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# PROCESS YIELD AND CAPABILITY INDICES

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## ABSTRACT

Capability indices measure the performance of a process. Although process yield is the primary focus on the performance criteria, the  $C_p''(u, v)$  indices combine process yield and process centering. With this compromise, there is no direct link between the process yield and these indices, but literature provides lower and upper bounds for the process yield. However errors in the proposed results limit the knowledge of these bounds to a few special cases. In this paper we give these bounds for any  $C_p''(u, v)$  index, allowing the user to choose the index which best suits his needs. An application on high-tech paint is also presented.

## KEYWORDS

Asymmetric tolerances, Process capability indices, Process centering, Process yield

## 1. INTRODUCTION

Process capability indices are widely used in manufacturing industries to measure the ability of a process to realize items that meet the tolerance limits  $[L; U]$ . The original reasons for introducing capability indices seem to be associated with the expected percentage of conforming items, that is, the probability of obtaining a value inside the tolerance limits. The first generation  $C_p$  and  $C_{pk}$  indices were defined in this objective. However these indices did not measure process centering, that is, process capability relative to  $T$ , the target value, and did not encourage process optimization. For this reason the  $C_{pm}$  and  $C_{pmk}$  indices were introduced. In order to generalize the four basic capability indices  $C_p$ ,  $C_{pk}$ ,  $C_{pm}$ , and  $C_{pmk}$ , Vännman (1995) proposed a superstructure containing these four basic indices as

$$C_p''(u, v) = \frac{d - u |\mu - m|}{3\sqrt{\sigma^2 + v(\mu - T)^2}},$$

where  $\mu$  and  $\sigma$  are the mean and the standard deviation of the variable of interest,  $m = (L + U)/2$  is the midpoint of the tolerance interval,  $d = (U - L)/2$  is the half-length of the tolerance interval, and  $u$  and  $v$  are two non-negative parameters. However, if indices  $C_p''(u, v)$  are well adapted to the case of symmetrical tolerances ( $T = m$ ), they have some undesirable properties when the tolerances are asymmetrical ( $T \neq m$ ) (see Boyles (1994)). To overcome

the problems with asymmetric tolerances, and to generalize the family  $C_p(u, v)$  to the case  $T \neq m$ , Chen and Pearn (2001) suggested to use the family

$$C_p''(u, v) = \frac{d^* - uA^*}{3\sqrt{\sigma^2 + vA^2}}, \quad (1)$$

in which  $A = \max\{d(\mu - T)/D_u, d(T - \mu)/D_l\}$ ,  $A^* = \max\{d^*(\mu - T)/D_u, d^*(T - \mu)/D_l\}$ ,  $D_u = U - T$ ,  $D_l = T - L$ , and  $d^* = \min\{D_u, D_l\}$ . The family  $C_p''(u, v)$  has an obvious interest since the choice of  $u$  and  $v$  allows to attach more or less importance either to the process yield, or to the process centering, which are the most important criteria to measure the process performance. However in order to enable the user to understand what these indices mean, it is necessary to explain the links which join the indices, the process yield and the process centering. Links between capability indices and process centering are known and given in the most widespread form by Chen and Pearn (2001). Links between capability indices and process yield have only been partly studied. See Juran, Gryna and Bingham (1974) for  $C_p$ , Boyles (1991) for  $C_{pk}$ , Boyles (1994), and Ruczinski (1996) for  $C_{pm}$ , Boyles (1994), and Chen and Hsu (1995) for  $C_{pmk}$ , Pearn and Chen (1998), Pearn, Lin and Chen (2004), and Chang and Wu (2008) for  $C_{pk}''$ , Pearn, Lin and Chen (1999) for  $C_{pmk}''$ , and Chen and Pearn (2001) for  $C_p''(u, v)$ . However some of these studies include errors or inaccuracies. Thus the purpose of this paper is to specify the relations between the  $C_p''(u, v)$  indices and the percentage of conforming or nonconforming items, and this for any  $u, v \geq 0$ . In the following section, the results found in the literature are recalled and the cases that have not been studied accurately are brought to the fore.

In section 3 we state several lemmas that will allow, for a given value of  $C_p''(u, v)$ , to study the variations of the conforming items proportion depending on the position of the mean process. In Section 4 we give the results of our study with six theorems specifying the minimum and maximum values of the proportion of nonconforming items. Finally in the last section we provide an example to show how the results obtained can be applied to a real industrial application.

## 2. EXISTING RESULTS

In this section, we recall the existing results concerning the links between  $C_p(u, v)$  or  $C_p''(u, v)$  indices and the process yield. These studies consider the most usual case where the variable of interest is normally distributed. In these conditions, the process yield, which we note *Yield*, is represented by the relation

$$Yield = \Phi((U - \mu)/\sigma) - \Phi((L - \mu)/\sigma), \quad (2)$$

in which  $\Phi$  is the cumulative function of the standard normal distribution. The user often prefers using the nonconforming items proportion, which we note *NC*, and which is obviously defined by the relation  $NC = 1 - Yield$ .

For  $C_p = C_p(0, 0) = (U - L)/6\sigma$ , first index introduced by Juran, Gryna and Bingham (1974), we have  $2\Phi(-3C_p) \leq NC \leq 1$ , the lower bound being reached only when the process is well centered, that is to say when  $\mu$  is on  $m$ .

For  $C_{pk} = C_p(1,0) = \min((U - \mu)/3\sigma, (\mu - L)/3\sigma)$ , index which takes into account the position of the mean inside the tolerance interval, we have  $\Phi(-3C_{pk}) \leq NC \leq 2\Phi(-3C_{pk})$  (Boyles (1991), Kotz and Johnson (1993)).

For  $C_{pm} = C_p(0,1) = (U - L)/6\sqrt{\sigma^2 + (\mu - T)^2}$ , and under the usual assumption that  $T = m$ , Rucinski (1996) shows that when  $C_{pm} < 1/3$  then  $2\Phi(-3C_{pm}) \leq NC \leq 1$ , when  $C_{pm} = 1/3$  then  $2\Phi(-3C_{pm}) = 2\Phi(-1) \leq NC \leq 1/2$ , when  $1/3 < C_{pm} < 1/\sqrt{3}$  then  $0 \leq NC \leq M$  where  $M$  is the solution of an equation which can be solved numerically, and finally when  $C_{pm} \geq 1/\sqrt{3}$ , then  $0 \leq NC \leq 2\Phi(-3C_{pm})$ .

For  $C_{pmk} = C_p(1,1) = C_{pk}C_{pm}/C_p$ , we have  $0 \leq NC \leq 2\Phi(-3C_{pmk})$  (Boyles (1994), Chen and Hsu (1995)).

Generally, when the tolerances are symmetrical, Vännman (1995) proposes the family  $C_p(u, v)$ , where  $u$  and  $v$  are two positive or null parameters. Kotz and Lovelace (1998, p.184) indicate that  $NC \leq 2\Phi(-3C_p(u, v))$  for all  $u$  and  $v$ , without taking into account the restrictions specified by Rucinski (1996) for  $C_{pm} = C_p(0,1)$ . Theorems 4, 5, and 6 will prove that this result is inaccurate when  $0 \leq u < 1$  and  $(u, v) \neq (0, 0)$ .

For asymmetrical tolerances, Chen and Pearn (2001) propose the family  $C_p''(u, v)$ . To study the process yield, these authors use the index  $S_{pk} = (1/3)\Phi^{-1}\{(1/2)\Phi((U - \mu)/\sigma) + (1/2)\Phi((\mu - L)/\sigma)\}$  suggested by Boyles (1994) which is directly related to the proportion of nonconforming items by the relation  $NC = 2\Phi(-3S_{pk})$ . After graphically noticing that  $C_p''(u, v) < S_{pk}$ , they conclude that if  $C_p''(u, v) = c$ , the process yield must be no less than that corresponding to  $S_{pk} = c$ . In other words, the proportion of nonconforming must not be greater than  $2\Phi(-3C_p''(u, v))$ . However it is possible to find values for which  $C_p''(u, v) > S_{pk}$ , which thus do not allow to obtain an upper bound of  $NC$ . For example, when  $(L, T, U) = (26, 50, 58)$ ,  $\mu = 59.3$ ,  $\sigma = 0.643$ , we have  $C_p''(0.5, 1) = 0.06$  and  $S_{pk} = 0.009$ . In these conditions, the proportion of nonconforming is equal to 0.98, a quantity which is not lower than  $2\Phi(-3C_p''(0.5, 1)) = 0.86$ .

In the particular case where  $(u, v) = (1, 1)$ , Pearn, Lin and Chen (1999) show that  $NC \leq 2\Phi(-3C_{pmk}'')$  supposing that  $C_{pmk}'' \leq C_{pmk}$ . However it is possible to find values for which  $C_{pmk}'' > C_{pmk}$ . For example, when  $(L, T, U) = (26, 50, 58)$ ,  $\mu = 49$ ,  $\sigma = 0.5$ , we have  $C_{pmk}'' = 3.07$  and  $C_{pmk} = 2.68$ .

In the particular case in which  $(u, v) = (1, 0)$ , Pearn and Chen (1998) use the fact that  $C_{pk}'' < S_{pk}$ , without proof, to show that  $NC \leq 2\Phi(-3C_{pk}'')$ . However, later, Pearn, Lin and Chen (2004), or Chang and Wu (2008), obtain a different result

$$NC \leq 2 - [\Phi(3C_{pk}'' / \min\{1, r\}) + \Phi(3C_{pk}'' \max\{1, r\})], \quad (3)$$

where  $r = D_l / D_u$ .

As we have just seen, the results evoked in the literature concerning the links between capability indices and process yield include some errors or inaccuracies. In the following section we give some necessary lemmas for a proper study of these links.

### 3. PRELIMINARY LEMMAS

To take into account the position of  $T$  in the interval  $]L;U[$ , we note  $T = m + \delta d$  where  $\delta \in ]-1;1[$ . Assuming that  $d_u = d / D_u$ , and  $d_l = d / D_l$ , we have

$$d_u = 1/(1 - \delta), \quad (4)$$

$$d_l = 1/(1 + \delta), \quad (5)$$

and

$$d^* / d = 1 - |\delta|. \quad (6)$$

To take into account the deviations of  $\mu$ , we assume that  $\mu = T + \lambda d$  where  $\lambda$  is unspecified. Links between capability indices and centering are given (Chen and Pearn (2001)) by the relation

$$T - \frac{(1-R)D_l}{3\sqrt{v}C_p''(u,v) + u(1-R)} < \mu < T + \frac{(1-R)D_u}{3\sqrt{v}C_p''(u,v) + u(1-R)}, \quad (7)$$

in which  $R = |1-r|/(1+r)$ . Since  $\lambda = (\mu - T)/d$  and  $d^*/d = 1 - R$ , the relation (7) can still be written in the form

$$-\frac{1}{d_l(\sqrt{v}d/\sigma_0 + u)} = \lambda_{\min} < \lambda < \lambda_{\max} = \frac{1}{d_u(\sqrt{v}d/\sigma_0 + u)}, \quad (8)$$

where  $\sigma_0 = d^*/(3C_p''(u,v))$ . Although it is not specified by the previous authors, note that the relation (7) is true for  $C_p''(u,v) > 0$ , and  $(u,v) \neq (0,0)$ . When  $(u,v) = (0,0)$ , the relation (8) remains true assuming that  $\lambda_{\min} = -\infty$  and  $\lambda_{\max} = +\infty$ . Thus in the following we assume that  $C_p''(u,v) > 0$  and  $u, v \geq 0$ . For given  $C_p''(u,v)$ , the relations (1) and (2) show that  $\sigma$  and *Yield* are functions of  $\mu$ , thus of  $\lambda$ . So, for given  $C_p''(u,v)$ , our purpose is to study the extrema of the process yield according to the values of  $\lambda$  defined in the relation (8). Since  $C_p''(u,v)$  is written differently depending on the sign of  $\mu - T$ , the extrema of the *Yield* function are to be searched separately in the intervals  $]\lambda_{\min}; 0]$  and  $[0; \lambda_{\max}[$ . Lemmas 1 and 3 give expressions of  $\sigma$  and *Yield* in these intervals. Lemma 2 will allow us to study the behaviour of the *Yield* function at the bounds of these intervals. Lemma 4 concerns the sign of the derivative of the *Yield* function which will enable us to obtain the extrema of this function in section 4.

#### Lemma 1 :

$$\sigma = \begin{cases} \sigma_u(\lambda) = (\sigma_0^2(1 - u\lambda d_u)^2 - v(\lambda d d_u)^2)^{1/2} & \text{if } 0 \leq \lambda < \lambda_{\max} \\ \sigma_l(\lambda) = (\sigma_0^2(1 + u\lambda d_l)^2 - v(\lambda d d_l)^2)^{1/2} & \text{if } \lambda_{\min} < \lambda \leq 0 \end{cases}.$$

#### Proof :

If  $0 \leq \lambda < \lambda_{\max}$ , then  $C_p''(u,v) = d^*(1 - u\lambda d_u) / \left(3(\sigma_u^2(\lambda) + v(\lambda d d_u)^2)^{1/2}\right)$ , thus  $\sigma_u(\lambda)$ .

If  $\lambda_{\min} < \lambda \leq 0$ , then  $C_p''(u,v) = d^*(1 + u\lambda d_l) / \left(3(\sigma_l^2(\lambda) + v(\lambda d d_l)^2)^{1/2}\right)$ , thus  $\sigma_l(\lambda)$ .

#### Lemma 2 :

If  $(u, v) \neq (0, 0)$ , then  $\lim_{\lambda \rightarrow \lambda_{\max}} \sigma_u(\lambda) = \lim_{\lambda \rightarrow \lambda_{\min}} \sigma_l(\lambda) = 0$ .

**Proof :**

Let  $(u, v) \neq (0, 0)$ . According to (8) and lemma 1, we have

$$\begin{aligned} \lim_{\lambda \rightarrow \lambda_{\max}} \sigma_u(\lambda) &= \lim_{\lambda \rightarrow 1/[d_u(\sqrt{v}d/\sigma_0 + u)]} \left( \sigma_0^2(1 - u\lambda d_u)^2 - v(\lambda d d_u)^2 \right)^{1/2} \\ &= \left[ \sigma_0^2 \left( 1 - u \frac{1}{d_u(\sqrt{v}d/\sigma_0 + u)} d_u \right)^2 - v \left( \frac{1}{d_u(\sqrt{v}d/\sigma_0 + u)} d d_u \right)^2 \right]^{1/2} \\ &= \left[ \left( \frac{\sqrt{v}d}{\sqrt{v}d/\sigma_0 + u} \right)^2 - \left( \frac{\sqrt{v}d}{\sqrt{v}d/\sigma_0 + u} \right)^2 \right]^{1/2} = 0 \end{aligned}$$

It is the same for  $\sigma_l(\lambda)$ .

**Lemma 3 :**

a) If  $(u, v) \neq (0, 0)$ , then

$$Yield = F(\lambda) = \begin{cases} F_u(\lambda) = \Phi(d(1 - \delta - \lambda)/\sigma_u(\lambda)) - \Phi(-d(1 + \delta + \lambda)/\sigma_u(\lambda)) & \text{if } 0 \leq \lambda < \lambda_{\max} \\ F_l(\lambda) = \Phi(d(1 - \delta - \lambda)/\sigma_l(\lambda)) - \Phi(-d(1 + \delta + \lambda)/\sigma_l(\lambda)) & \text{if } \lambda_{\min} < \lambda \leq 0 \end{cases}$$

b) If  $(u, v) = (0, 0)$ , then

$$Yield = F(\lambda) = \Phi(d(1 - \delta - \lambda)/\sigma_0) - \Phi(-d(1 + \delta + \lambda)/\sigma_0), \text{ for any } \lambda \in ]-\infty, +\infty[.$$

**Proof :**

Since  $\mu = T + \lambda d$  and  $T = m + \delta d$ , we have  $U - \mu = d(1 - \delta - \lambda)$ , and  $L - \mu = -d(1 + \delta + \lambda)$ , thus the lemma from (2).

**Lemma 4 :**

a) If  $(u, v) \neq (0, 0)$ , then  $F_u'(\lambda)$  has the sign of

$$Q_u(\lambda) = q_u(\lambda) + v\lambda d^2 d_u^2 - (k_u(\lambda) + (\delta + \lambda)q_u(\lambda) + \delta v\lambda d^2 d_u^2) \tanh(d^2(\delta + \lambda)/\sigma_u^2(\lambda)),$$

where  $k_u(\lambda) = \sigma_0^2(1 - u\lambda d_u)^2$  and  $q_u(\lambda) = u d_u \sigma_0^2(1 - u\lambda d_u)$ .

b) If  $(u, v) \neq (0, 0)$ , then  $F_l'(\lambda)$  has the sign of

$$Q_l(\lambda) = q_l(\lambda) + v\lambda d^2 d_l^2 - (k_l(\lambda) + (\delta + \lambda)q_l(\lambda) + \delta v\lambda d^2 d_l^2) \tanh(d^2(\delta + \lambda)/\sigma_l^2(\lambda)),$$

where  $k_l(\lambda) = \sigma_0^2(1 + u\lambda d_l)^2$  and  $q_l(\lambda) = -u d_l \sigma_0^2(1 + u\lambda d_l)$ .

c) If  $(u, v) = (0, 0)$ , then  $F'(\lambda)$  has the sign of  $Q(\lambda) = -\sinh(d^2(\delta + \lambda)/\sigma_0^2)$ .

**Proof :**

a) From lemma 3, we have  $F_u(\lambda) = \Phi(\psi_u(\lambda)) - \Phi(\varphi_u(\lambda))$ , where

$$\psi_u(\lambda) = d(1 - \delta - \lambda)/\sigma_u(\lambda), \text{ and } \varphi_u(\lambda) = -d(1 + \delta + \lambda)/\sigma_u(\lambda).$$

From lemma 1,  $\sigma_u'(\lambda) = -(u d_u \sigma_0^2(1 - u\lambda d_u) + v\lambda d^2 d_u^2)/\sigma_u(\lambda)$ , thus

$$\begin{aligned} \psi_u'(\lambda) &= [-d\sigma_u(\lambda) + d(1 - \delta - \lambda)\{u d_u \sigma_0^2(1 - u\lambda d_u) + v\lambda d^2 d_u^2\}/\sigma_u(\lambda)]/\sigma_u^2(\lambda) \\ &= [-d\sigma_u^2(\lambda) + d(1 - \delta - \lambda)\{u d_u \sigma_0^2(1 - u\lambda d_u) + v\lambda d^2 d_u^2\}]/\sigma_u^3(\lambda). \end{aligned}$$

From lemma 1, we obtain

$$\begin{aligned} \psi_u'(\lambda) &= [-d\{\sigma_0^2(1 - u\lambda d_u)^2 - v(\lambda d d_u)^2\} + d(1 - \delta - \lambda)\{u d_u \sigma_0^2(1 - u\lambda d_u) + v\lambda d^2 d_u^2\}]/\sigma_u^3(\lambda) \\ &= d[-\sigma_0^2(1 - u\lambda d_u)^2 + (1 - \delta - \lambda)u d_u \sigma_0^2(1 - u\lambda d_u) + (1 - \delta)v\lambda d^2 d_u^2]/\sigma_u^3(\lambda) \end{aligned}$$

$$\begin{aligned}
&= d[-k_u(\lambda) + (1 - \delta - \lambda)q_u(\lambda) + (1 - \delta)v\lambda d^2 d_u^2] / \sigma_u^3(\lambda) . \\
\varphi_u'(\lambda) &= [-d\sigma_u^2(\lambda) - d(1 + \delta + \lambda)\{ud_u\sigma_0^2(1 - u\lambda d_u) + v\lambda d^2 d_u^2\}] / \sigma_u^3(\lambda) \\
&= -d[\sigma_0^2(1 - u\lambda d_u)^2 - v(\lambda d d_u)^2 + (1 + \delta + \lambda)\{ud_u\sigma_0^2(1 - u\lambda d_u) + v\lambda d^2 d_u^2\}] / \sigma_u^3(\lambda) \\
&= -d[\sigma_0^2(1 - u\lambda d_u)^2 + (1 + \delta + \lambda)ud_u\sigma_0^2(1 - u\lambda d_u) + (1 + \delta)v\lambda d^2 d_u^2] / \sigma_u^3(\lambda) \\
&= -d[k_u(\lambda) + (1 + \delta + \lambda)q_u(\lambda) + (1 + \delta)v\lambda d^2 d_u^2] / \sigma_u^3(\lambda) .
\end{aligned}$$

Consequently,

$$\begin{aligned}
F_u'(\lambda) &= \Phi'(\psi_u(\lambda))\psi_u'(\lambda) - \Phi'(\varphi_u(\lambda))\varphi_u'(\lambda) \\
&= (2\pi)^{-1/2} e^{-d(1-\delta-\lambda)/\sigma_u(\lambda)^2/2} \psi_u'(\lambda) - (2\pi)^{-1/2} e^{-d(1+\delta+\lambda)/\sigma_u(\lambda)^2/2} \varphi_u'(\lambda) \\
&= (2\pi)^{-1/2} e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_u^2(\lambda))} e^{d^2(\delta+\lambda)/\sigma_u^2(\lambda)} \psi_u'(\lambda) - (2\pi)^{-1/2} e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_u^2(\lambda))} e^{-d^2(\delta+\lambda)/\sigma_u^2(\lambda)} \varphi_u'(\lambda) \\
&= (2\pi)^{-1/2} e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_u^2(\lambda))} [e^{d^2(\delta+\lambda)/\sigma_u^2(\lambda)} \psi_u'(\lambda) - e^{-d^2(\delta+\lambda)/\sigma_u^2(\lambda)} \varphi_u'(\lambda)] \\
&= (2\pi)^{-1/2} e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_u^2(\lambda))} \left[ e^{d^2(\delta+\lambda)/\sigma_u^2(\lambda)} d[-k_u(\lambda) + (1 - \delta - \lambda)q_u(\lambda) + (1 - \delta)v\lambda d^2 d_u^2] / \sigma_u^3(\lambda) \right. \\
&\quad \left. + e^{-d^2(\delta+\lambda)/\sigma_u^2(\lambda)} d(k_u(\lambda) + (1 + \delta + \lambda)q_u(\lambda) + (1 + \delta)v\lambda d^2 d_u^2) / \sigma_u^3(\lambda) \right] \\
&= d(2\pi)^{-1/2} \sigma_u^{-3}(\lambda) e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_u^2(\lambda))} \left[ (e^{d^2(\delta+\lambda)/\sigma_u^2(\lambda)} + e^{-d^2(\delta+\lambda)/\sigma_u^2(\lambda)})(q_u(\lambda) + v\lambda d^2 d_u^2) \right. \\
&\quad \left. - (e^{d^2(\delta+\lambda)/\sigma_u^2(\lambda)} - e^{-d^2(\delta+\lambda)/\sigma_u^2(\lambda)})(k_u(\lambda) + (\delta + \lambda)q_u(\lambda) + \delta v\lambda d^2 d_u^2) \right] \\
&= d\sqrt{2\pi}^{-1/2} \sigma_u^{-3}(\lambda) e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_u^2(\lambda))} \\
&\quad \times \left[ (q_u(\lambda) + v\lambda d^2 d_u^2) \cosh(d^2(\delta + \lambda) / \sigma_u^2(\lambda)) \right. \\
&\quad \left. - (k_u(\lambda) + (\delta + \lambda)q_u(\lambda) + \delta v\lambda d^2 d_u^2) \sinh(d^2(\delta + \lambda) / \sigma_u^2(\lambda)) \right] \\
&= d\sqrt{2\pi}^{-1/2} \sigma_u^{-3}(\lambda) e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_u^2(\lambda))} \cosh(d^2(\delta + \lambda) / \sigma_u^2(\lambda)) Q_u(\lambda) .
\end{aligned}$$

Now  $d\sqrt{2\pi}^{-1/2} \sigma_u^{-3}(\lambda) e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_u^2(\lambda))} \cosh(d^2(\delta + \lambda) / \sigma_u^2(\lambda)) > 0$ . Thus  $F_u'(\lambda)$  has the sign of  $Q_u(\lambda)$ .

b) The proof is similar for  $F_l'(\lambda)$ .

c) If  $(u, v) = (0, 0)$ , from the lemma 3,

$$\begin{aligned}
F'(\lambda) &= -(d / \sigma_0)[\Phi'(d(1 - \delta - \lambda) / \sigma_0) - \Phi'(-d(1 + \delta + \lambda) / \sigma_0)] \\
&= -(d / \sigma_0)[(2\pi)^{-1/2} e^{-d(1-\delta-\lambda)/\sigma_0^2/2} - (2\pi)^{-1/2} e^{-d(1+\delta+\lambda)/\sigma_0^2/2}] \\
&= -(d / \sigma_0)(2\pi)^{-1/2} e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_0^2)} [e^{d^2(\delta+\lambda)/\sigma_0^2} - e^{-d^2(\delta+\lambda)/\sigma_0^2}] \\
&= -(d / \sigma_0)\sqrt{2\pi}^{-1/2} e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_0^2)} \sinh(d^2(\delta + \lambda) / \sigma_0^2) = d\sqrt{2\pi}^{-1/2} \sigma_0^{-1} e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_0^2)} Q(\lambda) .
\end{aligned}$$

Now  $d\sqrt{2\pi}^{-1/2} \sigma_0^{-1} e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_0^2)} > 0$ , thus  $F'(\lambda)$  has the sign of  $Q(\lambda)$ .

#### 4. EXTREMA OF NONCONFORMING ITEMS PERCENTAGE

The following sub-sections explain the behaviour of the functions *Yield* or *NC*, more precisely the existence of maxima and minima, by distinguishing the various situations depending on the  $u$  and  $v$  values.

##### 4.1. Case $(u, v) = (0, 0)$

When  $(u, v) = (0, 0)$ , we have  $C_p''(u, v) = C_p''(0, 0) = C_p''$ .

**Theorem 1 :**

$$2\Phi(-3C_p''/(1-|\delta|)) \leq NC \leq 1.$$

**Proof :**

From the lemma 4,  $F'(\lambda)$  has the sign of  $Q(\lambda) = -\sinh(d^2(\delta + \lambda)/\sigma_0^2)$ .

$$\text{Thus } F'(\lambda) \begin{cases} > 0 & \text{if } \lambda < -\delta \\ = 0 & \text{if } \lambda = -\delta \\ < 0 & \text{if } \lambda > -\delta \end{cases}.$$

Consequently  $F(\lambda)$  has a unique maximum at  $\lambda = -\delta$ , and this maximum is equal to

$$F(-\delta) = 2\Phi\left(3\frac{d}{d^*}C_p''\right) - 1 = 2\Phi(3C_p''/(1-|\delta|)) - 1, \text{ from (6). On the other hand,}$$

$$\lim_{\lambda \rightarrow +\infty} F(\lambda) = \lim_{\lambda \rightarrow +\infty} [\Phi(d(1-\delta-\lambda)/\sigma_0) - \Phi(-d(1+\delta+\lambda)/\sigma_0)] = \Phi(-\infty) - \Phi(-\infty) = 0,$$

$$\lim_{\lambda \rightarrow -\infty} F(\lambda) = \lim_{\lambda \rightarrow -\infty} [\Phi(d(1-\delta-\lambda)/\sigma_0) - \Phi(-d(1+\delta+\lambda)/\sigma_0)] = \Phi(+\infty) - \Phi(+\infty) = 0.$$

Thus  $0 \leq F(\lambda) \leq 2\Phi(3C_p''/(1-|\delta|)) - 1$ , and the theorem since  $NC = 1 - \text{Yield}$ .

**Particular case:** If  $T = m$ , we have  $\delta = 0$ ,  $C_p'' = C_p$ , thus  $2\Phi(-3C_p) < NC \leq 1$ , result well known, given for example by Pearn and Kotz (2006, p.9).

#### 4.2. Case $(u, v) = (1, 0)$

When  $(u, v) = (1, 0)$ , we have  $C_p''(u, v) = C_p''(1, 0) = C_{pk}''$ .

**Theorem 2 :**

$$\Phi(-3C_{pk}''(1+|\delta|)/(1-|\delta|)) \leq NC \leq \Phi(-3C_{pk}'') + \Phi(-3C_{pk}''(1+|\delta|)/(1-|\delta|)).$$

**Proof :**

If  $(u, v) = (1, 0)$ , from (8),  $-1/d_l = \lambda_{\min} < \lambda < \lambda_{\max} = 1/d_u$ .

- Let  $0 \leq \lambda < 1/d_u$ . We have  $Q_u(\lambda) = q_u(\lambda) - (k_u(\lambda) + (\delta + \lambda)q_u(\lambda)) \tanh(d^2(\delta + \lambda)/\sigma_u^2(\lambda))$ , where  $k_u(\lambda) = \sigma_0^2(1 - \lambda d_u)^2$  and  $q_u(\lambda) = d_u \sigma_0^2(1 - \lambda d_u)$ .

$$\begin{aligned} Q_u(\lambda) &= d_u \sigma_0^2(1 - \lambda d_u) - \{\sigma_0^2(1 - \lambda d_u)^2 + (\delta + \lambda)d_u \sigma_0^2(1 - \lambda d_u)\} \tanh(d^2(\delta + \lambda)/\sigma_u^2(\lambda)) \\ &= \sigma_0^2(1 - \lambda d_u)[d_u - \{(1 - \lambda d_u) + (\delta + \lambda)d_u\} \tanh(d^2(\delta + \lambda)/\sigma_u^2(\lambda))] \\ &= \sigma_0^2(1 - \lambda d_u)[d_u - (1 + \delta d_u) \tanh(d^2(\delta + \lambda)/\sigma_u^2(\lambda))] \\ &= \sigma_0^2(1 - \lambda d_u)d_u[1 - \tanh(d^2(\delta + \lambda)/\sigma_u^2(\lambda))], \text{ from (4).} \end{aligned}$$

Now  $0 \leq \tanh(d^2(\delta + \lambda)/\sigma_u^2(\lambda)) < 1$ , and  $1 - \lambda d_u > 0$ , thus  $Q_u(\lambda) > 0$ , and from the lemma 4,  $F_u'(\lambda) > 0$ , when  $0 \leq \lambda < 1/d_u$ .

- Let  $-1/d_l < \lambda \leq 0$ . We have  $Q_l(\lambda) = q_l(\lambda) - (k_l(\lambda) + (\delta + \lambda)q_l(\lambda)) \tanh(d^2(\delta + \lambda)/\sigma_l^2(\lambda))$ , where  $k_l(\lambda) = \sigma_0^2(1 + \lambda d_l)^2$  and  $q_l(\lambda) = -d_l \sigma_0^2(1 + \lambda d_l)$ .

$$\begin{aligned} Q_l(\lambda) &= -d_l \sigma_0^2(1 + \lambda d_l) - \{\sigma_0^2(1 + \lambda d_l)^2 - (\delta + \lambda)d_l \sigma_0^2(1 + \lambda d_l)\} \tanh(d^2(\delta + \lambda)/\sigma_l^2(\lambda)) \\ &= \sigma_0^2(1 + \lambda d_l)[-d_l + \{-(1 + \lambda d_l) + (\delta + \lambda)d_l\} \tanh(d^2(\delta + \lambda)/\sigma_l^2(\lambda))] \\ &= \sigma_0^2(1 + \lambda d_l)[-d_l + (-1 + \delta d_l) \tanh(d^2(\delta + \lambda)/\sigma_l^2(\lambda))] \\ &= -d_l \sigma_0^2(1 + \lambda d_l)[1 + \tanh(d^2(\delta + \lambda)/\sigma_l^2(\lambda))], \text{ from (5).} \end{aligned}$$



Now  $0 \leq \tanh(d^2(\delta + \lambda) / \sigma_l^2(\lambda)) < 1$ , and  $1 + \lambda d_l > 0$ , thus  $Q_l(\lambda) < 0$ , and from the lemma 4,  $F_l'(\lambda) < 0$ , when  $-1/d_l < \lambda \leq 0$ .

From the study of  $F'(\lambda)$ , it results that  $F(\lambda)$  has a minimum at  $\lambda = 0$ , that is to say at  $\mu = T$ , equal to

$$\begin{aligned} F(0) &= F_u(0) = F_l(0) = \Phi(d(1-\delta)/\sigma_0) - \Phi(-d(1+\delta)/\sigma_0) \\ &= \Phi\left(3C_p''(u,v)(1-\delta)/(1-|\delta|)\right) - \Phi\left(-3C_p''(u,v)(1+\delta)/(1+|\delta|)\right) \text{ from (6). Thus} \\ F(0) &= \Phi\left(3C_p''(u,v)\right) - \Phi\left(-3C_p''(u,v)(1+|\delta|)/(1-|\delta|)\right). \end{aligned} \quad (9)$$

So, for  $(u, v) = (1, 0)$ , we have  $F(0) = \Phi\left(3C_{pk}''\right) - \Phi\left(-3C_{pk}''(1+|\delta|)/(1-|\delta|)\right)$ .

From the study of  $F'(\lambda)$ , it results that  $F(\lambda)$  has a maximum when  $\lambda \rightarrow \lambda_{\max}$  or  $\lambda \rightarrow \lambda_{\min}$ , equal to  $\max\left(\lim_{\lambda \rightarrow \lambda_{\max}} F_u(\lambda), \lim_{\lambda \rightarrow \lambda_{\min}} F_l(\lambda)\right)$ . Now when  $(u, v) = (1, 0)$ , from lemma 1 and (4),

$$\begin{aligned} \sigma_u(\lambda) &= \sigma_0(1 - \lambda d_u) = \sigma_0 d_u (1/d_u - \lambda) = \sigma_0 d_u (1 - \delta - \lambda), \text{ thus} \\ (1 - \delta - \lambda) / \sigma_u(\lambda) &= 1 / (\sigma_0 d_u), \end{aligned} \quad (10)$$

and from lemma 1 and (5),  $\sigma_l(\lambda) = \sigma_0(1 + \lambda d_l) = \sigma_0 d_l (1/d_l + \lambda) = \sigma_0 d_l (1 + \delta + \lambda)$ , thus

$$(1 + \delta + \lambda) / \sigma_l(\lambda) = 1 / (\sigma_0 d_l). \quad (11)$$

From (6), (10) and (11), and from lemmas 2 and 3, we have

$$\begin{aligned} \lim_{\lambda \rightarrow \lambda_{\max}} F_u(\lambda) &= \lim_{\lambda \rightarrow 1/d_u} \Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)) - \lim_{\lambda \rightarrow 1/d_u} \Phi(-d(1+\delta+\lambda)/\sigma_u(\lambda)) \\ &= \Phi(d/(\sigma_0 d_u)) - \Phi(-\infty) = \Phi(3C_{pk}''(1/d_u)(d/d^*)) = \Phi\left(3C_{pk}''(1-\delta)/(1-|\delta|)\right), \\ \lim_{\lambda \rightarrow \lambda_{\min}} F_l(\lambda) &= \lim_{\lambda \rightarrow -1/d_l} \Phi(d(1-\delta-\lambda)/\sigma_l(\lambda)) - \lim_{\lambda \rightarrow -1/d_l} \Phi(-d(1+\delta+\lambda)/\sigma_l(\lambda)) \\ &= \Phi(+\infty) - \Phi(-d/(\sigma_0 d_l)) = \Phi(3C_{pk}''(1/d_l)(d/d^*)) = \Phi\left(3C_{pk}''(1+\delta)/(1+|\delta|)\right). \end{aligned}$$

Finally  $F(\lambda)$  has an upper bound equal to

$$\max\left(\Phi\left(3C_{pk}''(1-\delta)/(1-|\delta|)\right), \Phi\left(3C_{pk}''(1+\delta)/(1+|\delta|)\right)\right) = \Phi\left(3C_{pk}''(1+|\delta|)/(1+|\delta|)\right).$$

Consequently  $\Phi\left(-3C_{pk}''(1+|\delta|)/(1+|\delta|)\right) \leq NC \leq \Phi\left(-3C_{pk}''\right) + \Phi\left(-3C_{pk}''(1+|\delta|)/(1+|\delta|)\right)$ .

The upper bound given in Theorem 2 is identical to the one given by Chang and Wu (2008) in the expression (3). To reach that conclusion, we just need to observe that  $r = (1 + \delta) / (1 - \delta)$ .

**Particular case :** If  $T = m$ , we have  $\delta = 0$ , thus  $\Phi(-3C_{pk}'') \leq NC \leq 2\Phi(-3C_{pk}'')$ , result well known, given by Boyles (1991).

### 4.3. Case $(u = 1, v > 0)$ , and $u > 1$

#### Theorem 3 :

When  $u = 1$  and  $v > 0$ , or when  $u > 1$ , we have

$$0 \leq NC \leq \Phi\left(-3C_p''(u,v)\right) + \Phi\left(-3C_p''(u,v)(1+|\delta|)/(1+|\delta|)\right).$$

#### Proof :

We have  $C_p''(u+x, v+y) \leq C_p''(u, v)$ , for any  $x, y \geq 0$ . Thus when  $u = 1$  and  $v > 0$ , or when  $u > 1$ ,  $C_p''(u, v) \leq C_p''(1, 0) = C_{pk}''$ . Thus from the theorem 2,

$$NC \leq \Phi\left(-3C_{pk}''\right) + \Phi\left(-3C_{pk}''(1+|\delta|)/(1+|\delta|)\right)$$

$\leq \Phi(-3C_p''(u, v)) + \Phi(-3C_p''(u, v)(1 + |\delta|)/(1 - |\delta|))$ , and from (9), this upper bound is reached at  $\lambda = 0$ . Moreover,  $F(\lambda)$  is always maximised by 1, value reached at  $\lambda_{\min}$  and  $\lambda_{\max}$  according to lemma 5 (see appendix). Thus  $NC$  is minimized by 0, and the theorem.

**Particular cases :** If  $T = m$ , then  $\delta = 0$ , and  $C_p''(u, v) = C_p''(u, v)$ . Thus when  $u = 1$ ,  $v > 0$ , or  $u > 1$ , we have  $0 \leq NC \leq 2\Phi(-3C_p''(u, v))$ . When  $(u, v) = (1, 1)$ , we obtain the result given by Boyles (1994). In addition, when  $T \neq m$ , and  $(u, v) = (1, 1)$ , then  $C_p''(1, 1) = C_{pmk}''$ , and we have  $NC \leq \Phi(-3C_{pmk}'') + \Phi(-3C_{pmk}''(1 + |\delta|)/(1 - |\delta|)) \leq 2\Phi(-3C_{pmk}'')$ . The result obtained by Pearn, Lin, and Chen (1999) is thus exact, although their proof is not true in all cases.

#### 4.4. Case $0 < u < 1$ , $v = 0$

##### Theorem 4 :

When  $0 < u < 1$  and  $v = 0$ ,

1)  $\delta > 0$

a) If  $C_p''(u, 0) \geq C_1$ , then  $M_1 \leq NC \leq 1$ ,

b) If  $C_p''(u, 0) < C_1$ , then  $\min(M_1, M_2) \leq NC \leq 1$ ,

where

$$C_1 = \frac{1 - |\delta|}{3} \sqrt{\frac{1}{|\delta|} \tanh^{-1}\left(\frac{u}{1 - |\delta|(1 - u)}\right)},$$

$$M_1 = \Phi\left(3C_p''(u, 0) \frac{\lambda_{l1} + |\delta| - 1}{(1 - |\delta|)(1 + u\lambda_{l1}/(1 + |\delta|))}\right) + \Phi\left(-3C_p''(u, 0) \frac{\lambda_{l1} + |\delta| + 1}{(1 - |\delta|)(1 + u\lambda_{l1}/(1 + |\delta|))}\right),$$

$$M_2 = \Phi\left(3C_p''(u, 0) \frac{\lambda_{u1} + |\delta| - 1}{(1 - |\delta|)(1 - u\lambda_{u1}/(1 - |\delta|))}\right) + \Phi\left(-3C_p''(u, 0) \frac{\lambda_{u1} + |\delta| + 1}{(1 - |\delta|)(1 - u\lambda_{u1}/(1 - |\delta|))}\right),$$

$$\lambda_{l1} = -\frac{1 + |\delta|}{u} - (3C_p''(u, 0))^2 \frac{1 - \sqrt{1 + 4 \frac{u(1 + |\delta|(1 - u))}{(3C_p''(u, 0))^2} \left(\frac{1 - |\delta|}{1 + |\delta|}\right)^2 \tanh^{-1}\left(\frac{u}{1 + |\delta|(1 - u)}\right)}}{2u^2 \left(\frac{1 - |\delta|}{1 + |\delta|}\right)^2 \tanh^{-1}\left(\frac{u}{1 + |\delta|(1 - u)}\right)},$$

$$\lambda_{u1} = \frac{1 - |\delta|}{u} + (3C_p''(u, 0))^2 \frac{1 - \sqrt{1 + 4 \frac{u(1 - |\delta|(1 - u))}{(3C_p''(u, 0))^2} \tanh^{-1}\left(\frac{u}{1 - |\delta|(1 - u)}\right)}}{2u^2 \tanh^{-1}\left(\frac{u}{1 - |\delta|(1 - u)}\right)}.$$

2)  $\delta < 0$

We have the same results as in 1) if  $M_1$  is replaced by  $M_2$ , and  $M_2$  by  $M_1$ .

3)  $\delta = 0$

$M_0 \leq NC \leq 1$ ,

where  $M_0 = \Phi(3C_p''(u, 0)(\lambda_0 - 1)/(1 - u\lambda_0)) + \Phi(-3C_p''(u, 0)(\lambda_0 + 1)/(1 - u\lambda_0))$ ,

$$\text{with } \lambda_0 = \frac{1}{u} + \frac{(3C_p''(u,0))^2}{2u^2 \tanh^{-1}(u)} \left( 1 - \sqrt{1 + \frac{4u \tanh^{-1}(u)}{(3C_p''(u,0))^2}} \right).$$

**Proof :**

Let  $\delta$  be positive or null. We obviously have  $0 \leq F(\lambda) \leq 1$ . According to lemma 6 (see appendix) the lower bound 0 is reached at  $\lambda_{\min}$  and  $\lambda_{\max}$ . If there is a maximum less or equal to 1, it is necessarily obtained for the values of  $\lambda \in ]\lambda_{\min}, \lambda_{\max}[$ , solutions of the equation  $F_u'(\lambda) = 0$  or  $F_l'(\lambda) = 0$ .

- Study of  $F_u'(\lambda)$ .

Let  $0 \leq \lambda < \lambda_{\max} = 1/ud_u$ . When  $v = 0$  and from lemma 4, we have

$$\begin{aligned} F_u'(\lambda) = 0 &\Leftrightarrow Q_u(\lambda) = 0 \Leftrightarrow q_u(\lambda) - (k_u(\lambda) + (\delta + \lambda)q_u(\lambda)) \tanh(d^2(\delta + \lambda)/\sigma_u^2(\lambda)) = 0 \\ &\Leftrightarrow ud_u \sigma_0^2(1 - u\lambda d_u) - (\sigma_0^2(1 - u\lambda d_u)^2 + (\delta + \lambda)ud_u \sigma_0^2(1 - u\lambda d_u)) \tanh(d^2(\delta + \lambda)/\sigma_u^2(\lambda)) = 0 \\ &\Leftrightarrow ud_u \sigma_0^2(1 - u\lambda d_u) - \sigma_0^2(1 - u\lambda d_u)(1 + \delta ud_u) \tanh(d^2(\delta + \lambda)/\sigma_u^2(\lambda)) = 0 \\ &\Leftrightarrow \sigma_0^2(1 - u\lambda d_u)[ud_u - (1 + \delta ud_u) \tanh(d^2(\delta + \lambda)/\sigma_u^2(\lambda))] = 0 \\ &\Leftrightarrow ud_u - (1 + \delta ud_u) \tanh(d^2(\delta + \lambda)/\sigma_u^2(\lambda)) = 0, \text{ since } 0 \leq \lambda < 1/ud_u, \\ &\Leftrightarrow \tanh(d^2(\delta + \lambda)/[\sigma_0^2(1 - u\lambda d_u)^2]) = ud_u/(1 + \delta ud_u), \text{ from lemma 1,} \\ &\Leftrightarrow d^2(\delta + \lambda)/(\sigma_0^2(1 - u\lambda d_u)^2) = \tanh^{-1}(ud_u/(1 + \delta ud_u)). \end{aligned}$$

Let  $t_u$  be the quantity  $\tanh^{-1}(ud_u/(1 + \delta ud_u))$ .  $t_u$  is positive and  $t_u = \tanh^{-1}(u/(1 - \delta(1 - u)))$  from (4). Now, we have

$$\begin{aligned} F_u'(\lambda) = 0 &\Leftrightarrow d^2(\delta + \lambda) = t_u \sigma_0^2(1 - u\lambda d_u)^2 \\ &\Leftrightarrow t_u \sigma_0^2 u^2 d_u^2 \lambda^2 - (2ud_u t_u \sigma_0^2 + d^2)\lambda - d^2 \delta + t_u \sigma_0^2 = 0, \end{aligned} \quad (12)$$

which is a second-degree polynomial of the variable  $\lambda$ .

Since  $\Delta_u = d^4 + 4t_u \sigma_0^2 ud_u d^2(1 + u\delta d_u) > 0$ , we have two roots,

$$\begin{aligned} \lambda_{u1} &= \frac{2ud_u t_u \sigma_0^2 + d^2 - \sqrt{\Delta_u}}{2t_u \sigma_0^2 u^2 d_u^2} = \frac{1}{ud_u} + \frac{d^2 - \sqrt{\Delta_u}}{2t_u \sigma_0^2 u^2 d_u^2}, \\ \lambda_{u2} &= \frac{2ud_u t_u \sigma_0^2 + d^2 + \sqrt{\Delta_u}}{2t_u \sigma_0^2 u^2 d_u^2} = \frac{1}{ud_u} + \frac{d^2 + \sqrt{\Delta_u}}{2t_u \sigma_0^2 u^2 d_u^2}. \end{aligned}$$

As can be seen,  $\lambda_{u2} > 1/ud_u = \lambda_{\max}$  is not suitable in the studied field. To make  $\lambda_{u1}$  become acceptable, we need to  $0 \leq \lambda_{u1} < \lambda_{\max}$ . Since  $\Delta_u > d^4$ , we have  $\lambda_{u1} < 1/ud_u = \lambda_{\max}$ . In addition, since  $\lambda_{u2} > 0$ , for  $\lambda_{u1}$  to be positive or null, the product of the roots of (12) has to be positive or null, or that  $-d^2 \delta + t_u \sigma_0^2 \geq 0 \Leftrightarrow -d^2 \delta + t_u (d^* / (3C_p''(u,0)))^2 \geq 0 \Leftrightarrow C_p''(u,0) \leq H$ , with

$$H = \sqrt{t_u / \delta} d^* / (3d) = \frac{1 - |\delta|}{3} \sqrt{\frac{1}{\delta} \tanh^{-1}\left(\frac{u}{1 - \delta(1 - u)}\right)}. \text{ Thus when } C_p''(u,0) \leq H, \text{ there exists}$$

$\lambda_{u1} \in [0, \lambda_{\max}[$  for which  $F_u'(\lambda) = 0$  and thus for which  $F_u(\lambda)$  is maximum.

Note that in the particular case where  $C_p''(u,0) = H$ , that is to say when  $-d^2 \delta + t_u \sigma_0^2 = 0$ , we have  $\lambda_{u1} = 0$  and  $t_u = d^2 \delta / \sigma_0^2$ . When  $v = 0$ , from lemma 4, we have

$$Q_u(0) = \sigma_0^2 [ud_u - (1 + \delta ud_u) \tanh(d^2 \delta / \sigma_0^2)] = \sigma_0^2 [ud_u - (1 + \delta ud_u) \tanh(t_u)]$$

$= \sigma_0^2 [ud_u - (1 + \delta ud_u)ud_u / (1 + \delta ud_u)] = 0$ . Consequently, according to lemma 4,  $F_u'(\lambda) = 0$ , and  $F_u(\lambda)$  is maximum for  $\lambda = \lambda_{u1} = 0$ .

When  $C_p''(u, 0) > H$ , we have  $\lambda_{u1} < 0$ . Thus there is no value of  $\lambda \in [0, \lambda_{\max}[$  for which  $F_u'(\lambda) = 0$ . On the other hand  $F_u(0) = \Phi(3C_p''(u, 0)) - \Phi(-3C_p''(u, 0)(1 + \delta)/(1 - |\delta|))$ , from (9). Consequently, we have  $F_u(0) > 0$ ,  $\lim_{\lambda \rightarrow \lambda_{\max}} F_u(\lambda) = 0$  from lemma 6, and  $F_u'(\lambda) \neq 0$  when  $\lambda \in [0, \lambda_{\max}[$ . Thus  $F_u(\lambda)$  is decreasing when  $\lambda \in [0, \lambda_{\max}[$  and maximum when  $\lambda = 0$ .

- Study of  $F_l'(\lambda)$

Let  $-1/ud_l = \lambda_{\min} < \lambda \leq 0$ . When  $v = 0$  and from lemma 4, we have

$$\begin{aligned} F_l'(\lambda) = 0 &\Leftrightarrow Q_l(\lambda) = 0 \Leftrightarrow q_l(\lambda) - (k_l(\lambda) + (\delta + \lambda)q_l(\lambda)) \tanh(d^2(\delta + \lambda)/\sigma_l^2(\lambda)) = 0 \\ &\Leftrightarrow -ud_l\sigma_0^2(1 + u\lambda d_l) - [\sigma_0^2(1 + u\lambda d_l)^2 - (\delta + \lambda)ud_l\sigma_0^2(1 + u\lambda d_l)] \tanh(d^2(\delta + \lambda)/\sigma_l^2(\lambda)) = 0 \\ &\Leftrightarrow \sigma_0^2(1 + u\lambda d_l)[-ud_l + (-1 + \delta ud_l) \tanh(d^2(\delta + \lambda)/\sigma_l^2(\lambda))] = 0 \\ &\Leftrightarrow -ud_l + (-1 + \delta ud_l) \tanh(d^2(\delta + \lambda)/\sigma_l^2(\lambda)) = 0, \text{ since } -1/ud_l = \lambda_{\min} < \lambda \leq 0. \end{aligned}$$

Thus from lemma 1, we have

$$F_l'(\lambda) = 0 \Leftrightarrow \tanh(d^2(\delta + \lambda)/[\sigma_0^2(1 + u\lambda d_l)^2]) = ud_l/(-1 + \delta ud_l). \quad (13)$$

From (5),  $ud_l/(-1 + \delta ud_l) = -u/(1 + \delta(1 - u)) < 0$ . Consequently the solutions of (13) can exist only for  $\lambda < -\delta$ . Let  $t_l$  be the quantity  $\tanh^{-1}(-u/(1 + \delta(1 - u)))$ . We have

$$\begin{aligned} F_l'(\lambda) = 0 &\Leftrightarrow d^2(\delta + \lambda)/[\sigma_0^2(1 + u\lambda d_l)^2] = t_l \Leftrightarrow d^2(\delta + \lambda) = t_l\sigma_0^2(1 + u\lambda d_l)^2 \\ &\Leftrightarrow t_l\sigma_0^2u^2d_l^2\lambda^2 + (2ud_l t_l\sigma_0^2 - d^2)\lambda - d^2\delta + t_l\sigma_0^2 = 0, \text{ which is a second-degree polynomial of} \\ &\text{the variable } \lambda. \text{ Since } t_l < 0 \text{ and } -1 + \delta ud_l < 0, \Delta_l = d^4 + 4t_l\sigma_0^2ud_l d^2(-1 + u\delta d_l) > 0, \text{ and we} \\ &\text{have two roots,} \end{aligned}$$

$$\begin{aligned} \lambda_{l1} &= \frac{d^2 - 2ud_l t_l\sigma_0^2 - \sqrt{\Delta_l}}{2t_l\sigma_0^2u^2d_l^2} = -\frac{1}{ud_l} + \frac{d^2 - \sqrt{\Delta_l}}{2t_l\sigma_0^2u^2d_l^2}, \\ \lambda_{l2} &= \frac{d^2 - 2ud_l t_l\sigma_0^2 + \sqrt{\Delta_l}}{2t_l\sigma_0^2u^2d_l^2} = -\frac{1}{ud_l} + \frac{d^2 + \sqrt{\Delta_l}}{2t_l\sigma_0^2u^2d_l^2}. \end{aligned}$$

As can be seen,  $\lambda_{l2} < -1/ud_l = \lambda_{\min}$  is not suitable in the studied field. To make  $\lambda_{l1}$  become acceptable, we need to  $\lambda_{\min} < \lambda_{l1} < -\delta \leq 0$ . Since  $\Delta_l > d^4$ , we have  $\lambda_{l1} > -1/ud_l = \lambda_{\min}$ . Furthermore, since  $\lambda_{l2} < 0$ , for  $\lambda_{l1}$  to be negative or null, the product of the roots has to be positive or null, or that  $-d^2\delta + t_l\sigma_0^2 \leq 0$ , which is always true. Thus  $F_l(\lambda)$  is maximum when  $\lambda_{l1} \in ]\lambda_{\min}; -\delta[$ .

In conclusion, from the study of  $F_u'(\lambda)$  and  $F_l'(\lambda)$ , we can deduce:

- If  $C_p''(u, 0) \geq H$ ,  $F_u(\lambda)$  is maximum when  $\lambda = 0$ , and  $F_l(\lambda_{l1}) > F_l(0) = F_u(0)$ . Thus  $F(\lambda)$  has an upper bound when  $\lambda_{l1} \in ]\lambda_{\min}; -\delta[$ . On the other hand, from lemma 6 (appendix),

$$\lim_{\lambda \rightarrow \lambda_{\max}} F_u(\lambda) = \lim_{\lambda \rightarrow \lambda_{\min}} F_l(\lambda) = 0. \text{ Thus } 0 \leq F(\lambda) \leq F_l(\lambda_{l1}) \text{ or } 1 - F_l(\lambda_{l1}) \leq NC \leq 1. \text{ From (4), (5),}$$

(6), lemmas 1 and 3, and assuming that  $M_l = 1 - F_l(\lambda_{l1})$ , we obtain

$$M_l = 1 - \Phi(d(1 - \delta - \lambda_{l1})/(\sigma_0(1 + u\lambda_{l1}d_l))) + \Phi(-d(1 + \delta + \lambda_{l1})/(\sigma_0(1 + u\lambda_{l1}d_l)))$$

$$= \Phi \left( 3C_p''(u, 0) \frac{\lambda_{l1} + \delta - 1}{(1 - |\delta|)(1 + u\lambda_{l1}/(1 + \delta))} \right) + \Phi \left( -3C_p''(u, 0) \frac{\lambda_{l1} + \delta + 1}{(1 - |\delta|)(1 + u\lambda_{l1}/(1 + \delta))} \right),$$

$$\text{and } \lambda_{l1} = -\frac{1}{ud_l} + \frac{d^2 - \sqrt{d^4 + 4t_l\sigma_0^2ud_l d^2(-1 + u\delta d_l)}}{2t_l\sigma_0^2u^2d_l^2}$$

$$= -\frac{1 + \delta}{u} - (3C_p''(u, 0))^2 \frac{1 - \sqrt{1 + 4 \frac{u(1 + \delta(1 - u))}{(3C_p''(u, 0))^2} \left( \frac{1 - |\delta|}{1 + \delta} \right)^2 \tanh^{-1} \left( \frac{u}{1 + \delta(1 - u)} \right)}}{2u^2 \left( \frac{1 - |\delta|}{1 + \delta} \right)^2 \tanh^{-1} \left( \frac{u}{1 + \delta(1 - u)} \right)}.$$

- If  $C_p''(u, 0) < H$ ,  $F_u(\lambda)$  is maximum when  $\lambda_{u1} \in [0, \lambda_{\max}[$ , and  $F_l(\lambda)$  is maximum when  $\lambda_{l1} \in ]\lambda_{\min}; -\delta[$ . Thus,  $F(\lambda)$  has an upper bound when  $\lambda \in ]\lambda_{\min}; \lambda_{\max}[$ , equal to  $\max(F_l(\lambda_{l1}); F_u(\lambda_{u1}))$ . On the other hand, from lemma 6 (appendix),

$$\lim_{\lambda \rightarrow \lambda_{\max}} F_u(\lambda) = \lim_{\lambda \rightarrow \lambda_{\min}} F_l(\lambda) = 0. \quad \text{Thus} \quad 0 \leq F(\lambda) \leq \max(F_l(\lambda_{l1}), F_u(\lambda_{u1})), \quad \text{or}$$

$\min(1 - F_l(\lambda_{l1}), 1 - F_u(\lambda_{u1})) \leq NC \leq 1$ . From (4), (5), (6), lemmas 1 and 3, and assuming that

$M_u = 1 - F_u(\lambda_{u1})$ , we obtain

$$M_u = 1 - \Phi \left( d(1 - \delta - \lambda_{u1}) / (\sigma_0(1 - u\lambda_{u1}d_u)) \right) + \Phi \left( -d(1 + \delta + \lambda_{u1}) / (\sigma_0(1 - u\lambda_{u1}d_u)) \right) \\ = \Phi \left( 3C_p''(u, 0) \frac{\lambda_{u1} + \delta - 1}{(1 - |\delta|)(1 - u\lambda_{u1}/(1 - \delta))} \right) + \Phi \left( -3C_p''(u, 0) \frac{\lambda_{u1} + \delta + 1}{(1 - |\delta|)(1 - u\lambda_{u1}/(1 - \delta))} \right),$$

$$\text{and } \lambda_{u1} = \frac{1}{ud_u} + \frac{d^2 - \sqrt{d^4 + 4t_u\sigma_0^2ud_u d^2(1 + u\delta d_u)}}{2t_u\sigma_0^2u^2d_u^2}$$

$$= \frac{1 - \delta}{u} + (3C_p''(u, 0))^2 \frac{1 - \sqrt{1 + 4 \frac{u(1 - \delta(1 - u))}{(3C_p''(u, 0))^2} \tanh^{-1} \left( \frac{u}{1 - \delta(1 - u)} \right)}}{2u^2 \tanh^{-1} \left( \frac{u}{1 - \delta(1 - u)} \right)}.$$

In the particular case where  $\delta = 0$ , the product of the roots of the second-degree polynomial (12) is equal to  $u^{-2}$ , therefore it is always positive. Consequently,  $F_u(\lambda)$  is maximum in  $\lambda_{u1} \in [0, \lambda_{\max}[$ , for any value of  $C_p''(u, 0) > 0$ . Since  $\delta = 0$ , we have  $d_u = d_l = 1$ ,  $\sigma_u(\lambda) = \sigma_l(-\lambda)$ , and  $F_u(\lambda) = F_l(-\lambda)$  for  $0 \leq \lambda < \lambda_{\max}$ . Thus  $M_0 \leq NC \leq 1$ , where  $M_0 = M_l(\lambda_{l1}) = M_u(\lambda_{u1})$ , with  $\lambda_0 = \lambda_{u1} = -\lambda_{l1}$ .

Now, we study the case  $\delta < 0$ . From (4), (5), and lemmas 1 and 3, if  $0 \leq \lambda < \lambda_{\max}$ , we have  $F_u(\lambda, \delta) = F_l(-\lambda, -\delta)$ . Thus we have the following results:  $H$ ,  $M_u$ ,  $M_l$ ,  $\lambda_{u1}$ ,  $\lambda_{l1}$  being functions of  $\lambda$ ,

- If  $C_p''(u, 0) \geq H(-\delta)$ , then  $M_u' \leq NC \leq 1$ , where  $M_u'(\delta) = 1 - F_u(\lambda_{u1}') = M_l(-\delta)$ , and  $\lambda_{u1}'(\delta) = -\lambda_{l1}(-\delta) \in ]-\delta; \lambda_{\max}[$ .

- If  $C_p''(u, 0) < H(-\delta)$ , then  $\min(M_u', M_l') \leq NC \leq 1$ , where  $M_l'(\delta) = 1 - F_l(\lambda_{l1}') = M_u(-\delta)$ , and  $\lambda_{l1}'(\delta) = -\lambda_{u1}(-\delta) \in ]\lambda_{\min}; 0[$ .

Let  $C_1$  be  $H(|\delta|)$ ,  $M_1$  be  $M_l(|\delta|)$  and  $M_2$  be  $M_u(|\delta|)$ . Thus the theorem ensues from the results obtained for  $\delta \geq 0$  and  $\delta < 0$ .

#### 4.5. Case $0 < u < 1$ , $v > 0$

##### Theorem 5 :

When  $0 < u < 1$  and  $v > 0$ ,

1)  $\delta > 0$

a) If  $C_p''(u, v) > \max(C_1, C_2)$ , then  $0 \leq NC \leq 1 - F_u(\lambda_u)$ .

b) If  $C_p''(u, v) < C_2$ , then  $\min(1 - F_l(\lambda_l), 1 - F_u(\lambda_u)) \leq NC \leq 1$ .

c) If  $C_2 < C_p''(u, v) \leq C_1$ , then  $0 \leq NC \leq \max(1 - F_u(\lambda_u), 1 - F(0))$ .

d) If  $C_p''(u, v) = C_2 < C_1$ , then  $1 - F_l(\lambda_l) \leq NC \leq \max(1 - F(0), 1/2)$ .

e) If  $C_1 < C_2 = C_p''(u, v)$ , then  $1 - F_l(\lambda_l) \leq NC \leq \max(1 - F_u(\lambda_u), 1/2)$ ,

where  $C_1 = \frac{1 - |\delta|}{3} \sqrt{\frac{1}{|\delta|} \tanh^{-1}\left(\frac{u}{1 - |\delta|(1 - u)}\right)}$ ,  $C_2 = \frac{(1 - u)(1 - |\delta|)}{3\sqrt{v}}$ ,  $1 - F(0)$

$= \Phi(-3C_p''(u, v)) + \Phi(-3C_p''(u, v)(1 + |\delta|)/(1 - |\delta|))$ , and  $\lambda_u$  and  $\lambda_l$ , if they exist, are solutions of the following equations (14) and (15).

2)  $\delta < 0$

We have the same results as in 1) if  $F_u(\lambda_u)$  is replaced by  $F_l(\lambda_l)$ , and  $F_l(\lambda_l)$  by  $F_u(\lambda_u)$ .

3)  $\delta = 0$

a) If  $C_p''(u, v) > C_2$ , then  $0 \leq NC \leq \max(1 - F(\lambda_0), 1 - F(0))$ .

b) If  $C_p''(u, v) < C_2$ , then  $1 - F(\lambda_0) \leq NC \leq 1$ .

c) If  $C_p''(u, v) = C_2$ , then  $1 - F(\lambda_0) \leq NC \leq \max(1 - F(0), 1/2)$ ,

where  $\lambda_0$ , if it exists, is the solution of the following equations (14) or (15).

##### Proof :

The extrema of the function  $F(\lambda)$  are obtained either at the study intervals bounds  $\lambda_{\min}$ ,  $0$ ,  $\lambda_{\max}$ , either for the  $\lambda$  values solutions of the equations  $F_u'(\lambda) = 0$  or  $F_l'(\lambda) = 0$ , that is to say according to lemma 4, of the equations

$$q_u(\lambda) + v\lambda d^2 d_u^2 - (k_u(\lambda) + (\delta + \lambda)q_u(\lambda) + \delta v\lambda d^2 d_u^2) \tanh(d^2(\delta + \lambda)/\sigma_u^2(\lambda)) = 0, \quad (14)$$

$$q_l(\lambda) + v\lambda d^2 d_l^2 - (k_l(\lambda) + (\delta + \lambda)q_l(\lambda) + \delta v\lambda d^2 d_l^2) \tanh(d^2(\delta + \lambda)/\sigma_l^2(\lambda)) = 0. \quad (15)$$

These solutions can only be obtained numerically.

When  $\delta \geq 0$ , the theorem is ensued from lemmas 7, 8, 9 and 10 in appendix. When  $\delta < 0$ , we use the fact that  $F_u(\lambda, \delta) = F_l(-\lambda, -\delta)$  for  $0 \leq \lambda < \lambda_{\max}$ .

#### 4.6. Case $u = 0$ , $v > 0$

##### Theorem 6 :

When  $u = 0$  and  $v > 0$ ,

1)  $\delta > 0$

a) If  $C_p''(u, v) > C_2$ , then  $0 \leq NC \leq 1 - F_u(\lambda_u)$ .

b) If  $C_p''(u, v) < C_2$ , then  $\min(1 - F_l(\lambda_l), 1 - F_u(\lambda_u)) \leq NC \leq 1$ .

c) If  $C_p''(u, v) = C_2$ , then  $1 - F_l(\lambda_l) \leq NC \leq \max(1 - F_u(\lambda_u), 1/2)$ .

where  $C_2 = (1 - u)(1 - |\delta|) / (3\sqrt{v})$ , and  $\lambda_u$  and  $\lambda_l$ , are solutions of the equations (14) and (15).

2)  $\delta < 0$

We have the same results as in 1) if  $F_u(\lambda_u)$  is replaced by  $F_l(\lambda_l)$ , and  $F_l(\lambda_l)$  by  $F_u(\lambda_u)$ .

3)  $\delta = 0$

a) If  $C_p''(u, v) > C_2$ , then  $0 \leq NC \leq \max(1 - F(\lambda_0), 1 - F(0))$ .

b) If  $C_p''(u, v) < C_2$ , then  $\min(1 - F(\lambda_0), 1 - F(0)) \leq NC \leq 1$ .

c) If  $C_p''(u, v) = C_2$ , then  $\min(1 - F(\lambda_0), 1 - F(0)) \leq NC \leq \max(1 - F(0), 1/2)$ ,

where  $\lambda_0$ , if it exists, is the solution of the equations (14) or (15).

### Proof :

When  $u = 0$ , we have  $C_1 = 0$ , thus  $C_p''(u, v) > C_1$ . The theorem is ensued from lemmas 7, 8, 9 and 10 in the appendix.

**Particular case :** When  $(u, v) = (0, 1)$ , and  $\delta = 0$ , we have  $C_p''(0, 1) = C_{pm}$ ,  $C_2 = 1/3$ , and  $1 - F(0) = 2\Phi(-3C_{pm})$ . The results we obtain thus are compatible with Rucinski's (1996).

## 5. APPLICATION EXAMPLE

A company of the Toyal group manufactures aluminium paste used for the fabrication of high-tech paint for cars, hi-fi, mobile telephony, cosmetics... The manufacturing process consists in crushing the raw material to which lubricant is added. The product is then conveyed into a mixer where a solvent is added in order to obtain a final product containing a constant non volatile percentage. A quality control is carried out at this stage of the production. It concerns the non volatile percentage which has a target of 67. The usual tolerances for the profession are  $\pm 1$ , but are difficult to hold for this type of product. The lower values being more prejudicial for the customer, the tolerances have been fixed at 66 and 69. We have  $m = 67.5$ ,  $d = 1.5$ , and  $\delta = (T - m) / d = 1/3$ . Suppose that the process is considered capable when  $C_p''(u, v)$  takes a value larger or equal to 1, the number of nonconforming items is smaller or equal to 1500 parts per million (ppm), and the process mean does not move away more than 20% of the distance between the target and the tolerances. From theorems 1 to 6 we can find the pairs  $(u, v)$  such as  $NC \leq 1500$  when  $C_p''(u, v) = 1$ . We limit our study to varying  $u$  and  $v$  with a step of 0.1. Table 1 gives the pairs  $(u, v)$  where the upper bound of  $NC$  (en ppm) is the nearest to 1500. From (6) and (8) we have  $-KD_l < \mu - T < KD_u$ , where  $K = (3\sqrt{v}C_p''(u, v) / (1 - |\delta|) + u)^{-1}$ . If we want the process mean not to move away more than 20% of the distance between the target and the tolerances, we must take  $K = 0.2$ . Thus, in table 1, we have written out the value of  $K$  when  $C_p''(u, v) = 1$ . So the  $C_p''(0.3, 1.1)$  index will meet our objectives in the best way.

## 6. CONCLUSION

The motives underlying the introduction of process capability indices seem quite clearly to be related to monitoring the proportion of nonconforming items. However various authors have addressed the practical importance of process centering as a component of process capability.

For these reasons, Vännman (1995) suggests the  $C_p(u, v)$  indices when the tolerances are symmetrical, then Chen and Pearn (2001) suggest the  $C_p''(u, v)$  indices when the tolerances are asymmetrical. The compromise between process yield and process centering is achieved by the choice of the parameters  $u$  and  $v$ . However, if the links between capability indices and process centering have already been studied, those between capability indices and process yield have only been accurately studied for some particular cases. In this paper we study the links between the process yield and the  $C_p''(u, v)$  indices. We find already known results for some particular cases, we correct inaccuracies found in the literature, and expand the study to any positive or null values of  $u$  and  $v$ . From these results, the practitioner can choose a pair  $(u, v)$ , so that the resulting index  $C_p''(u, v)$  will meet his objectives best. In order to illustrate how this reasoning can be applied, we present a real example on an aluminium paste manufacturing process.

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## APPENDIX

### Lemma 5:

When  $u = 1$  and  $v > 0$ , or when  $u > 1$ , then  $\lim_{\lambda \rightarrow \lambda_{\max}} F_u(\lambda) = \lim_{\lambda \rightarrow \lambda_{\min}} F_l(\lambda) = 1$ .

#### Proof :

From lemma 3, we have

$$\begin{aligned} \lim_{\lambda \rightarrow \lambda_{\max}} F_u(\lambda) &= \lim_{\lambda \rightarrow \lambda_{\max}} [\Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)) - \Phi(-d(1+\delta+\lambda)/\sigma_u(\lambda))] \\ &= \lim_{\lambda \rightarrow \lambda_{\max}} \Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)) - \Phi(-\infty) = \lim_{\lambda \rightarrow \lambda_{\max}} \Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)). \end{aligned}$$

On the other hand, from (4) and (8),  $d(1-\delta-\lambda_{\max}) = D_u(\sqrt{v}d + (u-1)\sigma_0)/(\sqrt{v}d + u\sigma_0)$ .

Since  $0 \leq \lambda < \lambda_{\max}$ , we have  $C_p''(u, v) > 0$  and thus  $\sigma_0 > 0$ . When  $u = 1$  and  $v > 0$ , or  $u > 1$ , thus we have  $d(1-\delta-\lambda_{\max}) > 0$ , from where  $\lim_{\lambda \rightarrow \lambda_{\max}} \Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)) = \Phi(+\infty) = 1$ , and

$$\lim_{\lambda \rightarrow \lambda_{\max}} F_u(\lambda) = 1.$$

From lemma 3, we have

$$\begin{aligned} \lim_{\lambda \rightarrow \lambda_{\min}} F_l(\lambda) &= \lim_{\lambda \rightarrow \lambda_{\min}} [\Phi(d(1-\delta-\lambda)/\sigma_l(\lambda)) - \Phi(-d(1+\delta+\lambda)/\sigma_l(\lambda))] \\ &= \Phi(+\infty) - \lim_{\lambda \rightarrow \lambda_{\min}} \Phi(-d(1+\delta+\lambda)/\sigma_l(\lambda)) = 1 - \lim_{\lambda \rightarrow \lambda_{\min}} \Phi(-d(1+\delta+\lambda)/\sigma_l(\lambda)). \end{aligned}$$

On the other hand, from (5) and (8),  $-d(1+\delta+\lambda_{\min}) = -D_l(\sqrt{v}d + (u-1)\sigma_0)/(\sqrt{v}d + u\sigma_0)$ .

Since  $\lambda_{\min} < \lambda \leq 0$ , we have  $C_p''(u, v) > 0$  and thus  $\sigma_0 > 0$ . When  $u = 1$  and  $v > 0$ , or  $u > 1$ , thus we have  $-d(1+\delta+\lambda_{\min}) < 0$ , from where  $\lim_{\lambda \rightarrow \lambda_{\min}} \Phi(-d(1+\delta+\lambda)/\sigma_l(\lambda)) = \Phi(-\infty) = 0$ , and  $\lim_{\lambda \rightarrow \lambda_{\min}} F_l(\lambda) = 1$ .

### Lemma 6 :

When  $0 < u < 1$  and  $v = 0$ , then  $\lim_{\lambda \rightarrow \lambda_{\max}} F_u(\lambda) = \lim_{\lambda \rightarrow \lambda_{\min}} F_l(\lambda) = 0$ .

#### Proof :

From lemma 3 we have

$$\begin{aligned} \lim_{\lambda \rightarrow \lambda_{\max}} F_u(\lambda) &= \lim_{\lambda \rightarrow \lambda_{\max}} [\Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)) - \Phi(-d(1+\delta+\lambda)/\sigma_u(\lambda))] \\ &= \lim_{\lambda \rightarrow \lambda_{\max}} \Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)) - \Phi(-\infty) = \lim_{\lambda \rightarrow \lambda_{\max}} \Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)). \end{aligned}$$

Now when  $0 < u < 1$  and  $v = 0$ , from (8) we have  $\lambda_{\max} = 1/ud_u$ , and from (4),  $d(1-\delta-\lambda_{\max}) = D_u(u-1)/u < 0$ , thus  $\lim_{\lambda \rightarrow \lambda_{\max}} F_u(\lambda) = \Phi(-\infty) = 0$ .

From lemma 3 we have

$$\begin{aligned} \lim_{\lambda \rightarrow \lambda_{\min}} F_l(\lambda) &= \lim_{\lambda \rightarrow \lambda_{\min}} [\Phi(d(1-\delta-\lambda)/\sigma_l(\lambda)) - \Phi(-d(1+\delta+\lambda)/\sigma_l(\lambda))] \\ &= \Phi(+\infty) - \lim_{\lambda \rightarrow \lambda_{\min}} \Phi(-d(1+\delta+\lambda)/\sigma_l(\lambda)) = 1 - \lim_{\lambda \rightarrow \lambda_{\min}} \Phi(-d(1+\delta+\lambda)/\sigma_l(\lambda)). \end{aligned}$$

Now when  $0 < u < 1$  and  $v = 0$ , from (8) we have  $\lambda_{\min} = -1/ud_l$ , and from (5),  $-d(1+\delta+\lambda_{\min}) = -D_l(u-1)/u > 0$ , thus  $\lim_{\lambda \rightarrow \lambda_{\min}} F_l(\lambda) = 1 - \Phi(+\infty) = 0$ .

### Lemma 7 :

When  $0 \leq u < 1$  and  $v > 0$ ,

a) If  $C_p''(u, v) > (1-u)(1-|\delta|)/(3\sqrt{v})$ , then  $\lim_{\lambda \rightarrow \lambda_{\max}} F_u(\lambda) = \lim_{\lambda \rightarrow \lambda_{\min}} F_l(\lambda) = 1$ .

b) If  $C_p''(u, v) = (1-u)(1-|\delta|)/(3\sqrt{v})$ , then  $\lim_{\lambda \rightarrow \lambda_{\max}} F_u(\lambda) = \lim_{\lambda \rightarrow \lambda_{\min}} F_l(\lambda) = 1/2$ .

c) If  $0 < C_p''(u, v) < (1-u)(1-|\delta|)/(3\sqrt{v})$ , then  $\lim_{\lambda \rightarrow \lambda_{\max}} F_u(\lambda) = \lim_{\lambda \rightarrow \lambda_{\min}} F_l(\lambda) = 0$ .

**Proof :**

From lemma 3 we have

$$\begin{aligned} \lim_{\lambda \rightarrow \lambda_{\max}} F_u(\lambda) &= \lim_{\lambda \rightarrow \lambda_{\max}} [\Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)) - \Phi(-d(1+\delta+\lambda)/\sigma_u(\lambda))] \\ &= \lim_{\lambda \rightarrow \lambda_{\max}} \Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)) - \Phi(-\infty) = \lim_{\lambda \rightarrow \lambda_{\max}} \Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)). \end{aligned}$$

From (4) and (8), we have

$$d(1-\delta-\lambda_{\max}) = dD_u(3\sqrt{v}C_p''(u, v) + (u-1)(1-|\delta|))/(3\sqrt{v}dC_p''(u, v) + ud^*). \text{ Thus}$$

a) If  $C_p''(u, v) > (1-u)(1-|\delta|)/(3\sqrt{v})$ , then  $d(1-\delta-\lambda_{\max}) > 0$ ,  $\lim_{\lambda \rightarrow \lambda_{\max}} \Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)) = \Phi(+\infty) = 1$ , and  $\lim_{\lambda \rightarrow \lambda_{\max}} F_u(\lambda) = 1$ .

b) If  $C_p''(u, v) = (1-u)(1-|\delta|)/(3\sqrt{v})$ , then  $d(1-\delta-\lambda_{\max}) = 0$ ,  $\lim_{\lambda \rightarrow \lambda_{\max}} \Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)) = \Phi(0) = 1/2$ , and  $\lim_{\lambda \rightarrow \lambda_{\max}} F_u(\lambda) = 1/2$ .

c) If  $0 < C_p''(u, v) < (1-u)(1-|\delta|)/(3\sqrt{v})$ , then  $d(1-\delta-\lambda_{\max}) < 0$ ,

$$\lim_{\lambda \rightarrow \lambda_{\max}} \Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)) = \Phi(-\infty) = 0, \text{ and } \lim_{\lambda \rightarrow \lambda_{\max}} F_u(\lambda) = 0.$$

In a similar way we have  $\lim_{\lambda \rightarrow \lambda_{\min}} F_l(\lambda) = 1 - \lim_{\lambda \rightarrow \lambda_{\min}} \Phi(-d(1+\delta+\lambda)/\sigma_l(\lambda))$ , and the lemma since from (5) and (8), we have

$$-d(1+\delta+\lambda_{\min}) = -dD_l(3\sqrt{v}C_p''(u, v) + (u-1)(1-|\delta|))/(3\sqrt{v}dC_p''(u, v) + ud^*).$$

**Lemma 8 :**

When  $0 \leq u < 1$  and  $v > 0$ , if  $C_p''(u, v) \geq (1-u)(1-|\delta|)/(3\sqrt{v})$ , then for any  $\lambda_u \in [0; \lambda_{\max}]$ , there exists  $\lambda_l = -\lambda_u d_u / d_l \in [\lambda_{\min}; 0]$  such as  $F_u(\lambda_u) \leq F_l(\lambda_l)$  when  $\delta > 0$ ,  $F_l(\lambda_l) \leq F_u(\lambda_u)$  when  $\delta < 0$ , and  $F_l(\lambda_l) = F_u(\lambda_u)$  when  $\delta = 0$ .

**Proof :**

The proof is given for  $\delta > 0$ . The case  $\delta < 0$  is similar, and the case  $\delta = 0$  is obvious.

Let  $\lambda_u \in [0; \lambda_{\max}]$ ,  $\lambda_l = -\lambda_u d_u / d_l \in [\lambda_{\min}; 0]$ ,  $E = \sigma_u(\lambda_u) / d$ ,  $a = (1-\delta-\lambda_u) / E$ ,

$b = (1+\delta+\lambda_l) / E$ , and  $x = (\lambda_u - \lambda_l) / E > 0$ .

If  $C_p''(u, v) \geq (1-u)(1-|\delta|)/(3\sqrt{v})$ , then  $3\sqrt{v}C_p''(u, v)/(1-|\delta|) + u \geq 1$ , and

$1/(d_u(3\sqrt{v}C_p''(u, v)/(1-|\delta|) + u)) \leq 1/d_u$ . From (4) and (8), we deduce

$0 \leq \lambda_u \leq \lambda_{\max} = 1/(d_u(3\sqrt{v}C_p''(u, v)/(1-|\delta|) + u)) \leq 1/d_u = 1-\delta$ , from where  $a \geq 0$  and  $1-\lambda_u d_u \geq 0$ . Consequently, since  $\delta > 0$ , we have

$d_u > d_l$ ,  $(1-\lambda_u d_u)/d_u \leq (1-\lambda_u d_u)/d_l$ , and  $1/d_u - \lambda_u \leq 1/d_l - \lambda_u d_u / d_l$ . Since  $\lambda_l = -\lambda_u d_u / d_l$ , from (4) and (5) we deduce  $a = (1-\delta-\lambda_u) / E \leq (1+\delta+\lambda_l) / E = b$ .

Let  $f$  the probability density function of the standard normal distribution  $N(0,1)$ . Since  $f$  is decreasing on  $[0; +\infty[$ , when  $x > 0$  and  $b \geq a \geq 0$ , we have  $\int_b^{b+x} f(x)dx \leq \int_a^{a+x} f(x)dx$ , thus

$$\int_{(1+\delta+\lambda_u)/E}^{(1+\delta+\lambda_u)/E} f(x)dx \leq \int_{(1-\delta-\lambda_u)/E}^{(1-\delta-\lambda_u)/E} f(x)dx \Leftrightarrow \int_{-(1+\delta+\lambda_u)/E}^{-(1+\delta+\lambda_u)/E} f(x)dx + \int_{(1-\delta-\lambda_u)/E}^{(1-\delta-\lambda_u)/E} f(x)dx \leq 0 \Leftrightarrow A + B \leq 0,$$

$$\text{where } A = \int_{-(1+\delta+\lambda_u)/E}^{-(1+\delta+\lambda_u)/E} f(x)dx = \Phi(-(1+\delta+\lambda_u)/E) - \Phi(-(1+\delta+\lambda_u)/E),$$

$$\text{and } B = \int_{(1-\delta-\lambda_u)/E}^{(1-\delta-\lambda_u)/E} f(x)dx = \Phi((1-\delta-\lambda_u)/E) - \Phi((1-\delta-\lambda_u)/E). \text{ Now}$$

$$\Phi((1-\delta-\lambda_u)/E) - \Phi(-(1+\delta+\lambda_u)/E) = \Phi((1-\delta-\lambda_u)/E) - \Phi(-(1+\delta+\lambda_u)/E) + A + B, \text{ thus}$$

$$\Phi((1-\delta-\lambda_u)/E) - \Phi(-(1+\delta+\lambda_u)/E) \leq \Phi((1-\delta-\lambda_u)/E) - \Phi(-(1+\delta+\lambda_u)/E).$$

Since  $\lambda_l = -\lambda_u d_u / d_l$ , from lemma 1, we have  $\sigma_l(\lambda_l) = \sigma_u(\lambda_u)$ , and  $E = \sigma_u(\lambda_u) / d = \sigma_l(\lambda_l) / d$ . Consequently,

$$\Phi(d(1-\delta-\lambda_u)/\sigma_u(\lambda_u)) - \Phi(-d(1+\delta+\lambda_u)/\sigma_u(\lambda_u))$$

$$\leq \Phi(d(1-\delta-\lambda_l)/\sigma_l(\lambda_l)) - \Phi(-d(1+\delta+\lambda_l)/\sigma_l(\lambda_l)),$$

$$\text{and } F_u(\lambda_u) \leq F_l(\lambda_l).$$

It should be noted that when  $C_p''(u, v) < (1-u)(1-|\delta|)/(3\sqrt{v})$  and  $\delta \neq 0$ , no general rule can be obtained, as shown in the graphic investigations which have been made, but which are not detailed here.

### Lemma 9 :

1) When  $0 < u < 1$  and  $v > 0$ , we have

if  $\delta = 0$ , then  $F(\lambda)$  has a relative minimum at  $\lambda = 0$ ,

if  $\delta > 0$ , then  $F(\lambda)$  decreases in the neighbourhood of  $\lambda = 0$  when  $C_p''(u, v) > C_1$ , and has a relative minimum at  $\lambda = 0$  when  $C_p''(u, v) \leq C_1$ ,

if  $\delta < 0$ , then  $F(\lambda)$  increases in the neighbourhood of  $\lambda = 0$  when  $C_p''(u, v) > C_1$ , and has a relative minimum at  $\lambda = 0$  when  $C_p''(u, v) \leq C_1$ ,

$$\text{where } C_1 = \frac{(1-|\delta|)}{3} \sqrt{\frac{1}{|\delta|} \tanh^{-1}\left(\frac{u}{1-|\delta|(1-u)}\right)}.$$

2) When  $u = 0$  and  $v > 0$ , we have

if  $\delta = 0$ , then  $F(\lambda)$  has a relative extremum at  $\lambda = 0$ ,

if  $\delta > 0$ , then  $F(\lambda)$  decreases in the neighbourhood of  $\lambda = 0$ .

if  $\delta < 0$ , then  $F(\lambda)$  increases in the neighbourhood of  $\lambda = 0$ .

### Proof :

If  $\delta = 0$ , we have  $\lim_{\lambda \rightarrow 0^+} Q_u(\lambda) = u\sigma_0^2$  and  $\lim_{\lambda \rightarrow 0^-} Q_l(\lambda) = -u\sigma_0^2$ , thus from lemma 4,  $F(\lambda)$  has a relative minimum at  $\lambda = 0$  when  $0 < u < 1$ , and has a relative extremum when  $u = 0$ .

Let  $\delta > 0$ . We have  $\lim_{\lambda \rightarrow 0^+} \sigma_u(\lambda) = \lim_{\lambda \rightarrow 0^-} \sigma_l(\lambda) = \sigma_0$ ,  $\lim_{\lambda \rightarrow 0^+} k_u(\lambda) = \lim_{\lambda \rightarrow 0^-} k_l(\lambda) = \sigma_0^2$ ,  $\lim_{\lambda \rightarrow 0^+} q_u(\lambda) = ud_u\sigma_0^2$ , and  $\lim_{\lambda \rightarrow 0^-} q_l(\lambda) = -ud_l\sigma_0^2$ . Thus from (4), (5), and (6), we have

$$\lim_{\lambda \rightarrow 0^-} Q_l(\lambda) = -\frac{\sigma_0^2}{1+\delta} \left[ u + (1+\delta-\delta u) \tanh\left((3C_p''(u, v))^2 \delta / (1-|\delta|)^2\right) \right], \text{ and}$$

$$\lim_{\lambda \rightarrow 0^+} Q_u(\lambda) = \frac{\sigma_0^2}{1-\delta} \left[ u - (1-\delta + \delta u) \tanh \left( (3C_p''(u, v))^2 \delta / (1-|\delta|)^2 \right) \right].$$

Since  $0 \leq u < 1$ ,  $\delta > 0$ , and  $1 + \delta - \delta u > 0$ , we have  $\lim_{\lambda \rightarrow 0^+} Q_u(\lambda) < 0$ . For  $u = 0$ ,  $\lim_{\lambda \rightarrow 0^+} Q_u(\lambda) < 0$ , thus  $F(\lambda)$  decreases in the neighbourhood of  $\lambda = 0$ . For  $0 < u < 1$ , since  $1 - \delta + \delta u > 0$ , we have  $\lim_{\lambda \rightarrow 0^+} Q_u(\lambda) > 0$  when  $C_p''(u, v) < C$ ,  $\lim_{\lambda \rightarrow 0^+} Q_u(\lambda) < 0$  when  $C_p''(u, v) > C$ , and  $\lim_{\lambda \rightarrow 0^+} Q_u(\lambda) = 0$  when  $C_p''(u, v) = C$ , where  $C = (1-|\delta|) \sqrt{(1/\delta) \tanh^{-1}(u/(1-\delta(1-u)))} / 3$ , which is equal to  $C_1$  since  $\delta > 0$ . For  $\delta < 0$ , we use the fact that  $F_u(\lambda, \delta) = F_l(-\lambda, -\delta)$  if  $0 \leq \lambda < \lambda_{\max}$ .

**Lemma 10 :**

1) When  $0 \leq u < 1$  and  $v > 0$ , if  $C_p''(u, v) = (1-u)(1-|\delta|)/(3\sqrt{v})$ , then

$-F_u(\lambda)$  is decreasing in the neighbourhood of  $\lambda = \lambda_{\max} = 1 - \delta$ , and cannot have more than two extrema for  $\lambda \in [0; \lambda_{\max}[$ .

$-F_l(\lambda)$  is increasing in the neighbourhood of  $\lambda = \lambda_{\min} = 1 + \delta$ , and cannot have more than two extrema for  $\lambda \in ]\lambda_{\min}; 0]$ .

2) When  $u = \delta = 0$  and  $v > 0$ , if  $C_p''(u, v) = (1-u)(1-|\delta|)/(3\sqrt{v})$ , then  $F_u(\lambda)$  cannot have more than one extremum for  $\lambda \in [0; \lambda_{\max}[$ , and  $F_l(\lambda)$  cannot have more than one extremum for  $\lambda \in [\lambda_{\min}; 0]$ .

**Proof :**

The proof is given for  $\delta \geq 0$ . The case  $\delta \leq 0$  is similar.

1) If  $C_p''(u, v) = (1-u)(1-|\delta|)/(3\sqrt{v})$ , we have  $\sigma_0 = \sqrt{vd}/(1-u)$ ,  $\lambda_{\max} = 1 - \delta$ ,

$$k_u(\lambda) = \frac{d^2 v (1-\delta - \lambda u)^2}{(1-u)^2 (1-\delta)^2}, \quad q_u(\lambda) = \frac{u d^2 v (1-\delta - u \lambda)}{(1-\delta)^2 (1-u)^2},$$

$$\sigma_u(\lambda) = \frac{\sqrt{vd} \left( (1-\delta - u \lambda)^2 - \lambda^2 (1-u)^2 \right)^{1/2}}{(1-\delta)(1-u)}, \quad \text{and} \quad Q_u(\lambda) = \frac{vd^2}{(1-u)^2 (1-\delta)^2} (A - B \tanh(x))$$

$$= \frac{vd^2 (A + B)}{(1-u)^2 (1-\delta)^2 (e^{2x} + 1)} \left( 1 + \frac{A - B}{A + B} e^{2x} \right), \quad \text{where} \quad A = u(1-\delta - u \lambda) + \lambda(1-u)^2,$$

$$B = (1-\delta - \lambda u)(1-\delta + \delta u) + \delta \lambda (1-u)^2, \quad \text{and} \quad x = \frac{(\delta + \lambda)(1-\delta)^2 (1-u)^2}{v \left( (1-\delta - u \lambda)^2 - \lambda^2 (1-u)^2 \right)}.$$
 Now,

$A - B = (1-\delta)(1-u)(\lambda - (1-\delta))$  which is negative since  $\lambda \in [0; \lambda_{\max}[$ , and

$A + B = (1-\delta - u \lambda)(1-\delta + u(1+\delta)) + \lambda(1-u)^2(1+\delta)$  which is positive. When  $\lambda$  tends to  $1 - \delta$ , 1 is negligible compared to  $e^{2x} (A - B)/(A + B)$ , thus  $Q_u(\lambda)$  is negative and  $F_u(\lambda)$  is decreasing in the neighbourhood of  $\lambda = 1 - \delta$ .

Now we show that  $F_u(\lambda)$  cannot have more than two extrema when  $\lambda \in [0; \lambda_{\max}[$ . Let

$$\lambda = \frac{(1-\delta)t}{1+t-u} \quad \text{with} \quad t \in [0; +\infty[. \quad \text{We have} \quad x = \frac{(t+1-u)(t+\delta(1-u))}{v(2t+1)} \quad \text{and}$$

$\frac{A-B}{A+B} = -\frac{(1-\delta)(1-u)}{[2t+1-\delta+u(1+\delta)]}$ . Thus  $Q_u(\lambda)$  has the sign of

$$G(t) = 1 - \frac{(1-\delta)(1-u)}{[2t+1-\delta+u(1+\delta)]} e^{2\frac{(t+1-u)(t+\delta(1-u))}{v(2t+1)}}.$$

Let  $y = \frac{2t+(1+\delta)(1-u)}{(1-u)(1-\delta)}$  with  $y \geq \frac{(1+\delta)}{(1-\delta)} \geq 1$ . We have  $G(y) = 1 - e^{\frac{(1-u)(1-\delta)(y^2-1)}{2v(y+k)}} / (y+2k)$ ,

where  $k = \frac{u-\delta(1-u)}{(1-\delta)(1-u)}$ .

Since  $G'(y) = \frac{e^{\frac{(1-u)(1-\delta)(y^2-1)}{2v(y+k)}}}{(y+2k)^2} \left( 1 - \frac{(1-u)(1-\delta)}{2v} \frac{y^3+4ky^2+(1+4k^2)y+2k}{(y+k)^2} \right)$ ,  $G'(y)$  has the sign

of  $H(y) = 1 - \frac{(1-u)(1-\delta)}{2v} \frac{y^3+4ky^2+(1+4k^2)y+2k}{(y+k)^2}$ . We have

$$H'(y) = -\frac{(1-u)(1-\delta)}{2v} \frac{y^3+3ky^2+(4k^2-1)y+4k^3-3k}{(y+k)^3}, \text{ and } y+k = \frac{2t+1}{(1-u)(1-\delta)} > 0 \text{ since}$$

$t \geq 0$ . Thus  $H'(y)$  has the opposite sign of  $K(y) = y^3+3ky^2+(4k^2-1)y+4k^3-3k$ .

For the particular case where  $u=0$  and  $\delta=0$ , we have  $k=0$  and  $K(y) > 0$  since  $y \geq (1+\delta)/(1-\delta) \geq 1$ .

For the case where  $u \neq 0$  or  $\delta \neq 0$ , we have  $K'(y) = 3y^2+6ky+(4k^2-1)$ , and the discriminant of the quadratic equation  $K'(y)$  is  $\Delta = 12(1-k^2)$ .

If  $|k| > 1$ , we have  $\Delta < 0$ , thus  $K'(y) > 0$  and  $K(y)$  is increasing. When  $k > 1$ , since  $K(0) = 4k^3-3k > 0$ , thus  $K(y) > 0$  for  $y \geq (1+\delta)/(1-\delta)$ . When  $k < -1$ , we have  $K(-2k) = -k > 0$ , and  $(1+\delta)/(1-\delta) - (-2k) = (1-\delta+u+\delta u)/((1-\delta)(1-u)) > 0$ . Thus  $K(y) > 0$  when  $y \geq (1+\delta)/(1-\delta)$ .

If  $|k| \leq 1$ , we have  $\Delta \geq 0$  and  $K'(y)$  has two real roots,  $y_1 = \left( -6k - \sqrt{12(1-k^2)} \right) / 6$  and

$$y_2 = \left( -6k + \sqrt{12(1-k^2)} \right) / 6. \text{ Now } \frac{(1+\delta)}{(1-\delta)} - y_2 = \frac{1}{(1-\delta)(1-u)} - \frac{\sqrt{12(1-k^2)}}{6} > 0, \text{ since}$$

$(1-\delta)^{-1}(1-u)^{-1} > 1$  and  $0 \leq \sqrt{12(1-k^2)} / 6 < 1$ . Thus  $K(y)$  is increasing when

$y \geq (1+\delta)/(1-\delta) > y_2 \geq y_1$ . Moreover,  $K(1) = 4k^2(k+1) > 0$ . Thus  $K(y) > 0$  when  $y \geq (1+\delta)/(1-\delta) \geq 1$ .

To conclude, for all cases we have  $K(y) > 0$ , thus  $H'(y) < 0$ , and  $H(y)$  is decreasing with  $\lim_{y \rightarrow +\infty} H(y) = -\infty$ . Therefore  $H(y)$  has at the maximum one zero,  $G(y)$  has at the maximum

two zeros, and  $F_u(\lambda)$  cannot have more than two extrema when  $\lambda \in [0; \lambda_{\max}[$ .

2) When  $u = \delta = 0$ , we have  $\lim_{y \rightarrow (1+\delta)/(1-\delta)} G(y) = 0$ , thus  $G(y)$  has at the maximum one zero,

and  $F_u(\lambda)$  cannot have more than one extremum when  $\lambda \in [0; \lambda_{\max}[$ .