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ABSTRACT

Capability indices were introduced to compare the performance of various processes independently of their tolerance interval. The concept of performance, related at first to the proportion of conforming items (process yield), quickly evolved to take into account the process position in relation to its target (process centering), as well. If the links between capability indices and process centering have already been studied, those between capability indices and processes yield have only been partly studied. In this paper we clarify the links between the process yield and the indices $C_p''(u,v)$.

KEYWORDS

Asymmetric tolerances, Process capability indices, Process centering, Process yield.

1. INTRODUCTION

In any manufacturing process, the variable of interest is associated with a target *T* and a tolerance interval $[L;U]$. The process performance can be evaluated by taking various criterions into consideration. The oldest one is the process yield or percentage of conforming items, which we note *Yield*. The user often prefers to express this performance by using the percentage of nonconforming items, which we note *NC*, which is obviously defined by the relation *NC* = 1 – *Yield*. In the following study we will consider the most usual case where the variable of interest is normally distributed with mean μ and standard deviation σ. In these conditions, the process yield is represented by the relation $Yield = \Phi\left((U-\mu)/\sigma\right) - \Phi\left((L-\mu)/\sigma\right)$, in which Φ is the cumulative function of the standard normal distribution.

However the process performance cannot only be related to the proportion of conforming items. Between two processes leading to the same number of conforming items, the more effective in the user's mind will be the one whose mean value is on the target. The process centering, that is to say the ability to cluster around the target is thus another important fact to measure the process performance. The four basic capability indices C_p, C_{pk}, C_{pm}, and C_{pmk} have been widely used as unitless measures, which combine natural process variability, manufacturing tolerances, and the process centering. Vännman (1995) proposed a superstructure containing these four basic indices as

$$C_p(u,v)=\frac{d-u\left|\mu-m\right|}{3\sqrt{\sigma^2+v\left(\mu-T\right)^2}},$$

where $m = (L + U)/2$ stands for the midpoint of the tolerance interval, $d=(U-L)/2$ stands for the half-length of the tolerance interval, and u and v are two non-negative parameters. However, if indices $C_p(u,v)$ are well adapted to the case of symmetrical tolerances ($T = m$), they have some undesirable properties when the tolerances are asymmetrical ($T \neq m$). To generalize the family $C_p(u,v)$ to the case $T \neq m$, Chen and Pearn (2001) suggested to use the family

$$C_p^{"}(u,v)=\frac{d^*-uA^*}{3\sqrt{\sigma^2+vA^2}},$$

in which $A=\max\left\{d(\mu-T)/D_u, d(T-\mu)/D_l\right\}$, $A^*=\max\left\{d^*(\mu-T)/D_u, d^*(T-\mu)/D_l\right\}$, $D_u=U-T$, $D_l=T-L$, $d^*=\min\left\{D_u, D_l\right\}$. Note that when $u\neq 0$, the calculation of $C_p^{"}(u,v)$ can lead to negative values, which is hardly satisfactory to measure the process performance. In addition for $u = 0$, $C_p^{"}(u,v)$ is obviously positive. Therefore, in the following part of this paper, we will only be interested in the case where $C_p^{"}(u,v)>0$, in order not to increase the heaviness of the paper with the particular case where $C_p^{"}(u,v)=0$.

The choice of u and v allows us to attach more or less importance to the process yield or to the process centering. In order to enable the user to choose the best index according to his needs, the links which join indices, process yield and process centering, have to be known precisely.

Links between capability indices and centering are given (Kotz and Lovelace, 1998, p.184) in the case of symmetrical tolerance by the relation

$$\left|\mu-T\right|<\frac{d}{3C_p(u,v)\sqrt{v}+u}. \qquad (1)$$

In the case of asymmetrical tolerances, Chen and Pearn (2001) suggest a generalization of this expression under the form

$$T-\frac{(1-R)D_l}{3\sqrt{v}C_p^{"}(u,v)+u(1-R)}<\mu<T+\frac{(1-R)D_u}{3\sqrt{v}C_p^{"}(u,v)+u(1-R)}, \qquad (2)$$

in which $R=|1-r|/(1+r)$, and $r=D_l/D_u$. Although it is not specified by the previous authors, note that the relations (1) and (2) are true for $C_p^{"}(u,v)>0$, and $(u,v)\neq(0,0)$. When $(u,v)=(0,0)$, these relations are expressed in the form $-\infty<\mu<+\infty$.

On the other hand, the links between capability indices and processes yield have only been partly studied. In the following paragraph we explain the various results found in the literature.

For $C_p=(U-L)/6\sigma$, first index introduced by Juran (1974), we have $2\Phi(-3C_p)\leq NC\leq 1$ (Pearn and Kotz, 2006, p.9), the lower bound being reached only when the process is well centered, that is to say when μ is on m.

For $C_{pk}=\min\left((U-\mu)/3\sigma,(\mu-L)/3\sigma\right)$, index which takes into account the position of the mean inside the tolerance interval, we have $\Phi(-3C_{pk})\leq NC\leq 2\Phi(-3C_{pk})$ (Pearn and Kotz, 2006, p.42).

For $C_{pm} = (U-L)/6\sqrt{\sigma^2+(\mu-T)^2}$, and under the usual assumption that $T = m$, Ruczinski (1996) shows that when $C_{pm} < 1/3$ then $2\Phi(-3C_{pm}) \le NC \le 1$, when $1/3 < C_{pm} < 1/\sqrt{3}$ then $0 \le NC \le M$ where M is the solution of an equation which can be solved numerically, and finally when $C_{pm} > 1/\sqrt{3}$, then $0 \le NC \le 2\Phi(-3C_{pm})$.

For $C_{pmk} = C_{pk}C_{pm}/C_p$, and $T = m$, Pearn and Kotz (2006, p.114) quote a working paper of Pearn and Lin (2005) showing that when $C_{pmk} \ge \sqrt{2}/3$, we have $0 \le NC \le 2\Phi(-3C_{pmk})$.

Generally, when the tolerances are symmetrical, Vännman (1995) proposes the family $C_p(u,v)$, where *u* and *v* are two positive or null parameters. Kotz and Lovelace (1998, p.184) indicate that $NC \le 2\Phi(-3C_p(u,v))$, without taking into account the restrictions specified by Ruczinski (1996) for C_{pm} and by Pearn and Lin (2005) for C_{pmk}.

For asymmetrical tolerances, Chen and Pearn (2001) propose the family $C_p''(u,v)$. To study the process yield, these authors use the index $S_{pk} = (1/3)\Phi^{-1}\{(1/2)\Phi((U-\mu)/\sigma) + (1/2)\Phi((\mu-L)/\sigma)\}$ suggested by Boyles (1994) which is directly related to the proportion of nonconforming items by the relation $NC = 2\Phi(-3S_{pk})$. After graphically noticing that $C_p''(u,v) < S_{pk}$, they conclude that if $C_p''(u,v) = c$, the process yield must be no less than that corresponding to $S_{pk} = c$. In other words, the proportion of nonconforming must not be greater than $2\Phi(-3C_p''(u,v))$. However it is possible to find values for which $C_p''(u,v) > S_{pk}$, which thus do not allow to obtain an upper bound of *NC*. For example, when $(L,T,U) = (26,50,58)$, μ = 59.3, σ = 0.643, we have $C_p''(0.5,1) = 0.06$ and $S_{pk} = 0.009$. In these conditions, the proportion of nonconforming is equal to 0.98, a quantity which is not lower than $2\Phi(-3C_p''(0.5,1))$ = 0.86.

In the particular case where $(u,v) = (1,1)$, Pearn, Lin and Chen (1999) show that $NC \le 2\Phi(-3C_{pmk}'')$ supposing that $C_{pmk}'' \le C_{pmk}$. However it is possible to find values for which $C_{pmk}'' > C_{pmk}$. For example, when $(L,T,U) = (26,50,58)$, μ = 49, σ = 0.5, we have $C_{pmk}'' = 3.07$ and $C_{pmk} = 2.68$.

Lastly, when $(u,v) = (1,0)$, Pearn, Lin and Chen (2004), or Chang and Wu (2008), show that

$$NC \le 2 - [\Phi(3C_{pk}''/\min\{1,r\}) + \Phi(3C_{pk}''\max\{1,r\})]. \qquad (3)$$

As we have just seen, the results evoked in the literature concerning the links between capability indices and process yield include some errors or inaccuracies. Thus the purpose of this paper is to specify the relations between the $C_p''(u,v)$ indices and the proportion of conforming or nonconforming items, and this for any *u*, *v* ≥ 0, that we can obtain. In the following section, we will just study the case where $T \in [m;U[$, since the case $T \in]L;m]$ is considered in a similar way.

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To take into account the position of T in the interval $[m;U[$, we note $T = m + \delta d$ where $\delta \in [0,1[$. To take into account the deviations of μ, we assume that $\mu = T + \lambda d$ where λ is unspecified. The relation (2) allows to see the field in which μ is located, more accurately. In addition, since $d^* / d = 1 - R$, (2) can still be written in the form

$$T - \frac{d^* D_l}{3\sqrt{v} d C_p^{"}(u,v) + u d^*} < \mu < T + \frac{d^* D_u}{3\sqrt{v} d C_p^{"}(u,v) + u d^*}. \quad (4)$$

The previous relation allows to specify the field of variation of $\lambda = (\mu - T)/d$,

$$-\frac{1}{d_l(\sqrt{v} d / \sigma_0 + u)} = \lambda_{\min} < \lambda < \lambda_{\max} = \frac{1}{d_u(\sqrt{v} d / \sigma_0 + u)}, \quad (5)$$

where $d_u = d / D_u$, $d_l = d / D_l$, and $\sigma_0 = d^* / (3C_p^{"}(u,v))$. As noted in the previous section, the relation (5) is true for $C_p^{"}(u,v) > 0$, $(u,v) \neq (0,0)$, and in the case when $(u,v) = (0,0)$, the relation (5) remains true assuming that $\lambda_{\min} = -\infty$ and $\lambda_{\max} = +\infty$. Reciprocally, it is not difficult to note that if $\lambda_{\min} < \lambda < \lambda_{\max}$, then $C_p^{"}(u,v) > 0$. This is obvious when $u = 0$. When $u \neq 0$, from (5) we have $-1/(ud_l) < \lambda < 1/(ud_u)$. Consequently, if $0 \leq \lambda < 1/(ud_u)$, $C_p^{"}(u,v) = \dfrac{d^*(1 - u\lambda d_u)}{3\sqrt{\sigma^2 + v(\lambda d d_u)^2}} > 0$, and if $-1/(ud_l) < \lambda \leq 0$, $C_p^{"}(u,v) = \dfrac{d^*(1 + u\lambda d_l)}{3\sqrt{\sigma^2 + v(\lambda d d_l)^2}} > 0$.

The four following lemmas allow us to study the variations of the process yield according to the values of λ defined in (5), in a general way. The following sections will enable us to explain the behaviour of the process yield, in particular the existence of maxima and minima more clearly, by the distinction of various situations depending on the *u* and *v* values.

Lemma 1 :

$$\sigma = \begin{cases} \sigma_u(\lambda) = \left(\sigma_0^2 (1 - u\lambda d_u)^2 - v(\lambda d d_u)^2\right)^{1/2} & \text{if} \quad 0 \leq \lambda < \lambda_{\max} \\ \sigma_l(\lambda) = \left(\sigma_0^2 (1 + u\lambda d_l)^2 - v(\lambda d d_l)^2\right)^{1/2} & \text{if} \quad \lambda_{\min} < \lambda \leq 0 \end{cases}.$$

Proof :

If $0 \leq \lambda < \lambda_{\max}$, then $C_p^{"}(u,v) = d^*(1 - u\lambda d_u) / \left(3\left(\sigma_u^2(\lambda) + v(\lambda d d_u)^2\right)^{1/2}\right)$, thus $\sigma_u(\lambda)$.

If $\lambda_{\min} < \lambda \leq 0$, then $C_p^{"}(u,v) = d^*(1 + u\lambda d_l) / \left(3\left(\sigma_l^2(\lambda) + v(\lambda d d_l)^2\right)^{1/2}\right)$, thus $\sigma_l(\lambda)$.

Lemma 2 :

If $(u,v) \neq (0,0)$, then $\lim\limits_{\lambda \to \lambda_{\max}} \sigma_u(\lambda) = \lim\limits_{\lambda \to \lambda_{\min}} \sigma_l(\lambda) = 0$.

Proof :

Let $(u,v) \neq (0,0)$. According to lemma 1, we have

$$\lim_{\lambda \to \lambda_{\max}} \sigma_u(\lambda) = \lim_{\lambda \to 1/\left[d_u(\sqrt{v} d / \sigma_0 + u)\right]} \left(\sigma_0^2 (1 - u\lambda d_u)^2 - v(\lambda d d_u)^2\right)^{1/2}$$

$$= \left(\sigma_0^2 \left(1 - u \frac{1}{d_u(\sqrt{v} d / \sigma_0 + u)} d_u \right)^2 - v \left(\frac{1}{d_u(\sqrt{v} d / \sigma_0 + u)} d d_u \right)^2 \right)^{1/2}$$

$$= \left(\left(\frac{\sqrt{v}d}{\sqrt{v}d/\sigma_0 + u} \right)^2 - \left(\frac{\sqrt{v}d}{\sqrt{v}d/\sigma_0 + u} \right)^2 \right)^{1/2} = 0$$

It is the same for $\sigma_l(\lambda)$.

Lemma 3 :

a) If $(u,v) \neq (0,0)$, then

$$Yield = F(\lambda) = \begin{cases} F_u(\lambda) = \Phi\left(d(1-\delta-\lambda)/\sigma_u(\lambda)\right) - \Phi\left(-d(1+\delta+\lambda)/\sigma_u(\lambda)\right) \ \ if \ \ 0 \le \lambda < \lambda_{\max} \\ F_l(\lambda) = \Phi\left(d(1-\delta-\lambda)/\sigma_l(\lambda)\right) - \Phi\left(-d(1+\delta+\lambda)/\sigma_l(\lambda)\right) \ \ if \ \ \lambda_{\min} < \lambda \le 0 \end{cases}.$$

b) If $(u,v) = (0,0)$, then

$Yield = F(\lambda) = \Phi\left(d(1-\delta-\lambda)/\sigma_0\right) - \Phi\left(-d(1+\delta+\lambda)/\sigma_0\right)$, for any $\lambda \in]-\infty, +\infty[$.

Proof :

Since $\mu = T + \lambda d$ and $T = m + \delta d$, we have $U - \mu = d(1-\delta-\lambda)$, and $L - \mu = -d(1+\delta+\lambda)$, thus the lemma since $Yield = \Phi\left((U-\mu)/\sigma\right) - \Phi\left((L-\mu)/\sigma\right)$.

Lemma 4 :

a) If $(u,v) \neq (0,0)$, then $F_u'(\lambda)$ has the sign of

$Q_u(\lambda) = q_u(\lambda) + v\lambda d^2 d_u^2 - (k_u(\lambda) + (\delta+\lambda)q_u(\lambda) + \delta v\lambda d^2 d_u^2)\tanh(d^2(\delta+\lambda)/\sigma_u^2(\lambda))$,

where $k_u(\lambda) = \sigma_0^2(1-u\lambda d_u)^2$ and $q_u(\lambda) = ud_u\sigma_0^2(1-u\lambda d_u)$.

b) If $(u,v) \neq (0,0)$, then $F_l'(\lambda)$ has the sign of

$Q_l(\lambda) = q_l(\lambda) + v\lambda d^2 d_l^2 - (k_l(\lambda) + (\delta+\lambda)q_l(\lambda) + \delta v\lambda d^2 d_l^2)\tanh(d^2(\delta+\lambda)/\sigma_l^2(\lambda))$,

where $k_l(\lambda) = \sigma_0^2(1+u\lambda d_l)^2$ and $q_l(\lambda) = -ud_l\sigma_0^2(1+u\lambda d_l)$.

c) If $(u,v) = (0,0)$, then $F'(\lambda)$ has the sign of $Q(\lambda) = -\sinh(d^2(\delta+\lambda)/\sigma_0^2)$.

Proof :

a) From lemma 3, we have $F_u(\lambda) = \Phi(\psi_u(\lambda)) - \Phi(\varphi_u(\lambda))$, where

$\psi_u(\lambda) = d(1-\delta-\lambda)/\sigma_u(\lambda)$, and $\varphi_u(\lambda) = -d(1+\delta+\lambda)/\sigma_u(\lambda)$.

From lemma 1, $\sigma_u'(\lambda) = -(ud_u\sigma_0^2(1-u\lambda d_u) + v\lambda d^2 d_u^2)/\sigma_u(\lambda)$, thus

$\psi_u'(\lambda) = [-d\sigma_u(\lambda) + d(1-\delta-\lambda)\{ud_u\sigma_0^2(1-u\lambda d_u) + v\lambda d^2 d_u^2\}/\sigma_u(\lambda)]/\sigma_u^2(\lambda)$

$= [-d\sigma_u^2(\lambda) + d(1-\delta-\lambda)\{ud_u\sigma_0^2(1-u\lambda d_u) + v\lambda d^2 d_u^2\}]/\sigma_u^3(\lambda)$.

From lemma 1, we obtain

$\psi_u'(\lambda) = [-d\{\sigma_0^2(1-u\lambda d_u)^2 - v(\lambda d d_u)^2\} + d(1-\delta-\lambda)\{ud_u\sigma_0^2(1-u\lambda d_u) + v\lambda d^2 d_u^2\}]/\sigma_u^3(\lambda)$

$= d[-\sigma_0^2(1-u\lambda d_u)^2 + (1-\delta-\lambda)ud_u\sigma_0^2(1-u\lambda d_u) + (1-\delta)v\lambda d^2 d_u^2]/\sigma_u^3(\lambda)$

$= d[-k_u(\lambda) + (1-\delta-\lambda)q_u(\lambda) + (1-\delta)v\lambda d^2 d_u^2]/\sigma_u^3(\lambda)$.

$\varphi_u'(\lambda) = [-d\sigma_u^2(\lambda) - d(1+\delta+\lambda)\{ud_u\sigma_0^2(1-u\lambda d_u) + v\lambda d^2 d_u^2\}]/\sigma_u^3(\lambda)$

$= -d[\sigma_0^2(1-u\lambda d_u)^2 - v(\lambda d d_u)^2 + (1+\delta+\lambda)\{ud_u\sigma_0^2(1-u\lambda d_u) + v\lambda d^2 d_u^2\}]/\sigma_u^3(\lambda)$

$= -d[\sigma_0^2(1-u\lambda d_u)^2 + (1+\delta+\lambda)ud_u\sigma_0^2(1-u\lambda d_u) + (1+\delta)v\lambda d^2 d_u^2]/\sigma_u^3(\lambda)$

$= -d[k_u(\lambda) + (1+\delta+\lambda)q_u(\lambda) + (1+\delta)v\lambda d^2 d_u^2]/\sigma_u^3(\lambda)$.

Consequently,

$F_u'(\lambda) = \Phi'(\psi_u(\lambda))\psi_u'(\lambda) - \Phi'(\varphi_u(\lambda))\varphi_u'(\lambda)$

$$= (2\pi)^{-1/2} e^{-\left(d(1-\delta-\lambda)/\sigma_u(\lambda)\right)^2/2} \psi_u^{'}(\lambda) - (2\pi)^{-1/2} e^{-\left(d(1+\delta+\lambda)/\sigma_u(\lambda)\right)^2/2} \varphi_u^{'}(\lambda)$$

$$= (2\pi)^{-1/2} e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_u^2(\lambda))} e^{d^2(\delta+\lambda)/\sigma_u^2(\lambda)} \psi_u^{'}(\lambda) - (2\pi)^{-1/2} e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_u^2(\lambda))} e^{-d^2(\delta+\lambda)/\sigma_u^2(\lambda)} \varphi_u^{'}(\lambda)$$

$$= (2\pi)^{-1/2} e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_u^2(\lambda))} [e^{d^2(\delta+\lambda)/\sigma_u^2(\lambda)} \psi_u^{'}(\lambda) - e^{-d^2(\delta+\lambda)/\sigma_u^2(\lambda)} \varphi_u^{'}(\lambda)]$$

$$= (2\pi)^{-1/2} e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_u^2(\lambda))} \Big[e^{d^2(\delta+\lambda)/\sigma_u^2(\lambda)} d(-k_u(\lambda) + (1-\delta-\lambda) q_u(\lambda) + (1-\delta) v\lambda d^2 d_u^2) / \sigma_u^3(\lambda)$$
$$+ e^{-d^2(\delta+\lambda)/\sigma_u^2(\lambda)} d(k_u(\lambda) + (1+\delta+\lambda) q_u(\lambda) + (1+\delta) v\lambda d^2 d_u^2) / \sigma_u^3(\lambda) \Big]$$

$$= d(2\pi)^{-1/2} \sigma_u^{-3}(\lambda) e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_u^2(\lambda))} \Big[(e^{d^2(\delta+\lambda)/\sigma_u^2(\lambda)} + e^{-d^2(\delta+\lambda)/\sigma_u^2(\lambda)})(q_u(\lambda) + v\lambda d^2 d_u^2)$$
$$- (e^{d^2(\delta+\lambda)/\sigma_u^2(\lambda)} - e^{-d^2(\delta+\lambda)/\sigma_u^2(\lambda)})(k_u(\lambda) + (\delta+\lambda) q_u(\lambda) + \delta v\lambda d^2 d_u^2) \Big]$$

$$= d\sqrt{2}\pi^{-1/2} \sigma_u^{-3}(\lambda) e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_u^2(\lambda))}$$
$$\times \Big[(q_u(\lambda) + v\lambda d^2 d_u^2) \cosh(d^2(\delta+\lambda)/\sigma_u^2(\lambda))$$
$$- (k_u(\lambda) + (\delta+\lambda) q_u(\lambda) + \delta v\lambda d^2 d_u^2) \sinh(d^2(\delta+\lambda)/\sigma_u^2(\lambda)) \Big]$$

$$= d\sqrt{2}\pi^{-1/2} \sigma_u^{-3}(\lambda) e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_u^2(\lambda))} \cosh(d^2(\delta+\lambda)/\sigma_u^2(\lambda)) Q_u(\lambda).$$

Now $d\sqrt{2}\pi^{-1/2} \sigma_u^{-3}(\lambda) e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_u^2(\lambda))} \cosh(d^2(\delta+\lambda)/\sigma_u^2(\lambda)) > 0$. Thus $F_u^{'}(\lambda)$ has the sign of $Q_u(\lambda)$.

b) The proof is similar for $F_l^{'}(\lambda)$.

c) If $(u,v) = (0,0)$, from the lemma 3,

$$F^{'}(\lambda) = -(d/\sigma_0)[\Phi'\left(d(1-\delta-\lambda)/\sigma_0\right) - \Phi'\left(-d(1+\delta+\lambda)/\sigma_0\right)]$$

$$= -(d/\sigma_0)[(2\pi)^{-1/2} e^{-\left(d(1-\delta-\lambda)/\sigma_0\right)^2/2} - (2\pi)^{-1/2} e^{-\left(d(1+\delta+\lambda)/\sigma_0\right)^2/2}]$$

$$= -(d/\sigma_0)(2\pi)^{-1/2} e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_0^2)} [e^{d^2(\delta+\lambda)/\sigma_0^2} - e^{-d^2(\delta+\lambda)/\sigma_0^2}]$$

$$= -(d/\sigma_0)\sqrt{2}\pi^{-1/2} e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_0^2)} \sinh(d^2(\delta+\lambda)/\sigma_0^2) = d\sqrt{2}\pi^{-1/2} \sigma_0^{-1} e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_0^2)} Q(\lambda).$$

Now $d\sqrt{2}\pi^{-1/2} \sigma_0^{-1} e^{-d^2(1+(\delta+\lambda)^2)/(2\sigma_0^2)} > 0$. thus $F^{'}(\lambda)$ has the sign of $Q(\lambda)$.

2.1. Case $(u,v) = (0,0)$

When $(u,v) = (0,0)$, we have $C_p^{''}(u,v) = C_p^{''}(0,0) = C_p^{''}$.

Theorem 1 :

$$2\Phi\left(-3\frac{d}{d^*} C_p^{''}\right) \le NC \le 1.$$

Proof: From the lemma 4, $F^{'}(\lambda)$ has the sign of $Q(\lambda) = -\sinh(d^2(\delta+\lambda)/\sigma_0^2)$.

Thus $F^{'}(\lambda) \begin{cases} > 0 \ \ si \ \ \lambda < -\delta \\ = 0 \ \ si \ \ \lambda = -\delta \\ < 0 \ \ si \ \ \lambda > -\delta \end{cases}$.

Consequently $F(\lambda)$ has a unique maximum at $\lambda = -\delta$, and this maximum is equal to $F(-\delta) = 2\Phi\left(3\frac{d}{d^*} C_p^{''}\right) - 1$. On the other hand,

$$\lim_{\lambda \to +\infty} F(\lambda) = \lim_{\lambda \to +\infty} [\Phi\left(d(1-\delta-\lambda)/\sigma_0\right) - \Phi\left(-d(1+\delta+\lambda)/\sigma_0\right)] = \Phi(-\infty) - \Phi(-\infty) = 0,$$

$\lim_{\lambda\to-\infty} F(\lambda) = \lim_{\lambda\to-\infty}[\Phi\left(d(1-\delta-\lambda)/\sigma_0\right) - \Phi\left(-d(1+\delta+\lambda)/\sigma_0\right)] = \Phi(+\infty) - \Phi(+\infty) = 0$.

Thus $0 \le F(\lambda) \le 2\Phi\left(3\frac{d}{d^*}C_p^{"}\right) - 1$, and $2\Phi\left(-3\frac{d}{d^*}C_p^{"}\right) \le NC \le 1$.

Particular case: If $T = m$, we have $d^* = d$, $C_p^{"} = C_p$, thus $2\Phi\left(-3C_p\right) < NC \le 1$, result well known, given for example by Pearn and Kotz (2006, p.9).

2.2. Case $(u,v) = (1,0)$

When $(u,v) = (1,0)$, we have $C_p^{"}(u,v) = C_p^{"}(1,0) = C_{pk}^{"}$.

Theorem 2 :

$$\min\left(\Phi\left(-3\frac{D_u}{d^*}C_{pk}^{"}\right), \Phi\left(-3\frac{D_l}{d^*}C_{pk}^{"}\right)\right) \le NC \le \Phi\left(-3\frac{D_u}{d^*}C_{pk}^{"}\right) + \Phi\left(-3\frac{D_l}{d^*}C_{pk}^{"}\right).$$

Proof :

If $(u,v) = (1,0)$, from (5), $-1/d_l = \lambda_{\min} < \lambda < \lambda_{\max} = 1/d_u$.

- Let $0 \le \lambda < 1/d_u$. We have $Q_u(\lambda) = q_u(\lambda) - (k_u(\lambda) + (\delta+\lambda)q_u(\lambda))\tanh(d^2(\delta+\lambda)/\sigma_u^2(\lambda))$, where $k_u(\lambda) = \sigma_0^2(1-\lambda d_u)^2$ and $q_u(\lambda) = d_u\sigma_0^2(1-\lambda d_u)$.

$Q_u(\lambda) = d_u\sigma_0^2(1-\lambda d_u) - \{\sigma_0^2(1-\lambda d_u)^2 + (\delta+\lambda)d_u\sigma_0^2(1-\lambda d_u)\}\tanh(d^2(\delta+\lambda)/\sigma_u^2(\lambda))$

$= \sigma_0^2(1-\lambda d_u)[d_u - \{(1-\lambda d_u) + (\delta+\lambda)d_u\}\tanh(d^2(\delta+\lambda)/\sigma_u^2(\lambda))]$

$= \sigma_0^2(1-\lambda d_u)[d_u - (1+\delta d_u)\tanh(d^2(\delta+\lambda)/\sigma_u^2(\lambda))]$

$= \sigma_0^2(1-\lambda d_u)d_u[1-\tanh(d^2(\delta+\lambda)/\sigma_u^2(\lambda))]$,

since $\delta = (T-m)/d = (T-U+U-m)/d = (-D_u+d)/d = 1-1/d_u$. (6)

Now $0 \le \tanh(d^2(\delta+\lambda)/\sigma_u^2(\lambda)) < 1$, and $1-\lambda d_u > 0$, thus $Q_u(\lambda) > 0$, and from the lemma 4, $F_u^{'}(\lambda) > 0$, when $0 \le \lambda < 1/d_u$.

- Let $-1/d_l < \lambda \le 0$. We have $Q_l(\lambda) = q_l(\lambda) - (k_l(\lambda) + (\delta+\lambda)q_l(\lambda))\tanh(d^2(\delta+\lambda)/\sigma_l^2(\lambda))$, where $k_l(\lambda) = \sigma_0^2(1+\lambda d_l)^2$ and $q_l(\lambda) = -d_l\sigma_0^2(1+\lambda d_l)$.

$Q_l(\lambda) = -d_l\sigma_0^2(1+\lambda d_l) - \{\sigma_0^2(1+\lambda d_l)^2 - (\delta+\lambda)d_l\sigma_0^2(1+\lambda d_l)\}\tanh(d^2(\delta+\lambda)/\sigma_l^2(\lambda))$

$= \sigma_0^2(1+\lambda d_l)[-d_l + \{-(1+\lambda d_l) + (\delta+\lambda)d_l\}\tanh(d^2(\delta+\lambda)/\sigma_l^2(\lambda))]$

$= \sigma_0^2(1+\lambda d_l)[-d_l + (-1+\delta d_l)\tanh(d^2(\delta+\lambda)/\sigma_l^2(\lambda))]$

$= -d_l\sigma_0^2(1+\lambda d_l)[1+\tanh(d^2(\delta+\lambda)/\sigma_l^2(\lambda))]$,

since $\delta = (T-m)/d = (T-L+L-m)/d = (D_l-d)/d = 1/d_l - 1$. (7)

Now $0 \le \tanh(d^2(\delta+\lambda)/\sigma_l^2(\lambda)) < 1$, and $1+\lambda d_l > 0$, thus $Q_l(\lambda) < 0$, and from the lemma 4, $F_l^{'}(\lambda) < 0$, when $-1/d_l < \lambda \le 0$.

From the study of $F^{'}(\lambda)$, it results that $F(\lambda)$ has a minimum at $\lambda = 0$, that is to say at $\mu = T$, equal to $F(0) = \Phi(3D_uC_{pk}^{"}/d^*) - \Phi(-3D_lC_{pk}^{"}/d^*)$ and a maximum when $\lambda \to \lambda_{\max}$ or $\lambda \to \lambda_{\min}$, equal to $\max\left(\lim_{\lambda\to\lambda_{\max}} F_u(\lambda), \lim_{\lambda\to\lambda_{\min}} F_l(\lambda)\right)$. Now when $(u,v) = (1,0)$, from lemma 1, $\sigma_u(\lambda) = \sigma_0(1-\lambda d_u) = \sigma_0 d_u(1/d_u - \lambda) = \sigma_0 d_u(1-\delta-\lambda)$ from (6), thus

$(1-\delta-\lambda)/\sigma_u(\lambda) = 1/(\sigma_0 d_u)$, (8)

and $\sigma_l(\lambda) = \sigma_0(1+\lambda d_l) = \sigma_0 d_l(1/d_l + \lambda) = \sigma_0 d_l(1+\delta+\lambda)$ from (7), thus

$(1+\delta+\lambda)/\sigma_l(\lambda) = 1/(\sigma_0 d_l)$. (9)

From (8), (9) and from lemmas 2 and 3, we have

$\lim_{\lambda\to\lambda_{\max}} F_u(\lambda) = \lim_{\lambda\to 1/d_u} \Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)) - \lim_{\lambda\to 1/d_u} \Phi(-d(1+\delta+\lambda)/\sigma_u(\lambda))$

$= \Phi(d/(\sigma_0 d_u)) - \Phi(-\infty) = \Phi(3D_u C''_{pk}/d^*)$,

$\lim_{\lambda\to\lambda_{\min}} F_l(\lambda) = \lim_{\lambda\to -1/d_l} \Phi(d(1-\delta-\lambda)/\sigma_l(\lambda)) - \lim_{\lambda\to -1/d_l} \Phi(-d(1+\delta+\lambda)/\sigma_l(\lambda))$

$= \Phi(+\infty) - \Phi(-d/(\sigma_0 d_l)) = \Phi(3D_l C''_{pk}/d^*)$.

Finally $F(\lambda)$ has a lower bound equal to $\max(\Phi(3D_u C''_{pk}/d^*), \Phi(3D_l C''_{pk}/d^*))$ $= \Phi(3D_l C''_{pk}/d^*)$, since $T \in [m;U[$.

Consequently $\Phi(-3D_l C''_{pk}/d^*) \le NC \le \Phi(-3D_u C''_{pk}/d^*) + \Phi(-3D_l C''_{pk}/d^*)$.

Note that the lower bound depends on D_l, only, and not on D_u. For $T \in]L;m]$, a similar proof gives a minimum equal to $\Phi(-3D_u C''_{pk}/d^*)$, thus the expression given in the theorem, valid for any position of $T \in]L;U[$.

The upper bound given in Theorem 2 is identical to the one given by Chang and Wu (2008) in the expression (3). To reach that conclusion, we just need to observe that if $D_u < D_l$, then $r > 1$, $d^* = D_u$, and if $D_u > D_l$, then $r < 1$, $d^* = D_l$.

Particular case : If $T = m$, we have $D_u = D_l = d^*$, thus $\Phi(-3C_{pk}) \le NC \le 2\Phi(-3C_{pk})$, result well known, given for example by Pearn and Kotz (2006, p.42).

2.3. Case $(u = 1, v > 0)$, and $u > 1$

Lemma 5:

When $u = 1$ and $v > 0$, or when $u > 1$, then $\lim_{\lambda\to\lambda_{\max}} F_u(\lambda) = \lim_{\lambda\to\lambda_{\min}} F_l(\lambda) = 1$.

Proof :

From lemma 3, we have

$\lim_{\lambda\to\lambda_{\max}} F_u(\lambda) = \lim_{\lambda\to\lambda_{\max}} [\Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)) - \Phi(-d(1+\delta+\lambda)/\sigma_u(\lambda))]$

$= \lim_{\lambda\to\lambda_{\max}} \Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)) - \Phi(-\infty) = \lim_{\lambda\to\lambda_{\max}} \Phi(d(1-\delta-\lambda)/\sigma_u(\lambda))$.

On the other hand, from (5) and (6), $d(1-\delta-\lambda_{\max}) = D_u(\sqrt{v}d + (u-1)\sigma_0)/(\sqrt{v}d + u\sigma_0)$.

Since $0 \le \lambda < \lambda_{\max}$, we have $C''_p(u,v) > 0$ and thus $\sigma_0 > 0$. When $u = 1$ and $v > 0$, or $u > 1$, thus we have $d(1-\delta-\lambda_{\max}) > 0$, from where $\lim_{\lambda\to\lambda_{\max}} \Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)) = \Phi(+\infty) = 1$, and $\lim_{\lambda\to\lambda_{\max}} F_u(\lambda) = 1$.

From lemma 3, we have

$\lim_{\lambda\to\lambda_{\min}} F_l(\lambda) = \lim_{\lambda\to\lambda_{\min}} [\Phi(d(1-\delta-\lambda)/\sigma_l(\lambda)) - \Phi(-d(1+\delta+\lambda)/\sigma_l(\lambda))]$

$= \Phi(+\infty) - \lim_{\lambda\to\lambda_{\min}} \Phi(-d(1+\delta+\lambda)/\sigma_l(\lambda)) = 1 - \lim_{\lambda\to\lambda_{\min}} \Phi(-d(1+\delta+\lambda)/\sigma_l(\lambda))$.

On the other hand, from (5) and (7), $-d(1+\delta+\lambda_{\min}) = -D_l(\sqrt{v}d + (u-1)\sigma_0)/(\sqrt{v}d + u\sigma_0)$.

Since $\lambda_{\min} < \lambda \le 0$, we have $C_p''(u,v) > 0$ and thus $\sigma_0 > 0$. When $u = 1$ and $v > 0$, or $u > 1$, thus we have $-d(1+\delta+\lambda_{\min}) < 0$, from where $\lim_{\lambda \to \lambda_{\min}} \Phi(-d(1+\delta+\lambda)/\sigma_l(\lambda)) = \Phi(-\infty) = 0$, and $\lim_{\lambda \to \lambda_{\min}} F_l(\lambda) = 1$.

Theorem 3 :
When $u = 1$ and $v > 0$, or when $u > 1$, we have
$$0 \le NC \le \Phi\left(-3\frac{D_u}{d^*}C_p''(u,v)\right) + \Phi\left(-3\frac{D_l}{d^*}C_p''(u,v)\right).$$
Proof :
We have $C_p''(u+x, v+y) \le C_p''(u,v)$, for any *x*, *y* ≥ 0. Thus when $u = 1$ and $v > 0$, or when $u > 1$, $C_p''(u,v) \le C_p''(1,0) = C_{pk}''$.
Thus from the theorem 2,
$NC \le \Phi(-3C_{pk}''D_u/d^*) + \Phi(-3C_{pk}''D_l/d^*) \le \Phi(-3C_p''(u,v)D_u/d^*) + \Phi(-3C_p''(u,v)D_l/d^*)$. This upper bound is reached at $\lambda = 0$ since in this case, $\sigma_u(\lambda) = \sigma_l(\lambda) = \sigma_0$ according to lemma 1, $F(0) = \Phi(d(1-\delta)/\sigma_0) - \Phi(-d(1+\delta)/\sigma_0)$ according to lemma 3, and from (6) and (7), $F(0) = \Phi(3C_p''(u,v)D_u/d^*) - \Phi(-3C_p''(u,v)D_l/d^*)$. Consequently, when $\lambda = 0$,
$NC = 1 - F(0) = \Phi(-3C_p''(u,v)D_u/d^*) + \Phi(-3C_p''(u,v)D_l/d^*)$.
Moreover, $F(\lambda)$ is always maximised by 1, value reached at $\lambda_{\min}$ and $\lambda_{\max}$ according to lemma 5. Thus NC is minimized by 0, and the theorem.

Particular cases : If $T = m$, then $D_u = D_l = d^*$, and $C_p''(u,v) = C_p(u,v)$. Thus when $u = 1$, $v > 0$, or $u > 1$, we have $0 \le NC \le 2\Phi(-3C_p(u,v))$. When $(u,v) = (1,1)$, this result supplements the one obtained by Pearn and Lin (2005) who restrict the relation to the values of $C_{pmk} \ge \sqrt{2}/3$. In addition, when $T \ne m$, and $(u,v) = (1,1)$, then $C_p''(1,1) = C_{pmk}''$ and we have $NC \le \Phi\left(-3\frac{D_u}{d^*}C_p''(u,v)\right) + \Phi\left(-3\frac{D_l}{d^*}C_p''(u,v)\right) \le 2\Phi(-3C_{pmk}'')$. The result obtained by Pearn, Lin, and Chen (1999) is thus exact, although their proof is not true in all cases.

2.4. Case $0 < u < 1$, $v = 0$

Lemma 6 :
When $0 < u < 1$ and $v = 0$, then $\lim_{\lambda \to \lambda_{\max}} F_u(\lambda) = \lim_{\lambda \to \lambda_{\min}} F_l(\lambda) = 0$.

Proof :
From lemma 3 we have
$\lim_{\lambda \to \lambda_{\max}} F_u(\lambda) = \lim_{\lambda \to \lambda_{\max}} [\Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)) - \Phi(-d(1+\delta+\lambda)/\sigma_u(\lambda))]$
$= \lim_{\lambda \to \lambda_{\max}} \Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)) - \Phi(-\infty) = \lim_{\lambda \to \lambda_{\max}} \Phi(d(1-\delta-\lambda)/\sigma_u(\lambda))$.
Now when $0 < u < 1$ and $v = 0$, from (5) we have $\lambda_{\max} = 1/ud_u$, and from (6), $d(1-\delta-\lambda_{\max})$ $= D_u(u-1)/u < 0$, thus $\lim_{\lambda \to \lambda_{\max}} F_u(\lambda) = \Phi(-\infty) = 0$.
From lemma 3 we have

$$\lim_{\lambda\to\lambda_{\min}} F_l(\lambda) = \lim_{\lambda\to\lambda_{\min}} [\Phi(d(1-\delta-\lambda)/\sigma_l(\lambda)) - \Phi(-d(1+\delta+\lambda)/\sigma_l(\lambda))]$$
$$= \Phi(+\infty) - \lim_{\lambda\to\lambda_{\min}} \Phi(-d(1+\delta+\lambda)/\sigma_l(\lambda)) = 1 - \lim_{\lambda\to\lambda_{\min}} \Phi(-d(1+\delta+\lambda)/\sigma_l(\lambda)).$$

Now when $0<u<1$ and $v=0$, from (5) we have $\lambda_{\min} = -1/ud_l$, and from (7), $-d(1+\delta+\lambda_{\min}) = -D_l(u-1)/u > 0$, thus $\lim_{\lambda\to\lambda_{\min}} F_l(\lambda) = 1-\Phi(+\infty) = 0$.

Theorem 4 :

When $0<u<1$ and $v=0$,

a) If $C_p^{"}(u,0) \geq \frac{d^*}{3d}\sqrt{\frac{1}{2\delta}\ln\left(\frac{1+ud_u/d_l}{1-u}\right)}$, then $M_l \leq NC \leq 1$,

b) If $C_p^{"}(u,0) < \frac{d^*}{3d}\sqrt{\frac{1}{2\delta}\ln\left(\frac{1+ud_u/d_l}{1-u}\right)}$, then $\max(M_l, M_u) \leq NC \leq 1$,

with

$$M_l = \Phi\left(3\frac{\lambda_{l1}d - D_u}{d^*(1+u\lambda_{l1}d_l)}C_p^{"}(u,0)\right) + \Phi\left(-3\frac{\lambda_{l1}d + D_l}{d^*(1+u\lambda_{l1}d_l)}C_p^{"}(u,0)\right),$$

$$M_u = \Phi\left(3\frac{\lambda_{u1}d - D_u}{d^*(1-u\lambda_{u1}d_u)}C_p^{"}(u,0)\right) + \Phi\left(-3\frac{\lambda_{u1}d + D_l}{d^*(1-u\lambda_{u1}d_u)}C_p^{"}(u,0)\right),$$

$$\lambda_{l1} = -\frac{1}{ud_l} + \frac{d^2 - \sqrt{d^4 - 2\left(\frac{d^*}{3C_p^{"}(u,0)}\right)^2 ud_l\, d^2(1-u\delta d_l)\ln\left(\frac{1-u}{1+ud_l/d_u}\right)}}{\left(\frac{d^*}{3C_p^{"}(u,0)}\right)^2 u^2 d_l^2 \ln\left(\frac{1-u}{1+ud_l/d_u}\right)},$$

$$\lambda_{u1} = \frac{1}{ud_u} + \frac{d^2 - \sqrt{d^4 + 2\left(\frac{d^*}{3C_p^{"}(u,0)}\right)^2 ud_u d^2(1+u\delta d_u)\ln\left(\frac{1+ud_u/d_l}{1-u}\right)}}{\left(\frac{d^*}{3C_p^{"}(u,0)}\right)^2 u^2 d_u^2 \ln\left(\frac{1+ud_u/d_l}{1-u}\right)}.$$

Proof :

We obviously have $0 \leq F(\lambda) \leq 1$. According to lemma 6 the lower bound 0 is reached at $\lambda_{\min}$ and $\lambda_{\max}$. If there is a maximum less or equal to 1, it is necessarily obtained for the values of $\lambda \in]\lambda_{\min}, \lambda_{\max}[$, solutions of the equation $F_u^{'}(\lambda) = 0$ or $F_l^{'}(\lambda) = 0$.

- Study of $F_u^{'}(\lambda)$.

Let $0 \leq \lambda < \lambda_{\max} = 1/ud_u$. When $v = 0$ and from lemma 4, we have

$F_u^{'}(\lambda) = 0 \Leftrightarrow Q_u(\lambda) = 0 \Leftrightarrow q_u(\lambda) - (k_u(\lambda) + (\delta+\lambda)q_u(\lambda))\tanh(d^2(\delta+\lambda)/\sigma_u^2(\lambda)) = 0$

$\Leftrightarrow ud_u\sigma_0^2(1-u\lambda d_u) - (\sigma_0^2(1-u\lambda d_u)^2 + (\delta+\lambda)ud_u\sigma_0^2(1-u\lambda d_u))\tanh(d^2(\delta+\lambda)/\sigma_u^2(\lambda)) = 0$

$\Leftrightarrow ud_u\sigma_0^2(1-u\lambda d_u) - \sigma_0^2(1-u\lambda d_u)(1+\delta ud_u)\tanh(d^2(\delta+\lambda)/\sigma_u^2(\lambda)) = 0$

$\Leftrightarrow \sigma_0^2(1-u\lambda d_u)[ud_u - (1+\delta ud_u)\tanh(d^2(\delta+\lambda)/\sigma_u^2(\lambda))] = 0$

$\Leftrightarrow ud_u - (1+\delta ud_u)\tanh(d^2(\delta+\lambda)/\sigma_u^2(\lambda)) = 0$, since $0 \leq \lambda < 1/ud_u$,

$\Leftrightarrow \tanh(d^2(\delta+\lambda)/[\sigma_0^2(1-u\lambda d_u)^2]) = ud_u/(1+\delta ud_u)$, from lemma 1,

$\Leftrightarrow d^2(\delta+\lambda)/(\sigma_0^2(1-u\lambda d_u)^2) = \tanh^{-1}(ud_u/(1+\delta ud_u))$.

Assuming that $t_u = \tanh^{-1}(ud_u/(1+\delta ud_u)) > 0$, we have

$F_u^{'}(\lambda) = 0 \Leftrightarrow d^2(\delta+\lambda) = t_u\sigma_0^2(1-u\lambda d_u)^2$

$\Leftrightarrow t_u\sigma_0^2u^2d_u^2\lambda^2 - (2ud_ut_u\sigma_0^2 + d^2)\lambda - d^2\delta + t_u\sigma_0^2 = 0$, (10)

which is a second-degree polynomial of the variable λ.

Since $\Delta_u = d^4 + 4t_u\sigma_0^2ud_ud^2(1+u\delta d_u) > 0$, we have two roots,

$$\lambda_{u1} = \frac{2ud_ut_u\sigma_0^2 + d^2 - \sqrt{\Delta_u}}{2t_u\sigma_0^2u^2d_u^2} = \frac{1}{ud_u} + \frac{d^2 - \sqrt{\Delta_u}}{2t_u\sigma_0^2u^2d_u^2},$$

$$\lambda_{u2} = \frac{2ud_ut_u\sigma_0^2 + d^2 + \sqrt{\Delta_u}}{2t_u\sigma_0^2u^2d_u^2} = \frac{1}{ud_u} + \frac{d^2 + \sqrt{\Delta_u}}{2t_u\sigma_0^2u^2d_u^2}.$$

As can be seen $\lambda_{u2} > 1/ud_u = \lambda_{\max}$ is not suitable in the studied field. To make λ_{u1} become acceptable, we need to $0 \le \lambda_{u1} < \lambda_{\max}$. Since $\Delta_u > d^4$, we have $\lambda_{u1} < 1/ud_u = \lambda_{\max}$. In addition, since $\lambda_{u2} > 0$, for λ_{u1} to be positive or null, the product of the roots of (10) has to be positive or null, or that $-d^2\delta + t_u\sigma_0^2 \ge 0 \Leftrightarrow C_p^{"}(u,0) \le \sqrt{t_u/\delta}\,d^*/(3d)$. Now

$$t_u = \tanh^{-1}(ud_u/(1+\delta ud_u)) = \frac{1}{2}\ln\left(\left(1+\frac{ud_u}{1+\delta ud_u}\right)\Big/\left(1-\frac{ud_u}{1+\delta ud_u}\right)\right) = \frac{1}{2}\ln\left(\frac{1+(1+\delta)ud_u}{1-(1-\delta)ud_u}\right)$$

$$= \frac{1}{2}\ln\left(\frac{1+ud_u/d_l}{1-u}\right)$$, from (6) and (7). Thus when $C_p^{"}(u,0) \le \frac{d^*}{3d}\sqrt{\frac{1}{2\delta}\ln\left(\frac{1+ud_u/d_l}{1-u}\right)}$, there exists $\lambda_{u1} \in [0, \lambda_{\max}[$ for which $F_u^{'}(\lambda) = 0$ and thus for which $F_u(\lambda)$ is maximum.

Note that in the particular case where $C_p^{"}(u,0) = \frac{d^*}{3d}\sqrt{\frac{1}{2\delta}\ln\left(\frac{1+ud_u/d_l}{1-u}\right)}$, that is to say when $-d^2\delta + t_u\sigma_0^2 = 0$, we have $\lambda_{u1} = 0$ and $t_u = d^2\delta/\sigma_0^2$. When $v = 0$, from lemma 4, we have

$Q_u(0) = \sigma_0^2[ud_u - (1+\delta ud_u)\tanh(d^2\delta/\sigma_0^2)] = \sigma_0^2[ud_u - (1+\delta ud_u)\tanh(t_u)]$

$= \sigma_0^2[ud_u - (1+\delta ud_u)ud_u/(1+\delta ud_u)] = 0$. Consequently, according to lemma 4, $F_u^{'}(\lambda) = 0$, and $F_u(\lambda)$ is maximum for $\lambda = \lambda_{u1} = 0$.

When $C_p^{"}(u,0) > \frac{d^*}{3d}\sqrt{\frac{1}{2\delta}\ln\left(\frac{1+ud_u/d_l}{1-u}\right)}$, we have $\lambda_{u1} < 0$. Thus there is no value of $\lambda \in [0, \lambda_{\max}[$ for which $F_u^{'}(\lambda) = 0$. On the other hand, from (6), (7) and lemma 3, $F_u(0) = \Phi(3C_p^{"}(u,0)D_u/d^*) - \Phi(-3C_p^{"}(u,0)D_l/d^*)$. Consequently, we have $F_u(0) > 0$, $\lim_{\lambda\to\lambda_{\max}} F_u(\lambda) = 0$ from lemma 6, and $F_u^{'}(\lambda) \ne 0$ when $\lambda \in [0, \lambda_{\max}[$. Thus $F_u(\lambda)$ is decreasing when $\lambda \in [0, \lambda_{\max}[$ and maximum when $\lambda = 0$.

- Study of $F_l^{'}(\lambda)$

Let $-1/ud_l = \lambda_{\min} < \lambda \le 0$. When $v = 0$ and from lemma 4, we have

$F_l^{'}(\lambda) = 0 \Leftrightarrow Q_l(\lambda) = 0 \Leftrightarrow q_l(\lambda) - (k_l(\lambda) + (\delta+\lambda)q_l(\lambda))\tanh(d^2(\delta+\lambda)/\sigma_l^2(\lambda)) = 0$

$\Leftrightarrow -ud_l\sigma_0^2(1+u\lambda d_l) - [\sigma_0^2(1+u\lambda d_l)^2 - (\delta+\lambda)ud_l\sigma_0^2(1+u\lambda d_l)]\tanh(d^2(\delta+\lambda)/\sigma_l^2(\lambda)) = 0$

$\Leftrightarrow \sigma_0^2(1+u\lambda d_l)[-ud_l + (-1+\delta ud_l)\tanh(d^2(\delta+\lambda)/\sigma_l^2(\lambda))] = 0$

$\Leftrightarrow -ud_l + (-1+\delta ud_l)\tanh(d^2(\delta+\lambda)/\sigma_l^2(\lambda)) = 0$, since $-1/ud_l = \lambda_{\min} < \lambda \le 0$.

Thus from lemma 1, we have

$$F_l^{'}(\lambda) = 0 \Leftrightarrow \tanh(d^2(\delta+\lambda)/[\sigma_0^2(1+u\lambda d_l)^2]) = ud_l/(-1+\delta ud_l). \quad (11)$$

From (7), $-1+\delta ud_l = -1+(1/d_l - 1)ud_l = -1+u-ud_l$, and since $u<1$, we have $-1+\delta ud_l < 0$. Consequently the solutions of (11) can exist only for $\lambda < -\delta$. In this case,

$F_l^{'}(\lambda) = 0 \Leftrightarrow d^2(\delta+\lambda)/[\sigma_0^2(1+u\lambda d_l)^2] = \tanh^{-1}(ud_l/(-1+\delta ud_l))$, and assuming that $t_l = \tanh^{-1}(ud_l/(-1+\delta ud_l)) < 0$, we have

$F_l^{'}(\lambda) = 0 \Leftrightarrow d^2(\delta+\lambda) = t_l\sigma_0^2(1+u\lambda d_l)^2 \Leftrightarrow t_l\sigma_0^2u^2d_l^2\lambda^2 + (2ud_lt_l\sigma_0^2 - d^2)\lambda - d^2\delta + t_l\sigma_0^2 = 0$,

which is a second-degree polynomial of the variable λ. Since $t_l < 0$ and $-1+\delta ud_l < 0$, $\Delta_l = d^4 + 4t_l\sigma_0^2ud_l\,d^2(-1+u\delta d_l) > 0$, and we have two roots,

$$\lambda_{l1} = \frac{d^2 - 2ud_lt_l\sigma_0^2 - \sqrt{\Delta_l}}{2t_l\sigma_0^2u^2d_l^2} = -\frac{1}{ud_l} + \frac{d^2 - \sqrt{\Delta_l}}{2t_l\sigma_0^2u^2d_l^2},$$

$$\lambda_{l2} = \frac{d^2 - 2ud_lt_l\sigma_0^2 + \sqrt{\Delta_l}}{2t_l\sigma_0^2u^2d_l^2} = -\frac{1}{ud_l} + \frac{d^2 + \sqrt{\Delta_l}}{2t_l\sigma_0^2u^2d_l^2}.$$

As can be seen, $\lambda_{l2} < -1/ud_l = \lambda_{\min}$ is not suitable in the studied field. To make λ_{l1} become acceptable, we need to $\lambda_{\min} < \lambda_{l1} < -\delta \le 0$. Since $\Delta_l > d^4$, we have $\lambda_{l1} > -1/ud_l = \lambda_{\min}$. Furthermore, since $\lambda_{l2} < 0$, for λ_{l1} to be negative or null, the product of the roots has to be positive or null, or that $-d^2\delta + t_l\sigma_0^2 \le 0$, which is always true. Thus $F_l(\lambda)$ is maximum when $\lambda_{l1} \in]\lambda_{\min}; -\delta[$.

In conclusion, from the study of $F_u^{'}(\lambda)$ and $F_l^{'}(\lambda)$, we can deduce:

- If $C_p^{''}(u,0) \ge \frac{d^*}{3d}\sqrt{\frac{1}{2\delta}\ln\left(\frac{1+ud_u/d_l}{1-u}\right)}$, $F_u(\lambda)$ is maximum when $\lambda = 0$, and $F_l(\lambda_{l1}) > F_l(0) = F_u(0)$. Thus $F(\lambda)$ has an upper bound when $\lambda_{l1} \in]\lambda_{\min}; -\delta[$. On the other hand, from lemma 6, $\lim_{\lambda\to\lambda_{\max}} F_u(\lambda) = \lim_{\lambda\to\lambda_{\min}} F_l(\lambda) = 0$. Thus $0 \le F(\lambda) \le F_l(\lambda_{l1})$ or $1 - F_l(\lambda_{l1}) \le NC \le 1$. From (6), (7) and lemmas 1 and 3, we obtain a) assuming that $M_l = 1 - F_l(\lambda_{l1})$.

- If $C_p^{''}(u,0) < \frac{d^*}{3d}\sqrt{\frac{1}{2\delta}\ln\left(\frac{1+ud_u/d_l}{1-u}\right)}$, $F_u(\lambda)$ is maximum when $\lambda_{u1} \in [0, \lambda_{\max}[$, and $F_l(\lambda)$ is maximum when $\lambda_{l1} \in]\lambda_{\min}; -\delta[$. Thus, $F(\lambda)$ has an upper bound when $\lambda \in]\lambda_{\min}; \lambda_{\max}[$, equal to $\max(F_l(\lambda_{l1}); F_u(\lambda_{u1}))$. On the other hand, from lemma 6, $\lim_{\lambda\to\lambda_{\max}} F_u(\lambda) = \lim_{\lambda\to\lambda_{\min}} F_l(\lambda) = 0$. Thus $0 \le F(\lambda) \le \max(F_l(\lambda_{l1}), F_u(\lambda_{u1}))$, or $\max(1-F_l(\lambda_{l1}), 1-F_u(\lambda_{u1})) \le NC \le 1$. From (6), (7) and lemmas 1 and 3, we obtain b) assuming that $M_u = 1 - F_u(\lambda_{u1})$.

Particular case : If $T = m$, then $\delta = 0$, $d_u = d_l = 1$, $d^* = d$, and $C_p^{''}(u,0) = C_p(u,0)$. The product of the roots of the second-degree polynomial (10) has the sign of t_u, therefore it is

always positive. Consequently, $F_u(\lambda)$ is maximum in $\lambda_{u1} \in [0, \lambda_{\max}[$, for any value of $C_p(u,0) > 0$. Thus theorem 4 is stated as follows:

When $0 < u < 1$, $v = 0$, then $M \le NC \le 1$,

where $M = M_u = M_l = \Phi(3C_p(u,0)(\lambda_0 - 1)/(1 - u\lambda_0)) + \Phi(-3C_p(u,0)(\lambda_0 + 1)/(1 - u\lambda_0))$,

with $\lambda_0 = \lambda_{u1} = -\lambda_{l1} = \dfrac{1}{u} + \left(1 - \sqrt{1 + \dfrac{2u}{(3C_p(u,0))^2} \ln\left(\dfrac{1+u}{1-u}\right)}\right) \Bigg/ \left(\left(\dfrac{u}{3C_p(u,0)}\right)^2 \ln\left(\dfrac{1+u}{1-u}\right)\right)$.

2.5. Case $0 \le u < 1$, $v > 0$

Lemma 7 :

When $0 \le u < 1$ and $v > 0$,

a) If $C_p^{"}(u,v) > \dfrac{(1-u)d^*}{3\sqrt{v}d}$, then $\lim\limits_{\lambda \to \lambda_{\max}} F_u(\lambda) = \lim\limits_{\lambda \to \lambda_{\min}} F_l(\lambda) = 1$.

b) If $C_p^{"}(u,v) = \dfrac{(1-u)d^*}{3\sqrt{v}d}$, then $\lim\limits_{\lambda \to \lambda_{\max}} F_u(\lambda) = \lim\limits_{\lambda \to \lambda_{\min}} F_l(\lambda) = 1/2$.

c) If $0 < C_p^{"}(u,v) < \dfrac{(1-u)d^*}{3\sqrt{v}d}$, then $\lim\limits_{\lambda \to \lambda_{\max}} F_u(\lambda) = \lim\limits_{\lambda \to \lambda_{\min}} F_l(\lambda) = 0$.

Proof :

From lemma 3 we have

$\lim\limits_{\lambda \to \lambda_{\max}} F_u(\lambda) = \lim\limits_{\lambda \to \lambda_{\max}} [\Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)) - \Phi(-d(1+\delta+\lambda)/\sigma_u(\lambda))]$

$= \lim\limits_{\lambda \to \lambda_{\max}} \Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)) - \Phi(-\infty) = \lim\limits_{\lambda \to \lambda_{\max}} \Phi(d(1-\delta-\lambda)/\sigma_u(\lambda))$.

From (5) and (6), we have

$d(1-\delta-\lambda_{\max}) = D_u(3\sqrt{v}dC_p^{"}(u,v) + (u-1)d^*)/(3\sqrt{v}dC_p^{"}(u,v) + ud^*)$. Thus

a) If $C_p^{"}(u,v) > (1-u)d^*/(3\sqrt{v}d)$, then $d(1-\delta-\lambda_{\max}) > 0$, $\lim\limits_{\lambda \to \lambda_{\max}} \Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)) = \Phi(+\infty) = 1$, and $\lim\limits_{\lambda \to \lambda_{\max}} F_u(\lambda) = 1$.

b) If $C_p^{"}(u,v) = (1-u)d^*/(3\sqrt{v}d)$, then $d(1-\delta-\lambda_{\max}) = 0$, $\lim\limits_{\lambda \to \lambda_{\max}} \Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)) = \Phi(0) = 1/2$, and $\lim\limits_{\lambda \to \lambda_{\max}} F_u(\lambda) = 1/2$.

c) If $0 < C_p^{"}(u,v) < (1-u)d^*/(3\sqrt{v}d)$, then $d(1-\delta-\lambda_{\max}) < 0$, $\lim\limits_{\lambda \to \lambda_{\max}} \Phi(d(1-\delta-\lambda)/\sigma_u(\lambda)) = \Phi(-\infty) = 0$, and $\lim\limits_{\lambda \to \lambda_{\max}} F_u(\lambda) = 0$.

In a similar way we have $\lim\limits_{\lambda \to \lambda_{\min}} F_l(\lambda) = 1 - \lim\limits_{\lambda \to \lambda_{\min}} \Phi(-d(1+\delta+\lambda)/\sigma_l(\lambda))$, and the lemma since from (5) and (7), $-d(1+\delta+\lambda_{\min}) = -D_l(3\sqrt{v}dC_p^{"}(u,v) + (u-1)d^*)/(3\sqrt{v}dC_p^{"}(u,v) + ud^*)$.

Lemma 8 :

When $0 \le u < 1$ and $v > 0$, if $C_p^{"}(u,v) \ge \dfrac{(1-u)d^*}{3\sqrt{v}d}$, then for any $\lambda_u \in [0; \lambda_{\max}]$, there exists $\lambda_l = -\dfrac{d_u}{d_l}\lambda_u \in [\lambda_{\min}; 0]$ such as $F_u(\lambda_u) \le F_l(\lambda_l)$.

Proof :

Let $\lambda_u \in \left[0;\lambda_{\max}\right]$, $\lambda_l = -\lambda_u d_u / d_l \in \left[\lambda_{\min};0\right]$, $E = \sigma_u(\lambda_u)/d$, $a = (1-\delta-\lambda_u)/E$, $b = (1+\delta+\lambda_l)/E$, and $x = (\lambda_u - \lambda_l)/E > 0$.

If $C_p^{"}(u,v) \geq d^*(1-u)/(3\sqrt{v}d)$, then $3\sqrt{v}dC_p^{"}(u,v) + ud^* \geq d^*$, and $d^*/(d_u(3\sqrt{v}dC_p^{"}(u,v) + ud^*)) \leq 1/d_u$. From (5) and (6), we deduce $0 \leq \lambda_u \leq \lambda_{\max} = d^*/(d_u(3C_p^{"}(u,v)\sqrt{v}d + ud^*)) \leq 1/d_u = 1-\delta$, from where $a \geq 0$ and $1-\lambda_u d_u \geq 0$. Consequently, since $T \in \left[m;U\right[$, we have $d_u > d_l$, $(1-\lambda_u d_u)/d_u \leq (1-\lambda_u d_u)/d_l$, and $1/d_u - \lambda_u \leq 1/d_l - d_u\lambda_u/d_l$. Since $\lambda_l = -\lambda_u d_u / d_l$, from (6) and (7) we deduce $a = (1-\delta-\lambda_u)/E \leq (1+\delta+\lambda_l)/E = b$.

Let f the probability density function of the standard normal distribution N(0,1). Since f is decreasing on $\left[0;+\infty\right[$, when $x > 0$ and $b \geq a \geq 0$, we have $\int_b^{b+x} f(x)dx \leq \int_a^{a+x} f(x)dx$, thus $\int_{(1+\delta+\lambda_l)/E}^{(1+\delta+\lambda_u)/E} f(x)dx \leq \int_{(1-\delta-\lambda_l)/E}^{(1-\delta-\lambda_u)/E} f(x)dx \Leftrightarrow \int_{-(1+\delta+\lambda_l)/E}^{-(1+\delta+\lambda_u)/E} f(x)dx + \int_{(1-\delta-\lambda_u)/E}^{(1-\delta-\lambda_l)/E} f(x)dx \leq 0 \Leftrightarrow A + B \leq 0$,

where $A = \int_{-(1+\delta+\lambda_l)/E}^{-(1+\delta+\lambda_u)/E} f(x)dx = \Phi(-(1+\delta+\lambda_l)/E) - \Phi(-(1+\delta+\lambda_u)/E)$,

and $B = \int_{(1-\delta-\lambda_u)/E}^{(1-\delta-\lambda_l)/E} f(x)dx = \Phi((1-\delta-\lambda_u)/E) - \Phi((1-\delta-\lambda_l)/E)$. Now

$\Phi((1-\delta-\lambda_u)/E) - \Phi(-(1+\delta+\lambda_u)/E) = \Phi((1-\delta-\lambda_l)/E) - \Phi(-(1+\delta+\lambda_l)/E) + A + B$, thus

$\Phi((1-\delta-\lambda_u)/E) - \Phi(-(1+\delta+\lambda_u)/E) \leq \Phi((1-\delta-\lambda_l)/E) - \Phi(-(1+\delta+\lambda_l)/E)$.

Since $\lambda_l = -\lambda_u d_u / d_l$, from lemma 1, we have $\sigma_l(\lambda_l) = \sigma_u(\lambda_u)$, and $E = \sigma_u(\lambda_u)/d = \sigma_l(\lambda_l)/d$. Consequently,

$$\Phi(d(1-\delta-\lambda_u)/\sigma_u(\lambda_u)) - \Phi(-d(1+\delta+\lambda_u)/\sigma_u(\lambda_u)) \leq \Phi(d(1-\delta-\lambda_l)/\sigma_l(\lambda_l)) - \Phi(-d(1+\delta+\lambda_l)/\sigma_l(\lambda_l)),$$

and $F_u(\lambda_u) \leq F_l(\lambda_l)$.

It should be noted that when $C_p^{"}(u,v) < d^*(1-u)/(3\sqrt{v}d)$, no general rule can be obtained, as shown in the graphic investigations which have been made, but which are not detailed here.

Theorem 5 :

When $0 \leq u < 1$ and $v > 0$,

a) If $C_p^{"}(u,v) > \dfrac{(1-u)d^*}{3\sqrt{v}d}$, then $\lambda_u \in \left[0;\lambda_{\max}\right]$ exists, a solution of the following equation (12) such as $0 \leq NC \leq 1 - F_u(\lambda_u)$.

b) If $C_p^{"}(u,v) = \dfrac{(1-u)d^*}{3\sqrt{v}d}$, then $\lambda_u \in \left[0;\lambda_{\max}\right]$ and $\lambda_l \in \left[\lambda_{\min};0\right]$ exist, solutions of the following equations (12) and (13) such as $1 - F_l(\lambda_l) \leq NC \leq 1 - F_u(\lambda_u)$.

c) If $0 < C_p^{"}(u,v) < \dfrac{(1-u)d^*}{3\sqrt{v}d}$, then $\lambda \in \left[\lambda_{\min};\lambda_{\max}\right]$ exists, a solution of the one of the following equations (12) and (13) such as $1 - F(\lambda) \leq NC \leq 1$.

Proof :

The existence of maxima and minima is established by lemmas 7 and 8. These maxima and minima are necessarily obtained for the values of λ solutions of the equations $F_u^{'}(\lambda)=0$ or $F_l^{'}(\lambda)=0$, that is to say according to lemma 4, of the equations

$$q_u(\lambda)+v\lambda d^2 d_u^2-(k_u(\lambda)+(\delta+\lambda)q_u(\lambda)+\delta v\lambda d^2 d_u^2)\tanh(d^2(\delta+\lambda)/\sigma_u^2(\lambda))=0 \quad (12)$$

$$q_l(\lambda)+v\lambda d^2 d_l^2-(k_l(\lambda)+(\delta+\lambda)q_l(\lambda)+\delta v\lambda d^2 d_l^2)\tanh(d^2(\delta+\lambda)/\sigma_l^2(\lambda))=0 \quad (13)$$

These solutions can only be obtained numerically.

Particular case : Let $u = 0$. Since $T\in[m;U[$, $d/D_u>1$, and $d^*=D_u$.

If $\mu>T$, it is easy to see that $C_p^{"}(0,v)\le C_p(0,v)$.

If $\mu<T$, the function $f(D_u)=d^4-D_l^2D_u^2=d^4-(2d-D_u)^2D_u^2$ being always positive or null when $D_u\in]0;d]$, we have $d^2/D_u^2D_l^2\ge 1/d^2$, and thus

$$C_p^{"}(0,v)=d^*/\left(3\sqrt{\sigma^2+vA^2}\right)=1/\left(3\left(\sigma^2/{D_u}^2+vd^2(\mu-T)^2/({D_u}^2{D_l}^2)\right)^{1/2}\right)$$

$$\le 1/\left(3\left(\sigma^2/d^2+v(\mu-T)^2/d^2\right)^{1/2}\right)=d/\left(3\sqrt{\sigma^2+v(\mu-T)^2}\right)=C_p(0,v).$$

Moreover, since $C_p^{"}(u+x,v+y)\le C_p^{"}(u,v)$, for any x, $y \ge 0$, we have $C_p(0,v)\le C_p(0,1)=C_{pm}$ when $v \ge 1$, and $C_p^{"}(u,v)\le C_p^{"}(0,v)$ when u $\ge$ 0. Therefore $C_p^{"}(u,v)\le C_{pm}$, when $u \ge 0$ and $v \ge 1$. Now when $C_{pm}>1/\sqrt{3}$, Ruczinski (1996) shows that $F(\lambda)$ is minimum at $\lambda=0$, and that the minimum is equal to $2\Phi(3C_{pm})-1$, which means that $NC\le 2\Phi(-3C_{pm})$. We can deduce that when $0\le u<1$, $v \ge 1$ and $C_p^{"}(u,v)>1/\sqrt{3}$, then $NC\le 2\Phi(-3C_{pm})\le 2\Phi(-3C_p^{"}(u,v))$. Furthermore, if $T = m$, the upper bound $2\Phi(-3C_p(u,v))$ is reached at $\lambda=0$. Actually, when $\lambda=0$ and $T = m$, $NC=1-F_u(0)=1-F_l(0)$ $=1-\Phi(d/\sigma_0)+\Phi(-d/\sigma_0)=2\Phi(-d/\sigma_0)=2\Phi(-3C_p(u,v))$.

3. CONCLUSION

The indices family $C_p(u,v)$ suggested by Vännman (1995) for symmetrical tolerances, then the family $C_p^{"}(u,v)$ suggested by Chen and Pearn (2001) for asymmetrical tolerances, give an algebraic generalization of the usual indices C_p, C_{pk}, C_{pm}, and C_{pmk}. If these generalizations have an obvious theoretical interest, however they do not make the user's work easier, since the choice of an index among the four standard indices is already confusing for him. Vännman (1995), for symmetrical tolerances, then Grau (2009) for asymmetrical tolerances, suggest to choose an index according to the properties of its estimator. In the work previously developed we give the user the theoretical elements bringing to the fore the links between the indices $C_p(u,v)$ or $C_p^{"}(u,v)$ and the process yield. The knowledge of these links as well as of those associated to the process centering, allows the user to choose an index according to the importance which he wishes to attach simultaneously to the centering and the proportion of nonconforming.

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