operatorname{Spec}(k) \to \operatorname{Spec}(R_i) \to \tilde{S}$ be the induced point. Then the following computation allows to conclude :

$$\alpha_{R_i,k} \overset{7.1.30}{=\!=} \tilde{\alpha}_{R_i,k} \overset{7.1.29}{=\!=} \tilde{\alpha} \otimes^{\flat} x_i$$

□

7.1.d. *Relative product.*

7.1.32. In this part, we construct a *relative product* which generalizes the pullback defined by Suslin et Voevodsky in [SV00b, 3.3.1]. Consider the situation of a diagram of cycles

$$\begin{array}{ccc} \beta & & X \\ \downarrow f & \subset & \downarrow \\ \alpha' \longrightarrow \alpha & & S' \longrightarrow S \end{array}$$

where the diagram on the right is the support of the one on the left. Let n be exponential characteristic of $\overline{\alpha'}$.

The relative product of β and α' over α will be a $\Lambda[1/n]$-cycle which fits into the following commutative diagram of cycles

$$\begin{array}{ccc} \beta \otimes_\alpha \alpha' \longrightarrow \beta & & X \times_S S' \longrightarrow X \\ \downarrow \quad \downarrow & \subset & \downarrow \quad \downarrow \\ \alpha' \longrightarrow \alpha & & S' \longrightarrow S \end{array}$$

where the right commutative square is again the support of the left one.

It will be defined under an assumption on β/α and is therefore non symmetric[33]. This assumption will imply that β/α is pre-special, and the first property of $\beta \otimes_\alpha \alpha'$ is that it is pre-special over α'.

We define this product in three steps in which the following properties[34] will be a guideline :

(P1) Let S_0 be the support of α and β_0 be the reduction of β/α as an S_0-cycle. Consider the canonical factorization $\alpha' \to S_0 \to \alpha$.
Then, $\beta \otimes_\alpha \alpha' = \beta_0 \otimes_{S_0} \alpha'$.

(P2) Consider a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}\,(E) & \longrightarrow & \mathrm{Spec}\,(R') & \longrightarrow & \mathrm{Spec}\,(R) \\ & & \downarrow & (*) & \downarrow \\ & & \alpha' & \longrightarrow & \alpha \end{array}$$

such that (R, E) (resp. (R', E)) is a fat point on α (resp. α').
Then, $(\beta \otimes_\alpha \alpha')_{R',E} = \beta_{R,E}$.

Assume $\alpha' \to \alpha = \langle S' \to S \rangle$.

(P3) If β is a Hilbert cycle over S, $\beta \otimes_S S' = \beta \otimes_S^\flat S'$.

(P4) Consider a factorization $S' \to U \xrightarrow{j} S$ such that j is an open immersion. Then $\beta \otimes_S S' = \beta|_U \otimes_U S'$.

(P5) Consider a factorization $S' \to \tilde{S} \xrightarrow{p} S$ such that p is a birational morphism. Then $\beta \otimes_S S' = \tilde{\beta} \otimes_{\tilde{S}} S'$.

Lemma 7.1.33. *Consider the hypothesis of 7.1.32 in the case where $\alpha' = \mathrm{Spec}\,(k)$ is a point x of α.*

We suppose that f is special at x.

Then the pre-special $\Lambda[1/n]$-cycle $\beta \otimes_\alpha k$ exists and is uniquely determined by property (P2) above. We also put $\beta_k := \beta \otimes_\alpha k$.

The properties (P1) to (P5) are fulfilled and in addition :
(P6) For any extension fields L/k, $\beta_L = \beta_k \otimes_k^\flat L$.

Proof. According to lemma 7.1.22 there always exists a fat point (R, E) over an extension of x. Thus the unicity statement follows from 7.1.17(1).

---

[33] See further 7.2.3 for this question.

[34] All these properties except (P3) will be particular cases of the associativity of the exterior product.

For the existence, we first consider the case where $\alpha = \langle S \rangle$ is a reduced scheme. Applying lemma 7.1.16, there exists a blow-up $p : \tilde{S} \to S$ such that the strict transform $\tilde{\beta}$ of β along p is a Hilbert cycle over $\tilde{S}$.

As p is surjective, the fiber $\tilde{S}_k$ is a non empty algebraic k-scheme. Thus, it admits a closed point given by a finite extension k'_0 of k. Let k'/k be a normal closure of k'_0/k and G be its Galois group. As β/S is special at x by hypothesis, lemma 7.1.31 implies that $\tilde{\beta} \otimes^{\flat}_{\tilde{S}} k'$ is G-invariant. Thus, applying lemma 7.1.17, there exists a unique cycle $\beta_k \subset X_k$ with coefficients in $\Lambda[1/p]$ such that $\beta_k \otimes^{\flat}_k k' = \tilde{\beta} \otimes^{\flat}_{\tilde{S}} k'$.

We prove (P2). Given a diagram $(*)$ with $\alpha' = \mathrm{Spec}\,(k)$, we first remark that $(\beta_k)_{R',E} = \beta_k \otimes^{\flat}_k E$. As p is proper birational, the dominant dash $\mathrm{Spec}\,(R) \to S$ lifts to a dominant dash $\mathrm{Spec}\,(R) \to \tilde{S}$. Let E'/k be a composite extension of k'/k and E/k. With these notations, we get the following computation :

$$\beta_{R,E} \otimes^{\flat}_E E' \overset{7.1.30}{=\!=} \tilde{\beta}_{R,E} \otimes^{\flat} E' \overset{7.1.29}{=\!=} \tilde{\beta} \otimes^{\flat}_{\tilde{S}} E' \overset{7.1.13}{=\!=} (\tilde{\beta} \otimes^{\flat}_{\tilde{S}} k') \otimes^{\flat}_E E' = \beta_k \otimes^{\flat}_k E',$$

so that we can conclude by applying 7.1.17(1).

In the general case, we consider he support S of α abd β_0/S the reduction of β/α. According to (P1), we are led to put $\beta_k := (\beta_0)_k$ with the help of the preceding case. Considering the definition of specialization along fat points, we easily check this cycle satisfies property (P2).

Finally, property (P6) (resp. (P3), (P5)) follows from the unicity statement applying lemmas 7.1.22, 7.1.17(1) (resp. and moreover lemma 7.1.29, 7.1.30). □

Remark 7.1.34. In the case where x is a dominant point, the cycle β_k defined in the previous proposition agrees with the one defined in 7.1.23(1).

Lemma 7.1.35. *Consider the hypothesis of 7.1.32 in the case where $\alpha' = \mathrm{Spec}\,(O)$ is a dash of α. Let K be the fraction field of O and x the corresponding point on α.*

We suppose that f is special at x.

Then the pre-special $\Lambda[1/n]$-cycle $\beta \otimes_\alpha O$ exists and is uniquely defined by the property $(\beta \otimes_\alpha O) \otimes^{\flat}_O K = \beta_K$ with the notations of the preceding lemma. We also put $\beta_O := \beta \otimes_\alpha O$.

The properties (P1) to (P5) are fulfilled and in addition :

(P6') For any extension O'/O of discrete valuation rings, $\beta_{O'} = \beta_O \otimes^{\flat}_O O'$.

Proof. Remark that, with the notation of definition 7.1.7, $\beta_O \otimes^{\flat}_O K = \beta_O|_{\mathrm{Spec}(K)}$. For the first statement, we simply apply lemma 7.1.8 and put $\beta_O = j_*(\beta_K)$ where $j : X_K \to X_O$ is the canonical open immersion.

Then properties (P1), (P3), (P4), (P5) and (P6') of the case considered in this lemma follows easily from the uniqueness statement and the corresponding properties in the preceding lemma (applying again 7.1.8).

It remains to prove (P2). According to (P1), we reduce to the case $\alpha = \langle S \rangle$ for a reduced scheme S. We choose a birational morphism $p : \tilde{S} \to S$ such that the proper transform $\tilde{\beta}$ is a Hilbert $\tilde{S}$-cycles. Consider a diagram of the form $(*)$ in this case. According to property (P3), we can assume $R' = O$.

Remark the dash $\mathrm{Spec}\,(R) \to S$ admits an extension $\mathrm{Spec}\,(R) \to \tilde{S}$ as p is proper. The point x admits an extension K'/K which lifts to a point $x' : \mathrm{Spec}\,(K') \to \tilde{S}$ – again $\tilde{S}_K$ is a non empty algebraic scheme. The discrete valuation corresponding to $O \subset K$ extends to a discrete valuation on K' as K'/K is finite. Let $O' \subset K'$ be the corresponding valuation ring. The corresponding dash $\mathrm{Spec}\,(O') \to S$ thus admits a lifting to $\tilde{S}$ corresponding to the point x' as p is proper. Considering a composite extension E'/K of K'/K and E/K, we have obtained a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}\,(E') & \longrightarrow & \mathrm{Spec}\,(O') & \longrightarrow & \mathrm{Spec}\,(R) \\ & & \| & & \downarrow \\ & & \mathrm{Spec}\,(O') & \longrightarrow & \tilde{S} \end{array}$$

which lifts our original diagram $(*)$. Let x_1 (resp. x_2) be the point $\mathrm{Spec}\,(E)' \to \tilde{S}$ corresponding to the the composite through the upper way (resp. lower way) in the preceding diagram.

Then, $\beta_{R,E} \otimes^{\flat}_{E} E' = \tilde{\beta}_{x_1}$. Moreover, we get

$$(\beta \otimes_S O)_{O,E} \otimes^{\flat}_{E} E' \overset{7.1.29}{=\!=} (\beta \otimes_S O) \otimes^{\flat}_{O} E' \overset{(P5)+(P6')}{=\!=\!=} (\tilde{\beta} \otimes_{\tilde{S}} O') \otimes^{\flat}_{O'} E' \overset{(P3)}{=\!=} \tilde{\beta}_{x_2}.$$

By hypothesis, β/α is special at $\mathrm{Spec}\,(K') \to S$. Thus lemma 7.1.31 concludes. □

Theorem 7.1.36. *Consider the hypothesis of 7.1.32.*
Assume f is special at the generic points of α'.
Then the pre-special $\Lambda[1/n]$-cycle $\beta \otimes_\alpha \alpha'$ exists and is uniquely determined by property (P2).
It satisfies all the properties (P1) to (P5).

Proof. According to lemma 7.1.22, for any point s of S' with residue field κ, there exists an extension E/κ and a fat point (R, E) (resp. (R', E)) of α (resp. α') over $\mathrm{Spec}\,(E) \to \alpha$ (resp. $\mathrm{Spec}\,(E) \to \alpha'$). The uniqueness statement follows by applying lemma 7.1.17(1).

For the existence, we write $\alpha' = \sum_{i\in I} n_i.\langle Z_i\rangle_{S'}$ in standard form.

For any $i \in I$, let K_i be the function field of Z_i and consider the canonical morphism $\mathrm{Spec}\,(K_i) \to \alpha$. Let $\beta_{K_i} \subset X_{K_i}$ be the $\Lambda[1/n]$-cycle defined in lemma 7.1.33. Let $j_i : X_{K_i} \to X'$ be the canonical immersion and put :

$$\beta \otimes_\alpha \alpha' = \sum_{i\in I} n_i.j_{i*}(\beta_{K_i}). \tag{7.1.36.1}$$

Then properties (P1), (P3), (P4) and (P5) are direct consequences of this definition and of the corresponding properties of lemma 7.1.33.

We check property (P2). Given a diagram of the form $(*)$, there exists a unique $i \in I$ such that $\mathrm{Spec}\,(R')$ dominates Z_i. Thus we get for this choice of $i \in I$ that $(\beta \otimes_\alpha \alpha')_{R',E} = \big(j_{i*}(\beta_{K_i})\big)_{R',E}$. Let K' be the fraction field of R' and consider the open immersion $j' : X_{K'} \to X_{R'}$. The following computation then concludes :

$$\big(j_{i*}(\beta_{K_i})\big)_{R',E} = j'_*\big(j_{i*}(\beta_{K_i})_{K'}\big) \otimes^{\flat}_{R'} E \overset{7.1.24}{=\!=} j'_*(\beta_{K'}) \otimes^{\flat}_{R'} E \overset{7.1.35}{=\!=} \beta_{R'} \otimes^{\flat}_{R'} E \overset{7.1.35(P2)}{=\!=\!=} \beta_{R,E}.$$

□

Definition 7.1.37. In the situation of the previous theorem, we call the $\Lambda[1/n]$-cycle $\beta \otimes_\alpha \alpha'$ the pullback of β/α by α'.

7.1.38. By construction, the cycle $\beta \otimes_\alpha \alpha'$ is bilinear with respect to addition of cycles in the following sense:

(P7) Consider the hypothesis of 7.1.32. Let α'_1, α'_2 be cycles with domain S' such that $\alpha = \alpha'_1 + \alpha'_2$. If β/α is special at the generic points of α_1 and α_2, then the following cycles are equal in $X \times_S S'$:

$$\beta \otimes_\alpha (\alpha'_1 + \alpha'_2) = \beta \otimes_\alpha \alpha'_1 + \beta \otimes_\alpha \alpha'_2.$$

(P7') Consider the hypothesis of 7.1.32. Let β_1, β_2 be cycles with domain X such that $\beta = \beta_1 + \beta_2$. If β_1 and β_2 are special over α at the generic points of α', then β/α is special at the generic points of α' and the following cycles are equal in $X \times_S S'$:

$$(\beta_1 + \beta_2) \otimes_\alpha \alpha' = \beta_1 \otimes_\alpha \alpha' + \beta_2 \otimes_\alpha \alpha'.$$

In the theorem above, we can assume that X (resp. S, S') is the support of β (resp. α, α'). Thus the support of $\beta \otimes_\alpha \alpha'$ is included in $X \times_S S'$. More precisely:

Lemma 7.1.39. *Consider the hypothesis of 7.1.32 and assume that X (resp. S, S') is the support of β (resp. α, α'). Then, if β/α is special at the generic points of α', we obtain:*

(i) Let $(X \times_S S')^{(0)}$ be the generic points of $X \times_S S'$. Then, we can write

$$\beta \otimes_\alpha \alpha' = \sum_{x\in(X\times_S S')^{(0)}} m_x.x$$

(ii) For any generic point x of $X \times_S S'$, if $m_x \neq 0$, the image of x in S' is a generic point s' and the multiplicity of s' in α' divides m_x in $\Lambda[1/n]$.

Proof. Point (ii) is just a traduction that $\beta \otimes_\alpha \alpha'$ is pre-special over α'. For point (i), we reduce easily to the case where α is the scheme S and S is reduced. We can also assume that α' is the spectrum of a field k. It is sufficient to check point (i) after an extension of k. Thus we can apply lemma 7.1.16 to reduce to that case where β is a Hilbert cycle over S. This case is obvious. □

Definition 7.1.40. In the situation of the previous lemma, we put

$$m^{SV}(x; \beta \otimes_\alpha \alpha') := m_x \in \Lambda[1/n]$$

and we call them the Suslin-Voevodsky's multiplicities (in the operation of relative product).

Remark 7.1.41. Consider the notations of the previous lemma:

(1) Assume that α is the spectrum of a field k. Then the product $\beta \otimes_k \alpha'$ is always defined and agrees with the classical *exterior product* (according to (P3)).
(2) According to the previous lemma, the irreducible components of $X \times_S S'$ which does not dominate an irreducible component of S' have multiplicity 0: they correspond to the "non proper components" with respect to the operation $\beta \otimes_\alpha \alpha'$.
(3) Assume $\alpha' \to \alpha = \langle S' \xrightarrow{p} S \rangle$, $\beta = \sum_{i \in I} n_i.x_i$. Let y be a generic point of $X \times_S S'$ lying over a generic point s' of S'. Let S'_0 be the irreductible component of S' corresponding to s'. Consider *any* irreductible component S_0 of S which contains $p(s')$ and let $\beta_0 = \sum_i n_i.x_i$ where the sums runs over the indexes i such that x_i lies over S_0. Then, according to (7.1.36.1),
$$m(y; \beta \otimes_S \langle S' \rangle) = m(y; \beta_0 \otimes_{S_0} \langle S'_0 \rangle).$$
This is a key property of the Suslin-Voevodsky's multiplicities which explains why we have to consider the property that β/α is special at s' (see 7.3.24 for a refined statement).

Lemma 7.1.42. *Consider a morphism of cycles $\alpha' \to \alpha$ and a pre-special morphism $f : \beta \to \alpha$ which is special at the generic points of α. Consider a commutative square*

$$\begin{array}{ccc} \mathrm{Spec}\,(k') & \xrightarrow{x'} & \alpha' \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,(k) & \xrightarrow{x} & \alpha \end{array}$$

such that k and k' are fields. Then the following conditions are equivalent :

(i) f is special at x.
(ii) $\beta \otimes_\alpha \alpha' \to \alpha'$ is special at x'.

Proof. This follows easily from lemma 7.1.22 and property (P2). □

Corollary 7.1.43. *Let $f : \beta \to \alpha$ be a special morphism.*
Then for any morphism $\alpha' \to \alpha$, $\beta \otimes_\alpha \alpha' \to \alpha'$ is special.

Definition 7.1.44. Let $f : \beta \to \alpha$ be a morphism of cycles and $x : \mathrm{Spec}\,(k) \to \alpha$ be a point. We say that f is Λ-universal at x if it is special at x and the cycle $\beta \otimes_\alpha k$ has coefficients in Λ.

In the situation of this definition, let s be the image of x in the support of α, and κ_s be its residue field. Then according to (P6), $\beta_k = \beta_{\kappa_s} \otimes^\flat_{\kappa_s} k$. Thus f is Λ-universal at x if and only if it is Λ-universal at s. Furthermore, the following lemma follows easily :

Lemma 7.1.45. *Let $f : \beta \to \alpha$ be a morphism of cycles. The following conditions are equivalent :*

(i) For any point $s \in \overline{\alpha}$, f is Λ-universal at s.
(ii) For any point $x : \mathrm{Spec}\,(k) \to \alpha$, f is Λ-universal at x.
(iii) For any morphism of cycles $\alpha' \to \alpha$, $\beta \otimes_\alpha \alpha'$ has coefficients in Λ.

Definition 7.1.46. We say that a morphism of cycles f is Λ-universal if it satisfies the equivalent properties of the preceding lemma.

Of course, Λ-universal morphisms are stable by base change.

7.2. **Intersection theoretic properties.**

7.2.a. *Commutativity.*

Lemma 7.2.1. *Consider morphisms of cycles with support in the left diagram*

$$\begin{array}{ccc} & \beta & \\ & \downarrow & \\ \gamma & \longrightarrow & \alpha \end{array} \qquad \subset \qquad \begin{array}{ccc} & X & \\ & \downarrow f & \\ T & \xrightarrow{g} & S \end{array}$$

such that β/α is pre-special and γ/α is pseudo-dominant.

Assume

$$\alpha = \sum_{i\in I} n_i.s_i, \quad \beta = \sum_{j\in J} m_j.x_j, \quad \gamma = \sum_{l\in\Lambda} p_l.t_l$$

and denote by κ_{s_i} (resp. κ_{x_j}, κ_{t_l}) the residue field of s_i (resp. x_j, t_l) in S (resp. X, T). Considering $(i,j,l) \in I\times J\times\Lambda$ such that $f(x_j) = g(t_l) = s_i$, we denote by $\nu_{j,l} : \mathrm{Spec}\left(\kappa_{x_j} \otimes_{\kappa_{s_i}} \kappa_{t_l}\right) \to X\times_S T$ the canonical immersion.

Then the followin assertion holds :

(i) β is special at the generic points of γ.
(ii) The cycle $\beta \otimes_\alpha \gamma$ has coefficients in Λ.
(iii) The following equality of cycles holds

$$\beta \otimes_\alpha \gamma = \sum_{i,j,l} \frac{m_j}{n_i} p_l.\nu_{j,l*}\big(\langle \mathrm{Spec}\left(\kappa_{y_j} \otimes_{\kappa_{x_i}} \kappa_{z_l}\right)\rangle\big)$$

where the sum runs over $(i,j,l) \in I \times J \times \Lambda$ such that $f(x_j) = g(t_j) = s_i$.

Proof. Assertion (i) is in fact the first point of 7.1.27. Assertion (ii) follows from assertion (iii), which is a consequence of the defining formula (7.1.36.1) and remark 7.1.34. □

Corollary 7.2.2. *Let $g : T \to S$ be a flat morphism and $\beta = \sum_{j\in J} m_j.\langle Z_j\rangle_X$ be a pre-special S-cycle written in standard form.*

Then β/S is pre-special at the generic points of T and

$$\beta \otimes_S \langle T\rangle = \sum_{j\in J} m_j.\langle Z_j \times_S T\rangle.$$

The external product is by nature non commutative. The previous lemma implies it is commutative whenever it makes sense :

Corollary 7.2.3. *Consider pre-special morphisms of cycles $\beta \to \alpha$ and $\gamma \to \alpha$.*

Then β (resp. γ) is special at the generic points of γ (resp. β) and the following equality holds: $\beta \otimes_\alpha \gamma = \gamma \otimes_\alpha \beta$.

7.2.b. *Associativity.*

Proposition 7.2.4. *Consider morphism of cycles $\beta \xrightarrow{f} \alpha$, $\alpha'' \to \alpha' \to \alpha$ such that f is special at the generic points of α' and of α''. Let n be the exponential characteristic of α''.*

Then the following assertions hold:

(i) The relative cycle $\beta \otimes_\alpha \alpha'/\alpha'$ is special at the generic points of α''.
(ii) The cycle $(\beta \otimes_\alpha \alpha') \otimes_{\alpha'} \alpha''$ has coefficients in $\Lambda[1/n]$.
(iii) $(\beta \otimes_\alpha \alpha') \otimes_{\alpha'} \alpha'' = \beta \otimes_\alpha \alpha''$.

Proof. Assertion (i) is a corollary of lemma 7.1.42. Assertion (ii) is in fact a corollary of assertion (iii), which in turn follows easily from the uniqueness statement in theorem 7.1.36. □

Lemma 7.2.5. *Let $\gamma \xrightarrow{g} \beta \xrightarrow{f} \alpha$ be two pre-special morphisms of cycles with domains $Y \to X \to S$. Consider a fat point (R,k) over α such that γ/β is special at the generic points of $\beta_{R,k}$.*

Then γ/α is pre-special and the following equality of cycles holds in Y_k:

$$\gamma_{R,k} = \gamma \otimes_\beta (\beta_{R,k}).$$

Proof. The first statement is obvious.

We first prove: $\gamma_R = \gamma \otimes_\beta \beta_R$.
Remark that $\beta_R \to \beta$ is pseudo-dominant. Thus γ/β is special at the generic points of β_R and the right hand side of the preceding equality is well defined. Moreover, according to lemma 7.2.1, we can restrict to the case where $\alpha = s$, $\beta = x$ and $\gamma = y$, with multiplicity 1. Let κ_s, κ_x, κ_y be the corresponding respective residue fields, and K be the fraction field of R.
Then, according to (7.1.25.1), $\gamma_R = \langle \kappa_y \otimes_{\kappa_s} K \rangle_{Y_R}$ and $\beta_R = \langle \kappa_x \otimes_{\kappa_s} K \rangle_{X_R}$. But lemma 7.2.1 implies that $\gamma \otimes_\beta \beta_R = \langle \kappa_y \otimes_{\kappa_x} (\kappa_x \otimes_{\kappa_s} K) \rangle_{X_R}$. Thus the associativity of the tensor product of fields allows to conclude.

From this equality and proposition 7.2.4, we deduce that:

$$\gamma_R \otimes_{\beta_R} \beta_{R,k} = (\gamma \otimes_\beta \beta_R) \otimes_{\beta_R} \beta_{R,k} = \gamma \otimes_\beta \beta_{R,k}.$$

Thus, the equality we have to prove can be written $\gamma_R \otimes^\flat_R k = \gamma_R \otimes_{\beta_R} (\beta_R \otimes^\flat_R k)$ and we are reduced to the case $\alpha = \mathrm{Spec}\,(R)$.
In this case, we can assume $\beta = \langle X \rangle$ with X integral. Let us consider a blow-up $\tilde{X} \xrightarrow{p} X$ such that the proper transform $\tilde{\gamma}$ of γ along p is a Hilbert cycle over $\tilde{X}$ (7.1.16). We easily get (from (P3) and 7.1.13) that

$$\tilde{\gamma}_k = \tilde{\gamma} \otimes_{\tilde{X}} \langle \tilde{X}_k \rangle.$$

Let Y (resp. $\tilde{Y}$) be the support of γ (resp. $\tilde{\gamma}$), $q : \tilde{Y} \to Y$ the canonical projection. We consider the cartesian square obtained by pullback along $\mathrm{Spec}\,(k) \to \mathrm{Spec}\,(R)$:

$$\begin{array}{ccc} \tilde{Y}_k & \xrightarrow{q_k} & Y_k \\ \downarrow & & \downarrow \\ \tilde{X}_k & \xrightarrow{p_k} & X_k. \end{array}$$

As $X_k \subset X$ (resp. $Y_k \subset Y$) is purely of codimension 1, the proper morphism p_k (resp. q_k) is still birational. As a consequence, $q_{k*}(\tilde{\gamma}) = \gamma$. Let y be a point in $\tilde{Y}_k^{(0)} \simeq Y_k^{(0)}$ which lies above a point x in $\tilde{X}_k^{(0)} \simeq X_k^{(0)}$ Then, according to (P5) and using the notations of 7.1.40, we get

$$m(y; \tilde{\gamma} \otimes_{\tilde{X}} \langle \tilde{X}_k \rangle) = m(y; \gamma \otimes_X \langle X_k \rangle).$$

This readily implies $q_{k*}(\tilde{\gamma} \otimes_{\tilde{X}} \langle \tilde{X}_k \rangle) = \gamma \otimes_X \langle X_k \rangle$ and allows us to conclude. □

As a corollary of this lemma using the uniqueness statement in theorem 7.1.36, we obtained :

Corollary 7.2.6. *Let $\gamma \xrightarrow{g} \beta \xrightarrow{f} \alpha$ be pre-special morphisms of cycles.*

Let $x : \mathrm{Spec}\,(k) \to \alpha$ be a point. If β/α is special at x and γ/β is special at the generic points of β_k, then γ/α is special at x.

Let $\alpha' \to \alpha$ be any morphism of cycles with domain $S' \to S$ and n be the exponential characteristic of α'. Then, whenever it is well defined, the following equality of $\Lambda[1/n]$-cycles holds:

$$\gamma \otimes_\beta (\beta \otimes_\alpha \alpha') = \gamma \otimes_\alpha \alpha'.$$

A consequence of the transitivity formulas is the associativity of our exterior product :

Corollary 7.2.7. *Suppose given the following morphisms of cycles*

$$\begin{array}{ccccc} \alpha & & \beta & & \gamma \\ & \searrow & \swarrow_f & \searrow & \swarrow_g \\ & & \delta & & \sigma \end{array}$$

such that f and g are pre-specials.

Then, whenever it is well defined, the following equality of cycles hold:

$$\gamma \otimes_\sigma (\beta \otimes_\delta \alpha) = (\gamma \otimes_\sigma \beta) \otimes_\delta \alpha$$

Proof. Indeed, by the transitivity formulas 7.2.4 and 7.2.6, both members of the equation are equal to $(\gamma \otimes_\sigma \beta) \otimes_\beta (\beta \otimes_\delta \alpha)$. □

7.2.c. *Projection formulas.*

Proposition 7.2.8. *Consider morphisms of cycles with support in the left diagram*

$$\begin{array}{ccc} \beta & & X \\ \downarrow & \subset & \downarrow \\ \alpha' \longrightarrow \alpha & & S' \xrightarrow{q} S \end{array}$$

such that β/α is special at the generic points of α'.
Consider a factorization $S' \xrightarrow{g} T \to S$.

Then β/α is special at the generic points of $h_*(\alpha)$ and the following equality of cycles holds in $X \times_S T$:

$$\beta \otimes_\alpha g_*(\alpha') = (1_X \times_S g)_*(\beta \otimes_\alpha \alpha').$$

Proof. The first assuption is obvious. By linearity, we can assume S' is integral and α' is the generic point s of S' with multiplicity 1. Let L (resp. E) be the residue field of s (resp. $g(s)$).

Consider the pullback square $\begin{array}{ccc} X_L & \xrightarrow{g_0} & X_E \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} \\ X \times_S S' & \xrightarrow{g_X} & X \times_S T \end{array}$ where i and j are the natural immersions.

Let d be the degree of L/E if it is finite and 0 otherwise. We are reduced to prove the equality $g_{X*}(j_*(\beta_L)) = d.i_*(\beta_E)$. Using the functoriality of pushout and property (P6), it is sufficient to prove the equality $g_{0*}(\beta_E \otimes^\flat_E L) = d.\beta_E$. If $d = 0$, the morphism g_0 induces an infinite extension of fields on any point of X_L which concludes. If L/E is finite, g_0 is finite flat and $\beta_E \otimes^\flat_E L$ is the usual pullback by g_0. Then the needed equality follows easily (see [Ful98, 1.7.4]). □

Lemma 7.2.9. *Let $\beta \to \alpha$ be a pre-special morphism of cycles with domain $X \xrightarrow{p} S$. Let (R, k) a fat point over α and $X \xrightarrow{f} Y \to S$ be a factorization of p. Let f_k be the pullback of f over $\mathrm{Spec}\,(k)$.*

Suppose that the support of β is proper with respect to f. Then $f_*(\beta)$ is pre-special over α and the equality of cycles $\big(f_*(\beta)\big)_{R,k} = f_{k*}(\beta_{R,k})$ holds in Y_k.

Proof. As usual, considering the support S of α, we reduce to the case where $\alpha = \langle S \rangle$. Let K be the fraction field of R. As $\mathrm{Spec}\,(K)$ maps to a generic point of S, we can assume S is integral. Let F be its function field. We can assume by linearity that β is a point x in X with multiplicity 1.

Let L (resp. E) be the residue field of x (resp. $y = f(x)$). Let d be the degree of L/E if it is finite and 0 otherwise. Consider the following pullback square

$$\begin{array}{ccc} \mathrm{Spec}\,(L \otimes_F K) & \xrightarrow{j} & X \times_S \mathrm{Spec}\,(R) = X_R \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_R} \\ \mathrm{Spec}\,(E \otimes_F K) & \xrightarrow{i} & Y \times_S \mathrm{Spec}\,(R) = Y_R. \end{array}$$

According to the formula (7.1.25.1), we obtain:

$$\begin{aligned} f_{R*}(\beta_R) &= f_{R*}j_*(\langle L \otimes_F K \rangle) = i_* f_{0*}(\langle L \otimes_F K \rangle) \\ &= i_* f_{0*}(f_0^*(\langle E \otimes_F K \rangle) = i_*(d.\langle E \otimes_F K \rangle) = \langle f_*(\beta) \rangle_R. \end{aligned}$$

We are finally reduced to the case $S = \mathrm{Spec}\,(R)$ and β is a Hilbert cycle over $\mathrm{Spec}\,(R)$. Note that $f_*(\beta)$ is still a Hilbert cycle over $\mathrm{Spec}\,(R)$. As $\beta_{R,k} = \beta \otimes^\flat_R k$, the result follows now from proposition 7.1.14. □

Corollary 7.2.10. *Consider morphisms of cycles with support in the left diagram*

$$\begin{array}{ccc} \beta & & X \\ \downarrow & \subset & \downarrow{\scriptstyle p} \\ \alpha' \longrightarrow \alpha & & S' \longrightarrow S \end{array}$$

such that β/α is special at the generic points of α' (resp. Λ-universal).

Consider a factorization $X \xrightarrow{f} Y \to S$ of p.

Suppose that the support of β is proper with respect to f. Then $f_*(\beta)/\alpha$ is special at the generic points of α' (resp. Λ-universal) and the following equality of cycles holds in $X \times_S S'$:

$$(f \times_S 1_{S'})_*(\beta \otimes_\alpha \alpha') = \big(f_*(\beta)\big) \otimes_\alpha \alpha'.$$

7.3. **Geometric properties.**

7.3.1. We introduce a notation which will come often in the next section. Let S be a scheme and $\alpha = \sum_{i\in I} n_i.\langle Z_i\rangle_X$ an S-cycle written in standard form.

Let s be a point of S and $\mathrm{Spec}\,(k) \xrightarrow{\bar{s}} S$ be a geometric point of S with k separably closed. Let S' be one of the following local schemes: the localization of S at s, the Hensel localization of S at s, the strict localization of S at $\bar{s}$.

We then define the cycle with coefficients in Λ and domain $X \times_S S'$ as:

$$\alpha|_{S'} = \sum_{i\in I} n_i \langle Z_i \times_S S'\rangle_{X\times_S S'}.$$

Remark 7.3.2. The canonical morphism $S' \to S$ is flat. In particular, α/S is special at the generic points of S' and we easily get: $\alpha|_{S'} = \alpha \otimes_S S'$.

7.3.a. *Constructibility.*

Definition 7.3.3. Let S be a scheme and $s \in S$ a point. We say that a pre-special S-cycle α is trivial at s if it is special at s and $\alpha \otimes_S s = 0$.

Naturally, we say that α is trivial if it is zero. Thus α is trivial if and only if it is trivial at the generic points of S.

Recall from [GD67, 1.9.6] that an ind-constructible subset of a noetherian scheme X is a union of locally closed subset of X.

Lemma 7.3.4. *Let S be a noetherian scheme, and α/S be a pre-special cycle. Then the set*

$$T = \big\{s \in S \mid \alpha/S \text{ is special (resp. trivial, } \Lambda\text{-universal) at } s\big\}$$

is ind-constructible in S.

Proof. Let s be a point of T, and Z be its closure in S with its reduced subscheme structure. Put $\alpha_Z = \alpha \otimes_S Z$, defined because α is special at the generic point of Z. Given any point t of Z, we know that α/S is special at t if and only if α_Z/Z is special at t (*cf.* 7.1.42). But there exists a dense open subset U_s of Z such that $\alpha_Z|_{U_Z}$ is a Hilbert cycle over U_Z. Thus, α/S is special at each point of U_s and $U_s \subset T$. This concludes and the same argument proves the respective statements. □

7.3.5. Let I be a left filtering category and $(S_i)_{i\in I}$ be a projective system of noetherian schemes with affine transition morphisms. We let S be the projective limit of (S_i) and we assume the followings:

(1) S is noetherian.
(2) There exists an index $i \in I$ such that the canonical projection $S \xrightarrow{p_i} S_i$ is dominant.

In this case, there exists an index j/i such that for any k/j, the map p_k induces an isomorphism $S^{(0)} \to S_k^{(0)}$ on the generic points (*cf.* [GD67, 8.4.1]). Thus, replacing I by I/j, we can assume that this property is satisfied for all index $i \in I$. As a consequence, the following properties are consequences of the previous ones:

(3) For any $i \in I$, $p_i : S \to S_i$ is pseudo-dominant and p_i induces an isomorphism $S^{(0)} \to S_i^{(0)}$.
(4) For any arrow $j \to i$ of I, $p_{ji} : S_j \to S_i$ is pseudo-dominant and p_{ji} induces an isomorphism $S_j^{(0)} \to S_i^{(0)}$.

Proposition 7.3.6. *Consider the notations and hypothesis above. Assume we are given a projective system of cycles $(\alpha_i)_{i\in I}$ such that α_i is a pre-special cycle over S_i and for any $j \to i$, $\alpha_j = \alpha_i \otimes_{S_i} S_j$. Put $\alpha = \alpha_i \otimes_{S_i} S$ for an index $i \in I$.*[35]

The following conditions are equivalent:

[35]The external product is well defined because of point (3) and (4) of the hypothesis above.

(i) α/S *is special (resp. trivial,* Λ*-universal).*
(ii) There exists $i \in I$ *such that* α_i/S_i *is special (resp. trivial,* Λ*-universal).*
(iii) There exists $i \in I$ *such that for all* j/i, α_j/S_j *is special (resp. trivial,* Λ*-universal).*

Let s *be point of* S *and* s_i *its image in* S_i*. Then the following conditions are equivalent:*

(i) α/S *is special (resp. trivial,* Λ*-universal) at* s*.*
(ii) There exists $i \in I$ *such that* α_i/S_i *is special (resp. trivial,* Λ*-universal) at* s_i*.*
(iii) There exists $i \in I$ *such that for all* j/i, α_j/S_j *is special (resp. trivial,* Λ*-universal) at* s_j*.*

Proof. Let P be one of the respective properties: "special", "trivial", "Λ-universal". Using the fact that being P at s is an ind-constructible property (from lemma 7.3.4), it is sufficient to apply [GD67, th. 8.3.2] to the following family of sets:

$$F_i = \{s_i \in S_i \mid \ \alpha_i \text{ satsifies } \mathsf{P} \text{ at } s_i\}, \quad F = \{s \in S \mid \ \alpha \text{ satsifies } \mathsf{P} \text{ at } s\}.$$

To get the two sets of equivalent conditions of the statement from *op. cit.* we have to prove the following relations:

$$(1) : \forall (j \to i) \in \mathrm{Fl}(I), f_{ji}^{-1}(F_i) \subset F_j,$$
$$(2) : F = \cup_{i \in I} f_i^{-1}(F_i).$$

We consider the case where P is the property "special". For relation (1), we apply 7.1.42 which implies the stronger relation $f_{ji}^{-1}(F_i) = F_j$. For relation (2), another application of 7.1.42 gives in fact the stronger relation $F = f_i^{-1}(F_i)$ for any $i \in I$.

Consider a point $s_j \in S$ and put $s_i = f_{ji}(s_j)$. Assume α_i is special at s_i. Then, applying 7.2.4 and (P3), we get:

$$\alpha_j \otimes_{S_j} s_j = (\alpha_i \otimes_{S_i} s_i) \otimes^{\flat}_{\kappa(s_i)} \kappa(s_j). \tag{7.3.6.1}$$

Similarly, given $s \in S_j$, $s_i = f_i(s)$, and assuming α_i is special at s_i, we get:

$$\alpha \otimes_S s = (\alpha_i \otimes_{S_i} s_i) \otimes^{\flat}_{\kappa(s_i)} \kappa(s). \tag{7.3.6.2}$$

We consider now the case where P is the property "trivial". Then relation (1) follows from (7.3.6.1). Relation (2) follows from (7.3.6.1) and 7.1.17(1).

We finally consider the case P is the property "Λ-universal". Relation (1) in this case is again a consequence of (7.3.6.1). According to (7.3.6.2), we get the inclusion $\cup_{i \in I} f_i^{-1}(F_i) \subset F$. We have to prove the reciprocal inclusion.
Consider a point $s \in S$ with residue field k such that α/S is Λ-universal at s. For any $i \in I$, we put $s_i = f_i(s)$ and denote by k_i its residue field. It is sufficient to find an index $i \in I$ such that $\alpha_i \otimes_{S_i} s_i$ has coefficients in Λ. Thus we are reduced to the following lemma:

Lemma 7.3.7. *Let* $(k_i)_{i \in I^{op}}$ *be an ind-field,* $k = \varinjlim_{i \in I^{op}} k_i$*.*
Consider a family $(\beta_i)_{i \in I}$ *such that* β_i *is a* k_i*-cycle of finite type with coefficients in* $\mathbf{Q}$ *and for any* j/i, $\beta_j = \beta_i \otimes^{\flat}_{k_i} k_j$*.*

If for an index $i \in I$, $\beta_i \otimes^{\flat}_{k_i} k$ *has coefficients in* Λ*, then there exists* j/i *such that* β_j *has coefficients in* Λ*.*

We can assume that for any j/i, β_j has positive coefficients. Let X_j (resp. X) be the support of β_j (resp. β). We obtain a pro-scheme $(X_j)_{j/i}$ such that $X = \varprojlim_{i \in I} X_i$. The transition maps of $(X_j)_{j/i}$ are dominant. Thus, by enlarging i, we can assume that for any j/i, the induced map $\pi_0(X_i) \to \pi_0(X_j)$ is a bijection. Thus we can consider each element of $\pi_0(X)$ separately and assume that all the X_i are integrals: for any j/i, $\beta_j = n_j.\langle X_j \rangle$ for a positive element $n_j \in \mathbf{Q}$. Arguing generically, we can further assume $X_j = \mathrm{Spec}\,(L_j)$ for a field extension of finite type L_j of k_j. By assumption now, for any j/i, $L_i \otimes_{k_i} k_j$ is an Artinian ring whose reduction is the field L_j. Moreover, $n_j = n_i.\mathrm{lg}(L_i \otimes_{k_i} k_j)$ and we know that $n := n_i.\mathrm{lg}(L_i \otimes_{k_i} k)$ belongs to Λ.

Let p be a prime not invertible in Λ such that $v_p(n_i) < 0$ where v_p denotes the p-adic valuation on $\mathbf{Q}$. It is sufficient to find an index j/i such that $v_p(n_j) \geq 0$. Let $L = (L_i \otimes_{k_i} k)_{red}$. Remark that $L = \varinjlim_{i \in I^{op}} L_i$. It is a field extension of finite type of k. Consider elements $a_1, ..., a_n$

algebraically independant over k such that L is a finite extension of $k(a_1,...,a_n)$. By enlarging i, we can assume that $a_1,...,a_n$ belongs to L_i. Thus L_i is a finite extension of $k_i(a_1,...,a_n)$: replacing k_i by $k_i(a_1,...,a_n)$, we can assume that L_i/k_i is finite.
Let L' be the subextension of L over k generated by the p-th roots of elements of k. As L/k is finite, L'/k is finite, generated by elements $b_1,...,b_r \in L$. consider an index j/i such that $b_1,...,b_r$ belongs to L_j. It follows that $v_p(\lg(L_i \otimes_{k_i} k_j)) = v_p(\lg(L_i \otimes_{k_i} k))$. Thus $v_p(n_j) = v_p(n) \geq 0$ and we are done. □

Corollary 7.3.8. *Let S be a scheme and α be a pre-special S-cycle.*
Let $\bar{s}$ be a geometric point of S, with image s in S, and S' be the strict localization of S at $\bar{s}$.
Then the following conditions are equivalent:

(i) α/S is special at s.
(i') α/S is special at $\bar{s}$.
(ii) $\big(\alpha|_{S'}\big)/S'$ is special at $\bar{s}$ (notation of 7.3.1).
(iii) There exists an étale neighbourhood V of $\bar{s}$ in S such that $(\alpha \otimes_S V)/V$ is special at $\bar{s}$.

Proof. The equivalence of (i) and (i') follows trivially from definition (*cf.* 7.1.27). Recall from 7.3.1 that $\alpha|_{S'} = \alpha \otimes_S S'$. Thus $(i') \Rightarrow (ii)$ is easy (see 7.1.42). Moreover, $(ii) \Rightarrow (iii)$ is a consequence of the previous proposition applied to the pro-scheme of étale neighbourhood of $\bar{s}$. Finally, $(iii) \Rightarrow (i)$ follows from lemma 7.1.42. □

Proposition 7.3.9. *Consider the notations and hypothesis of 7.3.5. Assume that S and S_i are reduced for any $i \in I$.*

Suppose given a projective system $(X_i)_{i\in I^{op}}$ of S_i-schemes of finite type such that for any j/i, $X_j = X_i \times_{S_i} S_j$. We let X be the projective limit of (X_i).

Then for any pre-special (resp. special, Λ-universal) S-cycle $\alpha \subset X$, there exists $i \in I$ and a pre-special (resp. special, Λ-universal) S_i-cycle $\alpha_i \subset X_i$ such that $\alpha = \alpha_i \otimes_{S_i} S$.[36]

Proof. Using proposition 7.3.6, we are reduced to consider the first of the respective cases of the proposition. Write $\alpha = \sum_{r\in\Theta} n_r.\langle Z_r\rangle_X$ in standard form.
Consider $r \in \Theta$. As X is noetherian, there exists an index $i \in I$ and a closed subscheme $Z_{r,i} \subset X_i$ such that $Z_r = Z_{r,i} \times_{S_i} S$. Moreover, replacing $Z_{r,i}$ by the reduced closure of the image of the canonical map $Z_r \xrightarrow{(*)} Z_{r,i}$, we can assume that the map $(*)$ is dominant. For any $j \in I/i$, we put $Z_{r,j} = Z_{r,i} \times_{S_i} S_j$. The limit of the pro-scheme $(Z_{r,j})_{j\in I/i^{op}}$ is the integral scheme Z_r. Thus, applying [GD67, 8.2.2], we see that by enlarging i, we can assume that for any $j \in I/i$, $Z_{r,j}$ is irreducible (but not necessarily reduced).
We repeat this construction for every $r \in \Theta$, enlarging i at each step. Fix now an element $j \in I/i$. The scheme $Z_{r,j}$ may not be reduced. However, its reduction $Z'_{r,j}$ is an integral scheme such that $Z'_{r,j} \times_{S_j} S = Z_r$. We put

$$\alpha_j = \sum_{r\in\Theta} n_r \langle Z'_{r,j}\rangle_{X_j}.$$

Let $z_{r,j}$ be the generic point of $Z'_{r,j}$, and $s_{r,j}$ be its image in S_j. It is a generic point and corresponds uniquely to a generic point s_r of S according to the point (3) of the hypothesis 7.3.5. Thus α_j/S_j is pre-special. Moreover, we get from the above that $\kappa(z_{r,j}) \otimes_{\kappa(s_{r,j})} \kappa(s_r) = \kappa(z_r)$ where z_r is the generic point of Z_r. Thus the relation $\alpha_j \otimes_{S_j} S = \alpha$ follows from lemma 7.2.1. □

7.3.b. *Samuel's multiplicities.*

7.3.10. We give some recall on Samuel's multiplicities, following as a general reference [Bou93, VIII.§7].
Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$. Let $M \neq 0$ be a A-module of finite type

[36] This external product is defined in any case because of point (3) of the hypothesis above.

and $\mathfrak{q} \subset \mathfrak{m}$ an ideal of A such that $M/\mathfrak{q}M$ has finite length. Let d be the dimension of the support of M. Recall from *loc. cit.* that *Samuel's multiplicity* of M at $\mathfrak{q}$ is defined as the integer:

$$e^A_{\mathfrak{q}}(M) := \lim_{n\to\infty} \left(\frac{d!}{n^d} \lg_A(M/\mathfrak{q}^n M) \right)$$

In the case $M = A$, we simply put $e_{\mathfrak{q}}(A) := e^A_{\mathfrak{q}}(A)$ and $e(A) := e^A_{\mathfrak{m}}(A)$.

We will use the following properties of these multiplicities that we recall for the convenience of the reader; let A be a local noetherian ring with maximal ideal $\mathfrak{m}$:

Let Φ be the generic points $\mathfrak{p}$ of $\mathrm{Spec}\,(A)$ such that $\dim(A/\mathfrak{p}A) = \dim A$. Then according to proposition 3 of *loc. cit.*:

$$(\mathcal{S}1) \qquad e_{\mathfrak{q}}(A) = \sum_{\mathfrak{p}\in\Phi} \lg(A_{\mathfrak{p}}).e_{\mathfrak{q}}(A/\mathfrak{p}).$$

Let B be a local flat A-algebra such that $B/\mathfrak{m}B$ has finite length over B. Then according to proposition 4 of *loc. cit.*:

$$(\mathcal{S}2) \qquad \frac{e_{\mathfrak{m}B}(B)}{e(A)} = \lg_B(B/\mathfrak{m}B).$$

Let B be a local flat A-algebra such that $\mathfrak{m}B$ is the maximal ideal of B. Let $\mathfrak{q} \subset A$ be an ideal such that $A/\mathfrak{q}A$ has finite length. Then according to the corollary of proposition 4 in *loc. cit.*:

$$(\mathcal{S}3) \qquad e_{\mathfrak{q}B}(B) = e_{\mathfrak{q}}(A).$$

Assume A is integral with fraction field K. Let B be a finite local A-algebra such that $B \supset A$. Let k_B/k_A be the extension of the residue fields of B/A. Then, according to proposition 5 and point b) of the corollary of proposition 4 in *loc. cit.*,

$$(\mathcal{S}4) \qquad \frac{e_{\mathfrak{m}B}(B)}{e(A)} = \frac{\dim_K(B \otimes_A K)}{[k_B : k_A]}.$$

Definition 7.3.11. (i) Let $S = \mathrm{Spec}\,(A)$ be a local scheme, $s = \mathfrak{m}$ the closed point of S.

Let Z be an S-scheme of finite type with special fiber Z_s. For any generic point z of Z_s, denoting by B the local ring of Z at z, we define the *Samuel multiplicity of Z at z over S* as the rational integer:

$$m^{\mathcal{S}}(z, Z/S) = \frac{e_{\mathfrak{m}B}(B)}{e(A)}.$$

In the case where Z is integral, we define the *Samuel specialization of the S-cycle $\langle Z\rangle$ at s* as the cycle with rational coefficients and domain Z_s:

$$\langle Z\rangle \otimes^{\mathcal{S}}_S s = \sum_{z\in Z_s^{(0)}} m^{\mathcal{S}}(z, Z/S).z.$$

Consider an S-cycle of finite type $\alpha = \sum_{i\in I} n_i.\langle Z_i\rangle_X$ written in standard form. We define the *Samuel specialization of the S-cycle α at s* as the cycle with domain X_s:

$$\alpha \otimes^{\mathcal{S}}_S s = \sum_{i\in I} n_i.\langle Z_i\rangle \otimes^{\mathcal{S}}_S s.$$

(ii) Let S be a scheme. For any point s of S, we let $S_{(s)}$ be the localized scheme of S at s.

Let $f : Z \to S$ be an S-scheme of finite type, and z a point of Z which is generic in its fiber. Put $s = f(z)$. We define the *Samuel multiplicity of Z/S at z* as the integer

$$m^{\mathcal{S}}(z, Z/S) := m^{\mathcal{S}}(z, Z \times_S S_{(s)}/S_{(s)}).$$

Consider an S-cycle of finite type α with domain X and a point s of S. We define the Samuel relative product of α and s over S as the cycle with rational coefficients:

$$\alpha \otimes^{\mathcal{S}}_S s = \left(\alpha|_{S_{(s)}}\right) \otimes^{\mathcal{S}}_{S_{(s)}} s.$$

Lemma 7.3.12. *Let S be a scheme, and $p : Z' \to Z$ an S-morphism which is a birational universal homeomorphism. Then for any point $s \in S$,*

$$\langle Z' \rangle \otimes_S^{\mathcal{S}} s = \langle Z \rangle \otimes_S^{\mathcal{S}} s$$

in $(Z'_s)_{red} = (Z_s)_{red}$.

Proof. By hypothesis, p induces an isomorphism $Z'^{(0)} \simeq Z^{(0)}$ between the generic points. Given any irreducible component T' of Z' corresponding to the irreducible component T of Z, we get by hypothesis:

$$T'_{red} \simeq T_{red} \text{ (as schemes)}, \ \lg\left(\mathcal{O}_{Z',T'}\right) = \lg\left(\mathcal{O}_{Z,T}\right).$$

Thus, we easily concludes from the definition. □

7.3.13. Let $Z \xrightarrow{f} S$ be a morphism of finite type and a z a point of Z, $s = f(z)$. Assume z is a generic point of Z_s. We introduce the following condition:

$$\mathcal{D}(z, Z/S) : \begin{cases} \text{For any irreducible component } T \text{ of } Z_{(z)}, \\ T_s = \emptyset \text{ or } \dim(T) = \dim(Z_{(z)}). \end{cases}$$

Remark 7.3.14. This condition is in particular satisfied if $Z_{(z)}$ is absolutely equidimensional (and a fortiori if Z is absolutely equidimensional).

An immediate translation of $(\mathcal{S}1)$ gives:

Lemma 7.3.15. *Let S be a local scheme with closed point s and Z be an S-scheme of finite type such that Z_s is irreducible with generic point z.*

If the condition $\mathcal{D}(z, Z/S)$ is satisfied, then $\langle Z \rangle \otimes_S^{\mathcal{S}} s = m^{\mathcal{S}}(z, Z/S).z$.

We get directly from $(\mathcal{S}2)$ the following lemma:

Lemma 7.3.16. *Let S be a scheme, s be a point of S, and $\alpha = \sum_{i \in I} n_i.\langle Z_i \rangle_X$ be an S-cycle in standard form such that Z_i is a flat S-scheme of finite type.*

Then α is a Hilbert S-cycle and $\alpha \otimes_S^{\mathcal{S}} s = \alpha \otimes_S^{\flat} s$.

With the notations of 7.3.1, we get from $(\mathcal{S}3)$:

Lemma 7.3.17. *Let S be a scheme, s a point of S with residue field k and α an S-cycle of finite type.*

(i) Let S' be the Hensel localization of S at s. Then, $\alpha \otimes_S^{\mathcal{S}} s = \left(\alpha|_{S'}\right) \otimes_{S'}^{\mathcal{S}} s$.

(ii) Let $\bar{k}$ a separable closure corresponding and $\bar{s}$ the corresponding geometric point of S. Let $S_{(\bar{s})}$ be the strict localization of S at $\bar{s}$. Then,

$$\left(\alpha \otimes_S^{\mathcal{S}} s\right) \otimes_k^{\flat} \bar{k} = \left(\alpha|_{S_{(\bar{s})}}\right) \otimes_{S_{(\bar{s})}}^{\mathcal{S}} \bar{s}.$$

Let us recall from [GD67, 13.3.2] the following definition:

Definition 7.3.18. Let $f : X \to S$ be a morphism of finite type between noetherian schemes, and x a point of X.

We say f is equidimensional at x if there exists an open neighbourhood U of x in X and a quasi-finite pseudo-dominant S-morphism $U \to \mathbf{A}_S^d$ for $d \in \mathbf{N}$.

We say f is equidimensional if it is equidimensional at every point of X.

Remark 7.3.19. A quasi-finite morphism is equidimensional if and only if it is pseudo-dominant.

Note finally that a direct translation of $(\mathcal{S}4)$ gives:

Lemma 7.3.20. *Let $S = \mathrm{Spec}\,(A)$ be an integral local scheme with closed point s and fraction field K. Let Z be a finite equidimensional scheme and z a generic point of Z_s. Let B be the local ring of Z at z.*

Then,

$$m^{\mathcal{S}}(z, Z/S) = \frac{\dim_K(B \otimes_A K)}{[\kappa(x) : \kappa(s)]}.$$

The following lemma is the most important computation of this subsection.

Lemma 7.3.21. *Consider a cartesian square*

$$\begin{array}{ccc} Z' & \xrightarrow{g'} & Z \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

and a point s' of S', $s = g(s')$. Let k (resp. k') be the residue field of s (resp. s'). We assume the following conditions:

(1) S (resp. S') is geometrically unibranch at s (resp. s').
(2) f and f' are equidimensional of dimension n.
(3) For any generic point z of Z_s (resp. z' of $Z_{s'}$) the condition $\mathcal{D}(z, Z/S)$ (resp. $\mathcal{D}(z', Z'/S')$) is satisfied.

Then, the following equality holds in $Z_{s'}$:

$$\langle Z' \rangle \otimes^{\mathcal{S}}_{S'} s' = (\langle Z \rangle \otimes^{\mathcal{S}}_{S} s) \otimes^{\flat}_{k} k'.$$

Proof. According to lemma 7.3.15, we have to prove the equality:

$$\sum_{z' \in Z'^{(0)}_{s'}} m^{\mathcal{S}}(z', Z'/S').z' = \sum_{z \in Z^{(0)}_{s}} m^{\mathcal{S}}(z, Z/S).\langle \mathrm{Spec}\,(\kappa(z) \otimes_k k') \rangle_{Z_{s'}}. \tag{7.3.21.1}$$

As f is equidimensional of dimension n, we can assume according to we can assume according to 7.3.18 that there exists a quasi-finite pseudo-dominant S-morphism $p : Z \to \mathbf{A}^n_S$. For any generic point z of Z_s, $t = p(z)$ is the generic point of $\mathbf{A}^n_s$. Thus applying $(\mathcal{S}3)$, we get:

$$m^{\mathcal{S}}(z, Z/S) = m^{\mathcal{S}}(z, Z/\mathbf{A}^n_S).$$

Consider the S' morphism $p' : Z' \to \mathbf{A}^n_{Z'}$ obtained by base change. It is quasi-finite. As Z'/S' is equidimensional of dimension n, p' must be pseudo-dominant. For any generic point z' of $Z_{s'}$, $t' = p'(z')$ is the generic point of $\mathbf{A}^n_{s'}$ and as in the preceding paragraph, we get

$$m^{\mathcal{S}}(z', Z'/S') = m^{\mathcal{S}}(z', Z'/\mathbf{A}^n_{S'}).$$

Moreover, the residue field κ_t of t (resp. $\kappa_{t'}$ of t') is $k(t_1, ..., t_n)$ (resp. $k'(t_1, ..., t_n)$ and this implies $\mathrm{Spec}\,(\kappa(z) \otimes_{\kappa_t} \kappa_{t'})$ is homeomorphic to $\mathrm{Spec}\,(\kappa(z) \otimes_k k')$ and has the same geometric multiplicities. Putting this and the two preceding relations in (7.3.21.1), we get reduced to the case $n = 0$ – indeed, according to [GD67, 14.4.1.1], $\mathbf{A}^n_S$ (resp. $\mathbf{A}^n_{S'}$) is geometrically unibranch at t (resp. t').

Assume now $n = 0$, so that f and f' are quasi-finite pseudo-dominant.

Let $\bar{k}$ be a separable closure of k and $\bar{k}'$ a separable closure of a composite of $\bar{k}$ and k'. It is sufficient to prove relation (7.3.21.1) after extension to $\bar{k}'$ (lemma 7.1.17). Thus according to 7.3.17 and hypothesis (3), we can assume S and S' are integral strictly local schemes.

For any $z \in Z^{(0)}_s$, the extension $\kappa(z)/k$ is totally inseparable. Moreover, z corresponds to a unique point $z' \in Z^{(0)}_{s'}$ and we have to prove for any $z \in \mathscr{Z}^{(0)}$:

$$m^{\mathcal{S}}(z', Z'/S') = m^{\mathcal{S}}(z, Z/S).\lg(\kappa(z) \otimes_k k').$$

Let $S = \mathrm{Spec}\,(A)$, $K = \mathrm{Frac}(A)$ and $B = \mathcal{O}_{Z,z}$ (resp. $S' = \mathrm{Spec}\,(A')$, $K' = \mathrm{Frac}(A')$ and $B' = \mathcal{O}_{Z',z'}$). As B is quasi-finite dominant over A and A is henselian, B/A is necessarily finite dominant. The same is true for B'/A' and $(\mathcal{S}4)$ gives the formulas:

$$m^{\mathcal{S}}(z, Z/S) = \frac{\dim_K(B \otimes_A K)}{[\kappa(z) : k]}, \qquad m^{\mathcal{S}}(z', Z'/S') = \frac{\dim_{K'}(B' \otimes_{A'} K')}{[\kappa(z') : k']}.$$

As $B' \otimes_{A'} K' = (B \otimes_A K) \otimes_K K'$, the numerator of these two rationals are the same. To conclude, we are reduced to the easy relation

$$[\kappa(z') : k'].\lg(\kappa(z) \otimes_k k') = [\kappa(z) : k].$$

□

Definition 7.3.22. Let S be a scheme and $\alpha = \sum_{i\in I} n_i.\langle Z_i\rangle_X$ be an S-cycle in standard form.

We say α/S is pseudo-equidimensional over s if it is pre-special and for any $i \in I$, the structual map $Z_i \to S$ is equidimensional at the generic points of the fiber $Z_{i,s}$.

Lemma 7.3.23. *Let S be a strictly local integral scheme with closed point s and residue field k and α be an S-cycle pseudo-equidimensional over s.*

Then for any extension $\mathrm{Spec}\,(k') \xrightarrow{s'} S$ *of s and any fat point (R, k') of S over s', the following relation holds:*

$$\alpha_{R,k'} = \big(\alpha \otimes^{\mathcal{S}}_{S} s\big) \otimes^{\flat}_{k} k'.$$

Proof. We put $S' = \mathrm{Spec}\,(R)$ and denote by s' its closed point.

Reductions.– By additivity, we reduce to the case $\alpha = \langle Z\rangle$, Z is integral and the structural morphism $f : Z \to S$ is equidimensional at the generic points of Z_s. Any generic points of $S'_{s'}$ dominantes a generic point of Z_s so that we can argue locally at each generic point x of Z_s. Thus we can assume Z_s is irreducible with generic point x. Moreover, as Z is equidimensional at x, we can assume according to 7.3.18 there exists a quasi-finite pseudo-dominant S-morphism

$$Z \xrightarrow{p} \mathbf{A}^n_S. \tag{7.3.23.1}$$

Note that S is geometrically unibranch at s. Thus, applying [GD67, 14.4.1] ("critère de Chevalley"), f is universally open at x. As S' is a dash whose close point goes to s in S, it follows from [GD67, 14.3.7] that the base change $f' : Z' \to S'$ of f along S'/S is pseudo-dominant.

Let T be an irreducible component of Z', with special fiber $T_{s'}$ and generic fiber $T_{K'}$ over S'. Then $T \to S'$ is a dominant morphism of finite type. Thus, according to [GD67, 14.3.10], either $T_{s'} = \emptyset$ or $\dim(T_{s'}) = \dim(T_{K'})$. Moreover, the dimension of T_η is equal to the transcendantal degree of the function field of T over K', which is equal to the transcendental degree of Z over K. This is n according to (7.3.23.1). Thus, in any case, T is equidimensional of dimension n over S' and this implies Z' is equidimensional of dimension n over S'. Moreover, either $T_{s'} = \emptyset$ or $\dim(T) = n + 1 = \dim(Z')$. Note this implies that for any generic point z' of $Z_{s'}$, the condition $\mathcal{D}(z', Z'/S')$ is satisfied.

Middle step.– We prove: $\alpha_{R,k} = \langle Z'\rangle \otimes^{\mathcal{S}}_{S'} s'$.
According to lemma 7.3.16,

$$\alpha_{R,k} = \langle \overline{Z'_K}\rangle \otimes^{\flat}_{R} k' = \langle \overline{Z'_K}\rangle \otimes^{\mathcal{S}}_{S'} s'.$$

But the canonical map $\overline{Z'_K} \to Z'$ is a birational universal homeomorphism so that we conclude this step by lemma 7.3.12.

Final step.– We have only to point out that the conditions of the preceding lemma are fulfilled for the obvious square, and this is precisely what we need. □

Corollary 7.3.24. *Let S be a reduced scheme, s a point of S and α an S-cycle pseudo-equidimensional over s.*

Let $\bar{s}$ be a geometric point of S with image s in S and S' be the strict localization of S at $\bar{s}$. We let $S' = \cup_{\lambda\in\Lambda} S'_\lambda$ be the irreducible components of S' and $\alpha_\lambda = \alpha \otimes_S S_\lambda$ be the restriction of α over S_λ.

Then the following conditions are equivalent:

(i) α/S is special at s.

(ii) the cycle $\alpha_\lambda \otimes^{\mathcal{S}}_{S'_\lambda} \bar{s}$ does not depend on $\lambda \in \Lambda$.

Moreover, when these conditions are fulfilled, $\alpha \otimes_S \bar{s} = \alpha_\lambda \otimes^{\mathcal{S}}_{S'_\lambda} \bar{s}$.

Proof. According to corollary 7.3.8, we reduce to the case $S = S'$. Then it follows directly from the preceding lemma. □

Corollary 7.3.25. *Let S be a reduced scheme, geometrically unibranch at a point $s \in S$, and α an S-cycle. The following conditions are equivalent:*

(i) α/S is pseudo-equidimensional over s.

(ii) α/S *is special at* s.
Under these conditions, $\alpha \otimes_S s = \alpha \otimes_S^{\mathcal{S}} s$.

Remark 7.3.26. In particular, over a reduced geometrically unibranch scheme S, every cycle whose support is equidimensional over S is special.

Corollary 7.3.27. *Let* S *be a reduced scheme and* $s \in S$ *a point such that* S *is geometrically unibranch at* s *and* $e(\mathcal{O}_{S,s}) = 1$. *Then for any* S*-cycle* α, *the following conditions are equivalent:*

(i) α/S *is pseudo-equidimensional over* s.
(ii) α/S *is* A*-universal at* s.

Remark 7.3.28. In particular, over a regular scheme S, every cycle whose support is equidimensional over S is A-universal. Remark also the following theorem:

Theorem 7.3.29. *Let* S *be an excellent scheme,* $s \in S$ *a point. The following conditions are equivalent:*

(i) S *is regular at* s.
(ii) S *is geometrically unibranch at* s *and* $e(\mathcal{O}_{S,s}) = 1$.
(iii) S *is unibranch at* s *and* $e(\mathcal{O}_{S,s}) = 1$.

Indication of proof. We can assume S is the spectrum of a regular local ring A with closed point s. The implication $(i) \Rightarrow (ii)$ follows from the fact that a normal local ring is geometrically unibranch (at its closed point) and from [Bou93, AC.VIII.§7, prop. 2]. $(ii) \Rightarrow (iii)$ is trivial. For the last implication, let $\hat{A}$ be the completion of the local ring A. We know from [Bou93, AC.VIII.108, ex. 24] that when $e(A) = 1$ and $\hat{A}$ is integral, A is regular. But according to [GD67, 7.8.3, (vii)], the second condition is equivalent to the fact A is unibranch.

Finally, we get the following theorem of Suslin and Voevodsky:

Theorem 7.3.30. *Let* S *be a scheme and* s *a point with residue field* κ_s *such that the local ring* A *of* S *at* s *is regular. Then for any equidimensional* S*-scheme* Z *and any generic point* z *of* Z_s,

$$m^{SV}(z, \langle Z \rangle \otimes_S s) = \sum_i (-1)^i \mathrm{lg}_A \mathrm{Tor}_i^A(\mathcal{O}_{Z,z}, \kappa_s).$$

Proof. We reduce to the case $S = \mathrm{Spec}\,(A)$. Then Z is absolutely equidimensional and we can apply lemma 7.3.15 together with corollary 7.3.25 to get that $m^{SV}(z, \langle Z \rangle \otimes_S s) = m^{\mathcal{S}}(z, Z/S)$. Then the result follows from the theorem of Serre [Ser58, IV.12, th. 1]. □

Remark 7.3.31. Let S be a regular scheme, X a smooth S-scheme and $\alpha \subset X$ an S-cycle whose support is equidimensional over S. Let s be a point of S and $i : X_s \to X$ the closed immersion of the fiber of X at s. Then the cycle $i^*(\alpha)$ of [Ser58, V-28, par. 7] is well defined and we get:

$$\alpha \otimes_S s = i^*(\alpha).$$

8. Finite correspondences

8.0. In this section, $\mathscr{S}$ is the category of all noetherian schemes, We fix an admissible class $\mathscr{P}$ of morphisms in $\mathscr{S}$ and assume in addition that $\mathscr{P}$ is contained in the class of separated morphisms of finite type.

Consider two S-schemes X and Y. To clarify certain formulas, we will denote $X \times_S Y$ simply by XY and let $p_{XY}^X : XY \to X$ be the canonical projection morphism.

We fix a ring of coefficients $\Lambda \subset \mathbf{Q}$.

8.1. **Definition and composition.**

8.1.1. Let S be a base scheme. For any $\mathscr{P}$-scheme X/S, we let $c_0(X/S, \Lambda)$ be the Λ-module made of the finite and Λ-universal S-cycles with domain X.[37] Consider a morphism $f : Y \to X$ of $\mathscr{P}$-schemes over S. Then the pushout of cycles induces a well defined morphism:

$$f_* : c_0(Y/S, \Lambda) \to c_0(X/S, \Lambda).$$

[37] With the notations of [SV00b], $c_0(X/S, \mathbf{Z}) = c_{equi}(X/S, 0)$ when S is reduced.

Indeed, consider a cycle $\alpha \in c_0(Y/S)$. Let us denote by Z its support in Y and by $f(Z) \subset X$ image of the latter by f. We consider these subsets as reduced subschemes. Note that $f(Z)$ is separated and of finite type over S because X/S is noetherian, separated, and of finite type, by assumption 8.0. Because Z/S is proper, [GD61, 5.4.3(ii)] shows that $f(Z)$ is indeed proper over S. Thus, the cycle $f_*(\alpha)$ is Λ-universal according to corollary 7.2.10. Finally, Z/S is finite, we deduce that $f(Z)$ is quasi-finite, thus finite, over S. This implies the result.

Definition 8.1.2. Let X and Y be two $\mathscr{P}$-schemes over S.

A finite S-correspondence from X to Y with coefficients in Λ is an element of

$$c_S(X,Y)_\Lambda := c_0(X \times_S Y/X).$$

We denote such a correspondence by the symbol $X \overset{\alpha}{\bullet\!\!\longrightarrow} Y$.

In the case $\Lambda = \mathbf{Z}$, we simply put $c_S(X,Y) := c_S(X,Y)_{\mathbf{Z}}$. Through the rest of this section, any cycle and any finite S-correspondence are assumed to have coefficients in Λ.

Remark 8.1.3. (1) According to properties (P7) and (P7') (*cf.* 7.1.38) of the relative exterior product, $c_S(X,Y)_\Lambda$ commutes with finite sums in X and Y.

(2) Consider $\alpha \in c_S(X,Y)_\Lambda$. Let Z be the support of α. Then, Z is finite pseudo-dominant over X (by definition 7.1.18). This means that Z is finite equidimensional over X.

When X is regular (resp. X is reduced geometrically unibranch and $\mathrm{char}(X) \subset \Lambda^\times$), any cycle $\alpha \subset X \times_S Y$ whose support is finite equidimensional over X defines a finite S-correspondence – *cf.* 7.3.28 (resp. 7.3.26).

Moreover, in each respective case, $c_S(X,Y)_\Lambda$ is the free Λ-module generated by the closed integral subschemes Z of $X \times_S Y$ which are finite equidimensional over X.

(3) Recall that in general, there is only an inclusion

$$c_S(X,Y) \otimes_{\mathbf{Z}} \Lambda \subset c_S(X,Y)_\Lambda .$$

This inclusion is an equality if X is regular (*cf.* 7.3.28) or $\mathrm{char}(X) \subset \Lambda^\times$.[38]

Example 8.1.4. (1) Let $f : X \to Y$ be a morphism in $\mathscr{P}/S$.

Because X/S is separated (assumption 8.0), the graph Γ_f of f is a closed subscheme of $X \times_S Y$. The canonical projection $\Gamma_f \to X$ is an isomorphism. Thus $\langle \Gamma_f \rangle_{XY}$ is a Hilbert cycle over X. In particular, it is Λ-universal and also finite over X, thus it defines a finite S-correspondence from X to Y.

(2) Let $f : Y \to X$ be finite S-morphism which is Λ-universal (as a morphism of the associated cycles). Then the graph Γ_f of f is closed in $X \times_S Y$ and the projection $\Gamma_f \to X$ is isomorphic to f. Thus the cycle $\langle \Gamma_f \rangle_{XY}$ is a finite Λ-universal cycle over X which therefore define a finite S-correspondence ${}^t f : X \bullet\!\!\longrightarrow Y$.

Suppose we are given finite S-correspondences $X \overset{\alpha}{\bullet\!\!\longrightarrow} Y \overset{\beta}{\bullet\!\!\longrightarrow} Z$. Consider the following diagram of cycles :

$$\begin{array}{ccccc} \beta \otimes_Y \alpha & \succ & \beta & \succ & Z. \\ \downarrow & & \downarrow & & \\ \alpha & \longrightarrow & Y & & \\ \downarrow & & & & \\ X & & & & \end{array}$$

The pullback cycle is well defined and has coefficients in Λ as β is Λ-universal over Y. Moreover, according to the definition of pullback (*cf.* 7.1.36) and corollary 7.2.6, $\beta \otimes_Y \alpha$ is a finite Λ-universal cycle over X with domain XYZ. Note finally that according to 8.1.1, the pushout of this latter cycle by p^{XZ}_{XYZ} is an element of $c_S(X,Z)_\Lambda$.

Definition 8.1.5. Using the preceding notations, we define the composition product of β and α as the finite S-correspondence

$$\beta \circ \alpha = p^{XZ}_{XYZ*}(\beta \otimes_Y \alpha) : X \bullet\!\!\longrightarrow Z.$$

[38] Recall indeed that the Suslin-Voeodsky's multiplicities of a cycle over a scheme X can only have denominators whose prime factors divide the residue characteristics of X according to 7.1.36.

8.1.6. In the case where S is regular and X, Y, Z are smooth over S, the composition product defined above agree with the one defined in [Dég07, 4.1.16] in terms of the Tor-formula of Serre. In fact, this is a direct consequence of 7.3.30 after reduction to the case where α and β are represented by closed integral subschemes (see also point (2) of remark 8.1.3).

We sum up the main properties of the composition for finite correspondences in the following proposition :

Proposition 8.1.7. *Let X, Y, Z be $\mathscr{P}$-schemes over S.*

(1) For any finite S-correspondences $X \overset{\alpha}{\bullet\!\!\rightarrow} Y \overset{\beta}{\bullet\!\!\rightarrow} Z \overset{\gamma}{\bullet\!\!\rightarrow} T$, we have
$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha)$.

(2) For any $X \overset{\alpha}{\bullet\!\!\rightarrow} Y \xrightarrow{g} Z$, $\langle \Gamma_g \rangle_{YZ} \circ \alpha = (1_X \times_S g)_*(\alpha)$.

(3) For any $X \xrightarrow{f} Y \overset{\beta}{\bullet\!\!\rightarrow} Z$, $\beta \circ \langle \Gamma_f \rangle_{XY} = \beta \otimes_Y \langle X \rangle$.
Moreover, if f is flat, $\beta \circ \langle \Gamma_f \rangle_{XY} = (f \times_S 1_Z)^*(\beta)$ considering the flat pullback of cycles in the classical sense.

(4) For any $X \xleftarrow{f} Y \overset{\beta}{\bullet\!\!\rightarrow} Z$ such that f is finite Λ-universal,
$\beta \circ {}^t f = (f \times_S 1_Z)_*(\beta)$.

(5) For any $X \overset{\alpha}{\bullet\!\!\rightarrow} Y \xleftarrow{g} Z$ such that g is finite Λ-universal,
${}^t g \circ \alpha = \langle Z \rangle \otimes_Y \alpha$.
If we suppose that g is finite flat, then ${}^t g \circ \alpha = (1_X \times_S g)^*(\alpha)$.

Proof. (1) Using respectively the projection formulas 7.2.10 and 7.2.8, we obtain

$$(\gamma \circ \beta) \circ \alpha = p^{XT}_{XYZT*}\big((\gamma \otimes_Z \beta) \otimes_Y \alpha\big)$$
$$\gamma \circ (\beta \circ \alpha) = p^{XT}_{XYZT*}\big(\gamma \otimes_Z (\beta \otimes_Y \alpha)\big).$$

Thus this formula is a direct consequence of the associativity 7.2.7.

(2) Let $\epsilon : \Gamma_g \to Y$ and $p^{XZ}_{X\Gamma_g} : X\Gamma_g \to XZ$ be the canonical projections. As ϵ is an isomorphism, we have tautologically $\langle Y \rangle = \epsilon_*(\langle \Gamma_g \rangle)$. We conclude by the following computation :

$$\begin{aligned}(1_X \times_S g)_*(\alpha) &= (1_X \times_S g)_*(\langle Y \rangle \otimes_Y \alpha) = (1_X \times_S g)_*(\epsilon_* \langle \Gamma_g \rangle \otimes_Y \alpha) \\ &\overset{(*)}{=} (1_X \times_S g)_*(1_X \times_S \epsilon)_*(\langle \Gamma_g \rangle \otimes_Y \alpha) = p^{XZ}_{X\Gamma_g *}(\langle \Gamma_g \rangle \otimes_Y \alpha) \\ &\overset{(*)}{=} p^{XZ}_{XYZ*}(\langle \Gamma_g \rangle_{YZ} \otimes_Y \alpha)\end{aligned}$$

The equalities labeled $(*)$ follow from the projection formula of 7.2.10.

(3) The first assertion follows from projection formula of 7.2.8 and the fact that Γ_f is isomorphic to X :

$$\beta \circ \langle \Gamma_f \rangle_{XY} = p^{XZ}_{XYZ*}(\beta \otimes_Y \langle \Gamma_f \rangle_{XY}) = \beta \otimes_Y p^X_{XY*}(\langle \Gamma_f \rangle_{XY}) = \beta \otimes_Y \langle X \rangle$$

The second assertion follows from corollary 7.2.2.

(4) and (5): The proof of these assertions is strictly similar to that of (2) and (3) instead that we use the projection formula of 7.2.8 (and do not need the commutativity 7.2.3). □

As a corollary, we obtain that the composition of S-morphisms coincide with the composition of the associated graph considered as finite S-correspondences. For any S-morphism $f : X \to Y$, we will still denote by $f : X \bullet\!\!\rightarrow Y$ the finite S-correspondence equal to $\langle \Gamma_f \rangle_{XY}$. Note moreover that for any $\mathscr{P}$-scheme X/S, the identity morphism of X is the neutral element for the composition of finite S-correspondences.

Definition 8.1.8. We let $\mathscr{P}^{cor}_{\Lambda,S}$ be the category of $\mathscr{P}$-schemes over S with morphisms the finite S-correspondences and the composition product of definition 8.1.5.

An object of $\mathscr{P}^{cor}_{\Lambda,S}$ will be denoted by $[X]$. The category $\mathscr{P}^{cor}_{\Lambda,S}$ is additive, and the direct sum is given by the disjoint union of $\mathscr{P}$-schemes over S. We have a canonical faithful functor

$$\gamma : \mathscr{P}/S \to \mathscr{P}^{cor}_{\Lambda,S} \tag{8.1.8.1}$$

which is the identity on objects and the graph on morphisms.

8.1.9. Given two S-morphisms $f : Y \to X$ and $g : X' \to X$ such that g is finite Λ-universal, we get from the previous proposition the equality of cycles in YX':

$$ {}^t g \circ f = \langle X' \rangle \otimes_X \langle Y \rangle_{YX}$$

where Y is seen as a closed subscheme of YX through the graph of f.

In particular, when either f or g is flat, we get (use property (P3) of 7.1.32 or corollary 7.2.2):

$$ {}^t g \circ f = \langle X' \times_X Y \rangle_{YX'}.$$

To state the next formulas (the generalized degree formulas), we introduce the following notion:

Definition 8.1.10. Let $f : X' \to X$ be a finite equidimensional morphism.

For any generic point x of X, we define the degree of f at x as the integer:

$$\deg_x(f) = \sum_{x'/x} [\kappa_{x'} : \kappa_x]$$

where the sums run over the generic points of X' lying above x.

Proposition 8.1.11. *Let X be a connected S-scheme and $f : X' \to X$ be a finite S-morphism.*

Then, if f is Λ-universal, there exists an integer $d \in \mathbf{N}^*$ such that for any generic point x of X, $\deg_x(f) = d$.

Moreover, $f \circ {}^t f = d.1_X$.

Proof. Let Δ' be the diagonal of X'/S. For any generic point x of X, we let Δ_x be the diagonal of the corresponding irreducible component of X, seen as a closed subscheme of X. According to proposition 8.1.7, and the definition of pushout, we get

$$\alpha := f \circ {}^t f = (f \times_S f)_*(\langle \Delta' \rangle_{X'X'}) = \sum_{x \in X^{(0)}} \deg_x(f).\langle \Delta_x \rangle_{XX}.$$

Considering generic points x, y of X, we prove $\deg_x(f) = \deg_y(f)$. By induction, we can reduce to the case where x and y have a common specialisation s in X. Then, as α/X is special, we get by definition of the relative product (see more precisely 7.1.41)

$$\alpha \otimes_S s = \deg_x(f).s = \deg_y(f).s$$

as required. The remaining assertion then follows. □

The previous proposition applies in particular when f is finite flat. Assuming X is connected, we call d the degree of f.

Proposition 8.1.12. *Let $f : X' \to X$ be an S-morphism which is finite, radicial and Λ-universal.*

Assume X is connected, and let d be the degree of f.

Then ${}^t f \circ f = d.1_{X'}$. In particular, if d is invertible in Λ, f is an isomorphism in $\mathscr{P}^{cor}_{\Lambda,S}$.

Proof. According to 8.1.9, ${}^t f \circ f = \langle X' \rangle \otimes_X \langle X' \rangle$ as cycles in $X'X'$. Let x be the generic point of X and k be its residue field. Let $\{x'_i, i \in I\}$ be the set of generic points of X, and for any $i \in I$, k'_i be the residue field of x'_i. According to 7.2.1, we thus obtain:

$$ {}^t f \circ f = \sum_{(i,j) \in I^2} \langle \mathrm{Spec}\left(k'_i \otimes_k k'_j\right) \rangle_{X'X'}.$$

The result now follows by the definition of the degree and the fact that for any $i \in I$, k'_i/k is radicial. □

8.2. **Monoidal structure.** Fix a base scheme S. Let X, X', Y, Y' be $\mathscr{P}$-schemes over S.

Consider finite S-correspondences $\alpha : X \bullet\!\!\to Y$ and $\beta : X' \bullet\!\!\to Y'$. Then $\alpha X' := \alpha \otimes_X \langle XX' \rangle$ and $\alpha' X := \alpha' \otimes_{X'} \langle XX' \rangle$ are both finite Λ-universal cycles over XX'. Using stability by composition of finite Λ-universal morphisms (*cf.* corollary 7.2.6), the cycle $(\alpha X') \otimes_{XX'} (\alpha' X)$ is finite Λ-universal over XX'.

Definition 8.2.1. Using the above notation, we define the tensor product of α and α' over S as the finite S-correspondence

$$\alpha \otimes_S^{tr} \alpha' = (\alpha X') \otimes_{XX'} (\alpha' X) : XX' \bullet\!\!\rightarrow YY'.$$

Let us first remark that this tensor product is commutative (use commutativity of the exterior product 7.2.3) and associative (use associativity of exterior product 7.2.7). Moreover, it is compatible with composition :

Lemma 8.2.2. *Suppose given finite S-correspondences :*
$\alpha : X \to Y,\ \beta : Y \to Z,\ \alpha' : X' \to Y',\ \beta' : Y' \to Z'$. *Then*

$$(\beta \circ \alpha) \otimes_S^{tr} (\beta' \circ \alpha') = (\beta \otimes_S^{tr} \beta') \circ (\alpha \otimes_S^{tr} \alpha').$$

Proof. We put $\alpha X' = \alpha \otimes_X \langle XX' \rangle$, $\alpha' X = \alpha' \otimes_X \langle XX' \rangle$ and $\beta Y' = \beta \otimes_Y \langle YY' \rangle$, $\beta' Y = \beta' \otimes_Y \langle YY' \rangle$. We can compute the right hand side of the above equation as follows :

$$\begin{aligned}
&p^{XX'ZZ'}_{XX'YY'ZZ'*}\Big((\beta Y' \otimes_{YY'} \beta' Y) \otimes_{YY'} (\alpha X' \otimes_{XX'} \alpha' X)\Big) \\
&\overset{(1)}{=} p^{XX'ZZ'}_{XX'YY'ZZ'*}\Big((\beta Y' \otimes_{YY'} \beta' Y) \otimes_{YY'} (\alpha' X \otimes_{XX'} \alpha X')\Big) \\
&\overset{(2)}{=} p^{XX'ZZ'}_{XX'YY'ZZ'*}\Big(\beta Y' \otimes_{YY'} ((\beta' Y \otimes_{YY'} \alpha' X) \otimes_{XX'} \alpha X')\Big) \\
&\overset{(3)}{=} p^{XX'ZZ'}_{XX'YY'ZZ'*}\Big((\beta Y' \otimes_{YY'} \alpha X') \otimes_{XX'} (\beta' Y \otimes_{YY'} \alpha' X))\Big).
\end{aligned}$$

Equality (1) follows from commutativity 7.2.3, equality (2) from associativity 7.2.7 and equality (3) by both commutativity and associativity.

For the left hand side, we note that using the projection formula 7.2.10, the left hand side is equal to

$$p^{XX'ZZ'}_{XX'YY'ZZ'*}\Big(\big((\beta \otimes_Y \alpha) \otimes_X \langle XX' \rangle\big) \otimes_{XX'} \big((\beta' \otimes_{Y'} \alpha') \otimes_{X'} \langle XX' \rangle\big)\Big).$$

We are left to remark that

$$(\beta \otimes_Y \alpha) \otimes_X \langle XX' \rangle = \big((\beta Y') \otimes_{YY'} \alpha\big) \otimes_X \langle XX' \rangle = \beta Y' \otimes_{YY'} \alpha X',$$

using transitivity 7.2.4 and associativity 7.2.7. We thus conclude by symmetry of the other part in the left hand side. □

Definition 8.2.3. We define a symmetric monoidal structure on the category $\mathscr{P}^{cor}_{\Lambda,S}$ by putting $[X] \otimes_S^{tr} [Y] = [X \times_S Y]$ on objects and using the tensor product of the previous definition for morphisms.

8.2.4. Note that the functor $\gamma : \mathscr{P}/S \to \mathscr{P}^{cor}_{\Lambda,S}$ is monoidal for the cartesian structure on the left hand category. Indeed, this is a consequence of property (P3) of the relative product (see 7.1.32) and the remark that for any morphisms $f : X \to Y$ and $f' : X' \to Y'$, $(\Gamma_f \times_S X') \times_{XX'} (\Gamma'_f \times_S X) = \Gamma_{f \times_S f'}$.

8.3. **Functoriality.** Fix a morphism of schemes $f : T \to S$. For any $\mathscr{P}$-scheme X/S, we put $X_T = X \times_S T$. For a pair of $\mathscr{P}$-schemes over S (resp. T-schemes) (X, Y), we put $XY = X \times_S Y$ (resp. $XY_T = X \times_T Y$).

8.3.a. *Base change.* Consider a finite S-correspondence $\alpha : X \bullet\!\!\rightarrow Y$. The cycle $\alpha \otimes_X \langle X_T \rangle$ defines a finite T-correspondence from X_T to Y_T denoted by α_T.

Lemma 8.3.1. *Consider finite S-correspondences $X \overset{\alpha}{\bullet\!\!\rightarrow} Y \overset{\beta}{\bullet\!\!\rightarrow} Y$.*
Then $(\beta \circ \alpha)_T = \beta_T \circ \alpha_T$.

Proof. This follows easily using the projection formula 7.2.10, the associativity formula 7.2.7 and the transitivity formula 7.2.4 :

$$p^{XZ}_{XYZ*}(\beta \otimes_Y \alpha) \otimes_X \langle X_T\rangle = p^{XZ_T}_{XYZ_T*}\big((\beta \otimes_Y \alpha) \otimes_X \langle X_T\rangle\big)$$
$$= p^{XZ_T}_{XYZ_T*}\big(\beta \otimes_Y (\alpha \otimes_X \langle X_T\rangle)\big) = p^{XZ_T}_{XYZ_T*}\big((\beta \otimes_Y \langle Y_T\rangle) \otimes_{Y_T} (\alpha \otimes_X \langle X_T\rangle)\big).$$

□

Definition 8.3.2. Let $f : T \to S$ be a morphism of schemes. Using the preceding lemma, we define the base change functor

$$\begin{array}{rcl} f^* : \mathscr{P}^{cor}_{\Lambda,S} & \to & \mathscr{P}^{cor}_{\Lambda,T} \\ [X/S] & \mapsto & [X_T/T] \\ c_S(X,Y)_\Lambda \ni \alpha & \mapsto & \alpha_T. \end{array}$$

We sum up the basic properties of the base change for correspondences in the following lemma.

Lemma 8.3.3. *Take the notation and hypothesis of the previous definition.*

(1) The functor f^* is symmetric monoidal.

(2) Let $f_0^* : \mathscr{P}/S \to \mathscr{P}/T$ be the classical base change functor on $\mathscr{P}$-schemes over S. Then the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{P}/S & \xrightarrow{\gamma_S} & \mathscr{P}^{cor}_{\Lambda,S} \\ {\scriptstyle f_0^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathscr{P}/T & \xrightarrow{\gamma_T} & \mathscr{P}^{cor}_{\Lambda,T}. \end{array}$$

(3) Let $\sigma : T' \to T$ be a morphism of schemes. Through the canonical isomorphisms $(X_T)_{T'} \simeq X_{T'}$, equality $(f \circ \sigma)^* = \sigma^* \circ f^*$ holds.

Proof. (1) This point follows easily using the associativity formula 7.2.7 and the transitivity formulas 7.2.4, 7.2.6.
(2) This point follows from the fact that for any S-morphism $f : X \to Y$, there is a canonical isomorphism $\Gamma_{f_T} \to \Gamma_f \times_S T$.
(3) This point is a direct application of the transitivity 7.2.4. □

Lemma 8.3.4. *Let $f : T \to S$ be a universal homeomorphism.*
Then $f^* : \mathscr{P}^{cor}_{\Lambda,S} \to \mathscr{P}^{cor}_{\Lambda,T}$ is fully faithful.

Proof. Let X and Y be $\mathscr{P}$-schemes over S. Then $X_T \to X$ is a universal homeomorphism. Any generic point x of X corresponds uniquely to a generic point of X_T. Let m_x (resp. m'_x) be the geometric multiplicity of x in X (resp. X_T). Consider a finite S-correspondence $\alpha = \sum_{i\in I} n_i.z_i$. For each $i \in I$, let x_i be the generic point of X dominated by z_i. Then we get by definition:

$$f^*(\alpha) = \sum_{i\in I} m'_{x_i} \frac{n_i}{m_{x_i}}.z_i$$

and the lemma is clear. □

8.3.b. *Restriction.* Consider a $\mathscr{P}$-morphism $p : T \to S$. For any pair of T-schemes (X,Y), we denote by $\delta_{XY} : X \times_T Y \to X \times_S Y$ the canonical closed immersion deduced by base change from the diagonal immersion of T/S.

Consider a finite T-correspondence $\alpha : X \bullet\!\!\to Y$. The cycle $\delta_{XY*}(\alpha)$ is the cycle α considered as a cycle in $X \times_S Y$. It defines a finite S-correspondence from X to Y.

Lemma 8.3.5. *Let X, Y and Z be T-schemes. The following relations are true :*

(1) For any T-morphism $f : X \to Y$, $\delta_{XY*}\big(\langle\Gamma_f\rangle_{XY_T}\big) = \langle\Gamma_f\rangle_{XY}$.

(2) For all $\alpha \in c_T(X,Y)_\Lambda$ and $\beta \in c_T(Y,Z)_\Lambda$,

$$\delta_{XZ*}(\beta \circ \alpha) = (\delta_{YZ*}(\beta)) \circ (\delta_{XY*}(\alpha)).$$

Proof. The first assertion is obvious.

The second assertion is a consequence of the projection formulas 7.2.8 and 7.2.10, and the functoriality of pushout :

$$\begin{aligned}(\delta_{YZ*}(\beta)) \circ (\delta_{XY*}(\alpha)) &= p^{XZ}_{XYZ*}\big(\delta_{YZ*}(\beta) \otimes_Y \delta_{XY*}(\alpha)\big) \\ &= p^{XZ}_{XYZ*}\delta_{XYZ*}(\beta \otimes_Y \alpha) = \delta_{XZ*}p^{XZ_T}_{XYZ_T*}(\beta \otimes_Y \alpha).\end{aligned}$$

□

Definition 8.3.6. Let $p : T \to S$ be a $\mathscr{P}$-morphism.

Using the preceding lemma, we define a functor

$$\begin{array}{rcl} p_\sharp : \mathscr{P}^{cor}_{\Lambda,T} & \to & \mathscr{P}^{cor}_{\Lambda,S} \\ [X \to T] & \mapsto & [X \to T \xrightarrow{p} S] \\ c_T\,(X,Y)_\Lambda \ni \alpha & \mapsto & \delta_{XY*}(\alpha). \end{array}$$

This functor enjoy the following properties:

Lemma 8.3.7. *Let $p : T \to S$ be a $\mathscr{P}$-morphism.*

(1) The functor $p_\sharp$ is left adjoint to the functor p^*.

(2) For any composable $\mathscr{P}$-morphisms $Z \xrightarrow{q} T \xrightarrow{p} S$, $(pq)_\sharp = p_\sharp q_\sharp$.

(3) Let $p^0_\sharp : \mathscr{P}/T \to \mathscr{P}/S$ be the functor induced by composition with p. Then the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{P}/T & \xrightarrow{\gamma_T} & \mathscr{P}^{cor}_{\Lambda,T} \\ {\scriptstyle p^0_\sharp}\downarrow & & \downarrow{\scriptstyle p_\sharp} \\ \mathscr{P}/S & \xrightarrow{\gamma_S} & \mathscr{P}^{cor}_{\Lambda,S}. \end{array}$$

Proof. For point (1), we have to construct for schemes X/T and Y/S a natural isomorphism $c_S\,(p_\sharp X, Y)_\Lambda \simeq c_T\,(X, p^*Y)_\Lambda$. It is induced by the canonical isomorphism of schemes $(p_\sharp X)\times_S Y \simeq X \times_T (p^*Y)$.

Point (2) follows from the associativity of the pushout functor on cycles. Note also that this identification is compatible with the transposition of the identification of 8.3.3(3) according to the adjunction property just obtained.

Point (3) is a reformulation of 8.3.5(2). □

8.3.c. *A finiteness property.*

8.3.8. We assume here that $\mathscr{P}$ is the class of all separated morphisms of finite type in $\mathscr{S}$.

Let I be a left filtering category and $(X_i)_{i\in I}$ be a projective system of separated S-schemes of finite type with affine dominant transition morphisms. We let $\mathscr{X}$ be the projective limit of $(X_i)_i$ and assume that $\mathscr{X}$ is Noetherian over S.

Proposition 8.3.9. *Let Y be a $\mathscr{P}$-scheme of finite type over S. Then the canonical morphism*

$$\varphi : \varinjlim_{i\in I^{op}} c_S\,(X_i, Y)_\Lambda \to c_0(\mathscr{X} \times_S Y/\mathscr{X}, \Lambda).$$

is an isomorphism.

Proof. Note that according to [AGV73, IV, 8.3.8(i)], we can assume the conditions (2) of 7.3.5 is verified for $(X_i)_{i\in I}$. Thus conditions (1) to (4) of *loc. cit.* are verified. Then the surjectivity of φ follows from 7.3.9 and the injectivity from 7.3.6. □

8.4. **The fibred category of correspondences.** We can summarize the preceding constructions:

Proposition 8.4.1. *The 2-functor*

$$S \mapsto \mathscr{P}^{cor}_{\Lambda,S}$$

equipped with the pullback defined in 8.3.2 and with the tensor product of 8.2.3 is a monoidal $\mathscr{P}$-fibred category such that the functor

$$\gamma : \mathscr{P} \to \mathscr{P}^{cor}_\Lambda$$

(see (8.1.8.1)*) is a morphism of monoidal $\mathscr{P}$-fibred category.*

Proof. According to lemma 8.3.7, for any $\mathscr{P}$ morphisms p, p^* admits a left adjoint $p_\sharp$. We have checked that γ is symmetric monoidal and commutes with f^* and $p_\sharp$ (see respectively 8.2.4, 8.3.3 and 8.3.7). But γ is essentially surjective. Thus, to prove the properties ($\mathscr{P}$-BC) and ($\mathscr{P}$-PF) for the fibred category $\mathscr{P}_\Lambda^{cor}$, we are reduced to the case of case of $\mathscr{P}$ which is easy (see example 1.1.27). This concludes. □

Remark 8.4.2. Consider the definition above.

(1) The category $\mathscr{P}_\Lambda^{cor}$ is Λ-linear. For any scheme S, $\mathscr{P}_{\Lambda,S}^{cor}$ is additive. For any finite family of schemes $(S_i)_{i\in I}$ which admits a sum S in $\mathscr{S}$, the canonical map

$$\mathscr{P}_{\Lambda,S}^{cor} \to \bigoplus_{i\in I} \mathscr{P}_{\Lambda,S_i}^{cor}$$

is an isomorphism.

(2) The functor $\gamma : \mathscr{P} \to \mathscr{P}_\Lambda^{cor}$ is nothing else than the canonical geometric sections of $\mathscr{P}_\Lambda^{cor}$ (see definition 1.1.34).

We will apply these definitions in the particular cases $\mathscr{P} = Sm$ (resp. $\mathscr{P} = \mathscr{S}^{ft}$) the class of smooth separated (resp. separated) morphisms of finite type. Note that we get a commutative square

$$\begin{array}{ccc} Sm & \xrightarrow{\gamma} & \mathscr{S}m_\Lambda^{cor} \\ \downarrow & & \downarrow \\ \mathscr{S}^{ft} & \xrightarrow{\gamma} & \mathscr{S}_\Lambda^{ft,cor} \end{array}$$

where the vertical maps are the obvious embeddings of monoidal Sm-fibred categories.

Remark 8.4.3. Let S be a regular scheme. We have already seen in remark 8.1.3 that

$$\mathscr{P}_{\Lambda,S}^{cor} = \mathscr{P}_{\mathbf{Z},S}^{cor} \otimes_{\mathbf{Z}} \mathbf{Z}.$$

Moreover, remark 8.1.6 implies that the category $\mathscr{S}m_{\mathbf{Z},S}^{cor}$ defined here coincide with the one introduced in [Dég07]. Finally, using again the computation of Suslin-Voevodsky's multiplicities in term of the Tor-formula (*cf.* 7.3.30), we can check that the operations τ^*, $\tau_\sharp$, and $\otimes^{tr}$ defined here coincide with that of [Dég07].

9. Sheaves with transfers

9.0. The category $\mathscr{S}$ is the category of noetherian schemes of finite dimension. We fix an admissible class $\mathscr{P}$ of morphisms in $\mathscr{S}$ satisfying the following assumptions:

(a) Any morphism in $\mathscr{P}$ is separated of finite type.
(b) Any étale separated morphism of finite type is in $\mathscr{P}$.

We fix a topology t on $\mathscr{S}$ which is $\mathscr{P}$-admissible and such that:

(c) For any scheme S, there is a class of covers $\mathcal{E}$ of the form $(p : W \to S)$ with p a $\mathscr{P}$-morphism such that t is the topology generated by $\mathcal{E}$ and the covers of the form $(U \to U \sqcup V, V \to U \sqcup V)$ for any schemes U and V in $\mathscr{S}$.

We fix a ring of coefficients $\Lambda \subset \mathbf{Q}$.

Note that in sections 9.4 and 9.5, we will apply the conventions of section 1.4 by replacing the class of smooth morphisms of finite type (resp. morphisms of finite type) there by the class of smooth separated morphisms of finite type (resp. separated morphisms of finite type).

9.1. **Presheaves with transfers.** We consider the additive category $\mathscr{P}^{cor}_{\Lambda,S}$ of definition 8.1.8 and the graph functor $\gamma : \mathscr{P}/S \to \mathscr{P}^{cor}_{\Lambda,S}$ of (8.1.8.1).

Definition 9.1.1. A presheaf with transfers F over S is an additive presheaf of abelian groups over $\mathscr{P}^{cor}_{\Lambda,S}$. We denote by $\mathrm{PSh}\left(\mathscr{P}^{cor}_{\Lambda,S}\right)$ the corresponding category.

If X is a $\mathscr{P}$-scheme over S, we denote by $\Lambda^{tr}_S(X)$ the presheaf with transfers represented by X.

We denote by $\hat{\gamma}_*$ the functor

$$\mathrm{PSh}\left(\mathscr{P}^{cor}_{\Lambda,S}\right) \to \mathrm{PSh}(\mathscr{P}/S, \Lambda), F \mapsto F \circ \gamma. \tag{9.1.1.1}$$

Note that $\mathrm{PSh}\left(\mathscr{P}^{cor}_{\Lambda,S}\right)$ is obviously a Grothendieck abelian category generated by the objects $\Lambda^{tr}_S(X)$ for a $\mathscr{P}$-scheme X/S. Moreover, the following proposition is straightforward:

Proposition 9.1.2. *There is an essentially unique Grothendieck abelian $\mathscr{P}$-premotivic category $\mathrm{PSh}\left(\mathscr{P}^{cor}_{\Lambda}\right)$ which is geometrically generated (cf. 1.1.40), whose fiber over a scheme S is $\mathrm{PSh}\left(\mathscr{P}^{cor}_{\Lambda,S}\right)$ and such that the functor Λ^{tr}_S induces a morphism of additive monoidal $\mathscr{P}$-fibred categories.*

$$\mathscr{P}^{cor}_{\Lambda} \to \mathrm{PSh}\left(\mathscr{P}^{cor}_{\Lambda}\right). \tag{9.1.2.1}$$

Moreover, the functor (9.1.1.1) *induces a morphism of abelian $\mathscr{P}$-premotivic categories*

$$\hat{\gamma}^* : \mathrm{PSh}(\mathscr{P}, \Lambda) \rightleftarrows \mathrm{PSh}\left(\mathscr{P}^{cor}_{\Lambda}\right) : \hat{\gamma}_*.$$

Proof. To help the reader, we recall the following consequence of Yoneda's lemma:

Lemma 9.1.3. *Let $F : (\mathscr{P}^{cor}_{\Lambda,S})^{op} \to \Lambda\text{-mod}$ be a presheaf with transfers. Let $\mathcal{I}$ be the category of representables preshaves with transfers over F. Then the canonical map*

$$\varinjlim_{\Lambda^{tr}_S(X)\to F} \Lambda^{tr}_S(X) \to F$$

is an isomorphism. The limit is taken in $\mathrm{PSh}\left(\mathscr{P}^{cor}_{\Lambda,S}\right)$ and runs over $\mathcal{I}$.

This lemma allows to define the structural left adjoint of $\mathrm{PSh}\left(Pmorc\right)$ (recall f^*, $p_\sharp$ for p a $\mathscr{P}$-morphism and the tensor product) because they are indeed determined by (9.1.2.1). The existence of the structural right adjoints is formal.

The same lemma allows to get the adjunction $(\hat{\gamma}^*, \hat{\gamma}_*)$. □

Remark 9.1.4. Note that for any presheaf with transfers F over S, and any morphism $f : T \to S$ (resp. $\mathscr{P}$-morphism $p : S \to S'$), we get as usual $f_*F = f \circ f^*$ (resp. $p^*F = F \circ p_\sharp$) where the functor f^* (resp. $p_\sharp$) on the right hand side is taken with respect to the $\mathscr{P}$-fibred category $\mathscr{P}^{cor}_{\Lambda}$.

9.2. **Sheaves with transfers.**

Definition 9.2.1. A t-sheaf with transfers over S is a presheaf with transfers F such that the functor $F \circ \gamma_S$ is a t-sheaf. We denote by $\mathrm{Sh}_t\left(\mathscr{P}^{cor}_{\Lambda,S}\right)$ the full subcategory of $\mathrm{PSh}(\mathscr{P}^{cor}_{\Lambda,S}, \Lambda)$ of sheaves with transfers.

According to this definition, we get a canonical faithful functor

$$\gamma_* : \mathrm{Sh}_t\left(\mathscr{P}^{cor}_{\Lambda,S}\right) \to \mathrm{Sh}_t(\mathscr{P}/S, \Lambda), F \mapsto F \circ \gamma.$$

Example 9.2.2. A particularly important case for us is the case when $t = \mathrm{Nis}$ is the Nisnevich topology. According to the original definition of Voevodsky, a Nisnevich sheaf with transfers will be called simply a *sheaf with transfers*.

Remark 9.2.3. Later on, in the case $\mathscr{P} = \mathscr{S}^{ft}$, we will use the notation $\underline{\Lambda}^{tr}_S(X)$ to denote the presheaf on the big site $\mathscr{S}^{ft,cor}_{\Lambda,S}$.

Proposition 9.2.4. *Let X be an $\mathscr{P}$-scheme over S.*

(1) The presheaf $\Lambda^{tr}_S(X)$ is an étale sheaf with transfers.

(2) If $\mathrm{char}(X) \subset \Lambda^\times$, $\Lambda^{tr}_S(X)$ is a qfh-sheaf with transfers.

Proof. For point (1), we follow the proof of [Dég07, 4.2.4]: the computation of the pullback by an étale map is given in our context by point (3) of proposition 8.1.7. Moreover, the property for a cycle α/Y to be A-universal is étale-local on Y according to 7.3.8. For point (2), we refer to [SV00b, 4.2.7]. □

We can actually describe explicitely representable presheaves with transfers in the following case:

Proposition 9.2.5. *Let S be a scheme and X be a finite étale S-scheme. Then for any $\mathscr{P}$-scheme Y over S,*

$$\Gamma(Y, \Lambda_S^{tr}(X)) = \pi_0(Y \times_S X).\Lambda.$$

This readily follows from the following lemma:

Lemma 9.2.6. *Let $f : X \to S$ be an étale separated morphism of finite type. Let $\pi_0^{finite}(X/S)$ be the set of connected components V of X such that $f(V)$ is equal to a connected component of S (*i.e. *f is finite over V).*

Then $c_0(X/S, \Lambda) = \pi_0^{finite}(X/S).\Lambda$.

Proof. We can assume that S is reduced and connected.

We first treat the case where $V = S$. Consider a finite Λ-universal S-cycle α with domain S. Write $\alpha = \sum_{i\in I} n_i.\langle Z_i\rangle_S$ in standard form. By definition, Z_i dominates an irreducible component of S thus Z_i is equal to that irreducible component.
Consider S_0 an irreducible component of S and an index $i \in I$ such that $S_0 \cap Z_i$ is not empty. Consider a point $s \in S_0 \cap Z_i$. We have obviously $\alpha_s = n_i.\langle \mathrm{Spec}\,(\kappa(s))\rangle \neq 0$. Thus there exists a component of α which dominates S_0 *i.e.* $\exists j \in I$ such that $Z_j = S_0$. Moreover, computing α_s using alternatively Z_i and Z_j gives $n_i = n_j$.
As S is noetherian, we see inductively $\{Z_i | i \in I\}$ is the set of irreducible components of S and for any $i, j \in I$, $n_i = n_j$. Thus $c_0(S/S, \Lambda) = \mathbf{Z}$.

Consider now the case of an étale S-scheme X. By additivity of c_0, we can assume that X is connected. Consider the following canonical map:

$$c_0(X/S, \Lambda) \to c_0(X \times_S X/X, \Lambda), \alpha \mapsto \alpha \otimes_S^\flat X.$$

Note that considering the projection $p : X \times_S X \to X$, by definition, $\alpha \otimes_S^\flat X = p^*(\alpha)$.
Consider the diagonal $\delta : X \to X \times_S X$ of X/S. Because X/S is étale and separated, δ is a direct factor of $X \times_S X$ and we can write $X \times_S X = X \sqcup U$. Because c_0 is additive,

$$c_0(X \times_S X/X, \Lambda) = c_0(X/X, \Lambda) \oplus c_0(U/X, \Lambda).$$

Moreover, the projection on the first factor is induced by the map δ^* on cycles. Because $\delta^* p^* = 1$, we deduce that $c_0(X/S, \Lambda)$ is a direct factor of $c_0(X/X, \Lambda)$. According to the preceding case, this latter group is the free group generated by the cycle $\langle X\rangle$. This latter cycle is Λ-universal over S, because X/S is flat. Thus, if X/S is finite, it is an element of $c_0(X/S, \Lambda)$ so that $c_0(X/S, \Lambda) = \Lambda$. Otherwise, not any of the Λ-linear combination of $\langle X\rangle$ belongs to $c_0(X/S, \Lambda)$ so that $c_0(X/S, \Lambda) = 0$. □

9.3. **Associated sheaf with transfers.**

9.3.1. Recall from 3.2.1 that we denote by $(\mathscr{P}/S)^{\amalg}$ the category of I-diagrams of objects in $\mathscr{P}/S$ indexed by a discrete category I. Given any simplicial object $\mathcal{X}$ of $(\mathscr{P}/S)^{\amalg}$, we will consider the complex $\Lambda_S^{tr}(\mathcal{X})$ of $\mathrm{PSh}\left(\mathscr{P}_{\Lambda,S}^{cor}\right)$ applying the definition of 5.1.8 to the Grothendieck $\mathscr{P}$-fibred category $\mathrm{PSh}\,(\mathscr{P})$.

Consider a t-cover $p : W \to X$ in $\mathscr{P}/X$. We denote by W_X^n the n-fold product of W over X (in the category $\mathscr{P}/X$). We denote by $\check{S}(W/X)$ the Čech simplicial object of $\mathscr{P}_{\Lambda,S}^{cor}$ such that $\check{S}_n(W/X) = W_X^{n+1}$. The canonical morphism $\check{S}(W/X) \to X$ is a t-hypercover according to definition 3.2.1. We will call these particular type of t-hypercoverings the *Čech t-hypercoverings* of X.

Definition 9.3.2. We will say that the admissible topology t on $\mathscr{P}$ is *compatible with transfers* (resp. *weakly compatible with transfers*) if for any scheme S and any t-hypercover (resp. any Čech t-hypercover) $\mathcal{X} \to X$ in the site $\mathscr{P}/S$, the canonical morphism of complexes

$$\Lambda_S^{tr}(\mathcal{X}) \to \Lambda_S^{tr}(X) \tag{9.3.2.1}$$

induces a quasi-isomorphism of the associated t-sheaves on $\mathscr{P}/S$.

Obviously, if t is compatible with transfers then it is weakly compatible with transfers.

Recall from 9.2.4 that, in the cases $t = \mathrm{Nis}, \text{ét}$, (9.3.2.1) is actually a morphism of complexes of t-sheaves with transfers. The following proposition is a generalisation of [Voe96, 3.1.3] but its proof is in fact the same.

Proposition 9.3.3. *The Nisnevich (resp étale) topology t on $\mathscr{P}$ is weakly compatible with transfers.*

Proof. We consider a t-cover $p : W \to X$, the associated Čech hypercover $\mathcal{X} = \check{S}(W/X)$ of X and we prove that the map (9.3.2.1) is a quasi-isomorphism of t-sheaves. Recall that a point of $\mathscr{P}_S$ for the topology t is given by an essentially affine pro-object $(V_i)_{i\in I}$ of $\mathscr{P}/S$. Moreover, its projective limit $\mathcal{V}$ in the category of scheme is in particular a local henselian noetherian scheme.
It will be sufficient to check that the fiber of (9.3.2.1) at the point $(V_i)_{i\in I}$ is a quasi-isomorphism. Thus, according to proposition 8.3.9, we can assume that $S = \mathcal{V}$ is a local henselian scheme and we are to reduce to prove that the complex of abelian groups

$$\ldots \to c_0(W \times_X W/S, \Lambda) \to c_0(W/S, \Lambda) \xrightarrow{p_*} c_0(X/S, \Lambda) \to 0$$

is acyclic. We denote this complex by C.

Recall that the abelian group $c_0(X/S)$ is covariantly functorial in X with respect to separated morphisms of finite type $f : X' \to X$ (*cf.* paragraph 8.1.1). Moreover, if f is an immersion, f_* is obviously injective.

Let $\mathcal{F}_0$ be the set, ordered by inclusion, of closed subschemes Z of X such that Z/S is finite. Given a closed subscheme Z in $\mathcal{F}_0$, we let C_Z be the complex of abelian groups

$$\ldots \to c_0(W_Z \times_Z W_Z/S, \Lambda) \to c_0(W_Z/S, \Lambda) \xrightarrow{p_{Z*}} c_0(Z/S, \Lambda) \to 0 \tag{9.3.3.1}$$

where p_Z is the t-cover obtained by pullback along $Z \to X$. From what we have just recalled, we can identify C_Z with a subcomplex of C. The set $\mathcal{F}_0$ can be ordered by inclusion, and C is the union of its subcomplexes C_Z. If $\mathcal{F}_0$ is empty, then $C = 0$ and the proposition is clear. Otherwise, $\mathcal{F}_0$ is filtered and we can write:

$$C = \varinjlim_{Z\in\mathcal{F}_0} C_Z.$$

Thus, it will be sufficient to prove that C_Z is acyclic for any $Z \in \mathcal{F}_0$. Because S is henselian and Z is finite over S, Z is indeed a finite sum of local henselian schemes. This implies that the t-cover p_Z, which is in particular étale surjective, admits a splitting $s : Z \to W_Z$. Then the complex (9.3.3.1) is contractible with contracting homotopy defined by the family

$$(s \times_Z 1_{W_Z^n})_* : c_0(W_Z^n/S, \Lambda) \to c_0(W_Z^{n+1}/S, \Lambda).$$

□

9.3.4. Considering an additive abelian presheaf G on $\mathscr{P}/S$, the natural transformation

$$X \mapsto \mathrm{Hom}_{\mathrm{PSh}(\mathscr{P}/S)}(\hat{\gamma}_*\Lambda_S^{tr}(X), G)$$

defines a presheaf with transfers over S.[39] We will denote by G_τ its restriction to the site $\mathscr{P}/S$. Note that this definition can be applied in the case where G is a t-sheaf on $\mathscr{P}/S$, because under the assumption 9.0 on t, it is in particular an additive presheaf.

Definition 9.3.5. We will say that t is mildly compatible with transfers if for any scheme S and any t-sheaf F on $\mathscr{P}/S$, F_τ is a t-sheaf on $\mathscr{P}/S$.

If t is weakly compatible with transfers then is it mildly compatible with transfers.

[39] Actually, this defines a right adjoint to the functor $\hat{\gamma}_*$.

Remark 9.3.6. Assume t is mildly compatible with transfers. Then for any scheme S, any t-cover $p : W \to X$ in $\mathscr{P}/S$ induces a morphism

$$p_* : \Lambda_S^{tr}(W) \to \Lambda_S^{tr}(X)$$

which is an epimorphism of the associated t-sheaves on $\mathscr{P}/S$. This means that for any correspondence $\alpha \in c_S(Y, X)$, there exists a t-cover $q : W' \to Y$ and a correspondence $\alpha' \in c_S(W', W)$ making the following diagram commutative:

$$\begin{array}{ccc} W' & \xrightarrow{\hat{\alpha}} & W \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{\alpha} & X \end{array} \tag{9.3.6.1}$$

Lemma 9.3.7. *Assume t is mildly compatible with transfers.*

Let S be a scheme and P^{tr} be a presheaf with transfers over S. We put $P = P^{tr} \circ \gamma$ as a presheaf on $\mathscr{P}/S$. We denote by F the t-sheaf associated with P and by $\eta : P \to F$ the canonical natural transformation.

Then there exists a unique pair (F^{tr}, η^{tr}) such that:

(1) F^{tr} is a sheaf with transfers over S such that $F^{tr} \circ \gamma = F$.

(2) $\eta^{tr} : P^{tr} \to F^{tr}$ is a natural transformation of presheaves with transfers such that the induced transformation

$$P = (P^{tr} \circ \gamma) \to (F^{tr} \circ \gamma) = F$$

coincides with η.

Proof. As a preliminary observation, we note that given a presheaf G on $\mathscr{P}/S$, the data of a presheaf with transfers G^{tr} such that $G^{tr} \circ \gamma = G$ is equivalent to the data for any $\mathscr{P}$-schemes X and Y of a bilinear product

$$G(X) \otimes_{\mathbf{Z}} c_S(Y, X) \to G(Y), \rho \otimes \alpha \mapsto \langle \rho, \alpha \rangle \tag{9.3.7.1}$$

such that:

(a) For any $\mathscr{P}$-morphism $f : Y' \to Y$, $f^*\langle \rho, \alpha \rangle = \langle \rho, \alpha \circ f \rangle$.
(b) For any $\mathscr{P}$-morphism $f : X \to X'$, if $\rho = f^*(\rho')$, $\langle f^*(\rho'), \alpha \rangle = \langle \rho, f \circ \alpha \rangle$.
(c) When $X = Y$, for any $\rho \in F(X)$, $\langle \rho, 1_X \rangle = \rho$.
(d) For any correspondence $\beta \in c_S(Z, Y)$, $\langle \langle \rho, \alpha \rangle, \beta \rangle = \langle \rho, \alpha \circ \beta \rangle$.

On the other hand, the data of products of the form (9.3.7.1) for any $\mathscr{P}$-schemes X and Y over S which satisfy the conditions (a) and (b) above is equivalent to the data of a natural transformation

$$\phi : G \to G_\tau$$

by putting $\langle \rho, \alpha \rangle_\phi = [\phi_X(\rho)]_Y.\alpha$.

Applying this to the presheaf with transfers P^{tr}, we obtain a canonical natural transformation

$$\psi : P \to P_\tau.$$

By assumption on t, F_τ is a t-sheaf. Thus there existe a unique natural transformation ψ such that the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\psi} & P_\tau \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a_\tau} \\ F & \xrightarrow{\phi} & F_\tau \end{array}$$

Thus we get products of the form 9.3.7.1 associated with ϕ which satisfies (a) and (b). The commutativity of the above diagram asserts they are compatible with the ones corresponding to P^{tr} and the unicity of the natural transformation ϕ implies the uniqueness statement of the lemma.

To finish the proof of the existence, we must show (c) and (d) for the product $\langle ., . \rangle_\phi$. Consider a couple $(\rho, \alpha) \in F(X) \times c_S(Y, X)$. Because F is the t-sheaf associated with P, there exists a

t-cover $p : W \to X$ and a section $\hat{\rho} \in P(W)$ such that $p^*(\rho) = a_W(\hat{\rho})$. Moreover, according to remark 9.3.6, we get a t-cover $q : W' \to Y$ and a correspondence $\hat{\alpha} \in c_S(W', W)$ making the diagram (9.3.6.1) commutative. Then we get using (a) and (b):

$$q^*\langle \rho, \alpha \rangle_\phi = \langle \rho, \alpha \circ q \rangle_\phi = \langle \rho, p \circ \hat{\alpha} \rangle_\phi = \langle p^*\rho, \hat{\alpha} \rangle_\phi = \langle a_V(\hat{\rho}), \hat{\alpha} \rangle_\phi = \langle \hat{\rho}, \hat{\alpha} \rangle_\psi.$$

Because $q^* : F(X) \to F(W)$ is injective, we deduce easily from this principle the properties (c) and (d) and this concludes. □

9.3.8. Let us consider the canonical adjunction

$$a_t^* : \mathrm{PSh}(\mathscr{P}/S, \Lambda) \rightleftarrows \mathrm{Sh}_t(\mathscr{P}/S, \Lambda) : a_{t,*}$$

where $a_{t,*}$ is the canonical forgetful functor.

We also denote by abuse $a_{t,*} : \mathrm{PSh}\left(\mathscr{P}^{cor}_{\Lambda,S}\right) \to \mathrm{Sh}_t\left(\mathscr{P}^{cor}_{\Lambda,S}\right)$ the obvious forgetful functor. Trivially, the following relation holds:

$$\hat{\gamma}_*\, a_{t,*} = a_{t,*}\, \gamma_*. \tag{9.3.8.1}$$

Proposition 9.3.9. *Using the notations above, the following condition on the admissible topology t are equivalent:*

(i) t is mildly compatible with transfers.

(ii) For any scheme S, the functor $\mathcal{O}_t^{tr}$ admits a left adjoint $a_t^* : \mathrm{PSh}\left(\mathscr{P}^{cor}_{\Lambda,S}\right) \to \mathrm{Sh}_t\left(\mathscr{P}^{cor}_{\Lambda,S}\right)$ which is exact and such that the exchange transformation

$$a_t^*\, \hat{\gamma}_* \to \gamma_*\, a_t^* \tag{9.3.9.1}$$

induced by the identification (9.3.8.1) *is an isomorphism.*

Under these conditions, the following properties hold for any scheme S:

(iii) The category $\mathrm{Sh}_t\left(\mathscr{P}^{cor}_{\Lambda,S}\right)$ is a Grothendieck abelian category.

(iv) The functor γ_* commutes with every limits and colimits.

Proof. The fact (i) implies (ii) follows from the preceding lemma as we can put $a_t^{tr}(F) = F^{tr}$ with the notation of the lemma. The fact this defines a functor, as well as the properties stated in (ii), follows from the uniqueness statement of *loc. cit.*

Let us assume (ii). Then (iii) follows formally because from (ii), from the existence, adjunction property and exactness of a_t^*, because $\mathrm{PSh}\left(\mathscr{P}^{cor}_{\Lambda,S}\right)$ is a Grothendieck abelian category. Moreover, we deduce from the isomorphism (9.3.9.1) that γ_* is exact: indeed, a_t^* and $\hat{\gamma}_*$ are exact. As γ_* commutes with arbitrary direct sums, we get (iv).

From this point, we deduce the existence of a right adjoint $\gamma^!$ to the functor γ_*. Using again the isomorphism (9.3.9.1), we obtain for any t-sheaves F on $\mathscr{P}/S$ and any $\mathscr{P}$-scheme X/S a canonical isomorphism $F_\tau(X) = \gamma^! F(X)$. This proves (i). □

9.3.10. Under the assumption of the previous proposition, given any $\mathscr{P}$-scheme X/S, we will put $\Lambda_S^{tr}(X)_t = a_t^{tr}\Lambda_S^{tr}(X)$. The above proposition shows that the family $\Lambda_S^{tr}(X)_t$ for $\mathscr{P}$-schemes X/S is a generating family in $\mathrm{Sh}_t\left(\mathscr{P}^{cor}_{\Lambda,S}\right)$. Moreover, we get easily the following corollary of the preceding proposition and proposition 9.1.2:

Corollary 9.3.11. *Assume that t is mildly compatible with transfers.*

Then there exists an essentially unique Grothendieck abelian $\mathscr{P}$-premotivic category $\mathrm{Sh}_t(\mathscr{P}^{cor}_\Lambda)$ which is geometrically generated, whose fiber over a scheme S is $\mathrm{Sh}_t\left(\mathscr{P}^{cor}_{\Lambda,S}\right)$ and such that the t-sheafification functor induces an adjunction of abelian $\mathscr{P}$-premotivic categories:

$$a_t^* : \mathrm{PSh}\left(\mathscr{P}^{cor}_\Lambda\right) \rightleftarrows \mathrm{Sh}_t(\mathscr{P}^{cor}_\Lambda) : a_{t,*}.$$

Moreover, the functor γ_* induces an adjunction of abelian $\mathscr{P}$-premotivic categories:

$$\gamma^* : \mathrm{Sh}_t(\mathscr{P}, \Lambda) \rightleftarrows \mathrm{Sh}_t(\mathscr{P}^{cor}_\Lambda) : \gamma_*. \tag{9.3.11.1}$$

Remark 9.3.12. Notice moreover that $\gamma^*\, a_t^* = a_t^*\, \hat{\gamma}^*$.

Proof. In fact, using the exactness of a_t^*, given any sheaf F with transfers F over$S$, we get a canonical isomorphism

$$F = \varinjlim_{\Lambda_S^{tr}(X)_t \to F} \Lambda_S^{tr}(X)_t$$

where the limit is taken in $\mathrm{Sh}_t\big(\mathscr{P}_{\Lambda,S}^{cor}\big)$ and runs over the representable t-sheaves with transfers over F. As in the proof of 9.1.2, this allows to define uniquely the structural left adjoints of $\mathrm{Sh}_t(\mathscr{P}_{\Lambda}^{cor})$. The existence (and uniqueness) of the structural right adjoints then follows formally. The same remark allows to construct the functor γ^*. □

9.3.13. Recall from definition 5.1.9 we say that the abelian $\mathscr{P}$-premotivic category $\mathrm{Sh}_t(\mathscr{P}_{\Lambda}^{cor})$ satisfies cohomological t-descent if for any scheme S, and any t-hypercover $\mathcal{X} \to X$ in $\mathscr{P}/S$, the induced morphism of complexes in $\mathrm{Sh}_t\big(\mathscr{P}_{\Lambda,S}^{cor}\big)$

$$\Lambda_S^{tr}(\mathcal{X})_t \to \Lambda_S^{tr}(X)_t$$

is a quasi-isomorphism. The preceding corollary thus gives the following one:

Corollary 9.3.14. *Assume t is mildly compatible with transfers.*
Then the following conditions are equivalent:

(i) The topology t is compatible with transfers.
(ii) The abelian $\mathscr{P}$-premotivic category $\mathrm{Sh}_t(\mathscr{P}_{\Lambda}^{cor})$ satisfies cohomological t-descent.
(iii) The abelian $\mathscr{P}$-premotivic category $\mathrm{Sh}_t(\mathscr{P}_{\Lambda}^{cor})$ is compatible with t (see 5.1.9).

Proof. The equivalence of (i) and (ii) follows easily from the isomorphism (9.3.9.1). The equivalence of (ii) and (iii) is proposition 5.1.25 applied to the adjunction (9.3.11.1), in view of 9.3.9(iv). □

9.3.15. Recall (2.1.14) that a cd-structure P on $\mathscr{S}$ is the data of a family of commutative squares, called P-distinguished, of the form

$$\begin{array}{ccc} B & \xrightarrow{k} & Y \\ {\scriptstyle g}\downarrow & Q & \downarrow{\scriptstyle f} \\ A & \xrightarrow[i]{} & X \end{array} \tag{9.3.15.1}$$

which is stable by isomorphisms. Further, we will consider the following assumptions on P:

(a) P is complete, regular and bounded in the sense of [Voe00b].
(b) Any P-distinguished square as above is made of $\mathscr{P}$-morphisms and k is an immersion.
(c) Any square as above which is cartesian and such that $X = A \sqcup Y$, i and f being the obvious immersions, is P-distinguished.

Then the topology t_P associated with P (see 2.1.14) is $\mathscr{P}$-admissible and satisfy assumption 9.0(c). Moreover, according to [Voe00b, 2.9], we obtain the following properties:

(d) A presheaf F on $\mathscr{P}/S$ is a t_P-sheaf if and only if $F(\varnothing) = 0$ and for any P-distinguished square (9.3.15.1) in $\mathscr{P}/S$, the sequence

$$0 \to F(X) \xrightarrow{f^*+e^*} F(Y) \oplus F(A) \xrightarrow{k^*-g^*} F(B)$$

is exact.

(e) For any P-distinguished square (9.3.15.1) the sequence of representable pre-sheaves on $\mathscr{P}/S$

$$0 \to \Lambda_S(B) \xrightarrow{k_*-g_*} \Lambda_S(Y) \oplus \Lambda_S(A) \xrightarrow{f_*+e_*} \Lambda_S(X) \to 0$$

becomes exact on the associated t_P-sheaves.

Corollary 9.3.16. *Consider a cd-structure P satisfying properties (a) and (b) above and assume that $t = t_P$ is the topology associated with P. Then the following conditions are equivalent:*

(i) The topology t is compatible with transfers.
(ii) The topology t is mildly compatible with transfers.

(iii) For any scheme S and any P-distinguished square (9.3.15.1) *in $\mathscr{P}/S$, the short sequence of presheaves with transfers over S*

$$0 \to \Lambda_S^{tr}(B) \xrightarrow{k_*-g_*} \Lambda_S^{tr}(Y) \oplus \Lambda_S^{tr}(A) \xrightarrow{f_*+e_*} \Lambda_S^{tr}(X) \to 0$$

becomes exact on the associated t-sheaves on $\mathscr{P}/S$.

Proof. The implication (i) $\Rightarrow$ (ii) is obvious.

The implication (ii) $\Rightarrow$ (iii) follows from point (e) above and the following facts: γ^* is right exact (corollary 9.3.11), $\gamma^* a_t = a_t^{tr}\hat{\gamma}^*$ (remark 9.3.12), $k_* : \Lambda_S^{tr}(B) \to \Lambda_S^{tr}(Y)$ is a monomorphism of presheaves with transfers (use 8.1.7(2) and the fact k is an immersion from assumption (b)).

Assume (iii). Then we obtain (ii) as a direct consequence of the point (d) above. Thus, to prove (i), we have only to prove that the abelian $\mathscr{P}$-premotivic category $\mathrm{Sh}_t(\mathscr{P}_\Lambda^{cor})$ satisfies cohomological t-descent according to 9.3.14.

Let S be a scheme. Recall that the category $\mathrm{D}(\mathrm{Sh}_t(\mathscr{P}/S, \Lambda)$ has a canonical DG-structure (see for example 5.0.17). The cohomological t-descent for $\mathrm{Sh}_t\big(\mathscr{P}_{\Lambda,S}^{cor}\big)$ can be reformulated by saying that for any complex K of t-sheaves on $\mathscr{P}/S$, and any t-hypercover $\mathcal{X} \to X$, the canonical map of $\mathrm{D}(\Lambda\text{-}\,\mathrm{mod})$

$$\mathbf{R}Hom_{\mathrm{D}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda))}(\gamma_*\Lambda_S^{tr}(X)_t, K) \to \mathbf{R}Hom_{\mathrm{D}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda))}(\gamma_*\Lambda_S^{tr}(\mathcal{X})_t, K)$$

is an isomorphism. Recall also there is the injective model structure on $\mathrm{C}(\mathrm{Sh}_t(\mathscr{P}/S, \Lambda))$ for which every object is cofibrant and with quasi-isomorphisms as weak equivalences (recall are given in [CD09, 2.1]). Replacing K by a fibrant resolution for the injective model structure, we get for any simplicial objects $\mathcal{X}$ of $\mathscr{P}/S^{\mathrm{II}}$ that:

$$\mathbf{R}Hom_{\mathrm{D}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda))}(\gamma_*\Lambda_S^{tr}(\mathcal{X})_t, K) = Hom_{\mathrm{D}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda))}(\gamma_*\Lambda_S^{tr}(\mathcal{X})_t, K).$$

Thus it is sufficient to prove that the presheaf

$$E : \mathscr{P}/S^{op} \to \mathrm{C}(\Lambda\text{-}\,\mathrm{mod}), X \mapsto Hom_{\mathrm{D}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda))}(\gamma_*\Lambda_S^{tr}(X)_t, K)$$

satisfies t-descent in the sense of 3.2.5.

We derive from (iii) that E sends a P-distinguished square to a homotopy cartesian square in $\mathrm{D}(\Lambda\text{-}\,\mathrm{mod})$. Thus the assertion follows from the arguments on t-descent from [Voe00a, Voe00b]. □

9.4. **Examples.**

9.4.1. Assume that t is the Nisnevich topology. According to lemma 9.3.3 and corollary 9.3.16, t is then compatible with transfers. With the notation of corollary 9.3.11, we get the following definition:

Definition 9.4.2. We denote by

$$\mathrm{Sh}^{tr}(-, \Lambda), \qquad \underline{\mathrm{Sh}}^{tr}(-, \Lambda)$$

the respective abelian premotivic and generalized abelian premotivic categories defined in corollary 9.3.11 in the respective cases $\mathscr{P} = \mathscr{S}m$, $\mathscr{P} = \mathcal{S}^{ft}$.

From now on, the objects of $\mathrm{Sh}^{tr}(S, \Lambda)$ (resp. $\underline{\mathrm{Sh}}^{tr}(S, \Lambda)$) are called the sheaves with transfers over S (resp. generalized sheaves with transfers over S). Let X be a separated S-scheme finite type. We let $\underline{\Lambda}_S^{tr}(X)$ be the generalized sheaf with transfers represented by X (*cf.* 9.2.4). If X is S-smooth, we denote by $\Lambda_S^{tr}(X)$ the t-sheaf with transfers represented by X. An important property of sheaves with transfers is that the abelian premotivic category $\mathrm{Sh}^{tr}(-, \Lambda)$ (resp. $\underline{\mathrm{Sh}}^{tr}(-, \Lambda)$) is compatible with the Nisnevich topology on $\mathscr{S}m$ (resp. $\mathcal{S}^{ft}$) according to proposition 9.3.16. Note moreover that it is compactly geometrically generated.

Remark 9.4.3. Consider an extension of rings $\mathbf{Z} \subset \Lambda \subset \Lambda' \subset \mathbf{Q}$. Then, the canonical functor

$$\mathrm{PSh}(\mathscr{P}_{\Lambda,S}^{cor}, \Lambda) \to \mathrm{PSh}(\mathscr{P}_{\Lambda',S}^{cor}, \Lambda'), F \mapsto F \otimes_\Lambda \Lambda'$$

induces a canonical functor

$$\mathrm{Sh}^{tr}(S, \Lambda) \to \mathrm{Sh}^{tr}(S, \Lambda').$$

This is evidently compatible with f^*, $p_\sharp$, and tensor product. Thus we have defined a morphism of abelian premotivic categories:

$$\mathrm{Sh}^{tr}(-,\Lambda) \rightleftarrows \mathrm{Sh}^{tr}(-,\Lambda'). \tag{9.4.3.1}$$

9.4.4. We also have obtain an adjunction (resp. generalized adjunction) of premotivic abelian categories

$$\gamma^* : \mathrm{Sh}(\mathscr{S}m,\Lambda) \rightleftarrows \mathrm{Sh}^{tr}(-,\Lambda) : \gamma_*$$
$$\gamma^* : \mathrm{Sh}\big(\mathcal{S}^{ft},\Lambda\big) \rightleftarrows \underline{\mathrm{Sh}}^{tr}(-,\Lambda) : \gamma_*.$$

Note that in each case γ_* is conservative and exact according to 9.3.9(iv).

9.4.5. Let S be a scheme. Consider the inclusion functor $\varphi : \mathscr{S}m^{cor}_{\Lambda,S} \to \mathscr{S}^{ft,cor}_{\Lambda,S}$. It induces a functor

$$\varphi^* : \underline{\mathrm{Sh}}^{tr}(S,\Lambda) \to \mathrm{Sh}^{tr}(S,\Lambda), F \mapsto F\circ\varphi$$

which is obviously exact and commute to arbitrary direct sums. In particular, it commutes with arbitrary colimits.

Lemma 9.4.6. *With the notations above, the functor φ^* admits a left adjoint $\varphi_!$ such that for any smooth S-scheme X, $\varphi_!(\Lambda^{tr}_S(X)) = \underline{\Lambda}^{tr}_S(X)$. The functor $\varphi_!$ is fully faithful.*

In other words, we have defined an enlargement of premotivic abelian categories (*cf.* definition 1.4.8)

$$\varphi_! : \mathrm{Sh}^{tr}(-,\Lambda) \to \underline{\mathrm{Sh}}^{tr}(-,\Lambda) : \varphi^*. \tag{9.4.6.1}$$

Proof. Let F be a sheaf with transfers. Let $\{X/F\}$ be the category of representable sheaf $\Lambda^{tr}_S(X)$ over F for a smooth S-scheme X. We put

$$\varphi_!(F) = \varinjlim_{\{X/F\}} \underline{\Lambda}^{tr}_S(X).$$

The adjunction property of $\varphi_!$ is immediate from the Yoneda lemma. We prove that for any sheaf with transfers F, the unit adjunction morphism $F \to \varphi^*\varphi_!(F)$ is an isomorphism. As already remarked, φ^* commutes with colimits so that we are restricted to the case where $F = \Lambda^{tr}_S(X)$ which follows by definition. □

9.4.7. Assume now that t = cdh is the cdh-topology, and $\mathscr{P} = \mathcal{S}^{ft}$ is the class of separated morphisms of finite type. In this case, we get the following result according to [SV00b, 4.3.3] combined with [SV00b, 4.2.9]:

Proposition 9.4.8. *The admissible topology* cdh *on $\mathcal{S}^{ft}$ if mildly compatible with transfers.*

As a corollary, we a generalized premotivic abelian category whose fiber over a scheme S is the category $\underline{\mathrm{Sh}}^{tr}_{\mathrm{cdh}}(S,\Lambda)$ of cdh-sheaves with transfers on $\mathcal{S}^{ft}$. Moreover, the restriction of a_{cdh} to $\underline{\mathrm{Sh}}^{tr}(S,\Lambda)$ induces a morphism of generalized premotivic categories; we get the following commutative diagram of such morphisms:

$$\begin{array}{ccc} \underline{\mathrm{Sh}}(-,\Lambda) & \xrightarrow{a^*_{\mathrm{cdh}}} & \underline{\mathrm{Sh}}_{\mathrm{cdh}}(-,\Lambda) \\ \downarrow{\scriptstyle \gamma^*} & & \downarrow{\scriptstyle \gamma^*_{\mathrm{cdh}}} \\ \underline{\mathrm{Sh}}^{tr}(-,\Lambda) & \xrightarrow{a^*_{\mathrm{cdh}}} & \underline{\mathrm{Sh}}^{tr}_{\mathrm{cdh}}(-,\Lambda) \end{array}$$

9.5. **Comparison results.**

9.5.a. *Representable* qfh*-sheaves.*

9.5.1. Let us denote by $\mathrm{Sh}_{\mathrm{qfh}}(S,\Lambda)$ the category of qfh-sheaves of Λ-modules over $\mathscr{S}_S^{ft}$. Remark that for an S-scheme X, the Λ-presheaf represented by X is not a sheaf for the qfh-topology. We denote the associated sheaf by $\Lambda_S^{\mathrm{qfh}}(X)$. We let a_{qfh} be the associated qfh-sheaf functor. Recall that for any S-scheme X, the graph functor (9.4.4) induces a morphism of sheaves

$$\underline{\Lambda}_S(X) \xrightarrow{\gamma_{X/S}} \underline{\Lambda}_S^{tr}(X).$$

We recall the following theorem of Suslin and Voevodsky (see [SV00b, 4.2.12]):

Theorem 9.5.2. *Let S be a scheme such that* $\mathrm{char}(S) \subset \Lambda^\times$*. Then, for any S-scheme X, the application of a_{qfh} to $\gamma_{X/S}$ gives an isomorphism in* $\mathrm{Sh}_{\mathrm{qfh}}(S,\Lambda)$*:*

$$\Lambda_S^{\mathrm{qfh}}(X) \xrightarrow{\gamma_{X/S}^{\mathrm{qfh}}} \underline{\Lambda}_S^{tr}(X).$$

9.5.3. Using the previous theorem, we associate to any qfh-sheaf $F \in \mathrm{Sh}_{\mathrm{qfh}}(S,\Lambda)$ a presheaf with transfers

$$\rho(F) : X \mapsto \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{qfh}}(S,\Lambda)}(\underline{\Lambda}_S^{tr}(X), F).$$

We obviously get $\gamma^*\rho(F) = F$ as a presheaf over $\mathscr{S}_S^{ft}$ so that $\rho(F)$ is a sheaf with transfers and we have defined a functor

$$\rho : \mathrm{Sh}_{\mathrm{qfh}}(S,\Lambda) \to \underline{\mathrm{Sh}}^{tr}(S,\Lambda).$$

For any S-scheme X, $\rho(\Lambda_S^{\mathrm{qfh}}(X)) = \underline{\Lambda}_S^{tr}(X)$ according to the previous proposition.

Corollary 9.5.4. *Assume* $\mathrm{char}(S) \subset \Lambda^\times$*. Let $f : X' \to X$ be an morphism of S-schemes. If f is a universal homeomorphism, $f_* : \underline{\Lambda}_S^{tr}(X') \to \underline{\Lambda}_S^{tr}(X)$ is an isomorphism in $\underline{\mathrm{Sh}}^{tr}(S,\Lambda)$.*

Proof. Indeed, according to [Voe96, 3.2.5], $\Lambda_S^{\mathrm{qfh}}(X') \to \Lambda_S^{\mathrm{qfh}}(X)$ is an isomorphism in $\mathrm{Sh}_{\mathrm{qfh}}(S,\Lambda)$ and we conclude by applying the functor ρ. □

9.5.b. qfh*-sheaves and transfers.*

Proposition 9.5.5. *Assume* $\mathrm{char}(S) \subset \Lambda^\times$*. Any* qfh*-sheaf of Λ-modules over the category of S-schemes of finite type is naturally endowed with a unique structure of a sheaf with transfers, and any morphism of such* qfh*-sheaves is a morphism of sheaves with transfers.*

In particular, the qfh*-sheafification functor defines a left exact functor left adjoint to the forgetful functor $\rho : \mathrm{Sh}_{\mathrm{qfh}}(S,\Lambda) \to \underline{\mathrm{Sh}}^{tr}(S,\Lambda)$ introduced in 9.5.3.*

Proof. It follows from theorem 9.5.2 that the category of Λ-linear finite correspondences is canonically equivalent to the full subcategory of the category of qfh-sheaves of Λ-modules spanned by the objects of shape $\Lambda_S^{\mathrm{qfh}}(X)$ for X separated of finite type over S. The first assertion is thus an immediate consequence of theorem 9.5.2 and of the (additive) Yoneda lemma. The fact that the qfh-sheafification functor defines a left adjoint to the restriction functor ρ is then obvious, while its left exactness is a consequence of the facts that it is left exact (at the level of sheaves without transfers) and that forgetting transfers defines a conservative and exact functor from the category of Nisnevich sheaves with transfers to the category of Nisnevich sheaves. □

Recall the following theorem:

Theorem 9.5.6. *Let F be an étale rational sheaf on $\mathscr{S}_S^{ft}$. Then for any S-scheme X, and any integer i, the canonical map*

$$H^i_{\mathrm{Nis}}(X,F) \to H^i_{\text{ét}}(X,F)$$

is an isomorphism.

Proof. Using the compatibility of étale cohomology with projective limits of schemes, we are reduced to prove that $H^i_{\text{ét}}(X,F) = 0$ whenever X is henselian local and $i > 0$. Let k be the residue field of X, G its absolute Galois group and F_0 the restriction of F to $\mathrm{Spec}\,(k)$. Then F_0 is a G-module and according to [AGV73, 8.6], $H^i_{\text{ét}}(X,F) = H^i(G,F_0)$. As G is profinite, this group must be torsion so that it vanishes by assumption. □

Remark 9.5.7. The preceding theorem also follows formally from theorem 3.3.22.

Proposition 9.5.8. *Let F be a rational* qfh*-sheaf on $\mathscr{S}_S^{ft}$. Then for any geometrically unibranch S-scheme of finite type X, and any integer i, the canonical map*

$$H^i_{\mathrm{Nis}}(X,F) \to H^i_{\mathrm{qfh}}(X,F)$$

is an isomorphism.

Proof. According to 9.5.6, $H^i_{\mathrm{Nis}}(X,F) = H^i_{\text{ét}}(X,F)$. Let $p : X' \to X$ be the normalization of X. As X is an excellent geometrically unibranch scheme, p is a finite universal homeomorphism. It follows from [AGV73, VII, 1.1] that $H^i_{\text{ét}}(X,F) = H^i_{\text{ét}}(X',F)$ and from [Voe96, 3.2.5] that $H^i_{\mathrm{qfh}}(X,F) = H^i_{\mathrm{qfh}}(X',F)$. Thus we can assume that X is normal, and the result is now exactly [Voe96, 3.4.1]. □

Corollary 9.5.9. *Let S be an excellent scheme.*

(1) Let X be a geometrically unibranch S-scheme of finite type. For any point x of X, the local henselian scheme X^h_x is a point for the category of sheaves $\mathrm{Sh}_{\mathrm{qfh}}(S,\mathbf{Q})$ *(*i.e. *evaluating at X^h_x defines an exact functor).*

(2) The family of points X^h_x of the previous type is a conservative family for $\mathrm{Sh}_{\mathrm{qfh}}(S,\mathbf{Q})$.

Proof. The first point follows from the previous proposition. For any excellent scheme X, the normalization morphism $X' \to X$ is a qfh-covering. Thus the category $\mathrm{Sh}_{\mathrm{qfh}}(S,\mathbf{Q})$ is equivalent to the category of qfh-sheaves on the site made of geometrically unibranch S-schemes of finite type. This implies the second assertion. □

9.5.10. Let S be a geometrically unibranch scheme. Considering the notations of 9.5.3 and 9.4.5, we introduce the composite functor:

$$\psi^* : \mathrm{Sh}_{\mathrm{qfh}}(S,\mathbf{Q}) \xrightarrow{\rho} \underline{\mathrm{Sh}}^{tr}(S,\mathbf{Q}) \xrightarrow{\varphi^*} \mathrm{Sh}^{tr}(S,\mathbf{Q}).$$

Proposition 9.5.11. *Considering the above notation, the following conditions are true :*

(i) For any S-scheme X of finite type, $\psi^*\big(\mathbf{Q}^{\mathrm{qfh}}_S(X)\big) = \mathbf{Q}^{tr}_S(X)$.

(ii) The functor ψ^* admits a left adjoint $\psi_!$.

(iii) For any smooth S-scheme X, $\psi_!\big(\mathbf{Q}^{tr}_S(X)\big) = \mathbf{Q}^{\mathrm{qfh}}_S(X)$.

(iv) The functor ψ^* is exact and preserves colimits.

(v) The functor $\psi_!$ is fully faithful.

According to property (iii), the functor $\psi_!$ commutes with pullbacks. In other words, we have defined an enlargement of abelian premotivic categories (*cf.* definition 1.4.8)

$$\psi_! : \mathrm{Sh}^{tr}(-,\mathbf{Q}) \rightleftarrows \mathrm{Sh}_{\mathrm{qfh}}(-,\mathbf{Q}) : \psi^* \tag{9.5.11.1}$$

Proof. Point (i) follows from the fact $\mathbf{Q}^{tr}_S(X) = \mathbf{Q}^{\mathrm{qfh}}_S(X)$. Recall the enlargement of (9.4.6.1):

$$\varphi_! : \mathrm{Sh}^{tr}(-,\mathbf{Q}) \to \underline{\mathrm{Sh}}^{tr}(-,\mathbf{Q}) : \varphi^*.$$

We define the functor $\psi_!$ as the composite :

$$\mathrm{Sh}^{tr}(S,\mathbf{Q}) \xrightarrow{\varphi_!} \underline{\mathrm{Sh}}^{tr}(S,\mathbf{Q}) \xrightarrow{\gamma^*} \underline{\mathrm{Sh}}(S,\mathbf{Q}) \xrightarrow{a_{\mathrm{qfh}}} \mathrm{Sh}_{\mathrm{qfh}}(S,\mathbf{Q})\,.$$

According to the properties of the functors in this composite, $\psi_!$ is exact and preserves colimits. Moreover, for any smooth S-scheme X, as $\mathbf{Q}^{tr}_S(X)$ is a qfh-sheaf over $\mathscr{S}_S^{ft}$ according to 9.2.4, $\psi_!(\mathbf{Q}^{tr}_S(X)) = \mathbf{Q}^{\mathrm{qfh}}_S(X)$ which proves (iii). Property (ii) follows from (iii) and the fact $\psi_!$ commutes with colimits, while the sheaves $\mathbf{Q}^{tr}_S(X)$ for X/S smooth generate $\mathrm{Sh}^{tr}(S,\mathbf{Q})$.

For any smooth S-scheme X, $\Gamma(X;\psi^*(F)) = F(X)$. Thus the exactness of ψ^* follows from corollary 9.5.9. As ψ^* obviously preserves direct sums, we get (iv).

To check that for any sheaf with transfers F the unit map $F \to \psi^*\psi_!(F)$ is an isomorphism, we thus are reduced to the case where $F = \mathbf{Q}^{tr}_S(X)$ for a smooth S-scheme X which follows from (i) and (iii). □

10. Nisnevich motivic complexes

10.0. In this section, $\mathscr{S}$ is an adequate category of Σ-schemes (*cf.* 2.0). Any scheme in $\mathscr{S}$ is assumed to be of finite dimension.

We fix a ring of coefficients $\Lambda \subset \mathbf{Q}$.

10.1. **Definition.** We can apply the general definition of section 5 to the abelian premotivic category $\mathrm{Sh}^{tr}(-,\Lambda)$ constructed in 9.4.2:

Definition 10.1.1. We define the (Λ-linear) category of (Nisnevich) motivic complexes (resp. stable motivic complexes) following definition 5.3.22 (resp. definition 5.2.16) as

$$\mathrm{DM}_\Lambda = \mathrm{D}_{\mathbf{A}^1}\left(\mathrm{Sh}^{tr}(-,\Lambda)\right)$$
$$\text{resp. } \mathrm{DM}_\Lambda^{\mathit{eff}} = \mathrm{D}_{\mathbf{A}^1}^{\mathit{eff}}\left(\mathrm{Sh}^{tr}(-,\Lambda)\right).$$

Recall we have an adjunction of triangulated premotivic categories

$$\Sigma^\infty : \mathrm{DM}_\Lambda^{\mathit{eff}} \rightleftarrows \mathrm{DM}_\Lambda : \Omega^\infty.$$

Usually, we denote by $\Lambda_S^{tr}(X)$ the effective motive of a smooth S-scheme X, as an object in $\mathrm{DM}_\Lambda^{\mathit{eff}}(S)$ and we put $M_S(X) = \Sigma^\infty \Lambda_S^{tr}(X)$ which is the motive of X. Similarly, we also denote by $\mathbb{1}_S$ the unit of the monoidal category $\mathrm{DM}_\Lambda(S)$.

Remark 10.1.2. Consider an extension of rings $\mathbf{Z} \subset \Lambda \subset \Lambda' \subset \mathbf{Q}$. Then we deduce from the adjunction (9.4.3.1) of abelian premotivic categories an adjunction:

$$\mathrm{DM}_\Lambda \rightleftarrows \mathrm{DM}_{\Lambda'}. \tag{10.1.2.1}$$

10.1.3. Let S be a scheme. Consider the triangulated subcategory $\mathscr{T}_S$ of $\mathrm{K}^b(\mathscr{S}m_{\Lambda,S}^{cor})$ generated by complexes of one the following forms :

(1) for any distinguished square $\begin{array}{ccc} W & \xrightarrow{k} & V \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & X \end{array}$ of smooth S-schemes,

$$[W] \xrightarrow{g_* - k_*} [U] \oplus [V] \xrightarrow{j^* + f^*} [X]$$

(2) for any smooth S-scheme X, $p : \mathbf{A}^1_X \to X$ the canonical projection.

$$[\mathbf{A}^1_X] \xrightarrow{p_*} [X].$$

Definition 10.1.4. We define the category $\mathrm{DM}_{c,\Lambda}^{\mathit{eff}}(S)$ of constructible effective motives over S as the pseudo-abelian envelope of the triangulated category

$$\mathrm{K}^b(\mathscr{S}m_{\Lambda,S}^{cor})/\mathscr{T}_S.$$

We define the category $\mathrm{DM}_{c,\Lambda}(S)$ of constructible motives over S as the triangulated category obtained from $\mathrm{DM}_{c,\Lambda}^{\mathit{eff}}(S)$ by formally inverting the Tate complex

$$[\mathbf{P}^1_S] \to [S].$$

Remark 10.1.5. If $S = \mathrm{Spec}\,(k)$ is the spectrum of a field, then, essentially by definition, the categories $\mathrm{DM}_c(S)$ and $\mathrm{DM}_c^{\mathit{eff}}(S)$ are respectively the categories $\mathrm{DM}_{gm}(k)$ and $\mathrm{DM}_{gm}^{\mathit{eff}}(k)$ introduced by Voevodsky in [VSF00].

We can use proposition 5.2.36 and corollary 5.3.31 to obtain the following commutative diagram

$$\begin{array}{ccc} \mathrm{DM}_{c,\Lambda}^{\mathit{eff}}(S) & \longrightarrow & \mathrm{DM}_\Lambda^{\mathit{eff}}(S) \\ \downarrow & & \downarrow{\scriptstyle \Sigma^\infty} \\ \mathrm{DM}_{c,\Lambda}(S) & \longrightarrow & \mathrm{DM}_\Lambda(S) \end{array}$$

where the horizontal maps are fully faithful, essentially surjective onto compact objects of the aim. Moreover, the triangulated categories $\mathrm{DM}_\Lambda^{\mathit{eff}}(S)$ and $\mathrm{DM}_\Lambda(S)$ are compactly generated.

Remark 10.1.6. Recall that in fact, $\mathrm{DM}_{c,\Lambda}$ (resp. $\mathrm{DM}_{c,\Lambda}^{\mathit{eff}}$) are Sm-fibred monoidal categories over $\mathscr{S}$, and that the horizontal maps in the preceding commutative diagram induce morphisms of Sm-fibred monoidal categories.

10.1.7. To get a generalized category, we simply apply the definitions of section 5 to the generalized abelian premotivic category $\underline{\mathrm{Sh}}^{tr}(-,\Lambda)$ constructed in 9.4.2:

Definition 10.1.8. We define the (Λ-linear) category of generalized motivic complexes (resp. stable motivic complexes) following definition 5.3.22 (resp. definition 5.2.16) as

$$\underline{\mathrm{DM}}_{\Lambda}^{\mathit{eff}} = \mathrm{D}_{\mathbf{A}^1}^{\mathit{eff}}\left(\underline{\mathrm{Sh}}^{tr}(-,\Lambda)\right)$$
$$\text{resp. } \underline{\mathrm{DM}}_{\Lambda} = \mathrm{D}_{\mathbf{A}^1}\left(\underline{\mathrm{Sh}}^{tr}(-,\Lambda)\right).$$

Given any separated S-scheme X of finite type, we will simply denote by $\underline{M}_S(X)$ the generalized premotive represented by X in $\underline{\mathrm{DM}}_{\Lambda}(S)$.

According to 6.1.9, we obtain an enlargement of premotivic triangulated categories:

$$\varphi_! : \mathrm{DM}_{\Lambda} \rightleftarrows \underline{\mathrm{DM}}_{\Lambda} : \varphi^*. \tag{10.1.8.1}$$

Let S be a scheme, and X a smooth separated S-scheme of finite type. Recall that according to *loc. cit.* the induced functor $\varphi_! : \mathrm{DM}_{\Lambda}(S) \to \underline{\mathrm{DM}}_{\Lambda}(S)$ is fully faithful. Moreover, $\varphi_!(M_S(X)) = \underline{M}_S(X)$. Recall also that $\varphi^*(\underline{M}_S(X)) = M_S(X)$.

Remark 10.1.9. The functor φ^* is far from being conservative. The following example was suggested by V.Vologodsky: let Z be a nowhere dense closed subscheme of S. Then $\rho^*(\underline{M}_S(Z)) = 0$.

In fact, one can see that $\mathrm{DM}_{\Lambda}(S)$ is a localization of the category $\underline{\mathrm{DM}}_{\Lambda}(S)$ with respect to the objects $\underline{\mathscr{M}}$ such that $\rho^*(\underline{\mathscr{M}}) = 0$.

Note also that this context also holds in the effective setting. More precisely, we have the following essentially commutative diagram

$$\begin{array}{ccc} \mathrm{DM}_{\Lambda}^{\mathit{eff}} & \xrightarrow{\Sigma^{\infty}} & \mathrm{DM}_{\Lambda} \\ \downarrow & & \downarrow{\scriptstyle \varphi_!} \\ \underline{\mathrm{DM}}_{\Lambda}^{\mathit{eff}} & \xrightarrow{\Sigma^{\infty}} & \underline{\mathrm{DM}}_{\Lambda} \end{array} \tag{10.1.9.1}$$

where the vertical maps are enlargement of triangulated premotivic categories induced by the enlargement (9.4.6.1), and the horizontal maps are the natural morphisms of premotivic (resp. generalized premotivic) triangulated categories.

10.2. **Nisnevich motivic cohomology.**

10.2.a. *Definition and functoriality.*

Definition 10.2.1. Let S be a scheme and $(n,m) \in \mathbf{Z}^2$ a couple of integer. We define the Nisnevich motivic cohomology of S in degree $n \in \mathbf{Z}$ and twist $m \in \mathbf{Z}$ with coefficients in Λ as the Λ-module

$$H^{n,m}_{\mathcal{M},\mathrm{Nis}}(S,\Lambda) = \mathrm{Hom}_{\mathrm{DM}_{\Lambda}(S)}\big(\mathbb{1}_S, \mathbb{1}_S(m)[n]\big).$$

Assuming $m \geq 0$, we define the *unstable* motivic cohomology of S in degree $n \in \mathbf{Z}$ and twist $m \in \mathbf{Z}$ with coefficients in Λ as the Λ-module

$$H^{n,m}_{\mathcal{M},\mathrm{Nis},eff}(S,\Lambda) = \mathrm{Hom}_{\mathrm{DM}_{\Lambda}^{\mathit{eff}}(S)}\big(\Lambda_S^{tr}, \Lambda_S^{tr}(m)[n]\big).$$

Motivic cohomology (resp. unstable motivic cohomology) is obviously contravariant with respect to morphisms of schemes. The monoidal structure allows to define easily a cup-product structure for this cohomology. According to proposition 5.3.30, we get

$$H^{n,m}_{\mathcal{M},\mathrm{Nis}}(S,\Lambda) = \varinjlim_{r>>0} \mathrm{Hom}_{\mathrm{DM}_{\Lambda}^{\mathit{eff}}(S)}\big(\Lambda_S^{tr}(r), \Lambda_S^{tr}(m+r)[n]\big).$$

Remark 10.2.2. Consider an extension of rings $\mathbf{Z} \subset \Lambda \subset \Lambda' \subset \mathbf{Q}$. Then using the left adjoint of the premotivic adjunction (10.1.2.1), we get a canonical morphism

$$H^{n,m}_{\mathcal{M},\mathrm{Nis}}(S,\Lambda) \to H^{n,m}_{\mathcal{M},\mathrm{Nis}}(S,\Lambda').$$

It is obviously compatible with pullbacks and the product structure.

10.2.3. Consider a separated morphism $p : X \to S$ of finite type. Recall from the $\mathcal{S}^{ft}$-fibred structure of $\underline{\mathrm{DM}}_\Lambda$ that $\underline{M}_S(X) = \mathbf{L}p_\sharp p^*(\mathbb{1}_S)$. Using the adjunction property of the pair $(\mathbf{L}p_\sharp, p^*)$, we easily get:

$$\begin{aligned} H^{n,m}_{\mathcal{M},\mathrm{Nis}}(X,\Lambda) &= \mathrm{Hom}_{\mathrm{DM}_\Lambda(X)}\big(\mathbb{1}_X, \mathbb{1}_X(m)[n]\big) = \mathrm{Hom}_{\underline{\mathrm{DM}}_\Lambda(X)}\big(\mathbb{1}_X, \mathbb{1}_X(m)[n]\big) \\ &= \mathrm{Hom}_{\mathrm{DM}_\Lambda(S)}\big(\underline{M}_S(X), \mathbb{1}_S(m)[n]\big). \end{aligned} \tag{10.2.3.1}$$

In particular, given any finite S-correspondence $\alpha : X \bullet\!\!\to Y$ between separated S-schemes of finite type, we obtain a pullback

$$\alpha^* : H^{n,m}_{\mathcal{M},\mathrm{Nis}}(Y,\Lambda) \to H^{n,m}_{\mathcal{M},\mathrm{Nis}}(X,\Lambda)$$

which is, among other properties, functorial with respect to composition of finite S-correspondences and extends the natural contravariant functoriality of motivic cohomology.

In particular, given any finite Λ-universal morphism $f : Y \to X$, we obtain a pushout

$$f_* : H^{n,m}_{\mathcal{M},\mathrm{Nis}}(Y,\Lambda) \to H^{n,m}_{\mathcal{M},\mathrm{Nis}}(X,\Lambda)$$

by considering the transpose of the graph of f.

Proposition 10.2.4. *Let $f : Y \to X$ be a finite Λ-universal morphism of scheme. Let $d > 0$ be its degree (cf. 8.1.10). Then for any element $x \in H^{n,m}_{\mathcal{M},\mathrm{Nis}}(X,\Lambda)$, $f_* f^*(x) = d.x$.*

This is a simple application of proposition 8.1.11. We left to the reader the exercice to state projection and base change formulas for this pushout.

Remark 10.2.5. Recall that, in the case X is regular, any finite morphism $f : Y \to X$ such that any irreducible component of Y dominates an irreducible component of Y is Λ-universal.

Another important application of the generalized Voevodsky premotivic category is obtained using the corollary 9.5.4:

Proposition 10.2.6. *Let $f : X' \to X$ be a separated universal homeomorphism of finite type. Assume that* $\mathrm{char}(X) \subset \Lambda^\times$. *Then the pullback functor*

$$H^{n,m}_{\mathcal{M},\mathrm{Nis}}(X,\Lambda) \to H^{n,m}_{\mathcal{M},\mathrm{Nis}}(X',\Lambda)$$

is an isomorphism.

Remark 10.2.7. The preceding considerations hold similarly for the unstable motivic cohomology.

10.2.b. *Motivic cohomology in weight* 0 *and* 1. Let S be a scheme and X a smooth S-scheme. For any subscheme Y of X, we denote by $\mathbf{Z}^{tr}_S(X/Y)$ the cokernel of the canonical morphism of sheaf with transfers $\mathbf{Z}^{tr}_S(Y) \to \mathbf{Z}^{tr}_S(X)$. As this morphism is a monomorphism, we obtain a canonical distinguished triangle in $\mathrm{DM}^{\mathit{eff}}_{\mathbf{Z}}(S)$

$$\mathbf{Z}^{tr}_S(Y) \to \mathbf{Z}^{tr}_S(X) \to \mathbf{Z}^{tr}_S(X/Y) \to \mathbf{Z}^{tr}_S(X)[1].$$

10.2.8. We will consider the Tate complex of sheaves with transfers over a scheme S to be defined by the formula

$$\mathbf{Z}^{tr}_S(1)[1] = \mathbf{Z}^{tr}_S(\mathbf{G}_m/\{1\}).$$

In the category of complexes, we thus obtain the canonical decomposition

$$\mathbf{Z}^{tr}_S(\mathbf{G}_m) = \mathbf{Z}^{tr}_S \oplus \mathbf{Z}^{tr}_S(1)[1].$$

For any complex C of sheaves with transfers, we will consider the complex $\underline{C}^*(C)$ introduced in 5.2.30 in the abstract case.

Proposition 10.2.9. *Suppose S is a normal scheme.*

Then there is a canonical isomorphism in $\mathrm{DM}_{\mathbf{Z}}^{\mathit{eff}}(S)$

$$\mathbf{G}_m \xrightarrow{\simeq} \mathbf{Z}_S^{tr}(1)[1].$$

Remark 10.2.10. Using lemma 5.2.33, this implies that for any smooth scheme X over S, we have isomorphisms

$$\mathrm{H}^i_{\mathrm{Nis}}(X, \mathbf{G}_m) \simeq \mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(S)}(\mathbf{Z}_S^{tr}(X), \mathbf{Z}_S^{tr}[i-1]) \simeq \mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(S)}(\mathbf{Z}_S^{tr}(X), \mathbf{Z}_S^{tr}(1)[1])$$

Proof. Let U be an open subscheme of $\mathbf{A}^1_S$ and X be a smooth S-scheme. Note that X is normal according to [GD67, 18.10.7]. Consider a cycle α of $X \times_S U$ which is finite equidimensional over X. Then it is a divisor in $X \times_S U$ and according to [GD67, 21.14.3], it is flat over X as X is normal. In particular, it is universal. As a consequence, we obtain the equality

$$H^i\Gamma(X; \underline{C}^*\mathbf{Z}_S^{tr}(U)) = H^{sing}_{-i}(X \times_S U/X)$$

where the left hand side is the singular cohomology groups of Suslin (*cf.* [SV00b]).

Suppose X and U are affine and let $Z = \mathbf{P}^1_S - U$. According to a theorem of Suslin and Voevodsky (*cf.* [SV00b]),

$$H^{sing}_{-i}(X \times_S U/X) = \begin{cases} \mathrm{Pic}(X \times_S \mathbf{P}^1_S, X \times_S Z) & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

This implies that $\underline{C}^*(\mathbf{Z}_S^{tr}[U])$ is $\mathbf{A}^1$-local, so that using lemma 5.2.33, we get

$$H^{sing}_{-i}(X \times_S U/X) \simeq \mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(S)}(\mathbf{Z}_S^{tr}(X), \underline{C}^*(\mathbf{Z}_S^{tr}(U))) \simeq \mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(S)}(\mathbf{Z}_S^{tr}(X), \mathbf{Z}_S^{tr}(U))$$

By definition, we have a distinguished triangle in $\mathrm{DM}_{\mathbf{Z}}^{\mathit{eff}}(S)$

$$\mathbf{Z}_S^{tr}(\mathbf{G}_m) \to \mathbf{Z}_S^{tr}(\mathbf{A}^1_S) \to \mathbf{Z}_S^{tr}(1)[2] \to \mathbf{Z}_S^{tr}(\mathbf{G}_m)[1].$$

Thus when X is affine, we obtain

$$H^{i+1}\Gamma(X; \underline{C}^*\mathbf{Z}_S^{tr}(1)) = \begin{cases} \mathbf{G}_m(X) & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

using that the evident map $\mathrm{Pic}(X \times_S \mathbf{P}^1_S, X_0 \sqcup X_\infty) \to \mathrm{Pic}(X \times_S \mathbf{P}^1_S, X_0)$ is surjective and has for kernel the group $\mathbf{G}_m(X_0)$. □

Proposition 10.2.11. *Let S be a scheme and $n \in \mathbf{Z}$ an integer.*

Then

(1)

$$\mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(S)}(\mathbf{Z}_S^{tr}, \mathbf{Z}_S^{tr}[n]) = \begin{cases} \mathbf{Z}^{\pi_0(S)} & \textit{if } n = 0 \\ 0 & \textit{otherwise} \end{cases}$$

(2) If S is normal,

$$\mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(S)}(\mathbf{Z}_S^{tr}, \mathbf{Z}_S^{tr}(1)[n]) = \begin{cases} \mathcal{O}_S(S)^\times & \textit{if } n = 1 \\ \mathrm{Pic}(S) & \textit{if } n = 2 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. The second case is a direct corollary of the previous proposition and the remark which follows it. For the first case, according to proposition 9.2.5, the sheaf $\mathbf{Z}_S^{tr}$ is Nisnevich local and $\mathbf{A}^1$-local as a complex of sheaves. It is obviously acyclic for the Nisnevich topology. Thus, we conclude using again 9.2.5 in the case $n = 0$. □

The following is a (very) weak cancellation result in $\mathrm{DM}^{\mathit{eff}}(S)$ over a normal base :

Proposition 10.2.12. *Let S be a normal scheme. Then*

$$\mathbf{R}\mathit{Hom}(\mathbf{Z}_S^{tr}(1), \mathbf{Z}_S^{tr}(1)) = \mathbf{Z}_S^{tr}.$$

Moreover, if $m = 0$ or $m = 1$, for any integer $n > m$,

$$\mathbf{R}\mathit{Hom}(\mathbf{Z}_S^{tr}(n), \mathbf{Z}_S^{tr}(m)) = 0.$$

Proof. We consider the first assertion. Any smooth S-scheme is normal. Hence it is sufficient to prove that for any connected normal scheme S, for any integer $n \in \mathbf{Z}$,

$$\mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(S)}(\mathbf{Z}^{tr}_S(1), \mathbf{Z}^{tr}_S(1)[n]) = \begin{cases} \mathbf{Z} & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Using the exact triangle

$$\mathbf{Z}^{tr}_S(\mathbf{G}_m) \to \mathbf{Z}^{tr}_S(\mathbf{A}^1) \to \mathbf{Z}^{tr}_S(1)[2] \xrightarrow{+1} \tag{10.2.12.1}$$

and the second case of the previous lemma, we obtain the following long exact sequence

$$\begin{aligned} \cdots \to \mathrm{Hom}(\mathbf{Z}^{tr}_S(\mathbf{A}^1), \mathbf{Z}^{tr}_S(1)[n]) \to \mathrm{Hom}(\mathbf{Z}^{tr}_S(\mathbf{G}_m), \mathbf{Z}^{tr}_S(1)[n]) \\ \to \mathrm{Hom}(\mathbf{Z}^{tr}_S(1), \mathbf{Z}^{tr}_S(1)[n-1]) \to \mathrm{Hom}(\mathbf{Z}^{tr}_S(\mathbf{A}^1), \mathbf{Z}^{tr}_S(1)[n+1]) \to \cdots \end{aligned}$$

Then we conclude using the previous lemma and the fact

$$\mathrm{Pic}(\mathbf{A}^1 \times S) = \mathrm{Pic}(\mathbf{G}_m \times S)$$

whenever S is normal.

For the last assertion, we are reduced to prove that if S is a normal scheme, for any integers $n > 0$ and i,

$$\mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(S)}(\mathbf{Z}^{tr}_S(n), \mathbf{Z}^{tr}_S[i]) = 0.$$

This is obviously implied by the case $n = 1$.

Consider the distinguished triangle (10.2.12.1) again. Then the long exact sequence attached to the cohomological functor $\mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}_{\mathbf{Z}}(S)}(-, \mathbf{Z}^{tr}_S)$ and applied to this triangle together with the first case of the previous lemma allows us to conclude. □

10.3. **Orientation and purity.** For any scheme S, we let $\mathbf{P}^\infty_S$ be the ind-scheme

$$S \to \mathbf{P}^1_S \to \cdots \to \mathbf{P}^n_S \to \mathbf{P}^{n+1}_S \to$$

made of the obvious closed immersions. This ind-scheme has a comultiplication given by the Segre embeddings

$$\mathbf{P}^\infty_S \times_S \mathbf{P}^\infty_S \to \mathbf{P}^\infty_S$$

Define $\mathbf{Z}^{tr}_S(\mathbf{P}^\infty) = \varinjlim \mathbf{Z}^{tr}_S(\mathbf{P}^n)$. Applying lemma 10.2.11 in the case $S = \mathbf{Z}$, we obtain

$$\mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(\mathbf{Z})}(\mathbf{Z}^{tr}(\mathbf{P}^\infty), \mathbf{Z}^{tr}(1)[2]) = \mathrm{Pic}(\mathbf{P}^\infty),$$

which is a free ring of power series in one variable. Considering the canonical dual invertible sheaf on $\mathbf{P}^\infty$, we obtain a canonical generator of the previous group which is a morphism of ind-objects of $\mathrm{DM}^{\mathit{eff}}(\mathbf{Z})$ denoted by

$$\mathbf{Z}^{tr}(\mathbf{P}^\infty) \to \mathbf{Z}^{tr}(1)[2].$$

For any scheme S, considering the canonical projection $f : S \to \mathrm{Spec}\,(\mathbf{Z})$, we obtain by pullback along f a morphism of $\mathrm{DM}^{\mathit{eff}}(S)$

$$\mathbf{Z}^{tr}_S(\mathbf{P}^\infty_S) \to \mathbf{Z}^{tr}_S(1)[2].$$

Applying the left adjoint of (10.1.2.1) in the case of the extension $\mathbf{Z} \subset \Lambda$, we deduce a canonical morphism

$$\mathfrak{c}_{1,S} : \Lambda^{tr}_S(\mathbf{P}^\infty_S) \to \Lambda^{tr}_S(1)[2].$$

Consider $\mathbf{G}_m$ as a sheaf of groups over Sm_S. Following [MV99, part 4], we introduce its classifying space $B\mathbf{G}_m$ as a simplicial sheaf over Sm_S. From proposition 1.16 of *loc. cit.*, we get $\mathrm{Hom}_{\mathscr{H}^s_\bullet(S)}(S_+, B\mathbf{G}_m) = \mathrm{Pic}(S)$. Moreover, in $\mathscr{H}_\bullet(S)$, we obtain a canonical isomorphism $B\mathbf{G}_m = \mathbf{P}^\infty_S$ of pointed simplicial sheaf (*cf. loc. cit.*, prop. 3.7). Thus finally, we obtain a canonical map of pointed sets

$$\begin{aligned} \mathrm{Pic}(S) = \mathrm{Hom}_{\mathscr{H}^s_\bullet(S)}(S_+, B\mathbf{G}_m) \to \mathrm{Hom}_{\mathscr{H}_\bullet(S)}(S_+, \mathbf{P}^\infty) \\ \to \mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}_\Lambda(S)}(\Lambda^{tr}_S, \Lambda^{tr}_S(\mathbf{P}^\infty/*)) \to \mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}_\Lambda(S)}(\Lambda^{tr}_S, \Lambda^{tr}_S(\mathbf{P}^\infty)). \end{aligned}$$

Definition 10.3.1. Consider the above notations. We define the first motivic Chern class as the following composite morphism

$$\begin{aligned}\mathrm{Pic}(S) &\longrightarrow \mathrm{Hom}_{\mathrm{DM}_\Lambda^{eff}(S)}(\Lambda_S^{tr}, \Lambda_S^{tr}(\mathbf{P}_S^\infty)) \xrightarrow{(\mathfrak{c}_{1,S})_*} \mathrm{Hom}_{\mathrm{DM}_\Lambda^{eff}(S)}(\Lambda_S^{tr}, \Lambda_S^{tr}(1)[2]) \\ &\longrightarrow \mathrm{Hom}_{\mathrm{DM}_\Lambda(S)}(\mathbb{1}_S, \mathbb{1}_S(1)[2]) = H^{2,1}_{\mathcal{M},\mathrm{Nis}}(S,\Lambda)\end{aligned}$$

10.3.2. The first Chern class is evidently compatible with pullback. By definition, the morphism

$$\mathfrak{c}_1 : \mathrm{Pic}(S) \to \mathrm{Hom}_{\mathrm{DM}_\Lambda^{eff}(S)}(\Lambda_S^{tr}, \Lambda_S^{tr}(1)[2])$$

is the morphism of lemma 10.2.11 when S is normal. In particular, it is additive in this case. One deduces (see below) that it is additive for any scheme S.

The triangulated category $\mathrm{DM}_\Lambda(S)$ thus satisfies all the axioms of [Dég08, 2.1] (see also 2.3.1 of *loc. cit.* in the regular case). In particular, we derive from the main results of *loc. cit.* the following facts:

(1) Let $p : P \to S$ be a projective bundle of rank n. Let $c : \mathbb{1}_S \to \mathbb{1}_S(1)[2]$ be the first Chern class of the canonical line bundle on P. Then the map

$$M_S(P) \xrightarrow{\sum_i p\otimes c^i} \oplus_{i=0}^n \mathbb{1}_S(i)[2i]$$

is an isomorphism. This gives the projective bundle theorem in motivic cohomology for any base scheme.

(2) Let $i : Z \to X$ be a closed immersion between smooth separated S-schemes of finite type. Assume i has pure codimension c and let j be the complementary open immersion. Then there is a canonical *purity isomorphism*:

$$\mathfrak{p}_{X,Z} : M_S(X/X-Z) \to M_S(Z)(c)[2c].$$

This defines in particular the *Gysin triangle*

$$M_S(X-Z) \xrightarrow{j_*} M_S(X) \xrightarrow{i^*} M_S(Z)(c)[2c] \xrightarrow{\partial_{X,Z}} M_S(X-Z)[1].$$

(3) Let $f : Y \to X$ be a projective morphism between smooth separated S-schemes of finite type. Assume f has pure relative dimension n. Then there is a Gysin morphism

$$f^* : M_S(X) \to M_S(Y)(n)[2n]$$

functorial in f. We refer the reader to *loc. cit* for various formulas involving the Gysin morphism (projection formula, excess intersection,...)

(4) For any smooth projective S-scheme X, the premotive $M_S(X)$ admits a strong dual. If X has pure relative dimension d over S, the strong dual of $M_S(X)$ is $M_S(X)(-d)[-2d]$.

Remark 10.3.3. The last property implies in particular that for any smooth projective S-schemes X and Y, if Y has pure relative dimension d over S,

$$\mathrm{Hom}_{\mathrm{DM}_\Lambda(S)}(M_S(X), M_S(Y)) = H^{2d,d}_{\mathcal{M},\mathrm{Nis}}(X \times_S Y, \Lambda).$$

From this, one can deduce that the analog of the category of pure motives relative to the base S build with the cohomology $H^{2d,d}_{\mathcal{M},\mathrm{Nis}}(-,\Lambda)$ instead of the Chow groups, admits a full embedding into the category $\mathrm{DM}_{c,\Lambda}(S)$, induced by the functor $X/S \mapsto M_S(X)$.

10.4. **Functoriality.**

10.4.1. Note that according to definition 9.4.2 and paragraph 9.4.4, we have an adjunction of abelian premotivic categories

$$\gamma^* : \mathrm{Sh}(-,\Lambda) \rightleftarrows \mathrm{Sh}^{tr}(-,\Lambda) : \gamma_*\,,$$

which induces, by 5.3.28, and adjunction of triangulated premotivic categories

$$\mathbf{L}\gamma^* : \mathrm{D}_{\mathbf{A}^1,\Lambda} \rightleftarrows \mathrm{DM}_\Lambda : \mathbf{R}\gamma_*\,.$$

The abelian premotivic category $\mathrm{Sh}^{tr}(-,\Lambda)$ satisfies the assumptions of 6.0. In particular, we deduce from corollaries 6.3.12 and 6.3.15 the following theorem:

Theorem 10.4.2. *The premotivic triangulated category* DM_Λ *satisfies the support property.*

Moreover, for any scheme S *and any closed immersion* $i : Z \to X$ *between smooth* S*-schemes,* DM_Λ *satisfies the localization with respect to* i*, (Loc*$_i$*).*

Thus we can apply corollary 2.2.12 to DM_Λ ; recall in particular that for any separated morphism $f : Y \to X$ of finite type, we get an adjunction

$$f_! : \mathrm{DM}_\Lambda(Y) \rightleftarrows \mathrm{DM}_\Lambda(X) : f^!$$

such that $f_! = f_*$ when f is proper and $f^! = f^*$ when f is an open immersion. For further properties of these adjoint functors, we refer the reader to *loc. cit.*

10.4.3. The qfh-sheafification functor (9.5.5) induces by 5.3.28 a premotivic adjunction

$$\underline{\alpha}^* : \underline{\mathrm{DM}}_{\mathbf{Q}} \rightleftarrows \underline{\mathrm{DM}}_{\mathrm{qfh},\mathbf{Q}} : \underline{\alpha}_* .$$

Theorem 10.4.4. *If* S *is a geometrically unibranch excellent noetherian scheme of finite dimension then the* qfh*-sheafification functor induces a fully faithful functor*

$$\mathrm{DM}_{\mathbf{Q}}(S) \to \underline{\mathrm{DM}}_{\mathrm{qfh},\mathbf{Q}}(S) .$$

Proof. Note that $\mathrm{DM}_{\mathbf{Q}}^{\mathit{eff}}(S)$ and $\mathrm{D}_{\mathbf{A}^1}^{\mathit{eff}}(\mathrm{Sh}_{\mathrm{qfh}}(S,\mathbf{Q}))$ are compactly generated; see example 5.1.28 and proposition 5.2.36. Hence this corollary follows from propositions 9.5.11 and 6.1.8. □

Part 4. Beilinson motives and algebraic K-theory

11.0. In all this part, $\mathscr{S}$ is assumed to be the category of noetherian schemes of finite dimension, except in the sections 14 and 16.

11. Stable homotopy theory of schemes

11.1. **Ring spectra.** Consider a base scheme S.

Recall that a ring spectrum E over S is a monoid object in the monoidal category $\mathrm{SH}(S)$. We say that E is commutative if it is commutative as a monoid in the symmetric monoidal category $\mathrm{SH}(S)$. In what follows, we will assume that all our ring spectra are commutative without mentioning it. The premotivic category is $\mathbf{Z}^2$-graded where the first index refers to the simplicial sphere and the second one to the Tate twist. According to our general convention, a cohomology theory represented in SH is $\mathbf{Z}^2$-graded accordingly: given such a ring spectrum E, for any smooth S-scheme X, and any integer $(i,n) \in \mathbf{Z}^2$, we get a bigraded ring:

$$E^{n,i}(X) = \mathrm{Hom}_{\mathrm{SH}(S)}\big(\Sigma^\infty X_+, E(i)[n]\big).$$

When X is a pointed smooth S-scheme, it defines a pointed sheaf of sets still denoted by X and we denote by $\tilde{E}^{n,i}(X)$ for the corresponding cohomology ring.

The *coefficient ring* associated with E is the cohomology of the base $E^{**} := E^{**}(S)$. The ring $E^{**}(X)$ (resp. $\tilde{E}^{**}(X)$) is in fact an E^{**}-algebra.

11.1.1. We say E is a *strict ring spectrum* if there exists a monoid in the category of symmetric Tate spectra E' and an isomorphism of ring spectra $E \simeq E'$ in $\mathrm{SH}(S)$. In this case, a module M over the monoid E in the monoidal category $\mathrm{SH}(S)$ will be said to be *strict* if there exists an E'-module M' in the category of symmetric Tate spectra, as well as an isomorphism of E-modules $M \simeq M'$ in $\mathrm{SH}(S)$.

11.2. **Orientation.**

11.2.1. Consider the infinite tower

$$\mathbf{P}^1_S \to \mathbf{P}^2_S \to \cdots \to \mathbf{P}^n_S \to \cdots$$

of schemes pointed by the infinity. We denote by $\mathbf{P}^\infty_S$ the limit of this tower as a pointed Nisnevich sheaf of sets and by $\iota : \mathbf{P}^1_S \to \mathbf{P}^\infty_S$ the induced inclusion. Recall the following definition, classically translated from topology:

Definition 11.2.2. Let E be a ring spectrum over S. An *orientation* of E is a cohomology class c in $\tilde{E}^{2,1}(\mathbf{P}^\infty_S)$ such that $\iota^*(c)$ is sent to the unit of the coefficient ring of E by the canonical isomorphism $\tilde{E}^{2,1}(\mathbf{P}^1_S) = E^{0,0}$.

We then say that (E, c) is an *oriented ring spectrum*. We shall say also that E is *orientable* if there exists an orientation c.

According to [MV99, 1.16 and 3.7], we get a canonical map for any smooth S-scheme X

$$\mathrm{Pic}(X) = H^1(X, \mathbf{G}_m) \to \mathrm{Hom}_{\mathscr{H}_\bullet(S)}(X_+, \mathbf{P}^\infty) \to \mathrm{Hom}_{\mathrm{SH}(S)}(\Sigma^\infty X_+, \Sigma^\infty \mathbf{P}^\infty)$$

(the first map is an isomorphism whenever S is regular (or even geometrically unibranch)). Given this map, an orientation c of a ring spectrum E defines a map of sets

$$c_{1,X} : \mathrm{Pic}(X) \to E^{2,1}(X)$$

which is natural in X (and from its construction in [MV99], one can check that $c = c_{1,\mathbf{P}^\infty_S}(\mathcal{O}(1))$). Usually, we put $c_1 = c_{1,X}$.

Example 11.2.3. The original example of an oriented ring spectrum is the algebraic cobordism spectrum *MGL* introduced by Voevodsky (*cf.* [Voe98]).

Remark 11.2.4. When E is a strict ring spectrum, the category E-mod satisfies the axioms of [Dég08, 2.1] (see example 2.12 of *loc.cit.*).

Recall the following result, which first appeared in [Vez01]:

Proposition 11.2.5 (Morel)**.** *Let (E, c) be an oriented ring spectrum.*

Then:

$$E^{**}(\mathbf{P}^\infty_S) = E^{**}[[c]]$$
$$E^{**}(\mathbf{P}^\infty_S \times \mathbf{P}^\infty_S) = E^{**}[[x, y]]$$

where x (resp. y) is the pullback of c along the first (resp. second) projection.

Remark 11.2.6. When E is a strict ring spectrum, this is [Dég08, 3.2] according to remark 11.2.4. The proof follows an argument of Morel ([Dég08, lemma 3.3]) and the arguments of *op.cit.*, p. 634, can be easily used to obtain the proposition arguing directly for the cohomology functor $X \mapsto E^{*,*}(X)$.

11.2.7. Recall the Segre embeddings

$$\mathbf{P}^n_S \times \mathbf{P}^m_S \to \mathbf{P}^{n+m+nm}_S$$

define a map

$$\sigma : \mathbf{P}^\infty_S \times \mathbf{P}^\infty_S \to \mathbf{P}^\infty_S.$$

It gives the structure of an H-group to $\mathbf{P}^\infty_S$ in the homotopy category $\mathscr{H}(S)$. Consider the hypothesis of the previous proposition. Then the pullack along σ in E-cohomology induces a map

$$E^{**}[[c]] \xrightarrow{\sigma^*} E^{**}[[x, y]]$$

and following Quillen, we check that the formal power series $\sigma^*(c)$ defines a formal group law over the ring E^{**}.

Definition 11.2.8. Let (E, c) be an oriented ring spectrum and consider the previous notation.

The formal group law $F_E(x, y) := \sigma^*(c)$ will be called the formal group law associated to (E, c).

Recall the formal group law has the form:

$$F_E(x, y) = x + y + \sum_{i+j>0} a_{ij}.x^i y^j$$

with $a_{ij} = a_{ji}$ in $E^{-2i-2j,-i-j}$.

The coefficients a_{ij} describe the failure of additivity of the first Chern class c_1. Indeed, assuming the previous definition, we get the following result:

Proposition 11.2.9. *Let X be a smooth S-scheme.*

(1) For any line bundle L/X, the class $c_1(L)$ is nilpotent in $E^{**}(X)$.
(2) Suppose X admits an ample line bundle. For any line bundles L, L' over X,

$$c_1(L_1 \otimes L_2) = F_E(c_1(L_1), c_1(L_2)) \in E^{2,1}(X).$$

We refer to [Dég08, 3.8] in the case where E is strict; the proof is the same in the general case. Recall the following theorem of Vezzosi (*cf.* [Vez01, 4.3]):

Theorem 11.2.10 (Vezzosi)**.** *Let (E, c) be an oriented spectra over S, with formal group law F_E. Then there exists a bijection between the following sets:*

(i) Orientation classes c' of E.
(ii) Morphisms of ring spectra $MGL \to E$ in $\mathrm{SH}(S)$.
(iii) Couples (F, φ) where F is a formal group law over E^{**} and φ is a power series over E^{**} which defines an isomorphism of formal group law: φ is invertible as a power series and $F_E(\varphi(x), \varphi(y)) = F(x, y)$.

11.3. **Rational category.** In what follows, we shall use frequently the equivalence of premotivic categories (see 5.3.37)

$$\mathrm{SH}_{\mathbf{Q}} \rightleftarrows \mathrm{D}_{\mathbf{A}^1,\Lambda}\,,$$

and shall identify freely any rational spectrum over a scheme S with an object of $\mathrm{D}_{\mathbf{A}^1}(S, \Lambda)$.

12. Algebraic K-theory

12.1. **The K-theory spectrum.** We consider the spectrum KGL_S which represents homotopy invariant K-theory in $\mathrm{SH}(S)$ according to Voevodsky (see [Voe98, 6.2], [Rio09, 5.2] and [PPR07]). It is characterized by the following properties:

(K1) For any morphism of schemes $f : T \to S$, there is an isomorphism $f^*KGL_S \simeq KGL_T$ in $\mathrm{SH}(T)$.
(K2) For any regular scheme S and any integer n, there exist an isomorphism

$$\mathrm{Hom}_{\mathrm{SH}(S)}(\mathbb{1}_S[n], KGL_S) \to K_n(S)$$

(where the right hand side is Quillen's algebraic K-theory) such that, for any morphism $f : T \to S$ of regular schemes, the following diagram is commutative:

$$\begin{array}{ccccc} \mathrm{Hom}(\mathbb{1}_S[n], KGL_S) & \to & \mathrm{Hom}(f^*\mathbb{1}_S[n], f^*KGL_S) & = & \mathrm{Hom}(\mathbb{1}_T[n], KGL_T) \\ \downarrow & & & & \downarrow \\ K_n(S) & & \xrightarrow{\qquad\qquad f^* \qquad\qquad} & & K_n(T) \end{array}$$

(where the lower horizontal map is the pullback in Quillen algebraic K-theory along the morphism f and the upper horizontal map is obtained by using the functor $f^* : \mathrm{SH}(S) \to \mathrm{SH}(T)$ and the identification (K1)).
(K3) For any scheme S, there exists a unique structure of a commutative monoid on KGL_S which is compatible with base change – using the identification (K1) – and induces the canonical ring structure on $K_0(S)$.

Thus, according to (K1) and (K3), the collection of the ring spectrum KGL_S for any scheme S form an absolute ring spectrum. As usual, when no confusion can arise, we will not indicate the base in the notation KGL.
Note that (K1) implies formally that the isomorphism of (K2) can be extended for any smooth S-scheme X (with S regular), giving a natural isomorphism:

$$\mathrm{Hom}_{\mathrm{SH}(S)}(\Sigma^\infty X_+[n], KGL) \to K_n(X)\,.$$

12.2. **Periodicity.**

12.2.1. Recall from the construction the following property of the spectrum KGL:

(K4) the spectrum KGL is a $\mathbf{P}^1$-periodic spectrum in the sense that there exists a canonical isomorphism

$$KGL \xrightarrow{\sim} \mathbf{R}Hom\left(\Sigma^\infty \mathbf{P}^1_S, KGL\right) = KGL(-1)[-2].$$

As usual, $\mathbf{P}^1_S$ is pointed by the infinite point.

This isomorphism is characterized uniquely by the fact that its adjoint isomorphism (obtained by tensoring with $\mathbb{1}_S(1)[2]$) is equal to the composite

$$\gamma_u : KGL(1)[2] \xrightarrow{1\otimes u} KGL \wedge KGL \xrightarrow{\mu} KGL. \tag{12.2.1.1}$$

where $u : \Sigma^\infty \mathbf{P}^1 \to KGL$ corresponds to the class $[\mathcal{O}(1)] - 1$ in $\tilde{K}_0(\mathbf{P}^1)$ through the isomorphism (K2) and μ is the structural map of monoid from (K3).

Using the isomorphism of (K4), the property (K1) can be extended as follows: for any smooth S-scheme X and any integers $(i, n) \in \mathbf{Z}^2$, there is a canonical isomorphism:

$$KGL^{n,i}(X) \xrightarrow{\sim} K_{2i-n}(X). \tag{12.2.1.2}$$

Remark 12.2.2. The element u is invertible in the ring $KGL^{*,*}(S)$. Its inverse is the *Bott element* $\beta \in KGL^{2,1}(S)$. If the ring spectrum KGL is oriented (*cf.* 11.2.2) by the class

$$\beta.([\mathcal{O}(1)] - 1) \in KGL^{2,1}(\mathbf{P}^\infty),$$

the corresponding formal group law is the multiplicative formal group law:

$$F(x, y) = x + y + \beta^{-1}.xy.$$

12.3. **Modules over algebraic K-theory.**

Theorem 12.3.1 (Röndigs, Spitzweck, Østvær)**.** *The spectrum KGL can be represented canonically by a cartesian monoid KGL', as well as by a homotopy cartesian commutative monoid KGL^β in the fibred model category of symmetric $\mathbf{P}^1$-spectra, in such way that there exists a morphism of monoids $KGL' \to KGL^\beta$ which is a termwise stable $\mathbf{A}^1$-equivalence.*

Proof. For any noetherian scheme of finite dimension S, one has a strict commutative ring spectrum KGL^β_S which is canonically isomorphic to KGL_S in $\mathrm{SH}(S)$ as ring spectra; see [RSØ09]. One can check that the objects KGL^β_S do form a commutative monoid over the diagram of all noetherian schemes of finite dimension (i.e. a commutative monoid in the category of sections of the fibred category of $\mathbf{P}^1$-spectra over the category of noetherian schemes of finite dimension), either by hand, by following the explicit construction of *loc. cit.*, either by modifying its construction very slightly as follows: one can perform *mutatis mutandis* the construction of *loc. cit.* in the $\mathbf{P}^1$-stabilization of the $\mathbf{A}^1$-localization of the model category of Nisnevich simplicial sheaves over (any essentially small adequate subcategory of) the category all noetherian schemes of finite dimension, and get an object KGL^β, whose restriction to each of the categories Sm/S is the object KGL^β_S. From this point of view, we clearly have canonical maps $f^*(KGL^\beta_S) \to KGL^\beta_T$ for any morphism of schemes $f : T \to S$. The object KGL^β is homotopy cartesian, as the composed map

$$\mathbf{L}f^*(KGL_S) \simeq \mathbf{L}f^*(KGL^\beta_S) \to f^*(KGL^\beta_S) \to KGL^\beta_T \simeq KGL_T$$

is an isomorphism in $\mathrm{SH}(T)$. Consider now a cofibrant resolution

$$KGL'_{\mathrm{Spec}(\mathbf{Z})} \to KGL^\beta_{\mathrm{Spec}(\mathbf{Z})}$$

in the model category of monoids of the category of symmetric $\mathbf{P}^1$-spectra over $\mathrm{Spec}\,(\mathbf{Z})$; see theorem 4.1.3. Then, we define, for each noetherian scheme of finite dimension S, the $\mathbf{P}^1$-spectrum KGL'_S as the pullback of $KGL'_{\mathrm{Spec}(\mathbf{Z})}$ along the map $f : S \to \mathrm{Spec}\,(\mathbf{Z})$. As the functor f^* is a left Quillen functor, the object KGL'_S is cofibrant (both as a monoid and as a $\mathbf{P}^1$-spectrum), so that we get, by construction, a termwise cofibrant cartesian strict $\mathbf{P}^1$-ring spectrum KGL', as well as a morphism $KGL' \to KGL^\beta$ which is a termwise stable $\mathbf{A}^1$-equivalence. □

12.3.2. For each noetherian scheme of finite dimension S, one can consider the model categories of modules over KGL'_S and KGL^β_S respectively; see 4.2.2. The change of scalars functor along the stable $\mathbf{A}^1$-equivalence $KGL'_S \to KGL^\beta_S$ defines a left Quillen equivalence, whence an equivalence of homotopy categories:

$$\mathrm{Ho}(KGL'_S\text{- mod}) \simeq \mathrm{Ho}(KGL^\beta_S\text{- mod}).$$

We put

$$\mathrm{Ho}(KGL\text{- mod})(S) = \mathrm{Ho}(KGL^\beta_S\text{- mod})\,,$$

and call this category the *homotopy category of KGL-modules over S*. By definition, for any smooth S-scheme X, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{SH}(S)}(\Sigma^\infty(X_+), KGL[n]) \simeq \mathrm{Hom}_{KGL}(KGL_S(X), KGL[n])$$

(where $KGL_S(X) = KGL_S \wedge^{\mathbf{L}}_S \Sigma^\infty(X_+)$, while Hom_{KGL} stands for $\mathrm{Hom}_{\mathrm{Ho}(KGL\text{- mod})(S)}$).

According to (K1) and (K3), for any regular scheme X, we thus get a canonical isomorphism:

$$\epsilon_S : \mathrm{Hom}_{KGL}(KGL_S[n], KGL_S) \xrightarrow{\sim} K_n(S). \tag{12.3.2.1}$$

Using Bott periodicity (K4), and the compatibility with base change, this isomorphism can be extended for any smooth S-scheme X and any pair $(n,m) \in \mathbf{Z}^2$:

$$\epsilon_{X/S} : \mathrm{Hom}_{KGL}(KGL_S(X), KGL_S(m)[n]) \xrightarrow{\sim} K_{2m-n}(X). \tag{12.3.2.2}$$

Corollary 12.3.3. *The categories* $\mathrm{Ho}(KGL\text{- mod})(S)$ *form a motivic category, and the functors*

$$\mathrm{SH}(S) \to \mathrm{Ho}(KGL\text{- mod})(S)\ , \quad M \mapsto KGL \wedge^{\mathbf{L}}_S M$$

define a morphism of motivic categories

$$\mathrm{SH} \to \mathrm{Ho}(KGL\text{- mod})$$

over the category of noetherian schemes of finite dimension.

Proof. This follows from the preceding theorem and from 4.2.11 and 4.2.16. □

12.4. **K-theory with support.**

12.4.1. Consider a closed immersion $i : Z \to S$ with complementary open immersion $j : U \to S$. Assume S is regular.

We use the definition of [Gil81, 2.13] for the K-theory of S with support in Z denoted by $K^Z_*(S)$. In other words, we define $K^Z(S)$ as the homotopy fiber of the restriction map

$$\mathbf{R}\Gamma(S, KGL_S) = K(S) \to K(U) = \mathbf{R}\Gamma(U, KGL_S)\,,$$

and put: $K^Z_n(S) = \pi_n(K^Z(S))$.

Applying the derived global section functor $\mathbf{R}\Gamma(S,-)$ to the homotopy fiber sequence

$$i_!\, i^!KGL_S \to KGL_S \to j_*\, j^*KGL_S\,, \tag{12.4.1.1}$$

we get a homotopy fiber sequence

$$\mathbf{R}\Gamma(S, i_!\, i^!KGL_S) \to \mathbf{R}\Gamma(S, KGL_S) \to \mathbf{R}\Gamma(U, KGL_S) \tag{12.4.1.2}$$

from which we deduce an isomorphism in the stable homotopy category of S^1-spectra:

$$\mathbf{R}\Gamma(Z, i^!KGL_S) = \mathbf{R}\Gamma(S, i_!\, i^!KGL_S) \simeq K^Z(S)\,. \tag{12.4.1.3}$$

We thus get the following property:

(K6) There is a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{SH}(S)}\left(\mathbb{1}_S[n], i_!i^!KGL_S\right) \to K^Z_n(S)$$

which satisfies the following compatiblities:

(K6a) the following diagram is commutative:

$$\begin{array}{ccccc} \mathrm{Hom}(\mathbb{1}[n+1], j_*j^*KGL_S) & \to & \mathrm{Hom}\big(\mathbb{1}[n], i_!i^!KGL_S\big) & \to & \mathrm{Hom}(\mathbb{1}[n], KGL_S) \\ \downarrow & & \downarrow & & \downarrow \\ K_{n+1}(U) & \longrightarrow & K_n^Z(S) & \longrightarrow & K_n(S) \end{array}$$

where the upper horizontal arrows are induced by the localization sequence (12.4.1.1), and the lower one is the canonical sequence of K-theory with support. The extreme left and right vertical maps are the isomorphisms of (K2);

(K6b) for any morphism $f : Y \to S$ of regular schemes, $k : T \to Y$ the pullback of i along f, the following diagram is commutative:

$$\begin{array}{ccccc} \mathrm{Hom}\left(\mathbb{1}[n], i_!i^!KGL_S\right) & \to & \mathrm{Hom}\left(f^*\mathbb{1}[n], f^*i_!i^!KGL_S\right) & \to & \mathrm{Hom}\left(\mathbb{1}[n], k_!k^!KGL_Y\right) \\ \downarrow & & & & \downarrow \\ K_n^Z(S) & & \xrightarrow{\quad f^*\quad} & & K_n^T(Y) \end{array}$$

where the lower horizontal map is given by the functoriality of relative K-theory (induced by the funtoriality of K-theory) and the left one is obtained using the functor f^* of SH, the canonical exchange morphism $f^*i_!i^! \to k_!k^!f^*$ and the identification (K1).

This property can be extended to the motivic category $\mathrm{Ho}(KGL\text{-}\mathrm{mod})$ and we get a canonical isomorphism

$$\epsilon_i : \mathrm{Hom}_{KGL}(KGL_S[n], i_!i^!KGL_S) \xrightarrow{\sim} K_n^Z(S) \tag{12.4.1.4}$$

satisfying the analog of (K6a) and (K6b).

12.5. **Fundamental class.**

12.5.1. Consider a cartesian square of regular schemes

$$\begin{array}{ccc} Z' & \xrightarrow{k} & S' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & S \end{array}$$

with i a closed immersion. We will say that this square is Tor-independant if Z and S' are Tor-independant over S in the sense of [BGI71, III, 1.5]: for any $i > 0$, $\mathrm{Tor}_i^S(\mathcal{O}_Z, \mathcal{O}_{S'}) = 0$.[40]

In this case, when we assume in addition that all the schemes in the previous square are regular and that i is a closed immersion we get from [TT90, 3.18][41] the formula

$$f^*i_* = k_*g^* : K_*(Z) \to K_*(S')$$

in Quillen K-theory. An important point for us is that there exists a *canonical homotopy* between these morphisms at the level of the Waldhausen spectra.[42] According to the localization theorem of Quillen [Qui73, 3.1], we get:

Theorem 12.5.2 (Quillen)**.** *For any closed immersion $i : Z \to S$ between regular schemes, there exists a canonical isomorphism*

$$\mathfrak{q}_i : K_n^Z(S) \to K_n(Z).$$

Moreover, this isomorphism is functorial with respect to the Tor-independant squares as above, with i a closed immersion and all the schemes regular.

---

[40] For example, when i is a regular closed immersion of codimension 1, this happens if and only if the above square is transversal.

[41] When all the schemes in the square admit ample line bundles, we can refer to [Qui73, 2.11].

[42] In the proof of Quillen, one can also trace back a canonical homotopy with the restriction mentioned in the preceding footnote.

Remark 12.5.3. In the condition of this theorem, the following diagram is commutative by construction:

$$\begin{array}{ccc} K_n^Z(S) & \longrightarrow & K_n(S) \\ {\scriptstyle \mathfrak{q}_i}\downarrow & \nearrow_{i_*} & \\ K_n(Z) & & \end{array}$$

where the non labelled map is the canonical one.

Definition 12.5.4. Let $i : Z \to S$ be a closed immersion between regular schemes.

We define the fundamental class associated with i as the morphism of KGL-modules:

$$\eta_i : i_* KGL_Z \to KGL_S$$

defined by the image of the unit element 1 through the following morphism:

$$K_0(Z) \xrightarrow{\mathfrak{q}_i^{-1}} K_0^Z(S) \xrightarrow{\epsilon_i^{-1}} \mathrm{Hom}_{KGL}(KGL_S, i_* i^! KGL_S) = \mathrm{Hom}_{KGL}(i_* KGL_Z, KGL_S).$$

We also denote by $\eta_i' : KGL_Z \to i^! KGL_S$ the morphism obtained by adjunction.

Remark 12.5.5. The fundamental class has the following functoriality properties.

(1) By definition, and applying remark 12.5.3, the composite map

$$KGL_S \to i_* i^*(KGL_S) = i_* KGL_Z \xrightarrow{\eta_i} KGL_S$$

corresponds via the isomorphism ϵ_S to $i_*(1) \in K_0(S)$. According to [BGI71, Exp. VII, 2.7], this class is equal to $\lambda_{-1}(N_i)$ where N_i is the conormal sheaf of the regular immersion i.

(2) In the situation of a Tor-independant square as in 12.5.1, remark that $f^* \eta_i = \eta_k$ through the canonical exchange isomorphism $f^* i_* = k_* g^*$ — apply the functoriality of ϵ_i from (K6b) and the one of $\mathfrak{q}_i$.

(3) Using the identification $i^! i_* = 1$, we get $\eta_i' = i^! \eta_i$. Consider a cartesian square as in 12.5.1 and assume f is smooth. Then the square is Tor-independant and we get $g^* \eta_i' = \eta_k'$ using the exchange isomorphism $g^* i^! = k^! f^*$.

12.6. **Absolute purity for K-theory.**

Proposition 12.6.1. *For any closed immersion $i : Z \to S$ between regular schemes, the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{Hom}_{KGL}(KGL_Z[n], KGL_Z) & \xrightarrow{\eta_i'} & \mathrm{Hom}_{KGL}(KGL_Z[n], i^! KGL_S) \\ {\scriptstyle \epsilon_Z}\downarrow & (*) & \downarrow{\scriptstyle \epsilon_i} \\ K_n(Z) & \xrightarrow{\mathfrak{q}_i^{-1}} & K_n^Z(S) \end{array}$$

Proof. In this proof, we denote by $[-,-]$ the bifunctor $\mathrm{Hom}_{KGL}(-,-)$.

Step 1: We assume that $i : Z \to S$ admits a retraction $p : S \to Z$.

Consider a KGL-linear map $\alpha : KGL_Z[n] \to KGL_Z$. Then, $\eta_i'(\alpha)$ corresponds by adjunction to the composition

$$i_* KGL_Z[n] \xrightarrow{i_*(\alpha)} i_* KGL_Z \xrightarrow{\eta_i} KGL_S.$$

Applying the projection formula for the motivic category $\mathrm{Ho}(KGL\text{-}\mathrm{mod})$, we get:

$$i_*(\alpha) = i_*(1 \otimes i^* p^*(\alpha)) = i_*(1) \otimes p^*(\alpha).$$

Here 1 stands for the identity morphism of the KGL-module KGL_Z. This shows that $\eta_i'(\alpha)$ corresponds by adjunction to the composite map:

$$\eta_i \otimes p^*(\alpha) : i_* KGL_Z[n] = i_* KGL_Z[n] \otimes KGL_S \to KGL_S \otimes KGL_S = KGL_S$$

(the tensor product is the KGL-linear one). By assumption, $i_* : K_*(Z) \to K_*(S)$ admits a retraction which implies the canonical map $\mathcal{O}_i : K_*^Z(S) \to K_*(S)$ admits a retraction (*cf.* remark 12.5.3). To check that the diagram $(*)$ is commutative, we can thus compose with $\mathcal{O}_i$.

Recall the first point of remark 12.5.5: applying property (K6a) and the fact the isomorphism $\epsilon_S : [KGL_S[n], KGL_S] \to K_n(S)$ is compatible with the algebra structures, we are finally reduced to prove that

$$i_*(\alpha) = i_*(1).p^*(\alpha) \in K_n(S).$$

This follows from the projection formula in K-theory (see [Qui73, 2.10] and [TT90, 3.17]).
Step 2: We shall reduce the general case to Step 1. We consider the following deformation to the normal cone diagram: let D be the blow-up of $\mathbf{A}^1_S$ in the closed subscheme $\{0\} \times Z$, P be the projective completion of the normal bundle of Z in S and s be the canonical section of P/Z; we get the following diagram of regular schemes:

$$\begin{array}{ccccc} Z & \xrightarrow{s_1} & \mathbf{A}^1_Z & \xleftarrow{s_0} & Z \\ {\scriptstyle i}\downarrow & & \downarrow & & \downarrow{\scriptstyle s} \\ S & \longrightarrow & D & \longleftarrow & P \end{array} \tag{12.6.1.1}$$

where s_0 (resp. s_1) is the zero (resp. unit) section of $\mathbf{A}^1_Z$ over Z. These squares are cartesian and Tor-independant in the sense of 12.5.1. The maps s_0 and s_1 induce isomorphisms in K-theory because Z is regular. Thus, the second point of remark 12.5.5 allows to reduce to the case of the immersion s which was done in Step 1. □

12.6.2. Consider a cartesian square

$$\begin{array}{ccc} T & \xrightarrow{k} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & S \end{array}$$

such that S and Z are regular, i is a closed immersion and f is smooth. In this case, the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Hom}_{KGL}(KGL_Z(T)[n], KGL_Z) & \xrightarrow{\eta'_i} & \mathrm{Hom}_{KGL}(KGL_Z(T)[n], i^!KGL_S) \\ \| & & \| \\ \mathrm{Hom}_{KGL}(KGL_T[n], KGL_T) & \xrightarrow{\eta'_k} & \mathrm{Hom}_{KGL}(KGL_T[n], k^!KGL_X) \end{array}$$

using the adjunction $(g_\sharp, g_*)$, the exchange isomorphism $g^*i^! \simeq k^!f^*$ (which uses relative purity for smooth morphisms) and the third point of remark 12.5.5. In particular, the preceding proposition has the following consequences:

Theorem 12.6.3 (Absolute purity)**.** *For any closed immersion $i : Z \to S$ between regular schemes, the map*

$$\eta'_i : KGL_Z \to i^!KGL_S$$

is an isomorphism in the category $\mathrm{Ho}(KGL\text{-}\mathrm{mod})(Z)$ *(or in* $\mathrm{SH}(Z)$*).*

Corollary 12.6.4. *Given a cartesian square as above, for any pair $(n, m) \in \mathbf{Z}^2$, the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{Hom}(KGL_S(X), i_*KGL_Z(m)[n]) & \xrightarrow{\eta_i} & \mathrm{Hom}(KGL_S(X), KGL_S(m)[n]) \\ \| & & \\ \mathrm{Hom}(KGL_Z(T), KGL_Z(m)[n]) & & \sim\downarrow{\scriptstyle \epsilon_{X/S}} \\ {\scriptstyle \epsilon_{T/Z}}\downarrow\sim & & \\ K_{2m-n}(T) & \xrightarrow{k_*} & K_{2m-n}(X) \end{array}$$

where the vertical maps are the isomorphisms (12.3.2.2).

12.7. **Trace maps.**

12.7.1. Let S be a regular scheme. Let Y be a smooth S-scheme. The canonical map

$$\mathrm{Pic}(Y) \to K_0(Y) \xrightarrow{\sim} \mathrm{Hom}_{KGL}(KGL_S(Y), KGL_S) \xrightarrow{\beta_*} \mathrm{Hom}_{KGL}(KGL_S(Y), KGL_S(1)[2])$$

defines representable Chern classes in the category $\mathrm{Ho}(KGL\text{-}\mathrm{mod})(S)$; they corresponds to the orientation defined in remark 12.2.2.
Let $p : P \to S$ be a projective bundle of rank n. Let $v = [\mathcal{O}(1)] - 1$ in $K_0(P)$. It corresponds to a map $\mathfrak{v} : KGL_S(P) \to KGL_S$. According to [Dég08, 3.2] and our choice of Chern classes, the following map

$$KGL_S(P) \xrightarrow{\sum_i \beta^i.\mathfrak{v}^i \boxtimes p_*} \bigoplus_{0\leq i\leq n} KGL_S(i)[2i]$$

is an isomorphism. As β is invertible, it follows that the map

$$\varphi_{P/S} : KGL_S(P) \xrightarrow{\sum_i \mathfrak{v}^i \boxtimes p_*} \bigoplus_{0\leq i\leq n} KGL_S \tag{12.7.1.1}$$

is an isomorphism as well. Using this formula, the map $\mathrm{Hom}(\varphi_{P/S}, KGL_S)$ is equal to the isomorphism of Quillen's projective bundle theorem in K-theory (*cf.* [Qui73, 4.3]):

$$f_{P/S} : \bigoplus_{i=0}^{n} K_*(S) \to K_*(P), (S_0, ..., S_n) \mapsto \sum_i p^*(S_i).v^i.$$

Let $p_* : K_*(P) \to K_*(S)$ be the pushout by the projective morphism p. According to the projection formula, it is $K_*(S)$-linear. In particular, it is determined by the $n+1$-uple $(a_0, ..., a_n)$ where $a_i = p_*(v^i) \in K_0(S)$ through the isomorphism $f_{P/S}$. Let $\mathfrak{a}_i : KGL_S \to KGL_S$ be the map corresponding to a_i.

Definition 12.7.2. Consider the previous notations. We define the *trace map* associated to the projection $p : P \to S$ as the morphism of KGL-modules

$$\mathrm{Tr}_p^{KGL} : p_*(KGL_P) = \mathbf{R}Hom(KGL_S(P), KGL_S) \xrightarrow{(\varphi_{P/S}^*)^{-1}} \bigoplus_{i=0}^{n} KGL_S \xrightarrow{(\mathfrak{a}_0,...,\mathfrak{a}_n)} KGL_S.$$

From this definition, it follows that Tr_p represents the pushout by p in K-theory:

$$\begin{array}{ccc}
\mathrm{Hom}_{KGL}(KGL_S[n], p_*KGL_P) & \xrightarrow{\mathrm{Tr}_{p*}^{KGL}} & \mathrm{Hom}_{KGL}(KGL_S[n], KGL_S) \\
\| & & \\
\mathrm{Hom}_{KGL}(KGL_P[n], KGL_P) & & \downarrow \epsilon_S \\
\epsilon_P \downarrow & & \\
K_n(P) & \xrightarrow{\qquad\qquad p_* \qquad\qquad} & K_n(S)
\end{array}$$

Consider moreover a cartesian square:

$$\begin{array}{ccc}
Q & \xrightarrow{q} & P \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p} \\
Y & \xrightarrow{f} & S
\end{array}$$

such that f is smooth. From the projective base change theorem, we get $f^*p_*p^* = q_*q^*g^*$. Using this identification, we easily obtain that $f^* \mathrm{Tr}_p^{KGL} = \mathrm{Tr}_q^{KGL}$. Thus, we conclude that the map

$$\mathrm{Hom}_{KGL}(KGL_S(Y)[n], p_*KGL_P) \xrightarrow{\mathrm{Tr}_p^{KGL}} \mathrm{Hom}_{KGL}(KGL_S(Y)[n], KGL_S)$$

represents the usual pushout map

$$q_* : K_n(Q) \to K_n(Y)$$

through the canonical isomorphisms (12.3.2.2).

12.7.3. Consider a projective morphism $f : T \to S$ between regular schemes and choose a factorization

$$T \xrightarrow{i} P \xrightarrow{p} S$$

where i is a closed immersion and p is the projection of a projective bundle. Let us define a morphism

$$\mathrm{Tr}^{KGL}_{(p,i)} : f_* KGL_T = p_* i_* KGL_T \xrightarrow{p_*\eta_i} p_* KGL_P \xrightarrow{\mathrm{Tr}^{KGL}_p} KGL_S.$$

According to 12.6.4 and the previous paragraph, for any cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ T & \xrightarrow{f} & S \end{array}$$

such that a is smooth, the following diagram is commutative.

$$\begin{array}{ccc} \mathrm{Hom}_{KGL}(KGL_S(X), f_*KGL_T(m)[n]) & \xrightarrow{\mathrm{Tr}^{KGL}_{(p,i)*}} & \mathrm{Hom}_{KGL}(KGL_S(X), KGL_S(m)[n]) \\ \| & & \\ \mathrm{Hom}_{KGL}(KGL_T(Y), KGL_Z(m)[n]) & & \simeq \downarrow {\scriptstyle \epsilon_{X/S}} \\ {\scriptstyle \epsilon_{Y/T}}\downarrow \simeq & & \\ K_{2m-n}(Y) & \xrightarrow{\qquad g_* \qquad} & K_{2m-n}(X) \end{array} \tag{12.7.3.1}$$

Definition 12.7.4. Considering the above notations, we define the *trace map* associated to f as the morphism

$$\mathrm{Tr}^{KGL}_f = \mathrm{Tr}^{KGL}_{(p,i)} : f_* f^* KGL_S \to KGL_S.$$

Remark 12.7.5. By definition, the trace map Tr^{KGL}_f is a morphism of KGL-modules. As a consequence, the map obtained by adjuntion

$$\eta'_f : KGL_T \simeq f^* KGL_S \to f^! KGL_S$$

is also a morphism of KGL-module. This implies that the morphism η'_f (and thus also Tr^{KGL}_f) is completely determined by the element

$$\eta'_f \in \mathrm{Hom}_{KGL}(KGL_T, f^! KGL_S) \simeq \mathrm{Hom}_{\mathrm{SH}(T)}(\mathbb{1}_T, f^! KGL_S)\,.$$

Moreover, as p is smooth, there is a canonical isomorphism $p^! KGL_S \simeq KGL_P$ (by relative purity for p and by periodicity; see [Rio09, lemma 6.1.3.3]). From there, we deduce from theorem 12.6.3 that we have a canonical isomorphism

$$f^! KGL_S \simeq i^! KGL_P \simeq KGL_T\,.$$

This implies that we have an isomorphism:

$$\mathrm{Hom}_{\mathrm{SH}(T)}(\mathbb{1}_T, f^! KGL_S) \simeq K_0(T)\,.$$

Hence the map η'_f is completely determined by a class in $K_0(T)$. The problem of the functoriality of trace maps in the motivic category $\mathrm{Ho}(KGL\text{-}\mathrm{mod})$ is thus a matter of functoriality of this element η'_f in K_0, which can be translated faithfully to the problem of the functoriality of pushforwards for K_0.

However, the only property of trace maps we shall use here is the following.

Proposition 12.7.6. *Let $f : T \to S$ be a finite flat morphism of regular schemes such that the $\mathcal{O}_S$-module $f_*\mathcal{O}_T$ is (globally) free of rank d. Then the following composite map*

$$KGL_S \to f_* f^* KGL_S \xrightarrow{\mathrm{Tr}^{KGL}_f} KGL_S$$

is equal to $d.1_{KGL_S}$ in $\mathrm{Ho}(KGL\text{-}\mathrm{mod})(S)$ (whence in $\mathrm{SH}(S)$).

Proof. Let φ be the composite map of $\mathrm{Ho}(KGL\text{-}\mathrm{mod})(S)$

$$KGL_S \to f_* f^* KGL_S \xrightarrow{\mathrm{Tr}_f} KGL_S\,.$$

As φ is KGL_S-linear by construction, it corresponds to an element

$$\varphi \in \mathrm{Hom}_{KGL}(KGL_S, KGL_S) \simeq \mathrm{Hom}_{\mathrm{SH}(S)}(\mathbb{1}_S, KGL_S) \simeq K_0(S)\,.$$

According to the commutative diagram (12.7.3.1), if we apply the functor $\mathrm{Hom}_{\mathrm{SH}(S)}(\mathbb{1}_S, -)$ to φ, we obtain through the evident canonical isomorphisms the composition of the usual pullback and pushforward by f in K-theory:

$$K_0(S) \xrightarrow{f^*} K_0(T) \xrightarrow{f_*} K_0(S).$$

With these notations, the element of $K_0(S)$ corresponding to φ is the pushforward of $1_T = f^*(1_S)$ by f, while the element corresponding to the identity of KGL_S is of course 1_S. Under our assumptions on f, it is obvious that we have the identity $f_*(1_T) = d.1_S \in K_0(S)$. This means that φ is d times the identity of KGL_S. □

13. Beilinson motives

13.1. The γ-filtration.

13.1.1. We denote by $KGL_{\mathbf{Q}}$ the $\mathbf{Q}$-localization of the absolute ring spectrum KGL, considered as a cartesian section of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}$. From [Rio09, 5.3.10], this spectrum has the following property:

(K5) For any scheme S, there exists a canonical decomposition

$$KGL_{\mathbf{Q},S} \simeq \bigoplus_{i\in\mathbf{Z}} {KGL^{(i)}}_S$$

compatible with base change and such that for any regular scheme S, the isomorphism of (K2) induces an isomorphism:

$$\mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})}\Big(\mathbf{Q}_S(X)[n], {KGL^{(i)}}_S\Big) \simeq K_n^{(i)}(S) := Gr_\gamma^i K_n(S)_{\mathbf{Q}}$$

where the right hand side is the i-th graded piece of the γ-filtration on K-theory groups.

We will denote by

$$\pi_i : KGL_{\mathbf{Q},S} \to {KGL^{(i)}}_S,$$
$$\text{resp. } \iota_i : {KGL^{(i)}}_S \to KGL_{\mathbf{Q},S}$$

the projection (resp. inclusion) defined by the decomposition (K3) and we put $p_i = \iota_i p_i$ for the corresponding projector on $KGL_{\mathbf{Q},S}$.

Definition 13.1.2 (Riou)**.** We define the *Beilinson motivic cohomology spectrum* as the rational Tate spectrum $H_{\mathrm{B},S} = {KGL^{(0)}}_S$.

Remark 13.1.3. Note that, by definition, for any morphism of schemes $f : T \to S$, we have $f^* H_{\mathrm{B},S} \simeq H_{\mathrm{B},T}$.

Lemma 13.1.4. *The isomorphism γ_u of* (12.2.1.1) *is homogenous of degree $+1$ with respect to the graduation (K5). In other words, for any integer $i \in \mathbf{Z}$, the following composite map is an isomorphism*

$$KGL^{(i)}(1)[2] \xrightarrow{\iota_i} KGL_{\mathbf{Q}}(1)[2] \xrightarrow{\gamma_u} KGL_{\mathbf{Q}} \xrightarrow{\pi_i} KGL^{(i+1)}.$$

For any integer $i \in \mathbf{Z}$, we thus get a canonical isomorphism

$$H_{\mathrm{B}}(i)[2i] \xrightarrow{\sim} KGL^{(i)}. \tag{13.1.4.1}$$

Proof. It is sufficient to check that, for $j \neq i+1$,

$$\begin{cases} p_j \circ \gamma_u \circ p_i = 0, \\ p_j \circ \gamma_u^{-1} \circ p_i = 0 \end{cases}$$

in $\mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})}(KGL_{\mathbf{Q}}, KGL_{\mathbf{Q}})$. But according to [Rio09, 5.3.1 and 5.3.6], we have only to check these equalities for the induced endomorphism of K_0 (seen as a presheaf on the category of smooth schemes over $\mathrm{Spec}\,(\mathbf{Z})$). This follows then from the compatibility of the projective bundle isomorphism with the γ-filtration; see [BGI71, Exp. VI, 5.6]. □

13.1.5. Recall from [NSØ08] that $KGL_{\mathbf{Q}}$ is canonically isomorphic (with respect to the orientation 12.2.2) to the universal oriented rational ring spectrum with multiplicative formal group law introduced in [NSØ08]. The isomorphism of the preceding corollary shows in particular that $H_{\text{Б}}$ is obtained from $KGL_{\mathbf{Q}}$ by killing the elements β^n for $n \neq 0$. In particular, this shows that $H_{\text{Б}}$ is canonically isomorphic to the spectrum denoted by $\mathsf{L}\mathbf{Q}$ in [NSØ08], which corresponds to the universal additive formal group law over $\mathbf{Q}$. This implies that $H_{\text{Б}}$ has a natural structure of a (commutative) ring spectrum.

Proposition 13.1.6. *The multiplication map*

$$\mu : H_{\text{Б}} \otimes H_{\text{Б}} \to H_{\text{Б}}$$

is an isomorphism.

This trivially implies that the following map is an isomorphism:

$$1 \otimes \eta : H_{\text{Б}} \to H_{\text{Б}} \otimes H_{\text{Б}}. \tag{13.1.6.1}$$

Proof. It is enough to treat the case $S = \mathrm{Spec}\,(\mathbf{Z})$. We will proove that the projector

$$\psi : H_{\text{Б}} \otimes H_{\text{Б}} \xrightarrow{\mu} H_{\text{Б}} \xrightarrow{1\otimes\eta} H_{\text{Б}} \otimes H_{\text{Б}}$$

is an isomorphism (in which case it is in fact the identity). We do that for the isomorphic ring spectrum $\mathsf{L}\mathbf{Q}$.

Let $\mathsf{H}^{top}\mathbf{Q}$ be the topological spectrum representing rational singular cohomology. In the terminology of [NSØ08], $\mathsf{L}\mathbf{Q}$ is a Tate spectrum representing the Landweber exact cohomology wich corresponds to the Adams graded MU_* algebra $\mathbf{Q}$ obtained by killing every generators of the Lazard ring MU_*. The corresponding topological spectrum is of course $\mathsf{H}^{top}\mathbf{Q}$.
According to [NSØ08, 9.2], the spectrum $E = \mathsf{L}\mathbf{Q}\otimes\mathsf{L}\mathbf{Q}$ is a Landweber exact spectrum corresponding to the MU_*-algebra $\mathbf{Q} \otimes_{MU_*} \mathbf{Q} = \mathbf{Q}$. In particular, the corresponding topological spectrum is simply $\mathsf{H}^{top}\mathbf{Q}$. Thus, according to [NSØ08, 9.7], applied with $F = E = \mathsf{L}\mathbf{Q} \otimes \mathsf{L}\mathbf{Q}$, we get an isomorphism of $\mathbf{Q}$-vector spaces

$$\mathrm{End}(\mathsf{L}\mathbf{Q} \otimes \mathsf{L}\mathbf{Q}) = \mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}, E_{**}) = \mathbf{Q}.$$

Thus $\psi = \lambda.Id$ for $\lambda \in \mathbf{Q}$. But $\lambda = 0$ is excluded because ψ is a projector on a non trivial factor, so that we can conclude. □

13.2. **Localization with respect to rational K-theory.**

Definition 13.2.1. Let S be any scheme.

We say that an object E of $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$ is $H_{\text{Б}}$-*acyclic* if $H_{\text{Б}} \otimes E = 0$ in $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$. A morphism of $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$ is an $H_{\text{Б}}$-*equivalence* if its cone is $H_{\text{Б}}$-acyclic (or, equivalently, if its tensor product with $H_{\text{Б}}$ is an isomorphism).

An object M of $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$ is $H_{\text{Б}}$-*local* if, for any $H_{\text{Б}}$-acyclic object E, the group $\mathrm{Hom}(E, M)$ vanishes.

We denote by $\mathrm{DM}_{\text{Б}}(S)$ the Verdier quotient of $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$ by the localizing subcategory made of $H_{\text{Б}}$-acyclic objects (i.e. the localization of $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$ by the class of $H_{\text{Б}}$-equivalences).

The objects of $\mathrm{DM}_{\text{Б}}(S)$ are called the *Beilinson motives*.

Proposition 13.2.2. *An object E of $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$ is $H_{\text{Б}}$-acyclic if and only if we have $KGL_{\mathbf{Q}} \otimes E = 0$.*

Proof. This follows immediately from property (K5) (see 13.1.1) and lemma 13.1.4. □

Proposition 13.2.3. *The localization functor* $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q}) \to \mathrm{DM}_{\text{Б}}(S)$ *admits a fully faithful right adjoint whose essential image in* $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$ *is the full subcategory spanned by* $H_{\text{Б}}$*-local objects. More precisely, there is a left Bousfield localization of the stable model category of symmetric Tate spectra* $\mathrm{Sp}(S,\mathbf{Q})$ *by a small set of maps whose homotopy category is precisely* $\mathrm{DM}_{\text{Б}}(S)$*.*

Proof. For each smooth S-scheme X and any integers $n, i \in \mathbf{Z}$, we have a functor with values in the category of $\mathbf{Q}$-vector spaces

$$F_{X,n,i} = \mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})}(\Sigma^\infty \mathbf{Q}_S(X), H_{\text{Б}} \otimes (-)(i)[n]) : \mathrm{Sp}(S,\mathbf{Q}) \to \mathbf{Q}\text{-}\mathrm{mod}$$

which preserves filtered colimits. We define the class of $H_{\text{Б}}$-weak equivalences as the class of maps of $\mathrm{Sp}(S,\mathbf{Q})$ whose image by $F_{X,n,i}$ is an isomorphism for all X and n, i. By virtue of [Bek00, Prop. 1.15 and 1.18], we can apply Smith's theorem [Bek00, Theorem 1.7] (with the class of cofibrations of $\mathrm{Sp}(S,\mathbf{Q})$), which implies the proposition. □

Remark 13.2.4. We shall often make the abuse of considering $\mathrm{DM}_{\text{Б}}(S)$ as a full subcategory in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$, with an implicit reference to the preceding proposition.

Note that $H_{\text{Б}}$-acyclic objects are stable by the operations f^* and $f_\sharp$, so that we obtain a premotivic category $\mathrm{DM}_{\text{Б}}$ together with a premotivic adjunction:

$$\beta^* : \mathrm{D}_{\mathbf{A}^1,\mathbf{Q}} \rightleftarrows \mathrm{DM}_{\text{Б}} : \beta_*.$$

Proposition 13.2.5. *The spectrum* $H_{\text{Б}}$ *is* $H_{\text{Б}}$*-local and the unit map* $\eta_{H_{\text{Б}}} : \mathbb{1} \to H_{\text{Б}}$ *is an* $H_{\text{Б}}$*-equivalence in* $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$*.*

Proof. The unit map $\eta : \mathbb{1}_S \to H_{\text{Б},S}$ is an $H_{\text{Б}}$-equivalence by 13.1.6.

Consider a rational spectrum E such that $E \otimes H_{\text{Б}} = 0$ and a map $f : E \to H_{\text{Б}}$. It follows trivially from the commutative diagram

$$\begin{array}{ccccc} E & \xrightarrow{f} & H_{\text{Б}} & & \\ {\scriptstyle 1\otimes\eta}\downarrow & & {\scriptstyle 1\otimes\eta}\downarrow & \searrow & \\ E \otimes H_{\text{Б}} & \xrightarrow{f\otimes 1} & H_{\text{Б}} \otimes H_{\text{Б}} & \xrightarrow{\mu} & H_{\text{Б}} \end{array}$$

that $f = 0$, which shows that $H_{\text{Б}}$ is $H_{\text{Б}}$-local. □

Corollary 13.2.6. *The family of ring spectra* $H_{\text{Б},S}$ *comes from a cofibrant cartesian commutative monoid (4.2.9) of the symmetric monoidal fibred model category of Tate spectra over the category of schemes.*

Proof. By virtue of proposition 13.2.5 and of corollary 4.1.9, there exists a cofibrant commutative monoid in the model category of symmetric Tate spectra over $\mathrm{Spec}\,(\mathbf{Z})$ which is canonically isomorphic to $H_{\text{Б},\mathbf{Z}}$ in $\mathrm{D}_{\mathbf{A}^1}(\mathrm{Spec}\,(\mathbf{Z}),\mathbf{Q})$ (as commutative ring spectrum). For a morphism of schemes $f : S \to \mathrm{Spec}\,(\mathbf{Z})$, we can then define $H_{\text{Б},S}$ as the pullback of $H_{\text{Б},\mathbf{Z}}$ (at the level of the model categories); using proposition 4.1.11, we see that this defines a cofibrant cartesian commutative monoid on the fibred category of spectra which is isomorphic to $H_{\text{Б},S}$ as commutative ring spectra in $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$. □

13.2.7. From now on, we shall assume that $H_{\text{Б}}$ is given by a cofibrant cartesian commutative monoid of the symmetric monoidal fibred model category of Tate spectra over the category of schemes. By virtue of propositions 4.2.10 and 4.2.16), we get the motivic category $\mathrm{Ho}(H_{\text{Б}}\text{-}\,\mathrm{mod})$ of $H_{\text{Б}}$-modules, together with a commutative diagram of morphisms of premotivic categories

$$\begin{array}{ccc} \mathrm{D}_{\mathbf{A}^1,\mathbf{Q}} & \xrightarrow{H_{\text{Б}}\otimes(-)} & \mathrm{Ho}(H_{\text{Б}}\text{-}\,\mathrm{mod}) \\ & {\scriptstyle\beta}\searrow \quad \nearrow{\scriptstyle\varphi} & \\ & \mathrm{DM}_{\text{Б}} & \end{array}$$

(any $H_{\text{Б}}$-acyclic object becomes null in the homotopy category of $H_{\text{Б}}$-modules by definition, so that $H_{\text{Б}} \otimes (-)$ factors uniquely through $\mathrm{DM}_{\text{Б}}$ by the universal property of localization).

Proposition 13.2.8. *The forgetful functor* $U : \mathrm{Ho}(H_{\Bbbk}\text{-}\mathrm{mod})(S) \to \mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$ *is fully faithful.*

Proof. We have to prove that, for any $H_{\Bbbk,S}$-module M, the map

$$H_{\Bbbk,S} \otimes M \to M$$

is an isomorphism in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$. As this is a natural transformation between exact functors which commute with small sums, and as $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}$ is a compactly generated triangulated category, it is sufficient to check this for $M = H_{\Bbbk,S} \otimes E$, with E a (compact) object of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$ (see 4.2.6). In this case, this follows immediately from the isomorphism (13.1.6.1). □

Theorem 13.2.9. *The functor* $\mathrm{DM}_{\Bbbk}(S) \to \mathrm{Ho}(H_{\Bbbk,S}\text{-}\mathrm{mod})$ *is an equivalence of categories.*

Proof. This follows formally from the preceding proposition by definition of $\mathrm{DM}_{\Bbbk}$ (see for instance [GZ67, Chap. I, Prop. 1.3]). □

Remark 13.2.10. The preceding theorem shows that the premotivic category $\mathrm{Ho}(H_{\Bbbk}\text{-}\mathrm{mod})$ as well as the morphism $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}} \to \mathrm{Ho}(H_{\Bbbk}\text{-}\mathrm{mod})$ are completely independent of the choice of the strictification of the (commutative) monoid structure on $H_{\Bbbk}$ given by corollary 13.2.6.

Corollary 13.2.11. $\mathrm{DM}_{\Bbbk} \simeq \mathrm{Ho}(H_{\Bbbk}\text{-}\mathrm{mod})$ *is a* $\mathbf{Q}$*-linear motivic category.*

Proof. This follows from proposition 4.2.16 and theorem 13.2.9 that $\mathrm{DM}_{\Bbbk}$ is pregeometric (because $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}$ is so, by 6.2.2). We conclude with corollary 2.4.18. □

Remark 13.2.12. One can prove also that $\mathrm{DM}_{\Bbbk}$ is motivic much more directly: this follows from the fact that $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}$ is motivic and that the six Grothendieck operations preserve $H_{\Bbbk}$-acyclic objects, so that all the properties of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}$ induce their analogs on $\mathrm{DM}_{\Bbbk}$ by the 2-universal property of localization (we leave this as an easy exercise for the reader).

Definition 13.2.13. For a scheme X, we define its Beilinson motivic cohomology by the formula

$$H^q(X, \mathbf{Q}(p)) = \mathrm{Hom}_{\mathrm{DM}_{\Bbbk}(X)}(\mathbb{1}_X, \mathbb{1}_X(p)[q])\,.$$

We can now justify the terminology of Beilinson motives:

Corollary 13.2.14. *For any regular scheme* X*, we have a canonical isomorphism*

$$H^q(X, \mathbf{Q}(p)) \simeq Gr^p_\gamma K_{2p-q}(X)_{\mathbf{Q}}\,.$$

Note also the following nice description of Beilinson motives:

Corollary 13.2.15. *Let* E *be a rational spectrum over* S*. The following conditions are equivalent:*

(i) E *is a Beilinson motive (i.e. is in the essential image of the right adjoint of the localization functor* $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}} \to \mathrm{DM}_{\Bbbk}$*);*
(ii) E *is* $H_{\Bbbk}$*-local;*
(iii) the map $\eta \otimes 1_E : E \to H_{\Bbbk} \otimes E$ *is an isomorphism;*
(iv) E *is an* $H_{\Bbbk}$*-module in* $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}$*;*
(v) E *is admits a strict* $H_{\Bbbk}$*-module structure.*

If, in addition, E *is a commutative ring spectrum, these conditions are equivalent to the following ones:*

(Ri) E *is orientable;*
(Rii) E *is an* $H_{\Bbbk}$*-algebra;*
(Riii) E *admits a unique structure of* $H_{\Bbbk}$*-algebra;*

And, if E *is a strict commutative ring spectrum, these conditions are equivalent to the following conditions:*

(Riv) there exists a morphism of commutative monoids $H_{\Bbbk} \to E$ *in the stable model category of Tate spectra;*
(Rv) there exists a unique morphism $H_{\Bbbk} \to E$ *in the homotopy category of commutative monoids of the category of Tate spectra.*

Proof. The equivalence between statements (i)–(v) follows immediately from 13.2.9. If E is a ring spectrum, the equivalence with (Ri), (Rii) and R(iii) is a consequence of 11.2.10 and of the fact that $MGL_{\mathbf{Q}}$ is $H_{\mathbb{B}}$-local; see [NSØ08, Cor. 10.6]. It remains to prove the equivalence with (Riv) and (Rv). Then, E is $H_{\mathbb{B}}$-local if and only if the map $E \to H_{\mathbb{B}} \otimes E$ is an isomorphism. But this map can be seen as a morphism of strict commutative ring spectra (using the model structure of 4.1.8 applied to the model category of Tate spectra) whose target is clearly an $H_{\mathbb{B}}$-algebra, so that (Riv) is equivalent to (ii). It remains to check that there is at most one strict $H_{\mathbb{B}}$-algebra structure on E (up to homotopy), which follows from the fact that $H_{\mathbb{B}}$ is the initial object in the homotopy category of commutative monoids of the model category given by theorem 4.1.8 applied to the model structure of proposition 13.2.3. □

Corollary 13.2.16. *One has the following properties.*

(1) The ring structure on the spectrum $H_{\mathbb{B}}$ is given by the following structural maps (with the notations of 13.1.1).

$$H_{\mathbb{B}} \otimes H_{\mathbb{B}} \xrightarrow{\iota_0 \otimes \iota_0} KGL_{\mathbf{Q}} \otimes KGL_{\mathbf{Q}} \xrightarrow{\mu_{KGL}} KGL_{\mathbf{Q}} \xrightarrow{\pi_0} H_{\mathbb{B}},$$
$$\mathbf{Q} \xrightarrow{\eta_{KGL}} KGL_{\mathbf{Q}} \xrightarrow{\pi_0} H_{\mathbb{B}}.$$

(2) The map $\imath_0 : H_{\mathbb{B}} \to KGL_{\mathbf{Q}}$ is compatible with the monoid structures.

(3) Let $H_{\mathbb{B}}[t, t^{-1}] = \bigoplus_{i \in \mathbf{Z}} H_{\mathbb{B}}(i)[2i]$ be the free $H_{\mathbb{B}}$-algebra generated by one invertible generator t of bidegree $(2, 1)$. Then the section $u : \mathbf{Q}(1)[2] \to KGL_{\mathbf{Q}}$ induces an isomorphism of $H_{\mathbb{B}}$-algebras:

$$\gamma'_u : H_{\mathbb{B}}[t, t^{-1}] \to KGL_{\mathbf{Q}}.$$

Proof. Property (1) follows from properties (2) and (3). Property (2) is a trivial consequence of the previous corollary. Using the isomorphisms (13.1.4.1) of lemma 13.1.4, we get a canonical isomorphism

$$H_{\mathbb{B},S}[t, t^{-1}] \xrightarrow{\sim} \bigoplus_{i \in \mathbf{Z}} KGL^{(i)}.$$

Through this isomorphism, the map γ'_u corresponds to the Adams decomposition (i.e. to the isomorphism (K5) of 13.1.1) from which we deduce property (3). □

Remark 13.2.17. One deduces easily, from the preceding proposition and from 13.1.6, another proof of the fact that $KGL_{\mathbf{Q}}$ is a strict commutative ring spectrum.

The isomorphism (3) is in fact compatible with the gradings of each term: the factor $H_{\mathbb{B}}.t^i$ is sent to the factor $KGL^{(i)}$. Recall also the parameter t corresponds to the unit β^{-1} in $KGL^{*,*}$.

Corollary 13.2.18. *The Adams decomposition is compatible with the monoid structure on $KGL_{\mathbf{Q}}$: for any integer i, j, l such that $l \neq i + j$, the following map is zero.*

$$KGL^{(i)} \otimes KGL^{(j)} \xrightarrow{\iota_i \otimes \iota_j} KGL_{\mathbf{Q}} \otimes KGL_{\mathbf{Q}} \xrightarrow{\mu} KGL_{\mathbf{Q}} \xrightarrow{\pi_l} KGL^{(l)}$$

13.3. **Motivic proper descent.**

Proposition 13.3.1. *The motivic category $\mathrm{DM}_{\mathbb{B}}$ is separated on the category of noetherian quasi-excellent schemes.*

Proof. According to 2.3.9, it is sufficient to check that, for any finite surjective morphism $f : T \to S$, the pullback functor

$$f^* : \mathrm{DM}_{\mathbb{B}}(S) \to \mathrm{DM}_{\mathbb{B}}(T)$$

is conservative.

We proceed by noetherian induction on S. Using the localization property of $\mathrm{DM}_{\mathbb{B}}$ and the induction hypothesis, we can replace S by any dense open subscheme U, and f by the finite surjective morphism $f \times_S U$. Thus, as S is quasi-excellent, we can assume it is regular. As f is finite surjective and T is quasi-excellent, we can even assume that T is regular. We can further assume that S and T are reduced (*cf.* 2.3.6). As f is generically flat, we may assume that f is flat. Reasonning on each connected component of T, we can even suppose that f is globally free

of rank d – *i.e.* there exists an isomorphism of $\mathcal{O}_S$-modules $f_*\mathcal{O}_T \simeq \mathcal{O}_S^d$. It is sufficient to prove that, for any Beilinson motive E over S, the adjunction map

$$E \to f_* f^*(E)$$

is a monomorphism in DM_{B}. Using the projection formula in DM_{B} applied to the finite morphism f (point (v) of theorem 2.4.21), this latter map is isomorphic to

$$\big(H_{\mathrm{B}} \to f_* f^*(H_{\mathrm{B}})\big) \otimes 1_E.$$

We are finally reduced to prove that the map $H_{\mathrm{B},S} \to f_* f^* H_{\mathrm{B},S}$ is a monomorphism in DM_{B} (any monomorphism of a triangulated category splits). As $H_{\mathrm{B},S}$ is a direct factor of $KGL_{\mathbf{Q},S}$, it is sufficient to find a retraction of the adjunction map

$$KGL_{\mathbf{Q},S} \to f_* f^* KGL_{\mathbf{Q},S}\,,$$

which follows from proposition 12.7.6. □

According to theorem 3.3.36, we deduce from the preceding proposition the following result:

Theorem 13.3.2. *The motivic category* DM_{B} *satisfies* h*-descent over noetherian quasi-excellent schemes of finite dimension: for any such scheme* X*, any* h*-hypercovering* $p : \mathscr{X} \to X$*, and for any Beilinson motive* E *over* X*, the map*

$$p^* : \mathbf{R}\Gamma(X, E) \to \mathbf{R}\Gamma(\mathscr{X}, E) = \mathbf{R}\varprojlim_n \mathbf{R}\Gamma(\mathscr{X}_n, E)$$

is an isomorphism in the derived category of the category of $\mathbf{Q}$*-vector spaces.*

13.4. **Motivic absolute purity.**

Theorem 13.4.1. *Let* $i : Z \to S$ *be a closed immersion between regular schemes. Assume* i *is of pure codimension* n*.*

Then, considering the notations of 13.1.1, definition 12.5.4, and the identification (13.1.4.1)*, the composed map*

$$H_{\mathrm{B},Z} \xrightarrow{\iota_0} KGL_{\mathbf{Q},Z} \xrightarrow{\eta_i'} i^! KGL_{\mathbf{Q},S} \xrightarrow{\pi_n} i^! H_{\mathrm{B},S}(n)[2n]$$

is an isomorphism.

Proof. We have only to check that the above composition induces an isomorphism after applying the functor $\mathrm{Hom}(\mathbf{Q}_S(X), -(a)[b])$ for a smooth S-scheme X and a couple of integers $(a, b) \in \mathbf{Z}^2$. Using Remark 12.5.5(3), this composition is compatible with smooth base change and we can assume $X = S$. Let us consider the projector

$$p_a : K_r^Z(S)_{\mathbf{Q}} = K_r(S/S - Z)_{\mathbf{Q}} \to K_r(S/S - Z)_{\mathbf{Q}}$$

induced by $\pi_a \circ \iota_a : KGL_{\mathbf{Q}} \to KGL_{\mathbf{Q}}$, and denote by $K_r^{(a)}(S/S-Z)$ (with $r = 2a-b$) its image. By virtue of Propostion 12.6.1, we only have to check that the following composite is an isomorphism:

$$\rho_i : K_r^{(a)}(Z) \xrightarrow{\iota_a} K_r(Z)_{\mathbf{Q}} \xrightarrow{\mathfrak{q}_i^{-1}} K_r(S/S - Z)_{\mathbf{Q}} \xrightarrow{\pi_a} K_r^{(a+n)}(S/S - Z).$$

From 12.5.2, the morphism ρ_i is functorial with respect to Tor-independant cartesian squares of regular schemes (*cf.* 12.5.1). Thus, using again the deformation diagram (12.6.1.1), we get a commutative diagram

$$\begin{array}{ccccc}
K_r^{(a)}(Z) & \longrightarrow & K_r^{(a)}(\mathbf{A}^1_Z) & \longleftarrow & K_r^{(a)}(Z) \\
\downarrow{\scriptstyle \rho_i} & & \downarrow & & \downarrow{\scriptstyle \rho_s} \\
K_r^{(a+n)}(S/S - Z) & \longrightarrow & K_r^{(a+n)}(D/D - \mathbf{A}^1_Z) & \longleftarrow & K_r^{(a+n)}(P/P - Z)
\end{array}$$

in which any of the horizontal maps is an isomorphism (as a direct factor of an isomorphism). Thus, we are reduced to the case of the closed immersion $s : Z \to P$, canonical section of the projectivisation of a vector bundle E (where E is the normal bundle of the closed immersion i). Moreover, as the assertion is local on Z, we may assume E is a trivial vector bundle.

Let $p : P \to Z$ be the canonical projection, $j : P-Z \to P$ the obvious open immersion. Consider the element $v = [\mathcal{O}(1)] - 1 \in K_0(P)$. Because v has virtual rank 0, it belongs to $K_0^{(1)}(P)$.
Recall that, according to the projective bundle formula, the horizontal lines in the following commutative diagram are split short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_r(P/P-Z)_{\mathbf{Q}} & \xrightarrow{\nu} & K_r(P)_{\mathbf{Q}} & \xrightarrow{j^*} & K_r(P-Z)_{\mathbf{Q}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_r^{(a+n)}(P/P-Z) & \xrightarrow{\nu'} & K_r^{(a+n)}(P) & \longrightarrow & K_r^{(a+n)}(P-Z) & \longrightarrow & 0. \end{array}$$

By assumption on E, v^n lies in the kernel of j^* and the diagram allows to identify the graded piece $K_r^{(a+n)}(P/P-Z)$ with the submodule of $K_r^{(a+n)}(P)$ of the form $K_r^{(a)}(Z).v^n$.

On the other hand, $j^*s_* = 0$: there exists a unique element $\epsilon \in K_0(Z)$ such that $s_*(1) = p^*(\epsilon).v^n$ in $K_0(P)$. From the relation $p_*s_*(1) = 1$, we obtain that ϵ is a unit in $K_0(Z)$, with inverse the element $p_*(v^n)$. By virtue of [BGI71, Exp. VI, Cor. 5.8], $p_*(v^n)$ belongs to the 0-th γ-graded part of $K_0(P)_{\mathbf{Q}}$ so that the same holds for its inverse ϵ. In the end, for any element $z \in K_r(Z)$, we get the following expression:

$$s_*(z) = s_*(1.s^*p^*(z)) = s_*(1).p^*(z) = p^*(\epsilon.z).v^n.$$

Thus, the commutative diagram

$$\begin{array}{ccccccc} K_r^{(a)}(Z) & \longrightarrow & K_r(Z)_{\mathbf{Q}} & \xrightarrow{\mathfrak{q}_s^{-1}} & K_r(P/P-Z)_{\mathbf{Q}} & \longrightarrow & K_r^{(a+n)}(P/P-Z) \\ & & & \searrow^{s_*} & \downarrow \nu & & \downarrow \nu' \\ & & & & K_r(P)_{\mathbf{Q}} & \longrightarrow & K_r^{(n)}(P) \end{array}$$

implies that the isomorphism $\mathfrak{q}_s^{-1}$ preserves the γ-filtrations (up to a shift by n). Hence it induces an isomorphism on the graded pieces by functoriality. □

14. Constructible motives

14.0. Consider as in 2.0 a base scheme $\mathcal{S}$ and an adequate category $\mathscr{S}$ of $\mathcal{S}$-schemes. In sections 14.1 and 14.3 we assume in addition:

(a) Any scheme in $\mathscr{S}$ is quasi-excellent.

We let as usual Sm be the adequate class of morphisms in $\mathscr{S}$ which are smooth of finite type, and we fix a stable combinatorial Sm-fibred model category $\mathscr{M}$ over $\mathscr{S}$ such that:

(b) $\mathrm{Ho}(\mathscr{M})$ is a motivic category over $\mathscr{S}$.

(c) $\mathrm{Ho}(\mathscr{M})$ is τ-generated where τ is a set of twists which is stable under negative Tate twists.

As usual, the geometric section of $\mathrm{Ho}(\mathscr{M})$ will be denoted by M.

This situation will be fixed in this entire section with a notable exception for paragraph 14.2.1 and proposition 14.2.2 which can be applied to an arbitrary category $\mathscr{S}$ endowed with an admissible class of morphisms $\mathscr{P}$, and to a cofibrantly generated $\mathscr{P}$-fibred model category $\mathscr{M}$ over $\mathscr{S}$.

14.1. **Finiteness theorems.** The aim of this section is to define the notion of τ-constructibility in $\mathrm{Ho}(\mathscr{M})$ and to study its stability properties under six operations of Grothendieck.

Definition 14.1.1. For a scheme X in $\mathscr{S}$, we denote by $\mathrm{Ho}(\mathscr{M})_c(X)$ the sub category of τ-constructible objects in $\mathrm{Ho}(\mathscr{M})(X)$; see 1.4.7.

Proposition 14.1.2. *Assume that, for any scheme X, the triangulated category $\mathrm{Ho}(\mathscr{M})(X)$ admits finite sums and that, for any smooth X scheme Y and any $n \in \tau$, the object $M_X(Y)\{n\}$ is compact. Then, an object of $\mathrm{Ho}(\mathscr{M})(X)$ is τ-constructible if and only if it is compact.*

Proof. If $\mathscr{T}$ is any compactly generated triangulated category, then, for any small family C of compact generators, the thick triangulated category of $\mathscr{T}$ generated by C consists exactly of the compact objects of $\mathscr{T}$. □

Example 14.1.3. If S is a noetherian scheme of finite dimension, then the constructible objects of $\mathrm{D}_{\mathbf{A}^1}(S,\Lambda)$ (resp. of $\mathrm{DM}_{\mathrm{B}}(S)$) are precisely the compact objects.

Proposition 14.1.4. *If M and N are τ-constructible in $\mathrm{Ho}(\mathscr{M})(X)$, so is $M \otimes_X^{\mathbf{L}} N$.*

Proof. For a fixed M, the full subcategory of $\mathrm{Ho}(\mathscr{M})(X)$ spanned by objects such that $M \otimes_X^{\mathbf{L}} N$ is τ-constructible is a thick triangulated subcategory of $\mathrm{Ho}(\mathscr{M})(X)$. In the case M is of shape $M_X(Y)\{n\}$ for Y smooth over X and $n \in \tau$, this proves that $M \otimes_X^{\mathbf{L}} N$ is τ-constructible whenever N is. By the same argument, using the symmetry of the tensor product, we get to the general case. □

Similarly, one has the following conservation property.

Proposition 14.1.5. *For any morphism $f : X \to Y$ of schemes, the functor*

$$\mathbf{L}f^* : \mathrm{Ho}(\mathscr{M})(Y) \to \mathrm{Ho}(\mathscr{M})(X)$$

preserves τ-constructible objects. If moreover f is smooth, the functor

$$\mathbf{L}f_\sharp : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(Y)$$

also preserves τ-constructible objects.

Corollary 14.1.6. *The categories $\mathrm{Ho}(\mathscr{M})_c(X)$ form a thick triangulated monoidal Sm-fibred subcategory of $\mathrm{Ho}(\mathscr{M})$.*

Proposition 14.1.7. *Let X a scheme, and $X = \bigcup_{i\in I} U_i$ a covering of X by open subschemes. An object M of $\mathrm{Ho}(\mathscr{M})(X)$ is τ-constructible if and only if its restriction to U_i is τ-constructible in $\mathrm{Ho}(\mathscr{M})(U_i)$ for all $i \in I$.*

Proof. This is a necessary condition by 14.1.5. For the converse, as X is noetherian, it is sufficient to treat the case where I is finite. Proceeding by induction on the cardinal of I it is sufficient to treat the case of a covering by two open subschemes $X = U \cup V$. For an open immersion $j : W \to X$, write $M_W = \mathbf{L}j_\sharp\, j^*(M)$. If the restrictions of M to U and V are τ-constructible, then so is its restriction to $U \cap V$, and we have a distinguished triangle

$$M_{U\cap V} \to M_U \oplus M_V \to M \to M_{U\cap V}[1]$$

in which M_W is constructible for $W = U, V, U \cap V$ (using 14.1.5 again), from which we deduce that M is τ-constructible. □

Corollary 14.1.8. *For any scheme X and any vector bundle E over X, the functors $\mathcal{T}h(E)$ and $\mathcal{T}h(-E)$ preserve τ-constructible objects in $\mathrm{Ho}(\mathscr{M})(X)$.*

Proof. To prove that $\mathcal{T}h(E)$ and $\mathcal{T}h(-E)$ preserves τ-constructible objects, by virtue of the preceding proposition, we may assume that E is trivial of rank r. It is thus sufficient to prove that $M(r)$ is τ-constructible whenever M is so for any integer r. For we may assume that $M = \mathbb{1}_X\{n\}$ for some $n \in \tau$ (using 14.1.5), this is true by assumption on τ; see 14.0(c). □

Corollary 14.1.9. *Let $f : X \to Y$ a morphism of finite type. The property that the functor*

$$\mathbf{R}f_* : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(Y)$$

preserves τ-constructible objects is local on Y with respect to the Zariski topology.

Proof. Consider a finite Zariski covering $\{v_i : Y_i \to Y\}_{i\in I}$, and write $f_i : X_i \to Y_i$ for the pullback of f along v_i for each i in I. Assume that the functors $\mathbf{R}f_{i,*}$ preserves τ-constructible objects; we shall prove that $\mathbf{R}f_*$ has the same property. Let M be a τ-constructible object in $\mathrm{Ho}(\mathscr{M})(X)$. Then for $i \in I$, using the smooth base change isomorphism (for open immersions), we see that the restriction of $\mathbf{R}f_*(M)$ to Y_i is isomorphic to the image by $\mathbf{R}f_{i,*}$ of the restriction of M to X_i, hence is τ-constructible. The preceding proposition thus implies that $\mathbf{R}f_*(M)$ is τ-constructible. □

Proposition 14.1.10. *For any closed immersion $i : Z \to X$, the functor*

$$\mathbf{R}i_* : \mathrm{Ho}(\mathscr{M})(Z) \to \mathrm{Ho}(\mathscr{M})(X)$$

preserves τ-constructible objects.

Proof. By virtue of corollary 14.1.9, we may assume that X is affine. The category $\mathrm{Ho}(\mathscr{M})_c(Z)$ is then the thick subcategory of $\mathrm{Ho}(\mathscr{M})(Z)$ generated by the objects of shape $M_Z(Y_0)\{n\}$ for Y_0 affine and smooth over Z, and $n \in \tau$. By virtue of [GD67, 18.1.1], for any affine and smooth Z-scheme Y_0, there exists a smooth and affine X-scheme Y whose pullback along i is isomorphic to Y_0. Therefore, the category $\mathrm{Ho}(\mathscr{M})_c(Z)$ is the the thick subcategory of $\mathrm{Ho}(\mathscr{M})(Z)$ generated by the objects of shape $\mathbf{L}i^*(M)$, where M is a τ-constructible object in $\mathrm{Ho}(\mathscr{M})(X)$. To prove the proposition, it is thus sufficient to prove that the functor $\mathbf{R}i_*\,\mathbf{L}i^*$ preserves τ-constructible objects. Let $j : U \to X$ be the complement open immersion. We then have distinguished triangles

$$\mathbf{L}j_\sharp\, j^*(M) \to M \to \mathbf{R}i_*\,\mathbf{L}i^*(M) \to \mathbf{L}j_\sharp\, j^*(M)[1]\,.$$

If M is τ-constructible, then so is $\mathbf{L}j_\sharp\, j^*(M)$ by virtue of 14.1.5, which concludes. □

Corollary 14.1.11. *Let $i : Z \to X$ be a closed immersion with open complement $j : U \to X$. an object M of $\mathrm{Ho}(\mathscr{M})(X)$ is τ-constructible if and only if $j^*(M)$ and $\mathbf{L}i^*(M)$ are τ-constructible in $\mathrm{Ho}(\mathscr{M})(U)$ and $\mathrm{Ho}(\mathscr{M})(Z)$ respectively.*

Proof. We have a distinguished triangle

$$\mathbf{L}j_\sharp\, j^*(M) \to M \to \mathbf{R}i_*\,\mathbf{L}i^*(M) \to \mathbf{L}j_\sharp\, j^*(M)[1]\,.$$

Hence this assertion follows from propositions 14.1.5 and 14.1.10. □

Proposition 14.1.12. *If $f : X \to Y$ is proper, then the functor*

$$\mathbf{R}f_* : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(Y)$$

preserves τ-constructible objects.

Proof. We shall first consider the case where f is projective. As this property is local on Y (corollary 14.1.9), we may assume that f factors as a closed immersion $i : X \to \mathbf{P}^n_Y$ followed by the canonical projection $p : \mathbf{P}^n_Y \to Y$. By virtue of proposition 14.1.10, we can assume that $f = p$. In this case, the functor $\mathbf{R}p_*$ is isomorphic to $\mathbf{L}p_\sharp$ composed with the quasi-inverse of the Thom endofunctor associated to the cotangent bundle of p; see 2.4.21 (iii). Therefore, the functor $\mathbf{R}p_*$ preserves τ-constructible objects by virtue of proposition 14.1.5 and of corollary 14.1.8. The case where f is proper follows easily from the projective case, using Chow's lemma and cdh-descent (the homotopy pullback squares (3.3.8.1)), by induction on the dimension of X. □

Corollary 14.1.13. *If $f : X \to Y$ is separated of finite type, then the functor*

$$f_! : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(Y)$$

preserves τ-constructible objects.

Proof. It is sufficient to treat the case where f is either an open immersion, either a proper morphism, which follows respectively from 14.1.5 and 14.1.12. □

Proposition 14.1.14. *Let X be a scheme. The category of τ-constructible objects in $\mathrm{Ho}(\mathscr{M})(X)$ is the smallest thick triangulated subcategory which contains the objects of shape $\mathbf{R}f_*(\mathbb{1}_{X'}\{n\})$, where $f : X' \to X$ is a (strictly) projective morphism, and $n \in \tau$.*

Proof. See [Ayo07a, lemma 2.2.23]. □

Theorem 14.1.15 (Gabber's weak local uniformisation)**.** *Let X be a scheme, and $Z \subset X$ a nowhere dense closed subscheme. Then there exists a finite h-covering $\{f_i : Y_i \to X\}_{i\in I}$ such that for all i in I, f_i is a morphism of finite type, the scheme Y_i is regular, and $f_i^{-1}(Z)$ is either empty or the support of a strict normal crossing divisor in Y_i.*

See [Gab05, Ill09] for a sketch of proof. A complete argument will be appear in [ILO]. Note that, if we are only interested by schemes of finite type over $\mathrm{Spec}\,(R)$, for R either a field, a complete discrete valuation ring, or a Dedekind domain whose field of functions is a global field, this is an immediate consequence of de Jong's resolution of singularities by alterations; see [dJ96]. One can also deduce the case of schemes of finite type over an excellent noetherian scheme of

dimension lesser or equal to 2 from [dJ97]; see theorem 14.3.5 and corollary 14.3.6 below for a precise statement.

Lemma 14.1.16. *Let $j : U \to X$ be a dense open immersion in $\mathscr{S}$. Then, there exists the following data:*

(i) a finite h-covering $\{f_i : Y_i \to X\}_{i\in I}$ such that for all i in I, f_i is a morphism of finite type, the scheme Y_i is regular, and $f_i^{-1}(U)$ is either Y_i itself or the complement of a strict normal crossing divisor in Y_i; we shall write

$$f : Y = \coprod_{i\in I} Y_i \to X$$

for the induced global h-covering;

(ii) a commutative diagram

$$\begin{array}{ccccc} X''' & & \xrightarrow{\quad g \quad} & & Y \\ {\scriptstyle q}\downarrow & & & & \downarrow{\scriptstyle f} \\ X'' & \xrightarrow{u} & X' & \xrightarrow{p} & X \end{array} \tag{14.1.16.1}$$

in which: p is a proper birational morphism, X' is normal, u is a Nisnevich covering, and q is a finite surjective morphism.

Let T (resp. T') be a closed subscheme of X (resp. X') and assume that for any irreducible component T_0 of T, the following inequality is satisfied:

$$\operatorname{codim}_{X'}(T') \geq \operatorname{codim}_X(T_0), \tag{14.1.16.2}$$

Then, possibly after shrinking X in an open neighbourhood of the generic points of T in X, one can replace X'' by an open cover and X''' by its pullback along this cover, in such a way that we have in addition the following properties:

(iii) $p(T') \subset T$ and the induced map $T' \to T$ is finite and pseudo-dominant;[43]

(iv) if we write $T'' = u^{-1}(T')$, the induced map $T'' \to T'$ is an isomorphism.

Proof. The existence of $f : Y \to X$ as in (i) follows from Gabber's weak uniformisation theorem, while the commutative diagram (14.1.16.1) satisfying property (ii) is ensured by lemma 3.3.27.

Consider moreover closed subschemes $T \subset X$ and $T' \subset X'$ satisfying (14.1.16.2).

We first show that, by shrinking X in an open neighbourhood of the generic points of T and by replacing the diagram (14.1.16.1) by its pullback over this neighbourhood, we can assume that condition (iii) is satisfied. Note that shrinking X in this way does not change the condition (14.1.16.2) because $\operatorname{codim}_X(T_0)$ does not change and $\operatorname{codim}_{X'}(T')$ can only increase.[44]

Note first that, by shrinking X, we can assume that any irreducible component T'_0 of T' dominates an irreducible component T_0 of T. In fact, given an irreducible component T'_0 which does not satisfies this condition, $p(T'_0)$ is a closed subscheme of X disjoint from the set of generic points of T and replacing X by $X - f(T'_0)$, we can throw out T'_0.

Further, shrinking X again, we can assume that for any pair (T'_0, T_0) as in the preceding paragraph, $p(T'_0) \subset T_0$. In fact, in any case, as $p(T'_0)$ is closed we get that $T_0 \subset p(T'_0)$. Let Z be the closure of $p(T'_0) - T_0$ in X. Then Z does not contain any generic point of T (because $p(T'_0)$ is irreducible), and $p(T'_0) \cap (X - Z) \subset T_0$. Thus it is sufficient to replace X by $X - Z$ to ensure this assumption.

[43] Recall from 7.1.3 that this means that any irreducible component of T' dominates an irreducible component of T.

[44] Remember that for any scheme X, $\operatorname{codim}_X(\varnothing) = +\infty$.

Consider again a pair (T'_0, T_0) as in the two preceding paragraphs and the induced commutative square:

$$\begin{array}{ccc} T'_0 & \longrightarrow & X' \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p} \\ T_0 & \longrightarrow & X \end{array} \tag{14.1.16.3}$$

We show that the map p_0 is generically finite. In fact, this will conclude the first step, because if it is true for any irreducible component T'_0 of T', we can shrink X again so that the dominant morphism $p_0 : T'_0 \to T_0$ becomes finite.

Denote by c' (resp. c) the codimension of T_0 in X (resp. T'_0 in X'). Note that (14.1.16.2) gives the inequality $c' \geq c$. Let t_0 be the generic point of T_0, Ω the localization of X at t_0 and consider the pullback of (14.1.16.3):

$$\begin{array}{ccc} W' & \longrightarrow & \Omega' \\ {\scriptstyle q_0}\downarrow & & \downarrow{\scriptstyle q} \\ \{t_0\} & \longrightarrow & \Omega. \end{array} \tag{14.1.16.4}$$

We have to prove that $\dim(W') = 0$. Consider an irreducible component Ω'_0 of Ω' containing W'. As q is still proper birational, Ω'_0 corresponds to a unique irreducible component Ω_0 of Ω such that q induces a proper birational map $\Omega'_0 \to \Omega_0$. According to [GD67, 5.6.6], we get the inequality

$$\dim(\Omega'_0) \leq \dim(\Omega_0).$$

Thus, we obtain the following inequalities:

$$\dim(W') \leq \dim(\Omega'_0) - \operatorname{codim}_{\Omega'_0}(W') \leq \dim(\Omega_0) - \operatorname{codim}_{\Omega'_0}(W') \leq \dim(\Omega) - \operatorname{codim}_{\Omega'_0}(W').$$

As this is true for any irreducible component Ω'_0 of Ω', we finally obtain:

$$\dim(W') \leq \dim(\Omega) - \operatorname{codim}_{\Omega'}(W') \leq c - c'$$

and this concludes the first step.

Keeping T' and T as above, as the map from T'' to T' is a Nisnevich covering, it is a split epimorphism in a neighbourhood of the generic points of T' in X'. Hence, as the map $X' \to X$ is proper and birational, we can find a neighbourhood of the generic points of T in X over which the map $T'' \to T'$ admits a section $s : T' \to T''$. Let S be a closed subset of X'' such that $T'' = s(T') \amalg S$ (which exists because $X'' \to X'$ is étale). The map $(X'' - T'') \amalg (X'' - S) \to X'$ is then a Nisnevich covering. Replacing X'' by $(X'' - T'') \amalg (X'' - S)$ (and X''' by the pullback of $X''' \to X''$ along $(X'' - T'') \amalg (X'' - S) \to X'$), we may assume that the induced map $T'' \to T'$ is an isomorphism, without modifying further the data f, p, T and T'. This gives property (iv) and ends the proof the lemma. □

14.1.17. Let $\mathscr{T}$ be a full *Open*-fibred subcategory of $\mathrm{Ho}(\mathscr{M})$ (where *Open* stands for the class of open immersions). We assume that $\mathscr{T}$ has the following properties.

(a) for any scheme X in $\mathscr{S}$, $\mathscr{T}(X)$ is a thick subcategory of $\mathrm{Ho}(\mathscr{M})(X)$ which contains the objects $\mathbb{1}_X\{n\}$, $n \in \tau$;
(b) for any separated morphism of finite type $f : X \to Y$ in $\mathscr{S}$, $\mathscr{T}$ is stable under $f_!$;
(c) for any dense open immersion $j : U \to X$, with X regular, which is the complement of a strict normal crossing divisor, $\mathbf{R}j_*(\mathbb{1}_U\{n\})$ is in $\mathscr{T}(U)$ for any $n \in \tau$.

Properties (a) and (b) have the following consequences: any τ-constructible object belongs to $\mathscr{T}$; given a closed immersion $i : Z \to X$ with complement open immersion $j : U \to X$, an object M of $\mathrm{Ho}(\mathscr{M})(X)$ belongs to $\mathscr{T}(X)$ is and only if $j^*(M)$ and $\mathbf{L}i^*(M)$ belongs to $\mathscr{T}(U)$ and $\mathscr{T}(Z)$ respectively; for any scheme X in $\mathscr{S}$, the condition that an object of $\mathrm{Ho}(\mathscr{M})(X)$ belongs to $\mathscr{T}(X)$ is local over X for the Zariski topology.

Theorem 14.1.18. *Under the assumptions of 14.1.17, if moreover* $\mathrm{Ho}(\mathscr{M})$ *is* $\mathbf{Q}$*-linear and separated, then, given a morphism of finite type* $f : X \to Y$*, for any* τ*-constructible object* M *of* $\mathrm{Ho}(\mathscr{M})(X)$*, the object* $\mathbf{R}f_*(M)$ *belongs to* $\mathscr{T}(Y)$*.*

Proof. It is sufficient to prove that, for any dense open immersion $j : U \to X$, and for any $n \in \tau$, the object $\mathbf{R}j_*(\mathbb{1}_U\{n\})$ is in $\mathscr{T}$. Indeed, assume this is known. We want to prove that $\mathbf{R}f_*(M)$ is in $\mathscr{T}(Y)$ whenever M is τ-constructible. We deduce from property (b) of 14.1.17 and from proposition 14.1.14 that it is sufficient to consider the case where $M = \mathbb{1}_X\{n\}$, with $n \in \tau$. Then, as this property is assumed to be known for dense open immersions, by an easy Mayer-Vietoris argument, we see that the condition that $\mathbf{R}f_*(\mathbb{1}_X\{n\})$ belongs to $\mathscr{T}$ is local on X with respect to the Zariski topology. Therefore, we may assume that f is separated. Consider a compactification of f, i.e. a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{j} & \bar{Y} \\ {\scriptstyle f}\downarrow & \swarrow_{\bar{f}} & \\ X & & \end{array}$$

with j a dense open immersion, and $\bar{f}$ proper. By property (b) of 14.1.17, we may assume that $f = j$ is a dense open immersion.

Let $j : U \to X$ be a dense open immersion. We shall prove by induction on the dimension of X that , for any $n \in \tau$, the object $\mathbf{R}j_*(\mathbb{1}_U\{n\})$ is in $\mathscr{T}$. The case where X is of dimension ≤ 0 follows from the fact the map j is then an isomorphism, which implies that $\mathbf{L}j_\sharp \simeq \mathbf{R}j_*$, and allows to conclude (because $\mathscr{T}$ is *Open*-fibred).

Assume that $\dim X > 0$. Following an argument used by Gabber [Gab05] in the context of ℓ-adic sheaves, we shall prove by induction on $c \geq 0$ that there exists a closed subscheme $T \subset X$ of codimension $> c$ such that, for any $n \in \tau$, the restriction of $\mathbf{R}j_*(\mathbb{1}_U\{n\})$ to $X - T$ is in $\mathscr{T}(X - T)$. As X is of finite dimension, this will obviously prove theorem 14.1.18.

The case where $c = 0$ is clear: we can choose T such that $X - T = U$. If $c > 0$, we choose a closed subscheme T of X, of codimension $> c - 1$, such that the restriction of $\mathbf{R}j_*(\mathbb{1}_U\{n\})$ to $X - T$ is in $\mathscr{T}$. It is then sufficient to find a dense open subscheme V of X, which contains all the generic points of T, and such that the restriction of $\mathbf{R}j_*(\mathbb{1}_U\{n\})$ to V is in $\mathscr{T}$: for such a V, we shall obtain that the restriction of $\mathbf{R}j_*(\mathbb{1}_U\{n\})$ to $V \cup (X - T)$ is in $\mathscr{T}$, the complement of $V \cup (X - T)$ being the support of a closed subscheme of codimension $> c$ in X. In particular, using the smooth base change isomorphism (for open immersions), we can always replace X by a generic neighbourhood of T. It is sufficient to prove that, possibly after shrinking X as above, the pullback of $\mathbf{R}j_*(\mathbb{1}_U\{n\})$ along $T \to X$ is in $\mathscr{T}$ (as we already know that its restriction to $X - T$ is in $\mathscr{T}$).

We may assume that T is purely of codimension c. We may assume that we have data as in points (i) and (ii) of lemma 14.1.16. We let $j' : U' \to X'$ denote the pullback of j along $p : X' \to X$. Then, we can find, by induction on c, a closed subscheme T' in X', of codimension $> c - 1$, such that the restriction of $\mathbf{R}j'_*(\mathbb{1}_{U'}\{n\})$ to $X' - T'$ is in $\mathscr{T}$. By shrinking X, we may assume that conditions (iii) and (iv) of lemma 14.1.16 are fulfilled as well.

For an X-scheme $w : W \to X$ and a closed subscheme $Z \subset W$, we shall write

$$\varphi(W, Z) = \mathbf{R}w_* \, \mathbf{R}i_* \, \mathbf{L}i^* \, \mathbf{R}j_{W,*} \, j_W^*(\mathbb{1}_W\{n\}) \, ,$$

where $i : Z \to W$ denotes the inclusion, and $j_W : W_U \to W$ stands for the pullback of j along w. This construction is functorial with respect to morphisms of pairs of X-schemes: if $W' \to W$ is a morphism of X-schemes, with Z' and Z two closed subschemes of W' and W respectively, such that Z' is sent to Z, then we get a natural map $\varphi(W, Z) \to \varphi(W', Z')$. Remember that we want to prove that $\varphi(X, T)$ is in $\mathscr{T}$. This will be done via the following lemmas (which hold assuming all the conditions stated in lemma 14.1.16 as well as our inductive assumptions).

Lemma 14.1.19. *The cone of the map* $\varphi(X, T) \to \varphi(X', T')$ *is in* $\mathscr{T}$*.*

The map $\varphi(X, T) \to \varphi(X', T')$ factors as

$$\varphi(X, T) \to \varphi(X', p^{-1}(T)) \to \varphi(X', T') \, .$$

By the octahedral axiom, it is sufficient to prove that each of these two maps has a cone in $\mathscr{T}$.

We shall prove first that the cone of the map $\varphi(X', p^{-1}(T)) \to \varphi(X', T')$ is in $\mathscr{T}$. Given an immersion $a : S \to X'$, we shall write

$$M_S = a_! \, \mathbf{L}a^*(M) \, .$$

We then have distinguished triangles

$$M_{p^{-1}(T)-T'} \to M_{p^{-1}(T)} \to M_{T'} \to M_{p^{-1}(T)-T'}[1] \, .$$

For $M = \mathbf{R}j'_*(\mathbb{1}_{U'}\{n\})$ (recall j' is the pullback of j along p) the image of this triangle by $\mathbf{R}p_*$ gives a distinguished triangle

$$\mathbf{R}p_*(M_{p^{-1}(T)-T'}) \to \varphi(X', p^{-1}(T)) \to \varphi(X', T') \to \mathbf{R}p_*(M_{p^{-1}(T)-T'})[1] \, .$$

As the restriction of $M = \mathbf{R}j'_*(\mathbb{1}_{U'}\{n\})$ to $X' - T'$ is in $\mathscr{T}$ by assumption on T', the object $M_{p^{-1}(T)-T'}$ is in $\mathscr{T}$ as well (by property (b) of 14.1.17 and because $\mathscr{T}$ is *Open*-fibred), from which we deduce that $\mathbf{R}p_*(M_{p^{-1}(T)-T'})$ is in $\mathscr{T}$ (using the condition (iii) of lemma 14.1.16 and the property (b) of 14.1.17).

Let V be a dense open subscheme of X such that $p^{-1}(V) \to V$ is an isomorphism. We may assume that $V \subset U$, and write $i : Z \to U$ for the complement closed immersion. Let $p_U : U' = p^{-1}(U) \to U$ be the pullback of p along j, and let $\bar{Z}$ be the reduced closure of Z in X. We thus get the commutative squares of immersions below,

$$\begin{array}{ccc} Z & \xrightarrow{k} & \bar{Z} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle l} \\ U & \xrightarrow[j]{} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} Z' & \xrightarrow{k'} & \bar{Z}' \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle l'} \\ U' & \xrightarrow[j']{} & X' \end{array}$$

where the square on the right is obtained from the one on the left by pulling back along $p : X' \to X$. As p is an isomorphism over V, we get by cdh-descent (proposition 3.3.9) the homotopy pullback square below.

$$\begin{array}{ccc} \mathbb{1}_U\{n\} & \longrightarrow & \mathbf{R}p_{U,*}(\mathbb{1}_{U'}\{n\}) \\ \downarrow & & \downarrow \\ \mathbf{R}i_* \, \mathbf{L}i^*(\mathbb{1}_Z\{n\}) & \longrightarrow & \mathbf{R}i_* \, \mathbf{L}i^* \, \mathbf{R}p_{U,*}(\mathbb{1}_{U'}\{n\}) \end{array}$$

If $a : T \to X$ denotes the inclusion, applying the functor $\mathbf{R}a_* \, \mathbf{L}a^* \, \mathbf{R}j_*$ to the commutative square above, we see from the proper base change formula and from the identification $\mathbf{R}j_* \, \mathbf{R}i_* \simeq \mathbf{R}l_* \, \mathbf{R}k_*$ that we get a commutative square isomorphic to the following one

$$\begin{array}{ccc} \varphi(X, T) & \longrightarrow & \varphi(X', p^{-1}(T)) \\ \downarrow & & \downarrow \\ \varphi(\bar{Z}, \bar{Z} \cap T) & \longrightarrow & \varphi(\bar{Z}', p^{-1}(\bar{Z} \cap T)) \, , \end{array}$$

which is thus homotopy cartesian as well. It is sufficient to prove that the two objects $\varphi(\bar{Z}, \bar{Z} \cap T)$ and $\varphi(\bar{Z}', p^{-1}(\bar{Z} \cap T))$ are in $\mathscr{T}$. It follows from the proper base change formula that the object $\varphi(\bar{Z}, \bar{Z} \cap T)$ is canonically isomorphic to the restriction to T of $\mathbf{R}l_* \, \mathbf{R}k_*(\mathbb{1}_Z\{n\})$. As $\dim \bar{Z} < \dim X$, we know that the object $\mathbf{R}k_*(\mathbb{1}_Z\{n\})$ is in $\mathscr{T}$. By property (b) of 14.1.17, we obtain that $\varphi(\bar{Z}, \bar{Z} \cap T)$ is in $\mathscr{T}$. Similarly, the object $\varphi(\bar{Z}', p^{-1}(\bar{Z} \cap T))$ is canonically isomorphic to the restriction of $\mathbf{R}p_* \, \mathbf{R}l'_* \, \mathbf{R}k'_*(\mathbb{1}_{Z'}\{n\})$ to T, and, as $\dim \bar{Z}' < \dim X'$ (because, p being an isomorphism over the dense open subscheme V of X, $\bar{Z}'$ does not contain any generic point of X'), $\mathbf{R}k'_*(\mathbb{1}_{Z'}\{n\})$ is in $\mathscr{T}$. We deduce again from property (b) of 14.1.17 that $\varphi(\bar{Z}', p^{-1}(\bar{Z} \cap T))$ is in $\mathscr{T}$ as well, which achieves the proof of the lemma.

Lemma 14.1.20. *The map $\varphi(X', T') \to \varphi(X'', T'')$ is an isomorphism in* $\mathrm{Ho}(\mathscr{M})(X)$.

Condition (iv) of lemma 14.1.16 can be reformulated by saying that we have the Nisnevich distinguished square below.

$$\begin{array}{ccc} X'' - T'' & \longrightarrow & X'' \\ \downarrow & & \downarrow{\scriptstyle v} \\ X' - T' & \longrightarrow & X' \end{array}$$

This lemma follows then by Nisnevich excision (proposition 3.3.4) and smooth base change (for étale maps).

Lemma 14.1.21. *Let T''' be the pullback of T'' along the finite surjective morphism $X''' \to X''$. The map $\varphi(X'', T'') \to \varphi(X''', T''')$ is a split monomorphism in* $\mathrm{Ho}(\mathscr{M})(X)$.

We have the following pullback squares

$$\begin{array}{ccccc} T''' & \xrightarrow{t} & X''' & \xleftarrow{j'''} & U''' \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q_U} \\ T'' & \xrightarrow{s} & X'' & \xleftarrow{j''} & U' \end{array}$$

in which j'' and j''' denote the pullback of j along pu and puq respectively, while s and t are the inclusions. By the proper base change formula applied to the left hand square, we see that the map $\varphi(X'', T'') \to \varphi(X''', T''')$ is isomorphic to the image of the map

$$\mathbf{R}j''_*(\mathbb{1}_{U''}\{n\}) \to \mathbf{R}q_*\,\mathbf{R}q^*\,\mathbf{R}j''_*(\mathbb{1}_{U''}\{n\}) \to \mathbf{R}q_*\,\mathbf{R}j'''_*(\mathbb{1}_{U'''}\{n\})\,.$$

by $\mathbf{R}f_*\,\mathbf{L}s^*$, where $f : T'' \to T$ is the map induced by p (note that f is proper as $T'' \simeq T'$ by assumption). As $\mathbf{R}q_*\,\mathbf{R}j'''_* \simeq \mathbf{R}j''_*\,\mathbf{R}q_{U,*}$, we are thus reduced to prove that the unit map

$$\mathbb{1}_{U''}\{n\} \to \mathbf{R}q_{U,*}(\mathbb{1}_{U'''}\{n\})$$

is a split monomorphism. As X'' is normal (because X' is so by assumption, while $X'' \to X'$ is étale), this follows immediately from corollary 3.3.39.

Now, we can finish the proof of theorem 14.1.18. Consider the Verdier quotient

$$D = \mathrm{Ho}(\mathscr{M})(X)/\mathscr{T}(X)\,.$$

We want to prove that, under the conditions stated in lemma 14.1.16, we have $\varphi(X, T) \simeq 0$ in D. Let $\pi : T''' \to T$ be the map induced by $puq : X''' \to X$. If $a : T''' \to Y$ denotes the map induced by $g : X''' \to Y$, and $j_Y : Y_U \to Y$ the pullback of j by f, we have the commutative diagram below.

$$\begin{array}{ccc} \varphi(X, T) & \longrightarrow & \varphi(X''', T''') \\ & \searrow \quad \nearrow & \\ & \mathbf{R}\pi_*\,\mathbf{L}a^*\,\mathbf{R}j_{Y,*}(\mathbb{1}_{Y_U}\{n\}) & \end{array}$$

By virtue of lemmas 14.1.19, 14.1.21, and 14.1.20, the horizontal map is a split monomorphism in D. It is thus sufficient to prove that this map vanishes in D, for which it will be sufficient to prove that $\mathbf{R}\pi_*\,\mathbf{L}a^*\,\mathbf{R}j_{Y,*}(\mathbb{1}_{Y_U}\{n\})$ is in $\mathscr{T}$. The morphism π is finite (by construction, the map $T'' \to T'$ is an isomorphism, while the maps $T''' \to T''$ and $T' \to T$ are finite). Under this condition, $\mathscr{T}$ is stable under the operations $\mathbf{R}\pi_*$ and $\mathbf{L}a^*$. To finish the proof of the theorem, it remains to check that $\mathbf{R}j_{Y,*}(\mathbb{1}_{Y_U}\{n\})$ is in $\mathscr{T}$, which follows from property (c) of 14.1.17 (and additivity). □

Definition 14.1.22. We shall say that $\mathrm{Ho}(\mathscr{M})$ is *weakly τ-pure* (or simply *weakly pure*) if it satisfies the following two conditions.

(a) For any closed immersion $i : Z \to X$ between regular schemes in $\mathscr{S}$, the image of $\mathbb{1}_X\{n\}$, $n \in \tau$, by the exceptional inverse image functor $i^! : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(Z)$ is τ-constructible.

(b) For any scheme X, any $n \in \tau$, and any τ-constructible object M in $\mathrm{Ho}(\mathscr{M})(X)$, the object $\mathbf{R}\mathrm{Hom}_X(\mathbb{1}_X\{n\}, M)$ is τ-constructible.

Remark 14.1.23. Condition (b) of the definition above will come essentially for free if the objects $\mathbb{1}_X\{n\}$ are $\otimes$-invertible with τ-constructible $\otimes$-quasi-inverse (which will hold in practice, essentially by definition).

Example 14.1.24. By virtue of theorem 13.4.1, the motivic category DM_{B} is weakly pure.

Lemma 14.1.25. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is weakly* τ*-pure. Let* $i : Z \to X$ *a closed immersion, with* X *regular, and* Z *the support of a strict normal crossing divisor. Then* $i^!(\mathbb{1}_X\{n\})$ *is* τ*-constructible for any* $n \in \tau$*. As a consequence, if* $j : U \to X$ *denotes the complement open immersion, then* $\mathbf{R}j_*(\mathbb{1}_U\{n\})$ *is* τ*-constructible for any* $n \in \tau$*.*

Proof. The first assertion follows easily by induction on the number of irreducible components of Z, using proposition 14.1.7. The second assertion follows from the distinguished triangles

$$\mathbf{R}i_*\, i^!(M) \to M \to \mathbf{R}j_*\, j^*(M) \to \mathbf{R}i_*\, i^!(M)[1]$$

and from lemma 14.1.10. □

Theorem 14.1.26. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is* $\mathbf{Q}$*-linear, separated, and weakly* τ*-pure. Then, for any morphism of finite type* $f : X \to Y$*, the functor*

$$\mathbf{R}f_* : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(Y)$$

preserves τ*-constructible objects.*

Proof. By virtue of propositions 14.1.5 and 14.1.12 as well as of lemma 14.1.25, if $\mathrm{Ho}(\mathscr{M})$ is weakly τ-pure, we can apply theorem 14.1.18, where $\mathscr{T}$ stands for the subcategory of τ-constructible objects. □

Corollary 14.1.27. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is* $\mathbf{Q}$*-linear, weakly* τ*-pure, and separated. For any scheme* X*, and for any couple of* τ*-constructible objects* M *and* N *in* $\mathrm{Ho}(\mathscr{M})(X)$*, the object* $\mathbf{R}Hom_X(M, N)$ *is* τ*-constructible.*

Proof. It is sufficient to treat the case where $M = \mathbf{L}f_\sharp(\mathbb{1}_Y\{n\})$, for $n \in \tau$ and $f : Y \to X$ a smooth morphism. But then, we have, by transposition of the Sm-projection formula, a natural isomorphism:

$$\mathbf{R}Hom_X(M, N) \simeq \mathbf{R}f_*\, \mathbf{R}Hom(\mathbb{1}_Y\{n\}, f^*(N))\,.$$

This corollary follows then immediately from proposition 14.1.5 and from theorem 14.1.26. □

Corollary 14.1.28. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is* $\mathbf{Q}$*-linear, weakly* τ*-pure, and separated. For any closed immersion* $i : Z \to X$*, the functor*

$$i^! : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(Z)$$

preserves τ*-constructible objects.*

Proof. Let $j : U \to X$ be the complement open immersion. For an object M of $\mathrm{Ho}(\mathscr{M})(X)$, we have the following distinguished triangle.

$$\mathbf{R}i_*\, i^!(M) \to M \to \mathbf{R}j_*\, j^*(M) \to \mathbf{R}i_*\, i^!(M)[1]\,.$$

By virtue of proposition 14.1.7 and theorem 14.1.26, if M is τ-constructible, then $\mathbf{R}j_*\, j^*(M)$ have the same property, which allows to conclude. □

Lemma 14.1.29. *Let* $f : X \to Y$ *be a separated morphism of finite type. The condition that the functor* $f^!$ *preserves* τ*-constructible objects in* $\mathrm{Ho}(\mathscr{M})$ *is local over* X *and over* Y *for the Zariski topology.*

Proof. If $u : X' \to X$ is a Zariski covering, then we have, by definition, $u^* = u^!$, so that, by proposition 14.1.7, the condition that $f^!$ preserves τ-constructibility is equivalent to the condition that $u^* f^! \simeq (fu)^!$ preserves τ-constructibility. Let $v : Y' \to Y$ be a Zariski covering, and consider the following pullback square.

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[v]{} & Y \end{array}$$

We then have a natural isomorphism $u^* f^! \simeq g^! v^*$, and, as u is still a Zariski covering, we deduce again from proposition 14.1.7 that, if $g^!$ preserves τ-constructibility, so does $f^!$. □

Proposition 14.1.30. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is* $\mathbf{Q}$*-linear, weakly* τ*-pure, and separated. Then, for any separated morphism of finite type* $f : X \to Y$*, the functor*

$$f^! : \mathrm{Ho}(\mathscr{M})(Y) \to \mathrm{Ho}(\mathscr{M})(X)$$

preserves τ*-constructible objects.*

Proof. By virtue of the preceding lemma, we may assume that f is affine. We can then factor f as an immersion $i : X \to \mathbf{A}^n_Y$ followed by the canonical projection $p : \mathbf{A}^n_Y \to Y$. The case of an immersion reduces to the case of an open immersion (14.1.5) and to the case of a closed immersion (14.1.28). Thus we may assume that $f = p$, in which case $p^! \simeq p^*(-)(n)[2n]$, so that we conclude by 14.1.5 and 14.1.10. □

In conclusion, we have proved:

Theorem 14.1.31. *If* $\mathrm{Ho}(\mathscr{M})$ *is* $\mathbf{Q}$*-linear, weakly* τ*-pure, and separated, then* τ*-constructible objects of* $\mathrm{Ho}(\mathscr{M})$ *are closed under the six operations of Grothendieck (induced by morphisms of finite type). In particular,* $\mathrm{Ho}(\mathscr{M})_c$ *is a motivic category.*

14.2. **Continuity.**

14.2.1. Let $\{S_\alpha\}_{\alpha\in A}$ be a pro-object in $\mathscr{S}$, with transition maps in $\mathscr{P}$, and such that $S = \varprojlim_{\alpha\in A} S_\alpha$ is representable in $\mathscr{S}$ (we assume that A is a partially ordered set to keep the notations simple). For each index α, we denote by $p_\alpha : S \to S_\alpha$ the canonical projection. Given an index $\alpha_0 \in A$ and an object E_{α_0} in $\mathscr{M}(S_{\alpha_0})$, we write E_α for the pullback of E_{α_0} along the map $S_\alpha \to S_{\alpha_0}$, and put $E = p^*_\alpha(E_\alpha)$.

For each index $\alpha \in A$, we choose a small set I_α (resp. J_α) of generating cofibrations (resp. of generating trivial cofibration) in $\mathrm{Ho}(\mathscr{M})(S_\alpha)$. We also choose a small set I (resp. J) of generating cofibrations (resp. of generating trivial cofibration) in $\mathrm{Ho}(\mathscr{M})(S)$.

Consider the following assumptions.

(a) We have $I \subset \bigcup_{\alpha\in A} p^*_\alpha(I_\alpha)$ and $J \subset \bigcup_{\alpha\in A} p^*_\alpha(J_\alpha)$.

(b) For any index α_0, if C_{α_0} and E_{α_0} are two objects of $\mathscr{M}(S_{\alpha_0})$, with C_{α_0} either a source or a target of a map in $I_{\alpha_0} \cup J_{\alpha_0}$, the natural map

$$\varinjlim_{\alpha\in A} \mathrm{Hom}_{\mathscr{M}(S_\alpha)}(C_\alpha, E_\alpha) \to \mathrm{Hom}_{\mathscr{M}(S)}(C, E)$$

is bijective.

Proposition 14.2.2. *Under the assumptions of 14.2.1, for any index* $\alpha_0 \in A$*, the pullback functor* $p^*_{\alpha_0} : \mathscr{M}(S_{\alpha_0}) \to \mathscr{M}(S)$ *preserves fibrations and trivial fibrations. Moreover, given an index* $\alpha_0 \in A$*, as well as two objects* C_{α_0} *and* E_{α_0} *in* $\mathscr{M}(S_{\alpha_0})$*, if* C_{α_0} *belongs to smallest full subcategory of* $\mathrm{Ho}(\mathscr{M})(S_{\alpha_0})$ *which is closed under finite homotopy colimits and which contains the source and targets of* I_{α_0}*, then, the canonical map*

$$\varinjlim_{\alpha\in A} \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(S_\alpha)}(C_\alpha, E_\alpha) \to \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(S)}(C, E)$$

is bijective.

Proof. We shall prove first that, for any index $\alpha_0 \in A$, the pullback functor $p_{\alpha_0}^*$ preserves fibrations and trivial fibrations. By assumption, for any $\alpha \geq \alpha_0$, the pullback functor along the $\mathscr{P}$-morphism $S_\alpha \to S_{\alpha_0}$ is both a left Quillen functor and a right Quillen functor. Let $E_{\alpha_0} \to F_{\alpha_0}$ be a trivial fibration (resp. a fibration) of $\mathscr{M}(S_{\alpha_0})$. Let $i : C \to D$ a generating cofibration (resp. a generating trivial cofibration) in $\mathscr{M}(S)$. By condition (a) of 14.2.1, we may assume that there exists $\alpha_1 \in A$, a cofibration (resp. a trivial cofibration) $i_{\alpha_1} : C_{\alpha_1} \to D_{\alpha_1}$, such that $i = p_{\alpha_1}^*(i_{\alpha_1})$. We want to prove that the map

$$\mathrm{Hom}(D, E) \to \mathrm{Hom}(C, E) \times_{\mathrm{Hom}(C,F)} \mathrm{Hom}(D, F)$$

is surjective. But, by condition (b) of 14.2.1, this map is isomorphic to the filtered colimit of the surjective maps

$$\mathrm{Hom}(D_\alpha, E_\alpha) \to \mathrm{Hom}(C_\alpha, E_\alpha) \times_{\mathrm{Hom}(C_\alpha,F_\alpha)} \mathrm{Hom}(D_\alpha, F_\alpha)$$

with $\alpha \geq \sup(\alpha_0, \alpha_1)$, which proves the first assertion. To prove the second assertions, we may assume that C_{α_0} is cofibrant and that E_{α_0} if fibrant. The set of maps from a cofibrant object to a fibrant object in the homotopy category of a model category can be described as homotopy classes of maps. Therefore, using the fact that $p_{\alpha_0}^*$ preserves cofibrations (and trivial cofibrations), fibrations and trivial cofibrations, we see it is sufficient to prove that the map

$$\varinjlim_{\alpha \in A} \mathrm{Hom}_{\mathscr{M}(S_\alpha)}(C_\alpha, E_\alpha) \to \mathrm{Hom}_{\mathscr{M}(S)}(C, E)$$

is bijective for some nice cofibrant replacement of C_{α_0}. But the assumptions on C_{α_0} imply that it is weakly equivalent to an object C'_{α_0} such that the map $\varnothing \to C'_{\alpha_0}$ belongs to the smallest class of maps in $\mathscr{M}(S_{\alpha_0})$, which contains I_{α_0}, and which is closed under pushouts and (finite) compositions. We may thus assume that $C_{\alpha_0} = C'_{\alpha_0}$. In that case, C_{α_0} is in particular contained in the smallest full subcategory of $\mathscr{M}(S_{\alpha_0})$ which is stable by finite colimits and which contains the source and targets of I_{α_0}. As filtered colimits commute with finite limits in the category of sets, we conclude by using again condition (a) of 14.2.1. □

Definition 14.2.3. We say that $\mathrm{Ho}(\mathscr{M})$ is *continuous* if, given any scheme S in $\mathscr{S}$, and any projective system of schemes $\{S_\alpha\}$ in $\mathscr{S}$, with affine transition maps, such that $S = \varprojlim_\alpha S_\alpha$, for any index α_0, and for any object E_{α_0} in $\mathrm{Ho}(\mathscr{M})(S_{\alpha_0})$, the map

$$\varinjlim_{\alpha \geq \alpha_0} \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(S_\alpha)}(\mathbb{1}_{S_\alpha}\{n\}, E_\alpha) \to \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(S)}(\mathbb{1}_S\{n\}, E)$$

is bijective for any $n \in \tau$.

The property of continuity allows to describe τ-constructible objects over S in terms of τ-constructible objects over the S_α's as follows.

Proposition 14.2.4. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is continuous. Consider a scheme* S *in* $\mathscr{S}$*, as well as a projective system of schemes* $\{S_\alpha\}$ *in* $\mathscr{S}$ *with affine transition maps, such that* $S = \varprojlim_\alpha S_\alpha$*. Then, for any index* α_0*, and for any objects* C_{α_0} *and* E_{α_0} *in* $\mathrm{Ho}(\mathscr{M})(S_{\alpha_0})$*, if* C_{α_0} *is* τ*-constructible, then the canonical map*

$$\varinjlim_{\alpha \geq \alpha_0} \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(S_\alpha)}(C_\alpha, E_\alpha) \to \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(S)}(C, E) \tag{14.2.4.1}$$

is bijective. Moreover, the canonical functor

$$2\text{-}\varinjlim_\alpha \mathrm{Ho}(\mathscr{M})_c(S_\alpha) \to \mathrm{Ho}(\mathscr{M})_c(S) \tag{14.2.4.2}$$

is an equivalence of monoidal triangulated categories.

Proof. To prove the first assertion, we may assume, without loss of generality, that $C_{\alpha_0} = M_{S_{\alpha_0}}(X_{\alpha_0})\{n\}$ for some some smooth S_{α_0}-scheme of finite type X_{α_0}, and $n \in \tau$. Consider an object E_{α_0} in $\mathrm{Ho}(\mathscr{M})(S_{\alpha_0})$. For $\alpha \geq \alpha_0$, write X_α (resp. E_α) for the pullback of X_{α_0} (resp. of E_{α_0}) along the map $S_\alpha \to S_{\alpha_0}$. Similarly, write X (resp. E) for the pullback of X_{α_0} (resp. of E_{α_0}) along the map $S \to S_{\alpha_0}$. We shall also write E'_α (resp. E') for the pullback of E_α (resp. E) along the smooth map $X_\alpha \to S_\alpha$ (resp. $X \to S$). Then, $\{X_\alpha\}$ is a projective system of schemes

in $\mathscr{S}$, with affine transition maps, such that $X = \varprojlim_\alpha X_\alpha$. Therefore, by continuity, we have the following natural isomorphism, which proves the first assertion.

$$\begin{aligned}\varinjlim_\alpha \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(S_\alpha)}(M_{S_\alpha}(X_\alpha)\{n\}, E_\alpha) &\simeq \varinjlim_\alpha \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(X_\alpha)}(\mathbb{1}_{X_\alpha}\{n\}, E'_\alpha)\\ &\simeq \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(X)}(\mathbb{1}_X\{n\}, E')\\ &\simeq \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(S)}(M_S(X)\{n\}, E)\end{aligned}$$

Note that the first assertion implies that the functor (14.2.4.2) is fully faithful. Note that pseudo-abelian triangulated categories are stable by filtered 2-colimits. In particular, the source of the functor (14.2.4.2) can be seen as a thick subcategory of $\mathrm{Ho}(\mathscr{M})(S)$. The essential surjectivity of (14.2.4.2) follows from the fact that, for any smooth S-scheme of finite type X, there exists some index α, and some smooth S_α-scheme X_α, such that $X \simeq S \times_{S_\alpha} X_\alpha$; see [GD67, 8.8.2 and 17.7.8]: this implies that the essential image of the fully faithful functor (14.2.4.2) contains all the objects of shape $M_S(X)\{n\}$ for $n \in \tau$ and X smooth over S, so that it contains $\mathrm{Ho}(\mathscr{M})_c(S)$, by definition. □

An example:

Theorem 14.2.5. *The motivic categories* $\mathrm{D}_{\mathbf{A}^1,\Lambda}$ *and* DM_Γ *are continuous.*

Proof. It is easy to check that, if $\mathrm{D}_{\mathbf{A}^1,\Lambda}$ is continuous (for $\Lambda = \mathbf{Q}$), then $\mathrm{Ho}(H_\Gamma\text{-}\mathrm{mod}) \simeq \mathrm{DM}_\Gamma$ is continuous as well. Hence it is sufficient to prove that $\mathrm{D}_{\mathbf{A}^1,\Lambda}$ is continuous. We have fully faithul functors

$$\mathrm{D}_{\mathbf{A}^1,\Lambda}(S) \to \underline{\mathrm{D}}_{\mathbf{A}^1,\Lambda}(S)$$

(see 6.1.9), so that is is sufficient to prove that $\underline{\mathrm{D}}_{\mathbf{A}^1,\Lambda}$ is continuous. Using proposition 5.3.30, it is even sufficient to prove that $\underline{\mathrm{D}}^{\mathit{eff}}_{\mathbf{A}^1,\Lambda}$ is continuous. Let $\underline{\mathrm{Sh}}_{\mathrm{Nis}}(S,\Lambda)$ be the abelian category of Nisnevich sheaves of Λ-modules over the category of affine S-schemes of finite type. We can see $\underline{\mathrm{D}}^{\mathit{eff}}_{\mathbf{A}^1,\Lambda}(S)$ as the full subcategory of the derived category of $\mathrm{D}(\underline{\mathrm{Sh}}_{\mathrm{Nis}}(S,\Lambda))$ spanned by $\mathbf{A}^1$-local complexes. As, for any morphism of schemes $f : T \to S$, the pullback functor

$$f^* : \mathrm{D}(\underline{\mathrm{Sh}}_{\mathrm{Nis}}(S,\Lambda)) \to \mathrm{D}(\underline{\mathrm{Sh}}_{\mathrm{Nis}}(S,\Lambda))$$

preserves $\mathbf{A}^1$-local complexes, it is sufficient to prove the property of continuity, as stated in definition 14.2.3, for the derived categories of Nisnevich sheaves. By Zariski descent, it is sufficient to prove the property stated in definition 14.2.3 in the case where S is affine. Consider the descent structure $(\mathcal{G},\mathcal{H})$ on the category $\underline{\mathrm{Sh}}_{\mathrm{Nis}}(S,\Lambda)$ of Nisnevich sheaves over the category of affine S-schemes of finite type (in the sense of [CD09, definition 2.2]), where $\mathcal{G}$ is the generating family made of objects of shape $\Lambda_S(X)$, for X affine of finite type over S, while $\mathcal{H}$ is made of the complexes of shape

$$\Lambda_S(W) \xrightarrow{g_*-l_*} \Lambda_S(U) \oplus \Lambda_S(V) \xrightarrow{j_*+f_*} \Lambda_S(X)$$

for any Nisnevich distinguished square made of affine S-schemes of finite type (*cf.* 2.1.15)

$$\begin{array}{ccc} W & \xrightarrow{l} & V \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & X. \end{array}$$

This descent structure defines a model category structure on the category $\mathrm{C}(\underline{\mathrm{Sh}}_{\mathrm{Nis}}(S,\Lambda))$, whose weak equivalences are the quasi-isomorphisms; see [CD09, 2.5]. The explicit description of the generating sets of cofibrations and of trivial cofibrations given by [CD09, 2.2 and 2.3] allow then to apply directly proposition 14.2.2, thanks to [GD67, 17.7.9]. □

Lemma 14.2.6. *Let $a : X \to Y$ be a morphism in $\mathscr{S}$. Assume that $X = \varprojlim_\alpha X_\alpha$, where $\{X_\alpha\}$ is a projective system of smooth affine Y-schemes. If* $\mathrm{Ho}(\mathscr{M})$ *is continuous, then, for any objects E and F in* $\mathrm{Ho}(\mathscr{M})(Y)$*, with E τ-constructible, there is a canonical isomorphism*

$$\mathbf{L}a^*\,\mathbf{R}\mathit{Hom}_Y(E,F) \simeq \mathbf{R}\mathit{Hom}_X(\mathbf{L}a^*(E), \mathbf{L}a^*(F))\,.$$

Proof. We have

$$\mathbf{R}a_* \, \mathbf{R}Hom_X(\mathbf{L}a^*(E), \mathbf{L}a^*(F)) \simeq \mathbf{R}Hom_Y(E, \mathbf{R}a_* \, \mathbf{L}a^*(F)) \, ,$$

so that the map $F \to \mathbf{R}a_* \, \mathbf{L}a^*(F))$ induces a map

$$\mathbf{R}Hom_Y(E, F) \to \mathbf{R}a_* \, \mathbf{R}Hom_X(\mathbf{L}a^*(E), \mathbf{L}a^*(F)) \, ,$$

hence, by adjunction, a map

$$\mathbf{L}a^* \, \mathbf{R}Hom_Y(E, F) \to \mathbf{R}Hom_X(\mathbf{L}a^*(E), \mathbf{L}a^*(F)) \, .$$

We already know that the later is an isomorphism whenever a is smooth.

Let us write $a_\alpha : X_\alpha \to Y$ for the structural maps. Let C be a τ-constructible object in $\mathrm{Ho}(\mathscr{M})(X)$. By proposition 14.2.4, we may assume that there exists an index α_0, and a τ-constructible object C_{α_0} in $\mathrm{Ho}(\mathscr{M})(X_{\alpha_0})$, such that, if we write C_α for the pullback of C_{α_0} along the map $X_\alpha \to X_{\alpha_0}$ for $\alpha \geq \alpha_0$, we have isomorphisms:

$$\begin{aligned}
\mathrm{Hom}(C, \mathbf{L}a^* \, \mathbf{R}Hom_Y(E, F)) &\simeq \varinjlim_\alpha \mathrm{Hom}(C_\alpha, \mathbf{L}a_\alpha^* \, \mathbf{R}Hom_Y(E, F)) \\
&\simeq \varinjlim_\alpha \mathrm{Hom}(C_\alpha, \mathbf{R}Hom_X(\mathbf{L}a_\alpha^*(E), \mathbf{L}a_\alpha^*(F))) \\
&\simeq \varinjlim_\alpha \mathrm{Hom}(C_\alpha \otimes^{\mathbf{L}}_{X_\alpha} \mathbf{L}a_\alpha^*(E), \mathbf{L}a_\alpha^*(F)) \\
&\simeq \mathrm{Hom}(C \otimes^{\mathbf{L}}_X \mathbf{L}a^*(E), \mathbf{L}a^*(F)) \\
&\simeq \mathrm{Hom}(C, \mathbf{R}Hom_X(\mathbf{L}a^*(E), \mathbf{L}a^*(F))) \, .
\end{aligned}$$

As τ-constructible objects generate $\mathrm{Ho}(\mathscr{M})(X)$, this proves the lemma. □

14.2.7. Let X be a scheme in $\mathscr{S}$. Assume that, for any point x of X, the corresponding morphism $i_x : \mathrm{Spec}\left(\mathscr{O}^h_{X,x}\right) \to X$ is in $\mathscr{S}$ (where $\mathscr{O}^h_{X,x}$ denotes the henselisation of $\mathscr{O}_{X,x}$). Consider at last a scheme of finite type Y over X, and write

$$a_x : Y_x = \mathrm{Spec}\left(\mathscr{O}^h_{X,x}\right) \times_X Y \to Y$$

for the morphism obtained by pullback. Finally, for an object E of $\mathrm{Ho}(\mathscr{M})(Y)$, let us write

$$E_x = \mathbf{L}a_x^*(E) \, .$$

Proposition 14.2.8. *Under the assumptions of 14.2.7, if moreover* $\mathrm{Ho}(\mathscr{M})$ *is continuous, then, the family of functors*

$$\mathrm{Ho}(\mathscr{M})(Y) \to \mathrm{Ho}(\mathscr{M})(Y_x) \ , \quad E \longmapsto E_x \ , \quad x \in X \, ,$$

is conservative.

Proof. Let E be an object of $\mathrm{Ho}(\mathscr{M})(Y)$ such that $E_x \simeq 0$ for any point x of X. For any τ-constructible object C of $\mathrm{Ho}(\mathscr{M})(Y)$, we have a presheaf of S^1-spectra on the small Nisnevich site of X:

$$F : U \longmapsto F(U) = \mathbf{R}\mathrm{Hom}(M_Y(U \times_X Y), \mathbf{R}Hom_Y(C, E)) \, .$$

It is sufficient to prove that $F(X)$ is acyclic. As $\mathrm{Ho}(\mathscr{M})$ satisfies Nisnevich descent (3.3.4), it is sufficient to prove that F is acyclic locally for the Nisnevich topology, i.e. that, for any point x of X, the spectrum $F(\mathrm{Spec}\left(\mathscr{O}^h_{X,x}\right))$ is acyclic. Writing $\mathrm{Spec}\left(\mathscr{O}^h_{X,x}\right)$ as the projective limit of the Nisnevich neighbourhoods of x in X, we see easily, using proposition 14.2.4 and lemma 14.2.6, that, for any integer i, $\pi_i(F(\mathrm{Spec}\left(\mathscr{O}^h_{X,x}\right)) \simeq \mathrm{Hom}(C_x, E_x[i]) \simeq 0$. □

Recall that a morphism of rings $u : A \to B$ is *regular* if it is flat, and if, for any prime ideal $\mathfrak{p}$ in A, with residue field $\kappa(\mathfrak{p})$, the $\kappa(\mathfrak{p})$-algebra $\kappa(\mathfrak{p}) \otimes_A B$ is geometrically regular (equivalently, this means that, for any prime ideal $\mathfrak{q}$ of B, the A-algebra $B_\mathfrak{q}$ is formally smooth for the $\mathfrak{q}$-adic topology). We recall the following great generalization of Neron's desingularisation theorem:

Theorem 14.2.9 (Popescu)**.** *A morphism of noetherian rings $u : A \to B$ is regular if and only if B is a filtered colimit of smooth A-algebras of finite type.*

Proof. See [Pop85, Spi99]. □

Proposition 14.2.10. *Let S be an excellent noetherian and henselian scheme. Write $\hat{S}$ for its completion along its closed point, and assume that both S and $\hat{S}$ are in $\mathscr{S}$. Consider an S-scheme of finite type X, and write $i : \hat{S} \times_S X \to X$ for the induced map. If $\mathrm{Ho}(\mathscr{M})$ is continuous, then the pullback functor*

$$\mathbf{L}i^* : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(\hat{S} \times_S X)$$

is conservative.

Proof. As S is excellent, the map $\hat{S} \to S$ is regular. By Popescu's theorem, we can then write $\hat{S} = \varprojlim_\alpha S_\alpha$, where $\{S_\alpha\}$ is a projective system of schemes with affine transition maps, and such that each scheme S_α is smooth over S. Moreover, as $\hat{S}$ and S have the same residue field, and as S is henselian, each map S_α has a section. Write $X_\alpha = S_\alpha \times_S X$, so that we have $X = \varprojlim_\alpha X_\alpha$. Consider a τ-constructible object C and an object E in $\mathrm{Ho}(\mathscr{M})(X)$. Then, as the maps $X_\alpha \to X$ have sections, it follows from the first assertion of proposition 14.2.4 that the map

$$\mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(X)}(C, E) \to \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(\hat{S} \times_S X)}(\mathbf{L}i^*(C), \mathbf{L}i^*(E))$$

is a monomorphism (as a filtered colimit of such things). Hence, if $\mathbf{L}i^*(E) \simeq 0$, for any τ-constructible object C in $\mathrm{Ho}(\mathscr{M})(X)$, we have $\mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(X)}(C, E) \simeq 0$. Therefore, as τ-constructible objects generate $\mathrm{Ho}(\mathscr{M})(X)$, we get $E \simeq 0$. □

Proposition 14.2.11. *Let $a : X \to Y$ be a regular morphism in $\mathscr{S}$. If $\mathrm{Ho}(\mathscr{M})$ is continuous, then, for any objects E and F in $\mathrm{Ho}(\mathscr{M})(Y)$, with E τ-constructible, there is a canonical isomorphism*

$$\mathbf{L}a^* \, \mathbf{R}Hom_Y(E, F) \simeq \mathbf{R}Hom_X(\mathbf{L}a^*(E), \mathbf{L}a^*(F)) \, .$$

Proof. We want to prove that the canonical map

$$\mathbf{L}a^* \, \mathbf{R}Hom_Y(E, F) \to \mathbf{R}Hom_X(\mathbf{L}a^*(E), \mathbf{L}a^*(F))$$

is an isomorphism, while we already know it is so whenever a is smooth. Therefore, to prove the general case, we see that the problem is local on X and on Y with respect to the Zariski topology. In particular, we may assume that both X and Y are affine. By Pospecu's theorem, we thus have $X = \varprojlim_\alpha X_\alpha$, where $\{X_\alpha\}$ is a projective system of smooth affine Y-schemes. We conclude by lemma 14.2.6. □

Proposition 14.2.12. *Consider the following pullback square in $\mathscr{S}$.*

$$\begin{array}{ccc} X' & \xrightarrow{a} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[b]{} & Y \end{array}$$

Assume that f is separated of finite type and that b is regular. Then, if $\mathrm{Ho}(\mathscr{M})$ is continuous, for any object E in $\mathrm{Ho}(\mathscr{M})(Y)$, there is a canonical isomorphism in $\mathrm{Ho}(\mathscr{M})(X')$:

$$\mathbf{L}a^* \, f^!(E) \simeq g^! \, \mathbf{L}b^*(E) \, .$$

Proof. We have a canonical map

$$f^!(E) \to \mathbf{R}a_* \, g^! \, \mathbf{L}b^*(E) \simeq f^! \, \mathbf{R}b_* \, \mathbf{L}b^*(E) \, ,$$

which gives, by adjunction, a natural morphism

$$\mathbf{L}a^* \, f^!(E) \to g^! \, \mathbf{L}b^*(E) \, .$$

The latter is invertible whenever b is smooth: this is obvious in the case of an open immersion, so that, by Zariski descent, it is sufficient to treat the case where b is smooth with trivial cotangent bundle of rank d; in this case, by relative purity (2.4.21 (iii)), this reduces to the canonical isomorphism $a^! f^! \simeq g^! b^!$ evaluated at $E(-d)[-2d]$. To prove the general case, as the condition is local on X and on Y for the Zariski topology, we may assume that f factors as an immersion $X \to \mathbf{P}^n_Y$, followed by the canonical projection $\mathbf{P}^n_Y \to Y$. We deduce from there that it is sufficient to treat the case where f is either a closed immersion, either a smooth morphism of finite type.

The case where f (hence also g) is smooth follows by relative purity (2.4.21): we can then replace $f^!$ and $g^!$ by f^* and g^* respectively, and the formula follows from the fact that $\mathbf{L}a^* f^* \simeq g^* \mathbf{L}b^*$. We may thus assume that f is a closed immersion. As g is a closed immersion as well, the functor $g_!$ is conservative (it is fully faithful). Therefore, it is sufficient to prove that the map

$$\mathbf{L}b^*\, f_!\, f^!(E) \simeq g_!\, \mathbf{L}a^*\, f^!(E) \to g_!\, g^!\, \mathbf{L}b^*(E)$$

is invertible. Then, using proposition 14.2.11 (which makes sense because $f_!$ preserves τ-constructibility by 14.1.12), and the projection formula, we have

$$\begin{aligned}
\mathbf{L}b^*\, f_!\, f^!(E) &\simeq \mathbf{L}b^* \mathbf{R}Hom_Y(f_!(\mathbb{1}_X), E) \\
&\simeq \mathbf{R}Hom_{Y'}(\mathbf{L}b^*\, f_!(\mathbb{1}_X), \mathbf{L}b^*(E)) \\
&\simeq \mathbf{R}Hom_{Y'}(g_!(\mathbb{1}_{X'}), \mathbf{L}b^*(E)) \\
&\simeq g_!\, g^!\, \mathbf{L}b^*(E)\,,
\end{aligned}$$

which achieves the proof. □

Lemma 14.2.13. *Let $f : X \to Y$ be a morphism in $\mathscr{S}$. Assume that $X = \varprojlim_\alpha X_\alpha$ and $Y = \varprojlim_\alpha Y_\alpha$, where $\{X_\alpha\}$ and $\{Y_\alpha\}$ are projective systems fo schemes with affine transition maps, while f is induced by a system of morphisms $f_\alpha : X_\alpha \to Y_\alpha$. Let α_0 be some index, C_{α_0} a τ-constructible object of $\mathrm{Ho}(\mathscr{M})(Y_{\alpha_0})$, and E_{α_0} an object of $\mathrm{Ho}(\mathscr{M})(X_{\alpha_0})$. If $\mathrm{Ho}(\mathscr{M})$ is continuous, then we have a natural isomorphism of abelian groups*

$$\varinjlim_{\alpha \geq \alpha_0} \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(Y_\alpha)}(C_\alpha, \mathbf{R}f_{\alpha,*}(E_\alpha)) \simeq \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(Y)}(C, \mathbf{R}f_*(E))\,.$$

Proof. By virtue of proposition 14.2.4, we have a natural isomorphism

$$\varinjlim_{\alpha \geq \alpha_0} \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(X_\alpha)}(\mathbf{L}f_\alpha^*(C_\alpha), E_\alpha) \simeq \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(Y)}(\mathbf{L}f^*(C), E)\,.$$

The expected formula follows by adjunction. □

Proposition 14.2.14. *Consider the following pullback square in $\mathscr{S}$.*

$$\begin{array}{ccc}
X' & \xrightarrow{a} & X \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\
Y' & \xrightarrow[b]{} & Y
\end{array}$$

with b regular. If $\mathrm{Ho}(\mathscr{M})$ is continuous, then, for any object E in $\mathrm{Ho}(\mathscr{M})(X)$, there is a canonical isomorphism in $\mathrm{Ho}(\mathscr{M})(Y')$:

$$\mathbf{L}b^*\, \mathbf{R}f_*(E) \simeq \mathbf{R}g_*\, \mathbf{L}a^*(E)\,.$$

Proof. This proposition is true in the case where b is smooth (by definition of Sm-fibred categories), from which we deduce, by Zariski separation, that this property is local on Y and on Y' for the Zariski topology. In particular, we may assume that both Y and Y' are affine. Then, by Popescu's theorem 14.2.9, we may assume that $Y' = \varprojlim_\alpha Y'_\alpha$, where $\{Y'_\alpha\}$ is a projective system of smooth Y-algebras. Then, using the preceding lemma as well as proposition 14.2.4, we reduce easily the proposition to the case where b is smooth. □

Proposition 14.2.15. *Assume that $\mathrm{Ho}(\mathscr{M})$ is continuous, $\mathbf{Q}$-linear and semi-separated, and consider a field k, with inseparable closure k', such that both $\mathrm{Spec}\,(k)$ and $\mathrm{Spec}\,(k')$ are in $\mathscr{S}$. Given a k-scheme X write $X' = k' \otimes_k X$, and $f : X' \to X$ for the canonical projection. Then the functor*

$$\mathbf{L}f^* : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(X')$$

is an equivalence of categories.

Proof. It follows immediately from proposition 14.2.4 and from proposition 2.1.13 that the functor

$$\mathbf{L}f^* : \mathrm{Ho}(\mathscr{M})_c(X) \to \mathrm{Ho}(\mathscr{M})_c(X')$$

is an equivalence of categories. Similarly, for any objects C and E in $\mathrm{Ho}(\mathscr{M})(X)$, if C is τ-constructible, the map

$$\mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(X)}(C, E) \to \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(X)}(\mathbf{L}f^*(C), \mathbf{L}f^*(E))$$

is bijective. As τ-constructible objects generate $\mathrm{Ho}(\mathscr{M})(X)$, this implies that the functor

$$\mathbf{L}f^* : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(X')$$

is fully faithful. As the latter is essentially surjective on a set of generators, this implies that it is an equivalence of categories (see 1.3.19). □

Here is a slightly more general version of proposition 13.3.1.

Proposition 14.2.16. *The motivic category* DM_{B} *is separated on noetherian schemes of finite dimension.*

Proof. As in the proof of 13.3.1, it is sufficient to prove that, given a finite surjective morphism $f : T \to S$, the pullback functor $f^* : \mathrm{DM}_{\mathrm{B}}(S) \to \mathrm{DM}_{\mathrm{B}}(T)$ is conservative. By virtue of proposition 14.2.8, we may assume that S is henselian. Using the localization property, we may even assume (by induction on the dimension) that S is the spectrum of a field. Replacing T by its reduction, we may thus assume that both S and T are regular. We can then conclude by a trace argument, as in the proof of proposition 13.3.1. □

Corollary 14.2.17. *The motivic category* DM_{B} *satisfies étale descent on noetherian schemes of finite dimension.*

Proof. This follows from the preceding proposition and from theorem 3.3.31. □

14.3. **Duality.** The aim of this section is to prove a local duality theorem in $\mathrm{Ho}(\mathscr{M})$ (see 14.3.28 and 14.3.31).

14.3.1. Recall that an *alteration* is a proper surjective morphism $p : X' \to X$ which is generically finite, i.e. such that there exists a dense open subscheme $U \subset X$ over which p is finite.

Definition 14.3.2 (de Jong)**.** Let X be a noetherian scheme endowed with an action of a finite group G. A *Galois alteration* of the couple (X, G) is the data of a finite group G', of a surjective morphism of groups $G' \to G$, of an alteration $X' \to X$, and of an action of G' on X', such that:

(i) the map $X' \to X$ is G'-equivariant;
(ii) for any irreducible component T of X, there exists a unique irreducible component T' of X' over T, and the corresponding finite field extension

$$k(T)^G \subset k(T')^{G'}$$

is purely inseparable.

In practice, we shall keep the morphism of groups $G' \to G$ implicit, and we shall say that $(X' \to X, G')$ is a Galois alteration of (X, G).

Given a noetherian scheme X, a *Galois alteration* of X is a Galois alteration $(X' \to X, G)$ of (X, e), where e denotes the trivial group. In this case, we shall say that $X' \to X$ is a *Galois alteration of X of group G.*

Remark 14.3.3. If $p : X' \to X$ is a Galois alteration of group G over X, then, if X and X' are normal, irreducible and quasi-excellent, p can be factored as a radicial finite surjective morphism $X'' \to X$, followed by a Galois alteration $X' \to X''$ of group G, such that $k(X'') = k(X')^G$ (just define X'' as the normalization of X in $k(X')^G$). In other words, up to a radicial finite surjective morphism, X is generically the quotient of X' under the action of G.

Definition 14.3.4. A noetherian scheme S *admits canonical dominant resolution of singularities up to quotient singularities* if, for any Galois alteration $S' \to S$ of group G, and for any G-equivariant nowhere dense closed subscheme $Z' \subset S'$, there exists a Galois alteration $(p : S'' \to S', G')$ of (S', G), such that S'' is regular and projective over S, and such that the inverse image of Z' in S'' is contained in a G'-equivariant strict normal crossing divisor (i.e. a strict normal crossing divisor whose irreducible components are stable under the action of G').

A noetherian scheme S admits *admits canonical resolution of singularities up to quotient singularities* if any integral closed subscheme of S admits canonical dominant resolution of singularities up to quotient singularities.

A noetherian scheme S admits *wide resolution of singularities up to quotient singularities* if, for any separated S-scheme of finite type X, and any nowhere dense closed subscheme $Z \subset X$, there exists a projective Galois alteration $p : X' \to X$ of group G, with X' regular, such that, in each connected component of X', $Z' = p^{-1}(Z)$ is either empty, either the support of a strict normal crossing divisor.

Theorem 14.3.5 (de Jong)**.** *If an excellent noetherian scheme of finite dimension S admits canonical resolution of singularities up to quotient singularities, then any separated S-scheme of finite type admits canonical resolution of singularities up to quotient singularities.*

Proof. Let X be a integral separated S-scheme of finite type. There exists a finite morphism $S' \to S$, with S' integral, an integral dominant S'-scheme X' and a radicial extension $X' \to X$ over S, such that X' has a geometrically irreducible generic fiber over S'. It follows then from (the proof of) [dJ97, theorem 5.13] that X' admits canonical dominant resolution of singularities up to quotient singularities, which implies that X has the same property. □

Corollary 14.3.6 (de Jong)**.** *Let S be an excellent noetherian scheme of dimension lesser or equal to 2. Then any separated scheme of finite type over S admits canonical resolution of singularities up to quotient singularities. In particular, S admits wide resolution of singularities up to quotient singularities.*

Proof. See [dJ97, corollary 5.15]. □

If we work with rational coefficients, resolution of singularities up to quotient singularities is almost as good as classical resolution of singularities: we have the following replacement of the blow-up formula.

Theorem 14.3.7. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is* $\mathbf{Q}$*-linear and separated. Let X be a scheme in $\mathscr{S}$. Consider a Galois alteration $p : X' \to X$ of group G, as well as a closed subscheme $Z \subset X$, such that $U = X - Z$ is normal, and such that the induced map $p_U : U' = p^{-1}(U) \to U$ is a finite morphism. Then the pullback square*

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & X' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{i} & X \end{array} \tag{14.3.7.1}$$

induces an homotopy pullback square

$$\begin{array}{ccc} M & \longrightarrow & (\mathbf{R}p_*\,\mathbf{L}p^*(M))^G \\ \downarrow & & \downarrow \\ \mathbf{R}i_*\,\mathbf{L}i^*(M) & \longrightarrow & (\mathbf{R}i_*\mathbf{R}q_*\,\mathbf{L}q^*\,\mathbf{L}i^*(M))^G \end{array} \tag{14.3.7.2}$$

for any object M of $\mathrm{Ho}(\mathscr{M})(X)$.

Proof. We already know that, for any object N of $\mathrm{Ho}(\mathscr{M})(U)$, the map

$$N \to (\mathbf{R}p_{U,*}\,\mathbf{L}p_U^*(N))^G$$

is an isomorphism (3.3.38). The proof is then similar to the proof of condition (iv) of theorem 3.3.36. □

Remark 14.3.8. Under the assumptions of the preceding theorem, applying the total derived functor $\mathbf{R}\mathrm{Hom}_X(-,E)$ to the homotopy pullback square (14.3.7.2) for $M=\mathbb{1}_X$, we obtain the homotopy pushout square

$$\begin{array}{ccc} (i_!\,q_!\,q^!\,i^!(E))_G & \longrightarrow & (p_!\,p^!(E))_G \\ \downarrow & & \downarrow \\ i_!\,i^!(E) & \longrightarrow & E \end{array} \tag{14.3.8.1}$$

for any object E of $\mathrm{Ho}(\mathscr{M})(X)$.

Corollary 14.3.9. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is* $\mathbf{Q}$*-linear and separated. Let* B *be a scheme in* $\mathscr{S}$*, admitting wide resolution of singularities up to quotient singularities. Consider a separated* B*-scheme of finite type* S*, endowed with a closed subscheme* $T\subset S$*. The category of* τ*-constructible objects in* $\mathrm{Ho}(\mathscr{M})(S)$ *is the smallest thick triangulated subcategory which contains the objects of shape* $\mathbf{R}f_*(\mathbb{1}_X\{n\})$ *for* $n\in\tau$*, and for* $f:X\to S$ *a projective morphism, with* X *regular and connected, such that* $f^{-1}(T)_{red}$ *is either empty, either* X *itself, either the support of a strict normal crossing divisor.*

Proof. Let $\mathrm{Ho}(\mathscr{M})(S)'$ be the smallest thick triangulated subcategory of $\mathrm{Ho}(\mathscr{M})(S)$ which contains the objects of shape $\mathbf{R}f_*(\mathbb{1}_X\{n\})$ for $n\in\tau$ and $f:X\to S$ a projective morphism with X regular and connected, while $f^{-1}(T)_{red}$ is empty, or X itself, or the support of a strict normal crossing divisor. We clearly have $\mathrm{Ho}(\mathscr{M})(S)'\subset\mathrm{Ho}(\mathscr{M})_c(S)$ (14.1.12). To prove the reverse inclusion, by virtue of 14.1.14, it is sufficient to prove that, for any $n\in\tau$, and any projective morphism $f:X\to S$, the object $\mathbf{R}f_*(\mathbb{1}_X\{n\})$ belongs to $\mathrm{Ho}(\mathscr{M})(S)'$. We shall proceed by induction on the dimension of X. If X is of dimension ≤ 0, we may replace it by its reduction, which is regular. If X is of dimension >0, by assumption on B, there exists a Galois alteration $p:X'\to X$ of group G, with X' regular and projective over S (and in which T becomes either empty, either X' itself, either the support of a strict normal crossing divisor, in each connected component of X'). Choose a closed subscheme $Z\subset X$, such that $U=X-Z$ is a normal dense open subscheme, and such that the induced map $r:U'=p^{-1}(U)\to U$ is a finite morphism, and consider the pullback square (14.3.7.1). As Z and $Z'=p^{-1}(Z)$ are of dimension smaller than the dimension of X, we conclude from the homotopy pullback square obtained by appyling the functor $\mathbf{R}f_*$ to (14.3.7.2) for $M=\mathbb{1}_X\{n\}$, $n\in\tau$. □

Definition 14.3.10. Let S be a scheme in $\mathscr{S}$. An object R of $\mathrm{Ho}(\mathscr{M})(S)$ is *τ-dualizing* if it satisfies the following conditions.

(i) The object R is τ-constructible.
(ii) For any τ-constructible object M of $\mathrm{Ho}(\mathscr{M})(S)$, the natural map

$$M\to\mathbf{R}\mathit{Hom}_S(\mathbf{R}\mathit{Hom}_S(M,R),R)$$

is an isomorphism.

Remark 14.3.11. If $\mathrm{Ho}(\mathscr{M})$ is weakly τ-pure, $\mathbf{Q}$-linear and separated, then, in particular, the six operations of Grothendieck preserve τ-constructibility in $\mathrm{Ho}(\mathscr{M})$ (14.1.31). Under this assumtion, for any scheme X in $\mathscr{S}$, and any $\otimes$-invertible object U in $\mathrm{Ho}(\mathscr{M})(X)$ which is τ-constructible, its quasi-inverse is τ-constructible: the quasi-inverse of U is simply its dual $U^\wedge=\mathbf{R}\mathit{Hom}(U,\mathbb{1}_X)$, which is τ-constructible by virtue of 14.1.27.

Proposition 14.3.12. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is weakly* τ*-pure,* $\mathbf{Q}$*-linear and separated, and consider a scheme* X *in* $\mathscr{S}$*.*

(i) Let R *be a* τ*-dualizing object, and* U *be a* τ*-constructible* $\otimes$*-invertible object in* $\mathrm{Ho}(\mathscr{M})(X)$*. Then* $U\otimes^{\mathbf{L}}_S R$ *is* τ*-dualizing.*

(ii) Let R and R' be two τ-dualizing objects in $\mathrm{Ho}(\mathscr{M})(X)$*. Then the evaluation map*

$$\mathbf{R}Hom_S(R,R')\otimes^{\mathbf{L}}_S R\to R'$$

is an isomorphism.

Proof. This follows immediately from [Ayo07a, 2.1.139]. □

Proposition 14.3.13. *Consider an open immersion $j:U\to X$ in $\mathscr{S}$. If R is a τ-dualizing object in* $\mathrm{Ho}(\mathscr{M})(X)$*, then $j^!(R)$ is τ-dualizing in* $\mathrm{Ho}(\mathscr{M})(U)$*.*

Proof. If M is a τ-constructible object in $\mathrm{Ho}(\mathscr{M})(U)$, then $j_!(M)$ is τ-constructible, and the map

$$j_!(M)\to\mathbf{R}Hom_X(\mathbf{R}Hom_X(j_!(M),R),R) \tag{14.3.13.1}$$

is an isomorphism. Using the isomorphisms of type

$$M\simeq j^*\,j_!(M)=j^!\,j_!(M)\quad\text{and}\quad j^*\mathbf{R}Hom_X(A,B)\simeq\mathbf{R}Hom_U(j^*(A),j^*(B))\,,$$

we see that the image of the map (14.3.13.1) by the functor $j^*=j^!$ is isomorphic to the map

$$M\to\mathbf{R}Hom_U(\mathbf{R}Hom_U(M,j^!(R)),j^!(R))\,, \tag{14.3.13.2}$$

which proves the proposition. □

Proposition 14.3.14. *Let X be a scheme in $\mathscr{S}$, and R an object in* $\mathrm{Ho}(\mathscr{M})(X)$*. Assume there exists an open covering $X=\bigcup_{i\in I}U_i$ such that the restriction of R on each of the open subschemes U_i is τ-dualizing in* $\mathrm{Ho}(\mathscr{M})(U_i)$*. Then R is τ-dualizing.*

Proof. We already know that the property of τ-constructibility is local with respect to the Zariski topology (14.1.7). Denote by $j_i:U_i\to X$ the corresponding open immersions, and put $R_i=j_i^!(R)$. Let M be a τ-constructible object in $\mathrm{Ho}(\mathscr{M})(X)$. Then, for all $i\in I$, the image by $j_i^*=j_i^!$ of the map

$$M\to\mathbf{R}Hom_X(\mathbf{R}Hom_X(M,R),R)$$

is isomorphic to the map

$$j_i^*(M)\to\mathbf{R}Hom_{U_i}(\mathbf{R}Hom_{U_i}(j_i^*(M),R_i),R_i)\,.$$

This proposition thus follows from the property of separation with respect to the Zariski topology. □

Corollary 14.3.15. *Let $f:X\to Y$ be a separated morphism of finite type in $\mathscr{S}$. Given an object R of* $\mathrm{Ho}(\mathscr{M})(Y)$*, the property for $f^!(R)$ of being a τ-dualizing object in* $\mathrm{Ho}(\mathscr{M})(X)$ *is local over X and over Y for the Zariski topology.*

Proposition 14.3.16. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is weakly τ-pure. Let $i:Z\to X$ be a closed immersion and R be a τ-dualizing object in* $\mathrm{Ho}(M)(X)$*. Then $i^!(R)$ is τ-dualizing in* $\mathrm{Ho}(M)(Z)$*.*

Proof. As $\mathrm{Ho}(\mathscr{M})$ is weakly τ-pure, we already know that $i^!(R)$ is τ-constructible. For any objects M and R of $\mathrm{Ho}(\mathscr{M})(Z)$ and $\mathrm{Ho}(\mathscr{M})(X)$ respectively, we have the identification:

$$i_!\,\mathbf{R}Hom_Z(M,i^!(R))\simeq\mathbf{R}Hom_X(i_!(M),R)\,.$$

Let $j:U\to X$ be the complement immersion. Then we have

$$j^!\mathbf{R}Hom_X(i_!(M),R)\simeq\mathbf{R}Hom_U(j^*\,i_!(M),j^!(R))\simeq 0\,,$$

so that

$$\mathbf{R}Hom_X(i_!(M),R)\simeq i_!\,\mathbf{L}i^*\mathbf{R}Hom_X(i_!(M),R)\,.$$

As $i_!$ is fully faithful, this provides a canonical isomorphism

$$\mathbf{L}i^*\mathbf{R}Hom_X(i_!(M),R)\simeq i^!\mathbf{R}Hom_X,(i_!(M),R)\,.$$

Under this identification, we see easily that the map

$$i_!(M)\to\mathbf{R}Hom_X(\mathbf{R}Hom_X(i_!(M),R),R)$$

is isomorphic to the image by $i_!$ of the map

$$M\to\mathbf{R}Hom_Z(\mathbf{R}Hom_Z(M,i^!(R)),i^!(R))\,.$$

As $i_!$ is fully faithful, it is conservative, and this ends the proof. □

Proposition 14.3.17. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is weakly* τ*-pure,* $\mathbf{Q}$*-linear and separated, and consider a scheme* B *in* $\mathscr{S}$ *which admits wide resolution of singularities up to quotient singularities. Consider a separated* B*-scheme of finite type* S*, and a* τ*-constructible object* R *in* $\mathrm{Ho}(\mathscr{M})(S)$*. The following conditions are equivalent.*

(i) For any separated morphism of finite type $f : X \to S$*, the object* $f^!(R)$ *is* τ*-dualizing.*
(ii) For any projective morphism $f : X \to S$*, the object* $f^!(R)$ *is* τ*-dualizing.*
(iii) For any projective morphism $f : X \to S$*, with* X *regular, the object* $f^!(R)$ *is* τ*-dualizing.*
(iv) For any projective morphism $f : X \to S$*, with* X *regular, and for any* $n \in \tau$*, the map*

$$\mathbb{1}_X\{n\} \to \mathbf{R}Hom_X(\mathbf{R}Hom_X(\mathbb{1}_X\{n\}, f^!(R)), f^!(R)) \tag{14.3.17.1}$$

is an isomorphism in $\mathrm{Ho}(\mathscr{M})(X)$*.*

If, furthermore, for any regular separated B*-scheme of finite type* X*, and for any* $n \in \tau$*, the object* $\mathbb{1}_X\{n\}$ *is* $\otimes$*-invertible, then these conditions are equivalent to the following one.*

(v) For any projective morphism $f : X \to S$*, with* X *regular, the map*

$$\mathbb{1}_X \to \mathbf{R}Hom_X(f^!(R)), f^!(R)) \tag{14.3.17.2}$$

is an isomorphism in $\mathrm{Ho}(\mathscr{M})(X)$*.*

Proof. It is clear that (i) implies (ii), which implies (iii), which implies (iv). Let us check that condition (ii) also implies condition (i). Let $f : X \to S$ be a morphism of separated B-schemes of finite type, with S regular. We want to prove that $f^!(\mathbb{1}_S)$ is τ-dualizing, while we already know it is true whenever f is projective. In the general case, by virtue of corollary 14.3.15, we may assume that f is quasi-projective, so that $f = pj$, where p is projective, and j is an open immersion. As $f^! \simeq j^! p^!$, we conclude with proposition 14.3.13. Under the additional assumption, the equivalence between (iv) and (v) is obvious. It thus remains to prove that (iv) implies (ii). It is in fact sufficient to prove that, under condition (iv), the object R itself is τ-dualizing. To prove that the map

$$M \to \mathbf{R}Hom_X(\mathbf{R}Hom_X(M, R), R) \tag{14.3.17.3}$$

is an isomorphism for any τ-constructible object M of $\mathrm{Ho}(\mathscr{M})(S)$, it is sufficient to consider the case where $M = \mathbf{R}f_*(\mathbb{1}_X\{n\}) = f_!(\mathbb{1}_X\{n\})$, where $n \in \tau$ and $f : X \to S$ is a projective morphism with X regular (14.3.9). For any object A of $\mathrm{Ho}(\mathscr{M})(X)$, we have canonical isomorphisms

$$\begin{aligned}\mathbf{R}Hom_S(f_!(A), R) &\simeq \mathbf{R}f_*\,\mathbf{R}Hom_X(A, f^!(R)) \\ &= f_!\,\mathbf{R}Hom_X(A, f^!(R))\,,\end{aligned}$$

from which we get a natural isomorphism:

$$\mathbf{R}Hom_S(\mathbf{R}Hom_S(f_!(A), R), R) \simeq f_!\,\mathbf{R}Hom_X(\mathbf{R}Hom_X(A, f^!(R)), f^!(R))\,.$$

Under these identifications, the map (14.3.17.3) for $M = f_!(\mathbb{1}_X\{n\})$ is the image of the map (14.3.17.1) by the functor $f_!$. As (14.3.17.1) is invertible by assumption, this proves that R is τ-dualizing. □

Lemma 14.3.18. *Let* X *be a scheme in* $\mathscr{S}$*, and* R *be an object of* $\mathrm{Ho}(\mathscr{M})(X)$*. The property for* R *of being* $\otimes$*-invertible is local over* X *with respect to the Zariski topology.*

Proof. Let $R^\wedge = \mathbf{R}Hom(R, \mathbb{1}_X)$ be the dual of R. The object R is $\otimes$-invertible if and only if the evaluation map

$$R^\wedge \otimes^{\mathbf{L}}_X R \to \mathbb{1}_X$$

is invertible. Let $j : U \to X$ be an open immersion. Then, for any objects M and N in $\mathrm{Ho}(\mathscr{M})(X)$, we have the identification

$$j^*\mathbf{R}Hom_X(M, N) \simeq \mathbf{R}Hom_U(j^*(M), j^*(N))\,.$$

In particular, we have $j^*(R^\wedge) \simeq j^*(R)^\wedge$. As j^* is monoidal, the lemma follows from the fact that $\mathrm{Ho}(\mathscr{M})$ has the property of separation with respect to the Zariski topology. □

Definition 14.3.19. We shall say that $\mathrm{Ho}(\mathscr{M})$ is *τ-pure* (or simply *pure*) if it satisfies the following conditions:

(i) $\mathrm{Ho}(\mathscr{M})$ is weakly τ-pure (14.1.22);
(ii) for any closed immersion between regular schemes $i : Z \to S$ in $\mathscr{S}$, the object $i^!(\mathbb{1}_S)$ is $\otimes$-invertible (i.e. the functor $i^!(\mathbb{1}_S) \otimes^{\mathbf{L}}_S (-)$ is an equivalence of categories);
(ii) for any regular scheme X in $\mathscr{S}$, and for any $n \in \tau$, the map

$$\mathbb{1}_X\{n\} \to \mathbf{R}Hom_X(\mathbf{R}Hom_X(\mathbb{1}_X\{n\}, \mathbb{1}_X), \mathbb{1}_X)$$

is an isomorphism.

Example 14.3.20. The motivic category DM_{B} is pure over excellent noetherian schemes (13.4.1).

Remark 14.3.21. Note that, whenever the objects $\mathbb{1}_X\{n\}$ are $\otimes$-invertible (which will be the case in practice), conditions (i) and (ii) of the preceding definition are equivalent to the condition that $i^!(\mathbb{1}_X)$ is τ-constructible and $\otimes$-invertible for any closed immersion i between regular separated schemes in $\mathscr{S}$, while condition (iii) is then automatic. This principle gives easily the property of τ-purity when $\mathscr{S}$ is made of schemes of finite type over some field:

Proposition 14.3.22. *Assume that $\mathscr{S}$ consists exactly of schemes of finite type over a field k, and that one of the following conditions is satisfied:*

(a) the field k is perfect;
(b) $\mathrm{Ho}(\mathscr{M})$ is semi-separated (2.1.11).

If the objects $\mathbb{1}\{n\}$ are $\otimes$-invertible in $\mathrm{Ho}(\mathscr{M})(\mathrm{Spec}\,(k))$ for all $n \in \tau$, then $\mathrm{Ho}(\mathscr{M})$ is τ-pure.

Proof. For any k-scheme of finite type $f : X \to \mathrm{Spec}\,(k)$, as the functor $\mathbf{L}f^*$ is symmetric monoidal, the objects $\mathbb{1}_X\{n\}$ are $\otimes$-invertible in $\mathrm{Ho}(\mathscr{M})(X)$ for all $n \in \tau$. Therefore, as stated in remark 14.3.21, we have only to prove that, for any closed immersion $i : Z \to X$ between regular k-schemes of finite type, the object $i^!(\mathbb{1}_X)$ is $\otimes$-invertible and τ-constructible. We may assume that X and Z are smooth (under condition (a), this is clear, and under condition (b), by virtue of proposition 2.1.13, we may replace k by any of its finite extensions). Using 14.3.18 and 14.1.7, we may also assume that X is quasi-projective and that Z is purely of codimension c in X, while the normal bundle of i is trivial. This proposition is then a consequence of relative purity (2.4.21), which gives a canonical isomorphism $i^!(\mathbb{1}_X) \simeq \mathbb{1}_Z(-c)[-2c]$. □

Proposition 14.3.23 (Poincaré duality)**.** *For any separated smooth morphism of finite type $f : Y \to X$, the object $f^!(\mathbb{1}_X)$ is $\otimes$-invertible in $\mathrm{Ho}(\mathscr{M})(Y)$, and we have an isomorphism*

$$f^!(\mathbb{1}_X) \otimes^{\mathbf{L}}_Y f^* \simeq f^! .$$

In particular, we have a canonical isomorphism

$$M_X(Y) \simeq f_! f^!(\mathbb{1}_X) .$$

If moreover f is proper, then, for any object M of $\mathrm{Ho}(\mathscr{M})(X)$, we have a natural isomorphism

$$\mathbf{R}Hom_X(f_! f^!(\mathbb{1}_X), M) \simeq \mathbf{R}f_*(\mathbb{1}_Y) \otimes^{\mathbf{L}}_X M .$$

In other words, $M_X(Y) \simeq f_! f^!(\mathbb{1}_X)$ is then a rigid object, with dual $f_!(\mathbb{1}_Y) = \mathbf{R}f_*(\mathbb{1}_Y)$.

Proof. To prove the first assertion, by virtue of lemma 14.3.18, we may assume that f is strictly quasi-projective, in which case, this follows from relative purity; see theorem 2.4.21 (iii). We have a natural map

$$f^!(\mathbb{1}_X) \otimes^{\mathbf{L}}_Y f^*(M) \to f^!(M)$$

defined as follows. Such a map corresponds by adjunction to a map

$$f^!(\mathbb{1}_X) \to \mathbf{R}Hom_Y(\mathbf{L}f^*(M), f^!(M)) \simeq f^!\mathbf{R}Hom_X(M, M) .$$

The latter morphism is defined as the image by $f^!$ of the unit map

$$M \to \mathbf{R}Hom_X(M, M) .$$

The property that the map $f^!(\mathbb{1}_X)\otimes^{\mathbf{L}}_Y f^*(M) \to f^!(M)$ is invertible is local for the Zariski topology on X and on Y, so that we assume that f is strictly quasi-projective with trivial cotangent bundle. We may then conclude again by relative purity.

Observe that, we also have:

$$\mathbf{R}f_*(f^*(M)) \simeq \mathbf{R}f_*Hom_Y(\mathbb{1}_Y, M) \simeq \mathbf{R}f_*\mathbf{R}Hom_Y(f^*(\mathbb{1}_X), f^*(M)) \simeq \mathbf{R}Hom_X(f_\sharp(\mathbb{1}_Y), M)\,.$$

On the other hand:

$$\mathbf{R}Hom_X(f_!f^!(\mathbb{1}_X), M) \simeq \mathbf{R}f_*\mathbf{R}Hom_Y(f^!(\mathbb{1}_X), f^!(M))\,.$$

As, $f^!(\mathbb{1}_X)\otimes^{\mathbf{L}}_Y f^*(M) \simeq f^!(M)$, and as $f^!(\mathbb{1}_X)$ is invertible, we have canonical isomorphisms

$$f^*(M) \simeq \mathbf{R}Hom_Y(\mathbb{1}_Y, f^*(M)) \simeq \mathbf{R}Hom_Y(f^!(\mathbb{1}_X), f^!(M))\,,$$

from which we get:

$$\mathbf{R}Hom_X(f_\sharp(\mathbb{1}_Y), M) \simeq \mathbf{R}Hom_X(f_!f^!(\mathbb{1}_X), M)\,.$$

By the (enriched) Yoneda lemma, we thus get: $M_X(Y) \simeq f_!f^!(\mathbb{1}_X)$.

Finally, if f is smooth and proper, we have $f_! \simeq \mathbf{R}f_*$, so that we also have the following canonical isomorphisms.

$$\mathbf{R}f_*(\mathbb{1}_Y)\otimes^{\mathbf{L}}_X M \simeq f_!(\mathbb{1}_Y)\otimes^{\mathbf{L}}_X M \simeq f_!(f^*(M)) \simeq \mathbf{R}f_*(f^*(M)) \simeq \mathbf{R}Hom_X(M_X(Y), M)\,.$$

This achieves the proof. □

Corollary 14.3.24. *Assume that $\mathscr{S}$ consists of schemes of finite type over a field k and that* $\mathrm{Ho}(\mathscr{M})$ *has the following properties:*

(a) it is τ-pure;
(b) for any $n \in \tau$, $\mathbb{1}\{n\}$ is rigid;
(c) either k is perfect, either $\mathrm{Ho}(\mathscr{M})$ *is continuous.*

Then, any τ-constructible object of $\mathrm{Ho}(\mathscr{M})(k)$ *is rigid.*

Proof. By 14.2.15, it is sufficient to treat the case where k is perfect. It is well known that rigid objects form a thick subcategory of $\mathrm{Ho}(\mathscr{M})$. Thus we conclude easily from corollary 14.3.9 and proposition 14.3.23. □

Lemma 14.3.25. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is τ-pure. Then, for any projective morphism $f : X \to S$ between regular schemes in $\mathscr{S}$, the object $f^!(\mathbb{1}_S)$ is $\otimes$-invertible and τ-constructible.*

Proof. As, for any open immersion $j : U \to X$, one has $j^* = j^!$, we deduce easily from lemma 14.3.18 (resp. proposition 14.1.7) that the property for $f^!(\mathbb{1}_S)$ of being $\otimes$-invertible (resp. τ-constructible) is local on S for the Zariski topology. Therefore, we may assume that S is separated over B and that f factors as a closed immersion $i : X \to \mathbf{P}^n_S$ followed by the canonical projection $p : \mathbf{P}^n_S \to S$. Using relative purity for p, we have the following computations:

$$f^!(\mathbb{1}_S) \simeq i^!\,p^!(\mathbb{1}_S) \simeq i^!(\mathbb{1}_{\mathbf{P}^n_S}(n)[2n]) \simeq i^!(\mathbb{1}_{\mathbf{P}^n_S})(n)[2n]\,.$$

As i is a closed immersion between regular schemes, the object $i^!(\mathbb{1}_{\mathbf{P}^n_S})$ is $\otimes$-invertible and τ-constructible by assumption on $\mathrm{Ho}(\mathscr{M})$, which implies that $f^!(\mathbb{1}_S)$ is $\otimes$-invertible and τ-constructible as well. □

Definition 14.3.26. Let B a scheme in $\mathscr{S}$. We shall say that *local duality holds over B in* $\mathrm{Ho}(\mathscr{M})$ if, for any separated morphism of finite type $f : X \to S$, with S regular and of finite type over B, the object $f^!(\mathbb{1}_S)$ is τ-dualizing in $\mathrm{Ho}(\mathscr{M})(X)$.

Remark 14.3.27. By definition, if $\mathrm{Ho}(\mathscr{M})$ is weakly τ-pure, and if local duality holds over B in $\mathrm{Ho}(\mathscr{M})$, then the restriction of $\mathrm{Ho}(\mathscr{M})$ to the category of B-schemes of finite type is τ-pure. A convenient sufficient condition for local duality to hold in $\mathrm{Ho}(\mathscr{M})$ is the following (in particular, using the result below as well as proposition 14.3.22, local duality holds almost systematically over fields).

Theorem 14.3.28. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is* τ*-pure,* $\mathbf{Q}$*-linear and separated, and consider a scheme* B *in* $\mathscr{S}$ *which admits wide resolution of singularities up to quotient singularities (e.g.* B *might be any scheme which is separated and of finite type over an excellent noetherian scheme of dimension lesser or equal to* 2 *in* $\mathscr{S}$*; see 14.3.6). Then local duality holds over* B *in* $\mathrm{Ho}(\mathscr{M})$*.*

Proof. Let S be a regular separated B-scheme of finite type. Then, for any separated morphism of finite type $f : X \to S$, the object $f^!(\mathbb{1}_S)$ is τ-dualizing: lemma 14.3.25 implies immediately condition (iv) of proposition 14.3.17. The general case (without the separation assumption on S) follows easily from corollary 14.3.14. □

Proposition 14.3.29. *Consider a scheme* B *in* $\mathscr{S}$*. Assume that* $\mathrm{Ho}(\mathscr{M})$ *is* τ*-pure, and that local duality holds over* B *in* $\mathrm{Ho}(\mathscr{M})$*. Consider a regular* B*-scheme of finite type* S*.*

(i) An object of $\mathrm{Ho}(\mathscr{M})(S)$ *is* τ*-dualizing if and only if it is* τ*-constructible and* $\otimes$*-invertible.*

(ii) For any separated morphism of S*-schemes of finite type* $f : X \to Y$*, and for any* τ*-dualizing object* R *in* $\mathrm{Ho}(\mathscr{M})(Y)$*, the object* $f^!(R)$ *is* τ*-dualizing in* $\mathrm{Ho}(\mathscr{M})(X)$*.*

Proof. As the unit of $\mathrm{Ho}(\mathscr{M})(S)$ is τ-dualizing by assumption, proposition 14.3.12 implies that an object of $\mathrm{Ho}(\mathscr{M})(S)$ is τ-dualizing if and only if it is τ-constructible and $\otimes$-invertible.

Consider a regular B-scheme of finite type S, as well as a separated morphism of S-schemes of finite type $f : X \to Y$, as well as a τ-dualizing object R in $\mathrm{Ho}(\mathscr{M})(Y)$. To prove that $f^!(R)$ is τ-dualizing, by virtue of corollary 14.3.14, we may assume that Y is separated over S. Denote by u and v the structural maps from X and Y to S respectively. As we already know that $v^!(\mathbb{1}_S)$ is τ-dualizing, by virtue of proposition 14.3.12, there exists a τ-constructible and $\otimes$-invertible object U in $\mathrm{Ho}(\mathscr{M})(Y)$ such that $U \otimes^{\mathbf{L}}_Y R \simeq v^!(\mathbb{1}_S)$. As the functor $\mathbf{L}f^*$ is symmetric monoidal, it preserves $\otimes$-invertible objects and their duals, from which we deduce the following isomorphisms:

$$\begin{aligned} u^!(\mathbb{1}_S) &\simeq f^! \, v^!(\mathbb{1}_S) \\ &\simeq f^!(U \otimes^{\mathbf{L}}_Y R) \\ &\simeq f^! \, \mathbf{R}Hom_Y(U^\wedge, R) \\ &\simeq \mathbf{R}Hom_X(\mathbf{L}f^*(U^\wedge), f^!(R)) \\ &\simeq \mathbf{R}Hom_X(\mathbf{L}f^*(U)^\wedge, f^!(R)) \\ &\simeq \mathbf{L}f^*(U) \otimes^{\mathbf{L}}_X f^!(R)\,. \end{aligned}$$

The object $a^!(\mathbb{1}_S)$ being τ-dualizing, while $\mathbf{L}f^*(U)$ is τ-constructible and invertible, we deduce from proposition 14.3.12 that $f^!(R)$ is τ-dualizing as well. □

14.3.30. Assume that $\mathrm{Ho}(\mathscr{M})$ is τ-pure, $\mathbf{Q}$-linear and separated, and consider a scheme B in $\mathscr{S}$, such that local duality holds over B in $\mathrm{Ho}(\mathscr{M})$. Consider a fixed regular B-scheme of finite type S, as well as a τ-constructible and $\otimes$-invertible object R in $\mathrm{Ho}(\mathscr{M})(S)$ (in the case S is of pure dimension d, it might be wise to consider $R = \mathbb{1}_S(d)[2d]$, but an arbitrary R as above is eligible by 14.3.29). Then, for any separated S-scheme of finite type $f : X \to S$, we define the *local duality functor*

$$D_X : \mathrm{Ho}(\mathscr{M})(X)^{op} \to \mathrm{Ho}(\mathscr{M})(X)$$

by the formula

$$D_X(M) = \mathbf{R}Hom_X(M, f^!(R))\,.$$

This functor D_X is right adjoint to itself.

Corollary 14.3.31. *Under the assumptions of 14.3.30, we have the following properties.*

(a) For any separated S*-scheme of finite type* X*, the functor* D_X *preserves* τ*-constructible objects.*

(b) For any separated S*-scheme of finite type* X*, the natural map*

$$M \to D_X(D_X(M))$$

is an isomorphism for any τ*-constructible object* M *in* $\mathrm{Ho}(\mathscr{M})(X)$*.*

(c) *For any separated S-scheme of finite type X, and for any objects M and N in $\mathrm{Ho}(\mathscr{M})(X)$, if N is τ-constructible, then we have a canonical isomorphism*

$$D_X(M \otimes_X^{\mathbf{L}} D_X(N)) \simeq \mathbf{R}Hom_X(M, N)\,.$$

(d) *For any morphism between separated S-schemes of finite type $f : Y \to X$, we have natural isomorphisms*

$$D_Y(f^*(M)) \simeq f^!(D_X(M))$$
$$f^*(D_X(M)) \simeq D_Y(f^!(M))$$
$$D_X(f_!(N)) \simeq f_*(D_Y(N))$$
$$f_!(D_Y(N)) \simeq D_X(f_*(N))$$

for any τ-constructible objects M and N in $\mathrm{Ho}(\mathscr{M})(X)$ and $\mathrm{Ho}(\mathscr{M})(Y)$ respectively.

Proof. Assertions (a) and (b) are only stated for the record[45]; see 14.1.27. To prove (c), we see that we have an obvious isomorphism

$$D_X(M \otimes_X^{\mathbf{L}} P) \simeq \mathbf{R}Hom_X(M, D_X(P))$$

for any objects M and P. If N is τ-constructible, we may replace P by $D_X(N)$ and get the expected formula using (b). The identification $D_Y\, f^* \simeq f^!\, D_X$ is a special case of the formula

$$\mathbf{R}Hom_Y(f^*(A), f^!(B)) \simeq f^!\, \mathbf{R}Hom_X(A, B)\,.$$

Therefore, we also get:

$$f^*\, D_X \simeq D_Y^2\, f^*\, D_X \simeq D_Y\, f^!\, D_X^2 \simeq D_Y\, f^!\,.$$

The two other formulas of (d) follow by adjunction. □

15. Comparison theorems

15.1. **Comparison with Voevodsky motives.**

15.1.1. We consider the premotivic adjunction of 10.4.1

$$\gamma^* : \mathrm{D}_{\mathbf{A}^1,\mathbf{Q}} \rightleftarrows \mathrm{DM}_{\mathbf{Q}} : \gamma_*\,. \tag{15.1.1.1}$$

For a scheme S, $\gamma_*(\mathbb{1}_S)$ is a (strict) commutative ring spectrum, and, for any object M of $\mathrm{DM}_{\mathbf{Q}}(S)$, $\gamma_*(M)$ is naturally endowed with a structure of $\gamma_*(\mathbb{1}_S)$-module. On the other hand, as we have the projective bundle formula in $\mathrm{DM}_{\mathbf{Q}}(S)$ (10.3.2), $\gamma_*(\mathbb{1}_S)$ is orientable (11.2.10), which implies that, for any object M of $\mathrm{DM}_{\mathbf{Q}}(S)$, $\gamma_*(M)$ is an $H_{\Bbbk,S}$-module, whence is $H_{\Bbbk}$-local (13.2.15). As consequence, we get a canonical factorization of (15.1.1.1):

$$\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}} \xrightarrow{\beta^*} \mathrm{DM}_{\Bbbk} \xrightarrow{\varphi^*} \mathrm{DM}_{\mathbf{Q}}. \tag{15.1.1.2}$$

Consider the commutative diagram of premotivic categories

$$\begin{array}{ccc} \mathrm{D}_{\mathbf{A}^1,\mathbf{Q}} & \xrightarrow{\gamma^*} & \mathrm{DM}_{\mathbf{Q}} \\ {\scriptstyle \rho_\sharp}\downarrow & & \downarrow{\scriptstyle \psi_\sharp} \\ \underline{\mathrm{D}}_{\mathbf{A}^1,\mathbf{Q}} & \xrightarrow{\underline{\gamma}^*} & \underline{\mathrm{DM}}_{\mathbf{Q}} \end{array} \tag{15.1.1.3}$$

in which the two vertical maps are the canonical enlargements, and, in particular, are fully faithful (see 6.1.8).

[45] We have put to much assumptions here: in fact, if $\mathrm{Ho}(\mathscr{M})$ is τ-pure and if local duality holds over B in $\mathrm{Ho}(\mathscr{M})$, the six Grothendieck operations preserve τ-constructible objects on the restriction of $\mathrm{Ho}(\mathscr{M})$ to B-schemes of finite type; we leave this as a formal exercice for the reader.

Let t denotes either the qfh-topology or the h-topology. We also have the following commutative triangle

$$\underline{\mathrm{D}}_{\mathbf{A}^1,\mathbf{Q}} \xrightarrow{\underline{\gamma}^*} \underline{\mathrm{DM}}_{\mathbf{Q}} \xrightarrow{\underline{\alpha}^*} \underline{\mathrm{DM}}_{t,\mathbf{Q}}, \quad \underline{a}^* : \underline{\mathrm{D}}_{\mathbf{A}^1,\mathbf{Q}} \to \underline{\mathrm{DM}}_{t,\mathbf{Q}} \tag{15.1.1.4}$$

in which both $\underline{a}^*$ and $\underline{\alpha}^*$ are induced by the t-sheafification functor; see 5.3.34 and 10.4.3. We obtain from (15.1.1.2), (15.1.1.3), and (15.1.1.4) the commutative diagram of premotivic categories below, in which $\chi_\sharp = \varphi^* \underline{\alpha}^* \psi_\sharp$.

$$\begin{array}{ccc} \mathrm{D}_{\mathbf{A}^1,\mathbf{Q}} & \xrightarrow{\beta^*} & \mathrm{DM}_{\mathrm{B}} \\ {\scriptstyle \rho_\sharp}\downarrow & & \downarrow{\scriptstyle \chi_\sharp} \\ \underline{\mathrm{D}}_{\mathbf{A}^1,\mathbf{Q}} & \xrightarrow{\underline{a}^*} & \underline{\mathrm{DM}}_{t,\mathbf{Q}} \end{array} \tag{15.1.1.5}$$

From now on, we shall fix an excellent noetherian scheme of finite dimension S.

Theorem 15.1.2. *We have canonical equivalences of categories*

$$\mathrm{DM}_{\mathrm{B}}(S) \simeq \mathrm{DM}_{\mathrm{qfh},\mathbf{Q}}(S) \simeq \mathrm{DM}_{\mathrm{h},\mathbf{Q}}(S)$$

(recall that, for $t = \mathrm{qfh}, \mathrm{h}$, $\mathrm{DM}_{t,\mathbf{Q}}(S)$ stands for the localizing subcategory of $\underline{\mathrm{DM}}_{t,\mathbf{Q}}(S)$, spanned by the objects of shape $\Sigma^\infty \mathbf{Q}_S(X)(n)$, where X runs over the family of smooth S-schemes, and $n \leq 0$ is an integer; see 5.3.34).

Proof. Let t denote the qfh-topology or the h-topology. We shall prove that the functor

$$\chi_\sharp : \mathrm{DM}_{\mathrm{B}}(S) \to \underline{\mathrm{DM}}_{t,\mathbf{Q}}(S)$$

is fully faithful, and that its essential image is precisely $\mathrm{DM}_{t,\mathbf{Q}}$. The functor

$$\beta_* : \mathrm{DM}_{\mathrm{B}} \to \mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$$

is fully faithful, so that its composition with its left adjoint β^* is canonically isomorphic to the identity. In particular, we get isomorphisms of functors:

$$\chi_\sharp \simeq \chi_\sharp\, \beta^*\, \beta_* \simeq \underline{a}^*\, \rho_\sharp\, \beta_* .$$

The right adjoint of $\underline{a}^*$ is fully faithful, and its essential image consists of the objects of $\underline{\mathrm{D}}_{\mathbf{A}^1,\mathbf{Q}}(S)$ which satisfy t-descent (5.3.32). On the other hand, the functor $\rho_\sharp$ is fully faithful, and an object of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$ satisfies t-descent if and only if its image by $\rho_\sharp$ satisfies t-descent (6.1.11). By virtue of theorem 13.3.2, this implies immediately that $\chi_\sharp$ is fully faithful. Let $\mathrm{DM}_{t,\mathbf{Q}}(S)$ be the localizing subcategory of $DM_{t,\mathbf{Q}}(S)$ spanned by the objects of shape $\Sigma^\infty \mathbf{Q}(X)(n)$, where X runs over the family of smooth S-schemes, and $n \leq 0$ is an integer (5.3.34). We know that $\mathrm{DM}_{t,\mathbf{Q}}(S)$ is compactly generated (see 5.1.28, 5.2.36 and 5.3.31), and that $\chi_\sharp$ is a fully faithful exact functor which preserves small sums as well as compact objects from $\mathrm{DM}_{\mathrm{B}}(S)$ to $\mathrm{DM}_{t,\mathbf{Q}}(S)$. As, by construction, there exists a generating family of compact objects of $\mathrm{DM}_{t,\mathbf{Q}}(S)$ in the essential image of $\chi_\sharp$, this implies that $\chi_\sharp$ induces an equivalence of triangulated categories $\mathrm{DM}_{\mathrm{B}}(S) \simeq \mathrm{DM}_{t,\mathbf{Q}}(S)$ (see 1.3.19). □

Let us underline the following result which completes corollary 13.2.15:

Theorem 15.1.3. *Let E be an object of $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$. The following conditions are equivalent:*

(i) E is a Beilinson motive;
(ii) E satisfies h*-descent;*
(iii) E satisfies qfh*-descent;*

Proof. We already know that condition (i) implies condition (ii) (theorem 13.3.2), and condition (ii) implies obviously condition (iii). It is thus sufficient to prove that condition (iii) implies condition (i). If E satisfies qfh-descent, then $\rho_\sharp(E)$ satisfies qfh-descent in $\underline{\mathrm{DM}}(S, \mathbf{Q})$ as well. The commutativity of (15.1.1.4) implies then that $\rho_\sharp(E)$ belongs to the essential image of $\underline{\gamma}_*$ (the right adjoint of $\underline{\gamma}^*$). As $\rho_\sharp$ is fully faithful, the commutativity of (15.1.1.3) thus implies that E itself

belongs to the essential image of γ_* (the right adjoint to γ^*). In particular, E is then a module over the ring spectrum $\gamma_*(\mathbb{1}_S)$, which is itself an H_{B}-algebra. We conclude by corollary 13.2.15. □

Theorem 15.1.4. *If S is geometrically unibranch (e.g. normal), then the comparison functor*

$$\varphi^* : \mathrm{DM}_{\mathrm{B}}(S) \to \mathrm{DM}_{\mathbf{Q}}(S)$$

is an equivalence of symmetric monoidal triangulated categories.

Proof. If S is geometrically unibranch, then we know that the composed functor

$$\mathrm{DM}_{\mathbf{Q}}(S) \xrightarrow{\psi_\sharp} \underline{\mathrm{DM}}_{\mathbf{Q}}(S) \xrightarrow{\underline{\alpha}^*} \underline{\mathrm{DM}}_{\mathrm{qfh},\mathbf{Q}}(S)$$

is fully faithful (10.4.4). The commutative diagram

$$\begin{array}{ccccc} \mathrm{DM}_{\mathrm{B}}(S) & \xrightarrow{\varphi^*} & \mathrm{DM}_{\mathbf{Q}}(S) & \xrightarrow{\underline{\alpha}^*\psi_\sharp} & \underline{\mathrm{DM}}_{\mathrm{qfh},\mathbf{Q}}(S) \\ & \searrow^{\chi_\sharp} & & \nearrow & \end{array}$$

and theorem 15.1.2 imply that φ^* is fully faithful. As φ^* is exact, preserves small sums as well as compact objects, and as $\mathrm{DM}_{\mathbf{Q}}(S)$ has a generating family of compact objects in the essential image of φ^*, the functor φ^* has to be an equivalence of categories (1.3.19). □

Remark 15.1.5. Some version of the preceding theorem (the one obtained by replacing DM_{B} by $\mathrm{Ho}(H_{\mathrm{B}}\text{-}\mathrm{mod})$) was already known in the case where S is the spectrum of a perfect field; see [RØ08, theorem 68]. The proof used de Jong's resolution of singularities by alterations and Poincaré duality in a crucial way. The proof of the preceding theorem we gave here relies on proper descent but does not use any kind of resolution of singularities.

We point out the following important fact about Voevodsky's motivic cohomology spectrum $H_{\mathcal{M},S} = \gamma_*(\mathbb{1}_S)$ with rational coefficients:

Corollary 15.1.6. *(1) For any geometrically unibranch excellent scheme S, the canonical map*

$$H_{\mathrm{B},S} \to H_{\mathcal{M},S} \otimes \mathbf{Q}$$

is an isomorphism of ring spectra.

(2) For any morphism $f : T \to S$ of excellent geometrically unibranch schemes, the canonical map

$$f^* H_{\mathcal{M},S} \otimes \mathbf{Q} \to H_{\mathcal{M},T} \otimes \mathbf{Q}$$

is an isomorphism of ring spectra.

The second part is the last conjecture of Voevodsky's paper [Voe02] with rational coefficients (and geometrically unibranch schemes).

Proof. The first part is a trivial consequence of the previous theorem, and the second follows from the first, as the Beilinson motivic cohomology spectrum is stable by pullbacks. □

15.2. **Comparison with Morel motives.**

15.2.1. Let S be a scheme. The permutation isomorphism

$$\tau : \mathbf{Q}(1)[1] \otimes^{\mathbf{L}}_{\mathbf{Q}} \mathbf{Q}(1)[1] \to \mathbf{Q}(1)[1] \otimes^{\mathbf{L}}_{\mathbf{Q}} \mathbf{Q}(1)[1] \tag{15.2.1.1}$$

satisfies the equation $\tau^2 = 1$ in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$. Hence it defines an element ϵ in $\mathrm{End}_{\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)}(\mathbf{Q})$ which also satisfies the relation $\epsilon^2 = 1$. We define two projectors

$$e_+ = \frac{\epsilon - 1}{2} \quad \text{and} \quad e_- = \frac{\epsilon + 1}{2}\,. \tag{15.2.1.2}$$

As the triangulated category $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$ is pseudo abelian, we can define two objects by the formulæ:

$$\mathbf{Q}_+ = \mathrm{Im}\, e_+ \quad \text{and} \quad \mathbf{Q}_- = \mathrm{Im}\, e_-\,. \tag{15.2.1.3}$$

Then for an object M of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$, we set

$$M_+ = \mathbf{Q}_+ \otimes^{\mathbf{L}}_{\mathbf{Q}} M \quad \text{and} \quad M_- = \mathbf{Q}_- \otimes^{\mathbf{L}}_{\mathbf{Q}} M\,. \tag{15.2.1.4}$$

It is obvious that for any objects M and N of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$, one has

$$\mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)}(M_i, N_j) = 0 \quad \text{for } i,j \in \{+,-\} \text{ with } i \neq j. \tag{15.2.1.5}$$

Denote by $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)_+$ (resp. $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)_-$) the full subcategory of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$ made of objects which are isomorphic to some M_+ (resp. some M_-) for an object M in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$. Then (15.2.1.5) implies that the direct sum functor $(M_+, M_-) \mapsto M_+ \oplus M_-$ induces an equivalence of triangulated categories

$$(\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)_+) \times (\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)_-) \simeq \mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)\,. \tag{15.2.1.6}$$

We shall call $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)_+$ the *category of Morel motives over* S. The aim of this section is to compare this category with $\mathrm{DM}_{\mathrm{B}}(S)$ (see theorem 15.2.13). This will consist essentially to prove that $\mathbf{Q}_+$ is nothing else than Beilinson's motivic spectrum H_{B} (which was announced by Morel in [Mor06b]). The main ingredients of the proof are the description of $\mathrm{DM}_{\mathrm{B}}(S)$ as full subcategory of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$, the homotopy t-structure on $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$, and Morel's computation of the endomorphism ring of the motivic sphere spectrum in terms of Milnor-Witt K-theory [Mor03, Mor04a, Mor04b, Mor06a].

15.2.2. For a little while, we shall assume that S is the spectrum of a field k.

Recall that the *algebraic Hopf fibration* is the map

$$\mathbf{A}^2 - \{0\} \to \mathbf{P}^1\ , \quad (x,y) \mapsto [x,y]\,.$$

This defines, by desuspension, a morphism

$$\eta : \mathbf{Q}(1)[1] \to \mathbf{Q}$$

in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$; see [Mor03, 6.2] (recall that we identify $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$ with $\mathrm{SH}_{\mathbf{Q}}(S)$ and that, under this identification, $\mathbf{Q}(1)[1]$ corresponds to $\Sigma^\infty(\mathbf{G}_m)$).

Lemma 15.2.3. *We have $\eta = \epsilon\eta$ in* $\mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)}(\mathbf{Q}(1)[1], \mathbf{Q})$.

Proof. See [Mor03, 6.2.3]. □

15.2.4. Recall the homotopy t-structure on $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$; see [Mor03, 5.2]. To remain close to the conventions of *loc. cit.*, we shall adopt homological notations, so that, for any object M of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$, we have the following truncation triangle

$$\tau_{>0}M \to M \to \tau_{\leq 0}M \to \tau_{>0}M[1]\,.$$

We whall write H_0 for the zeroth homology functor in the sense of this t-structure. This t-structure can be described in terms of generators, as in [Ayo07a, definition 2.2.41]: the category $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)_{\geq 0}$ is the smallest full subcategory of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$ which contains the objects of shape $\mathbf{Q}_S(X)(m)[m]$ for X smooth over S, $m \in \mathbf{Z}$, and which satisfies the following stability conditions:

(a) $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)_{\geq 0}$ is stable under suspension; i.e. for any object M in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)_{\geq 0}$, $M[1]$ is in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)_{\geq 0}$;

(b) $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)_{\geq 0}$ is closed under extensions: for any distinguished triangle

$$M' \to M \to M'' \to M'[1]\,,$$

if M' and M'' are in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)_{\geq 0}$, so is M;

(c) $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)_{\geq 0}$ is closed under small sums.

With this description, it is easy to see that $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)_{\geq 0}$ is also closed under tensor product (because the class of generators has this property). The category $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)_{\leq 0}$ is the full subcategory of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$ which consists of objects M such that

$$\mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)}(\mathbf{Q}_S(X)(m)[m+n], M) \simeq 0$$

for X/S smooth, $m \in \mathbf{Z}$, and $n > 0$; see [Ayo07a, 2.1.72].

Note that the heart of the homotopy t-structure is symmetric monoidal, with tensor product $\otimes^h$ defined by the formula:

$$F \otimes^h G = H_0(F \otimes_S^{\mathbf{L}} G)$$

(the unit object is $H_0(\mathbf{Q})$).

We shall still write $\eta : H_0(\mathbf{Q}(1)[1]) \to H_0(\mathbf{Q})$ for the map induced by the algebraic Hopf fibration.

Proposition 15.2.5. *Tensoring by $\mathbf{Q}(n)[n]$ defines a t-exact endofunctor of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$ for any integer n.*

Proof. As tensoring by $\mathbf{Q}(n)[n]$ is an equivalence of categories, it is sufficient to prove this for $n \geq 0$. This is then a particular case of [Ayo07a, 2.2.51]. □

Proposition 15.2.6. *For any smooth S-scheme X of dimension d, and for any object M of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$, the map*

$$\mathrm{Hom}(\mathbf{Q}_S(X), M) \to \mathrm{Hom}(\mathbf{Q}_S(X), M_{\leq n})$$

is an isomorphism for $n > d$.

Proof. Using [Mor03, lemma 5.2.5], it sufficient to prove the analog for the homotopy t-structure on $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,\mathbf{Q}}(S)$, which follows from [Mor05, lemma 3.3.3]. □

Proposition 15.2.7. *The homotopy t-structure is non-degenerated. Even better, for any object M of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$, we have canonical isomorphisms*

$$\mathbf{L}\varinjlim_n \tau_{>n} M \simeq M \quad \textit{and} \quad \mathbf{R}\varprojlim_n \tau_{>n} M \simeq 0\,,$$

as well as isomorphisms

$$\mathbf{L}\varinjlim_n \tau_{\leq n} M \simeq 0 \quad \textit{and} \quad M \simeq \mathbf{R}\varprojlim_n \tau_{\leq n} M\,.$$

Proof. The first assertion is a direct consequence of propositions 15.2.5 and 15.2.6 (because the objects of shape $\mathbf{Q}_S(X)(m)[i]$, for X/S smooth, and $m, i \in \mathbf{Z}$, form a generating family). As the objects $\mathbf{Q}_S(X)(m)[m+n]$ are compact in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$, the category $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)_{\leq 0}$ is closed under small sums. As $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)_{\geq 0}$ is also closed under small sums, we deduce easily that the truncation functors $\tau_{>0}$ and $\tau_{\leq 0}$ preserve small sums, which implies that the homology functor H_0 has the same property. Moreover, if

$$C_0 \to \cdots \to C_n \to C_{n+1} \to \cdots$$

is a sequence of maps in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$, then $C = \mathbf{L}\varinjlim_n C_n$ fits in a distinguished triangle of shape

$$\bigoplus_n C_n \overset{1-s}{\to} \bigoplus_n C_n \to C \to \bigoplus_n C_n[1]\,,$$

where s is the map induced by the maps $C_n \to C_{n+1}$. This implies that, for any integer i, we have

$$\varinjlim_n H_i(C_n) \simeq H_i(C)$$

(where the colimit is taken in the heart of the homotopy t-structure). As the homotopy t-structure is non-degenerated, this proves the two formulas

$$\mathbf{L}\varinjlim_n \tau_{>n} M \simeq M \quad \text{and} \quad \mathbf{L}\varinjlim_n \tau_{\leq n} M \simeq 0\,.$$

Let X be a smooth S-scheme of finite type, and p, q be some integer. To prove that the map

$$\mathrm{Hom}(\mathbf{Q}_S(X)(m)[i], M) \to \mathrm{Hom}(\mathbf{Q}_S(X)(m)[i], \mathbf{R}\varprojlim_n \tau_{\leq n} M)$$

is bijective, we may assume that $m = 0$ (replacing M by $M(-m)[-m]$ and i by $i - m$, and using proposition 15.2.5). Consider the Milnor short exact sequence below, with $A = \mathbf{Q}_S(X)[i]$:

$$0 \to \varprojlim_n{}^1 \mathrm{Hom}(A[1], \tau_{\leq n} M) \to \mathrm{Hom}(A, \mathbf{R}\varprojlim_n \tau_{\leq n} M) \to \varprojlim_n \mathrm{Hom}(A, \tau_{\leq n} M) \to 0\,.$$

Using proposition 15.2.6, as $\varprojlim^1$ of a constant functor vanishes, we get that the map

$$\mathrm{Hom}(A, M) \to \mathrm{Hom}(A, \mathbf{R}\varprojlim_n \tau_{\leq n} M)$$

is an isomorphism. This gives the isomorphism

$$M \simeq \mathbf{R}\varprojlim_n \tau_{\leq n} M\,.$$

Using the previous isomorphism, and by contemplating the homotopy limit of the homotopy cofiber sequences

$$\tau_{>n} M \to M \to \tau_{\leq n} M\,,$$

we deduce the isomorphism $\mathbf{R}\varprojlim_n \tau_{>n} M \simeq 0$. □

Lemma 15.2.8. *We have $H_{\text{Þ}} \in \mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)_{\geq 0}$, so that we have a canonical map*

$$H_{\text{Þ}} \to H_0(H_{\text{Þ}})$$

in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$. In particular, for any object M in the heart of the homotopy t-structure, if M is endowed with an action of the monoid $H_0(H_{\text{Þ}})$, then M has a natural structure of $H_{\text{Þ}}$-module in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$.

Proof. As $H_{\text{Þ}}$ is isomorphic to the motivic cohomology spectrum in the sense of Voevodsky (15.1.6), the first assertion is the first assertion of [Mor03, theorem 5.3.2]. Therefore, the truncation triangle for $H_{\text{Þ}}$ gives a triangle

$$\tau_{>0} H_{\text{Þ}} \to H_{\text{Þ}} \to H_0(H_{\text{Þ}}) \to \tau_{>0} H_{\text{Þ}}[1]\,,$$

which gives the second assertion. For the third assertion, consider an object M in the heart of the homotopy t-structure, endowed with an action of $H_0(H_{\text{Þ}})$. Note that $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)_{\geq 0}$ is closed under tensor product, so that $H_{\text{Þ}} \otimes^{\mathbf{L}}_S M$ is in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)_{\geq 0}$. Hence we have natural maps

$$H_{\text{Þ}} \otimes^{\mathbf{L}}_S M \to H_0(H_{\text{Þ}} \otimes^{\mathbf{L}}_S M) \to H_0(H_0(H_{\text{Þ}}) \otimes^{\mathbf{L}}_S M) = H_0(H_{\text{Þ}}) \otimes^h M\,.$$

Then the structural map $H_0(H_{\text{Þ}}) \otimes^h M \to M$ defines a map $H_{\text{Þ}} \otimes^{\mathbf{L}}_S M \to M$ which gives the expected action (observe that, as we already know that $H_{\text{Þ}}$-modules do form a thick subcategory of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$ (13.2.8), we don't even need to check all the axioms of an internal module: it is sufficient to check that the unit $\mathbf{Q} \to H_{\text{Þ}}$ induces a section $M \to H_{\text{Þ}} \otimes^{\mathbf{L}}_S M$ of the map constructed above). □

Lemma 15.2.9. *We have the following exact sequence in the heart of the homotopy t-structure.*

$$H_0(\mathbf{Q}(1)[1]) \xrightarrow{\eta} H_0(\mathbf{Q}) \to H_0(H_{\text{Þ}}) \to 0$$

Proof. Using the equivalence of categories from the heart of the homotopy t-structure to the category of homotopy modules in the sense of [Mor03, definition 5.2.4], we know that $H_0(H_{\text{Þ}})$ corresponds to the homotopy module $\underline{K}^M_* \otimes \mathbf{Q}$ associated to Milnor K-theory (see [Mor03, theorem 5.3.2]), while $H_0(\mathbf{Q})$ corresponds to the homotopy module $\underline{K}^{MW}_* \otimes \mathbf{Q}$ associated to Milnor-Witt K-theory (which follows easily from [Mor06a, theorem 3.40]). Considering $\underline{K}^M_*$ and $\underline{K}^{MW}_*$ as unramified sheaves in the sense of [Mor06a], this lemma is then a reformulation of the isomorphism

$$K^{MW}_*(F)/\eta \simeq K^M_*(F)$$

for any field F; see [Mor06a, remark 2.2]. □

Proposition 15.2.10. *We have $H_{\text{Þ}+} \simeq H_{\text{Þ}}$, and the induced map $\mathbf{Q}_+ \to H_{\text{Þ}}$ gives a canonical isomorphism $H_0(\mathbf{Q}_+) \simeq H_0(H_{\text{Þ}})$.*

Proof. The map $\epsilon(1)[1] : \mathbf{Q}(1)[1] \to \mathbf{Q}(1)[1]$ can be described geometrically as the morphism associated to the pointed morphism

$$\imath : \mathbf{G}_m \to \mathbf{G}_m\ , \quad t \mapsto t^{-1}$$

(see the second assertion of [Mor03, lemma 6.1.1]). In the decomposition

$$K_1(\mathbf{G}_m) \simeq k[t, t^{-1}]^\times \simeq k^\times \oplus \mathbf{Z}\,,$$

the map $\imath$ induces multiplication by -1 on $\mathbf{Z}$. Using the periodicity isomorphism $KGL(1)[2] \simeq KGL$, we get the identifications:

$$K_1(\mathbf{G}_m) \supset \mathrm{Hom}_{\mathrm{SH}(k)}(\Sigma^\infty(\mathbf{G}_m)[1], KGL) \simeq \mathrm{Hom}_{KGL}(KGL, KGL) \simeq K_0(k) \simeq \mathbf{Z}\,.$$

Therefore, ϵ acts as the multiplication by -1 on the spectrum $KGL_{\mathbf{Q}}$, whence on H_{B} as well. This means precisely that $H_{\mathrm{B}+} \simeq H_{\mathrm{B}}$. By lemma 15.2.3, the class 2η vanishes in $\mathbf{Q}_+$, so that, appyling the (t-exact) functor $M \mapsto M_+$ to the exact sequence of lemma 15.2.9, we get an isomorphism $H_0(\mathbf{Q}_+) \simeq H_0(H_{\mathrm{B}+}) \simeq H_0(H_{\mathrm{B}})$. □

Corollary 15.2.11. *For any object M in the heart of the homotopy t-structure, M_+ is a Beilinson motive.*

Proof. The object M is an $H_0(\mathbf{Q})$-module, so that M_+ is an $H_0(\mathbf{Q}_+)$-module. By virtue of proposition 15.2.10, M_+ is then a module over $H_0(H_{\mathrm{B}})$, so that, by lemma 15.2.8, M_+ is naturally endowed with an action of H_{B}. □

Remark 15.2.12. Until now, we didn't really use the fact we are in a $\mathbf{Q}$-linear context (replacing H_{B} by Voevodsky's motivic spectrum, we just needed 2 to be invertible in the preceding corollary). However, the following result really uses $\mathbf{Q}$-linearity (because, in the proof, we see $\mathrm{DM}_{\mathrm{B}}(S) \simeq \mathrm{DM}_{\mathbf{Q}}(S)$ as a full subcategory of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$).

Theorem 15.2.13 (Morel)**.** *For any noetherian scheme of finite dimension S, the map $\mathbf{Q}_+ \to H_{\mathrm{B}}$ is an isomorphism in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$. As a consequence, we have a canonical equivalence of categories*

$$\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)_+ \simeq \mathrm{DM}_{\mathrm{B}}(S)\,.$$

Proof. Observe that, if ever $\mathbf{Q}_+ \simeq H_{\mathrm{B}}$, we have $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)_+ \simeq \mathrm{DM}_{\mathrm{B}}(S)$: this follows from the fact that an object M of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$ belongs to $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)_+$ (resp. to $\mathrm{DM}_{\mathrm{B}}(S)$) if and only if there exists an isomorphism $M \simeq M_+$ (resp. $M \simeq H_{\mathrm{B}} \otimes^{\mathbf{L}}_S M$; see 13.2.15). It is thus sufficient to prove the first assertion.

As both $\mathbf{Q}_+$ and H_{B} are stable by pullback, it is sufficient to treat the case where $S = \mathrm{Spec}\,(\mathbf{Z})$. Using proposition 14.2.8, we may replace S by any of its henselisations, so that, by the localization property, it is sufficient to treat the case where S is the spectrum of a (perfect) field k.

We shall prove directly that, for any object M of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$, M_+ is an H_{B}-module (or, equivalently, is H_{B}-local). Note that $\mathrm{DM}_{\mathrm{B}}(S)$ is closed under homotopy limits and homotopy colimits in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$: indeed the inclusion functor $\mathrm{DM}_{\mathrm{B}} \to \mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}$ has a left adjoint which preserves a family of compact generators, whence it also has a left adjoint (1.3.18). By virtue of proposition 15.2.7, we may thus assume that M is bounded with respect to the homotopy t-structure. As $\mathrm{DM}_{\mathrm{B}}(S)$ is certainly closed under extensions in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$, we may even assume that M belongs to the heart the homotopy t-structure. We conclude with corollary 15.2.11. □

Corollary 15.2.14 (Morel)**.** *For any noetherian scheme of finite dimension S, if -1 is a sum of squares in all the residue fields of S (e.g. if S is a scheme over a finite field), then $\mathbf{Q}_- \simeq 0$ in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$, and we have a canonical equivalence of categories*

$$\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S) \simeq \mathrm{DM}_{\mathrm{B}}(S)\,.$$

Proof. It is sufficient to prove that, under this assumption, $\mathbf{Q}_- \simeq 0$. As in the preceding proof, we may replace S by any of its henselisations (14.2.8), so that, by the localization property (and by induction on the dimension), it is sufficient to treat the case where S is the spectrum of a field k. We have to check that, if -1 is a sum of squares in k, then we have $\epsilon = -1$. Using [Mor03, remark 6.3.5 and lemma 6.3.7], we see that, if k is of characteristic 2, we always have $\epsilon = -1$, while, if the characteristic of k is distinct from 2, we have a morphism of rings

$$GW(k) \to \mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(\mathrm{Spec}(k))}(\mathbf{Q},\mathbf{Q})\,,$$

where $GW(k)$ denotes the Grothendieck group of quadratic forms over k, such that $-\epsilon$ is the image of the class of the quadratic form $-X^2$, which proves the result. (For a more precise version of this, with integral coefficients, see [Mor06a, proposition 2.13].) □

15.2.15. Consider the $\mathbf{Q}$-linear *étale motivic category* $\mathrm{D}_{\mathbf{A}^1,\text{ét}}(-,\mathbf{Q})$, defined by

$$\mathrm{D}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Q}) = \mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_{\text{ét}}(Sm/S,\mathbf{Q}))$$

(see 5.3.34). The étale sheafification functor induces a morphism of motivic categories

$$\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S) \to \mathrm{D}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Q})\,. \tag{15.2.15.1}$$

We shall prove the following result, as an application of Morel's theorem 15.2.13.

Theorem 15.2.16. *For any noetherian scheme of finite dimension S, there is a canonical equivalence of categories*

$$\mathrm{DM}_{\mathrm{B}}(S) \simeq \mathrm{D}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Q})\,.$$

In order prove theorem 15.2.16, we shall study the behaviour of the decomposition (15.2.1.3) in $\mathrm{D}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Q})$:

Lemma 15.2.17. *We have $\mathbf{Q}_- \simeq 0$ in $\mathrm{D}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Q})$.*

Proof. Proceeding as in the proof of theorem 15.2.13, we may assume that S is the spectrum of a perfect field k. By étale descent, we see that we may replace k by any of its finite extension. In particular, we may assume that -1 is a sum of squares in k. But then, by virtue of corollary 15.2.14, $\mathbf{Q}_- \simeq 0$ in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$, so that, by functoriality, $\mathbf{Q}_- \simeq 0$ in $\mathrm{D}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Q})$. □

Proof of theorem 15.2.16. Note that the functor (15.2.15.1) has a fully faithful right adjoint, whose essential image consists of objects of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$ which satisfy étale descent. As any Beilinson motive satisfies étale descent (14.2.17), $\mathrm{DM}_{\mathrm{B}}(S)$ can be seen naturally as a full subcategory of $\mathrm{D}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Q})$. On the other hand, by virtue of the preceding lemma, any object of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$ which satisfies étale descent belongs to $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)_+$. Hence, by theorem 15.2.13, any object of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$ which satisfies étale descent is a Beilinson motive. This achieves the proof. □

Remark 15.2.18. If S is excellent, and if all the residue fields of S are of characteristic zero, one can prove theorem 15.2.16 independently of Morel's theorem: this follows then directly from a descent argument, namely from corollary 3.3.37 and from theorem 15.1.3.

Corollary 15.2.19. *For any regular noetherian scheme of finite Krull dimension S, we have canonical isomorphisms*

$$\mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Q})}(\mathbf{Q}_S, \mathbf{Q}_S(p)[q]) \simeq Gr^p_\gamma K_{2p-q}(S)_{\mathbf{Q}}\,.$$

Proof. This follows immediately from theorem 15.2.16, by definition of DM_{B} (13.2.14). □

Corollary 15.2.20. *For any geometrically unibranch excellent noetherian scheme of finite Krull dimension S, there is a canonical equivalence of symmetric monoidal triangulated categories*

$$\mathrm{D}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Q}) \simeq \mathrm{DM}_{\mathbf{Q}}(S)\,.$$

Proof. This follows from theorems 15.1.4 and 15.2.16. □

Remark 15.2.21. The corollary above is also proved by Ayoub and Zucker [AZ09] in the case where S is the spectrum of a field.

Corollary 15.2.22. *Let S be an excellent noetherian scheme of finite dimension. An object of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$ satisfies* h*-descent if and only if it satisfies étale descent.*

Proof. This follows from theorems 15.1.3 and 15.2.16. □

16. Realizations

16.1. **Tilting.**

16.1.1. Let $\mathscr{M}$ be a stable perfect symmetric monoidal *Sm*-fibred combinatorial model category over an adequate category of $\mathcal{S}$-schemes $\mathscr{S}$, such that $\mathrm{Ho}(\mathscr{M})$ is motivic, with generating set of twists τ.

Consider a homotopy cartesian commutative monoid $\mathcal{E}$ in $\mathscr{M}$. Then $\mathcal{E}$-mod is an *Sm*-fibred model category, such that $\mathrm{Ho}(\mathcal{E}\text{-mod})$ is motivic, and we have a morphism of motivic categories (see 4.2.11 and 4.2.16)

$$\mathrm{Ho}(\mathscr{M}) \to \mathrm{Ho}(\mathcal{E}\text{-mod})\ , \quad M \mapsto \mathcal{E} \otimes^{\mathbf{L}} M\,.$$

Moreover this change of scalars functor commutes with the six operations of Grothendieck. In practice, all the realization functors are obtained in this way (at least over fields), which can be formulated as follows (for simplicity, we shall work here in a **Q**-linear context, but, if we are ready to consider higher categorical constructions, there is no reason to make such an assumption).

16.1.2. Consider a quasi-excellent noetherian scheme of finite dimension S, as well as two stable symmetric monoidal Sm-fibred combinatorial model categories $\mathscr{M}$ and $\mathscr{M}'$ over the category of S-schemes of finite type such that $\mathrm{Ho}(\mathscr{M})$ and $\mathrm{Ho}(\mathscr{M}')$ are motivic. We also assume that both $\mathrm{Ho}(\mathscr{M})$ and $\mathrm{Ho}(\mathscr{M}')$ are **Q**-linear and separated, and are generated by their Tate twists (1.1.40).

Consider a Quillen adjunction

$$\varphi^* : \mathscr{M} \rightleftarrows \mathscr{M}' : \varphi_* , \tag{16.1.2.1}$$

inducing a morphism of Sm-fibred categories

$$\mathbf{L}\varphi^* : \mathrm{Ho}(\mathscr{M}) \to \mathrm{Ho}(\mathscr{M}') . \tag{16.1.2.2}$$

Note that the functor $\mathbf{L}\varphi^*$ preserves constructible objects, whence defines a morphism of motivic categories

$$\mathbf{L}\varphi^* : \mathrm{Ho}(\mathscr{M})_c \to \mathrm{Ho}(\mathscr{M}')_c . \tag{16.1.2.3}$$

Proposition 16.1.3. *Under the assumptions of 16.1.2, if, for any regular S-scheme of finite type X, and for any integers p and q, the map*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(X)}(\mathbb{1}_X, \mathbb{1}_X(p)[q]) \to \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M}')(X)}(\mathbb{1}_X, \mathbb{1}_X(p)[q])$$

is bijective, then the morphism (16.1.2.3) *is an equivalence of motivic categories. Moreover, if both* $\mathrm{Ho}(\mathscr{M})$ *and* $\mathrm{Ho}(\mathscr{M}')$ *are compactly generated by their Tate twists, then the morphism 16.1.2.2 is an equivalence of categories.*

Proof. Note first that, for any S-scheme of finite type X, and for any integers p and q, the map

$$\mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(X)}(\mathbb{1}_X, \mathbb{1}_X(p)[q]) \to \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M}')(X)}(\mathbb{1}_X, \mathbb{1}_X(p)[q])$$

is bijective: by h-descent (3.3.36) and by virtue of Gabber's weak uniformization theorem 14.1.15, it is sufficient to treat the case where X is regular, which is done by assumption. Let S be a S-scheme of finite type. To prove that the functor

$$\mathbf{L}\varphi^* : \mathrm{Ho}(\mathscr{M})_c(S) \to \mathrm{Ho}_c(\mathscr{M}')(S)$$

is an equivalence, by virtue of theorem 14.1.18, it is sufficient to prove that, for any projective morphisms $f : X \to S$ and $g : Y \to S$, and for any integers p and q, the map

$$\mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(X)}(\mathbf{R}f_*(\mathbb{1}_X), \mathbf{R}g_*(\mathbb{1}_Y)(p)[q]) \to \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M}')(X)}(\mathbf{R}f_*(\mathbb{1}_X), \mathbf{R}g_*(\mathbb{1}_Y)(p)[q])$$

is an isomorphism. Consider the pullback square

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{pr_2} & Y \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & S \end{array}$$

Considering the isomorphisms (obtained by adjunction and proper base change)

$$\begin{aligned} \mathrm{Hom}(\mathbf{R}f_*(\mathbb{1}_X), \mathbf{R}g_*(\mathbb{1}_Y)(p)[q]) &\simeq \mathrm{Hom}(\mathbf{L}g^*\,\mathbf{R}f_*(\mathbb{1}_X), \mathbb{1}_X(p)[q]) \\ &\simeq \mathrm{Hom}(\mathbf{R}pr_{2,*}\,\mathbf{L}pr_1^*(\mathbb{1}_X), \mathbb{1}_X(p)[q]) \\ &\simeq \mathrm{Hom}(\mathbf{R}pr_{2,*}(\mathbb{1}_{X\times_S Y}), \mathbb{1}_X(p)[q]) , \end{aligned}$$

we conclude easily from proposition 2.4.17 that (16.1.2.3) is an equivalence of motivic categories. If both $\mathrm{Ho}(\mathscr{M})$ and $\mathrm{Ho}(\mathscr{M}')$ are compactly generated by their Tate twists, then the sum preserving exact functor

$$\mathbf{L}\varphi^* : \mathrm{Ho}(\mathscr{M})(S) \to \mathrm{Ho}(\mathscr{M}')(S)$$

is an equivalence at the level of compact objects, whence it is an equivalence of categories (1.3.19). □

16.1.4. Under the assumptions of 16.1.2, assume that $\mathscr{M}$ and $\mathscr{M}'$ are strongly $\mathbf{Q}$-linear (4.1.4), left proper, tractable, satisfy the monoid axiom, and have cofibrant unit objects. Let $\mathcal{E}'$ be a fibrant resolution of $\mathbb{1}$ in $\mathscr{M}'(\mathrm{Spec}\,(k))$. By virtue of theorem 4.1.8, we may assume that $\mathcal{E}'$ is a fibrant and cofibrant commutative monoid in $\mathscr{M}'$. Then $\mathbf{R}\varphi_*(\mathbb{1}) = \varphi_*(\mathcal{E}')$. is a commutative monoid in $\mathscr{M}$. Let $\mathcal{E}$ be a cofibrant resolution of $\varphi_*(\mathcal{E}')$ in $\mathscr{M}(\mathrm{Spec}\,(k))$. Using theorem 4.1.8, we may assume that $\mathcal{E}$ is a fibrant and cofibrant commutative monoid, and that the map

$$\mathcal{E} \to \mathbf{R}\varphi_*(\mathcal{E}')$$

is a morphism of commutative monoids (and a weak equivalence by construction). We can see $\mathcal{E}$ and $\mathcal{E}'$ as cartesian commutative monoids in $\mathscr{M}$ and $\mathscr{M}'$ respectively (by considering their pullbacks along morphisms of finite type $f : X \to \mathrm{Spec}\,(k)$). We obtain the essentially commutative diagram of left Quillen functors below (in which the lower horizontal map is the functor induced by φ^* and by the change of scalars functor along the map $\varphi^*(\mathcal{E}) \to \mathcal{E}'$).

$$\begin{array}{ccc} \mathscr{M} & \longrightarrow & \mathscr{M}' \\ \downarrow & & \downarrow \\ \mathcal{E}\text{-}\mathrm{mod} & \longrightarrow & \mathcal{E}'\text{-}\mathrm{mod} \end{array} \tag{16.1.4.1}$$

Note furthermore that the right hand vertical left Quillen functor is a Quillen equivalence by construction (identifying $\mathscr{M}'(X)$ with $\mathbb{1}_X$-modules, and using the fact that the morphism of monoids $\mathbb{1}_X \to \mathcal{E}'_X$ is a weak equivalence in $\mathscr{M}'(X)$).

Theorem 16.1.5. *Consider the assumptions of 16.1.4, with $S = \mathrm{Spec}\,(k)$ the spectrum of a field k. We suppose furthermore that one of the following conditions is verified.*

(i) The field k is perfect.
(ii) The motivic categories $\mathrm{Ho}(\mathscr{M})$ and $\mathrm{Ho}(\mathscr{M}')$ are continuous and semi-separated.

Then the morphism

$$\mathrm{Ho}(\mathcal{E}\text{-}\mathrm{mod})_c \to \mathrm{Ho}(\mathcal{E}'\text{-}\mathrm{mod})_c \simeq \mathrm{Ho}(\mathscr{M}')_c$$

is an equivalence of motivic categories. Under these identifications, the morphism (16.1.2.3) *corresponds to the change of scalar functor*

$$\mathrm{Ho}(\mathscr{M})_c \to \mathrm{Ho}(\mathcal{E}\text{-}\mathrm{mod})_c\ , \quad M \mapsto \mathcal{E} \otimes^{\mathbf{L}} M\,.$$

If moreover both $\mathrm{Ho}(\mathscr{M})$ and $\mathrm{Ho}(\mathscr{M}')$ are compactly generated by their Tate twists, then these identifications extend to non-constructible objects, so that, in particular, the morphism (16.1.2.2) *corresponds to the change of scalar functor*

$$\mathrm{Ho}(\mathscr{M}) \to \mathrm{Ho}(\mathscr{M}') \simeq \mathrm{Ho}(\mathcal{E}\text{-}\mathrm{mod})\ , \quad M \mapsto \mathcal{E} \otimes^{\mathbf{L}} M\,.$$

Proof. For any regular k-scheme of finite type X, and for any integers p and q, the map

$$\mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(X)}(\mathbb{1}_X, \mathcal{E}_X(p)[q]) \to \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M}')(X)}(\mathbb{1}_X, \mathcal{E}'_X(p)[q])$$

is bijective: this is easy to check whenever X is smooth over k, which proves the assertion under condition (i), while, under condition (ii), we see immediately from proposition 14.2.15 that we may assume condition (i). The first assertion is then a special case of the first assertion of proposition 16.1.3. Similarly, by proposition 4.2.6, the second assertion follows from the second assertion of proposition 16.1.3. □

Example 16.1.6. Let $\mathscr{M}$ be the stable Sm-fibred model category of Tate spectra, so that $\mathrm{Ho}(\mathscr{M}) = \mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}$, and write $\mathscr{M}_{\mathrm{B}}$ for the left Bousfield localization of $\mathscr{M}$ by the class of H_{B}-equivalences (see 13.2.3), so that $\mathrm{Ho}(\mathscr{M}_{\mathrm{B}}) = \mathrm{DM}_{\mathrm{B}}$.

Let k be a field of characteristic zero, endowed with an embedding $\sigma : k \to \mathbf{C}$. Given a complex analytic manifold X, let $\mathscr{M}_{an}(X)$ be the category of complexes of sheaves of $\mathbf{Q}$-vector spaces on the smooth analytic site of X (i.e. on the category of smooth analytic X-manifolds, endowed with the Grothendieck topology corresponding to open coverings), endowed with its local model structure (see [Ayo07b, 4.4.16] and [Ayo08]). We shall write $\mathscr{M}_{Betti}^{\mathit{eff}}(X)$ for the stable left Bousfield

localization of $\mathscr{M}_{an}(X)$ by the maps of shape $\mathbf{Q}(U \times \mathbf{D}^1) \to \mathbf{Q}(U)$ for any analytic smooth $X(\mathbf{C})$-manifold U (where $\mathbf{D}^1 = \{z \in \mathbf{C},\ |z| < 1\}$ denotes the open disc). We define at last $\mathscr{M}_{Betti}(X)$ as the stable model category of analytic $\mathbf{G}_m$-spectra over $X(\mathbf{C})$ (see [Ayo08, section 1]).

Given a k-scheme of finite type X, we shall write

$$\mathrm{D}_{Betti}(X) = \mathrm{Ho}(\mathscr{M}_{Betti}(X)) \tag{16.1.6.1}$$

(where the topological space $X(\mathbf{C})$ is endowed with its canonical analytic structure). According to [Ayo08, 1.8 and 1.10], there exists a canonical equivalences of categories

$$\mathrm{D}_{Betti}(X) \simeq \mathrm{Ho}(\mathscr{M}_{Betti}^{\mathit{eff}}(X)) \simeq \mathrm{D}(X(\mathbf{C}), \mathbf{Q})\,, \tag{16.1.6.2}$$

where $\mathrm{D}(X(\mathbf{C}), \mathbf{Q})$ stands for the (unbounded) derived category of the abelian category of sheaves of $\mathbf{Q}$-vector spaces on the small site of $X(\mathbf{C})$. By virtue of [Ayo08, section 2], there exists a symmetric monoidal left Quillen morphism of monoidal Sm-fibred model categories over the category of k-schemes of finite type

$$An^* : \mathscr{M} \to \mathscr{M}_{Betti}\,, \tag{16.1.6.3}$$

which induces a morphism of motivic categories over the category of k-schemes of finite type. Hence $\mathbf{R}An_*(\mathbb{1})$ is a ring spectrum in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(\mathrm{Spec}\,(k))$ which represents Betti cohomology of smooth k-schemes. As D_{Betti} satisfies étale descent, it follows from corollary 3.3.37 that it satisfies h-descent, from which, by virtue of theorem 15.1.3, the morphism (16.1.6.3) defines a left Quillen functor

$$An^* : \mathscr{M}_{\mathrm{B}} \to \mathscr{M}_{Betti}\,, \tag{16.1.6.4}$$

hence gives rise to a morphism of motivic categories

$$\mathrm{DM}_{\mathrm{B}} \to \mathrm{D}_{Betti}\ . \tag{16.1.6.5}$$

Appyling theorem 16.1.5 to (16.1.6.4), we obtain a commutative ring spectrum $\mathcal{E}_{Betti} = \mathbf{R}An_*(\mathbb{1})$ which represents Betti cohomology of smooth k-schemes, such that the restriction of the functor (16.1.6.5) to constructible objects corresponds to the change of scalars functors

$$\mathrm{DM}_{\mathrm{B},c}(X) \to \mathrm{Ho}(\mathcal{E}_{Betti}\text{-}\,\mathrm{mod})_c(X) \simeq \mathrm{D}_c^b(X(\mathbf{C}), \mathbf{Q})\ , \quad M \mapsto \mathcal{E}_{Betti} \otimes^{\mathbf{L}} M\,. \tag{16.1.6.6}$$

In other words, once Betti cohomology of smooth k-schemes is known, one can reconstruct canonically the bounded derived categories of constructible sheaves on $X(\mathbf{C})$ for any k-scheme of finite type X, from the theory of mixed motives. We expect all the realization functors to be of this shape (which should follow from (some variant of) theorem 16.1.5): the (absolute) cohomology of smooth k-schemes with constant coefficients determines the derived categories of constructible sheaves over any k-scheme of finite type, whatever this means. For instance, the whole theory of variations of mixed Hodge structures should be obtained from Deligne cohomology, seen as a ring spectrum in $\mathrm{DM}_{\mathrm{B}}(k)$ (or, more precisely, in $\mathscr{M}_{\mathrm{B}}(k)$).

16.2. **Mixed Weil cohomologies.** Let S be an excellent (regular) noetherian scheme of finite dimension, and $\mathbf{K}$ a field of characteristic zero, called the *field of coefficients*.

16.2.1. Let E be a Nisnevich sheaf of commutative differential graded $\mathbf{K}$-algebras (i.e. is a commutative monoid in the category of sheaves of complexes of $\mathbf{K}$-vector spaces). We shall write

$$H^n(X, E) = \mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}^{\mathit{eff}}(X)}(\mathbf{Q}_X, E[n])$$

for any smooth S-scheme of finite type X, and for any integer n (note that, if E satisfies Nisnevich descent and is $\mathbf{A}^1$-homotopy invariant, which we can always assume, using 4.1.8, then $H^n(X, E) = H^n(E(X))$).

We introduce the following axioms :

W1 *Dimension.*— $H^i(S, E) \simeq \begin{cases} \mathbf{K} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$

W2 *Stability.*— $\dim_{\mathbf{K}} H^i(\mathbf{G}_m, E) = \begin{cases} 1 & \text{if } i = 0 \text{ or } i = 1, \\ 0 & \text{otherwise.} \end{cases}$

W3 *Künneth formula.*— For any smooth S-schemes X and Y, the exterior cup product induces an isomorphism

$$\bigoplus_{p+q=n} H^p(X,E) \otimes_{\mathbf{K}} H^q(Y,E) \xrightarrow{\sim} H^n(X \times_S Y, E) \ .$$

W3′ *Weak Künneth formula.*— For any smooth S-scheme X, the exterior cup product induces an isomorphism

$$\bigoplus_{p+q=n} H^p(X,E) \otimes_{\mathbf{K}} H^q(\mathbf{G}_m,E) \xrightarrow{\sim} H^n(X \times_S \mathbf{G}_m, E) \ .$$

16.2.2. Under assumptions W1 and W2, we will call any non zero element $c \in H^1(\mathbf{G}_m, E)$ a *stability class*. Note that such a class corresponds to a non trivial map

$$c : \mathbf{Q}_S(1) \to E$$

in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,\mathbf{Q}}(S)$ (using the decomposition $\mathbf{Q}(\mathbf{G}_m) = \mathbf{Q} \oplus \mathbf{Q}(1)[1]$). In particular, possibly after replacing E by a fibrant resolution (so that E is homotopy invariant and satisfies Nisnevich descent), such a stability class can be lifted to an actual map of complexes of presheaves. Such a lift will be called a *stability structure* on E.

Definition 16.2.3. A sheaf of commutative differential graded $\mathbf{K}$-algebras E as above is a *mixed Weil cohomology* (resp. a *stable cohomology*) if it satisfies the properties W1, W2 and W3 (resp. W1, W2 and W3′) stated above.

Proposition 16.2.4. *Let E be a stable cohomology. There exists a commutative ring spectrum $\mathcal{E}$ in $\mathrm{DM}_{\mathrm{B}}(S)$ with the following properties.*

(i) For any smooth S-scheme X, and any integer i, there is a canonical isomorphism of $\mathbf{K}$-vector spaces

$$H^i(X,E) \simeq \mathrm{Hom}_{\mathrm{DM}_{\mathrm{B}}(S)}(M_S(X), \mathcal{E}[i]) \, .$$

(ii) Any choice of a stability structure on E defines a map $\mathbf{Q}(1) \to \mathcal{E}$ in $\mathrm{DM}_{\mathrm{B}}(S)$, which induces an $\mathcal{E}$-linear isomorphism $\mathcal{E}(1) \simeq \mathcal{E}$.

Proof. By [CD07, Proposition 2.1.6], there exists a commutative ring spectrum $\mathcal{E}$ in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$ such that, for any smooth S-scheme X, and any integer i, there is a canonical isomorphism of $\mathbf{K}$-vector spaces

$$H^i(X,E) \simeq \mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)}(M_S(X), \mathcal{E}[i]) \, ,$$

and such that any choice of a stability structure on E defines an isomorphism $\mathcal{E}(1) \simeq \mathcal{E}$. By virtue of [CD07, corollary 2.2.8] and of theorem 11.2.10, this ring spectrum $\mathcal{E}$ is oriented, so that, by corollary 13.2.15, $\mathcal{E}$ is an H_{B}-module, i.e. belongs to $\mathrm{DM}_{\mathrm{B}}(S)$. □

16.2.5. Given a stable cohomology E and its associated ring spectrum $\mathcal{E}$, we can see $\mathcal{E}$ as a cartesian commutative monoid: we define, for an S-scheme X, with structural map $f : X \to S$:

$$\mathcal{E}_X = \mathbf{L}f^*(\mathcal{E}) \, ,$$

and put

$$\mathrm{D}(X,\mathcal{E}) = \mathrm{Ho}(\mathcal{E}\text{-}\mathrm{mod})(X) = \mathrm{Ho}(\mathcal{E}_X\text{-}\mathrm{mod}) \, . \tag{16.2.5.1}$$

We thus have realization functors

$$\mathrm{DM}_{\mathrm{B}}(X) \to \mathrm{D}(X,\mathcal{E}) \ , \quad M \mapsto \mathcal{E}_X \otimes^{\mathbf{L}}_X M \tag{16.2.5.2}$$

which commute with the six operations of Grothendieck. Furthermore, $\mathrm{D}(-,\mathcal{E})$ is a motivic category which is $\mathbf{Q}$-linear (in fact $\mathbf{K}$-linear), separated, and continuous. Observe furthermore that, if S is the spectrum of a field , then $\mathrm{D}(-,\mathcal{E})$ is also pure (14.3.22), so that the six Grothendieck operations preserves constructible objects in $\mathrm{D}(-,\mathcal{E})$ (14.1.31).

For an S-scheme X, define

$$H^q(X, E(p)) = \mathrm{Hom}_{\mathrm{DM}_{\mathrm{B}}(X)}(\mathbf{Q}_X, \mathcal{E}(p)[q]) \simeq \mathrm{Hom}_{\mathrm{D}(X,\mathcal{E})}(\mathcal{E}_X, \mathcal{E}_X(p)[q])$$

(this notation is compatible with 16.2.1 by virtue of proposition 16.2.4).

Corollary 16.2.6. *Any stable cohomology (in particular, any mixed Weil cohomology) extends naturally to S-schemes of finite type, and this extension satisfies cohomological* h*-descent (in particular, étale descent as well as proper descent).*

Proof. This follows immediately from the construction above and from theorem 13.3.2. □

16.2.7. We denote by $\mathrm{D}^{\vee}(S,\mathcal{E})$ the localizing subcategory of $\mathrm{D}(S,\mathcal{E})$ generated by its rigid objects (i.e. by the objects which have strong duals). For instance, by Poincaré duality, for any smooth and proper S-scheme X, $\mathcal{E}(X)=\mathcal{E}\otimes^{\mathbf{L}}_{S}M_S(X)$ belongs to $\mathrm{D}^{\vee}(S,\mathcal{E})$; see 14.3.23.

If we denote by $\mathrm{D}(\mathbf{K})$ the (unbounded) derived category of the abelian category of $\mathbf{K}$-vector spaces, we get the following interpretation of the Künneth formula.

Theorem 16.2.8. *If E is a mixed Weil cohomology, then the functor*

$$\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E},-):\mathrm{D}^{\vee}(S,\mathcal{E})\to\mathrm{D}(\mathbf{K})$$

is an equivalence of symmetric monoidal triangulated categories.

Proof. This is [CD07, theorem 2.6.2]. □

Theorem 16.2.9. *If S is the spectrum of a field, then $\mathrm{D}^{\vee}(S,\mathcal{E})=\mathrm{D}(S,\mathcal{E})$.*

Proof. This follows then from corollary 14.3.24. □

Remark 16.2.10. It is not reasonnable to expect the analog of theorem 16.2.9 to hold whenever S is of dimension >0; see (the proof of) [CD07, corollary 3.2.7]. Heuristically, for higher dimensional schemes X, the rigid objects of $\mathrm{D}(X,\mathcal{E})$ are extensions of some kind of locally constant sheaves (in the ℓ-adic setting, these correspond to $\mathbf{Q}_{\ell}$*-faisceaux lisses*).

Corollary 16.2.11. *If E is a mixed Weil cohomology, and if S is the spectrum of a field, then the functor*

$$\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E},-):\mathrm{D}(S,\mathcal{E})\to\mathrm{D}(\mathbf{K})$$

is an equivalence of symmetric monoidal triangulated categories.

16.2.12. Assume that E is a mixed Weil cohomology, and that S is the spectrum of a field k. For each k-scheme of finite type X, denote by $\mathrm{D}_c(X,\mathcal{E})$ the category of constructible objects of $\mathrm{D}(X,\mathcal{E})$: by definition, this is the thick triangulated subcategory of $\mathrm{D}(X,\mathcal{E})$ generated by objects of shape $\mathcal{E}(Y)=\mathcal{E}\otimes^{\mathbf{L}}_{X}M_X(Y)$ for Y smooth over X (we can drop Tate twists because of 16.2.4 (ii)). The category $\mathrm{D}_c(X,\mathcal{E})$ also coincides with the category of compact objects in $\mathrm{D}(X,\mathcal{E})$; see 14.1.2. Write $\mathrm{D}^b(\mathbf{K})$ for the bounded derived category of the abelian category of finite dimensional $\mathbf{K}$-vector spaces. Note that $\mathrm{D}^b(\mathbf{K})$ is canonically equivalent to the homotopy category of perfect complexes of $\mathbf{K}$-modules, i.e. to the category of compact objects of $\mathrm{D}(\mathbf{K})$.

Corollary 16.2.13. *Under the assumptions of 16.2.12, we have a canonical equivalence of symmetric monoidal triangulated categories*

$$\mathrm{D}_c(\mathrm{Spec}\,(k),\mathcal{E})\simeq\mathrm{D}^b(\mathbf{K})\,.$$

Proof. This follows from 16.2.11 and from the fact that equivalences of categories preserve compact objects. □

Corollary 16.2.14. *Under the assumptions of 16.2.12, if E' is another $\mathbf{K}$-linear stable cohomology with associated ring spectrum $\mathcal{E}'$, any morphism of presheaves of commutative differential $\mathbf{K}$-algebras $E\to E'$ inducing an isomorphism $H^1(\mathbf{G}_m,E)\simeq H^1(\mathbf{G}_m,E')$ gives a canonical isomorphism $\mathcal{E}\simeq\mathcal{E}'$ in the homotopy category of commutative ring spectra. In particular, we get canonical equivalences of categories*

$$\mathrm{D}(X,\mathcal{E})\simeq\mathrm{D}(X,\mathcal{E}')$$

for any k-scheme of finite type X (and these are compatible with the six operations of Grothendieck, as well as with the realization functors).

Proof. This follows from theorem 16.2.9 and from [CD07, theorem 2.6.5]. □

Remark 16.2.15. The preceding result can be stated in the following way: if $\mathcal{E}$ and $\mathcal{E}'$ are two (strict) commutative ring spectra associated to $\mathbf{K}$-linear mixed Weil cohomologies defined on smooth k-schemes, then any morphism $\mathcal{E} \to \mathcal{E}'$ in the homotopy category of (commutative) monoids in the model category of $\mathbf{K}$-linear Tate spectra is invertible.

Theorem 16.2.16. *Under the assumptions of 16.2.12, the six operations of Grothendieck preserve constructibility in* $\mathrm{D}(-,\mathcal{E})$.

Proof. Observe that $\mathrm{D}(-,\mathcal{E})$ is $\mathbf{Q}$-linear and separated (because DM_{B} is so, see 4.2.16), as well as pure (by proposition 14.3.22). We conclude with 14.1.31. □

16.2.17. As a consequence, we have, for any k-scheme of finite type X, a realization functor

$$\mathrm{DM}_{\mathrm{B},c}(X) \to \mathrm{D}_c(X,\mathcal{E})$$

which preserves the six operations of Grothendieck. For $X = \mathrm{Spec}\,(k)$, by virtue of corollary 16.2.13, this corresponds to a symmetric monoidal exact realization functor

$$R : \mathrm{DM}_{\mathrm{B},c}(\mathrm{Spec}\,(k)) \to \mathrm{D}^b(\mathbf{K})\,.$$

This leads to a finiteness result:

Corollary 16.2.18. *Under the assumptions of 16.2.12, for any k-scheme of finite type X, and for any objects M and N in $\mathrm{D}_c(X,\mathcal{E})$, $\mathrm{Hom}_{\mathcal{E}}(M,N[n])$ is a finite dimensional $\mathbf{K}$-vector space, and it is trivial for all but a finite number of values of n.*

Proof. Let $f : X \to \mathrm{Spec}\,(k)$ be the structural map. By virtue of 16.2.16, as M and N are constructible, the object $\mathbf{R}f_*\,\mathbf{R}\mathit{Hom}_X(M,N)$ is constructible as well, i.e. is a compact object of $\mathrm{D}(\mathrm{Spec}\,(k)\,,\mathcal{E})$. But $\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(M,N)$ is nothing else than the image of $\mathbf{R}f_*\,\mathbf{R}\mathit{Hom}_X(M,N)$ by the equivalence of categories given by corollary 16.2.11. Hence $\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(M,N)$ is a compact object of $\mathrm{D}(\mathbf{K})$, which means that it belongs to $\mathrm{D}^b(\mathbf{K})$. □

16.2.19. For a $\mathbf{K}$-vector space V and an integer n, define

$$V(n) = \begin{cases} V \otimes_{\mathbf{K}} \mathrm{Hom}_{\mathbf{K}}(H^1(\mathbf{G}_m,E)^{\otimes n},\mathbf{K}) & \text{if } n > 0, \\ V \otimes_{\mathbf{K}} H^1(\mathbf{G}_m,E)^{\otimes(-n)} & \text{if } n \leq 0. \end{cases}$$

Any choice of a generator in $\mathbf{K}(-1) = H^1(\mathbf{G}_m,E) \simeq H^2(\mathbf{P}^1_k,E)$ defines a natural isomorphism $V(n) \simeq V$ for any integer n. We have canonical isomorphisms

$$H^q(X,E(p)) \simeq H^q(X,E)(p)$$

(using the fact that the equivalence of corollary 16.2.13 is monoidal). The realization functors (16.2.5.2) induce in particular cycle class maps

$$cl_X : H^q(X,\mathbf{Q}(p)) \to H^q(X,E)(p)$$

(and similarly for cohomology with compact support, for homology, and for Borel-Moore homology).

Example 16.2.20. Let k be a field of characteristic zero. We then have a mixed Weil cohomology E_{dR} defined by the algebraic de Rham complex

$$E_{dR}(X) = \Omega^*_{A/k}$$

for any smooth affine k-scheme of finite type $X = \mathrm{Spec}\,(A)$ (algebraic de Rham cohomology of smooth k-schemes of finite type is obtained by Zariski descent); see [CD07, 3.1.5]. We obtain from 16.2.4 a commutative ring spectrum $\mathcal{E}_{dR}$, and, for a k-scheme of finite type X, we define

$$\mathrm{D}_{dR}(X) = \mathrm{D}_c(X,\mathcal{E}_{dR})\,.$$

We thus get a motivic category D_{dR}, and we have a natural definition of algebraic de Rham cohomology of k-schemes of finite type, given by

$$H^n_{dR}(X) = \mathrm{Hom}_{\mathrm{D}_{dR}(X)}(\mathcal{E}_{dR,X},\mathcal{E}_{dR,X}[n])\,.$$

This definition coincides with the usual one: this is true by definition for smooth k-schemes of finite type, while the general case follows from h-descent (16.2.6) and from de Jong's theorem 14.3.6 (or resolution of singularities à la Hironaka). We have, by construction, a realization functor

$$R_{dR} : \mathrm{DM}_{\mathrm{B},c}(X) \to \mathrm{D}_{dR}(X)$$

which preserves the six operations of Grothendieck. In particular, we have cycle class maps

$$H^q(X, \mathbf{Q}(p)) \to H^q_{dR}(X)(p)\,.$$

Note that, for any field extension k'/k, we have natural isomorphisms

$$H^n_{dR}(X) \otimes_k k' \simeq H^n_{dR}(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}\,(k'))\,.$$

Example 16.2.21. Let k be a field of characteristic zero, which is algebraically closed and complete with respect to some valuation (archimedian or not). We can then define a stable cohomology $E_{dR,an}$ as analytic de Rham cohomology of X^{an}, for any smooth k-scheme of finite type X; see [CD07, 3.1.7]. As above, we get a ring spectrum $\mathcal{E}_{dR,an}$, and for any k-scheme of finite type, a category of coefficients

$$\mathrm{D}_{dR,an}(X) = \mathrm{D}_c(X, \mathcal{E}_{dR,an})\,,$$

which allows to define the analytic de Rham cohomology of any k-scheme of finite type X by

$$H^n_{dR,an}(X) = \mathrm{Hom}_{\mathrm{D}_{dR,an}(X)}(\mathcal{E}_{dR,an,X}, \mathcal{E}_{dR,an,X}[n])\,.$$

We also have a realization functor

$$R_{dR,an} : \mathrm{DM}_{\mathrm{B},c}(X) \to \mathrm{D}_{dR,an}(X)$$

which preserves the six operations of Grothendieck.

We then have a morphism of stable cohomologies

$$E_{dR} \to E_{dR,an}$$

which happens to be a quasi-isomorphism locally for the Nisnevich topology (this is Grothendieck's theorem in the case where K is archimedian, and Kiehl's theorem in the case where K is non-archimedian; anyway, one obtains this directly from corollary 16.2.14). This induces a canonical isomorphism

$$\mathcal{E}_{dR} \simeq \mathcal{E}_{dR,an}$$

in the homotopy category of commutative ring spectra. In particular, $E_{dR,an}$ is a mixed Weil cohomology, and, for any k-scheme of finite type, we have natural equivalences of categories

$$\mathrm{D}_{dR}(X) \to \mathrm{D}_{dR,an}(X)\ , \quad M \mapsto \mathcal{E}_{dR,an} \otimes^{\mathbf{L}}_{\mathcal{E}_{dR}} M$$

which commute with the six operations of Grothendieck and are compatible with the realization functors.

Note that, in the case $k = \mathbf{C}$, $\mathcal{E}_{dR,an}$ coincides with Betti cohomology (after tensorization by $\mathbf{C}$), so that we have canonical fully faithful functors

$$\mathrm{D}_{Betti,c}(X) \otimes_{\mathbf{Q}} \mathbf{C} \to \mathrm{D}_{dR,an}(X)$$

which are compatible with the realization functors. More precisely, we have equivalences

$$\mathrm{D}^b_c(X(\mathbf{C}), \mathbf{C}) \simeq \mathrm{Ho}(\mathcal{E}_{Betti} \otimes_{\mathbf{Q}} \mathbf{C}\text{-}\,\mathrm{mod})_c(X) \simeq \mathrm{D}_{dR,an}(X)\,.$$

In particular, by the Riemann-Hilbert correspondence, $\mathrm{D}_{dR,an}(X)$ is equivalent to the bounded derived category of analytic regular holonomic $\mathcal{D}$-modules on X. (A purely algebraic proof of this equivalence would furnish a new proof of the Riemann-Hilbert correspondence, using corollary 16.2.14.)

Example 16.2.22. Let V be a complete discrete valuation ring of mixed characteristic with perfect residue field k and field of functions K. The Monsky-Washnitzer complex defines a stable cohomology E_{MW} over smooth V-schemes of finite type, defined by

$$E_{MW}(X) = \Omega^*_{A^\dagger/V} \otimes_V K$$

for any affine smooth V-scheme $X = \mathrm{Spec}\,(A)$ (the case of a smooth V-scheme of finite type is obtained by Zariski descent); see [CD07, 3.2.3]. Let $\mathcal{E}_{MW}$ be the corresponding ring spectrum in

$\mathrm{DM}_{\mathrm{B}}(\mathrm{Spec}\,(V))$, and write $j : \mathrm{Spec}\,(K) \to \mathrm{Spec}\,(V)$ and $i : \mathrm{Spec}\,(k) \to \mathrm{Spec}\,(V)$ for the canonical immersions. As we obviously have $j^*\mathcal{E}_{MW} = 0$ (the Monsky-Washnitzer cohomology of a smooth V-scheme with empty special fiber vanishes), we have a canonical isomorphism

$$\mathcal{E}_{MW} \simeq \mathbf{R}i_* \mathbf{L}i^*\mathcal{E}_{MW}\,.$$

We define the rigid cohomology spectrum $\mathcal{E}_{rig}$ in $\mathrm{DM}_{\mathrm{B}}(\mathrm{Spec}\,(k))$ by the formula

$$\mathcal{E}_{rig} = \mathbf{L}i^*\mathcal{E}_{MW}\,.$$

This is a ring spectrum associated to a K-linear mixed Weil cohomology: cohomology with coefficients in $\mathcal{E}_{rig}$ coincides with rigid cohomology in the sense of Berthelot, and the Künneth formula for rigid cohomology holds for smooth and projective k-schemes (as rigid cohomology coincides then with cristalline cohomology), from which we deduce the Künneth formula for smooth k-schemes of finite type; see [CD07, 3.2.10]. As before, we define

$$\mathrm{D}_{rig}(X) = \mathrm{D}_c(X, \mathcal{E}_{rig})$$

for any k-scheme of finite type X, and put

$$H^n_{rig}(X) = \mathrm{Hom}_{\mathrm{D}_{rig}(X)}(\mathcal{E}_{rig,X}, \mathcal{E}_{rig,X}[n])\,.$$

Here again, we have, by construction, realization functors

$$R_{rig} : \mathrm{DM}_{\mathrm{B},c}(X) \to \mathrm{D}_{rig}(X)$$

which preserve the six operations of Grothendieck, as well as cycle class maps

$$H^q(X, \mathbf{Q}(p)) \to H^q_{rig}(X)(p)\,.$$

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