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# Random sampling of lattice paths with constraints, via transportation

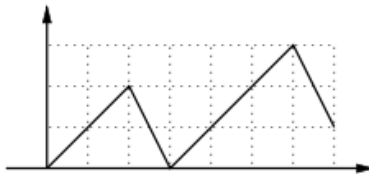
Lucas Gerin

February 5, 2010

## Abstract

We investigate Monte Carlo Markov Chain (MCMC) procedures for the random sampling of some one-dimensional lattice paths with constraints, for various constraints. We will see that an approach inspired by *optimal transport* allows us to efficiently bound the mixing time of the associated Markov chain. The algorithm is robust and easy to implement, and samples an "almost" uniform path of length  $n$  in  $n^{3+\varepsilon}$  steps. This bound makes use of a certain *contraction property* of the Markov chain, and is also used to derive a bound for the running time of Propp-Wilson's *Coupling From The Past* algorithm.

## 1 Lattice Paths with Constraints



We are interested in this paper in some families of one-dimensional lattices paths. We fix three integers  $n, a, b > 0$ , and consider the paths of length  $n$ , with steps  $+a/-b$ , that is, the words of  $n$  letters taken in the alphabet  $\{a, -b\}$ . Such a word  $s = (s_1, s_2, \dots, s_n)$  is identified to the path  $S = (S_1, \dots, S_n) := (s_1, s_1 + s_2, \dots, s_1 + s_2 + \dots + s_n)$ . On the right, one sees the lattice path  $S = (1, 2, 0, 1, 2, 3, 1)$  associated to the word  $s = (1, 1, -2, 1, 1, 1, -2)$ . The problem we discuss here is to efficiently sample uniform (or *almost* uniform) paths in a sub-family  $\mathcal{A}_n$  of paths, with Markov chains.

To illustrate the methods and the results, we focus on three particular sub-families.

1. Discrete *meanders*, denoted by  $\mathcal{M}_n$ , which are simply the non-negative paths:  $S \in \mathcal{M}_n$  if for any  $i \leq n$  we have  $S_i \geq 0$ . This example is mainly illustrative because the combinatorial properties of meanders make it possible to perform exact sampling very efficiently (an algorithm running in  $\mathcal{O}(n^{1+\varepsilon})$  steps is given in [3], an order that we cannot get in the present paper).
2. Paths with *walls*. A path with a wall of height  $h$  between  $r$  and  $s$  is a path such that  $S_i \geq h$  for any  $r \leq i \leq s$  (see Fig. 1 for an example). These are denoted by  $\mathcal{W}_n = \mathcal{W}_n(h, r, s)$ , they appear in statistical mechanics as toy models for the analysis of random interfaces and polymers (see examples in [6]).
3. *Culminating paths*, denoted by  $\mathcal{C}_n$ , which are non-negative paths whose maximum is attained at the last step: for any  $i$  we have  $0 \leq S_i \leq S_n$ . They have been introduced in [3], motivated in particular by the analysis of some algorithms in bioinformatics.

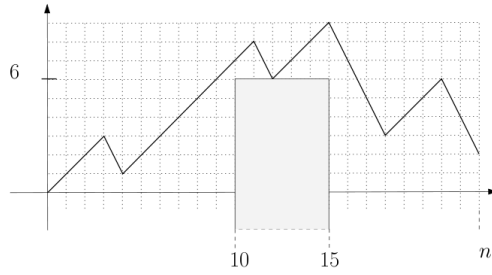


Figure 1: A path of steps  $+1/-2$ , with a wall of height  $h = 6$  between  $i = 10$  and  $j = 15$ .

**Remark 1.** *The methods discussed here apply to any values of  $(a, b)$ , but we have in mind the challenging case  $b > a$ : for our three families the ratio  $\text{card}(\mathcal{A}_n)/\text{card}(\mathcal{P}_n)$  decreases exponentially fast, making impossible a naive rejection algorithm.*

## 2 Sampling with Markov chains

We will consider Markov chains in a family  $\mathcal{A}_n$ , where all the probability transitions are symmetric. For a modern introduction to Markov chains, we refer to [7]. Hence we are given a transition matrix  $(p_{i,j})$  of size  $|\mathcal{A}_n| \times |\mathcal{A}_n|$  with

$$p_{i,j} = p_{j,i} \text{ whenever } i \neq j,$$

$$p_{i,i} = 1 - \sum_{j \neq i} p_{i,j}.$$

**Lemma 2.** *If such a Markov chain is irreducible, then it admits as unique stationnary distribution the uniform distribution  $\pi = \pi(\mathcal{A}_n)$  on  $\mathcal{A}_n$ .*

*Proof.* The equality  $\pi(i)p_{i,j} = \pi(j)p_{j,i}$  holds for any two vertices  $i, j$ . This shows that the probability distribution  $\pi$  is reversible for  $(p_{i,j})$ , and hence stationnary. It is unique if the chain is irreducible.  $\square$

This lemma already provides us with a scheme for sampling an almost uniform path in  $\mathcal{A}_n$ , without knowing much about  $\mathcal{A}_n$ . To do so, we define a “flip” operator on paths. Fix an integer  $i \in \{1, 2, \dots, n-1\}$  and a path  $S = (S_1, \dots, S_n)$ ; let  $(s_1, \dots, s_n)$  be the corresponding word.

The path  $\phi(\mathbf{S}, i, \uparrow)$  is defined as follows : if  $(s_i, s_{i+1}) = (-b, a) = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$  then these two steps are exchanged into  $(a, -b) = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array}$ . The  $n-2$  other steps remain unchanged. For the case  $i = n$ ,  $\phi(\mathbf{S}, n, \uparrow)$  is simply the path associated to the word

$$(s_1, \dots, s_{n-1}, a).$$

The path  $\phi(\mathbf{S}, i, \downarrow)$  is defined equally: if  $(s_i, s_{i+1}) = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array}$ , it turns into  $\begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$ . The path  $\phi(\mathbf{S}, n, \downarrow)$  is the path associated to  $(s_1, \dots, s_{n-1}, -b)$ .

For the family  $\mathcal{A}_n = \mathcal{C}_n$ , we have to use another definition of  $\phi(\mathbf{S}, n, \uparrow)$  and  $\phi(\mathbf{S}, n, \downarrow)$ , if we want the chain to be irreducible. Notice that since the maximum is reached at  $n$ , the  $\lceil b/a \rceil + 1$  last steps are necessarily

$$(a, a, \dots, a) \text{ or } (-b, a, \dots, a).$$

We thus define  $\phi(\mathbf{S}, n, \uparrow)$  as the path obtained by changing the  $\lceil b/a \rceil + 1$  last steps into  $(a, a, \dots, a)$  (regardless of their initial values in  $\mathbf{S}$ ) and  $\phi(\mathbf{S}, n, \downarrow)$  as the path obtained by changing the  $\lceil b/a \rceil + 1$  last steps into  $(-b, a, \dots, a)$

We are also given a probability distribution  $\mathbf{p} = (p_i)_{1 \leq i \leq n}$ , and we assume that  $p_i > 0$  for each  $i$ . We will consider a particular sequence  $\mathbf{p}$  later on, but at this point we can take the uniform distribution in  $\{1, \dots, n\}$ . We can now describe the algorithm.

---

**Algorithm 1** MCMC: Approximate sampling of a path in  $\mathcal{A}_n$

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```

initialize  $\mathbf{S} \in \mathcal{A}_n$ 
 $I_1, I_2, \dots \leftarrow$  i.i.d. r.v. with law  $\mathbf{p}$ 
 $\varepsilon_1, \varepsilon_2, \dots \leftarrow$  i.i.d. uniform r.v. in  $\{\uparrow, \downarrow\}$ 
for  $t = 1$  to  $T$  do
  if  $\phi(\mathbf{S}, I_t, \varepsilon_t)$  is in  $\mathcal{A}_n$  then
     $\mathbf{S} \leftarrow \phi(\mathbf{S}, I_t, \varepsilon_t)$ 
  end if
end for

```

---

In other words, this algorithm performs the Markov chain in  $\mathcal{A}_n$  with transition matrix  $P = (P_{\mathbf{R}, \mathbf{S}})_{\mathbf{R}, \mathbf{S} \in \mathcal{A}_n}$  defined as follows:

$$\begin{cases} P_{\mathbf{R}, \mathbf{S}} &= p_i/2, \text{ if } \mathbf{S} = \phi(\mathbf{R}, i, \varepsilon) \text{ for some } \varepsilon \text{ and } 0 \text{ otherwise,} \\ P_{\mathbf{R}, \mathbf{R}} &= 1 - \sum_{\mathbf{S} \neq \mathbf{R}} P_{\mathbf{R}, \mathbf{S}}. \end{cases}$$

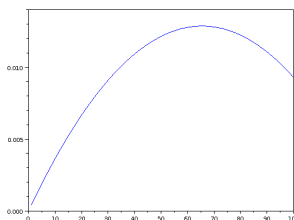
**Proposition 3.** Denote by  $S(t)$  the path sampled by the  $t$ -th run of the loop in Algorithm 1. When  $t \rightarrow \infty$ , the sequence  $S(t)$  converges in law to the uniform distribution in  $\mathcal{A}_n$ .

*Proof.* We have to check that the chain is aperiodic and irreducible. Aperiodicity comes from the (many) loops. Irreducibility will follow from Lemma 5.  $\square$

We choose now the distribution  $(p_i)$ . Instead of  $p_i = 1/n$ , we will use the weights defined by (see the plot of  $i \mapsto p_i$  for  $n = 100$  below):

$$p_i := \frac{4i}{n(n+1)} - \kappa_0 \frac{i(i+1)}{2} \quad (\text{for } i = 1, \dots, n), \quad (1)$$

where  $\kappa_0 = \frac{6}{n(n+1)(n+2)} \sim 6n^{-3}$ . We let the reader check that  $(p_i)_{i \leq n}$  is indeed a probability distribution.



The reason for which we use this particular distribution will appear in the proof of Proposition 6. We will then need the following relation: for each  $1 \leq i \leq n-1$ ,

$$2p_i - p_{i-1} - p_{i+1} = \kappa_0. \quad (2)$$

**Remark 4.** There are obviously many other Markov chains which are reversible with respect to the uniform measure, and some of them may seem more natural to the reader. However, such Markov chains are in general neither monotonous (see later Section 4) nor of positive Ricci curvature (Section 3). The latter condition is essential for our purpose.

## 2.1 Analysis of Algorithm 1

We could deduce from a brief glance at Algorithm 1 that time-complexity is always linear in  $T$ , but we have to pay attention to what is hidden behind each run of the `for` loop.

- If  $I_t < n$ , the time needed for the test " $\phi(\mathbf{S}, I_t, \varepsilon_t)$  is in  $\mathcal{A}_n$ " can be considered as constant, since we only have to compare  $0, S(i), S(n)$ .
- If  $I_t = n$ , the new value  $S(n)$  is compared with the maximum of  $S$ , which can be done in  $\mathcal{O}(n)$ . Fortunately, this occurs with probability  $p_n = \mathcal{O}(n^{-1})$ , so that the time-complexity of each loop is, on average, a  $\mathcal{O}(1)$ .

For Algorithm 1 to be efficient, we need to know how  $S(T)$  is close in law to  $\pi$ . This question is related to the spectral properties of the matrix  $P$ . In particular, the speed of convergence is governed by the spectral gap (*i.e.*  $1 - \lambda$ , where  $\lambda$  is the largest of the modulus of the eigenvalues different from one, see [9] for example), but this quantity is not known in general. Some geometrical methods [5] allow to bound from below  $1 - \lambda$ , but they assume a precise knowledge of the structure of the graph defined by the chain  $P$ . It seems that such results do not apply here.

Instead, we will study the metric properties of the chain  $P$  with respect to a natural distance on  $\mathcal{A}_n$ , and show that it satisfies a certain *contraction property*.

## 3 Error estimates with contraction

Going back to a more general setting, we consider a Markov chain in a finite set  $V$ , endowed with a metric  $d$ . For a vertice  $x \in V$  and a transition matrix  $P$ , we denote by  $P\delta_x$  (resp.  $P^t\delta_x$ ) the law of the Markov chain associated to  $P$  at time 1 (resp.  $t$ ), when starting from  $x$ . For  $x, y \in V$ , the main assumption made on  $P$  is that there is a coupling between  $P\delta_x, P\delta_y$  (that is, a random variable  $(X_1, Y_1)$  with  $X_1 \stackrel{\text{law}}{=} P\delta_x, Y_1 \stackrel{\text{law}}{=} P\delta_y$ ) such that

$$\mathbb{E}[d(X_1, Y_1)] \leq (1 - \kappa)d(x, y), \quad (3)$$

for some  $\kappa > 0$ , which is called the *Ricci curvature* of the chain, by analogy with the Ricci curvature in differential geometry<sup>1</sup>. If the inequality holds, then it implies that  $P$  admits a unique stationary measure  $\pi$  and that, for any  $x$ ,

$$\|P^t\delta_x - \pi\|_{\text{TV}} \leq (1 - \kappa)^t \text{diam}(V), \quad (4)$$

where  $\text{diam}(V)$  is the diameter of the graph with vertices  $V$  induced by the Markov chain and  $\|\cdot\|_{\text{TV}}$  stands, as usual, for the *Total Variation* distance

$$\|\mu_1 - \mu_2\|_{\text{TV}} := \sup_{A \subset V} |\mu_1(A) - \mu_2(A)|.$$

Hence, a positive Ricci curvature gives the exponential convergence to the stationary measure, with an exact (again, exact means non-asymptotic) bound. In many situations, a smart choice for the coupling between  $X_1, X_2$  gives a sharp rate of convergence in eq. (4) (see [10]).

### 3.1 Metric properties of $P$

To apply the Ricci curvature machinery, we endow each  $\mathcal{A}_n$  with the  $L^1$ -distance

$$d_1(S, S') = \frac{1}{a+b} \sum_{i=0}^n |S_i - S'_i|.$$

---

<sup>1</sup>The Ricci curvature is actually the largest positive number such that (3) holds, for all the couplings of  $P\delta_x, P\delta_y$ ; here we should rather say that Ricci curvature is larger than  $\kappa$ .

(Notice that  $|S_i - S'_i|$  is always a multiple of  $a + b$ .) For our purpose, it is fundamental that this metric space is *geodesic*.

**Lemma 5** (Families  $\mathcal{A}_n$  are geodesic). *Each  $\mathcal{A}_n$ , equipped with the distance  $d_1$ , is geodesic in the following sense: for any two  $S, T \in \mathcal{A}_n$  with  $d_1(S, T) = k$ , there exist  $k + 1$  paths  $S_0 = S, S_1, \dots, S_k = T$  in  $\mathcal{A}_n$  such that for each  $i$*

- $d_1(S_i, S_{i+1}) = 1$  ;
- $S_i$  and  $S_{i+1}$  are neighbours in the Markov chain  $P$ .

This implies in particular that  $P$  is irreducible and that the diameter of  $P$  is smaller than  $\max_{S, T} d_1(S, T) \leq n(n + 1)/2$ .

*Proof.* The proof goes by induction on  $k$ . We fix  $S \neq T$  (and denote by  $s, t$  the corresponding words) ; we want to decrease  $d_1(S, T)$  by one, by applying the operator  $\phi(\cdot, i, \varepsilon)$  with proper  $i, \varepsilon$ . We denote by  $i_0 \in \{1, \dots, n\}$  the first index for which  $S \neq T$ :

$$S_0 = T_0, S_1 = T_1, \dots, S_{i_0-1} = T_{i_0-1}, S_{i_0} \neq T_{i_0}.$$

For instance we have  $T_{i_0} = S_{i_0} + a + b$ . Let  $j$  be the position of the left-most peak in  $T$  in  $\{i_0 + 1, i_0 + 2, \dots, n\}$ , if such a peak exists. Then  $S' := \phi(\mathbf{T}, j, \downarrow)$  is also in  $\mathcal{A}_n$ : it is immediate for the three families  $\mathcal{M}_n, \mathcal{W}_n, \mathcal{C}_n$ . We have  $d_1(S, S') = k - 1$ .

If there is no peak in  $T$  after  $i_0$ , then  $(t_{i_0+1}, t_{i_0+2}, \dots, t_n) = (a, a, \dots, a)$ . Hence we try to increase the final steps of  $S$  by one. To do so, we choose  $S' := \phi(\mathbf{S}, n, \uparrow)$  if  $S \neq \phi(\mathbf{S}, n, \uparrow)$ , or  $S' = \phi(\mathbf{S}, j, \uparrow)$  where  $j$  is the position of the right-most valley otherwise. □

We will show that Ricci curvature of  $P$  w.r.t. this distance is (at least) of order  $1/n^3$ .

**Proposition 6.** *For the three families  $\mathcal{M}_n, \mathcal{C}_n, \mathcal{W}_n$ , the Ricci curvature of the associated Markov chain, with weights defined as in (1), is larger than  $\kappa_0$ .*

*Proof.* Fix  $\mathbf{S}, \mathbf{T}$  in  $\mathcal{A}_n \in \{\mathcal{M}_n, \mathcal{C}_n, \mathcal{W}_n\}$ , we first assume that  $\mathbf{S}, \mathbf{T}$  are neighbours, for instance  $\mathbf{T} = \phi(\mathbf{S}, i, \uparrow)$  for some  $i$ , as in Fig. 2. Let  $(\mathbf{S}^1, \mathbf{S}^2)$  be the random variable in  $\mathcal{A}_n \times \mathcal{A}_n$

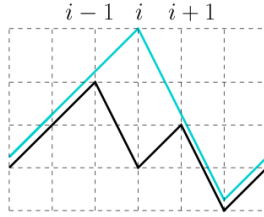


Figure 2: The paths  $\mathbf{S}, \mathbf{T}$  differ at  $i$ .

whose law is defined by

$$(\mathbf{S}^1, \mathbf{S}^2) \stackrel{(\text{law})}{=} (\phi(\mathbf{S}, \mathcal{I}, E), \phi(\mathbf{T}, \mathcal{I}, E)),$$

where  $\mathcal{I}$  is a r.v.  $\{1, \dots, n\}$  with distribution  $\mathbf{p}$  and  $E$  is uniform in  $\{\uparrow, \downarrow\}$ . In other words, we run one loop of Algorithm 1 simultaneously on the both paths.

We want to show that  $\mathbf{S}^1, \mathbf{S}^2$  are, on average, closer than  $\mathbf{S}, \mathbf{T}$ . Three cases may occur:

**Case 1.**  $\mathcal{I} = i$  This occurs with probability  $p_i$  and, no matter the value of  $E$ , we have  $\mathbf{S}^1 = \mathbf{S}^2$ .

**Case 2.**  $\mathcal{I} = i - 1$  or  $i + 1$ . We consider the case  $i - 1$ . Since  $\mathbf{S}$  and  $\mathbf{T}$  coincide everywhere but in  $i$ , we necessarily have one of these two cases:

- there is a peak in  $\mathbf{S}$  at  $i - 1$  and neither peak nor valley in  $\mathbf{T}$  at  $i - 1$  (as in the figure on the right) ;
- there is a valley in  $\mathbf{T}$  at  $i - 1$  and neither peak nor valley in  $\mathbf{S}$  at  $i - 1$ .

In the first case for instance, then we may have  $d_1(\mathbf{S}^1, \mathbf{S}^2) = 2$  if  $E = \downarrow$ , while the distance remains unchanged if  $E = \uparrow$ . The case  $\mathcal{I} = i + 1$  is identical. This shows that with a probability smaller than  $p_{i-1}/2 + p_{i+1}/2$  we have  $d_1(\mathbf{S}^1, \mathbf{S}^2) = 2$ .

**Case 3.**  $\mathcal{I} \neq i - 1, i, i + 1$  In this case,  $\mathbf{S}$  and  $\mathbf{T}$  are possibly modified in  $\mathcal{I}$ , but if there is a modification it occurs in both paths. It is immediate for the families  $\mathcal{M}_n, \mathcal{W}_n$ , less apparent for  $\mathcal{C}_n$ . In the latter we have to check that if  $\phi(\mathbf{T}, \mathcal{I}, \uparrow)$  is in  $\mathcal{C}_n$ , so is  $\phi(\mathbf{S}, \mathcal{I}, \uparrow)$ . But this is true because we have  $\max_j S_j = S_n = T_n$ . Hence a flip in  $\mathbf{S}$  at  $\mathcal{I}$  does not violate the *maximum-at-last-position* condition, because it does not violate this condition for  $\mathbf{T}$ .

Thus, we have proven that when  $\mathbf{S}, \mathbf{T}$  only differ at  $i$

$$\begin{aligned} \mathbb{E} [d_1(\mathbf{S}^1, \mathbf{S}^2)] &\leq 2 \times (p_{i-1}/2 + p_{i+1}/2) + 0 \times p_i + 1 \times (1 - p_i - p_{i-1}/2 - p_{i+1}/2) \\ &\leq (1 - \kappa_0) \times 1 = (1 - \kappa_0)d_1(S, T). \end{aligned} \quad (5)$$

What makes Ricci curvature very useful is that if this inequality holds for pairs of neighbours then it holds for any pair, as noticed in [4]. Indeed, take  $k$  paths  $S_0 = S, S_1, \dots, S_k = T$  as in Lemma 5 and apply the triangular inequality for  $d_1$ :

$$\begin{aligned} \mathbb{E} [d_1(\phi(S, \mathcal{I}, E), \phi(T, \mathcal{I}, E))] &\leq \sum_{i=0}^{k-1} \mathbb{E} [d_1(\phi(S_i, \mathcal{I}, E), \phi(S_{i+1}, \mathcal{I}, E))] \\ &\leq (1 - \kappa_0)k = (1 - \kappa_0)d_1(S, T). \end{aligned}$$

□

**Remark 7.** *It is easy to exhibit some  $S, T$  such that ineq. (5) is in fact an equality. In the case where  $p_i = 1/n$ , this equality reads  $\mathbb{E} [d_1(\mathbf{S}^1, \mathbf{S}^2)] = d_1(S, T)$ , and we cannot obtain a positive Ricci curvature (though this does not prove that there is not another coupling or another distance for which we could get a  $\kappa > 0$  in the case  $p_i = 1/n$ ).*

Combining Proposition 6 with Eq. (4) gives our main result.

**Theorem 8.** *For each family  $\mathcal{A}_n$ , Algorithm 1 returns an almost uniform sample of  $\pi$ , as soon as  $T \gg n^3$ :*

$$\| \mathbf{S}(T) - \pi \|_{\text{TV}} \leq n^2/2 \exp \left( -\frac{6T}{n(n+1)(n+2)} \right).$$

### 3.2 Related works

Bounding mixing times via a contraction property over the transportation metric is quite a standard technique, the main ideas dating back to Dobrushin (1950's). A modern introduction is made in [9]. For geodesic spaces, this technique has been developed in [4] under the name *path coupling*.

The Markov chain  $P$  on lattice paths has in fact already been introduced<sup>2</sup> by D. Wilson [12] for lattice paths with a fixed end-point (as a first step for the sampling of random tilings), with uniform weights  $p_i = 1/n$ . The author also proves a mixing time of order  $n^3 \log n$ , by showing that (3) holds with a different distance (namely, a kind of Fourier transform of the heights of the paths). It does not seem to us that his method can be applied for paths with our kinds of constraints (when the end-point is not fixed).

<sup>2</sup>It is considered in [12] 2d-paths from  $(0, 0)$  to  $(x, y)$  with steps East/North. These are, up to a linear transformation, one-dimensional paths of length  $x + y$  with steps  $+x/-y$ , starting and ending at zero.

More generally, we do believe that it is difficult to build a Markov chain for these kinds of lattice paths which has a mixing time much smaller than  $n^3$ , with the constraint that each step of the chain is fast to compute (in addition to [12], see also [2] for some related results in the context of quasicrystals: the weights are also uniform with yet another distance).

## 4 Coupling From The Past with $P$

Propp-Wilson's Coupling From The Past (CFTP) [11] is a very general procedure for the exact sampling of the stationary distribution of a Markov chain. It is efficient if the chain is monotonous with respect to a certain order relation  $\preceq$  on the set  $V$  of vertices, with two extremal points denoted  $\hat{0}, \hat{1}$  (*i.e.* such that  $\hat{0} \preceq x \preceq \hat{1}$  for any vertex  $x$ ). This is the case here for our three families, with the partial order

$$\mathbf{S} \preceq \mathbf{T} \text{ iff } S_i \leq T_i \text{ for any } i.$$

For the family  $\mathcal{M}_{10}$  with  $a = 1, b = -2$  for instance, we have

$$\begin{aligned} \hat{0} = \hat{0}_{\text{meanders}} &= (1, 1, -2, 1, 1, -2, 1, 1, -2, 1), \\ \hat{1} = \hat{1}_{\text{meanders}} &= (1, 1, 1, 1, 1, 1, 1, 1, 1, 1). \end{aligned}$$

It is easy to check that for each  $n$ , families  $\mathcal{C}_n$  and  $\mathcal{W}_n$  also admit extremal points  $\hat{0}, \hat{1}$ .

We describe CFTP, with our notations, in Algorithm 2.

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**Algorithm 2** CFTP: Exact sampling of a path in  $\mathcal{A}_n$

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```

S  $\leftarrow$   $\hat{0}$ , T  $\leftarrow$   $\hat{1}$ 
... ,  $I_{-2}, I_{-1} \leftarrow$  i.i.d. r.v. with law p
... ,  $\varepsilon_{-2}, \varepsilon_{-1} \leftarrow$  i.i.d. uniform r.v. in  $\{\uparrow, \downarrow\}$ 
 $\tau = 1$ 
repeat
  S  $\leftarrow$   $\hat{0}$ , T  $\leftarrow$   $\hat{1}$ 
  for  $t = -\tau$  to 0 do
    if  $\phi(\mathbf{S}, I_t, \varepsilon_t)$  is in  $\mathcal{A}_n$  then S  $\leftarrow$   $\phi(\mathbf{S}, I_t, \varepsilon_t)$ 
    if  $\phi(\mathbf{T}, I_t, \varepsilon_t)$  is in  $\mathcal{A}_n$  then T  $\leftarrow$   $\phi(\mathbf{T}, I_t, \varepsilon_t)$ 
  end for
   $\tau \leftarrow 2\tau$ 
until S = T

```

---

We refer to ([7],Chap.10) for a very clear introduction to CFTP, and we only outline here the reasons why this indeed gives an exact sampling of the stationary distribution.

- The output of the algorithm (if it ever ends!) is the state of the chain  $P$  which has been running "since time  $-\infty$ ", and thus has reached stationnarity.
- The exit condition  $\mathbf{S} = \mathbf{T}$  ensures that it is not worth running the chain from  $T$  steps earlier, since the trajectory of any lattice path  $\hat{0} \preceq \mathbf{R} \preceq \hat{1}$  is "sandwiched" between those of  $\hat{0}, \hat{1}$ , and therefore ends at the same value.

**Proposition 9.** *Algorithm 2 ends with probability 1 and returns a exact sample of the uniform distribution over  $\mathcal{A}_n$ . This takes on average  $\mathcal{O}(n^3(\log n)^2)$  time units.*

We recall that CFTP has a major drawback compared to MCMC. For the algorithm to be correct, we have to reuse the same random variables  $I_t, \varepsilon_t$ , so that space-complexity is in fact linear in  $n^3(\log n)^2$ . This may become an issue when  $n$  is large.



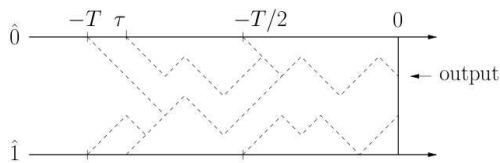


Figure 3: A sketchy representation of CFTP : trajectories starting from  $\hat{0}, \hat{1}$  at time  $-T/2$  don't meet before time zero, while those starting at time  $-T$  do.

*Proof.* It is shown in [11] that Algorithm 2 returns an exact sampling in  $\mathcal{O}(t_{\text{mix}} \log H)$  runs of the chain, with

$$t_{\text{mix}} := \left\{ t \geq 0 ; \sup_{v \in V} \| P^t \delta_v - \pi \|_{\text{TV}} \leq e^{-1} \right\}.$$

The number  $H$  is the length of the longest ordered chain of states between  $\hat{0}$  and  $\hat{1}$ . It is a consequence of the proof of Lemma 5 that  $H = \mathcal{O}(n^2)$ . Regarding  $t_{\text{mix}}$ , we have, for instance by ([9], p.189),

$$t_{\text{mix}} \leq \frac{1}{\kappa_0} (\log(\text{diam}V) + 1),$$

hence  $t_{\text{mix}} = \mathcal{O}(n^3 \log n)$ . (Recall that under Section 2.1, each test in Algorithm 2 takes, on average,  $\mathcal{O}(1)$  time units.)  $\square$

## 5 Concluding remarks and simulations

1. In Fig.4, we show simulations of the three kinds of paths, for  $n = 600$ . The final height of the culminating path is very low (about 30), it would be interesting to use our algorithms to investigate the behaviour of this height when  $n \rightarrow \infty$  ; this question was left open in [3].

2. One may wonder to what extent this work applies to other families  $\mathcal{A}_n$  of paths. The main assumption is that the family of paths should be a geodesic space w.r.t. distance  $d_1$ . This is true for example if the following condition on  $\mathcal{A}_n$  is fulfilled:

$$(R, T \in \mathcal{A}_n \text{ and } R \preceq S \preceq T) \Rightarrow S \in \mathcal{A}_n.$$

Notice however that this is quite a strong requirement, and it is not verified for culminating paths for instance.

3. A motivation to sample random paths is to make and test guesses for some functionals of these paths, taken on average over  $\mathcal{A}_n$ . Consider a function  $f : \mathcal{A}_n \rightarrow \mathbb{R}$ , we want an approximate value of  $\pi(f) := \text{card}(\mathcal{A}_n)^{-1} \sum_{s \in \mathcal{A}_n} f(s)$ , if the exact value is out of reach by calculation. We estimate this quantity by

$$\hat{\pi}(f) := \frac{1}{T} \sum_{t=1}^T f(\mathbf{S}(t)), \quad (6)$$

(recall that  $\mathbf{S}(t)$  is the value of the chain at time  $t$ ). For Algorithm 1 to be efficient in practice, we have to bound

$$\mathbb{P}(|\pi(f) - \hat{\pi}(f)| > r), \quad (7)$$

for any fixed  $r > 0$ , by a non-asymptotic (in  $T$ ) quantity. This can be done with ([8], Th.4-5), in which one can find concentration inequalities for (7). The sharpness of these inequalities depends on  $\kappa$  and on the geometrical structure of  $\mathcal{A}_n$ .

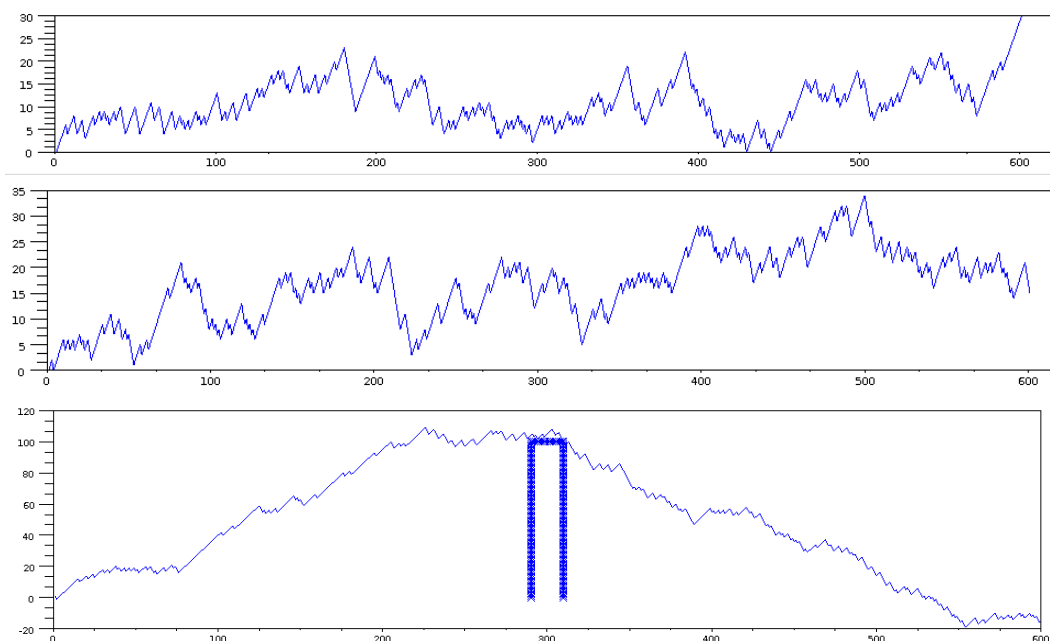


Figure 4: (Almost) uniform paths of length 600. From top to bottom: a culminating path, a meander, a path with wall (shown by an arch).

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