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Radjesvarane Alexandre, Y. Morimoto, Seiji Ukai, Chao-Jiang Xu, Tong Yang

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**GLOBAL EXISTENCE AND FULL REGULARITY
OF THE BOLTZMANN EQUATION WITHOUT ANGULAR CUTOFF
PART I : MAXWELLIAN CASE AND SMALL SINGULARITY**

R. ALEXANDRE, Y. MORIMOTO, S. UKAI, C.-J. XU, AND T. YANG

ABSTRACT. The global existence and uniqueness of classical solutions to the Boltzmann equation without angular cut-off is proved under the condition that the initial data is in some Sobolev space. In addition, the solutions thus obtained are shown to be positive and C^∞ in all variables. One of the key observations is the introduction of a new important norm related to the singularity in the cross section of the collision operator. This norm captures the essential property of the singularity and yields precisely the dissipation description of the linearized collision operator through the celebrated H-theorem.

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1. INTRODUCTION

Consider the Cauchy problem for the Boltzmann equation

$$(1.1) \quad f_t + v \cdot \nabla_x f = Q(f, f), \quad f|_{t=0} = f_0,$$

where $f = f(t, x, v)$ is the density distribution function of particles with position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$ at time t . The right hand side of (1.1) is given by the Boltzmann bilinear collision operator

$$Q(g, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma) \{g'_* f' - g_* f\} d\sigma dv_*,$$

where $f'_* = f(t, x, v'_*)$, $f' = f(t, x, v')$, $f_* = f(t, x, v_*)$, $f = f(t, x, v)$, and for $\sigma \in \mathbb{S}^2$,

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma,$$

which give the relations between the post and pre collisional velocities.

As a fundamental equation in kinetic theories and a key stone in statistical physics, the Boltzmann equation has attracted, and is still attracting, a lot of research investigations since its derivation in 1872.

A huge number of mathematical works have been performed under the Grad's cutoff assumption, avoiding the non-integrable angular singularity in the cross-sections.

However, except for the hard sphere model, for most of the other molecule interaction potentials such as the inverse power laws, the cross section $B(v - v_*, \sigma)$ has a non-integrable angular singularity when the Grad's angular cut-off assumption is not imposed. This implies that the collision operator is a singular integral operator on the microscopic velocity variable v .

Thus, it is believed that this kind of singularity should lead to smoothing effect on solutions, that is, the solutions have higher regularity than the initial data; it should be similar to the case when one replaces the collision operator in the Boltzmann equation by the Laplacian to some fractional power.

The mathematical study on this kind of regularization problems for the Boltzmann equation can be traced back at least to the work by Pao in 1970s [45] (though that work was concerned with other aspects of Boltzmann equation) and then many others afterwards, see for example the references quoted in the reviews [1, 50].

However, most of the available results so far are limited to the space homogeneous problems. In fact, even though there are some works on the space inhomogeneous problems, a full mathematical theory is still at an incomplete stage. The difficulties come from the coupling of the convection operator and the collision operator, and the commutators of the differential (pseudo-differential) operators with the collision operator. In this direction, in [9], we prove the C^∞ regularity of solutions in all variables for the space inhomogeneous problem when some regularity on the solutions is imposed to start with, together with some local existence results [9, 10, 11].

Notice that the non-negative cross section $B(z, \sigma)$ depends only on $|z|$ and the scalar product $\langle \frac{z}{|z|}, \sigma \rangle$. In most cases, the B cannot be expressed explicitly, but to capture its

main properties, we can assume that it takes the form

$$(1.2) \quad B(|v - v_*|, \cos \theta) = \Phi(|v - v_*|)b(\cos \theta), \quad \cos \theta = \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

Recall that for the potential of the inverse power law $\frac{1}{\rho^r}$, $r > 1$, ρ being the distance between two particles, the cross section in the above form has a kinetic factor related to the relative velocity given by

$$(1.3) \quad \Phi(|v - v_*|) \approx |v - v_*|^\gamma, \quad \gamma = 1 - \frac{4}{r},$$

and a factor related to the collision angle containing a singularity,

$$(1.4) \quad \theta b(\cos \theta) \approx K\theta^{-2-\nu} \text{ when } \theta \rightarrow 0+,$$

for some constants $K > 0$ and $0 < \nu = \frac{2}{r} < 2$. The cases $1 < r < 4$, $r = 4$ and $r > 4$ correspond to so-called soft, Maxwellian molecule and hard potentials respectively. Notice that the Boltzmann collision operator is not well-defined for the case $s = 1$ which corresponds to the Coulomb potential. It is then obvious that the cross section is not integrable over the sphere so that the cancellations between the gain and loss terms in the collision operator must be taken into account. In the following discussion, we will consider this type of cross sections with the parameters γ and ν given above.

Let us now review some related works about the singularity of the non-cutoff cross-sections in the Boltzmann equation. For much more references and comments, the reader is referred to [1, 50]. Since the work of Pao in [45] in 1970s, the existence of weak solutions to the spatially homogeneous case was proved by Arkeryd in [13] for the mild singularity case, that is, when $0 < s < \frac{1}{2}$. By using an abstract Cauchy-Kovalevskaya theorem, Ukai in [46] proved the local existence of solutions in the function space that is analytic in x and Gevrey in v in 1980s. The study on the gain of regularity in all variables came much later until Desvillettes obtained some regularization results for some simplified kinetic models in [21].

Around the same period, a systematic approach by using the entropy dissipation and/or the smoothing property of the gain part in the collision operator was initiated and has been developed to an almost optimal stage through the efforts by many researchers, such as Alexandre, Bouchut, Desvillettes, Golse, P.-L. Lions, Villani, Wennberg.

And the underlying tools have proven to be very useful for the study on the mathematical theory regarding the regularizing effect for the spatially homogeneous problems for which now the theory has become quite satisfactory, cf. [5, 12, 6, 22, 23, 25, 34, 43, 48], and the references therein, see also [1].

Compared to the spatially homogeneous problems, the original spatially inhomogeneous Boltzmann equation is of course physically more interesting and mathematically more challenging. However, for the spatially inhomogeneous problems, the mathematical theory is so far not satisfactory. One of the main reasons is that the singular integral collision operator is on the velocity variable only and the gain of the regularity in the space variable comes from the coupling of the transport operator with the collision operator. If the collision operator was a differential operator, such as the Laplacian to some fractional power, many theories and analytic tools have been developed, see for instance [20]. However, for the Boltzmann equation, the collision operator is definitively similar to a singular integral, so that the analysis is more involved. It required some harmonic tools which were introduced initially in [3], and then developed in [34, 1].

As for the Cauchy problem, the local existence of solutions between two moving Maxwellians was proven in [4] by constructing upper and lower solutions. The global existence of

renormalized solutions with defect measures was shown in [12] these solutions becoming weak solutions if the defect measures vanish.

Some results related to linear kinetic equations were also given in [8] and [16]. In particular, a generalized uncertainty principle in the view point of Fefferman [28] (see also [39, 40, 41]) was introduced in [8] for the study of smoothing effects of the linearized and spatially inhomogeneous Boltzmann equation with non-cutoff cross sections, and partial smoothing effects for nonlinear Boltzmann equation were also proven there.

Recently, by using some suitable pseudo-differential operators and some harmonic analytic tools, we have developed in [11] the analysis of the coercivity and upper bounds of the collision operators, together with the estimation of the commutators of these (pseudo-) differential operators with the collision operator. Through this analysis, we began to consider the space inhomogeneous problem. More precisely, in [9, 11], for solutions as perturbation of vacuum states, by sharpening the coercivity and upper bound estimates for the collision operator and by analyzing the commutators between the collision operator and some weighted pseudo-differential operators, we established the hypo-ellipticity of the Boltzmann operator, using a generalized version of the uncertainty principle. Based on these tools, the local existence and the regularizing effect in all (time, space and velocity) variables on solutions in some weighted Sobolev space were also proven.

The present work is a continuation of our collaborative program since the 2006's and in particular of our series of works [9, 10, 11].

In this paper, we will obtain the global existence of classical positive solutions for the full space Boltzmann equation, near global equilibrium and prove the full regularity C^∞ regularization for positive t in all variables for the solutions in some weighted Sobolev space. As mentioned in the abstract, one of the main ingredients in the proof is the introduction of a new non-isotropic norm which captures the main feature of the singularity in the cross-section. This norm not only works extremely well for the description of the dissipative effect of the linearized collision operator through the H-theorem, but is also well fitted for the upper bound estimations of the nonlinear collision operator. Here, we would like to mention the work by C. Mouhot [44] about the gain of moment in a linearized context due to the singularity in the cross-section. Such a gain of moment is in fact really important for the proof of global existence and full regularity of the solution. In addition, such a property is also well described by the new non-isotropic norm.

Finally in this review of some of these previous works, we would like to mention two aspects of solution's behavior which are well studied for the spatially homogeneous problems, but not so well understood for the spatially inhomogeneous problems. The first one is about the gain of the full regularity which is guaranteed by the natural bounds, that is, the bounds on mass, energy and entropy for the space homogeneous case, cf. [34]; and the second one is about the gain of Gevrey regularity which has been proved in [42] also for the spatially homogeneous case.

We now come back to the problem considered in this paper. To concentrate on the analysis on the singularity of the grazing effect when θ is close to zero, in this paper, we study the Maxwellian molecule type cross-sections, that is, we assume that the cross-section behaves like

$$(1.5) \quad B(|v - v_*|, \cos \theta) = b(\cos \theta), \quad \cos \theta = \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle, \quad 0 \leq \theta \leq \frac{\pi}{2},$$

and

$$(1.6) \quad b(\cos \theta) \approx K\theta^{-2-2s}, \quad \theta \rightarrow 0^+,$$

with $0 < s < 1$.

In order to prove the global existence, we need to fully use the dissipative effect of the collision operator. As in similar studies but for angular cutoff case, such dissipative effect can be fully represented by the dissipation of the linearized collision operator on the microscopic component of the solution through the celebrated H-theorem. Thus, as usual, we linearize the Boltzmann equation around a normalized Maxwellian distribution

$$\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}.$$

Since μ is the global equilibrium state satisfying $Q(\mu, \mu) = 0$, by setting $f = \mu + \sqrt{\mu}g$, we have

$$Q(\mu + \sqrt{\mu}g, \mu + \sqrt{\mu}g) = Q(\mu, \sqrt{\mu}g) + Q(\sqrt{\mu}g, \mu) + Q(\sqrt{\mu}g, \sqrt{\mu}g).$$

Denote

$$(1.7) \quad \Gamma(g, h) = \mu^{-1/2} Q(\sqrt{\mu}g, \sqrt{\mu}h).$$

Then the linearized Boltzmann operator takes the form

$$(1.8) \quad \mathcal{L}g = \mathcal{L}_1 g + \mathcal{L}_2 g = -\Gamma(\sqrt{\mu}, g) - \Gamma(g, \sqrt{\mu}).$$

And the original problem (1.1) is now reduced to the study on the perturbation g in the following Cauchy problem

$$(1.9) \quad \begin{cases} g_t + v \cdot \nabla_x g + \mathcal{L}g = \Gamma(g, g), & t > 0; \\ g|_{t=0} = g_0. \end{cases}$$

Throughout this paper, we shall consider this Cauchy problem in some weighted Sobolev space. For $k, \ell \in \mathbb{R}$, set

$$H_\ell^k(\mathbb{R}_{x,v}^6) = \left\{ f \in \mathcal{S}'(\mathbb{R}_{x,v}^6); W^\ell f \in H^k(\mathbb{R}_{x,v}^6) \right\},$$

where $\mathbb{R}_{x,v}^6 = \mathbb{R}_x^3 \times \mathbb{R}_v^3$ and $W^\ell(v) = \langle v \rangle^\ell = (1 + |v|^2)^{\ell/2}$ is the weight in the velocity variable $v \in \mathbb{R}_v^3$.

The main theorem of this paper can be stated as follows.

Theorem 1.1. *Assume that the cross-section satisfies (1.6) with $0 < s < 1/2$. Let $g_0 \in H_\ell^k(\mathbb{R}^6)$ for some $k \geq 3, \ell \geq 3$ and*

$$f_0(x, v) = \mu + \sqrt{\mu} g_0(x, v) \geq 0.$$

Then there exists $\varepsilon_0 > 0$, such that if $\|g_0\|_{H_\ell^k(\mathbb{R}^6)} \leq \varepsilon_0$, the Cauchy problem (1.9) admits a unique global solution $g \in L^\infty([0, +\infty[; H_\ell^k(\mathbb{R}^6))$ with $f(t, x, v) = \mu + \sqrt{\mu} g(t, x, v) \geq 0$ and

$$g \in C^\infty([0, +\infty[\times \mathbb{R}^6).$$

In fact, for the uniqueness of solution, we shall also prove the following stronger result, which is also of independent interest, where we do not need to assume that f is a small perturbation of μ .

Theorem 1.2. *Assume that $0 < s < 1/2, 0 < T \leq +\infty$ and $l > 2s + 3/2 + 2$. Let $f_0 \geq 0, f_0 \in L^\infty(\mathbb{R}_x^3; H_{l+2}^{2s}(\mathbb{R}_v^3))$. Suppose that the Cauchy problem (1.1) admits two solutions $f_1, f_2 \in L^\infty([0, T] \times \mathbb{R}_x^3; H_{l+2}^{2s}(\mathbb{R}_v^3))$. If one solution is non-negative then $f_1 \equiv f_2$.*

The rest of the paper will be organized as follows. In the next section, we will introduce our new isotropic norm and prove some essential coercivity and upper bound estimates on the collision operators with respect to this new norm. In order to study the gain of regularity of the solution, we shall need to clarify the application of some pseudo-differential operators on the Boltzmann equation. For this purpose, we need to study the commutators

of the collision operators with those pseudo-differential operators. These estimates on the commutators will be given in the third section. In Section 4, we will apply the energy method for the Boltzmann equation and obtain the local existence theorem. In Section 5, we will study the uniqueness and the non-negativity of the solutions. Finally, the full regularity and the global existence of the solution will be given in the last two sections respectively.

2. NON ISOTROPIC NORMS

In this section we perform the harmonic analysis of the bilinear collision operator,

$$Q(g, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos \theta) \{g'_* f' - g_* f\} d\sigma dv_*.$$

Since the collision operator acts only w.r.t. the velocity variable $v \in \mathbb{R}^3$, in this section (t, x) are considered as parameters and therefore do not appear if not necessary.

2.1. Coercivity and upper bound estimates. Let $g \geq 0$, $g \neq 0$, $g \in L^1_2 \cap L \log L(\mathbb{R}^3_v)$. It was shown in [5] that there exists a constant $c_g > 0$ depending only on the values of $\|g\|_{L^1_2}$ and $\|g\|_{L \log L}$ such that for any smooth function $f \in H^s(\mathbb{R}^3_v)$, we have

$$(2.1.1) \quad c_g \|f\|_{H^s(\mathbb{R}^3_v)}^2 \leq (-Q(g, f), f)_{L^2(\mathbb{R}^3_v)} + C \|g\|_{L^1(\mathbb{R}^3_v)} \|f\|_{L^2(\mathbb{R}^3_v)}^2.$$

We shall need some functional estimates on the Boltzmann collision operators in the proof. The first one, given below, is about the boundedness of the collision operator in some weighted Sobolev spaces, see [1, 34].

Theorem 2.1. *Assume that the cross-section satisfies (1.6) with $0 < s < 1$. Then for any $m \in \mathbb{R}$ and any $\alpha \in \mathbb{R}$, there exists $C > 0$ such that*

$$(2.1.2) \quad \|Q(f, g)\|_{H^m_\alpha(\mathbb{R}^3_v)} \leq C \|f\|_{L^1_{\alpha+2s}(\mathbb{R}^3_v)} \|g\|_{H^{m+2s}_{(\alpha+2s)^+}(\mathbb{R}^3_v)}$$

for all $f \in L^1_{\alpha+2s}(\mathbb{R}^3_v)$ and $g \in H^{m+2s}_{(\alpha+2s)^+}(\mathbb{R}^3_v)$.

It is well-known that \mathcal{L} (acting w.r.t. velocity variable) is an unbounded symmetric operator on $L^2(\mathbb{R}^3_v)$, such that its Dirichlet form satisfies

$$(\mathcal{L}g, g)_{L^2(\mathbb{R}^3_v)} \geq 0,$$

and that $(\mathcal{L}g, g)_{L^2(\mathbb{R}^3_v)} = 0$ if and only if $\mathbf{P}g = g$ where

$$\mathbf{P}g = (a + b \cdot v + c|v|^2) \sqrt{\mu}$$

with $a, c \in \mathbb{R}$, $b \in \mathbb{R}^3$, \mathbf{P} being the L^2 -orthogonal projection onto the null space

$$(2.1.3) \quad \text{Ker}(\mathcal{L}) = \text{Span} \{ \sqrt{\mu}, v_1 \sqrt{\mu}, v_2 \sqrt{\mu}, v_3 \sqrt{\mu}, |v|^2 \sqrt{\mu} \}.$$

We shall also need the following coercive type result in this linearized framework

Theorem 2.2. *(Theorem 1.1 of [44])*

Assume that the cross-section satisfies (1.6) with $0 < s < 1$. Then there exists a constant $C > 0$ such that

$$(2.1.4) \quad (\mathcal{L}g, g)_{L^2(\mathbb{R}^3_v)} \geq C \|(g - \mathbf{P}g)\|_{L^2_s(\mathbb{R}^3_v)}^2.$$

We give now two formula for the bilinear operator $\Gamma(\cdot, \cdot)$. First of all, by using the conservation of the momentum,

$$|v'_*|^2 + |v'|^2 = |v_*|^2 + |v|^2,$$

we have $\mu(v_*) = \mu^{-1}(v)\mu(v'_*)\mu(v')$. Then suitable functions f, g , it holds that

$$\begin{aligned} \Gamma(f, g)(v) &= \mu^{-1/2} \iint b(\cos \theta) (\sqrt{\mu'_*} f'_* \sqrt{\mu'} g' - \sqrt{\mu_*} f_* \sqrt{\mu} g) dv_* d\sigma \\ (2.1.5) \quad &= \iint b(\cos \theta) \sqrt{\mu_*} (f'_* g' - f_* g) dv_* d\sigma \\ &= Q(\sqrt{\mu} f, g) + \iint b(\cos \theta) (\sqrt{\mu_*} - \sqrt{\mu'_*}) f'_* g' dv_* d\sigma. \end{aligned}$$

Therefore, by using the classical change of variables $(v, v_*) \rightarrow (v', v'_*)$, we have

$$\begin{aligned} (\Gamma(f, g), h)_{L^2(\mathbb{R}_v^3)} &= \iiint b(\cos \theta) \sqrt{\mu_*} (f'_* g' - f_* g) h \\ &= \iiint b(\cos \theta) \sqrt{\mu'_*} (f_* g - f'_* g') h'. \end{aligned}$$

Thus

$$(2.1.6) \quad (\Gamma(f, g), h)_{L^2(\mathbb{R}_v^3)} = \frac{1}{2} \iiint b(\cos \theta) (f'_* g' - f_* g) (\sqrt{\mu_*} h - \sqrt{\mu'_*} h').$$

The following result shows that the linear parts are somehow controlled by one part only

Lemma 2.3.

$$(2.1.7) \quad (\mathcal{L}g, g)_{L^2(\mathbb{R}_v^3)} \geq \frac{1}{2} (\mathcal{L}g, g)_{L^2(\mathbb{R}_v^3)}.$$

Proof.

$$\begin{aligned} (\mathcal{L}g, g)_{L^2(\mathbb{R}_v^3)} &= -(\Gamma(\sqrt{\mu}, g), g)_{L^2(\mathbb{R}_v^3)} \\ &= \frac{1}{2} \iiint b(\cos \theta) (\mu'_*{}^{1/2} g' - \mu_*{}^{1/2} g)^2 dv_* d\sigma dv \\ &= \frac{1}{2} \iiint b(\cos \theta) (\mu'^{1/2} g'_* - \mu^{1/2} g_*)^2 dv_* d\sigma dv \\ &= \frac{1}{4} \iiint b(\cos \theta) \left\{ (\mu'_*{}^{1/2} g' - \mu_*{}^{1/2} g)^2 + (\mu'^{1/2} g'_* - \mu^{1/2} g_*)^2 \right\} dv_* d\sigma dv. \end{aligned}$$

And changing variables as in (2.1.6) implies

$$\begin{aligned}
(\mathcal{L}g, g)_{L^2(\mathbb{R}_v^3)} &= -(\Gamma(\sqrt{\mu}, g) + \Gamma(g, \sqrt{\mu}), g)_{L^2(\mathbb{R}_v^3)} \\
&= \iiint b(\cos \theta) (\mu_*^{1/2} g - (\mu'_*)^{1/2} g' + g_* (\mu)^{1/2} - g'_* (\mu')^{1/2}) (\mu_*)^{1/2} g dv_* d\sigma dv \\
&= \iiint b(\cos \theta) ((\mu'_*)^{1/2} g' - (\mu_*)^{1/2} g + g'_* (\mu')^{1/2} - g_* (\mu)^{1/2}) (\mu'_*)^{1/2} g' dv_* d\sigma dv \\
&= \iiint b(\cos \theta) (\mu^{1/2} g_* - (\mu')^{1/2} g'_* + g (\mu_*)^{1/2} - g' (\mu'_*)^{1/2}) (\mu)^{1/2} g_* dv_* d\sigma dv \\
&= \iiint b(\cos \theta) ((\mu')^{1/2} g'_* - (\mu)^{1/2} g_* + g' (\mu'_*)^{1/2} - g (\mu_*)^{1/2}) (\mu')^{1/2} g'_* dv_* d\sigma dv \\
&= \frac{1}{4} \iiint b(\cos \theta) \left((\mu_*^{1/2} g - (\mu'_*)^{1/2} g') + (\mu^{1/2} g_* - (\mu')^{1/2} g'_*) \right)^2 dv_* d\sigma dv.
\end{aligned}$$

Therefore (2.1.7) is proven, by using $(A + B)^2 \leq 2(A^2 + B^2)$. \square

2.2. The norm associated with collision operators. Our non-isotropic norm associated with the cross-section $b(\cos \theta)$ will be defined as follows

$$(2.2.1) \quad \|g\|^2 = \iiint b(\cos \theta) \mu_* (g' - g)^2 + \iiint b(\cos \theta) g_*^2 (\sqrt{\mu'} - \sqrt{\mu})^2$$

where the integration \iiint is over $\mathbb{R}_v^3 \times \mathbb{R}_{v_*}^3 \times \mathbb{S}_\sigma^2$. Thus, it is a norm w.r.t. the velocity variable $v \in \mathbb{R}^3$, and we still consider (t, x) as parameter for the moment.

This norm is termed non-isotropic since it mixes both the derivative of order s and also a weight of order s coming from the singularity of cross-section $b(\cos \theta)$, as it should be clear from our previous works.

Lemma 2.4. *Assume that the collision kernel $b(\cos \theta)$ satisfies (1.6) with $0 < s < 1$. Then there exists $C > 0$ such that*

$$(2.2.2) \quad \|g\|^2 \leq C \|g\|_{H_s^s}^2$$

for any $g \in H_s^s(\mathbb{R}_v^3)$.

Proof. Applying (2.1.2) with $\alpha = -s$ and $m = -s$, we have

$$(2.2.3) \quad \left| (Q(f^2, g), g)_{L^2(\mathbb{R}_v^3)} \right| \leq C \|f^2\|_{L_{2s}^1} \|g\|_{H_s^s} \|g\|_{H_s^s} \leq C \|f\|_{L_{2s}^2}^2 \|g\|_{H_s^s}^2.$$

On the other hand

$$\begin{aligned}
(Q(f^2, g), g)_{L^2(\mathbb{R}_v^3)} &= \iiint b(\cos \theta) (f_*^{2'} g' - f_*^2 g) g \\
&= \iiint b(\cos \theta) f_*^{2'} (g' - g) g + \int g^2 \iiint b(\cos \theta) (f_*^{2'} - f_*^2)
\end{aligned}$$

For the first term, if we set $\alpha = g'$ and $\beta = g$ then using $\beta(\alpha - \beta) = \frac{1}{2}(\alpha^2 - \beta^2) - \frac{1}{2}(\alpha - \beta)^2$, we find

$$\begin{aligned}
(Q(f^2, g), g)_{L^2(\mathbb{R}_v^3)} &= \frac{1}{2} \iiint b(\cos \theta) f_*^{2'} (g^{2'} - g^2) \\
&\quad - \frac{1}{2} \iiint b(\cos \theta) f_*^{2'} (g' - g)^2 + \int g^2 \iiint b(\cos \theta) (f_*^{2'} - f_*^2),
\end{aligned}$$

By the changing of variable $(v'_*, v') \rightarrow (v_*, v)$, the first term is also $\frac{1}{2} \iiint b g^2 (f_*^2 - f_*'^2)$. Thus it follows that

$$(2.2.4) \quad (Q(f^2, g), g)_{L^2(\mathbb{R}_v^3)} = -\frac{1}{2} \iiint b f_*'^2 (g' - g)^2 + \frac{1}{2} \int g^2 \iint b (f_*'^2 - f_*^2).$$

And thus

$$\iiint b f_*'^2 (g' - g)^2 \leq 2|(Q(f^2, g), g)| + \iiint b g^2 (f_*'^2 - f_*^2)$$

Using (2.2.3) and the Cancellation lemma of [5] for the second term, we get that

$$(2.2.5) \quad \iiint b f_*'^2 (g' - g)^2 \leq C \|f\|_{L^2_s}^2 \|g\|_{H^s_s}^2 + C \|g\|_{L^2_s}^2 \|f\|_{L^2_s}^2 \leq C \|f\|_{L^2_s}^2 \|g\|_{H^s_s}^2.$$

Thus

$$\|g\|^2 \leq C (\|\sqrt{\mu}\|_{L^2_s}^2 \|g\|_{H^s_s}^2 + \|g\|_{L^2_s}^2 \|\sqrt{\mu}\|_{H^s_s}^2) \leq C \|g\|_{H^s_s}^2.$$

□

Lemma 2.5. *Assume that the cross-section satisfies (1.6) with $0 < s < 1/2$. Then for any $\phi \in C_b^1$, we have*

$$(2.2.6) \quad \int_{\sigma} b(\cos \theta) |\phi(v_*) - \phi(v'_*)| d\sigma \leq C_{\phi} |v - v_*|^{2s} \leq C \langle v \rangle^{2s} \langle v_* \rangle^{2s},$$

where C_{ϕ} depends on $\|\phi\|_{C_b^1} = \|\phi\|_{L^{\infty}} + \|\nabla \phi\|_{L^{\infty}}$.

Proof. By using Taylor's formula, it follows

$$(2.2.7) \quad |\phi(v_*) - \phi(v'_*)| \leq C_{\phi} |v_* - v'_*| \leq C_{\phi} \sin\left(\frac{\theta}{2}\right) |v - v_*|$$

and $|\phi(v_*) - \phi(v'_*)| \leq C_{\phi}$. Then for any $\delta \in (0, \pi/2)$,

$$\begin{aligned} \int_{\sigma} b(\cos \theta) |\phi(v_*) - \phi(v'_*)| d\sigma &\leq C_{\phi} \left\{ |v - v_*| \int_0^{\delta} \frac{\sin(\theta/2)}{\theta^{1+2s}} d\theta + \int_{\delta}^{\pi/2} \frac{1}{\theta^{1+2s}} d\theta \right\} \\ &\leq C_{\phi} \left\{ |v - v_*| \delta^{-2s+1} + \delta^{-2s} \right\}. \end{aligned}$$

If $|v - v_*|^{-1} \leq \frac{\pi}{2}$, by choosing $\delta = |v - v_*|^{-1}$ we get

$$\int_{\sigma} b(\cos \theta) |\phi(v_*) - \phi(v'_*)| d\sigma \leq C_{\phi} |v - v_*|^{2s} \leq C \langle v \rangle^{2s} \langle v_* \rangle^{2s},$$

If $|v - v_*| \leq \frac{2}{\pi}$, we have

$$\int_{\sigma} b(\cos \theta) |\phi(v_*) - \phi(v'_*)| d\sigma \leq C_{\phi} |v - v_*| \leq C_{\phi} \frac{2}{\pi} \leq C \langle v \rangle^{2s} \langle v_* \rangle^{2s}.$$

□

Lemma 2.6. *Assume that the cross-section satisfies (1.6) with $0 < s < 1/2$. Then there exists $C_0, C_1 > 0$ such that*

$$(2.2.8) \quad C_0 \|g - \mathbf{P}g\|^2 \leq (\mathcal{L}g, g)_{L^2(\mathbb{R}_v^3)} \leq C_1 \|g\|^2.$$

Proof. We study firstly the terms linked with \mathcal{L}_2 . By using (2.1.5), we have

$$\begin{aligned} -(\mathcal{L}_2 g, g)_{L^2(\mathbb{R}_v^3)} &= (\Gamma(g, \sqrt{\mu}), g)_{L^2(\mathbb{R}^3)} \\ &= (Q(\sqrt{\mu}g, \sqrt{\mu}), g)_{L^2(\mathbb{R}_v^3)} + \iiint b(\cos \theta) (\sqrt{\mu_*} - \sqrt{\mu'_*}) g'_* \sqrt{\mu'} g. \end{aligned}$$

Using (2.1.2) with $\alpha = 0, m = 0$ we have

$$\begin{aligned} \left| (Q(\sqrt{\mu}g, \sqrt{\mu}), g)_{L^2(\mathbb{R}_v^3)} \right| &\leq \|Q(\sqrt{\mu}g, \sqrt{\mu})\|_{L^2} \|g\|_{L^2} \\ &\leq C \|\sqrt{\mu}g\|_{L^1_{2s}} \|\sqrt{\mu}\|_{H^2_{2s}} \|g\|_{L^2} \leq C \|g\|_{L^2}^2. \end{aligned}$$

For the second term, we have

$$\begin{aligned} &\iiint b(\cos \theta) (\sqrt{\mu_*} - \sqrt{\mu'_*}) g'_* \sqrt{\mu'} g \, dv dv_* d\sigma \\ &= \iiint b(\cos \theta) (\sqrt{\mu_*} - \sqrt{\mu'_*}) g'_*(\mu')^{1/4} (\mu')^{1/4} - (\mu)^{1/4} g \\ &+ \iiint b(\cos \theta) (\sqrt{\mu_*} - \sqrt{\mu'_*}) g'_*(\mu')^{1/4} (\mu)^{1/4} g. \end{aligned}$$

Thus

$$\begin{aligned} &\left| \iiint b(\cos \theta) (\sqrt{\mu_*} - \sqrt{\mu'_*}) g'_* \sqrt{\mu'} g \right| \\ &\leq \left(\iiint b(\cos \theta) (\sqrt{\mu_*} - \sqrt{\mu'_*})^2 |g|^2 (\mu')^{1/4} \right)^{1/2} \\ &\quad \times \left(\iiint b(\cos \theta) (\mu')^{1/4} - (\mu)^{1/4})^2 |g'_*|^2 (\mu')^{1/4} \right)^{1/2} \\ &+ \left(\iiint b(\cos \theta) |\sqrt{\mu_*} - \sqrt{\mu'_*}| |g|^2 (\mu')^{1/4} (\mu)^{1/4} \right)^{1/2} \\ &\quad \times \left(\iiint b(\cos \theta) |\sqrt{\mu_*} - \sqrt{\mu'_*}| |g'_*|^2 (\mu')^{1/4} (\mu)^{1/4} \right)^{1/2} \\ &\leq I_1^{1/2} \times I_2^{1/2} + I_3^{1/2} \times I_4^{1/2}. \end{aligned}$$

Using Lemma 2.5 with $\phi = \mu^{1/4}$, we obtain

$$\int_{\sigma} b(\cos \theta) |(\mu'_*)^{1/4} - (\mu_*)^{1/4}| d\sigma \leq C |v - v_*|^{2s} \leq C \langle v \rangle^{2s} \langle v_* \rangle^{2s}$$

and $(\mu'_*)^{1/4} (\mu')^{1/2} = (\mu'_*)^{1/4} (\mu')^{1/4} (\mu')^{1/4} = (\mu_*)^{1/4} \mu^{1/4} (\mu')^{1/4}$, we get

$$\begin{aligned} I_1 + I_3 &\leq C \iiint b(\cos \theta) |(\mu_*)^{1/2} - (\mu'_*)^{1/2}| |g|^2 (\mu')^{1/2} \, dv dv_* d\sigma \\ &\leq C \iiint b(\cos \theta) |(\mu_*)^{1/4} - (\mu'_*)^{1/4}| ((\mu_*)^{1/4} + (\mu'_*)^{1/4}) |g|^2 (\mu')^{1/2} \\ &\leq C \iiint b(\cos \theta) |(\mu_*)^{1/4} - (\mu'_*)^{1/4}| (\mu_*)^{1/4} |g|^2 \\ &+ C \iiint b(\cos \theta) |(\mu_*)^{1/4} - (\mu'_*)^{1/4}| (\mu'_*)^{1/4} (\mu')^{1/2} |g|^2 \\ &\leq C \iint (\langle v_* \rangle^{2s} (\mu_*)^{1/4} \langle v \rangle^{2s} |g|^2 + \langle v_* \rangle^{2s} (\mu_*)^{1/4} \langle v \rangle^{2s} \mu^{1/4} |g|^2) \, dv dv_* \\ &\leq C (\|g\|_{L^2_{2s}(\mathbb{R}^3)}^2 + \|g\|_{L^2(\mathbb{R}^3)}^2). \end{aligned}$$

For the term I_2 , using the change of variables $v \rightarrow v_*$ and $v' \rightarrow v$ implies that

$$\begin{aligned} & \iiint b(\cos \theta) \left((\mu')^{1/4} - (\mu)^{1/4} \right)^2 |g'_*|^2 (\mu')^{1/4} \\ &= \iiint b(\cos \theta) \left((\mu'_*)^{1/4} - (\mu_*)^{1/4} \right)^2 |g|^2 (\mu_*)^{1/4} \\ &\leq C \iint \left(\langle v_* \rangle^{2s} (\mu_*)^{1/4} \langle v \rangle^{2s} |g|^2 \right) dv dv_* \leq C \|g\|_{L^2_s(\mathbb{R}^3)}^2. \end{aligned}$$

For the term I_4 , using the change of variables $v' \rightarrow v$ implies that

$$\begin{aligned} & \iiint b(\cos \theta) \left| \sqrt{\mu_*} - \sqrt{\mu'_*} \right| |g'_*|^2 (\mu')^{1/4} (\mu)^{1/4} \\ &= \iiint b(\cos \theta) \left| \sqrt{\mu_*} - \sqrt{\mu'_*} \right| |g_*|^2 (\mu')^{1/4} (\mu)^{1/4} \\ &\leq C \iint \left(\langle v \rangle^{2s} (\mu)^{1/4} \langle v_* \rangle^{2s} |g_*|^2 \right) dv dv_* \leq C \|g\|_{L^2_s(\mathbb{R}^3)}^2. \end{aligned}$$

Thus we have proved that

$$(2.2.9) \quad |(\mathcal{L}_2 g, g)| \leq C \|g\|_{L^2_s}^2.$$

Now from (2.1.6), we have

$$\begin{aligned} (\mathcal{L}_1 g, g)_{L^2(\mathbb{R}^3_+)} &= -(\Gamma(\sqrt{\mu}, g), g)_{L^2(\mathbb{R}^3_+)} \\ &= \frac{1}{2} \iiint b(\cos \theta) \left((\mu'_*)^{1/2} g' - (\mu_*)^{1/2} g \right)^2 \\ &= \frac{1}{2} \iiint b(\cos \theta) \left((\mu'_*)^{1/2} (g' - g) + g \left((\mu_*)^{1/2} - (\mu'_*)^{1/2} \right) \right)^2 \\ &\geq \frac{1}{4} \iiint b(\cos \theta) \mu'_* (g' - g)^2 - \frac{1}{2} \iiint b(\cos \theta) g^2 \left((\mu'_*)^{1/2} - (\mu_*)^{1/2} \right)^2, \end{aligned}$$

where we used the inequality $(A + B)^2 \geq \frac{1}{2}A^2 - B^2$. Then

$$\begin{aligned} (\mathcal{L}_1 g, g)_{L^2(\mathbb{R}^3_+)} &\geq \frac{1}{4} \left(\iiint b(\cos \theta) \mu'_* (g' - g)^2 + \iiint b(\cos \theta) g^2 \left((\mu'_*)^{1/2} - (\mu_*)^{1/2} \right)^2 \right) \\ &\quad - \frac{3}{4} \iiint b(\cos \theta) g^2 \left((\mu'_*)^{1/2} - (\mu_*)^{1/2} \right)^2. \end{aligned}$$

We apply now (2.2.5) with $f \sim g$, $g \sim \mu^{1/2}$ and the changing of variables $v \rightarrow v_*$, to get

$$\iiint b(\cos \theta) g^2 \left((\mu'_*)^{1/2} - (\mu_*)^{1/2} \right)^2 \leq C \|g\|_{L^2_s}^2 \|\sqrt{\mu}\|_{H^3_s}^2 \leq C \|g\|_{L^2_s}^2.$$

Therefore

$$(2.2.10) \quad (\mathcal{L}_1 g, g)_{L^2} \geq \frac{1}{4} \|g\|^2 - C \|g\|_{L^2_s}^2.$$

Using the fact that $\mathbf{P}\mathcal{L}g = 0$ and $\mathcal{L}\mathbf{P}g = 0$, we have

$$\begin{aligned} (\mathcal{L}g, g)_{L^2} &= (\mathcal{L}(\mathbf{I} - \mathbf{P})g, (\mathbf{I} - \mathbf{P})g)_{L^2} \\ &= (\mathcal{L}_1(\mathbf{I} - \mathbf{P})g, (\mathbf{I} - \mathbf{P})g)_{L^2} + (\mathcal{L}_2(\mathbf{I} - \mathbf{P})g, (\mathbf{I} - \mathbf{P})g)_{L^2} \\ &\geq \frac{1}{4} \|(\mathbf{I} - \mathbf{P})g\|^2 - C \|(\mathbf{I} - \mathbf{P})g\|_{L^2_s}^2, \end{aligned}$$

so that using Theorem 2.2, we can deduce

$$\| (I - P)g \|^2 \leq 4(\mathcal{L}g, g)_{L^2} + C\|(\mathbf{I} - \mathbf{P})g\|_{L^2_s}^2 \leq \tilde{C}(\mathcal{L}g, g)_{L^2}.$$

We have therefore proved the lower bound estimation.

For the upper bound estimation, we compute

$$\begin{aligned} (\mathcal{L}_1 g, g)_{L^2(\mathbb{R}_v^3)} &= \frac{1}{2} \iiint b(\cos \theta) \left((\mu'_*)^{1/2} g' - (\mu_*)^{1/2} g \right)^2 \\ &= \frac{1}{2} \iiint b(\cos \theta) \left((\mu'_*)^{1/2} (g' - g) + g \left((\mu'_*)^{1/2} - (\mu_*)^{1/2} \right) \right)^2 \\ &\leq \iiint b(\cos \theta) \mu'_* (g' - g)^2 + \iiint b(\cos \theta) g^2 \left((\mu'_*)^{1/2} - (\mu_*)^{1/2} \right)^2 \\ &\leq \|g\|^2, \end{aligned}$$

thus

$$(2.2.11) \quad (\mathcal{L}_1 g, g)_{L^2(\mathbb{R}_v^3)} \leq \|g\|^2,$$

and

$$\begin{aligned} (\mathcal{L}_2 g, g)_{L^2(\mathbb{R}_v^3)} &= \frac{1}{2} \iiint b(\cos \theta) \left(g'_*(\mu')^{1/2} - g_*(\mu)^{1/2} \right) \left((\mu'_*)^{1/2} g' - (\mu_*)^{1/2} g \right) \\ &\leq \frac{1}{4} \iiint b(\cos \theta) \left(g'_*(\mu')^{1/2} - g_*(\mu)^{1/2} \right)^2 + \frac{1}{4} \iiint b(\cos \theta) \left((\mu'_*)^{1/2} g' - (\mu_*)^{1/2} g \right)^2 \\ &\leq \frac{1}{4} \iiint b(\cos \theta) \left(g'_*(\mu')^{1/2} - (\mu)^{1/2} \right) + (\mu)^{1/2} (g'_* - g_*) \right)^2 \\ &\quad + \frac{1}{4} \iiint b(\cos \theta) \left((\mu'_*)^{1/2} (g' - g) + g \left((\mu'_*)^{1/2} - (\mu_*)^{1/2} \right) \right)^2 \\ &\leq \|g\|^2 \end{aligned}$$

so that all in all

$$(2.2.12) \quad (\mathcal{L}_2 g, g)_{L^2(\mathbb{R}_v^3)} \leq \|g\|^2,$$

The proof of Lemma 2.6 is now complete. \square

Lemma 2.7.

$$(2.2.13) \quad C_1 \|g\|_{H^s}^2 + C_2 \|g\|_{L^2_s}^2 \leq \|g\|^2.$$

Proof. By using the Borylev formula and some of the computations in [5], one has

$$\begin{aligned} A &= \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} b(k \cdot \sigma) \mu_* \left(g(v) - g(v') \right)^2 d\sigma dv_* dv \\ &= (2\pi)^{-3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \left\{ \hat{\mu}(0) |\hat{g}(\xi)|^2 + \hat{\mu}(0) |\hat{g}(\xi^+)|^2 \right. \\ &\quad \left. - 2\operatorname{Re} \hat{\mu}(\xi^-) \hat{g}(\xi^+) \bar{\hat{g}}(\xi) \right\} d\sigma d\xi \\ &\geq \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} |\hat{g}(\xi)|^2 \left\{ \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) (\hat{\mu}(0) - |\hat{\mu}(\xi^-)|) d\sigma \right\} d\xi \\ &\geq C_1 \int_{|\xi| \geq 1} |\xi|^{2s} |\hat{g}(\xi)|^2 d\xi \geq C_1 2^{-2s} \int_{|\xi| \geq 1} (1 + |\xi|^2)^s |\hat{g}(\xi)|^2 d\xi \\ &\geq C_1 2^{-2s} \|g\|_{H^s(\mathbb{R}_v^3)}^2 - C_1 \|g\|_{L^2(\mathbb{R}_v^3)}^2, \end{aligned}$$

where we used the fact that

$$\int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) (\hat{\mu}(0) - |\hat{\mu}(\xi^-)|) d\sigma d\xi \geq C_1 |\xi|^{2s}, \quad \forall |\xi| \geq 1.$$

On the other hand,

$$\begin{aligned} B &= \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} b(k \cdot \sigma) g_*^2(\mu^{1/2}(v) - \mu^{1/2}(v'))^2 d\sigma dv_* dv \\ &= (2\pi)^{-3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \left\{ \widehat{g^2}(0) |\widehat{\mu^{1/2}}(\xi)|^2 + \widehat{g^2}(0) |\widehat{\mu^{1/2}}(\xi^+)|^2 \right. \\ &\quad \left. - 2 \operatorname{Re} \widehat{g^2}(\xi^-) \widehat{\mu^{1/2}}(\xi^+) \overline{\widehat{\mu^{1/2}}(\xi)} \right\} d\sigma d\xi \\ &= \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \widehat{g^2}(0) |\widehat{\mu^{1/2}}(\xi^+) - \widehat{\mu^{1/2}}(\xi)|^2 d\sigma d\xi \\ &\quad + \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) (\widehat{g^2}(0) - \operatorname{Re} \widehat{g^2}(\xi^-)) \overline{\widehat{\mu^{1/2}}(\xi)} \widehat{\mu^{1/2}}(\xi^+) d\sigma d\xi \\ &= B_1 + B_2. \end{aligned}$$

For the first term on the right hand side,

$$\begin{aligned} B_1 &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \widehat{g^2}(0) |\widehat{\mu^{1/2}}(\xi^+) - \widehat{\mu^{1/2}}(\xi)|^2 d\sigma d\xi \\ &= \|g\|_{L^2(\mathbb{R}_v^3)}^2 \int_{\mathbb{R}_\xi^3} \mu(\xi) \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) |\widehat{\mu^{1/2}}(\xi^-) - 1|^2 d\sigma d\xi \\ &\geq C_2 \|g\|_{L^2(\mathbb{R}_v^3)}^2, \end{aligned}$$

where

$$C_2 = \int_{\mathbb{R}_\xi^3} \mu(\xi) \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) |\widehat{\mu^{1/2}}(\xi^-) - 1|^2 d\sigma d\xi > 0.$$

For the second term on the right hand side, by using

$$\overline{\widehat{\mu^{1/2}}(\xi)} \widehat{\mu^{1/2}}(\xi^+) \geq \mu(\xi),$$

we have

$$\begin{aligned} B_2 &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) (\widehat{g^2}(0) - \operatorname{Re} \widehat{g^2}(\xi^-)) \overline{\widehat{\mu^{1/2}}(\xi)} \widehat{\mu^{1/2}}(\xi^+) d\sigma d\xi \\ &\geq \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) (\widehat{g^2}(0) - \operatorname{Re} \widehat{g^2}(\xi^-)) \mu(\xi) d\sigma d\xi \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \int_{\mathbb{R}_v^3} g^2(v) (1 - \cos(\xi^- \cdot v)) dv \mu(\xi) d\sigma d\xi \\ &= \int_{\mathbb{R}_v^3} g^2(v) \left(\int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\left(\frac{v}{|v|} \cdot \sigma\right) (1 - \cos(\xi \cdot v^-)) \mu(\xi) d\sigma d\xi \right) dv \\ &= \int_{\mathbb{R}_v^3} g^2(v) \left(\int_{\mathbb{S}^2} b\left(\frac{v}{|v|} \cdot \sigma\right) (\widehat{\mu}(0) - \widehat{\mu}(v^-)) d\sigma \right) dv \\ &\geq C_3 \int_{|v| \geq 1} g^2(v) |v|^{2s} dv \geq C_3 2^{-2s} \|g\|_{L_v^2(\mathbb{R}_v^3)}^2 - C_3 \|g\|_{L^2(\mathbb{R}_v^3)}^2, \end{aligned}$$

where we have used the exchange $\frac{v}{|v|} \cdot \sigma \leftrightarrow \frac{\xi}{|\xi|} \cdot \sigma$ and Proposition 2 of [5] with ξ replaced by v .

Finally, for $0 < \lambda_1, \lambda_2 < 1$,

$$\begin{aligned} \|g\|^2 &= A + B_1 + B_2 \geq \lambda_1 A + B_1 + \lambda_2 B_2 \\ &\geq \lambda_1 C_1 2^{-2s} \|g\|_{H^s(\mathbb{R}^3)}^2 - \lambda_1 C_1 \|g\|_{L^2(\mathbb{R}^3)}^2 \\ &\quad + C_2 \|g\|_{L^2(\mathbb{R}^3)}^2 + \lambda_2 C_3 2^{-2s} \|g\|_{L^2_s(\mathbb{R}^3)}^2 - \lambda_2 C_3 \|g\|_{L^2(\mathbb{R}^3)}^2 \\ &\geq \lambda_1 C_1 2^{-2s} \|g\|_{H^s(\mathbb{R}^3)}^2 + (C_2 - \lambda_1 C_1 - \lambda_2 C_3) \|g\|_{L^2(\mathbb{R}^3)}^2 + \lambda_2 C_3 2^{-2s} \|g\|_{L^2_s(\mathbb{R}^3)}^2, \end{aligned}$$

choosing λ_1, λ_2 such that

$$C_2 - \lambda_1 C_1 - \lambda_2 C_3 \geq \frac{1}{2} C_2,$$

we can conclude the proof of the Lemma. \square

2.3. Upper bound estimates.

Lemma 2.8.

$$(2.3.1) \quad \iiint b(\cos \theta) f_*^2 (g' - g)^2 \leq C \|f\|_{L^2_s}^2 \|g\|^2.$$

Proof. Since we want to estimate the right hand side of (2.3.1) by our triple norm, we cannot use (2.2.4) as in the proof of Lemma 2.4. So we need a different argument which will be based on Bolyev formula, which gives

$$\begin{aligned} &\iiint b(\cos \theta) \mu_*(g' - g)^2 dv_* d\sigma dv \\ &= \frac{1}{(2\pi)^3} \iint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) (\hat{\mu}(0) |\hat{g}(\xi)|^2 + |\hat{g}(\xi^+)|^2 - 2\operatorname{Re} \hat{\mu}(\xi^-) \hat{g}(\xi^+) \bar{\hat{g}}(\xi)) d\xi d\sigma \\ &= \frac{1}{(2\pi)^3} \iint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) (\hat{\mu}(0) |\hat{g}(\xi) - \hat{g}(\xi^+)|^2 + 2\operatorname{Re} (\hat{\mu}(0) - \hat{\mu}(\xi^-)) \hat{g}(\xi^+) \bar{\hat{g}}(\xi)) d\xi d\sigma, \end{aligned}$$

and

$$\begin{aligned} &\iiint b(\cos \theta) f_*^2 (g' - g)^2 dv_* d\sigma dv \\ &= \frac{1}{(2\pi)^3} \iint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) (\widehat{f^2}(0) |\hat{g}(\xi) - \hat{g}(\xi^+)|^2 + 2\operatorname{Re} (\widehat{f^2}(0) - \widehat{f^2}(\xi^-)) \hat{g}(\xi^+) \bar{\hat{g}}(\xi)) d\xi d\sigma. \end{aligned}$$

Since $\hat{\mu}(0) = 1$, $\widehat{f^2}(0) = \|f\|_{L^2}^2$, we obtain

$$\begin{aligned} &\iiint b(\cos \theta) f_*^2 (g' - g)^2 dv_* d\sigma dv \\ &= \|f\|_{L^2}^2 \iiint b(\cos \theta) \mu_*(g' - g)^2 dv_* d\sigma dv \\ &\quad - \frac{2}{(2\pi)^3} \|f\|_{L^2}^2 \iint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \operatorname{Re} (\hat{\mu}(0) - \hat{\mu}(\xi^-)) \hat{g}(\xi^+) \bar{\hat{g}}(\xi) d\xi d\sigma \\ &\quad + \frac{2}{(2\pi)^3} \iint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \operatorname{Re} (\widehat{f^2}(0) - \widehat{f^2}(\xi^-)) \hat{g}(\xi^+) \bar{\hat{g}}(\xi) d\xi d\sigma. \end{aligned}$$

Let us study the last term

$$\begin{aligned} & \iint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \operatorname{Re}\left(\widehat{f^2}(0) - \widehat{f^2}(\xi^-)\right) \widehat{g}(\xi^+) \overline{\widehat{g}(\xi)} d\xi d\sigma \\ & \leq C \iint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \left|\widehat{f^2}(0) - \widehat{f^2}(\xi^-)\right| \left(|\widehat{g}(\xi^+)|^2 + |\widehat{g}(\xi)|^2\right) d\xi d\sigma \\ & \leq C \iint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \left|\widehat{f^2}(0) - \widehat{f^2}(\xi^-)\right| |\widehat{g}(\xi)|^2 d\xi d\sigma, \end{aligned}$$

where we used the change of variables $\xi^+ \rightarrow \xi$ which is regular. Next

$$\int b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \left|\widehat{f^2}(0) - \widehat{f^2}(\xi^-)\right| d\sigma \leq \int b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \left(\int_{\mathbb{S}^2} f^2(v) \left|1 - e^{-iv \cdot \xi^-}\right| dv\right) d\sigma.$$

So we need to study the following quantity

$$\int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \left|1 - e^{-iv \cdot \xi^-}\right| d\sigma.$$

If $|v \cdot \xi| \geq \frac{2}{\pi}$, we choose $\delta = \frac{1}{|\xi| |v|}$ ($\leq \pi/2$), then $|1 - e^{-iv \cdot \xi^-}| \leq |v \cdot \xi| \sin \theta$ for any $0 \leq \theta \leq \delta$, and if $\frac{\pi}{2} \geq \theta \geq \delta$, we use $|1 - e^{-iv \cdot \xi^-}| \leq 2$. Then

$$\begin{aligned} \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \left|1 - e^{-iv \cdot \xi^-}\right| d\sigma & \leq C |v \cdot \xi| \int_0^\delta \frac{1}{\theta^{1+2s}} \sin \theta d\theta + C' \int_\delta^{\pi/2} \frac{1}{\theta^{1+2s}} d\theta \\ & \leq C |v \cdot \xi| \delta^{-2s+1} + C' \delta^{-2s} \leq \tilde{C} |v|^{2s} |\xi|^{2s}, \end{aligned}$$

If $|v \cdot \xi| \leq \frac{2}{\pi}$, we have directly

$$\begin{aligned} \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \left|1 - e^{-iv \cdot \xi^-}\right| d\sigma & \leq C \int_0^{\pi/2} \frac{1}{\theta^{1+2s}} |v \cdot \xi| \sin \theta d\theta \\ & \leq \tilde{C} |v \cdot \xi| \leq \tilde{C} |v|^{2s} |\xi|^{2s}. \end{aligned}$$

Thus, we have proved

$$\left| \iint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \operatorname{Re}\left(\widehat{f^2}(0) - \widehat{f^2}(\xi^-)\right) \widehat{g}(\xi^+) \overline{\widehat{g}(\xi)} d\xi d\sigma \right| \leq C \|f\|_{L_x^2}^2 \|g\|_{H^s}^2,$$

and similarly

$$\left| \iint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \operatorname{Re}\left(\widehat{\mu}(0) - \widehat{\mu}(\xi^-)\right) \widehat{g}(\xi^+) \overline{\widehat{g}(\xi)} d\xi d\sigma \right| \leq C \|\sqrt{\mu}\|_{L_x^2}^2 \|g\|_{H^s}^2.$$

Thus, we have proved (2.3.1) by using (2.2.13). \square

Proposition 2.9.

$$(2.3.2) \quad \left| (\Gamma(f, g), h)_{L^2(\mathbb{R}^3)} \right| \leq C \left(\|f\|_{L_x^2} \|g\| + \|g\|_{L_x^2} \|f\| \right) \|h\|.$$

Proof.

$$\begin{aligned}
(\Gamma(f, g), h)_{L^2(\mathbb{R}^3)} &= (\mu^{-1/2} Q(\mu^{1/2} f, \mu^{1/2} g), h)_{L^2(\mathbb{R}^3)} \\
&= \iiint b(\cos \theta) \mu_*^{1/2} (f'_* g' - f_* g) h \\
&= \frac{1}{2} \iiint b(\cos \theta) (f'_* g' - f_* g) (\mu_*^{1/2} h - \mu_*^{1/2'} h') \\
&\leq \frac{1}{2} \left(\iiint (\cos \theta) (f'_* g' - f_* g)^2 \right)^{1/2} \\
&\quad \times \left(\iiint b(\cos \theta) (\mu_*^{1/2} h - \mu_*^{1/2'} h')^2 \right)^{1/2} \\
&\leq \frac{1}{2} (A)^{1/2} \times (B)^{1/2}.
\end{aligned}$$

One has

$$\begin{aligned}
(B) &= \iiint b(\cos \theta) \left((\mu_*')^{1/2} (h' - h) + h((\mu_*')^{1/2} - (\mu_*)^{1/2}) \right)^2 \\
&\leq 2 \iiint b(\cos \theta) \left\{ \mu_*' (h' - h)^2 + h^2 ((\mu_*')^{1/2} - (\mu_*)^{1/2})^2 \right\} \\
&\leq 2 \iiint b(\cos \theta) \mu_* (h' - h)^2 + 2 \iiint b(\cos \theta) h^2 ((\mu_*')^{1/2} - \mu^{1/2})^2 = 2 \|h\|^2,
\end{aligned}$$

where we used the change of variables $v \rightarrow v'$ for the first term and $v \rightarrow v_*$ for the second term. And similarly

$$\begin{aligned}
(A) &= \iiint b(\cos \theta) \left(f'_* (g' - g) + g(f'_* - f_*) \right)^2 \\
&\leq 2 \iiint b(\cos \theta) \left\{ f_*'^2 (g' - g)^2 + g^2 (f'_* - f_*)^2 \right\} \\
&\leq 2 \iiint b(\cos \theta) f_*'^2 (g' - g)^2 + 2 \iiint b(\cos \theta) g_*'^2 (f' - f)^2.
\end{aligned}$$

Using now (2.3.1), we have

$$(A) \leq C \left(\|f\|_{L^2_x}^2 \|g\|^2 + \|g\|_{L^2_x}^2 \|f\|^2 \right).$$

□

3. COMMUTATORS ESTIMATES

3.1. Non isotropic norms in full space. We now define the norm associated with the collision operator on the full space of variables (x, v) and consider t as a parameter in this section. For $m \in \mathbb{N}, \ell \in \mathbb{R}$, we define

$$\mathcal{B}_\ell^m(\mathbb{R}_{x,v}^6) = \left\{ g \in \mathcal{S}'(\mathbb{R}_{x,v}^6); \|g\|_{\mathcal{B}_\ell^m(\mathbb{R}^6)}^2 = \sum_{|\alpha| \leq m} \int_{\mathbb{R}^3} \|W^\ell \partial_{x,v}^\alpha g(x, \cdot)\|^2 dx < +\infty \right\}$$

where $\|\cdot\|$ is norm w.r.t. the velocity variable $v \in \mathbb{R}_v^3$ defined in (2.2.1).

We have firstly

Lemma 3.1. For any $\ell \geq 0, \alpha, \beta \in \mathbb{N}^3$,

$$(3.1.1) \quad \|\|W^\ell \partial_x^\alpha \partial_v^\beta \mathbf{P}g\|\|_{\mathcal{B}_0^0(\mathbb{R}^6)} + \|\| \mathbf{P}(W^\ell \partial_x^\alpha \partial_v^\beta g)\|\|_{\mathcal{B}_0^0(\mathbb{R}^6)} \leq C_{\ell, \beta} \|\| \partial_x^\alpha g\|\|_{L^2(\mathbb{R}^6)}^2,$$

$$(3.1.2) \quad C_0 \|\|g\|\|_{\mathcal{B}_0^0(\mathbb{R}^6)}^2 - C_2 \|g\|_{L^2(\mathbb{R}^6)}^2 \leq (\mathcal{L}g, g)_{L^2(\mathbb{R}_{x,v}^6)} \leq C_3 \|\|g\|\|_{\mathcal{B}_0^0(\mathbb{R}^6)}^2$$

and

$$(3.1.3) \quad \|g\|_{L_x^2(\mathbb{R}^6)}^2 + \|g\|_{L^2(\mathbb{R}_x^3, H^s(\mathbb{R}_v^3))}^2 \leq C \|\|g\|\|_{\mathcal{B}_0^0(\mathbb{R}^6)}^2 \leq C \|g\|_{L^2(\mathbb{R}_x^3, H^s(\mathbb{R}_v^3))}^2.$$

Proof. By definition of the projection operator \mathbf{P} , we have

$$\mathbf{P}g = a_g(t, x)\mu^{1/2} + \sum_{j=1}^3 b_{g,j}(t, x) v_j \mu^{1/2} + c_g(t, x)|v|^2 \mu^{1/2}$$

with

$$a_g(t, x) = \int_{\mathbb{R}_v^3} g(t, x, v) \mu^{1/2}(v) dv, \quad c_g(t, x) = \int_v g(t, x, v) |v|^2 \mu^{1/2}(v) dv,$$

and

$$b_{g,j}(t, x) = \int_v g(t, x, v) v_j \mu^{1/2}(v) dv, \quad j = 1, 2, 3.$$

Thus (3.1.1) can be obtained by integration by parts. For (3.1.2), we use (2.2.2) and (2.2.8)

$$\begin{aligned} \|\|g\|\|_{\mathcal{B}_0^0(\mathbb{R}^6)}^2 &\geq C(\mathcal{L}g, g)_{L^2(\mathbb{R}_{x,v}^6)} \geq C_0 \|\|g - \mathbf{P}g\|\|_{\mathcal{B}_0^0(\mathbb{R}^6)}^2 \\ &\geq \frac{C_0}{2} \|\|g\|\|_{\mathcal{B}_0^0(\mathbb{R}^6)}^2 - C_0 \|\|\mathbf{P}g\|\|_{\mathcal{B}_0^0(\mathbb{R}^6)}^2 \\ &\geq \frac{C_0}{2} \|\|g\|\|_{\mathcal{B}_0^0(\mathbb{R}^6)}^2 - C_2 \|g\|_{L^2(\mathbb{R}^6)}^2. \end{aligned}$$

Finally, (3.1.3) is obtained by integration of (2.2.2) and (2.2.13). \square

We introduce the following notation, for $\gamma \in \mathbb{N}^3$,

$$(3.1.4) \quad \mathcal{T}(F, G, \mu_\gamma) = Q(\mu_\gamma F, G) + \iint b(\cos \theta) ((\mu_\gamma)_* - (\mu_\gamma)'_*) F'_* G' dv_* d\sigma$$

where $\mu_\gamma = p_\gamma(v) \sqrt{\mu(v)} = \partial^\gamma(\sqrt{\mu})$ is a Maxwellian type function of variable v . Then (2.1.5) implies

$$\mathcal{T}(f, g, \sqrt{\mu}) = \Gamma(f, g).$$

We have also the Leibniz formula

$$(3.1.5) \quad \partial_x^\alpha \partial_v^\beta \Gamma(f, g) = \sum_{\alpha_1 + \alpha_2 = \alpha, \beta_1 + \beta_2 + \beta_3 = \beta} C_{\beta_1, \beta_2, \beta_3}^{\alpha_1, \alpha_2} \mathcal{T}(\partial_x^{\alpha_1} \partial_v^{\beta_1} f, \partial_x^{\alpha_2} \partial_v^{\beta_2} g, \mu_{\beta_3}).$$

3.2. Estimates of commutators with weighted. First of all, let us recall the following Lemma of [9],

Lemma 3.2. Let $\ell \geq 0, 0 < s < 1/2$, then there exists $C > 0$ such that

$$(3.2.1) \quad \left| \left((W^\ell Q(f, g) - Q(f, W^\ell g)), h \right)_{L^2(\mathbb{R}^3)} \right| \leq C \|f\|_{L_t^1(\mathbb{R}^3)} \|g\|_{L_t^2(\mathbb{R}^3)} \|h\|_{L^2(\mathbb{R}^3)}.$$

Proposition 3.3. For $\ell \geq 0$,

$$(3.2.2) \quad \left(W^\ell \mathcal{T}(F, G, \mu_\gamma) - \mathcal{T}(F, W^\ell G, \mu_\gamma), h \right)_{L^2(\mathbb{R}^3)} \leq C \|F\|_{L_t^2} \|G\|_{L_t^2} \|h\|_{L_t^2}.$$

Proof. We have by (3.1.4),

$$\begin{aligned} & \left(W^\ell \mathcal{T}(F, G, \mu_\gamma) - \mathcal{T}(F, W^\ell G, \mu_\gamma), h \right)_{L^2(\mathbb{R}_v^3)} \\ &= \left(W^\ell Q(\mu_\gamma F, G) - Q(\mu_\gamma F, W^\ell G), h \right)_{L^2(\mathbb{R}_v^3)} \\ &+ \iiint b(\cos \theta) (\mu_{\gamma_*} - \mu'_{\gamma_*}) F'_* G' (W^\ell - W^{\ell'}) h \\ &= (B_1) + (B_2). \end{aligned}$$

Using Lemma 3.2, we have

$$(B_1) \leq C \|\mu_\gamma F\|_{L_t^1} \|G\|_{L_t^2} \|h\|_{L^2} \leq C \|F\|_{L^2} \|G\|_{L_t^2} \|h\|_{L^2}.$$

For (B_2) , using $0 < s < 1/2$, we have

$$(B_2) \leq \left(\iiint b(\cos \theta) |F'_*|^2 |G'|^2 \frac{|W^\ell - W^{\ell'}|^2}{\sin \theta} \right)^{1/2} \left(\iiint b(\cos \theta) \sin \theta (\mu_{\gamma_*} - \mu'_{\gamma_*})^2 |h|^2 \right)^{1/2}.$$

Using (2.2.5), we get

$$\iiint b(\cos \theta) (\mu_{\gamma_*} - \mu'_{\gamma_*})^2 |h|^2 \leq C \|\mu_\gamma\|_{H_x^s}^2 \|h\|_{L^2}^2.$$

Using

$$|W^\ell - W^{\ell'}|^2 \leq \sin^2 \theta \left((W_*^\ell)^2 + (W^{\ell'})^2 \right) \leq \sin^2 \theta (W_*^\ell)^2 (W^{\ell'})^2,$$

we get

$$\begin{aligned} \iiint b(\cos \theta) |F'_*|^2 |G'|^2 \frac{|W^\ell - W^{\ell'}|^2}{\sin \theta} &\leq \iiint b(\cos \theta) \sin \theta (W^\ell F)_*^2 (W^\ell G)^2 \\ &\leq C \|F\|_{L_t^2}^2 \|G\|_{L_t^2}^2, \end{aligned}$$

thus ending the proof of Proposition 3.3. \square

Proposition 3.4.

$$(3.2.3) \quad \left| \left(\mathcal{T}(F, G, \mu_\gamma), h \right)_{L^2(\mathbb{R}_v^3)} \right| \leq C \left(\|F\|_{L_x^2} \|G\| + \|G\|_{L_x^2} \|F\| \right) \|h\|$$

Proof.

$$\begin{aligned} \left(\mathcal{T}(F, G, \mu_\gamma), h \right)_{L^2(\mathbb{R}_v^3)} &= \iiint b(\cos \theta) (\mu_{\gamma_*})^{1/2} (F'_* G' - F_* G) h \\ &= \frac{1}{2} \iiint b(\cos \theta) (F'_* G' - F_* G) \left((\mu_{\gamma_*})^{1/2} h - (\mu'_{\gamma_*})^{1/2} h' \right) \\ &\leq \frac{1}{2} \left(\iiint (\cos \theta) (F'_* G' - F_* G)^2 \right)^{1/2} \\ &\quad \times \left(\iiint b(\cos \theta) \left((\mu_{\gamma_*})^{1/2} h - (\mu'_{\gamma_*})^{1/2} h' \right)^2 \right)^{1/2} \\ &\leq \frac{1}{2} (\tilde{A})^{1/2} \times (\tilde{B})^{1/2}. \end{aligned}$$

By using the estimate of term (A) in the proof of Proposition 2.9.

$$(\tilde{A}) \leq C \left(\|F\|_{L_x^2}^2 \|G\|^2 + \|G\|_{L_x^2}^2 \|F\|^2 \right)$$

and

$$(\tilde{B}) \leq C \left(\|\mu_\gamma\|_{L_x^2}^2 \|h\|^2 + \|h\|_{L_x^2}^2 \|\mu_\gamma\|^2 \right) \leq C \|h\|^2.$$

□

Proposition 3.5. *For any $\ell \geq 3$, and $N \geq 3$, we have, for all $\beta \in \mathbb{N}^6$, $|\beta| \leq N$,*

$$(3.2.4) \quad \left| \left(W^\ell \partial_{x,v}^\beta \Gamma(f, g), h \right)_{L^2(\mathbb{R}_{x,v}^6)} \right| \leq C \|f\|_{H_t^N(\mathbb{R}^6)} \|g\|_{\mathcal{B}_t^N(\mathbb{R}^6)} \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)}.$$

Remark 3.6. *In fact, for this proposition, we need to assume the condition $\ell > \frac{3}{2} + 2s$, and $N > \frac{3}{2} + 2s$. But in order to simplify the notations, we choose integer indexes ℓ, N , for $0 < s < 1/2$, such that $\ell \geq 3, N \geq 3$.*

Proof. Using Leibniz formula (3.1.5),

$$(3.2.5) \quad \begin{aligned} \left(W^\ell \partial_{x,v}^\beta \Gamma(f, g), h \right)_{L^2(\mathbb{R}_{x,v}^6)} &= \sum C_{\beta_2, \beta_3}^{\beta_1} \left(\mathcal{T}(\partial^{\beta_1} f, W^\ell \partial^{\beta_2} g, \mu_{\beta_3}), h \right)_{L^2(\mathbb{R}_{x,v}^6)} \\ &+ \sum C_{\beta_2, \beta_3}^{\beta_1} \left(W^\ell \mathcal{T}(\partial^{\beta_1} f, \partial^{\beta_2} g, \mu_{\beta_3}) - \mathcal{T}(\partial^{\beta_1} f, W^\ell \partial^{\beta_2} g, \mu_{\beta_3}), h \right)_{L^2(\mathbb{R}_{x,v}^6)} \end{aligned}$$

Then (3.2.2) implies

$$\begin{aligned} &\left| \left(W^\ell \mathcal{T}(\partial^{\beta_1} f, \partial^{\beta_2} g, \mu_{\beta_3}) - \mathcal{T}(\partial^{\beta_1} f, W^\ell \partial^{\beta_2} g, \mu_{\beta_3}), h \right)_{L^2(\mathbb{R}_{x,v}^6)} \right| \\ &\leq C \left(\int_{\mathbb{R}_x^3} \|\partial^{\beta_1} f\|_{L_t^\infty(\mathbb{R}_v^3)}^2 \|\partial^{\beta_2} g\|_{L_t^2(\mathbb{R}_v^3)}^2 dx \right)^{1/2} \|h\|_{L_s^2(\mathbb{R}_{x,v}^6)} \\ &\leq C \begin{cases} \|\partial^{\beta_1} f\|_{L^\infty(\mathbb{R}_x^3; L_s^2(\mathbb{R}_v^3))} \|\partial^{\beta_2} g\|_{L_t^2(\mathbb{R}_{x,v}^6)} \|h\|_{L_s^2(\mathbb{R}_{x,v}^6)} & \text{if } |\beta_1| \leq 1; \\ \|\partial^{\beta_1} f\|_{L_t^2(\mathbb{R}_{x,v}^6)} \|\partial^{\beta_2} g\|_{L^\infty(\mathbb{R}_x^3; L_s^2(\mathbb{R}_v^3))} \|h\|_{L_s^2(\mathbb{R}_{x,v}^6)} & \text{if } |\beta_1| \geq 2. \end{cases} \end{aligned}$$

Since $|\beta_1| \leq 1$ implies $|\beta_1| + 3/2 < 3 \leq N$ and $|\beta_1| \geq 2$ implies $|\beta_2| + 3/2 < |\beta|$, thus, we have

$$(3.2.6) \quad \begin{aligned} &\left| \left(W^\ell \mathcal{T}(\partial^{\beta_1} f, \partial^{\beta_2} g, \mu_{\beta_3}) - \mathcal{T}(\partial^{\beta_1} f, W^\ell \partial^{\beta_2} g, \mu_{\beta_3}), h \right)_{L^2(\mathbb{R}_{x,v}^6)} \right| \\ &\leq C \|f\|_{H_t^N(\mathbb{R}^6)} \|g\|_{H_t^{|\beta_1|}(\mathbb{R}^6)} \|h\|_{\mathcal{B}_0^0(\mathbb{R}_{x,v}^6)}. \end{aligned}$$

On the other hand, (3.2.3) implies that, if $|\beta_1| \leq 1$, then $|\beta_1| + \frac{3}{2} + s < 3 \leq N$,

$$\begin{aligned} &\left| \left(\mathcal{T}(\partial^{\beta_1} f, W^\ell \partial^{\beta_2} g, \mu_{\beta_3}), h \right)_{L^2(\mathbb{R}_{x,v}^6)} \right| \\ &\leq C \left(\int_{\mathbb{R}_x^3} \|\partial^{\beta_1} f\|_{L_s^2(\mathbb{R}_v^3)}^2 \left(\|W^\ell \partial^{\beta_2} g\|^2 + \|W^\ell \partial^{\beta_2} g\|_{H^s(\mathbb{R}_v^3)}^2 \right) \right. \\ &\quad \left. + \int_{\mathbb{R}_x^3} \|W^\ell \partial^{\beta_2} g\|_{L_s^2(\mathbb{R}_v^3)}^2 \left(\|\partial^{\beta_1} f\|^2 + \|\partial^{\beta_1} f\|_{H^s(\mathbb{R}_v^3)}^2 \right) dx \right)^{1/2} \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)} \\ &\leq C \left(\|\partial^{\beta_1} f\|_{L^\infty(\mathbb{R}_x^3; L_s^2(\mathbb{R}_v^3))} + \|\partial^{\beta_1} f\|_{L^\infty(\mathbb{R}_x^3; H_s^s(\mathbb{R}_v^3))} \right) \|g\|_{\mathcal{B}_t^{|\beta_2|}(\mathbb{R}^6)} \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)} \\ &\leq C \|f\|_{H_s^{|\beta_1|+3/2+s+\epsilon}(\mathbb{R}^6)} \|g\|_{\mathcal{B}_t^{|\beta_2|}(\mathbb{R}^6)} \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)}, \end{aligned}$$

so that, for $|\beta_1| \leq 1$,

$$(3.2.7) \quad \left| \left(\mathcal{T}(\partial^{\beta_1} f, W^\ell \partial^{\beta_2} g, \mu_{\beta_3}), h \right)_{L^2(\mathbb{R}_{x,v}^6)} \right| \leq C \|f\|_{H_s^3(\mathbb{R}^6)} \|g\|_{\mathcal{B}_t^{|\beta_2|}(\mathbb{R}^6)} \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)}.$$

Now for $|\beta_1| \geq 2$, there are two cases: $2 \leq |\beta_1| \leq |\beta| - 1$ then $|\beta_2| = |\beta| - |\beta_1| - |\beta_3| \leq |\beta| - 2$,

$$\begin{aligned}
& \left| (\mathcal{T}(\partial^{\beta_1} f, W^\ell \partial^{\beta_2} g, \mu_{\beta_3}), h)_{L^2(\mathbb{R}_{x,v}^6)} \right| \\
& \leq C \left(\|f\|_{H_s^{|\beta_1|}(\mathbb{R}^6)} \|W^\ell \partial^{\beta_2} g\|_{L^\infty(\mathbb{R}_x^3; H_s^2(\mathbb{R}_v^3))} \right. \\
& \quad \left. + \|W^\ell \partial^{\beta_2} g\|_{L^\infty(\mathbb{R}_x^3; L_s^2(\mathbb{R}_v^3))} \|\partial^{\beta_1} f\|_{L^2(\mathbb{R}_x^3; H_s^2(\mathbb{R}_v^3))} \right) \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)} \\
& \leq C \|f\|_{H_s^{|\beta_1|+s}(\mathbb{R}^6)} \|W^\ell \Lambda_x^{3/2+\epsilon} \Lambda_v^s \partial^{\beta_2} g\|_{L_s^2(\mathbb{R}^6)}^2 \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)} \\
& \leq C \|f\|_{H_s^{|\beta_1|+s}(\mathbb{R}^6)} \|g\|_{\mathcal{B}_t^{|\beta_1|-2+3/2+s+\epsilon}(\mathbb{R}^6)} \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)} \\
& \leq C \|f\|_{H_s^{|\beta_1|}(\mathbb{R}^6)} \|g\|_{\mathcal{B}_t^{|\beta_1|}(\mathbb{R}^6)} \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)}.
\end{aligned}$$

We study now the case: $\beta_1 = \beta$. Then we have

$$(\mathcal{T}(\partial^\beta f, W^\ell g, \sqrt{\mu}), h)_{L^2(\mathbb{R}_{x,v}^6)} = (\Gamma(\partial^\beta f, W^\ell g), h)_{L^2(\mathbb{R}_{x,v}^6)}$$

Since we don't admits the trip norm of f in the left hand side, we can't use the estimate (2.3.2) to close the proof.

We use (2.1.5),

$$\begin{aligned}
& (\Gamma(\partial^\beta f, W^\ell g), h)_{L^2(\mathbb{R}_{x,v}^6)} = (Q(\sqrt{\mu} \partial^\beta f, W^\ell g), h)_{L^2(\mathbb{R}_{x,v}^6)} \\
& + \int \iiint b(\cos \theta) (\sqrt{\mu_*} - \sqrt{\mu'_*}) (\partial^\beta f)'_* (W^\ell g)' h dv_* d\sigma dv dx.
\end{aligned}$$

Then (2.1.2) with $m = 0, \alpha = -s$,

$$\begin{aligned}
\left| (Q(\sqrt{\mu} \partial^\beta f, W^\ell g), h)_{L^2(\mathbb{R}_{x,v}^6)} \right| & \leq C \|h\|_{L_s^2(\mathbb{R}^6)} \|\sqrt{\mu} \partial^\beta f\|_{L^2(\mathbb{R}_x^3; L_s^1(\mathbb{R}_v^3))} \|W^\ell g\|_{L^\infty(\mathbb{R}_x^3; H_s^{2s}(\mathbb{R}_v^3))} \\
& \leq C \|f\|_{H^{|\beta|}(\mathbb{R}^6)} \|W^\ell g\|_{H_s^{3/2+2s+\epsilon}(\mathbb{R}^6)} \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \int \iiint b(\cos \theta) (\sqrt{\mu_*} - \sqrt{\mu'_*}) (\partial^\beta f)'_* (W^\ell g)' h dv_* d\sigma dv dx \\
& = \int \iiint b(\cos \theta) (\sqrt{\mu_*} - \sqrt{\mu'_*}) (\partial^\beta f)'_* (W^\ell g)' (h - h') dv_* d\sigma dv dx \\
& + \int \iiint b(\cos \theta) (\sqrt{\mu_*} - \sqrt{\mu'_*}) (\partial^\beta f)'_* (W^\ell g)' h' dv_* d\sigma dv dx \\
& = (B_1) + (B_2).
\end{aligned}$$

$$\begin{aligned}
|(B_1)| & \leq \left(\int \iiint b(\cos \theta) |(\partial^\beta f)'_*|^2 |(W^\ell g)'|^2 (\mu_*^{1/4} - (\mu'_*)^{1/4})^2 dv_* d\sigma dv dx \right)^{1/2} \\
& \quad \times \left(\int \iiint b(\cos \theta) (\mu_*^{1/4} + (\mu'_*)^{1/4})^2 (h - h')^2 dv_* d\sigma dv dx \right)^{1/2}.
\end{aligned}$$

Lemma 2.5 gives

$$\begin{aligned}
& \int \iiint b(\cos \theta) |(\partial^\beta f)'_*|^2 |(W^\ell g)'|^2 (\mu_*^{1/4} - (\mu'_*)^{1/4})^2 dv_* d\sigma dv dx \\
& \leq C \int_{\mathbb{R}_x^3} \int_{\mathbb{R}_{v,v_*}^6} |(\partial^\beta f)'_*|^2 |(W^\ell g)'|^2 \langle v \rangle^{2s} \langle v_* \rangle^{2s} dv_* ddv dx \\
& \leq C \int_{\mathbb{R}_x^3} \|\partial^\beta f\|_{L^2_s(\mathbb{R}_v^3)}^2 \|W^\ell g\|_{L^2_s(\mathbb{R}_v^3)}^2 dx \leq C \|\partial^\beta f\|_{L^2_s(\mathbb{R}^6)}^2 \|W^\ell g\|_{L^\infty(\mathbb{R}_x^3; L^2_s(\mathbb{R}_v^3))}^2 \\
& \leq C \|f\|_{H^N_s(\mathbb{R}^6)}^2 \|W^\ell \Lambda_x^{3/2+\epsilon} g\|_{L^2_s(\mathbb{R}^6)}^2 \leq C \|f\|_{H^N_s(\mathbb{R}^6)}^2 \|g\|_{\mathcal{B}^2_t(\mathbb{R}^6)}^2,
\end{aligned}$$

and Lemma 2.8 gives

$$\begin{aligned}
& \int \iiint b(\cos \theta) (\mu_*^{1/4} + (\mu'_*)^{1/4})^2 (h - h')^2 dv_* d\sigma dv dx \\
& \leq 4 \int \iiint b(\cos \theta) \mu_*^{1/2} (h - h')^2 dv_* d\sigma dv dx \leq C \|h\|_{\mathcal{B}^0_0(\mathbb{R}^6)}^2,
\end{aligned}$$

so that

$$|(B_1)| \leq C \|f\|_{H^N_s(\mathbb{R}^6)} \|g\|_{\mathcal{B}^2_t(\mathbb{R}^6)} \|h\|_{\mathcal{B}^0_0(\mathbb{R}^6)}.$$

For the term (B_2) , we have

$$\begin{aligned}
& \left| \int \iiint b(\cos \theta) (\sqrt{\mu_*} - \sqrt{\mu'_*}) (\partial^\beta f)'_* (W^\ell g)' h' dv_* d\sigma dv dx \right| \\
& = \left| \int \iiint b(\cos \theta) (\sqrt{\mu_*} - \sqrt{\mu'_*}) (\partial^\beta f)_* (W^\ell g) h dv_* d\sigma dv dx \right| \\
& \leq C \int_{\mathbb{R}_x^3} \int_{\mathbb{R}_{v,v_*}^6} |(\partial^\beta f)_*| |W^\ell g| |h| \langle v \rangle^{2s} \langle v_* \rangle^{2s} dv_* ddv dx \\
& \leq C \int_{\mathbb{R}_x^3} \|\partial^\beta f\|_{L^1_{2s}(\mathbb{R}_v^3)} \|W^\ell g\|_{L^2_s(\mathbb{R}_v^3)} \|h\|_{L^2_s(\mathbb{R}_v^3)} dx \\
& \leq C \|\partial^\beta f\|_{L^2(\mathbb{R}_x^3; L^2_{3/2+2s+\epsilon}(\mathbb{R}_v^3))} \|W^\ell g\|_{L^\infty(\mathbb{R}_x^3; L^2_s(\mathbb{R}_v^3))} \|h\|_{L^2_s(\mathbb{R}^6)}.
\end{aligned}$$

We get also

$$|(B_2)| \leq C \|f\|_{H^N_{3/2+2s+\epsilon}(\mathbb{R}^6)} \|g\|_{\mathcal{B}^2_t(\mathbb{R}^6)} \|h\|_{\mathcal{B}^0_0(\mathbb{R}^6)}.$$

So that

$$(3.2.8) \quad \left| \left(\Gamma(\partial^\beta f, W^\ell g), h \right)_{L^2(\mathbb{R}_{x,v}^6)} \right| \leq C \|f\|_{H^N_{3/2+2s+\epsilon}(\mathbb{R}^6)} \|g\|_{\mathcal{B}^2_t(\mathbb{R}^6)} \|h\|_{\mathcal{B}^0_0(\mathbb{R}^6)}.$$

Finally, for $|\beta_1| \geq 2$, by using $3/2 + 2s < 3 \leq N$, we have also

$$(3.2.9) \quad \left| \left(\mathcal{T}(\partial^{\beta_1} f, W^\ell \partial^{\beta_2} g, \mu_{\beta_3}), h \right)_{L^2(\mathbb{R}_{x,v}^6)} \right| \leq C \|f\|_{H^N_t(\mathbb{R}^6)} \|g\|_{\mathcal{B}^N_t(\mathbb{R}^6)} \|h\|_{\mathcal{B}^0_0(\mathbb{R}^6)}.$$

The proof of Proposition 3.5 is now ended. \square

By using the same arguments as in the proof of Proposition 3.5, we get also,

Corollary 3.7. *For any $\ell \geq 3$, we have, for all $\beta \in \mathbb{N}^6$, $|\beta| \leq 2$,*

$$(3.2.10) \quad \left| \left(W^\ell \partial_{x,v}^\beta \Gamma(f, g), h \right)_{L^2(\mathbb{R}_{x,v}^6)} \right| \leq C \|f\|_{H^3_t(\mathbb{R}^6)} \|g\|_{\mathcal{B}^3_t(\mathbb{R}^6)} \|h\|_{\mathcal{B}^0_0(\mathbb{R}^6)}.$$

For the linear linear term, we have the following result

Proposition 3.8. For $\ell \geq 3$, we have for any $\beta \in \mathbb{N}^6$,

$$(3.2.11) \quad \left| \left(W^\ell \partial_{x,v}^\beta \mathcal{L}_2(f), h \right)_{L^2(\mathbb{R}^6)} \right| \leq C_{|\beta|,\ell} \|f\|_{H_t^{|\beta|}(\mathbb{R}^6)} \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)}.$$

If $1 \leq |\beta|$, we have

$$(3.2.12) \quad \left| \left(\mathcal{L}_1(W^\ell \partial_{x,v}^\beta g) - W^\ell \partial_{x,v}^\beta \mathcal{L}_1(g), h \right)_{L^2(\mathbb{R}^6)} \right| \\ \leq C_{|\beta|,\ell} \left(\|g\|_{H_t^{|\beta|}(\mathbb{R}^6)} + \|g\|_{\mathcal{B}_t^{|\beta|-1}(\mathbb{R}^6)} \right) \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)},$$

and for $|\beta| = 0$,

$$(3.2.13) \quad \left| \left(\mathcal{L}_1(W^\ell g) - W^\ell \mathcal{L}_1(g), h \right)_{L^2(\mathbb{R}^6)} \right| \leq C \|g\|_{L_t^2(\mathbb{R}^6)} \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)}.$$

Remark 3.9. In the right hand side of (3.2.10), there is a main term $\|g\|_{\mathcal{B}_t^3(\mathbb{R}^6)}$. It is due to the Sobolev embedding

$$L^\infty(\mathbb{R}_x^3; H_s^{2s}(\mathbb{R}_v^3)) \subset H_s^{3/2+2s+\epsilon}(\mathbb{R}^6) \subset \mathcal{B}_0^3(\mathbb{R}^6).$$

This is also an important property of the non linear operator $\Gamma(\cdot, \cdot)$. For the linear operators in (3.2.12) and (3.2.13), this problem does not appear.

Proof. For (3.2.11), since $\mathcal{L}_2(f) = \Gamma(f, \sqrt{\mu})$, using Leibniz formula (3.1.5), we have

$$\begin{aligned} & \left(W^\ell \partial_{x,v}^\beta \mathcal{L}_2(f), h \right)_{L^2(\mathbb{R}_{x,v}^6)} = \left(W^\ell \partial_{x,v}^\beta \Gamma(f, \sqrt{\mu}), h \right)_{L^2(\mathbb{R}_{x,v}^6)} \\ & = \sum C_{\beta_2, \beta_3}^{\beta_1} \left(\mathcal{T}(\partial^{\beta_1} f, W^\ell \partial^{\beta_2} \sqrt{\mu}, \mu_{\beta_3}), h \right)_{L^2(\mathbb{R}_{x,v}^6)} \\ & \quad + \sum C_{\beta_2, \beta_3}^{\beta_1} \left(W^\ell \mathcal{T}(\partial^{\beta_1} f, \partial^{\beta_2} \sqrt{\mu}, \mu_{\beta_3}) - \mathcal{T}(\partial^{\beta_1} f, W^\ell \partial^{\beta_2} \sqrt{\mu}, \mu_{\beta_3}), h \right)_{L^2(\mathbb{R}_{x,v}^6)} \\ & = (C_1) + (C_2). \end{aligned}$$

Then (3.2.6) implies

$$\begin{aligned} |(C_2)| & = \left| \sum C_{\beta_2, \beta_3}^{\beta_1} \left(W^\ell \mathcal{T}(\partial^{\beta_1} f, \partial^{\beta_2} \sqrt{\mu}, \mu_{\beta_3}) - \mathcal{T}(\partial^{\beta_1} f, W^\ell \partial^{\beta_2} \sqrt{\mu}, \mu_{\beta_3}), h \right)_{L^2(\mathbb{R}_{x,v}^6)} \right| \\ & \leq C \|f\|_{H_t^{|\beta|}(\mathbb{R}^6)} \|\sqrt{\mu}\|_{H_t^{|\beta|}(\mathbb{R}^3)} \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)} \leq C \|f\|_{H_t^{|\beta|}(\mathbb{R}^6)} \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)}, \end{aligned}$$

and (3.2.3) implies also,

$$\begin{aligned} |(C_1)| & = \left| \sum C_{\beta_2, \beta_3}^{\beta_1} \left(\mathcal{T}(\partial^{\beta_1} f, W^\ell \partial^{\beta_2} \sqrt{\mu}, \mu_{\beta_3}), h \right)_{L^2(\mathbb{R}_{x,v}^6)} \right| \\ & \leq C \|f\|_{H_t^{|\beta|}(\mathbb{R}^6)} \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)}, \end{aligned}$$

where for the term of order $\beta_1 = \beta$, we needed to use (3.2.8).

For (3.2.12), since $\mathcal{L}_1(g) = \Gamma(\sqrt{\mu}, g)$, using again Leibniz formula (3.1.5), we have

$$\begin{aligned} & \left(W^\ell \partial_{x,v}^\beta \mathcal{L}_1(g) - \mathcal{L}_1(W^\ell \partial_{x,v}^\beta g), h \right)_{L^2(\mathbb{R}^6)} \\ & = \left(W^\ell \partial_{x,v}^\beta \Gamma(\sqrt{\mu}, g) - \Gamma(\sqrt{\mu}, W^\ell \partial_{x,v}^\beta g), h \right)_{L^2(\mathbb{R}^6)} \\ & = \sum_{|\beta_2| \leq |\beta|-1} C_{\beta_2, \beta_3}^{\beta_1} \left(\mathcal{T}(\partial^{\beta_1} \sqrt{\mu}, W^\ell \partial^{\beta_2} g, \mu_{\beta_3}), h \right)_{L^2(\mathbb{R}_{x,v}^6)} \\ & \quad + \sum C_{\beta_2, \beta_3}^{\beta_1} \left(W^\ell \mathcal{T}(\partial^{\beta_1} \sqrt{\mu}, \partial^{\beta_2} g, \mu_{\beta_3}) - \mathcal{T}(\partial^{\beta_1} \sqrt{\mu}, W^\ell \partial^{\beta_2} g, \mu_{\beta_3}), h \right)_{L^2(\mathbb{R}_{x,v}^6)} \\ & = (D_1) + (D_2). \end{aligned}$$

Then (3.2.6) implies

$$\begin{aligned} |(D_2)| &= \left| \sum_{\beta_2, \beta_3} C_{\beta_2, \beta_3}^{\beta_1} \left(W^\ell \mathcal{T}(\partial^{\beta_1} \sqrt{\mu}, \partial^{\beta_2} g, \mu_{\beta_3}) - \mathcal{T}(\partial^{\beta_1} \sqrt{\mu}, W^\ell \partial^{\beta_2} g, \mu_{\beta_3}), h \right)_{L^2(\mathbb{R}_{x,v}^6)} \right| \\ &\leq C \| \sqrt{\mu} \|_{H_t^{\beta_1}(\mathbb{R}^3)} \| g \|_{H_t^{\beta_1}(\mathbb{R}^6)} \| \| h \| \|_{\mathcal{B}_0^0(\mathbb{R}_{x,v}^6)} \leq C \| g \|_{H_t^{\beta_1}(\mathbb{R}^6)} \| \| h \| \|_{\mathcal{B}_0^0(\mathbb{R}_{x,v}^6)}, \end{aligned}$$

giving also (3.2.13).

On the other hand, (3.2.3) implies that, for $|\beta_2| \leq |\beta| - 1$,

$$\begin{aligned} &\left| \left(\mathcal{T}(\partial^{\beta_1} \sqrt{\mu}, W^\ell \partial^{\beta_2} g, \mu_{\beta_3}), h \right)_{L^2(\mathbb{R}_{x,v}^6)} \right| \\ &\leq C \left(\int_{\mathbb{R}_x^3} \|\partial^{\beta_1} \sqrt{\mu}\|_{L_x^2(\mathbb{R}_v^3)}^2 \left(\| \| W^\ell \partial^{\beta_2} g \| \|^2 + \| W^\ell \partial^{\beta_2} g \|_{H^s(\mathbb{R}_v^3)}^2 \right) \right. \\ &\quad \left. + \int_{\mathbb{R}_x^3} \| W^\ell \partial^{\beta_2} g \|_{L_x^2(\mathbb{R}_v^3)}^2 \left(\| \partial^{\beta_1} \sqrt{\mu} \| \|^2 + \| \partial^{\beta_1} \sqrt{\mu} \|_{H^s(\mathbb{R}_v^3)}^2 \right) dx \right)^{1/2} \| \| h \| \|_{\mathcal{B}_0^0(\mathbb{R}^6)} \\ &\leq C \| \sqrt{\mu} \|_{H_t^{\beta_1+s}(\mathbb{R}_x^3)} \| \| g \| \|_{\mathcal{B}_t^{\beta_2}(\mathbb{R}^6)} \| \| h \| \|_{\mathcal{B}_0^0(\mathbb{R}^6)} \\ &\leq C \| \| g \| \|_{\mathcal{B}_t^{\beta_1-1}(\mathbb{R}^6)} \| \| h \| \|_{\mathcal{B}_0^0(\mathbb{R}^6)}, \end{aligned}$$

thus ending the proof of Proposition 3.8. \square

4. LOCAL EXISTENCE

4.1. Energy estimates for the linearized Boltzmann equation. We now consider the following linearized Boltzmann equation,

$$(4.1.1) \quad \partial_t g + v \cdot \nabla_x g + \mathcal{L}_1 g = \Gamma(f, g) - \mathcal{L}_2 f, \quad g|_{t=0} = g_0.$$

Note that it is equivalent to the following Cauchy problem :

$$(4.1.2) \quad \partial_t G + v \cdot \nabla_x G = Q(F, G), \quad F|_{t=0} = F_0$$

with $F = \mu + \sqrt{\mu} f$ and $G = \mu + \sqrt{\mu} g$.

We now study the energy estimations of equation (4.1.1) in H_t^N for the smooth solution g .

For $N \geq 3$, $\ell \geq 3$ and $\beta \in \mathbb{N}^6$, $|\beta| \leq N$, we take

$$\varphi(t, x, v) = (-1)^{|\beta|} \left(\partial_{x,v}^\beta W^{2\ell} \partial_{x,v}^\beta g \right)(t, x, v)$$

as a test function on $\mathbb{R}_x^3 \times \mathbb{R}_v^3$, and we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \| \partial^\beta g \|_{L^2(\mathbb{R}^6)}^2 + \left(W^\ell \left[\partial_{x,v}^\beta, v \right] \cdot \nabla_x g, W^\ell \partial_{x,v}^\beta g \right)_{L^2(\mathbb{R}^6)} + \left(W^\ell \partial_{x,v}^\beta \mathcal{L}_1(g), W^\ell \partial_{x,v}^\beta g \right)_{L^2(\mathbb{R}^6)} \\ &= \left(W^\ell \partial_{x,v}^\beta \Gamma(f, g), W^\ell \partial_{x,v}^\beta g \right)_{L^2(\mathbb{R}^6)} - \left(W^\ell \partial_{x,v}^\beta \mathcal{L}_2(f), W^\ell \partial_{x,v}^\beta g \right)_{L^2(\mathbb{R}^6)}, \end{aligned}$$

where we used the fact that

$$\left(v \cdot \nabla_x \left(W^\ell \partial_{x,v}^\beta g \right), W^\ell \partial_{x,v}^\beta g \right)_{L^2(\mathbb{R}^6)} = 0.$$

Applying now Proposition 3.5 and Proposition 3.8, we get for any $3 \leq k \leq N$ and $|\beta| \leq k$,

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \| \partial^\beta g \|_{L^2(\mathbb{R}^6)}^2 + \left(\mathcal{L}_1 \left(W^\ell \partial_{x,v}^\beta g \right), W^\ell \partial_{x,v}^\beta g \right)_{L^2(\mathbb{R}^6)} \\ &\leq C \left\{ \| f \|_{H_t^k(\mathbb{R}^6)} \| \| g \| \|_{\mathcal{B}_t^k(\mathbb{R}^6)}^2 + \| g \|_{H_t^k(\mathbb{R}^6)}^2 + \| f \|_{H_t^k(\mathbb{R}^6)} \| \| g \| \|_{\mathcal{B}_t^k(\mathbb{R}^6)} \right. \\ &\quad \left. + \left(\| g \|_{H_t^k(\mathbb{R}^6)} + \| \| g \| \|_{\mathcal{B}_t^{k-1}(\mathbb{R}^6)} \right) \| \| g \| \|_{\mathcal{B}_t^k(\mathbb{R}^6)} \right\}. \end{aligned}$$

Taking summation over $|\beta| \leq k$, then Lemma 2.3, (3.1.2) and Cauchy inequality imply that

$$(4.1.3) \quad \begin{aligned} \frac{d}{dt} \|g\|_{H_t^k(\mathbb{R}^6)}^2 + \frac{C_0}{2} \|g\|_{\mathcal{B}_t^k(\mathbb{R}^6)}^2 &\leq C_{k,\ell} \|f\|_{H_t^k(\mathbb{R}^6)} \|g\|_{\mathcal{B}_t^k(\mathbb{R}^6)}^2 \\ &+ C_{k,\ell} \left(\|g\|_{H_t^k(\mathbb{R}^6)}^2 + \|f\|_{H_t^k(\mathbb{R}^6)}^2 + \|g\|_{\mathcal{B}_t^{k-1}(\mathbb{R}^6)}^2 \right), \quad 3 \leq k \leq N. \end{aligned}$$

For $k = 1, 2$, we use Corollary 3.7 to get

$$(4.1.4) \quad \begin{aligned} \frac{d}{dt} \|g\|_{H_t^k(\mathbb{R}^6)}^2 + \frac{C_0}{2} \|g\|_{\mathcal{B}_t^k(\mathbb{R}^6)}^2 &\leq C_{k,\ell} \|f\|_{H_t^k(\mathbb{R}^6)} \|g\|_{\mathcal{B}_t^k(\mathbb{R}^6)}^2 \\ &+ C_{k,\ell} \left(\|g\|_{H_t^k(\mathbb{R}^6)}^2 + \|f\|_{H_t^k(\mathbb{R}^6)}^2 + \|g\|_{\mathcal{B}_t^{k-1}(\mathbb{R}^6)}^2 \right), \end{aligned}$$

while for $k = 0$

$$(4.1.5) \quad \begin{aligned} \frac{d}{dt} \|g\|_{L_t^2(\mathbb{R}^6)}^2 + \frac{C_0}{2} \|g\|_{\mathcal{B}_t^0(\mathbb{R}^6)}^2 &\leq C_{0,\ell} \|f\|_{H_t^3(\mathbb{R}^6)} \|g\|_{\mathcal{B}_t^0(\mathbb{R}^6)}^2 \\ &+ C_{0,\ell} \left(\|g\|_{L_t^2(\mathbb{R}^6)}^2 + \|f\|_{H_t^3(\mathbb{R}^6)}^2 \right), \end{aligned}$$

where C_0 is the constant in (3.1.2) which is independent on k, ℓ and N .

Take the summation of (4.1.3)(or (4.1.4)) with k and $k - 1$ for $N \geq 3$, we have

$$\begin{aligned} &\frac{d}{dt} \left(\|g\|_{H_t^{k-1}(\mathbb{R}^6)}^2 + \frac{C_0}{2C_{k,\ell}} \|g\|_{H_t^k(\mathbb{R}^6)}^2 \right) + \frac{C_0^2}{2^2 C_{k,\ell}} \|g\|_{\mathcal{B}_t^k(\mathbb{R}^6)}^2 \\ &\leq \frac{C_0}{2} \left(\|f\|_{H_t^N(\mathbb{R}^6)} \|g\|_{\mathcal{B}_t^N(\mathbb{R}^6)}^2 + \|g\|_{H_t^N(\mathbb{R}^6)}^2 + \|f\|_{H_t^N(\mathbb{R}^6)}^2 \right) \\ &\quad + \frac{\partial}{\partial t} \|g\|_{H_t^{k-1}(\mathbb{R}^6)}^2 + \frac{C_0}{2} \|g\|_{\mathcal{B}_t^{k-1}(\mathbb{R}^6)}^2 \\ &\leq \frac{C_0}{2} \left(\|f\|_{H_t^N(\mathbb{R}^6)} \|g\|_{\mathcal{B}_t^N(\mathbb{R}^6)}^2 + \|g\|_{H_t^N(\mathbb{R}^6)}^2 + \|f\|_{H_t^N(\mathbb{R}^6)}^2 \right) \\ &\quad + C_{k-1,\ell} \left(\|f\|_{H_t^N(\mathbb{R}^6)} \|g\|_{\mathcal{B}_t^N(\mathbb{R}^6)}^2 + \|g\|_{H_t^N(\mathbb{R}^6)}^2 + \|f\|_{H_t^N(\mathbb{R}^6)}^2 + \|g\|_{\mathcal{B}_t^{k-2}(\mathbb{R}^6)}^2 \right). \end{aligned}$$

By induction, and using (4.1.5) for the last term, we have therefore obtained the following estimate

$$(4.1.6) \quad \begin{aligned} \frac{d}{dt} \left(\sum_{0 \leq k \leq N} c_{k,\ell} \|g\|_{H_t^k(\mathbb{R}^6)}^2 \right) + \widetilde{C}_0 \|g\|_{\mathcal{B}_t^N(\mathbb{R}^6)}^2 \\ \leq \widetilde{C}_{N,\ell} \left(\|f\|_{H_t^N(\mathbb{R}^6)} \|g\|_{\mathcal{B}_t^N(\mathbb{R}^6)}^2 + \|g\|_{H_t^N(\mathbb{R}^6)}^2 + \|f\|_{H_t^N(\mathbb{R}^6)}^2 \right) \end{aligned}$$

where $0 < \widetilde{C}_0 < C_0$ and $c_{k,\ell} > 0$ are the constants depending on k, ℓ , such that there exists $c_0, c_1 > 0$ such that

$$(4.1.7) \quad c_0 \|g\|_{H_t^N(\mathbb{R}^6)}^2 \leq \sum_{0 \leq k \leq N} c_{k,\ell} \|g\|_{H_t^k(\mathbb{R}^6)}^2 \leq c_1 \|g\|_{H_t^N(\mathbb{R}^6)}^2.$$

Thus we have proved the following theorem :

Theorem 4.1. *Let $N \geq 3, \ell \geq 3$. Assume that $g_0 \in H_t^N(\mathbb{R}^6)$ and $f \in L^\infty([0, T]; H_t^N(\mathbb{R}^6))$. If $g \in L^\infty([0, T]; H_t^N(\mathbb{R}^6)) \cap L^2([0, T]; \mathcal{B}_t^N(\mathbb{R}^6))$ is a solution of Cauchy problem (4.1.1), then there exists $\epsilon_0 > 0$ such that if*

$$(4.1.8) \quad \|f\|_{L^\infty([0, T]; H_t^N(\mathbb{R}^6))} \leq \epsilon_0, \quad \text{and} \quad \|f\|_{L^\infty([0, T]; H_t^N(\mathbb{R}^6))} \leq \|g\|_{L^\infty([0, T]; H_t^N(\mathbb{R}^6))},$$

we have

$$(4.1.9) \quad \|g\|_{L^\infty([0,T]; H_\ell^N(\mathbb{R}^6))}^2 + \|g\|_{L^2([0,T]; \mathcal{B}_\ell^N(\mathbb{R}^6))}^2 \leq C_1 e^{C_2 T} \|g_0\|_{H_\ell^N(\mathbb{R}^6)}^2$$

for constants C_1, C_2 depending only on N, ℓ .

Proof. Choosing $\epsilon_0 = \frac{\widetilde{C}_0}{2\widetilde{C}_{N,\ell}}$, we have, from (4.1.6),

$$\begin{aligned} & \frac{d}{dt} \left(\sum_{0 \leq k \leq N} c_{k,\ell} \|g\|_{H_\ell^k(\mathbb{R}^6)}^2 \right) + \frac{\widetilde{C}_0}{2} \|g\|_{\mathcal{B}_\ell^N(\mathbb{R}^6)}^2 \\ & \leq 2\widetilde{C}_{N,\ell} \|g\|_{H_\ell^N(\mathbb{R}^6)}^2 \leq 2c_0^{-1} \widetilde{C}_{N,\ell} \sum_{0 \leq k \leq N} c_{k,\ell} \|g\|_{H_\ell^k(\mathbb{R}^6)}^2, \end{aligned}$$

and

$$\frac{d}{dt} \left(e^{-2t c_0^{-1} \widetilde{C}_{N,\ell}} \sum_{0 \leq k \leq N} c_{k,\ell} \|g\|_{H_\ell^k(\mathbb{R}^6)}^2 \right) + \frac{\widetilde{C}_0}{2} e^{-2t c_0^{-1} \widetilde{C}_{N,\ell}} \|g\|_{\mathcal{B}_\ell^N(\mathbb{R}^6)}^2 \leq 0.$$

Thus

$$c_0 \|g\|_{L^\infty([0,T]; H_\ell^N(\mathbb{R}^6))}^2 + \frac{\widetilde{C}_0}{2} \|g\|_{L^2([0,T]; \mathcal{B}_\ell^N(\mathbb{R}^6))}^2 \leq c_1 e^{2T c_0^{-1} \widetilde{C}_{N,\ell}} \|g_0\|_{H_\ell^N(\mathbb{R}^6)}^2.$$

Therefore, we get (4.1.9) with

$$C_1 = c_1 \left(\min \left\{ c_0, \frac{\widetilde{C}_0}{2} \right\} \right)^{-1}, \quad C_2 = 2c_0^{-1} \widetilde{C}_{N,\ell}.$$

□

4.2. Linearized Boltzmann equation.

Theorem 4.2. *Let $\ell \geq 3, N \geq 3$ and $g_0 \in H_\ell^N(\mathbb{R}^6)$. There exists $\epsilon_0 > 0$ such that if*

$$\|f\|_{L^\infty([0,T]; H_\ell^N(\mathbb{R}^6))} \leq \epsilon_0,$$

then the Cauchy problem (4.1.1) admits a unique solution

$$g \in L^\infty([0, T]; H_\ell^N(\mathbb{R}^6)) \cap L^2([0, T]; \mathcal{B}_\ell^N(\mathbb{R}^6)).$$

Proof. We consider the following Cauchy problem :

$$(4.2.1) \quad \mathcal{P}g \equiv \partial_t g + v \cdot \nabla_x g + \mathcal{L}_1 g - \Gamma(f, g) = F, \quad g(0) = g_0.$$

For $h \in C^\infty([0, T]; \mathcal{S}(\mathbb{R}_{x,v}^6))$ with $h(T) = 0$, we define

$$\left(g, \mathcal{P}_{N,\ell}^* h \right)_{L^2([0,T]; H_\ell^N(\mathbb{R}^6))} = \left(\mathcal{P}g, h \right)_{L^2([0,T]; H_\ell^N(\mathbb{R}^6))},$$

so that $\mathcal{P}_{N,\ell}^*$ is the adjoint of the linear operator \mathcal{P} in the Hilbert space $L^2([0, T]; H_\ell^N(\mathbb{R}^6))$.

Setting

$$\mathbb{W} = \left\{ w = \mathcal{P}_{N,\ell}^* h; h \in C^\infty([0, T]; \mathcal{S}(\mathbb{R}_{x,v}^6)) \text{ with } h(T) = 0 \right\},$$

\mathbb{W} is a dense subspace of $L^2([0, T]; H_\ell^N(\mathbb{R}^6))$, and we have also

$$\mathcal{P}_{N,\ell}^*(h) = -\partial_t h + (v \cdot \nabla_x)^* h + \mathcal{L}_1^* h + \Gamma^*(f, h).$$

Then

$$\begin{aligned} \left(h, \mathcal{P}_{N,\ell}^* h \right)_{H_\ell^N(\mathbb{R}^6)} &= -\frac{1}{2} \frac{d}{dt} \|h(t)\|_{H_\ell^N(\mathbb{R}^6)}^2 + (v \cdot \nabla_x h, h)_{H_\ell^N(\mathbb{R}^6)} \\ &\quad + (\mathcal{L}_1(h), h)_{H_\ell^N(\mathbb{R}^6)} - (\Gamma(f, h), h)_{H_\ell^N(\mathbb{R}^6)}, \end{aligned}$$

Using the same arguments as in the proof of Theorem 4.1, for $\|f\|_{L^\infty([0,T]; H_\ell^N(\mathbb{R}^6))} \leq \epsilon_0$, we have

$$\int_t^T e^{2C(s-t)} \left| (h, \mathcal{P}_{N,\ell}^* h)_{H_\ell^N(\mathbb{R}^6)} \right| dt \geq \frac{1}{2} \|h(t)\|_{H_\ell^N(\mathbb{R}^6)}^2 + \int_t^T C_1 e^{2C(s-t)} \|h(s)\|_{\mathcal{B}_\ell^N(\mathbb{R}^6)}^2 ds.$$

Thus, for all $0 < t < T$,

$$\begin{aligned} \|h(t)\|_{H_\ell^N(\mathbb{R}^6)}^2 + C_1 \|h\|_{L^2([t,T]; \mathcal{B}_\ell^N(\mathbb{R}^6))}^2 &\leq C (h, \mathcal{P}_{N,\ell}^* h)_{L^2([t,T]; H_\ell^N(\mathbb{R}^6))} \\ &\leq C \|\mathcal{P}_{N,\ell}^*(h)\|_{L^2([t,T]; H_\ell^N(\mathbb{R}^6))} \|h\|_{L^2([t,T]; H_\ell^N(\mathbb{R}^6))}. \end{aligned}$$

We get finally

$$(4.2.2) \quad \|h\|_{L^2([0,T]; H_\ell^N(\mathbb{R}^6))} + \|h\|_{L^\infty([0,T]; H_\ell^N(\mathbb{R}^6))} \leq CT \|\mathcal{P}_{N,\ell}^*(h)\|_{L^2([0,T]; H_\ell^N(\mathbb{R}^6))},$$

and

$$\|h\|_{L^2([t,T]; H_\ell^N(\mathbb{R}^6))} \leq C \|h\|_{L^2([t,T]; \mathcal{B}_\ell^N(\mathbb{R}^6))}$$

giving also

$$(4.2.3) \quad \|h\|_{L^2([0,T]; \mathcal{B}_\ell^N(\mathbb{R}^6))} \leq C \|\mathcal{P}_{N,\ell}^*(h)\|_{L^2([0,T]; H_\ell^N(\mathbb{R}^6))}.$$

Next, we define a functional \mathcal{G} on \mathbb{W} as follows

$$\mathcal{G}(w) = (F, h)_{L^2([0,T]; H_\ell^N(\mathbb{R}^6))} + (g_0, h(0))_{H_\ell^N(\mathbb{R}^6)}.$$

Then, if $F \in L^2([0, T]; H_{\ell-s}^N(\mathbb{R}^6))$, (4.2.3) gives

$$\begin{aligned} |\mathcal{G}(w)| &\leq \|F\|_{L^2([0,T]; H_{\ell-s}^N(\mathbb{R}^6))} \|h\|_{L^2([0,T]; H_{\ell+s}^N(\mathbb{R}^6))} + \|g_0\|_{H_\ell^N(\mathbb{R}^6)} \|h(0)\|_{H_\ell^N(\mathbb{R}^6)} \\ &\leq C \|F\|_{L^2([0,T]; H_{\ell-s}^N(\mathbb{R}^6))} \|h\|_{L^2([0,T]; \mathcal{B}_\ell^N(\mathbb{R}^6))} + \|g_0\|_{H_\ell^N(\mathbb{R}^6)} \|h(0)\|_{H_\ell^N(\mathbb{R}^6)} \\ &\leq C \|\mathcal{P}_{N,\ell}^*(h)\|_{L^2([0,T]; H_\ell^N(\mathbb{R}^6))} \leq C \|w\|_{L^2([0,T]; H_\ell^N(\mathbb{R}^6))}, \end{aligned}$$

where we used (4.2.2) to get the estimation of the term $\|g_0\|_{H_\ell^N(\mathbb{R}^6)} \|h(0)\|_{H_\ell^N(\mathbb{R}^6)}$. Thus \mathcal{G} is a continuous linear functional on $(\mathbb{W}; \|\cdot\|_{L^2([0,T]; H_\ell^N(\mathbb{R}^6))})$. Now, there exists $g \in L^2([0, T]; H_\ell^N(\mathbb{R}^6))$ such that for any $w \in \mathbb{W}$,

$$\mathcal{G}(w) = (g, w)_{L^2([0,T]; H_\ell^N(\mathbb{R}^6))}$$

by Hahn-Banach Theorem. For any $h \in C^\infty([0, T]; \mathcal{S}(\mathbb{R}_{x,v}^6))$ with $h(T) = 0$, we have

$$(g, \mathcal{P}_{N,\ell}^* h)_{L^2([0,T]; H_\ell^N(\mathbb{R}^6))} = (F, h)_{L^2([0,T]; H_\ell^N(\mathbb{R}^6))} + (g_0, h(0))_{H_\ell^N(\mathbb{R}^6)},$$

and by the definition of the operator $\mathcal{P}_{N,\ell}^*$, we have also

$$(\mathcal{P}g, \tilde{h})_{L^2([0,T]; L^2(\mathbb{R}^6))} = (F, \tilde{h})_{L^2([0,T]; L^2(\mathbb{R}^6))} + (g_0, \tilde{h}(0))_{L^2(\mathbb{R}^6)},$$

where

$$\tilde{h} = \Lambda^N W^{2\ell} \Lambda^N h \in C^\infty([0, T]; \mathcal{S}(\mathbb{R}^6)) \text{ with } \tilde{h}(T) = 0.$$

Since $\Lambda^N W^{2\ell} \Lambda^N$ is an isomorphism on $\{h \in C^\infty([0, T]; \mathcal{S}(\mathbb{R}^6)) \text{ with } h(T) = 0\}$, we have proved that, if $F \in L^2([0, T]; H_{\ell-s}^N(\mathbb{R}^6))$, then $g \in L^2([0, T]; H_\ell^N(\mathbb{R}^6))$ is a solution of Cauchy problem (4.2.1).

To finish the proof of Theorem 4.2, we take

$$F = -\mathcal{L}_2(f) = \Gamma(f, \sqrt{\mu}),$$

to get

$$\left| (F, h)_{L^2([0,T]; H_\ell^N(\mathbb{R}^6))} \right| \leq C \|f\|_{L^2([0,T]; H_\ell^N(\mathbb{R}^6))} \|h\|_{L^2([0,T]; \mathcal{B}_\ell^N(\mathbb{R}^6))}.$$

Then \mathcal{G} is also continuous on \mathbb{W} , ending the proof of Theorem 4.2. \square

4.3. Convergence of an approximate sequence. We prove next the local existence results.

Theorem 4.3. *Let $N \geq 3, \ell \geq 3$. There exists $\epsilon_1 > 0$ and $0 < T < +\infty$ such that if $g_0 \in H_\ell^N(\mathbb{R}^6)$ and*

$$\|g_0\|_{H_\ell^N(\mathbb{R}^6)} \leq \epsilon_1,$$

then the Cauchy problem (1.9) admits a solution

$$g \in L^\infty([0, T]; H_\ell^N(\mathbb{R}^6)) \cap L^2([0, T]; \mathcal{B}_\ell^N(\mathbb{R}^6)).$$

Remark 4.4. *From equation (1.9), we have, for $0 < 2s < 1$,*

$$\partial_t g \in L^\infty([0, T]; H_{\ell-1}^{N-1}(\mathbb{R}^6)),$$

and thus

$$g \in H_{\ell-1}^N([0, T] \times \mathbb{R}^6).$$

We consider the following approximation sequence defined through the following Cauchy problem $n \in \mathbb{N}$

$$\partial_t f^{n+1} + v \cdot \nabla_x f^{n+1} = Q(f^n, f^{n+1}), \quad f^{n+1}|_{t=0} = f_0,$$

where $f^n = \mu + \mu^{1/2} g^n$ and $f^0 = f_0$. This is also equivalent to

$$(4.3.1) \quad \partial_t g^{n+1} + v \cdot \nabla_x g^{n+1} + \mathcal{L}_1 g^{n+1} - \Gamma(g^n, g^{n+1}) = -\mathcal{L}_2 g^n, \quad g^{n+1}|_{t=0} = g_0.$$

Proposition 4.5. *Let $N \geq 3, \ell \geq 3$. There exists $\epsilon_1 > 0$ and $T > 0$ such that if $g_0 \in H_\ell^N(\mathbb{R}^6)$ and*

$$(4.3.2) \quad \|g_0\|_{H_\ell^N(\mathbb{R}^6)} \leq \epsilon_1,$$

the Cauchy problem (4.3.1) admits a sequence of solutions

$$\{g^n, n \in \mathbb{N}\} \subset L^\infty([0, T]; H_\ell^N(\mathbb{R}^6)) \cap L^2([0, T]; \mathcal{B}_\ell^N(\mathbb{R}^6)).$$

Moreover, there exists $E > 0$ such that for all $n \in \mathbb{N}$,

$$(4.3.3) \quad \|g^n\|_{L^\infty([0, T]; H_\ell^N(\mathbb{R}^6))} + \|g^n\|_{L^2([0, T]; \mathcal{B}_\ell^N(\mathbb{R}^6))} \leq E.$$

Proof. We prove (4.3.3) by induction on n . We have firstly

$$\partial_t g^1 + v \cdot \nabla_x g^1 + \mathcal{L}_1 g^1 - \Gamma(g_0, g^1) = -\mathcal{L}_2 g_0, \quad g^1|_{t=0} = g_0.$$

Then Theorem 4.2 asserts the existence of

$$g^1 \in L^\infty([0, T]; H_\ell^N(\mathbb{R}^6)) \cap L^2([0, T]; \mathcal{B}_\ell^N(\mathbb{R}^6)),$$

and Theorem 4.1 deduces

$$\|g^1\|_{L^\infty([0, T]; H_\ell^N(\mathbb{R}^6))} + \|g^1\|_{L^2([0, T]; \mathcal{B}_\ell^N(\mathbb{R}^6))} \leq C_1 e^{C_2 T} \|g_0\|_{H_\ell^N(\mathbb{R}^6)},$$

for any $0 < T < +\infty$. Thus (4.3.3) is proven for $n = 1$ with $E = C_1 e^{C_2 T} \|g_0\|_{H_\ell^N(\mathbb{R}^6)}$.

Now we fix $T > 0$ and choose $\epsilon_1 > 0$ small enough such that

$$(4.3.4) \quad E = C_1 e^{C_2 T} \|g_0\|_{H_\ell^N(\mathbb{R}^6)} \leq C_1 e^{C_2 T} \epsilon_1 = \epsilon_0$$

where ϵ_0 is the constant in Theorem 4.1 and Theorem 4.2.

Assuming now that

$$\|g^n\|_{L^\infty([0, T]; H_\ell^N(\mathbb{R}^6))} + \|g^n\|_{L^2([0, T]; \mathcal{B}_\ell^N(\mathbb{R}^6))} \leq E \leq \epsilon_0,$$

then Theorem 4.2 asserts the existence of

$$g^{n+1} \in L^\infty([0, T]; H_\ell^N(\mathbb{R}^6)) \cap L^2([0, T]; \mathcal{B}_\ell^N(\mathbb{R}^6)),$$

and from Theorem 4.1, we deduce

$$\|g^{n+1}\|_{L^\infty([0,T]; H_\ell^N(\mathbb{R}^6))} + \|g^{n+1}\|_{L^2([0,T]; \mathcal{B}_\ell^N(\mathbb{R}^6))} \leq C_1 e^{C_2 T} \|g_0\|_{H_\ell^N(\mathbb{R}^6)} \leq E.$$

Thus we have proved (4.3.3) for all $n \in \mathbb{N}$, and ended the proof of Proposition 4.5. \square

Let us note the following Remark.

Remark 4.6. From (4.3.4), we observe that the smallness of the initial data ϵ_1 depends on the life time of solution T . That means that if we fix the smallness initial data such that $\epsilon_1 = \frac{\epsilon_0}{2C_1}$, we have to take $T > 0$ small enough so that $e^{C_2 T} \leq 2$. Thus, it is impossible to get global solution from here.

To conclude, by a standard weak compactness procedure, we may extract a subsequence such that

$$\begin{aligned} g^n &\rightharpoonup g \in L^\infty([0, T]; H_\ell^N(\mathbb{R}^6)) \quad \text{weakly*}, \\ g^n &\rightharpoonup g \in L^2([0, T]; \mathcal{B}_\ell^N(\mathbb{R}^6)) \quad \text{weakly}, \end{aligned}$$

hold for a limit function g . Furthermore, by using equation (4.3.1) and Proposition 4.5, we see that $\{\partial_t g^n\}$ is uniformly bounded in $L^\infty([0, T]; H_{\ell-1}^{N-1})$, so that it is a compact set in

$$C^{1-\delta}([0, T_*[; H_{\ell-1}^{N-1-\delta}(\Omega \times \mathbb{R}_v^3)).$$

Now we can take the limit in equation (4.3.1) and thus g is a solution of Cauchy problem (1.9).

5. POSITIVITY AND UNIQUENESS OF THE SOLUTIONS

5.1. Non-negativity.

Theorem 5.1. Let $N \geq 3, \ell \geq 3$. There exists $\epsilon_1 > 0$ such that if $g_0 \in H_\ell^N(\mathbb{R}^6)$ with $\mu + \mu^{1/2} g_0 \geq 0$, $\|g_0\|_{H_\ell^N(\mathbb{R}^6)} \leq \epsilon_1$ and $g \in L^\infty([0, T]; H_\ell^N(\mathbb{R}^6))$ is a solution of Cauchy problem (1.9), then we have $\mu + \mu^{1/2} g \geq 0$ on $[0, T] \times \mathbb{R}^6$.

Proof. By using the uniqueness of the solution for the above Cauchy problem, it is enough to prove the non-negativity of approximated solutions given by Proposition 4.5 :

$$(5.1.1) \quad f^n = \mu + \mu^{1/2} g^n \geq 0, \quad n \in \mathbb{N},$$

which will be proved by induction. It is true for $n = 0$, and we assume that it is true for some n and prove now that (5.1.1) will be true for $n + 1$.

From (4.3.1), $f^{n+1} = \mu + \mu^{1/2} g^{n+1}$ is the solution of the following Cauchy problem :

$$(5.1.2) \quad \begin{cases} \partial_t f^{n+1} + v \cdot \nabla_x f^{n+1} = Q(f^n, f^{n+1}) \\ f^{n+1}|_{t=0} = f_0 = \mu + \mu^{1/2} g_0 \geq 0 \end{cases}$$

Take the convex function $\beta(s) = \frac{1}{2}(s^-)^2 = \frac{1}{2}s(s^-)$ with $s^- = \min\{s, 0\}$. Setting $\phi(x, v) = (1 + |x|^2 + |v|^2)^{-2}$ and noticing that

$$\beta_s(f^{n+1})\phi(x, v) = \left(\frac{d}{ds} \beta\right)(f^{n+1})\phi(x, v) = \min\{f^{n+1}, 0\}\phi(x, v) \in L^\infty([0, T]; L^1(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))),$$

we have by (5.1.2),

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^6} \beta(f^{n+1})\phi dx dv &= \int_{\mathbb{R}^6} Q(f^n, f^{n+1}) \beta_s(f^{n+1})\phi dx dv \\ &\quad - \int_{\mathbb{R}^6} v \cdot \nabla_x (\beta(f^{n+1})\phi) dx dv - \int_{\mathbb{R}^6} (\phi^{-1} v \cdot \nabla_x \phi) \beta(f^{n+1})\phi dx dv, \end{aligned}$$

where the first term on the right hand side is well defined by Theorem 2.1 because f^n belongs to $L^\infty([0, T] \times \mathbb{R}_x^3; L^1_{2s} \cap H^{2s}_{2s}(\mathbb{R}_v^3))$. Since the second term vanishes and $|\nu \cdot \nabla_x \phi| \leq C\phi$, we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^6} \beta(f^{n+1}) \phi dx dv \leq \int_{\mathbb{R}^6} Q(f^n, f^{n+1}) \beta_s(f^{n+1}) \phi dx dv + C \int_{\mathbb{R}^6} \beta(f^{n+1}) \phi dx dv.$$

For the first term on the right hand side, we have

$$\begin{aligned} & \int_{\mathbb{R}^6} Q(f^n, f^{n+1}) \beta_s(f^{n+1}) \phi dx dv \\ &= \int_{\mathbb{R}_{x,v}^6} \int_{\mathbb{R}_{v_*}^3 \times \mathbb{S}_\sigma^2} b(\cos \theta) (f_*^{n'} f^{n+1'} - f_*^n f^{n+1}) \beta_s(f^{n+1}) \phi \\ &= \int_{\mathbb{R}_{x,v}^6} \int_{\mathbb{R}_{v_*}^3 \times \mathbb{S}_\sigma^2} b(\cos \theta) f_*^{n'} (f^{n+1'} - f^{n+1}) \beta_s(f^{n+1}) \phi \\ & \quad + \int_{\mathbb{R}_{x,v}^6} \beta_s(f^{n+1}) f^{n+1} \phi \int_{\mathbb{R}_{v_*}^3 \times \mathbb{S}_\sigma^2} b(\cos \theta) (f_*^{n'} - f_*^n) \\ &= I + II. \end{aligned}$$

From (4.3.3), we have, for any $n \in \mathbb{N}$,

$$\begin{aligned} \|f^n\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L^1(\mathbb{R}_v^3))} &\leq 1 + \|\sqrt{\mu} g^n\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L^1(\mathbb{R}_v^3))} \\ &\leq 1 + \|g^n\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L^2(\mathbb{R}_v^3))} \leq 1 + E, \end{aligned}$$

so that the cancellation lemma from [5] implies that

$$\int_{\mathbb{R}_{v_*}^3 \times \mathbb{S}_\sigma^2} b(\cos \theta) (f_*^{n'} - f_*^n) = C \int_{\mathbb{R}_v^3} f^n(t, x, v) dv \leq C \|f^n\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L^1(\mathbb{R}_v^3))} \leq C,$$

while $\beta_s(s)s = 2\beta(s)$ implies that

$$|II| \leq C \int_{\mathbb{R}^6} \beta(f^{n+1}) \phi dx dv.$$

On the other hand, by the convexity of β ,

$$\beta_s(a)(b - a) \leq \beta(b) - \beta(a),$$

and since $f_*^{n'} \geq 0$ by assumption, we get

$$\begin{aligned} I &= \int_{\mathbb{R}_{x,v}^6} \int_{\mathbb{R}_{v_*}^3 \times \mathbb{S}_\sigma^2} b(\cos \theta) f_*^{n'} (f^{n+1'} - f^{n+1}) \beta_s(f^{n+1}) \phi \\ &\leq \int_{\mathbb{R}_{x,v}^6} \int_{\mathbb{R}_{v_*}^3 \times \mathbb{S}_\sigma^2} b(\cos \theta) f_*^{n'} (\beta(f^{n+1'}) - \beta(f^{n+1})) \phi \\ &\leq \int_{\mathbb{R}_{x,v}^6} \int_{\mathbb{R}_{v_*}^3 \times \mathbb{S}_\sigma^2} b(\cos \theta) (f_*^{n'} \beta(f^{n+1'}) - f_*^n \beta(f^{n+1})) \phi \\ & \quad - \int_{\mathbb{R}_{x,v}^6} \int_{\mathbb{R}_{v_*}^3 \times \mathbb{S}_\sigma^2} b(\cos \theta) \beta(f^{n+1}) (f_*^{n'} - f_*^n) \phi \\ &\leq \int_{\mathbb{R}_{x,v}^6} \int_{\mathbb{R}_{v_*}^3 \times \mathbb{S}_\sigma^2} b(\cos \theta) f_*^{n'} \beta(f^{n+1}) (\phi' - \phi) + C \int_{\mathbb{R}^6} \beta(f^{n+1}) \phi dx dv \\ &= I_1 + I_2. \end{aligned}$$

Using Taylor's formula for the first term

$$I_1 = \int_0^1 d\tau \int_{\mathbb{R}_{x,v}^6} \int_{\mathbb{R}_{v_*}^3 \times \mathbb{S}_\sigma^2} b(\cos \theta) f_*^n \beta(f^{n+1})(v' - v) \cdot \nabla_v \phi(v + \tau(v' - v)).$$

Using

$$(5.1.3) \quad |v' - v| = |v - v_*| \sin\left(\frac{\theta}{2}\right) \leq \langle v \rangle \langle v_* \rangle \sin\left(\frac{\theta}{2}\right),$$

and with $v_\tau = v + \tau(v' - v)$, $0 < \tau < 1$, $0 \leq \theta \leq \pi/2$,

$$|v| \leq |v_\tau| + |v' - v| \leq |v_\tau| + \sin\left(\frac{\theta}{2}\right)(|v| + |v_*|) \leq |v_\tau| + \frac{\sqrt{2}}{2}|v| + |v_*|,$$

then

$$(1 + |x|^2 + |v|^2) \leq C(1 + |x|^2 + |v_\tau|^2)(1 + |v_*|^2),$$

which implies

$$|\nabla_v \phi(x, v_\tau)| \leq (1 + |x|^2 + |v_\tau|^2)^{-5/2} \leq C\phi(x, v) \frac{\langle v_* \rangle^5}{\langle v \rangle},$$

we obtain

$$|I_1| \leq C \|f^n\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L^1_6(\mathbb{R}_v^3))} \int_{\mathbb{R}^6} \beta(f^{n+1}) \phi dx dv.$$

Again from (4.3.3), we have, for any $n \in \mathbb{N}$,

$$\|f^n\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L^1_6(\mathbb{R}_v^3))} \leq \|\mu^{1/2}\|_{L^1_6(\mathbb{R}^3)} + \|\mu^{1/2} g^n\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L^1_6(\mathbb{R}_v^3))} \leq C(1 + E).$$

Finally, we have obtained, for $0 < t < T$,

$$\frac{d}{dt} \int_{\mathbb{R}^6} \beta(f^{n+1}) \phi dx dv \leq C \int_{\mathbb{R}^6} \beta(f^{n+1}) \phi dx dv, \quad \beta(f^{n+1})|_{t=0} = 0.$$

Therefore, for $0 < t < T$, and by continuity,

$$\int_{\mathbb{R}^6} \beta(f^{n+1}(t)) \phi dx dv = 0$$

which implies that, $f^{n+1}(t, x, v) \geq 0$ for $(t, x, v) \in [0, T] \times \mathbb{R}_{x,v}^6$, thus ending the proof of Theorem 5.1. \square

Remark 5.2. *This theorem can be also extended in the strong singularity case.*

Indeed, writing

$$\begin{aligned} I_1 &= \int_{\mathbb{R}_{x,v}^6} \int_{\mathbb{R}_{v_*}^3 \times \mathbb{S}_\sigma^2} b(\cos \theta) f_*^n \beta(f^{n+1})(v' - v) \cdot \nabla_v \phi(v) \\ &\quad + \frac{1}{2} \int_0^1 d\tau \int_{\mathbb{R}_{x,v}^6} \int_{\mathbb{R}_{v_*}^3 \times \mathbb{S}_\sigma^2} b(\cos \theta) f_*^n \beta(f^{n+1})(v' - v)^2 \nabla_v^2 \phi(v + \tau(v' - v)) \\ &= I_{11} + I_{12}, \end{aligned}$$

since we have

$$|\nabla_v^2 \phi(x, v_\tau)| \leq (1 + |x|^2 + |v_\tau|^2)^{-3} \leq C\phi(x, v) \frac{\langle v_* \rangle^6}{\langle v \rangle^2},$$

it follows from (5.1.3) that

$$|I_{12}| \leq C \|f^n\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L^1_8(\mathbb{R}_v^3))} \int_{\mathbb{R}^6} \beta(f^{n+1}) \phi dx dv.$$

On the other hand, setting $\mathbf{k} = \frac{v-v_*}{|v-v_*|}$ and writing

$$v' - v = \frac{1}{2}|v - v_*|(\sigma - (\sigma \cdot \mathbf{k})\mathbf{k}) + \frac{1}{2}((\sigma \cdot \mathbf{k}) - 1)(v - v_*)$$

we have

$$I_{11} = \frac{1}{2} \int_{\mathbb{R}_{x,v}^6} \int_{\mathbb{R}_{v_*}^3 \times \mathbb{S}_{\sigma}^2} b(\cos \theta) f_*^n \beta(f^{n+1}) (\cos \theta - 1)(v - v_*) \cdot \nabla_v \phi(v)$$

because it follows by symmetry that $\int_{\mathbb{S}^2} b(\sigma \cdot \mathbf{k})(\sigma - (\sigma \cdot \mathbf{k})\mathbf{k}) d\sigma = 0$. Therefore we have

$$|I_{11}| \leq C \|f^n\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L^1_6(\mathbb{R}_v^3))} \int_{\mathbb{R}^6} \beta(f^{n+1}) \phi dx dv,$$

and the same conclusion as Theorem 5.1 holds also in the strong singularity case.

5.2. Uniqueness of solutions. We now prove Theorem 1.2. To this end, we need a preliminary preparation by the next two lemmas .

Set $\varphi(v, x) = \langle v, x \rangle^2 = 1 + |v|^2 + |x|^2$ and

$$W_{\varphi, l} = \frac{\langle v \rangle^l}{\varphi(v, x)} = \frac{(1 + |v|^2)^{l/2}}{1 + |v|^2 + |x|^2}$$

Lemma 5.3. *Let $\mathbb{N} \ni l \geq 4$. Then*

$$\begin{aligned} |W_{\varphi, l} - W'_{\varphi, l}| &\leq C \sin\left(\frac{\theta}{2}\right) \left(\frac{W'_l + W'_{l-3} W'_{3,*}}{\varphi(v'_*, x)} + \sin^{l-3}\left(\frac{\theta}{2}\right) W'_{\varphi, l,*} \right) \\ (5.2.1) \quad &\leq C \left(\theta W'_l W'_{\varphi, 3,*} + \theta^{l-2} W'_{\varphi, l,*} \right), \end{aligned}$$

where $W'_{\varphi, l,*} = \frac{W'_{l,*}}{\varphi(v'_*, x)}$, and also simply

$$(5.2.2) \quad |W_{\varphi, l} - W'_{\varphi, l}| \leq C \sin\left(\frac{\theta}{2}\right) \frac{W'_l + W'_{l,*}}{\varphi(v', x)} \leq C \theta \frac{W'_l W'_{l,*}}{\varphi(v', x)}.$$

Proof. For $k \geq 0, a \geq 0$, set

$$F_k(\lambda) = \frac{\lambda^k}{\lambda + a}.$$

Then for $\lambda \in [1, \infty)$, we have $F'_k(\lambda) \geq 0$ if $k \geq 1$ and $F''_k(\lambda) \geq 0$ if $k \geq 2$. Since $F'_k(\lambda)$ is positive and increasing in $[1, \infty)$ if $k \geq 2$, it follows from the mean value theorem that for $\lambda, \lambda' \geq 1$

$$|F_k(\lambda) - F_k(\lambda')| \leq F'_k(\lambda + |\lambda - \lambda'|) |\lambda - \lambda'|.$$

Setting $\lambda = \langle v \rangle^2, \lambda' = \langle v' \rangle^2$, we have

$$\begin{aligned} |F_k(\langle v \rangle^2) - F_k(\langle v' \rangle^2)| &\leq F'_k(2(\langle v \rangle^2 + |v - v'|^2)) (2|v| + |v - v'|) |v - v'| \\ &\leq 2k F_{k-1/2}(2(\langle v \rangle^2 + |v - v'|^2)) |v - v'| \end{aligned}$$

because $|\lambda - \lambda'| \leq 2|v - v'| |v| + |v - v'|^2 \leq |v|^2 + 2|v - v'|^2$ and $\sqrt{\lambda} F'_k(\lambda) \leq k F_{k-1/2}(\lambda)$. Therefore, we obtain choosing $a = |x|^2$

$$\begin{aligned} |W_{\varphi, l} - W'_{\varphi, l}| &\leq C_l \left(\frac{\langle v \rangle^{l-1} |v - v'|}{\langle v \rangle^2 + |v - v'|^2 + a} + \frac{|v - v'|^l}{\langle v \rangle^2 + |v - v'|^2 + a} \right) \\ (5.2.3) \quad &\leq C_l |v - v'| \langle v \rangle^{l-3} F_1(\langle v \rangle^2) + C_l \frac{|v - v'|^l}{\langle v \rangle^2 + |v - v'|^2 + a} \\ &= I + II. \end{aligned}$$

Note that $\langle v \rangle^2 \leq 2\langle v_* \rangle^2 + 2|v - v_*|^2$. Since F_1 is increasing, we have

$$\begin{aligned} I &\leq C_l |v - v'| \langle v \rangle^{l-3} \frac{\langle v_* \rangle^2 + |v - v_*|^2}{\langle v_* \rangle^2 + |v - v_*|^2 + |x|^2} \\ &\leq C_l \sin\left(\frac{\theta}{2}\right) \frac{W_l + W_{l-3} W_{3,*}}{\varphi(v_*, x)}. \end{aligned}$$

On the other hand

$$\begin{aligned} II &\leq C_l \frac{|v - v_*|^l \sin^l\left(\frac{\theta}{2}\right)}{1 + (1 - \sin^2\left(\frac{\theta}{2}\right))|v|^2 + \frac{1}{2} \sin^2\left(\frac{\theta}{2}\right)|v_*|^2 + |x|^2} \\ &\leq C_l \sin^{l-2}\left(\frac{\theta}{2}\right) \frac{W_l + W_{l,*}}{\varphi(v_*, x)}. \end{aligned}$$

Since v and v' are symmetric, we get the first conclusion. The second is an easy consequence of the first inequality of (5.2.3). \square

Lemma 5.4. *Let $l \in \mathbb{N}$. If $0 < s < 1/2$, there exists $C > 0$ such that*

$$(5.2.4) \quad \left| \left((W_{\varphi,l} Q(f, g) - Q(f, W_{\varphi,l} g)), h \right)_{L^2(\mathbb{R}^6)} \right| \leq C \|f\|_{L^\infty(\mathbb{R}_x^3; L^1(\mathbb{R}_v^3))} \|W_{\varphi,l} g\|_{L^2(\mathbb{R}^6)} \|h\|_{L^2(\mathbb{R}^6)}.$$

Moreover, if $l \geq 5$ then

$$(5.2.5) \quad \left| \left((W_{\varphi,l} Q(f, g) - Q(f, W_{\varphi,l} g)), h \right)_{L^2(\mathbb{R}^6)} \right| \leq C \|W_{\varphi,l} f\|_{L^2(\mathbb{R}^6)} \|g\|_{L^\infty(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))} \|h\|_{L^2(\mathbb{R}^6)}.$$

Proof. It follows from (5.2.2) that

$$\begin{aligned} &\left| \left((W_{\varphi,l} Q(f, g) - Q(f, W_{\varphi,l} g)), h \right)_{L^2(\mathbb{R}^6)} \right| \\ &= \left| \iiint \iiint b f_*' g' (W_{\varphi,l}' - W_{\varphi,l}) h \, dv dv_* d\sigma dx \right| \\ &\leq C \iiint \iiint b |\theta| |(W_l f)'_*| |(W_{\varphi,l} g)'| |h| \, dv dv_* d\sigma dx \\ &= C \iiint \iiint b |\theta| |(W_l f)'_*| |(W_{\varphi,l} g)| |h'| \, dv dv_* d\sigma dx \\ &\leq C \left(\iiint \iiint b |\theta| |(W_l f)'_*| |(W_{\varphi,l} g)|^2 \, dv dv_* d\sigma dx \right)^{1/2} \\ &\quad \times \left(\iiint \iiint b |\theta| |(W_l f)'_*| |h'|^2 \, dv dv_* d\sigma dx \right)^{1/2} \\ &= C J_1 \times J_2. \end{aligned}$$

Clearly, one has

$$J_1^2 \leq C \|f\|_{L^\infty(\mathbb{R}_x^3; L^1(\mathbb{R}_v^3))} \|W_{\varphi,l} g\|_{L^2(\mathbb{R}^6)}^2 \int_{\mathbb{S}^2} b(\cos \theta) |\theta| \, d\sigma \leq C \|f\|_{L^\infty(\mathbb{R}_x^3; L^1(\mathbb{R}_v^3))} \|W_{\varphi,l} g\|_{L^2(\mathbb{R}^6)}^2.$$

Next, by the regular change of variables $v \rightarrow v'$, cf. [5, 12], we have

$$J_2^2 = \iiint D_0(v_*, v') |(W_l f)'_*| |h'|^2 \, dv_* dv' dx,$$

where

$$D_0(v, v') = 2 \int_{\mathbb{S}^2} \frac{\theta(v_*, v', \sigma)}{\cos^2(\theta(v_*, v', \sigma)/2)} b(\cos \theta(v_*, v', \sigma)) d\sigma \leq C \int_0^{\pi/4} \psi^{-1-2s} \sin \psi d\psi,$$

and

$$\cos \psi = \frac{v' - v_*}{|v' - v_*|} \cdot \sigma, \quad \psi = \theta/2, \quad d\sigma = \sin \psi d\psi d\phi.$$

Thus,

$$J_2^2 \leq C \|f\|_{L^\infty(\mathbb{R}_x^3; L^1(\mathbb{R}_v^3))} \|h\|_{L^2(\mathbb{R}^6)}^2,$$

and this together with the estimate on J_1 gives (5.2.4).

We now prove (5.2.5) by using (5.2.1) instead of (5.2.2). For this purpose, we have

$$\begin{aligned} & \left| \left((W_{\varphi, l} Q(f, g) - Q(f, W_{\varphi, l} g)), h \right)_{L^2(\mathbb{R}^6)} \right| \\ & \leq C \left\{ \iiint b |\theta| |(W_{\varphi, 3} f)'_*| |(W_l g)'| |h| dv dv_* d\sigma dx \right. \\ & \quad \left. + \iiint b |\theta|^{l-2} |(W_{\varphi, l} f)'_*| |g'| |h| dv dv_* d\sigma dx \right\} \\ & = \mathcal{M}_1 + \mathcal{M}_2. \end{aligned}$$

The estimation on \mathcal{M}_1 can be obtained by the similar method used to prove (5.2.4) except for the x variable. Indeed,

$$\begin{aligned} \mathcal{M}_1 & \leq C \int \left(\iiint b |\theta| |(W_{\varphi, 3} f)'_*| |(W_l g)|^2 dv dv_* d\sigma \right)^{1/2} \\ & \quad \times \left(\iiint b |\theta| |(W_{\varphi, 3} f)'_*| |h'|^2 dv dv_* d\sigma \right)^{1/2} dx \\ & \leq C \|g\|_{L^\infty(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))} \int \|W_{\varphi, 3} f\|_{L^1(\mathbb{R}_v^3)} \|h\|_{L^2(\mathbb{R}_v^3)} dx \\ & \leq C \|g\|_{L^\infty(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))} \left(\int \|W_{\varphi, 5} f\|_{L^2(\mathbb{R}_v^3)}^2 dx \right)^{1/2} \left(\int \|h\|_{L^2(\mathbb{R}_v^3)}^2 dx \right)^{1/2} \\ & \leq C \|W_{\varphi, 5} f\|_{L^2(\mathbb{R}^6)} \|g\|_{L^\infty(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))} \|h\|_{L^2(\mathbb{R}^6)}. \end{aligned}$$

\mathcal{M}_2 can be estimated as follows. Firstly, we have

$$\begin{aligned} \mathcal{M}_2^2 & = C^2 \left(\iiint b |\theta|^{l-2} |(W_{\varphi, l} f)'_*| |g| |h'| dv dv_* d\sigma dx \right)^2 \\ & \leq C^2 \iiint b |\theta|^{l-2-\frac{3}{2}} |g| |(W_{\varphi, l} f)'_*|^2 dv dv_* d\sigma dx \\ & \quad \times \iiint b |\theta|^{l-2+\frac{3}{2}} |g| |h'|^2 dv dv_* d\sigma dx \\ & = \mathcal{M}_{2,1} \times \mathcal{M}_{2,2}. \end{aligned}$$

Then, if $l - 2 - \frac{3}{2} - 2s - 1 > -1$, that is, $l > 2s + \frac{3}{2} + 2$, we have

$$\mathcal{M}_{2,1} \leq C \|g\|_{L^\infty(\mathbb{R}_x^3; L^1(\mathbb{R}_v^3))} \|W_{\varphi, l} f\|_{L^2(\mathbb{R}^6)}^2.$$

On the other hand, for $\mathcal{M}_{2,2}$ we need to apply the singular change of variables $v_* \rightarrow v'$. The Jacobian of this transform is

$$(5.2.6) \quad \left| \frac{\partial v_*}{\partial v'} \right| = \frac{8}{|I - \mathbf{k} \otimes \sigma|} = \frac{8}{|1 - \mathbf{k} \cdot \sigma|} = \frac{4}{\sin^2(\theta/2)} \leq 16\theta^{-2}, \quad \theta \in [0, \pi/2].$$

Notice that this gives rise to an additional singularity in the angle θ around 0. Actually, the situation is even worse in the following sense. Recall that θ is no longer a legitimate polar angle. In this case, the best choice of the pole is $\mathbf{k}'' = (v' - v)/|v' - v|$ for which polar angle ψ defined by $\cos \psi = \mathbf{k}'' \cdot \sigma$ satisfies (cf. [5, Fig. 1])

$$\psi = \frac{\pi - \theta}{2}, \quad d\sigma = \sin \psi d\psi d\phi, \quad \psi \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right].$$

This measure does not cancel any of the singularity of $b(\cos \theta)$, unlike the case in the usual polar coordinates. Nevertheless, this singular change of variables yields

$$\begin{aligned} \mathcal{M}_{1,2} &= C \iiint \iiint b |\theta|^{l+\frac{3}{2}} |(g)| |h'|^2 dv dv_* d\sigma dx \\ &\leq C \iiint D_1(v, v') |(g)| |h'|^2 dv dv' dx, \end{aligned}$$

when $l - 2 > \frac{3}{2} + 2s$ because

$$D_1(v, v') = \int_{\mathbb{S}^2} \theta^{l-2+\frac{3}{2}-2} b(\cos \theta) d\sigma \leq C \int_{\pi/4}^{\pi/2} \left(\frac{\pi}{2} - \psi\right)^{-2-2s+l-2+\frac{3}{2}-2} d\psi \leq C.$$

Therefore,

$$\mathcal{M}_{2,2} \leq C \|g\|_{L^\infty(\mathbb{R}_x^3; L^1(\mathbb{R}_v^3))} \|h\|_{L^2(\mathbb{R}^6)}^2.$$

Now the proof of (5.2.1) is completed by using the embedding estimate for $l > \frac{3}{2}$,

$$\|g\|_{L^1(\mathbb{R}_x^3)} \leq C \|g\|_{L^2(\mathbb{R}_x^3)}.$$

□

Proof of Theorem 1.2 : Set $F = f_1 - f_2$. Then we have

$$(5.2.7) \quad \begin{cases} F_t + v \cdot \nabla_x F = Q(f_1, F) + Q(F, f_2), \\ F|_{t=0} = 0. \end{cases}$$

Let $S(\tau) \in C_0^\infty(\mathbb{R})$ satisfy $0 \leq S \leq 1$ and

$$S(\tau) = 1, \quad |\tau| \leq 1; \quad S(\tau) = 0, \quad |\tau| \geq 2.$$

Set $S_N(D_x) = S(2^{-2N}|D_x|^2)$ and multiply $W_{\varphi,l} S_N(D_x)^2 W_{\varphi,l} F$ by (5.2.7). Integrating and letting $N \rightarrow \infty$, we have

$$(5.2.8) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|W_{\varphi,l} F(t)\|_{L^2(\mathbb{R}^6)}^2 &= \left(W_{\varphi,l} Q(f_1, F) + W_{\varphi,l} Q(F, f_2), W_{\varphi,l} F \right)_{L^2(\mathbb{R}^6)} \\ &\quad - (v \cdot \nabla_x (\varphi^{-1}) W_l F, W_{\varphi,l} F)_{L^2(\mathbb{R}^6)}, \end{aligned}$$

because $(v \cdot \nabla_x S_N(D_x) W_{\varphi,l} F, S_N(D_x) W_{\varphi,l} F)_{L^2(\mathbb{R}^6)} = 0$. The second term on the right hand side is estimated by $\|W_{\varphi,l} F\|_{L^2(\mathbb{R}^6)}^2$. Since $f_1 \geq 0$, it follows from the coercivity of $-(Q(f_1, g), g)$, we have

$$\left(Q(f_1, W_{\varphi,l} F), W_{\varphi,l} F \right)_{L^2(\mathbb{R}^6)} \leq C \|f_1(t)\|_{L^\infty(\mathbb{R}_x^3; L^1(\mathbb{R}_v^3))} \|W_{\varphi,l} F(t)\|_{L^2(\mathbb{R}_{x,v}^6)}^2.$$

By Lemma 5.4, we have

$$\begin{aligned} &\left| \left((W_{\varphi,l} Q(f_1, F) - Q(f_1, W_{\varphi,l} F)), W_{\varphi,l} F \right)_{L^2(\mathbb{R}^6)} \right| \\ &\leq C \|f_1\|_{L^\infty(\mathbb{R}_x^3; L^1(\mathbb{R}_v^3))} \|W_{\varphi,l} F\|_{L^2(\mathbb{R}^6)}^2 \end{aligned}$$

and

$$\begin{aligned} & \left| \left((W_{\varphi,l} Q(F, f_2) - Q(F, W_{\varphi,l} f_2)), W_{\varphi,l} F \right)_{L^2(\mathbb{R}^6)} \right| \\ & \leq C \|f_2\|_{L^\infty(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))} \|W_{\varphi,l} F\|_{L^2(\mathbb{R}^6)}^2. \end{aligned}$$

Finally, for $l > 3/2 + 2s + 2$, we have

$$\begin{aligned} & \left| \left(Q(F, W_{\varphi,l} f_2), W_{\varphi,l} F \right)_{L^2(\mathbb{R}^6)} \right| \leq C \|Q(F, W_{\varphi,l} f_2)\|_{L^2(\mathbb{R}^6)} \|W_{\varphi,l} F(t)\|_{L^2(\mathbb{R}^6)} \\ & \leq C \|W_{\varphi,l} F(t)\|_{L^2(\mathbb{R}^6)} \left(\int_{\mathbb{R}_x^3} \left\| \frac{F(t, x, \cdot)}{\langle x \rangle^2} \right\|_{L^1_{2s}(\mathbb{R}_v^3)}^2 \left\| \frac{\langle x \rangle^2}{\varphi} f_2(t, x, \cdot) \right\|_{H^{2s}_{1+2s}(\mathbb{R}_v^3)}^2 dx \right)^{1/2} \\ & \leq C \|W_{\varphi,l} F(t)\|_{L^2(\mathbb{R}^6)}^2 \|f_2(t)\|_{L^\infty(\mathbb{R}_x^3; H^{2s}_{1+2s}(\mathbb{R}_v^3))}, \end{aligned}$$

because $\langle x \rangle^{-2} \leq W_{\varphi,2}$ and $\langle x \rangle^2/\varphi$ is a bounded operator on H^{2s} uniformly with respect to x . Thus, we have, for any $0 < t < T$

$$\begin{aligned} & \frac{d}{dt} \|W_{\varphi,l} F(t)\|_{L^2(\mathbb{R}^6)}^2 \\ & \leq C \left(\|f_1\|_{L^\infty([0,T] \times \mathbb{R}_x^3; L^1(\mathbb{R}_v^3))} + \|f_2\|_{L^\infty([0,T] \times \mathbb{R}_x^3; H^{2s}_{1+2s}(\mathbb{R}_v^3))} \right) \|W_{\varphi,l} F(t)\|_{L^2(\mathbb{R}^6)}^2. \end{aligned}$$

Therefore, $\|W_{\varphi,l} F(0)\|_{L^2(\mathbb{R}^6)} = 0$ implies $\|W_{\varphi,l} F(t)\|_{L^2(\mathbb{R}^6)} = 0$ for all $t \in [0, T]$.

The uniqueness of g in Theorem 1.1 is a direct consequence of Theorem 1.2 because $g_0 \in H_\ell^k$ and $g \in L^\infty([0, T] \times H_\ell^k)$ with $k, \ell \geq 3$ imply $f_0 = \mu + \sqrt{\mu} g_0 \in L^\infty(\mathbb{R}_x^3; H_m^{2s})$ and $f = \mu + \sqrt{\mu} g \in L^\infty([0, T] \times \mathbb{R}_x^3; H_m^{2s})$ for any m , respectively, because $k > 3/2 + 2s$.

6. FULL REGULARITY OF SOLUTIONS

6.1. Initialization. We now prove the smoothness effect of Cauchy problem for the non-cut off Boltzmann equation.

Theorem 6.1. *Assume that $0 < s < 1/2$. There exists $\epsilon_1 > 0$ such that if $g_0 \in H_3^3(\mathbb{R}^6)$ with $\mu + \mu^{1/2} g_0 \geq 0$, $\|g_0\|_{H_3^3(\mathbb{R}^6)} \leq \epsilon_1$ and $g \in L^\infty([0, T]; H_3^3(\mathbb{R}^6))$ is a solution of Cauchy problem (1.9), then we have $g \in C^\infty([0, T] \times \mathbb{R}^6)$.*

We recall that $H_\ell^k(\mathbb{R}_{t,x}^7)$, $H_\ell^k(\mathbb{R}_{x,v}^6)$ and $H_\ell^k(\mathbb{R}_v^3)$ denote the Sobolev space with weight w.r.t. variable v . The regularity result is local in space and time, and thus for convenience, we define the following local version of weighted Sobolev space. For $0 \leq T_1 < T_2 < +\infty$, and any given open domain $\Omega \subset \mathbb{R}_x^3$, define

$$\begin{aligned} \mathcal{H}_\ell^m([T_1, T_2] \times \Omega \times \mathbb{R}_v^3) &= \left\{ f \in \mathcal{D}'([T_1, T_2] \times \Omega \times \mathbb{R}_v^3); \right. \\ & \left. \varphi(t)\psi(x)f \in H_\ell^m(\mathbb{R}^7), \forall \varphi \in C_0^\infty([T_1, T_2]), \psi \in C_0^\infty(\Omega) \right\}. \end{aligned}$$

The main result of [9] is following :

Theorem 6.2. *Assume that $0 < s < 1$, $0 \leq T_1 < T_2 < +\infty$, $\Omega \subset \mathbb{R}_x^3$ is an open domain. Let f be a non-negative solution of the Boltzmann equation (1.1) satisfying $f \in \mathcal{H}_\ell^5([T_1, T_2] \times \Omega \times \mathbb{R}_v^3)$ for all $\ell \in \mathbb{N}$. Furthermore, if f satisfies the non-vacuum condition*

$$(6.1.1) \quad \|f(t, x, \cdot)\|_{L^1(\mathbb{R}_v^3)} > 0,$$

for all $(t, x) \in]T_1, T_2[\times \Omega$, then we have

$$f \in \mathcal{H}_\ell^{+\infty}([T_1, T_2] \times \Omega \times \mathbb{R}_v^3) \subset C^{+\infty}([T_1, T_2] \times \Omega \times \mathbb{R}_v^3),$$

for any $\ell \in \mathbb{N}$.

Using Theorem 4.3 and Theorem 5.1, under the assumption of Theorem 6.1, the unique solution of the Cauchy problem (1.9) satisfies

$$\|g\|_{L^\infty([0, T]; H^3(\mathbb{R}^6))} \leq \epsilon_0, \quad \mu + \sqrt{\mu} g \geq 0.$$

Then $f = \mu + \sqrt{\mu} g \geq 0$ is a solution of Boltzmann equation (1.1). On the other hand, we choose $\epsilon_0 > 0$ small enough such that

$$\|\sqrt{\mu} g\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L^1(\mathbb{R}_v^3))} \leq C \|g\|_{L^\infty([0, T]; H^2(\mathbb{R}^6))} \leq C \epsilon_0 < 1$$

where C is the Sobolev constant of the embedding $H^2(\mathbb{R}_x^3) \subset L^\infty(\mathbb{R}_x^3)$. Then for any $(t, x) \in]0, T[\times \mathbb{R}_x^3$,

$$(6.1.2) \quad \int_{\mathbb{R}_v^3} f(t, x, v) dv = 1 + \int_{\mathbb{R}_v^3} \sqrt{\mu} g(t, x, v) dv \geq 1 - \|\sqrt{\mu} g\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L^1(\mathbb{R}_v^3))} > 0,$$

so that $f = \mu + \sqrt{\mu} g$ satisfies the condition (6.1.1).

By using equation (1.1) and Remark 4.4, we have also, for any $\ell \in \mathbb{N}$, any $0 < T_1 < T_2 < T$, and any bounded open domain $\Omega \subset \mathbb{R}_x^3$,

$$f = \mu + \sqrt{\mu} g \in \mathcal{H}_\ell^3([T_1, T_2] \times \Omega \times \mathbb{R}_v^3).$$

At this point, note that we cannot apply directly Theorem 6.2, since in our case, the initially assumed regularity of f is only $\mathcal{H}_\ell^3([T_1, T_2] \times \Omega \times \mathbb{R}_v^3)$.

Thus we prove here the following improved version of Theorem 6.2, and therefore finished the proof of Theorem 6.1.

Theorem 6.3. *Under the assumption of Theorem 6.1, we have, for any $0 < T_1 < T_2 < T$ and any bounded open domain $\Omega \subset \mathbb{R}_x^3$,*

$$f = \mu + \sqrt{\mu} g \in \mathcal{H}_\ell^5([T_1, T_2] \times \Omega \times \mathbb{R}_v^3).$$

for all $\ell \in \mathbb{N}$.

The proof of this Theorem is similar and in fact easier than that of Theorem 6.2 in [9]. In fact, since we have

$$g^n \rightarrow g,$$

by mollifying the initial data and using the uniqueness of solution, we do not need at all to mollify the solution as in [9]. We just need to prove the a priori estimates for smooth solution.

To make the paper self-contained, we give a proof here but using the notations and results of [9]. We recall that we are in Maxwellian case and mild singularity of kernel $0 < s < 1/2$.

Here and below, ϕ denotes a cutoff function satisfying $\phi \in C_0^\infty$ and $0 \leq \phi \leq 1$. Notation $\phi_1 \ll \phi_2$ stands for two cutoff functions such that $\phi_2 = 1$ on the support of ϕ_1 .

Take the smooth cutoff functions $\varphi, \varphi_2, \varphi_3 \in C_0^\infty([T_1, T_2])$ and $\psi, \psi_2, \psi_3 \in C_0^\infty(\Omega)$ such that $\varphi \ll \varphi_2 \ll \varphi_3$ and $\psi \ll \psi_2 \ll \psi_3$. Set $f_1 = \varphi(t)\psi(x)f$, $f_2 = \varphi_2(t)\psi_2(x)f$ and $f_3 = \varphi_3(t)\psi_3(x)f$. For $\alpha \in \mathbb{N}^6, |\alpha| \leq 3$, define

$$g = \partial_{x,v}^\alpha (\varphi(t)\psi(x)f) \in L^\infty([T_1, T_2]; L_\ell^2(\mathbb{R}^6)).$$

Then the Leibniz formula yields the following equation :

$$(6.1.3) \quad g_t + v \cdot \partial_x g = Q(f_2, g) + G, \quad (t, x, v) \in \mathbb{R}^7,$$

where

$$(6.1.4) \quad \begin{aligned} G &= \sum_{\alpha_1+\alpha_2=\alpha, 1\leq|\alpha_1|} C_{\alpha_2}^{\alpha_1} \mathcal{Q}(\partial^{\alpha_1} f_2, \partial^{\alpha_2} f_1) \\ &+ \partial^\alpha (\varphi_t \psi(x) f + v \cdot \psi_x(x) \varphi(t) f) + [\partial^\alpha, v \cdot \partial_x](\varphi(t) \psi(x) f) \\ &\equiv (A) + (B) + (C). \end{aligned}$$

Then we can take $W^{2\ell} g$ as a test function for equation of (6.1.3). It follows by integration by parts on $\mathbb{R}^7 = \mathbb{R}_t^1 \times \mathbb{R}_x^3 \times \mathbb{R}_v^3$ that

$$(6.1.5) \quad 0 = \left(W^\ell \mathcal{Q}(f_2, g), W^\ell g \right)_{L^2(\mathbb{R}^7)} + \left(G, W^{2\ell} g \right)_{L^2(\mathbb{R}^7)},$$

which is enough to get the required initial regularity.

6.2. Gain of regularity in v .

Proposition 6.4. *Under the assumption of Theorem 6.1, for any $0 < T_1 < T_2 < T$ and any bounded open domain $\Omega \subset \mathbb{R}_x^3$, one has,*

$$(6.2.1) \quad \Lambda_v^s f_1 \in L^2(\mathbb{R}_t; H_t^3(\mathbb{R}^6)),$$

for any $\ell \in \mathbb{N}$, where $f_1 = \varphi(t) \psi(x) f$ with $\varphi \in C_0^\infty([T_1, T_2])$, $\psi \in C^\infty(\Omega)$.

Proof. Firstly, the local positive lower bound (6.1.2) implies that

$$\inf_{(t,x) \in \text{Supp } \varphi \times \text{Supp } \psi_1} \|f_2(t, x, \cdot)\|_{L^1(\mathbb{R}_v^3)} = c_0 > 0.$$

Thus, the coercivity estimate (2.1.1) gives

$$\begin{aligned} -\left(\mathcal{Q}(f_2, W^\ell g), W^\ell g \right)_{L^2(\mathbb{R}^7)} &= - \int_{t \in \text{Supp } \varphi} \int_{x \in \text{Supp } \psi_1} \left(\mathcal{Q}(f_2, W^\ell g), W^\ell g \right)_{L^2(\mathbb{R}_v^3)} dx dt \\ &\geq \int_{\mathbb{R}_t} \int_{\mathbb{R}_x^3} \left(C_0 \|W^\ell g(t, x, \cdot)\|_{H^s(\mathbb{R}_v^3)}^2 - C \|f_2(t, x, \cdot)\|_{L^1(\mathbb{R}_v^3)} \|W^\ell g(t, x, \cdot)\|_{L^2(\mathbb{R}_v^3)}^2 \right) dx dt \\ &\geq C_0 \|\Lambda_v^s W^\ell g\|_{L^2(\mathbb{R}^7)}^2 - CT \|f_2\|_{L^\infty(\mathbb{R}_t^4; L^1(\mathbb{R}_x^3))} \|W^\ell g\|_{L^\infty([0, T]; L^2(\mathbb{R}^6))}^2, \end{aligned}$$

since

$$\|f_2\|_{L^\infty(\mathbb{R}_t^4; L^1(\mathbb{R}_x^3))} \leq C \|f_2\|_{L^\infty([0, T]; H_t^{3/2+\epsilon}(\mathbb{R}^6))},$$

and

$$\|W^\ell g\|_{L^\infty([0, T]; L^2(\mathbb{R}^6))}^2 \leq C \|f_1\|_{L^\infty([0, T]; H_t^3(\mathbb{R}^6))}^2.$$

Hence, we get, for $l > 3/2$,

$$(6.2.2) \quad \begin{aligned} \|\Lambda_v^s W^\ell g\|_{L^2(\mathbb{R}^7)}^2 &\leq C \|f_2\|_{L^\infty([0, T]; H_t^3(\mathbb{R}^6))}^2 + \left| \left(G, W^{2\ell} g \right)_{L^2(\mathbb{R}^7)} \right| \\ &\quad + \left| \left(W^\ell \mathcal{Q}(f_2, g) - \mathcal{Q}(f_2, W^\ell g), W^\ell g \right)_{L^2(\mathbb{R}^7)} \right|. \end{aligned}$$

By applying Lemma 3.2, we have

$$\begin{aligned} &\left| \left((W^\ell \mathcal{Q}(f_2, g) - \mathcal{Q}(f_2, W^\ell g)), W^\ell g \right)_{L^2(\mathbb{R}^7)} \right| \\ &\leq CT \|f_2\|_{L^\infty([0, T]; H_t^{3/2+\epsilon}(\mathbb{R}^6))} \|g\|_{L^\infty([0, T]; L_t^2(\mathbb{R}^6))} \|g\|_{L_t^2(\mathbb{R}^7)} \\ &\leq CT^2 \|f_2\|_{L^\infty([0, T]; H_t^{3/2+\epsilon}(\mathbb{R}^6))} \|f_1\|_{L^\infty([0, T]; H_t^3(\mathbb{R}^6))}^2. \end{aligned}$$

For the second term in (6.2.2), we shall prove the following claim: for $0 < s < 1/2$, one has

$$(6.2.3) \quad \left| \left(G, W^{2\ell} g \right)_{L^2(\mathbb{R}^7)} \right| \leq C \left(\|f_3\|_{L^\infty([0, T]; H_t^3(\mathbb{R}^6))} + \|f_2\|_{L^\infty([0, T]; H_t^3(\mathbb{R}^6))} \right) \|\Lambda_v^s W^\ell g\|_{L^2(\mathbb{R}_{t,x,v}^7)}.$$

In fact, for the term in (6.1.4), it is easy to get

$$\|(B)\|_{L_t^2(\mathbb{R}^7)}^2 + \|(C)\|_{L_t^2(\mathbb{R}^7)}^2 \leq CT \|f_3\|_{L^\infty([0,T]; H_{\ell+1}^3(\mathbb{R}^6))}^2.$$

For the term (A), recall that $\alpha_1 + \alpha_2 = \alpha$, $|\alpha| \leq 3$ and $|\alpha_2| \leq 2$. In the following, we will apply Theorem 2.1 with $m = -s$. We separate the discussion into two cases.

Case 1. If $|\alpha_1| = 1$, we have

$$\begin{aligned} & \int_{\mathbb{R}_t} \int_{\mathbb{R}_x^3} \|Q(\partial^{\alpha_1} f_2, \partial^{\alpha_2} f_1)(t, x, \cdot)\|_{H_t^s(\mathbb{R}_v^3)}^2 dx dt \\ & \leq C \int_{\mathbb{R}_t} \int_{\mathbb{R}_x^3} \|\partial^{\alpha_1} f_2(t, x, \cdot)\|_{L_{\ell+2s}^1(\mathbb{R}_v^3)}^2 \|\partial^{\alpha_2} f_1(t, x, \cdot)\|_{H_{\ell+2s}^s(\mathbb{R}_v^3)}^2 dx dt \\ & \leq C \|\partial^{\alpha_1} f_2\|_{L^\infty(\mathbb{R}_{t,x}^4; L_{\ell+2s}^1(\mathbb{R}_v^3))}^2 \int_{\mathbb{R}_t} \int_{\mathbb{R}_x^3} \|\partial^{\alpha_2} f_1(t, x, \cdot)\|_{H_{\ell+2s}^s(\mathbb{R}_v^3)}^2 dx dt \\ & \leq CT \|f_2\|_{L^\infty([0,T]; H_{\ell+3}^3(\mathbb{R}^6))}^2 \|f_1\|_{L^\infty([0,T]; H_{\ell+2s}^3(\mathbb{R}^6))}^2. \end{aligned}$$

Case 2. If $|\alpha_1| \geq 2$, then $|\alpha_2| \leq 1$, it follows that

$$\begin{aligned} & \int_{\mathbb{R}_t} \int_{\mathbb{R}_x^3} \|\partial^{\alpha_1} f_2(t, x, \cdot)\|_{L_{\ell+2s}^1(\mathbb{R}_v^3)}^2 \|\partial^{\alpha_2} f_1(t, x, \cdot)\|_{H_{\ell+2s}^s(\mathbb{R}_v^3)}^2 dx dt \\ & \leq C \|\partial^{\alpha_2} f_1\|_{L^\infty(\mathbb{R}_{t,x}^4; H_{\ell+2s}^s(\mathbb{R}_v^3))}^2 \int_{\mathbb{R}_t} \int_{\mathbb{R}_x^3} \|\partial^{\alpha_1} f_2(t, x, \cdot)\|_{L_{\ell+3/2+\delta+2s}^2(\mathbb{R}_v^3)}^2 dx dt \\ & \leq CT \|f_1\|_{L^\infty([0,T]; H_{\ell+2s}^{1+3/2+\epsilon}(\mathbb{R}^6))}^2 \|f_2\|_{L^\infty([0,T]; H_{\ell+3}^3(\mathbb{R}^6))}^2. \end{aligned}$$

By combining these two cases, we have proved

$$\left| (G, W^{2\ell} g)_{L^2(\mathbb{R}^7)} \right| \leq CT \left(\|f_2\|_{L^\infty([0,T]; H_{\ell+1}^3(\mathbb{R}^6))}^2 + \|f_2\|_{L^\infty([0,T]; H_{\ell+3}^3(\mathbb{R}^6))}^2 \|\Lambda_v^s W^\ell g\|_{L^2(\mathbb{R}_{t,x,v}^7)} \right).$$

Therefore, we have proved

$$\|\Lambda_v^s W^\ell g\|_{L^2(\mathbb{R}^7)}^2 \leq C \left(1 + \|f_3\|_{L^\infty([0,T]; H_\ell^3(\mathbb{R}^6))} \right)^4.$$

The proof of Proposition 6.4 is completed. \square

6.3. Gain of regularity in x . First of all, let us consider a transport equation in the form of

$$(6.3.1) \quad f_t + v \cdot \nabla_x f = g \in \mathcal{D}'(\mathbb{R}^{2n+1}),$$

where $(t, x, v) \in \mathbb{R}^{1+n+n} = \mathbb{R}^{2n+1}$. In [7], by using a generalized uncertainty principle, we proved the following hypo-elliptic estimate.

Lemma 6.5. *Assume that $g \in H^{-s'}(\mathbb{R}^{2n+1})$, for some $0 \leq s' < 1$. Let $f \in L^2(\mathbb{R}^{2n+1})$ be a weak solution of the transport equation (6.3.1), such that $\Lambda_v^s f \in L^2(\mathbb{R}^{2n+1})$ for some $0 < s \leq 1$. Then it follows that*

$$\Lambda_x^{s(1-s')/(s+1)} f \in L_{-\frac{s'}{s+1}}^2(\mathbb{R}^{2n+1}), \quad \Lambda_t^{s(1-s')/(s+1)} f \in L_{-\frac{s}{s+1}}^2(\mathbb{R}^{2n+1}),$$

where $\Lambda_\bullet = (1 + |\mathbf{D}_\bullet|^2)^{1/2}$.

We recall a Leibniz type formula for fractional derivatives with respect to variable x (see [9]). Let $0 < \lambda < 1$. Then there exists a positive constant $C_\lambda \neq 0$ such that

$$(6.3.2) \quad \begin{aligned} |D_x|^\lambda Q(f, g) &= Q(|D_x|^\lambda f, g) + Q(f, |D_x|^\lambda g) \\ &\quad + C_\lambda \int_{\mathbb{R}^3} |h|^{-3-\lambda} Q(f_h, g_h) dh, \end{aligned}$$

with

$$f_h(t, x, v) = f(t, x, v) - f(t, x + h, v), \quad h \in \mathbb{R}_x^3.$$

We have the following proposition on the gain of regularity in the variable x through the above uncertainty principle.

Proposition 6.6. *Under the hypothesis of Theorem 6.1, one has*

$$(6.3.3) \quad \Lambda_x^{s_0} f_1 \in L^2(\mathbb{R}_t; H_\ell^3(\mathbb{R}^6)),$$

for any $\ell \in \mathbb{N}$ and $0 < s_0 = \frac{s(1-s)}{(s+1)}$.

Proof. In fact, for any $\ell \in \mathbb{N}$, it follows from Proposition 6.4 that

$$\Lambda_v^s W^\ell g \in L^2(\mathbb{R}^7).$$

Then the upper bound estimation given by Theorem 2.1 with $m = -s$ implies that

$$W^\ell Q(f_2, g) \in L^2(\mathbb{R}_{t,x}^4; H^{-s}(\mathbb{R}_v^3)).$$

On the other hand, (6.2.3) gives

$$W^\ell G \in L^2(\mathbb{R}_{t,x}^4; H^{-s}(\mathbb{R}_v^3)).$$

By using equation (6.1.3), it follows that

$$(6.3.4) \quad \partial_t(W_1 g) + v \cdot \partial_x(W_1 g) = W_1 Q(f_2, g) + W_1 G \in H^{-s}(\mathbb{R}^7).$$

Finally, by using Lemma 6.5 with $s' = s$, we can conclude (6.3.3) and this completes the proof of the proposition. \square

Therefore, under the hypothesis $f \in L^\infty([0, T]; H_\ell^3(\mathbb{R}^6))$ for all $\ell \in \mathbb{N}$, it follows that for any $\ell \in \mathbb{N}$ we have

$$(6.3.5) \quad \Lambda_v^s(\varphi(t)\psi(x)f) \in L^2(\mathbb{R}_t; H_\ell^3(\mathbb{R}^6)), \quad \Lambda_x^{s_0}(\varphi(t)\psi(x)f) \in L^2(\mathbb{R}_t; H_\ell^3(\mathbb{R}^6)).$$

We now try to improve this partial regularity in x variable.

Proposition 6.7. *Let $0 < \lambda < 1$ and $f \in L^\infty([0, T]; H_\ell^3(\mathbb{R}^6))$ be a solution of the equation (1.1). Suppose that, for all $\ell \in \mathbb{N}$ and for any cutoff functions φ, ψ , we have*

$$(6.3.6) \quad \Lambda_v^s(\varphi(t)\psi(x)f) \in H_\ell^3(\mathbb{R}^7), \quad \Lambda_x^\lambda(\varphi(t)\psi(x)f) \in H_\ell^3(\mathbb{R}^7).$$

Then, one has

$$(6.3.7) \quad \|\Lambda_v^s \Lambda_x^\lambda(\varphi(t)\psi(x)f)\|_{L^2(\mathbb{R}_t; H_\ell^3(\mathbb{R}^6))} \leq C \|f\|_{L^\infty([0, T]; H_{\ell+3}^2(\mathbb{R}^6))} \\ \times \left(\|\Lambda_v^s(\varphi(t)\psi(x)f)\|_{H_{\ell+2s}^3(\mathbb{R}^7)} + \|\Lambda_x^\lambda(\varphi(t)\psi(x)f)\|_{H_{\ell+2s}^3(\mathbb{R}^7)} \right),$$

for any $\ell \in \mathbb{N}$ and any cutoff functions φ, ψ .

Remark 6.8. *We can replace, in the left hand side of (6.3.7), the factor $\|f\|_{L^\infty([0, T]; H_{\ell+3}^2(\mathbb{R}^6))}$ by $\|(\varphi(t)\psi(x)f)\|_{H_{\ell+5}^3(\mathbb{R}^7)}$, for some $\delta > 0$. We have to use this point when we shall prove the regularization from $H_{\ell+5}^3(\mathbb{R}^7)$ to $H_\ell^5(\mathbb{R}^7)$.*

Proof. Set $g = \partial_{x,v}^\alpha(\varphi(t)\psi(x)f)$ and $\alpha \in \mathbb{N}^6, |\alpha| \leq 3$. We choose $W^\ell |D_x|^\lambda \psi_2^2(x) |D_x|^\lambda W^\ell g$ as a test function for equation (6.1.3). Then

$$(6.3.8) \quad \left((v \cdot \partial_x \psi_2(x)) |D_x|^\lambda W^\ell g, \psi_2(x) |D_x|^\lambda W^\ell g \right)_{L^2(\mathbb{R}^7)} \\ = \left(\psi_2(x) |D_x|^\lambda W^\ell Q(f_2, g), \psi_2(x) |D_x|^\lambda W^\ell g \right)_{L^2(\mathbb{R}^7)} \\ + \left(\psi_2(x) |D_x|^\lambda W^\ell G, \psi_2(x) |D_x|^\lambda W^\ell g \right)_{L^2(\mathbb{R}^7)}.$$

We have that

$$\left| \left((v \cdot \partial_x \psi_2(x)) |D_x|^\lambda W^\ell g, \psi_2(x) |D_x|^\lambda W^\ell g \right)_{L^2(\mathbb{R}^7)} \right| \leq C \| \Lambda_x^\lambda \partial^\alpha (\varphi(t) \psi(x) f) \|_{L^2_{t+1}(\mathbb{R}^7)},$$

and the same estimation for the linear term of G in (6.1.4)

$$\left| \left(\psi_2(x) |D_x|^\lambda W^\ell ((B) + (C)), \psi_2(x) |D_x|^\lambda W^\ell g \right)_{L^2(\mathbb{R}^7)} \right| \leq C \| \Lambda_x^\lambda \partial^\alpha (\varphi(t) \psi(x) f) \|_{L^2_{t+1}(\mathbb{R}^7)}.$$

For the nonlinear terms of G in (6.1.4), we shall use the Leibniz formula (6.3.2). First of all, the coercivity estimate (2.1.1) gives, as in (6.2.2),

$$\begin{aligned} & - \left(Q(f_2, \psi_2(x) |D_x|^\lambda W^\ell g), \psi_2(x) |D_x|^\lambda W^\ell g \right)_{L^2(\mathbb{R}^7)} \\ & \geq C_0 \| \Lambda_v^s \psi_2(x) |D_x|^\lambda W^\ell g \|_{L^2(\mathbb{R}^7)}^2 - CT \| f_2 \|_{L^\infty(\mathbb{R}^4_x; L^1(\mathbb{R}^3))} \| \psi_2(x) |D_x|^\lambda W^\ell g \|_{L^\infty([0,T]; L^2(\mathbb{R}^6))}^2. \end{aligned}$$

On the other hand, the upper estimate of Theorem 2.1 with $m = -s$ gives,

$$\begin{aligned} & \left| \left(Q(|D_x|^\lambda f_2, \psi_1(x) W^\ell g), \psi_1(x) |D_x|^\lambda W^\ell g \right)_{L^2(\mathbb{R}^7)} \right| \\ & \leq C \| |D_x|^\lambda f_2 \|_{L^\infty(\mathbb{R}^4_x; L^1_{2s}(\mathbb{R}^3))} \| \psi_2(x) \Lambda_v^s W^\ell g \|_{L^2_s(\mathbb{R}^7)} \| \psi_2(x) |D_x|^\lambda \Lambda_v^s W^\ell g \|_{L^2(\mathbb{R}^7)} \\ & \leq C \| f_2 \|_{L^\infty([0,T]; H^{3/2+4+\epsilon}_{3/2+2s+\epsilon}(\mathbb{R}^6))} \| \psi_2(x) \Lambda_v^s W^\ell g \|_{L^2_s(\mathbb{R}^7)} \| \psi_2(x) |D_x|^\lambda \Lambda_v^s W^\ell g \|_{L^2(\mathbb{R}^7)} \\ & \leq \delta \| \psi_2(x) |D_x|^\lambda \Lambda_v^s W^\ell g \|_{L^2(\mathbb{R}^7)}^2 + C_\delta \| f_2 \|_{L^\infty([0,T]; H^3_3(\mathbb{R}^6))} \| \Lambda_v^s g \|_{L^2_{t+2s}(\mathbb{R}^7)}^2. \end{aligned}$$

For the cross term coming from the decomposition (6.3.2), by using again Theorem 2.1 with $m = -s$, we get

$$\begin{aligned} & \int_{\mathbb{R}^3} |h|^{-3-\lambda} \left| \left(Q((f_2)_h, (W^\ell g)_h), \psi_2^2(x) |D_x|^\lambda W^\ell g \right)_{L^2(\mathbb{R}^7)} \right| dh \\ & \leq |C_\lambda| \| \psi_2(x) |D_x|^\lambda \Lambda_v^s W^\ell g \|_{L^2(\mathbb{R}^7)} \\ & \quad \times \int_{\mathbb{R}^3} |h|^{-3-\lambda} \| (f_2)_h \|_{L^\infty(\mathbb{R}^4_x; L^1_{2s}(\mathbb{R}^3))} \| \Lambda_v^s (W^\ell g)_h \|_{L^2_s(\mathbb{R}^7)} dh \\ & \leq \delta \| \psi_1(x) |D_x|^\lambda \Lambda_v^s W^\ell g \|_{L^2(\mathbb{R}^7)}^2 + C_\delta \| f_2 \|_{L^\infty([0,T]; H^3_3(\mathbb{R}^6))} \| \Lambda_v^s g \|_{L^2_{t+2s}(\mathbb{R}^7)}^2. \end{aligned}$$

Hence, we have

$$\begin{aligned} & \left| \left(|D_x|^\lambda Q(f_2, \psi_2(x) W^\ell g) - Q(|D_x|^\lambda f_2, \psi_2(x) W^\ell g), \psi_2(x) |D_x|^\lambda W^\ell g \right)_{L^2(\mathbb{R}^7)} \right| \\ & \leq \delta \| \psi_2(x) |D_x|^\lambda \Lambda_v^s W^\ell g \|_{L^2(\mathbb{R}^7)}^2 + C_\delta \| f_2 \|_{L^\infty([0,T]; H^3_3(\mathbb{R}^6))} \| \Lambda_v^s g \|_{L^2_{t+2s}(\mathbb{R}^7)}^2. \end{aligned}$$

In conclusion, we get

$$\begin{aligned} & \| \psi_2(x) |D_x|^\lambda \Lambda_v^s W^\ell g \|_{L^2(\mathbb{R}^7)}^2 \\ & \leq C \| f_2 \|_{L^\infty([0,T]; H^3_3(\mathbb{R}^6))}^2 \left(\| |D_x|^\lambda g \|_{L^2_{t+2s}(\mathbb{R}^7)}^2 + \| \Lambda_v^s g \|_{L^2_{t+2s}(\mathbb{R}^7)}^2 \right) \\ & \quad + \left| \left(|D_x|^\lambda (W^\ell Q(f_2, g) - Q(f_2, W^\ell g)), \psi_2^2(x) |D_x|^\lambda W^\ell g \right)_{L^2(\mathbb{R}^7)} \right| \\ & \quad + \left| \left(|D_x|^\lambda W^\ell (A), \psi_2^2(x) |D_x|^\lambda W^\ell g \right)_{L^2(\mathbb{R}^7)} \right| \\ & = \text{(I) + (II) + (III)}. \end{aligned}$$

For the term (II), again, formula (6.3.2) yields,

$$\begin{aligned}
& \left(|D_x|^\lambda \left(W^\ell \mathcal{Q}(f_2, g) - \mathcal{Q}(f_2, W^\ell g) \right), \psi_2^2(x) |D_x|^\lambda W^\ell g \right)_{L^2(\mathbb{R}^7)} \\
&= \left(W^\ell \mathcal{Q}(|D_x|^\lambda f_2, g) - \mathcal{Q}(|D_x|^\lambda f_2, W^\ell g), \psi_2^2(x) |D_x|^\lambda W^\ell g \right)_{L^2(\mathbb{R}^7)} \\
&+ \left(W^\ell \mathcal{Q}(f_2, |D_{t,x}|^\lambda g) - \mathcal{Q}(f_2, W^\ell |D_x|^\lambda g), \psi_2^2(x) |D_x|^\lambda W^\ell g \right)_{L^2(\mathbb{R}^7)} \\
&+ C_\lambda \int_{\mathbb{R}^3} |h|^{-3-\lambda} \left((W^\ell \mathcal{Q}((f_2)_h, g_h) - \mathcal{Q}((f_2)_h, W^\ell g_h)), \psi_2^2(x) |D_x|^\lambda W^\ell g \right)_{L^2(\mathbb{R}^7)} dh.
\end{aligned}$$

Since $0 < s < 1/2$, Lemma 3.2 implies

$$\begin{aligned}
& \left| \left((W^\ell \mathcal{Q}(|D_x|^\lambda f_2, g) - \mathcal{Q}(|D_x|^\lambda f_2, W^\ell g)), \psi_2^2(x) |D_x|^\lambda W^\ell g \right)_{L^2(\mathbb{R}^7)} \right| \\
&\leq C \| |D_x|^\lambda f_2 \|_{L^\infty(\mathbb{R}_{t,x}^4, L_t^1(\mathbb{R}_v^3))} \|g\|_{L^2(\mathbb{R}_{t,x}^4, L_t^2(\mathbb{R}_v^3))} \| |D_x|^\lambda g \|_{L_t^2(\mathbb{R}^7)} \\
&\leq C \|f_2\|_{L^\infty([0,T]; H_{t+3/2+\epsilon}^{3/2+\epsilon}(\mathbb{R}^6))} \|g\|_{L_t^2(\mathbb{R}^3)} \| |D_x|^\lambda g \|_{L_t^2(\mathbb{R}^7)},
\end{aligned}$$

and

$$\begin{aligned}
& \left| \left((W^\ell \mathcal{Q}(f_2, |D_x|^\lambda g) - \mathcal{Q}(f_2, W^\ell |D_x|^\lambda g)), \psi_2^2(x) |D_x|^\lambda W^\ell g \right)_{L^2(\mathbb{R}^7)} \right| \\
&\leq C \|f_2\|_{L^\infty(\mathbb{R}_{t,x}^4, L_t^1(\mathbb{R}_v^3))} \| |D_x|^\lambda g \|_{L^2(\mathbb{R}_{t,x}^4, L_t^2(\mathbb{R}_v^3))} \| |D_x|^\lambda g \|_{L_t^2(\mathbb{R}^7)} \\
&\leq C \|f_2\|_{L^\infty([0,T]; H_{t+3/2+\epsilon}^{1+3/2+\epsilon}(\mathbb{R}^6))} \| |D_x|^\lambda g \|_{L_t^2(\mathbb{R}^3)}^2.
\end{aligned}$$

For the cross term,

$$\begin{aligned}
& \left| \int_{\mathbb{R}^3} |h|^{-3-\lambda} \left((W^\ell \mathcal{Q}((f_2)_h, g_h) - \mathcal{Q}((f_2)_h, W^\ell g_h)), \psi_2^2(x) |D_x|^\lambda W^\ell g \right)_{L^2(\mathbb{R}^7)} dh \right| \\
&\leq C \|f_2\|_{L^\infty([0,T]; H_{t+3/2+\epsilon}^{1+3/2+\epsilon}(\mathbb{R}^6))} \|g\|_{L_t^2(\mathbb{R}^3)} \| |D_x|^\lambda g \|_{L_t^2(\mathbb{R}^7)}.
\end{aligned}$$

Thus, we have

$$(II) \leq C \|f_2\|_{L^\infty([0,T]; H_{t+2}^3(\mathbb{R}^6))} \| |D_x|^\lambda g \|_{L_t^2(\mathbb{R}^3)}^2.$$

We now consider the last term (III). Recall that (A) stands for the nonlinear terms from G

$$(A) = \sum_{\alpha_1 + \alpha_2 = \alpha, \alpha_1 \neq 0} C_{\alpha_2}^{\alpha_1} \mathcal{Q}(\partial^{\alpha_1} f_2, \partial^{\alpha_2} f_1).$$

We have

$$\begin{aligned}
& \left| \left(|D_x|^\lambda \left(\mathcal{Q}(\partial^{\alpha_1} f_2, \partial^{\alpha_2} f_1) \right), W^\ell \psi_2^2(x) |D_x|^\lambda W^\ell g \right)_{L^2(\mathbb{R}^7)} \right| \\
&\leq C \|\Lambda_v^s \psi_1(x) |D_x|^\lambda W^\ell g\|_{L^2(\mathbb{R}^7)} \times \\
&\times \left\{ \left\| \mathcal{Q}(|D_{t,x}|^\lambda \partial^{\alpha_1} f_2, \partial^{\alpha_2} f_1) \right\|_{L^2(\mathbb{R}_{t,x}^4, H_t^{-s}(\mathbb{R}_v^3))} + \left\| \mathcal{Q}(\partial^{\alpha_1} f_2, |D_x|^\lambda \partial^{\alpha_2} f_1) \right\|_{L^2(\mathbb{R}_{t,x}^4, H_t^{-s}(\mathbb{R}_v^3))} \right. \\
&\left. + \left\| \int h^{-3-\lambda} \mathcal{Q}(\partial^{\alpha_1} (f_2)_h, \partial^{\alpha_2} (f_1)_h) dh \right\|_{L^2(\mathbb{R}_{t,x}^4, H_{|h|/2}^m(\mathbb{R}_v^3))} \right\}.
\end{aligned}$$

We divide the discussion into two cases.

Case 1. $|\alpha_1| = 1$ (then $|\alpha_2| \leq 2$). Applying (2.1.2) with $m = -s$. We have

$$\begin{aligned} & \left\| \mathcal{Q}(|D_x|^\lambda \partial^{\alpha_1} f_2, \partial^{\alpha_2} f_1) \right\|_{L^2(\mathbb{R}_{t,x}^4; H_\ell^{-s}(\mathbb{R}_v^3))} \\ & \leq C \|\Lambda_x^\lambda \partial^{\alpha_1} f_2\|_{L^2(\mathbb{R}_t; (L^\infty(\mathbb{R}_x^3; L_{\ell+2s}^1(\mathbb{R}_v^3)))} \|\Lambda_v^s \partial^{\alpha_2} f_1\|_{L^\infty(\mathbb{R}_t; L_{\ell+2s}^2(\mathbb{R}_v^6))} \\ & \leq C \|\Lambda_x^\lambda f_2\|_{H_{\ell+3/2+\epsilon+2s}^{1+3/2+\epsilon}(\mathbb{R}^7)} \|\Lambda_v^s f_1\|_{H_{\ell+2s}^{2+1/2+\epsilon}(\mathbb{R}^7)}, \\ & \left\| \mathcal{Q}(\partial^{\alpha_1} f_2, |D_x|^\lambda \partial^{\alpha_2} f_1) \right\|_{L^2(\mathbb{R}_{t,x}^4; H_\ell^{-s}(\mathbb{R}_v^3))} \\ & \leq C \|\partial^{\alpha_1} f_2\|_{L^\infty(\mathbb{R}_{t,x}^4; L_{\ell+2s}^1(\mathbb{R}_v^3))} \|\Lambda_v^s \Lambda_x^\lambda \partial^{\alpha_2} f_1\|_{L_{\ell+2s}^2(\mathbb{R}^7)} \\ & \leq C \|f\|_{L^\infty([0,T]; H_{\ell+3/2+\epsilon+2s}^{1+3/2+\epsilon}(\mathbb{R}^6))} \|\Lambda_v^s f_1\|_{H_{\ell+2s}^3(\mathbb{R}^7)}, \end{aligned}$$

and

$$\begin{aligned} & \left\| \int_{\mathbb{R}^3} h^{-3-\lambda} \mathcal{Q}(\partial^{\alpha_1}(f_2)_h, \partial^{\alpha_2}(f_1)_h) dh \right\|_{L^2(\mathbb{R}_{t,x}^4; H_\ell^{-s}(\mathbb{R}_v^3))} \\ & \leq C \int |h|^{-3-\lambda} \|\partial^{\alpha_1}(f_2)_h\|_{L^\infty(\mathbb{R}_{t,x}^4; L_{\ell+2s}^1(\mathbb{R}_v^3))} \|\Lambda_v^s \partial^{\alpha_2}(f_1)_h\|_{L_{\ell+2s}^2(\mathbb{R}^7)} dh \\ & \leq C \|\partial^{\alpha_1} f_2\|_{L^\infty(\mathbb{R}_{t,x}^4; L_{\ell+2s}^1(\mathbb{R}_v^3))} \|\Lambda_v^s \partial^{\alpha_2} \nabla_x f_1\|_{L_{\ell+2s}^2(\mathbb{R}^7)} \\ & \leq C \|f\|_{L^\infty([0,T]; H_{\ell+3/2+\epsilon+2s}^{1+3/2+\epsilon}(\mathbb{R}^6))} \|\Lambda_v^s f_1\|_{H_{\ell+2s}^3(\mathbb{R}^7)}. \end{aligned}$$

Case 2. $|\alpha_1| \geq 2$. By the same argument as above, one has

$$\begin{aligned} & \left\| \mathcal{Q}(|D_x|^\lambda \partial^{\alpha_1} f_2, \partial^{\alpha_2} f_1) \right\|_{L^2(\mathbb{R}_{t,x}^4; H_\ell^{-s}(\mathbb{R}_v^3))} + \left\| \mathcal{Q}(\partial^{\alpha_1} f_2, |D_x|^\lambda \partial^{\alpha_2} f_1) \right\|_{L^2(\mathbb{R}_{t,x}^4; H_\ell^{-s}(\mathbb{R}_v^3))} \\ & \leq C \|f\|_{L^\infty([0,T]; H_{\ell+3/2+\epsilon+2s}^{1+3/2+\epsilon}(\mathbb{R}^6))} \left(\|\Lambda_v^s f_1\|_{H_{\ell+2s}^3(\mathbb{R}^7)} + \|\Lambda_x^\lambda f_1\|_{H_{\ell+2s}^3(\mathbb{R}^7)} \right). \end{aligned}$$

When $|\alpha_1| = 2$, we have

$$\begin{aligned} & \left\| \int_{\mathbb{R}^3} h^{-3-\lambda} \mathcal{Q}(\partial^{\alpha_1}(f_2)_h, \partial^{\alpha_2}(f_1)_h) dh \right\|_{L^2(\mathbb{R}_{t,x}^4; H_\ell^{-s}(\mathbb{R}_v^3))} \\ & \leq C \int |h|^{-3-\lambda} \|\partial^{\alpha_1}(f_2)_h\|_{L^\infty(\mathbb{R}_t; (L^2(\mathbb{R}_x^3; L_{\ell+2s}^1(\mathbb{R}_v^3)))} \|\Lambda_v^s \partial^{\alpha_2}(f_1)_h\|_{L^2(\mathbb{R}_t; (L^\infty(\mathbb{R}_x^3; L_{\ell+2s}^2(\mathbb{R}_v^3)))} dh \\ & \leq C \|\nabla_x \partial^{\alpha_1} f_2\|_{L^\infty(\mathbb{R}_t; (L^2(\mathbb{R}_x^3; L_{\ell+2s}^1(\mathbb{R}_v^3)))} \|\Lambda_v^s \partial^{\alpha_2} f_1\|_{L^2(\mathbb{R}_t; (L^\infty(\mathbb{R}_x^3; L_{\ell+2s}^2(\mathbb{R}_v^3)))} \\ & \leq C \|f\|_{L^\infty([0,T]; H_{\ell+3/2+\epsilon+2s}^3(\mathbb{R}^6))} \|\Lambda_v^s f_1\|_{H_{\ell+2s}^3(\mathbb{R}^7)}, \end{aligned}$$

while when $|\alpha_1| = 3$, we have

$$\begin{aligned} & \left\| \int_{\mathbb{R}^3} h^{-3-\lambda} \mathcal{Q}(\partial^\alpha(f_2)_h, (f_1)_h) dh \right\|_{L^2(\mathbb{R}_{t,x}^4; H_\ell^{-s}(\mathbb{R}_v^3))} \\ & \leq C \int |h|^{-3-\lambda} \|\partial^\alpha(f_2)_h\|_{L^2(\mathbb{R}_{t,x}^4; L_{\ell+2s}^1(\mathbb{R}_v^3))} \|\Lambda_v^s (f_1)_h\|_{L^\infty(\mathbb{R}_{t,x}^4; L_{\ell+2s}^2(\mathbb{R}_v^3))} dh \\ & \leq C \|\partial^\alpha f_2\|_{L^2(\mathbb{R}_{t,x}^4; L_{\ell+2s}^1(\mathbb{R}_v^3))} \|\Lambda_v^s \nabla_x f_1\|_{L^\infty(\mathbb{R}_{t,x}^4; L_{\ell+2s}^2(\mathbb{R}_v^3))} \\ & \leq C \|f\|_{L^\infty([0,T]; H_{\ell+3/2+\epsilon+2s}^3(\mathbb{R}^6))}^2. \end{aligned}$$

Thus, by Cauchy Schwarz inequality, we obtain

$$(III) \leq \delta \|\Lambda_v^s \psi_1(x) |D_x|^\lambda W^\ell g\|_{L^2(\mathbb{R}^7)}^2 + C_\delta \|f\|_{L^\infty([0,T]; H_{\ell+3}^3(\mathbb{R}^6))}^2 \left(\|\Lambda_v^s f_1\|_{H_{\ell+2s}^3(\mathbb{R}^7)}^2 + \|\Lambda_x^\lambda f_1\|_{H_{\ell+2s}^3(\mathbb{R}^7)}^2 \right).$$

Finally, we get

$$\|\Lambda_v^s \psi_1(x) |D_x|^\lambda W^\ell g\|_{L^2(\mathbb{R}^7)}^2 \leq C_\delta \|f\|_{L^\infty([0,T]; H_{\ell+3}^3(\mathbb{R}^6))}^2 \left(\|\Lambda_v^s f_1\|_{H_{\ell+2s}^3(\mathbb{R}^7)}^2 + \|\Lambda_x^\lambda f_1\|_{H_{\ell+2s}^3(\mathbb{R}^7)}^2 \right).$$

Therefore, we have completed the proof of Proposition 6.7. \square

We are now ready to prove the following regularity result on the solution with respect to the x variable.

Proposition 6.9. *Under the hypothesis of Theorem 6.1, one has*

$$(6.3.9) \quad \Lambda_x^{1+\varepsilon}(\varphi(t)\psi(x)f) \in H_\ell^3(\mathbb{R}^7),$$

for any $\ell \in \mathbb{N}$ and some $\varepsilon > 0$.

Proof. By fixing $s_0 = \frac{s(1-s)}{(s+1)}$, then (6.3.5) and Proposition 6.7 with $\lambda = s_0$ imply

$$\Lambda_v^s \Lambda_x^{s_0} g \in H_\ell^3(\mathbb{R}^7).$$

It follows that,

$$(\Lambda_x^{s_0} g)_t + v \cdot \partial_x (\Lambda_x^{s_0} g) = \Lambda_x^{s_0} Q(f_2, g) + \Lambda_x^{s_0} G \in H_\ell^{-s}(\mathbb{R}^7).$$

By applying Lemma 6.5 with $s' = s$, we can deduce that

$$\Lambda_x^{s_0+s_0}(\varphi(t)\psi(x)f) \in H_\ell^3(\mathbb{R}^7),$$

for any $\ell \in \mathbb{N}$. If $2s_0 < 1$, using again Proposition 6.7 with $\lambda = 2s_0$ and Lemma 6.5 with $s' = s$, we have

$$\Lambda_v^s(\varphi(t)\psi(x)f), \Lambda_x^{2s_0}(\varphi(t)\psi(x)f) \in H_\ell^3(\mathbb{R}^7) \Rightarrow \Lambda_x^{3s_0}(\varphi(t)\psi(x)f) \in H_\ell^3(\mathbb{R}^7).$$

Choose $k_0 \in \mathbb{N}$ such that

$$k_0 s_0 < 1, \quad (k_0 + 1)s_0 = 1 + \varepsilon > 1.$$

Finally, (6.3.9) follows from Proposition 6.7 with $\lambda = k_0 s_0$ by induction. And this completes the proof of the proposition 6.9. \square

6.4. High order regularity. In this subsection, we give the proof of Theorem 6.3 using the above preparations. From Proposition 6.4, Proposition 6.9 and equation (1.1), it follows that for any $\ell \in \mathbb{N}$,

$$(6.4.1) \quad \Lambda_v^s(\varphi(t)\psi(x)f), \quad \nabla_x(\varphi(t)\psi(x)f) \in H_\ell^3(\mathbb{R}^7).$$

These facts will be used to get the high order regularity with respect to the variable v .

Proposition 6.10. *Let $0 < \lambda < 1$. Suppose that, for any cutoff functions $\varphi \in C_0^\infty([T_1, T_2])$, $\psi \in C_0^\infty(\Omega)$ and all $\ell \in \mathbb{N}$,*

$$(6.4.2) \quad \Lambda_v^\lambda(\varphi(t)\psi(x)f), \quad \nabla_x(\varphi(t)\psi(x)f) \in H_\ell^3(\mathbb{R}^7).$$

Then, for any cutoff function and any $\ell \in \mathbb{N}$,

$$(6.4.3) \quad \Lambda_v^{\lambda+s}(\varphi(t)\psi(x)f) \in H_\ell^3(\mathbb{R}^7).$$

We choose $W^\ell \Lambda_v^{2\lambda} W^\ell g$ as a test function for equation (6.1.3). The proof is similar to that of Proposition 6.7. The only difference is in the estimation of the linear commutators terms

$$\left| \left([\Lambda_v^\lambda, v] \cdot \partial_x W^\ell g, \Lambda_v^\lambda W^\ell g \right)_{L^2(\mathbb{R}^7)} \right| \leq C \|\Lambda_v^\lambda g\|_{L_\ell^2(\mathbb{R}^7)} \|\nabla_x g\|_{L_\ell^2(\mathbb{R}^7)},$$

since

$$[\Lambda_v^\lambda, v] \cdot \partial_x = \lambda \Lambda_v^{\lambda-2} \partial_v \cdot \partial_x,$$

and $\Lambda_v^{\lambda-2} \partial_v$ are bounded operators in L^2 . This is the reason why we prove in the first step the regularity of order 1 with respect to x .

Exactly as Proposition 6.3.9, we can get

$$(6.4.4) \quad \Lambda_v^{1+\varepsilon}(\varphi(t)\psi(x)f) \in H_\ell^3(\mathbb{R}^7),$$

for any $\ell \in \mathbb{N}$ and some $\varepsilon > 0$.

In conclusion, we have proved that there exists $\varepsilon > 0$ such that for any $\ell \in \mathbb{N}$, and any cutoff functions $\varphi(t)$ and $\psi(x)$,

$$\varphi(t)\psi(x)f \in H_t^{4+\varepsilon}(\mathbb{R}^7).$$

Using Remark 6.8, the proof of

$$f \in \mathcal{H}_\ell^{4+\varepsilon}(]T_1, T_2[\times \Omega \times \mathbb{R}_v^3), \forall \ell \in \mathbb{N} \implies f \in \mathcal{H}_\ell^5(]T_1, T_2[\times \Omega \times \mathbb{R}_v^3), \forall \ell \in \mathbb{N}.$$

is easier and therefore omitted, ending the proof of Theorem 6.3 .

7. GLOBAL EXISTENCE

We shall establish a global energy estimate for the Cauchy problem (1.9). Below, generic constants will be dropped out from the estimates, unless essential, in order for the inequality to have a simpler appearance.

7.1. Macroscopic Energy Estimate. As in [31, 32], we introduce the macro-micro decomposition near an absolute Maxwellian μ :

$$(7.1.1) \quad g = \mathbf{P}g + (\mathbf{I} - \mathbf{P})g = g_1 + g_2, \quad \mathbf{P}g = g_1 = (a + b \cdot v + c|v|^2)\mu^{1/2},$$

which reduces (1.9) to

$$\begin{aligned} \partial_t \{a + b \cdot v + c|v|^2\} \mu^{1/2} + v \cdot \nabla_x \{a + b \cdot v + |v|^2 c\} \mu^{1/2} \\ = -(\partial_t + v \cdot \nabla_x)g_2 + \mathcal{L}g_2 + \Gamma(g, g). \end{aligned}$$

Using

$$v \cdot \nabla_x b \cdot v = \sum_{i,j} v_i v_j \partial_i b_j = \sum_i v_i^2 \partial_i b_i + \sum_{i>j} v_i v_j (\partial_i b_j + \partial_j b_i)$$

we deduce

$$(7.1.2) \quad \begin{cases} \text{(i)} & v_i |v|^2 \mu^{1/2} : & \nabla_x c & = -\partial_t r_c + l_c + h_c, \\ \text{(ii)} & v_i^2 \mu^{1/2} : & \partial_t c + \partial_i b_i & = -\partial_t r_i + l_i + h_i, \\ \text{(iii)} & v_i v_j \mu^{1/2} : & \partial_i b_j + \partial_j b_i & = -\partial_t r_{ij} + l_{ij} + h_{ij}, \quad i \neq j, \\ \text{(iv)} & v_i \mu^{1/2} : & \partial_t b_i + \partial_i a & = -\partial_t r_{bi} + l_{bi} + h_{bi}, \\ \text{(v)} & \mu^{1/2} : & \partial_t a & = -\partial_t r_a + l_a + h_a, \end{cases}$$

where

$$r = (g_2, e)_{L_v^2}, \quad l = -(v \cdot \nabla_x g_2 + \mathcal{L}g_2, e)_{L_v^2}, \quad h = (\Gamma(g, g), e)_{L^2(\mathbb{R}_v^3)},$$

stand for r_c, \dots, h_a , while

$$e \in \text{span}\{v_i |v|^2 \mu^{1/2}, v_i^2 \mu^{1/2}, v_i v_j \mu^{1/2}, v_i \mu^{1/2}, \mu^{1/2}\}.$$

Lemma 7.1. *Let $\partial^\alpha = \partial_x^\alpha$, $\alpha \in \mathbb{N}^3$, $|\alpha| \leq m$, $m \geq 3$. Then,*

$$(7.1.3) \quad \|\partial^\alpha (a, b, c)^2\|_{L^2(\mathbb{R}_x^3)} \leq \|\nabla_x (a, b, c)\|_{H^{m-1}(\mathbb{R}_x^3)} \|(a, b, c)\|_{H^{m-1}(\mathbb{R}_x^3)}.$$

Proof. Let $k = |\alpha|$.

$$(k = 0) \quad \|(a, b, c)^2\|_{L^2(\mathbb{R}_x^3)} \leq \|(a, b, c)\|_{L^6(\mathbb{R}_x^3)} \|(a, b, c)\|_{L^3(\mathbb{R}_x^3)} \leq \|\nabla_x (a, b, c)\|_{L^2(\mathbb{R}_x^3)} \|(a, b, c)\|_{H^2(\mathbb{R}_x^3)},$$

because

$$\begin{aligned}
& \|ab\|_{L^2(\mathbb{R}_x^3)} \leq \|a\|_{L^6(\mathbb{R}_x^3)} \|b\|_{L^3(\mathbb{R}_x^3)} \leq \|\nabla_x a\|_{L^2(\mathbb{R}_x^3)} \|b\|_{L^\infty(\mathbb{R}_x^3)}^{1/3} \|b\|_{L^2(\mathbb{R}_x^3)}^{2/3} \leq \|\nabla_x a\|_{L^2(\mathbb{R}_x^3)} \|b\|_{H^2(\mathbb{R}_x^3)}. \\
(k=1) \quad & \|\partial(a, b, c)^2\|_{L^2(\mathbb{R}_x^3)} \leq \|(a, b, c)\partial(a, b, c)\|_{L^2(\mathbb{R}_x^3)} \leq \|(a, b, c)\|_{L^\infty(\mathbb{R}_x^3)} \|\nabla_x(a, b, c)\|_{L^2(\mathbb{R}_x^3)} \\
& \leq \|(a, b, c)\|_{H^{m-1}(\mathbb{R}_x^3)} \|\nabla_x(a, b, c)\|_{L^2(\mathbb{R}_x^3)}. \\
(k \geq 2) \quad & \|\partial^\alpha(a, b, c)^2\|_{L^2(\mathbb{R}_x^3)} \leq \sum_{k' \leq \frac{k}{2}} \|\partial_x^{k'}(a, b, c)\partial_x^{k-k'}(a, b, c)\|_{L^2(\mathbb{R}_x^3)} \\
& \leq \sum \|\partial_x^{k'}(a, b, c)\|_{L^\infty(\mathbb{R}_x^3)} \|\partial_x^{k-k'}(a, b, c)\|_{L^2(\mathbb{R}_x^3)} \leq C \|(a, b, c)\|_{H^{m-1}(\mathbb{R}_x^3)} \|\nabla_x(a, b, c)\|_{H^{m-1}(\mathbb{R}_x^3)}.
\end{aligned}$$

□

Lemma 7.2. (Estimate of r, l, h).

Let $\partial^\alpha = \partial_x^\alpha$, $\partial_i = \partial_{x_i}$, $|\alpha| \leq N-1$, $N \geq 3$.

$$(7.1.4) \quad \|\partial_i \partial^\alpha r\|_{L^2(\mathbb{R}_x^3)} + \|\partial^\alpha l\|_{L_x^2} \leq \|g_2\|_{H^N(\mathbb{R}_x^3, L^2(\mathbb{R}_v^3))} \equiv A_1.$$

$$(7.1.5) \quad \|\partial^\alpha h\|_{L^2(\mathbb{R}_x^3)} \leq \|\nabla_x(a, b, c)\|_{H^{N-2}(\mathbb{R}_x^3)} \|(a, b, c)\|_{H^{N-1}(\mathbb{R}_x^3)} \\ + \|(a, b, c)\|_{H^{N-1}(\mathbb{R}_x^3)} \|g_2\|_{H^{N-1}(\mathbb{R}_x^3, L^2(\mathbb{R}_v^3))} + \|g_2\|_{H^{N-1}(\mathbb{R}_x^3, L^2(\mathbb{R}_v^3))}^2 \equiv A_2.$$

Proof. (7.1.4) follows from

$$\|\partial_i \partial^\alpha r\|_{L^2(\mathbb{R}_x^3)} \leq \|(\partial_i \partial^\alpha g_2, e)\|_{L^2(\mathbb{R}_v^3)} \|L^2(\mathbb{R}_x^3)\| \leq \|\partial_i \partial^\alpha g_2\|_{L^2(\mathbb{R}_{x,v}^6)},$$

and

$$\|\partial^\alpha l\|_{L_x^2} \leq \|(\nabla_x \partial^\alpha g_2, ve)\|_{L^2(\mathbb{R}_v^3)} + (\partial^\alpha g_2, \mathcal{L}^* e)\|_{L^2(\mathbb{R}_v^3)} \|L^2(\mathbb{R}_x^3)\| \leq \|\partial_x^\alpha g_2\|_{H^1(\mathbb{R}_x^3, L^2(\mathbb{R}_v^3))}.$$

We prove (7.1.5) as follows

$$\begin{aligned}
h &= \int \int \int b(\cos \theta) \mu_*^{1/2} (g'_* g' - g_* g) e dv dv_* d\sigma \\
&= \int \int \int b(\cos \theta) g g_* (\mu^{1/2})'_* e' - \mu_*^{1/2} e) dv dv_* d\sigma \\
&= \int \int \int b(\cos \theta) (\mu^{1/2} g) (\mu^{1/2} g)_* (q(v') - q(v)) dv dv_* d\sigma \\
&\equiv \Phi(g, g) = \sum_{i,j=1}^2 \Phi(g_i, g_j) = \sum_{i,j=1}^2 \Phi^{(ij)}.
\end{aligned}$$

where $q = q(v)$ is some polynomial of v . Firstly, we have

$$\Phi^{(11)} \sim (a, b, c)^2 \Phi(\psi_j, \psi_k),$$

where

$$(7.1.6) \quad \psi_j(v) = \mu^{1/2} p_j(v), \quad p_j(v) \in \text{span}\{1, v, |v|^2\} \quad (\text{collision invariants}).$$

Clearly, $|\Phi(\psi_j, \psi_k)| < \infty$, so that by virtue of Lemma 7.1,

$$\|\partial^\alpha \Phi^{(11)}\|_{L^2(\mathbb{R}_x^3)} \leq \|\partial^\alpha(a, b, c)^2\|_{L^2(\mathbb{R}_x^3)} \leq \|\nabla_x(a, b, c)\|_{H^{N-2}(\mathbb{R}_x^3)} \|(a, b, c)\|_{H^{N-1}(\mathbb{R}_x^3)}.$$

On the other hand,

$$\|\Phi(g, f)\|_{L^2(\mathbb{R}_x^3)} \leq \|\mu^{1/2} g\|_{L^2(\mathbb{R}_x^3, L^1_3(\mathbb{R}_v^3))} \|\mu^{1/2} f\|_{L^2(\mathbb{R}_x^3, L^1_3(\mathbb{R}_v^3))} \leq \|g\|_{L^2(\mathbb{R}_{x,v}^6)} \|f\|_{L^2(\mathbb{R}_{x,v}^6)}$$

which yields for $|\alpha| \leq N - 1$,

$$\begin{aligned} \|\partial^\alpha \Phi^{(12)}\|_{L^2(\mathbb{R}_x^3)} &\leq \|\partial^\alpha(a, b, c)\|_{L^2(\mathbb{R}_x^3)} \|\partial^\alpha g_2\|_{L^2(\mathbb{R}_{x,v}^6)} \leq \|(a, b, c)\|_{H^{N-1}(\mathbb{R}_x^3)} \|g_2\|_{H^{N-1}(\mathbb{R}_x^3, L^2(\mathbb{R}_v^3))}, \\ \|\partial^\alpha \Phi^{(21)}\|_{L^2(\mathbb{R}_x^3)} &\leq \|\partial^\alpha g_2\|_{L^2(\mathbb{R}_{x,v}^6)} \|\partial^\alpha(a, b, c)\|_{L^2(\mathbb{R}_x^3)} \leq \|g_2\|_{H^{N-1}(\mathbb{R}_x^3, L^2(\mathbb{R}_v^3))} \|(a, b, c)\|_{H^{N-1}(\mathbb{R}_x^3)}, \\ \|\partial^\alpha \Phi^{(22)}\|_{L^2(\mathbb{R}_x^3)} &\leq \|\partial^\alpha g_2\|_{L^2(\mathbb{R}_{x,v}^6)}^2 \leq \|g_2\|_{H^{N-1}(\mathbb{R}_x^3, L^2(\mathbb{R}_v^3))}^2. \end{aligned}$$

Now the proof of (7.1.5) is completed. \square

Lemma 7.3. (Macro-energy estimate) *Let $|\alpha| \leq N - 1$.*

$$(7.1.7) \quad \|\nabla_x \partial^\alpha(a, b, c)\|_{L^2(\mathbb{R}_x^3)}^2 \leq -\frac{d}{dt} \left\{ (\partial^\alpha r, \nabla_x \partial^\alpha(a, b, c))_{L^2(\mathbb{R}_x^3)} + (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}_x^3)} \right\} \\ + C \|g_2\|_{H^N(\mathbb{R}_x^3, L^2(\mathbb{R}_v^3))}^2 + \mathcal{D}_1 \mathcal{E}_1,$$

where

$$(7.1.8) \quad \mathcal{D}_1 = \|\nabla_x(a, b, c)\|_{H^{N-1}(\mathbb{R}_x^3)}^2 + \|g_2\|_{H^N(\mathbb{R}_x^3, L^2(\mathbb{R}_v^3))}^2$$

is a dissipation rate and

$$(7.1.9) \quad \mathcal{E}_1 = \|(a, b, c)\|_{H^{N-1}(\mathbb{R}_x^3)}^2 + \|g_2\|_{H^{N-1}(\mathbb{R}_x^3, L^2(\mathbb{R}_v^3))}^2 = \|g\|_{H^{N-1}(\mathbb{R}_x^3, L^2(\mathbb{R}_v^3))}^2$$

is a temporal energy functional.

Proof. (a) Estimate of $\nabla_x \partial^\alpha a$. Let A_1, A_2 be as in Lemma 7.2. From (7.1.2) (iv),

$$\begin{aligned} \|\nabla_x \partial^\alpha a\|_{L^2(\mathbb{R}_x^3)}^2 &= (\nabla_x \partial^\alpha a, \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}_x^3)} \\ &= (\partial^\alpha(-\partial_t b - \partial_t r + l + h), \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}_x^3)} \\ &\leq R_1 + C_\eta(A_1^2 + A_2^2) + \eta \|\nabla_x \partial^\alpha a\|_{L^2(\mathbb{R}_x^3)}^2. \end{aligned}$$

$$\begin{aligned} R_1 &= -(\partial^\alpha \partial_t b + \partial^\alpha \partial_t r, \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}_x^3)} \\ &= -\frac{d}{dt} (\partial^\alpha(b + r), \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}_x^3)} + (\nabla_x \partial^\alpha(b + r), \partial_t \partial^\alpha a)_{L^2(\mathbb{R}_x^3)} \\ &\leq -\frac{d}{dt} (\partial^\alpha(b + r), \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}_x^3)} + C_\eta (\|\nabla_x \partial^\alpha b\|_{L^2(\mathbb{R}_x^3)}^2 + A_1^2) + \eta \|\partial_t \partial^\alpha a\|_{L^2(\mathbb{R}_x^3)}^2, \end{aligned}$$

(b) Estimate of $\nabla_x \partial^\alpha b$. From (7.1.2) (iii) and (ii),

$$\begin{aligned} \Delta_x \partial^\alpha b_i + \partial_i^2 \partial^\alpha b_i &= \sum_{j \neq i} \partial_j \partial^\alpha (\partial_j b_i + \partial_i b_j) + \partial_i \partial^\alpha (2\partial_i b_i - \sum_{j \neq i} \partial_i b_j) \\ &= \partial_i \partial^\alpha (-\partial_t r + l + h), \\ \|\nabla_x \partial^\alpha b\|_{L^2(\mathbb{R}_x^3)}^2 + \|\partial_i \partial^\alpha b\|_{L^2(\mathbb{R}_x^3)}^2 &= -(\Delta_x \partial^\alpha b_i + \partial_i^2 \partial^\alpha b_i, \partial^\alpha b)_{L^2(\mathbb{R}_x^3)} = R_2 + R_3 + R_4, \end{aligned}$$

where

$$\begin{aligned} R_2 &= (\partial_t \partial^\alpha r, \partial_i \partial^\alpha b)_{L^2(\mathbb{R}_x^3)} = \frac{d}{dt} (\partial^\alpha r, \partial_i \partial^\alpha b)_{L^2(\mathbb{R}_x^3)} + (\partial_i \partial^\alpha r, \partial_t \partial^\alpha b)_{L^2(\mathbb{R}_x^3)} \\ &\leq \frac{d}{dt} (\partial^\alpha r, \partial_i \partial^\alpha b)_{L^2(\mathbb{R}_x^3)} + C_\eta A_1^2 + \eta \|\partial_t \partial^\alpha b\|_{L^2(\mathbb{R}_x^3)}^2, \\ R_3 &= -(\partial^\alpha l, \partial_i \partial^\alpha b)_{L^2(\mathbb{R}_x^3)} \leq C_\eta A_1^2 + \eta \|\partial_i \partial^\alpha b\|_{L^2(\mathbb{R}_x^3)}^2, \\ R_4 &= -(\partial^\alpha h, \partial_i \partial^\alpha b)_{L^2(\mathbb{R}_x^3)} \leq C_\eta A_2^2 + \eta \|\partial_i \partial^\alpha b\|_{L^2(\mathbb{R}_x^3)}^2. \end{aligned}$$

(c) Estimate of $\nabla_x \partial^\alpha c$. From (7.1.2) (i),

$$\begin{aligned} \|\nabla_x \partial^\alpha c\|_{L^2(\mathbb{R}_x^3)}^2 &= (\nabla_x \partial^\alpha c, \nabla_x \partial^\alpha c)_{L^2(\mathbb{R}_x^3)} = (\partial^\alpha (-\partial_t r + l + h), \nabla_x \partial^\alpha c)_{L^2(\mathbb{R}_x^3)} \\ &\leq R_5 + C_\eta (A_2^2 + A_3^2) + \eta \|\nabla_x \partial^\alpha c\|_{L^2(\mathbb{R}_x^3)}^2 \\ R_5 &= -(\partial^\alpha \partial_t r, \nabla_x \partial^\alpha c)_{L^2(\mathbb{R}_x^3)} = -\frac{d}{dt} (\partial^\alpha r, \nabla_x \partial^\alpha c)_{L^2(\mathbb{R}_x^3)} + (\nabla_x \partial^\alpha r, \partial_t \partial^\alpha c)_{L^2(\mathbb{R}_x^3)} \\ &\leq -\frac{d}{dt} (\partial^\alpha r, \nabla_x \partial^\alpha c)_{L^2(\mathbb{R}_x^3)} + C_\eta A_1^2 + \eta \|\partial_t \partial^\alpha c\|_{L^2(\mathbb{R}_x^3)}^2, \end{aligned}$$

(d) Estimate of $\partial_t \partial^\alpha (a, b, c)$.

$$\begin{aligned} (7.1.10) \quad \|\partial_t \partial^\alpha (a, b, c)\|_{L^2(\mathbb{R}_x^3)} &= \|\partial^\alpha \partial_t P g\|_{L^2(\mathbb{R}_{x,v}^6)} \\ &= \|\partial^\alpha P(-v \cdot \nabla_x g - \mathcal{L}g + \Gamma(g, g))\|_{L^2(\mathbb{R}_{x,v}^6)} \\ &= \|\partial^\alpha P(v \cdot \nabla_x g)\|_{L^2(\mathbb{R}_{x,v}^6)} \leq \|\nabla_x \partial^\alpha (a, b, c)\|_{L^2(\mathbb{R}_x^3)} + \|\nabla_x \partial^\alpha g_2\|_{L^2(\mathbb{R}_{x,v}^6)}. \end{aligned}$$

Combining all the above estimates and taking $\eta > 0$ sufficiently small, we deduce

$$(7.1.11) \quad \|\nabla_x \partial^\alpha (a, b, c)\|_{L^2(\mathbb{R}_x^3)}^2 \leq -\frac{d}{dt} \left\{ (\partial^\alpha r, \nabla_x \partial^\alpha (a, b, c))_{L^2(\mathbb{R}_x^3)} + (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}_x^3)} \right\} + A_1^2 + A_2^2 + \eta \|\nabla_x \partial^\alpha g_2\|_{L^2(\mathbb{R}_{x,v}^6)}^2.$$

Now we choose $|\alpha| \leq N - 1$. Using Lemma (7.2),

$$A_1^2 + A_2^2 + \eta \|\nabla_x \partial^\alpha g_2\|_{L^2(\mathbb{R}_{x,v}^6)}^2 \leq \mathcal{D}_1 \mathcal{E}_1 + \eta \|g_2\|_{H_x^N(L^2(\mathbb{R}_v^3))}^2,$$

which completes the proof of Lemma 7.1.7. \square

7.2. Microscopic Energy Estimate. We shall use Lemma 2.6 and Proposition 3.5.

Step 1. Let $\alpha \in \mathbb{N}^3$, $|\alpha| \leq N$, and apply $\partial^\alpha = \partial_x^\alpha$ to (1.9),

$$(7.2.1) \quad \partial_t(\partial^\alpha g) + v \cdot \nabla_x(\partial^\alpha g) + \mathcal{L}(\partial^\alpha g) = \partial^\alpha \Gamma(g, g),$$

and take the $L^2(\mathbb{R}_{x,v}^6)$ inner product with $\partial^\alpha g$. By Lemma 2.6

$$(7.2.2) \quad \frac{1}{2} \frac{d}{dt} \|\partial^\alpha g\|_{L^2(\mathbb{R}_{x,v}^6)}^2 + D_1 \leq J,$$

where D_1 is a dissipation rate

$$D_1 = \int_{\mathbb{R}^3} \|\partial^\alpha g_2\|^2 dx = \|\partial^\alpha g_2\|_{\mathcal{B}_0^0(\mathbb{R}^6)}^2,$$

and J is given by

$$\begin{aligned} J &= (\partial^\alpha \Gamma(g, g), \partial^\alpha g)_{L^2(\mathbb{R}^6)} = \sum_{i,j=1}^2 (\partial^\alpha \Gamma(g_i, g_j), \partial^\alpha g_2)_{L^2(\mathbb{R}^6)} \\ &= \sum_{i,j=1}^2 J^{(ij)}, \end{aligned}$$

with

$$g = \mathbf{P}g + (\mathbf{I} - \mathbf{P})g = g_1 + g_2.$$

First, consider $J^{(11)}$. We have, with ψ in (7.1.6),

$$\begin{aligned}
(7.2.3) \quad |J^{(11)}| &\leq \|\partial^\alpha \Gamma(g_1, g_1)\|_{L^2(\mathbb{R}^6)} \|\partial^\alpha g_2\|_{L^2(\mathbb{R}^6)} \\
&\leq \left(\|\partial^\alpha \Gamma(g_1, g_1)\|_{L^2(\mathbb{R}_x^3)} \right)_{L^2(\mathbb{R}_x^3)} \|\partial^\alpha g_2\|_{L^2(\mathbb{R}^6)}, \\
\|\partial^\alpha \Gamma(g_1, g_1)\|_{L^2(\mathbb{R}_x^3)} &\sim \partial^\alpha(a, b, c)^2 \|\Gamma(\psi_j, \psi_k)\|_{L^2(\mathbb{R}_x^3)}, \\
\|\Gamma(\psi_j, \psi_k)\|_{L^2(\mathbb{R}_x^3)}^2 &= \int \left(\int \int b(\cos \theta) \mu_*^{1/2} \{(\psi_j)'_* \psi_k' - (\psi_j)_* \psi_k\} dv_* d\sigma \right)^2 dv \\
&= \int \mu \left(\int \int b(\cos \theta) \mu_* \{(p_j)'_* p_k' - (p_j)_* p_k\} dv_* d\sigma \right)^2 dv < \infty.
\end{aligned}$$

Consequently, by virtue of Lemma 7.1,

$$\begin{aligned}
|J^{(11)}| &\leq \|\partial^\alpha(a, b, c)^2\|_{L^2(\mathbb{R}_x^3)} \|\partial^\alpha g_2\|_{L^2(\mathbb{R}^6)} \\
&\leq \|\nabla_x(a, b, c)\|_{H^{N-2}(\mathbb{R}_x^3)} \|(a, b, c)\|_{H^{N-1}(\mathbb{R}_x^3)} \|g_2\|_{H^N(\mathbb{R}_x^3, L^2(\mathbb{R}_v^3))} \\
&\leq \|(a, b, c)\|_{H^{N-1}(\mathbb{R}_x^3)} \left(\|\nabla_x(a, b, c)\|_{H^{N-2}(\mathbb{R}_x^3)}^2 + \|g_2\|_{\mathcal{B}_0^N(\mathbb{R}^6)}^2 \right).
\end{aligned}$$

On the other hand, using Proposition 3.5,

$$\begin{aligned}
|J^{(12)}| &\leq \|g_1\|_{H_0^N(\mathbb{R}^6)} \|g_2\|_{\mathcal{B}_0^N(\mathbb{R}^6)} \|g_2\|_{\mathcal{B}_0^N(\mathbb{R}^6)} \\
&\leq C \|(a, b, c)\|_{H^N(\mathbb{R}_x^3)} \|g_2\|_{\mathcal{B}_0^N(\mathbb{R}^6)}^2, \\
|J^{(21)}| &\leq \|g_2\|_{H_0^N(\mathbb{R}^6)} \|g_1\|_{\mathcal{B}_0^N(\mathbb{R}^6)} \|g_2\|_{\mathcal{B}_0^N(\mathbb{R}^6)} \\
&\leq C \|g_2\|_{H_0^N(\mathbb{R}^6)} \|(a, b, c)\|_{H^N(\mathbb{R}_x^3)} \|g_2\|_{\mathcal{B}_0^N(\mathbb{R}^6)}, \\
|J^{(22)}| &\leq \|g_2\|_{H_0^N(\mathbb{R}^6)} \|g_2\|_{\mathcal{B}_0^N(\mathbb{R}^6)} \|g_2\|_{\mathcal{B}_0^N(\mathbb{R}^6)} \\
&\leq C \|g_2\|_{H_0^N(\mathbb{R}^6)} \|g_2\|_{\mathcal{B}_0^N(\mathbb{R}^6)}^2.
\end{aligned}$$

Taking the summation of (7.2.2) for $|\alpha| \leq N$, $N \geq 3$, we may conclude to get

Lemma 7.4. *Let $N \geq 3$. Then,*

$$(7.2.4) \quad \frac{d}{dt} \|g\|_{H^N(\mathbb{R}_x^3, L^2(\mathbb{R}_v^3))}^2 + \| \partial^\alpha g_2 \|_{\mathcal{B}_0^N(\mathbb{R}^6)}^2 \leq \mathcal{D}_2 \mathcal{E}_2^{1/2},$$

where

$$\begin{aligned}
\mathcal{D}_2 &= \|\nabla_x(a, b, c)\|_{H^{N-1}(\mathbb{R}_x^3)}^2 + \|g_2\|_{\mathcal{B}_0^N(\mathbb{R}^6)}^2, \\
\mathcal{E}_2 &= \|g\|_{H^N(\mathbb{R}_x^3, L^2(\mathbb{R}_v^3))}^2 = \|(a, b, c)\|_{H^N(\mathbb{R}_x^3)}^2 + \|g_2\|_{H^N(\mathbb{R}_x^3, L^2(\mathbb{R}_v^3))}^2.
\end{aligned}$$

Step 2. Let

$$\partial^\alpha = \partial_x^\alpha, \quad 1 \leq |\alpha| \leq N, \quad N \geq 3$$

and apply $W^\ell \partial^\alpha$ to (1.9). We have, with $g = \mathbf{P}g + (\mathbf{I} - \mathbf{P})g = g_1 + g_2$,

$$(7.2.5) \quad \partial_t(W^\ell \partial^\alpha g) + v \cdot \nabla_x(W^\ell \partial^\alpha g) + \mathcal{L}(W^\ell \partial^\alpha g) = S_1 + S_2,$$

where

$$\begin{aligned}
S_1 &= W^\ell \partial^\alpha \Gamma(g, g) \\
S_2 &= [\mathcal{L}, W^\ell](\partial^\alpha g).
\end{aligned}$$

Take the $L^2(\mathbb{R}_{x,v}^6)$ inner product with $W^\ell \partial^\alpha g$ to deduce

$$(7.2.6) \quad \frac{1}{2} \frac{d}{dt} \|W^\ell \partial^\alpha g\|_{L^2(\mathbb{R}^6)}^2 + D_2 \leq G,$$

where D_2 is a dissipation rate

$$\begin{aligned} D_2 &= \int_{\mathbb{R}^3} \|(\mathbf{I} - \mathbf{P})W^\ell \partial^\alpha g\|^2 dx \\ &\geq \frac{1}{2} \|\partial^\alpha g\|_{\mathcal{B}_\ell^0(\mathbb{R}^6)}^2 - C(\|\nabla_x(a, b, c)\|_{H^{N-1}(\mathbb{R}_x^3)} + \|g_2\|_{H^N(\mathbb{R}_x^3, L^2(\mathbb{R}_v^3))}). \end{aligned}$$

Here we used, with ψ as in (7.1.6) and for $1 \leq |\alpha| \leq N$,

$$\begin{aligned} \int_{\mathbb{R}^3} \|PW^\ell \partial^\alpha g\|^2 dx &= \|(W^\ell \psi, \partial^\alpha g)_{L_v^2}\|_{L^2(\mathbb{R}_x^3)}^2 \|\psi\|^2 \leq \|\partial^\alpha g\|_{L^2(\mathbb{R}_{x,v}^6)}^2 \\ &\leq \|\nabla_x(a, b, c)\|_{H^{N-1}(\mathbb{R}_x^3)}^2 + \|g_2\|_{H^N(\mathbb{R}_x^3, L^2(\mathbb{R}_v^3))}^2 \end{aligned}$$

On the other hand G is defined by

$$G = G_1 + G_2, \quad G_i = (S_i, W^\ell \partial^\alpha g)_{L^2(\mathbb{R}^6)}, \quad i = 1, 2.$$

Lemma 7.5. *Let $N \geq 3$. Then,*

$$\begin{aligned} G_1 &\leq (\|(a, b, c)\|_{H^N(\mathbb{R}_x^3)} + \|g_2\|_{H_\ell^N(\mathbb{R}^6)}) \\ &\quad \times (\|\nabla_x(a, b, c)\|_{H^{N-1}(\mathbb{R}_x^3)}^2 + \|g_2\|_{\mathcal{B}_\ell^N(\mathbb{R}^6)}^2). \\ &\leq \mathcal{E}_3^{1/2} \mathcal{D}_3 \end{aligned}$$

Proof. First, write

$$G_1 = \sum_{i,j=1,2} (W^\ell \partial^\alpha \Gamma(g_i, g_j), W^\ell \partial^\alpha g)_{L^2(\mathbb{R}^6)} = \sum_{i,j=1,2} G_1^{(ij)}.$$

Proceeding as in (7.2.3),

$$\begin{aligned} (7.2.7) \quad |G^{(11)}| &\leq \|W^{2\ell} \partial^\alpha \Gamma(g_1, g_1)\|_{L^2(\mathbb{R}^6)} \|\partial^\alpha g\|_{L^2(\mathbb{R}^6)} \\ &\leq \|(\|W^{2\ell} \partial^\alpha \Gamma(g_1, g_1)\|_{L^2(\mathbb{R}_v^3)})\|_{L^2(\mathbb{R}_x^3)} \|\partial^\alpha g\|_{L^2(\mathbb{R}^6)}, \\ &\sim \|\partial^\alpha \{(a, b, c)^2\}\|_{L^2(\mathbb{R}_x^3)} \|W^{2\ell} \Gamma(\psi_j, \psi_k)\|_{L^2(\mathbb{R}_v^3)} \|\partial^\alpha g\|_{L^2(\mathbb{R}^6)}, \\ \|W^{2\ell} \Gamma(\psi_j, \psi_k)\|_{L^2(\mathbb{R}_v^3)}^2 &= \int (W^{2\ell} \int \int b(\cos \theta) \mu_*^{1/2} \{(\psi_j)'_* \psi_k' - (\psi_j)_* \psi_k\} dv_* d\sigma)^2 dv \\ &= \int \mu W^{4\ell} \left(\int \int b(\cos \theta) \mu_* \{(p_j)'_* p_k' - (p_j)_* p_k\} dv_* d\sigma \right)^2 dv < \infty. \end{aligned}$$

Since $1 \leq |\alpha| \leq N$,

$$(7.2.8) \quad |G^{(11)}| \leq \|\nabla_x(a, b, c)\|_{H^{N-1}(\mathbb{R}_x^3)} \|(a, b, c)\|_{H^{N-1}(\mathbb{R}_x^3)} \\ \times (\|\nabla_x(a, b, c)\|_{H^{N-1}(\mathbb{R}_x^3)} + \|g_2\|_{H^N(\mathbb{R}_x^3, L^2(\mathbb{R}_v^3))}).$$

On the other hand, we have

$$\|g_1\|_{\mathcal{B}_\ell^N(\mathbb{R}^6)}^2 = \sum_{|\beta| \leq N} \int_{\mathbb{R}_x^3} \|W^\ell \partial_{x,v}^\beta g_1(x, \cdot)\|^2 dx \leq \|(a, b, c)\|_{H^N(\mathbb{R}_x^3)}^2.$$

This fact and Proposition 3.5 yield

$$\begin{aligned}
|G_1^{(12)}| &\leq \|g_1\|_{H_t^N(\mathbb{R}^6)} \|g_2\|_{\mathcal{B}_t^N(\mathbb{R}^6)} \|W^\ell \partial^\alpha g\|_{\mathcal{B}_0^0(\mathbb{R}^6)} \\
&\leq \|(a, b, c)\|_{H^N(\mathbb{R}_x^3)} \|g_2\|_{\mathcal{B}_t^N(\mathbb{R}^6)} \|\partial^\alpha g\|_{\mathcal{B}_t^0(\mathbb{R}^6)} \\
|G_1^{(21)}| &\leq \|g_2\|_{H_t^N(\mathbb{R}^6)} \|g_1\|_{\mathcal{B}_t^N(\mathbb{R}^6)} \|W^\ell \partial^\alpha g\|_{\mathcal{B}_0^0(\mathbb{R}^6)} \\
&\leq \|g_2\|_{H_t^N(\mathbb{R}^6)} \|(a, b, c)\|_{H^N(\mathbb{R}_x^3)} \|\partial^\alpha g\|_{\mathcal{B}_t^0(\mathbb{R}^6)}, \\
|G_1^{(22)}| &\leq \|g_2\|_{H_t^N(\mathbb{R}^6)} \|g_2\|_{\mathcal{B}_t^N(\mathbb{R}^6)} \|W^\ell \partial^\alpha g\|_{\mathcal{B}_0^0(\mathbb{R}^6)} \\
&\leq \|g_2\|_{H_t^N(\mathbb{R}^6)} \|g_2\|_{\mathcal{B}_t^N(\mathbb{R}^6)} \|\partial^\alpha g\|_{\mathcal{B}_t^0(\mathbb{R}^6)}.
\end{aligned}$$

Noticing that for $1 \leq |\alpha| \leq N$,

$$\begin{aligned}
\|\partial^\alpha g\|_{\mathcal{B}_t^0(\mathbb{R}^6)}^2 &\leq \int_{\mathbb{R}_x^3} \|W^\ell \partial^\alpha g_1(x, \cdot)\|^2 dx + \int_{\mathbb{R}_x^3} \|W^\ell \partial^\alpha g_2(x, \cdot)\|^2 dx \\
&\leq \|\nabla_x(a, b, c)\|_{H^{N-1}(\mathbb{R}_x^3)}^2 + \|g_2\|_{\mathcal{B}_t^N(\mathbb{R}^6)}^2,
\end{aligned}$$

we can now conclude Lemma 7.5. \square

We shall evaluate G_2 . In view of Proposition 3.8,

$$\begin{aligned}
G_2 &\leq \left| \left([\mathcal{L}_1, W^\ell] \partial^\alpha g, W^\ell \partial^\alpha g \right)_{L^2(\mathbb{R}^6)} \right| \\
&\quad + \left| \left(W^\ell \mathcal{L}_2(\partial^\alpha g), W^\ell \partial^\alpha g \right)_{L^2(\mathbb{R}^6)} \right| + \left| \left(\mathcal{L}_2(W^\ell \partial^\alpha g), W^\ell \partial^\alpha g \right)_{L^2(\mathbb{R}^6)} \right| \\
&\leq 3 \|\partial^\alpha g\|_{L_t^2(\mathbb{R}^6)} \|\partial^\alpha g\|_{\mathcal{B}_t^0(\mathbb{R}^6)} \\
&\leq (\|\nabla_x(a, b, c)\|_{H^{N-1}(\mathbb{R}_x^3)} + \|\partial^\alpha g_2\|_{L_t^2(\mathbb{R}^6)}) \|\partial^\alpha g\|_{\mathcal{B}_t^0(\mathbb{R}^6)} \\
&\leq \|\nabla_x(a, b, c)\|_{H^{N-1}(\mathbb{R}_x^3)}^2 + \|\partial^\alpha g_2\|_{L^2(\mathbb{R}^6)}^2 + \eta \|\partial^\alpha g\|_{\mathcal{B}_t^0(\mathbb{R}^6)}^2 \quad (\eta > 0).
\end{aligned}$$

Thus, we have established

Lemma 7.6. *Let $1 \leq |\alpha| \leq N, N \geq 3$. Then,*

$$\begin{aligned}
(7.2.9) \quad &\frac{d}{dt} \|\partial^\alpha g\|_{L_t^2(\mathbb{R}^6)}^2 + \|\partial^\alpha g\|_{\mathcal{B}_t^0(\mathbb{R}^6)}^2 \\
&\leq \mathcal{E}_3 \mathcal{D}_3 + \|\partial^\alpha g_2\|_{L^2(\mathbb{R}^6)}^2 + \|\nabla_x(a, b, c)\|_{H^{N-1}(\mathbb{R}_x^3)}^2.
\end{aligned}$$

Step 3 . We need also to estimate $W^\ell g_2$. Apply $W^\ell(\mathbf{I} - \mathbf{P})$ to (1.9)

$$\begin{aligned}
(7.2.10) \quad &\partial_t(W^\ell g_2) + v \cdot \nabla_x(W^\ell g_2) + \mathcal{L}(W^\ell g_2) \\
&= W^\ell \Gamma(g, g) + W^\ell [v \cdot \nabla_x, \mathbf{P}]g + [\mathcal{L}, W^\ell]g_2,
\end{aligned}$$

and take the inner product with $W^\ell g_2$:

$$\frac{d}{dt} \|W^\ell g_2\|_{L^2(\mathbb{R}^6)}^2 + D_3 \leq H,$$

where

$$\begin{aligned}
D_3 &= \int_{\mathbb{R}^3} \|(\mathbf{I} - \mathbf{P})W^\ell g_2\|^2 dx \\
&\geq \frac{1}{2} \|g_2\|_{\mathcal{B}_t^0(\mathbb{R}^6)}^2 - C \|g_2\|_{L^2(\mathbb{R}^3)}^2,
\end{aligned}$$

while

$$\begin{aligned}
H &= H_1 + H_2 + H_3, \\
H_1 &= (W^\ell \Gamma(g, g), W^\ell g_2)_{L^2(\mathbb{R}^6)}, \\
H_2 &= (W^\ell [v \cdot \nabla_x, \mathbf{P}]g, W^\ell g_2)_{L^2(\mathbb{R}^6)}, \\
H_3 &= ([\mathcal{L}, W^\ell]g_2, W^\ell g_2)_{L^2(\mathbb{R}^6)}. \\
H_1 &= \sum_{i,j=2} (W^\ell \Gamma(g_i, g_j), W^\ell g_2)_{L^2(\mathbb{R}^6)} = \sum_{i,j=2} H_1^{(ij)}.
\end{aligned}$$

Proceeding as in (7.2.7),

$$\begin{aligned}
(7.2.11) \quad |H^{(11)}| &\leq \|W^{2\ell} \Gamma(g_1, g_1)\|_{L^2(\mathbb{R}^6)} \|g_2\|_{L^2(\mathbb{R}^6)} \\
&\leq \|(\|W^{2\ell} \Gamma(g_1, g_1)\|_{L^2(\mathbb{R}_v^3)})\|_{L^2(\mathbb{R}_x^3)} \|g_2\|_{L^2(\mathbb{R}^6)}, \\
&\sim \|(a, b, c)\|_{L^2(\mathbb{R}_x^3)} \|W^{2\ell} \Gamma(\psi_j, \psi_k)\|_{L^2(\mathbb{R}_v^3)} \|g_2\|_{L^2(\mathbb{R}^6)}, \\
\|W^{2\ell} \Gamma(\psi_j, \psi_k)\|_{L^2(\mathbb{R}_v^3)}^2 &= \int (W^{2\ell} \int \int b(\cos \theta) \mu_*^{1/2} \{(\psi_j)'_* \psi_k' - (\psi_j)_* \psi_k\} dv_* d\sigma)^2 dv \\
&= \int \mu W^{4\ell} \left(\int \int b(\cos \theta) \mu_* \{(p_j)'_* p_k' - (p_j)_* p_k\} dv_* d\sigma \right)^2 dv < \infty.
\end{aligned}$$

Then we have, by using Lemma 7.1,

$$(7.2.12) \quad |H^{(11)}| \leq \|\nabla_x(a, b, c)\|_{L^2(\mathbb{R}_x^3)} \|(a, b, c)\|_{H^1(\mathbb{R}_x^3)} \|g_2\|_{L^2(\mathbb{R}^6)}.$$

On the other hand, we have

$$\|g_1\|_{\mathcal{B}_t^0(\mathbb{R}^6)}^2 = \int_{\mathbb{R}_x^3} \|W^\ell g_1(x, \cdot)\|_{\mathcal{B}_t^0(\mathbb{R}^6)}^2 dx \leq \|(a, b, c)\|_{L^2(\mathbb{R}_x^3)}^2.$$

This fact and Proposition 3.5 yield

$$\begin{aligned}
H_1^{(12)} &\leq \|g_1\|_{H_t^N(\mathbb{R}^6)} \|g_2\|_{\mathcal{B}_t^N(\mathbb{R}^6)} \|W^\ell g_2\|_{\mathcal{B}_0^0(\mathbb{R}^6)} \\
&\leq \|(a, b, c)\|_{H^N(\mathbb{R}_x^3)} \|g_2\|_{\mathcal{B}_t^N(\mathbb{R}^6)}^2, \\
H_1^{(21)} &\leq \|g_2\|_{H_t^N(\mathbb{R}^6)} \|g_1\|_{\mathcal{B}_t^N(\mathbb{R}^6)} \|W^\ell g_2\|_{\mathcal{B}_0^0(\mathbb{R}^6)} \\
&\leq \|g_2\|_{H_t^N(\mathbb{R}^6)} \|(a, b, c)\|_{H^N(\mathbb{R}_x^3)} \|g_2\|_{\mathcal{B}_0^0(\mathbb{R}^6)}, \\
H_1^{(22)} &\leq \|g_2\|_{H_t^N(\mathbb{R}^6)} \|g_2\|_{\mathcal{B}_t^N(\mathbb{R}^6)} \|W^\ell g_2\|_{\mathcal{B}_0^0(\mathbb{R}^6)} \\
&\leq \|g_2\|_{H_t^N(\mathbb{R}^6)} \|g_2\|_{\mathcal{B}_t^N(\mathbb{R}^6)}^2
\end{aligned}$$

and H_2 is evaluated as follows

$$\begin{aligned}
|H_2| &\leq |(W^{2\ell} [v \cdot \nabla_x, \mathbf{P}]g, g_2)_{L^2(\mathbb{R}^6)}| \leq \|\nabla_x g\|_{L^2(\mathbb{R}^6)} \|g_2\|_{L^2(\mathbb{R}^6)} \\
&\leq \|\nabla_x(a, b, c)\|_{L^2(\mathbb{R}^6)}^2 + \|g_2\|_{H^1(\mathbb{R}_x^3)(L^2(\mathbb{R}_v^3))}^2.
\end{aligned}$$

Finally, in view of Proposition 3.8 ,

$$\begin{aligned}
H_3 &\leq \left| ([\mathcal{L}_1, W^\ell]g_2, W^\ell g_2)_{L^2(\mathbb{R}^6)} \right| \\
&\quad + \left| (W^\ell \mathcal{L}_2(g_2), W^\ell g_2)_{L^2(\mathbb{R}^6)} \right| + \left| (\mathcal{L}_2(W^\ell g_2), W^\ell g_2)_{L^2(\mathbb{R}^6)} \right| \\
&\leq 3 \|g_2\|_{L_t^2(\mathbb{R}^6)} \|g_2\|_{\mathcal{B}_t^0(\mathbb{R}^6)}.
\end{aligned}$$

Since it holds by interpolation inequality that

$$(7.2.13) \quad \|g_2\|_{L_t^2(\mathbb{R}^6)} \leq \eta \|g_2\|_{L_{t+s}^2(\mathbb{R}^6)} + C_\eta \|g_2\|_{L^2(\mathbb{R}^6)} \leq \eta \|g_2\|_{\mathcal{B}_t^0(\mathbb{R}^6)} + C_\eta \|g_2\|_{L^2(\mathbb{R}^6)},$$

for any small enough $\eta > 0$, we have established

Lemma 7.7.

$$(7.2.14) \quad \begin{aligned} & \frac{d}{dt} \|g_2\|_{L_t^2(\mathbb{R}^6)}^2 + \|g_2\|_{\mathcal{B}_t^0(\mathbb{R}^6)}^2 \\ & \leq \mathcal{E}_3^{1/2} \mathcal{D}_3 + \|g_2\|_{L^2(\mathbb{R}^6)}^2 + \|\nabla_x(a, b, c)\|_{L^2(\mathbb{R}_x^3)}^2. \end{aligned}$$

Step 4 . Let

$$\partial^\beta = \partial_{x,v}^\beta = \partial^\alpha \partial^\gamma, \quad \partial^\alpha = \partial_x^\alpha, \quad \partial^\gamma = \partial_v^\gamma, \quad |\beta| = |\alpha| + |\gamma| \leq N, \gamma \neq 0, N \geq 3,$$

apply $W^\ell \partial^\beta(\mathbf{I} - \mathbf{P})$ to (7.2.5),

$$(7.2.15) \quad \begin{aligned} & \partial_t(W^\ell \partial^\beta g_2) + v \cdot \nabla_x(W^\ell \partial^\beta g_2) + \mathcal{L}(W^\ell \partial^\beta g_2) \\ & = W^\ell \partial^\beta \Gamma(g, g) + [v \cdot \nabla_x, W^\ell \partial^\beta] g_2 \\ & \quad - W^\ell \partial^\beta [\mathbf{P}, v \cdot \nabla] g \\ & \quad + [\mathcal{L}, W^\ell \partial^\beta] g_2 + W^\ell \partial^\beta (\partial_t + v \cdot \nabla_x) g_1, \end{aligned}$$

and take the $L^2(\mathbb{R}_{x,v}^6)$ inner product with $W^\ell \partial^\beta g_2$.

$$(7.2.16) \quad \frac{1}{2} \frac{d}{dt} \|\partial^\beta g_2\|_{L_t^2(\mathbb{R}^6)}^2 + D_3 \leq K.$$

Here D_2 is a dissipation rate

$$\begin{aligned} D_3 & = \int_{\mathbb{R}^3} \|(\mathbf{I} - \mathbf{P}) W^\ell \partial^\beta g_2\|^2 dx \\ & \geq \frac{1}{2} \|\partial^\beta g_2\|_{\mathcal{B}_t^0(\mathbb{R}^6)}^2 - C \|\partial^\alpha g_2\|_{L^2(\mathbb{R}^6)}^2. \end{aligned}$$

where we used, with ψ as in (7.1.6) and $\tilde{\psi} = (-1)^{|\gamma|} \partial^\gamma(W^\ell \psi)$,

$$\|PW^\ell \partial^\beta g_2\|^2 = \|\partial^\alpha(\psi, W^\ell \partial^\gamma g_2)_{L^2(\mathbb{R}_v^3)} \psi\|^2 = \|(\tilde{\psi}, \partial^\alpha g_2)_{L_v^2}\|_{L^2(\mathbb{R}_x^3)}^2 \|\psi\|^2 \leq \|\partial^\alpha g_2\|_{L^2(\mathbb{R}_{x,v}^6)}^2.$$

On the other hand, K is given by

$$\begin{aligned} K & = (W^\ell \partial^\beta \Gamma(g, g), W^\ell \partial^\beta g_2)_{L^2(\mathbb{R}^6)} \\ & \quad + ([v \cdot \nabla_x, W^\ell \partial^\beta] g_2, W^\ell \partial^\beta g_2)_{L^2(\mathbb{R}^6)} - (W^\ell \partial^\beta [\mathbf{P}, v \cdot \nabla] g, W^\ell \partial^\beta g_2)_{L^2(\mathbb{R}^6)} \\ & \quad + ([\mathcal{L}, W^\ell \partial^\beta] g_2, W^\ell \partial^\beta g_2)_{L^2(\mathbb{R}^6)} \\ & \quad + (W^\ell \partial^\beta (\partial_t + v \cdot \nabla_x) g_1, W^\ell \partial^\beta g_2)_{L^2(\mathbb{R}^6)} \\ & = K_1 + K_2 + K_3 + K_4 + K_5. \end{aligned}$$

Lemma 7.8. *Let $N \geq 3$. Then,*

$$\begin{aligned} K_1 & \leq \left(\|(a, b, c)\|_{H^N(\mathbb{R}_x^3)} + \|g_2\|_{H_t^N(\mathbb{R}^6)} \right) \\ & \quad \times \left(\|\nabla_x(a, b, c)\|_{H^{N-1}(\mathbb{R}_x^3)}^2 + \|g_2\|_{\mathcal{B}_t^N(\mathbb{R}^6)}^2 \right) \\ & \leq \mathcal{E}_3^{1/2} \mathcal{D}_3. \end{aligned}$$

Proof. First, write

$$K_1 = \sum_{i,j=2} (W^\ell \partial^\beta \Gamma(g_i, g_j), W^\ell \partial^\beta g_2)_{L^2(\mathbb{R}^6)} = \sum_{i,j=2} K_1^{(ij)}.$$

In view of Lemma 7.1,

$$\begin{aligned} K_1^{(11)} &= (W^\ell \partial^\beta \Gamma(g_1, g_1), W^\ell \partial^\beta g_2)_{L^2(\mathbb{R}^6)} \\ &\leq B_2 \|\partial^\alpha(a, b, c)\|_{L^2(\mathbb{R}_x^3)}^2 \|W^\ell \partial^\beta g_2\|_{L^2(\mathbb{R}^6)} \\ &\leq \|\nabla_x(a, b, c)\|_{H_x^{N-1}} \|(a, b, c)\|_{H_x^{N-1}} \|W^\ell \partial^\beta g_2\|_{L^2(\mathbb{R}^6)}, \end{aligned}$$

where, with ψ as in (7.1.6),

$$\begin{aligned} B_2^2 &\sim \|W^\ell \partial_v^\gamma \Gamma(\psi_j, \psi_k)\|_{L^2(\mathbb{R}_v^3)}^2 \\ &= \int (W^\ell \int \int b(\cos \theta) (\partial_v^{\gamma_1} \mu)_*^2 \\ &\quad \times \{(\partial_v^{\gamma_2} \psi_j)'_*(\partial_v^{\gamma_3} \psi_k)' - (\partial_v^{\gamma_2} \psi_j)_*(\partial_v^{\gamma_3} \psi_k)\} dv_* d\sigma)^2 dv \\ &= \int \mu W^{2\ell} \left(\int \int b(\cos \theta) \mu_* q_* \{(q_j)'_*(q_k)' - (q_j)_*(q_k)\} dv_* d\sigma \right)^2 dv < \infty, \end{aligned}$$

q, q_j, q_k being polynomials of v .

On the other hand, we have

$$\|g_1\|_{\mathcal{B}_\ell^N(\mathbb{R}^6)}^2 = \sum_{|\beta| \leq N} \int_{\mathbb{R}_x^3} \|W^\ell \partial^\beta g_1(x, \cdot)\|^2 dx \leq \|(a, b, c)\|_{H^N(\mathbb{R}_x^3)}^2.$$

This point and Proposition 3.5 yield

$$\begin{aligned} K_1^{(12)} &\leq \|g_1\|_{H_\ell^N(\mathbb{R}^6)} \|g_2\|_{\mathcal{B}_\ell^N(\mathbb{R}^6)} \|W^\ell \partial^\beta g_2\|_{\mathcal{B}_0^0(\mathbb{R}^6)} \\ &\leq \|(a, b, c)\|_{H^N(\mathbb{R}_x^3)} \|g_2\|_{\mathcal{B}_\ell^N(\mathbb{R}^6)}^2, \\ K_1^{(21)} &= \|g_2\|_{H_\ell^N(\mathbb{R}^6)} \|g_1\|_{\mathcal{B}_\ell^N(\mathbb{R}^6)} \|W^\ell \partial^\beta g_2\|_{\mathcal{B}_0^0(\mathbb{R}^6)} \\ &\leq \|g_2\|_{H_\ell^N(\mathbb{R}^6)} \|(a, b, c)\|_{H^N(\mathbb{R}_x^3)} \|g_2\|_{\mathcal{B}_\ell^N(\mathbb{R}^6)}, \\ K_1^{(22)} &= \|g_2\|_{H_\ell^N(\mathbb{R}^6)} \|g_2\|_{\mathcal{B}_\ell^N(\mathbb{R}^6)} \|W^\ell \partial^\beta g_2\|_{\mathcal{B}_0^0(\mathbb{R}^6)} \\ &\leq \|g_2\|_{H_\ell^N(\mathbb{R}^6)} \|g_2\|_{\mathcal{B}_\ell^N(\mathbb{R}^6)}^2. \end{aligned}$$

Now the proof of the lemma is completed. \square

K_2, K_4, K_5 are estimated as follows. We have, for $|\beta| = |\alpha| + |\gamma| \leq N, \gamma \neq 0$,

$$\begin{aligned} K_2 &= ([v \cdot \nabla_x, W^\ell \partial^\beta] g_2, W^\ell \partial^\beta g_2)_{L^2(\mathbb{R}^6)} \\ &\leq \|W^\ell \partial_x^{\alpha+1} \partial_v^{\gamma-1} g_2\|_{L^2(\mathbb{R}^6)} \|W^\ell \partial^\beta g_2\|_{L^2(\mathbb{R}^6)} \\ &\leq C_\eta \|W^\ell \partial_x^{\alpha+1} \partial_v^{\gamma-1} g_2\|_{L^2(\mathbb{R}^6)}^2 + \eta \|W^\ell \partial^\beta g_2\|_{L^2(\mathbb{R}^6)}^2. \end{aligned}$$

Note that

$$\begin{aligned} |K_3| &= |(W^\ell \partial^\beta [\mathbf{P}, v \cdot \nabla] g, W^\ell \partial^\beta g_2)_{L^2(\mathbb{R}^6)}, (W^\ell \partial^\beta g_2)_{L^2(\mathbb{R}^6)}| \\ &\leq |(\partial^\gamma \{W^{2\ell} \partial^\beta [\mathbf{P}, v \cdot \nabla] g\}, \partial^\alpha g_2)_{L^2(\mathbb{R}^6)}| \\ &\leq \left(\|\nabla_x \partial^\alpha(a, b, c)\|_{L^2(\mathbb{R}_x^3)} + \|\nabla_x \partial^\alpha g_2\|_{L^2(\mathbb{R}_{x,v}^6)} \right) \|\partial^\alpha g_2\|_{L^2(\mathbb{R}_{x,v}^6)} \\ &\leq \|\nabla_x(a, b, c)\|_{H^{N-1}(\mathbb{R}_x^3)}^2 + \|g_2\|_{H^N(\mathbb{R}_x^3, L^2(\mathbb{R}_v^3))}^2. \end{aligned}$$

In view of Proposition 3.8,

$$\begin{aligned}
K_4 &= ([\mathcal{L}, W^\ell \partial^\beta] g_2, W^\ell \partial^\beta g_2)_{L^2(\mathbb{R}^6)} \\
&\leq \left| ([\mathcal{L}_1, W^\ell \partial_{x,v}^\beta] g_2, W^\ell \partial^\beta g_2)_{L^2(\mathbb{R}^6)} \right| \\
&\quad + \left| (W^\ell \partial_{x,v}^\beta \mathcal{L}_2(g_2), W^\ell \partial^\beta g_2)_{L^2(\mathbb{R}^6)} \right| + \left| (\mathcal{L}_2(W^\ell \partial_{x,v}^\beta g_2), W^\ell \partial^\beta g_2)_{L^2(\mathbb{R}^6)} \right| \\
&\leq (\|g_2\|_{H_t^{|\beta|}(\mathbb{R}^6)} + \|g_2\|_{\mathcal{B}_t^{|\beta|-1}(\mathbb{R}^6)}) \|W^\ell \partial^\beta g_2\|_{\mathcal{B}_0^0(\mathbb{R}^6)}
\end{aligned}$$

Hence

$$K_4 \leq C_\eta (\|g_2\|_{H_t^{|\beta|}(\mathbb{R}^6)}^2 + \|(I-P)g\|_{\mathcal{B}_t^{|\beta|-1}(\mathbb{R}^6)}^2) + \eta \|W^\ell \partial^\beta (\mathbf{I}-\mathbf{P})g\|_{\mathcal{B}_0^0(\mathbb{R}^6)}^2$$

Finally, recalling (7.1.10),

$$\begin{aligned}
K_5 &= (W^\ell \partial^\beta (\partial_t + v \cdot \nabla_x) g_1, W^\ell \partial^\beta g_2)_{L^2(\mathbb{R}^6)} \\
&\leq |(\partial^\gamma \{W^{2\ell} \partial^\beta (\partial_t + v \cdot \nabla_x) g_1\}, \partial^\alpha g_2)_{L^2(\mathbb{R}^6)}| \\
&\leq \|(\partial^\alpha (\partial_t + \nabla_x)(a, b, c))\|_{L^2(\mathbb{R}_x^3)} \|\partial^\alpha g_2\|_{L^2(\mathbb{R}^6)}, \quad (|\alpha| \leq N-1, |\gamma| \geq 1) \\
&\leq (\|\nabla_x \partial^\alpha(a, b, c)\|_{L^2(\mathbb{R}_x^3)} + \|\nabla_x \partial^\alpha g_2\|_{L^2(\mathbb{R}_{x,v}^6)}) \|\partial^\alpha g_2\|_{L^2(\mathbb{R}_{x,v}^6)} \\
&\leq \|\nabla_x(a, b, c)\|_{H_x^{N-1}}^2 + \|g_2\|_{H^N(\mathbb{R}_x^3, L^2(\mathbb{R}_v^3))}^2.
\end{aligned}$$

Now using (7.2.13) we conclude the

Lemma 7.9. *Let $|\beta| = |\alpha + \gamma| \leq N$, $|\alpha| \leq N-1$, $|\gamma| \geq 1$, $N \geq 3$. Then,*

$$\begin{aligned}
(7.2.17) \quad & \frac{d}{dt} \|\partial^\beta g_2\|_{L_t^2(\mathbb{R}^6)}^2 + \|\partial^\beta g_2\|_{\mathcal{B}_t^0(\mathbb{R}^6)}^2 \\
& \leq \mathcal{E}_3 \mathcal{D}_3 + \|g_2\|_{H_t^{|\beta|}(\mathbb{R}^6)}^2 \\
& \quad + \|\partial^\beta g_2\|_{\mathcal{B}_t^{|\alpha+\gamma|-1}(\mathbb{R}^6)}^2 + \|\nabla_x(a, b, c)\|_{H^{N-1}(\mathbb{R}_x^3)}^2 + \|g_2\|_{H^N(\mathbb{R}_x^3, L^2(\mathbb{R}_v^3))}^2.
\end{aligned}$$

7.3. A Priori Estimate. We take the linear combination

$$\begin{aligned}
& \sum_{|\alpha| \leq N-1} C_\alpha^{(1)}(7.1.7)_\alpha + \sum_{|\alpha| \leq N} C_\alpha^{(2)}(7.2.2)_\alpha + \sum_{1 \leq |\alpha| \leq N} C_\alpha^{(3)}(7.2.9)_\alpha \\
& + C^{(4)}(7.2.14) + \sum_{|\beta| = |\alpha + \gamma| \leq N, |\alpha| \leq N-1, |\gamma| \geq 1} C_{\alpha, \gamma}^{(3)}(7.2.17)_{\alpha, \gamma}
\end{aligned}$$

With a suitable choice of the coefficients $C_\alpha^{(1)}$, $C_\alpha^{(2)}$, $C_\alpha^{(3)}$, $C^{(4)}$, $C_{\alpha, \gamma}^{(5)}$, we get

$$(7.3.1) \quad \frac{d}{dt} \tilde{\mathcal{E}} + \mathcal{D} \leq \mathcal{H},$$

where

$$\begin{aligned}
\tilde{\mathcal{E}} &= - \sum_{|\alpha| \leq N-1} C_\alpha^{(1)} \left[(\partial^\alpha r, \nabla_x \partial^\alpha (a, -b, c))_{L^2(\mathbb{R}_x^3)} + (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}_x^3)} \right] \\
&\quad + \sum_{|\alpha| \leq N} C_\alpha^{(2)} \|\partial_x^\alpha g\|_{L^2(\mathbb{R}^6)}^2 + \sum_{1 \leq |\alpha| \leq N} C_\alpha^{(3)} \|\partial_x^\alpha g\|_{L_\ell^2(\mathbb{R}^6)}^2 \\
&\quad + C^{(4)} \|g_2\|_{L_\ell^2(\mathbb{R}^6)}^2 + \sum_{|\beta|=|\alpha+\gamma| \leq N, |\alpha| \leq N-1, |\gamma| \geq 1} C_{\alpha,\gamma}^{(5)} \|W^\ell \partial_{x,v}^\beta g_2\|_{L^2(\mathbb{R}^6)}^2, \\
\mathcal{D} &= \sum_{|\alpha| \leq N-1} C_\alpha^{(1)} \|\nabla_x \partial_x^\alpha (a, b, c)\|_{L^2(\mathbb{R}_x^3)}^2 + \sum_{|\alpha| \leq N} C_\alpha^{(2)} \|\partial_x^\alpha g_2\|_{\mathcal{B}_0^0(\mathbb{R}^6)}^2 \\
&\quad + \sum_{1 \leq |\alpha| \leq N} C_\alpha^{(3)} \|\partial_x^\alpha g_2\|_{\mathcal{B}_\ell^0(\mathbb{R}^6)}^2 + C^{(4)} \|g_2\|_{\mathcal{B}_\ell^0(\mathbb{R}^6)}^2 \\
&\quad + \sum_{|\beta|=|\alpha+\gamma| \leq N, |\alpha| \leq N-1, |\gamma| \geq 1} C_{\alpha,\gamma}^{(3)} \|\partial_{x,v}^\beta g_2\|_{\mathcal{B}_\ell^0(\mathbb{R}^6)}^2, \\
\mathcal{H} &= \mathcal{D}_1 \mathcal{E}_1 + \mathcal{D}_2 \mathcal{E}_2 + \mathcal{D}_3 \mathcal{E}_3.
\end{aligned}$$

Define the temporal energy functional

$$(7.3.2) \quad \mathcal{E} = \sum_{|\alpha| \leq N} \|\partial_x^\alpha g\|_{L_\ell^2(\mathbb{R}^6)}^2 + \sum_{|\beta|=|\alpha+\gamma| \leq N, |\alpha| \leq N-1, |\gamma| \geq 1} \|\partial_{x,v}^\beta g_2\|_{L_\ell^2(\mathbb{R}^6)}^2,$$

Clearly

$$c_1 \mathcal{E} \leq \tilde{\mathcal{E}} \leq c_2 \mathcal{E}$$

holds for some constants $c_1, c_2 > 0$, and

$$\mathcal{H} \leq \mathcal{D} \mathcal{E}.$$

Now (7.3.1) gives

$$(7.3.3) \quad \mathcal{E}(t) + c_2 \left[1 - \sup_{0 \leq \tau \leq t} \mathcal{E}(\tau) \right] \int_0^t \mathcal{D}(\tau) d\tau \leq \mathcal{E}(0),$$

which leads to the following a priori estimate on the solution g .

Proposition 7.10. *Let $N, \ell \geq 3$. Let $T > 0$ and suppose that g is a classical solution to the Cauchy problem (1.9) on $[0, T]$. There exist constant $M_0, M_1 > 0$ such that if*

$$\max_{0 \leq t \leq T} \mathcal{E}(t) \leq M_0,$$

then g enjoys the estimate

$$\mathcal{E}(t) + \int_0^t \mathcal{D}(\tau) d\tau \leq M_1 \mathcal{E}(0),$$

for any $t \in [0, T]$.

Now, the proof of Theorem 1.1 is completed by the usual continuation argument based on Theorem 4.3 and Proposition 7.10.

Last remark: While finishing the redaction of this paper to be posted to ArXiv and Hal, P. T. Gressman and R. M. Strain communicate to us their recent work [33], where they show the global existence of solution on the torus. But the conditions, results and methods are different from ours. In particular, they managed to develop their own functional tools for the study of the collision operators, thus avoiding the use of our own ones, developed and published from the 2006's.

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R. ALEXANDRE, IRENAV RESEARCH INSTITUTE, FRENCH NAVAL ACADEMY BREST-LANVÉOC 29290, FRANCE
E-mail address: radjesvarane.alexandre@ecole-navale.fr

Y. MORIMOTO, GRADUATE SCHOOL OF HUMAN AND ENVIRONMENTAL STUDIES, KYOTO UNIVERSITY
KYOTO, 606-8501, JAPAN
E-mail address: morimoto@math.h.kyoto-u.ac.jp

S. UKAI, 17-26 IWASAKI-CHO, HODOGAYA-KU, YOKOHAMA 240-0015, JAPAN
E-mail address: ukai@kurims.kyoto-u.ac.jp

C.-J. XU, SCHOOL OF MATHEMATICS, WUHAN UNIVERSITY 430072, WUHAN, P. R. CHINA
AND
UNIVERSITÉ DE ROUEN, UMR 6085-CNRS, MATHÉMATIQUES
AVENUE DE L'UNIVERSITÉ, BP.12, 76801 SAINT ÉTIENNE DU ROUVRAY, FRANCE
E-mail address: Chao-Jiang.Xu@univ-rouen.fr

T. YANG, DEPARTMENT OF MATHEMATICS, CITY UNIVERSITY OF HONG KONG, HONG KONG, P. R. CHINA
E-mail address: matyang@cityu.edu.hk