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Abstract

The interface problem describing the scattering of time-harmonic electromagnetic waves by a dielectric body is often formulated as a pair of coupled boundary integral equations for the electric and magnetic current densities on the interface Γ. In this paper, following an idea developped by R. Kleinman and P. Martin [18] for acoustic scattering problems, we consider methods for solving the dielectric scattering problem using a single integral equation over Γ for a single unknown density. One knows that such boundary integral formulations of the Maxwell equations are not uniquely solvable when the exterior wave number is an eigenvalue of an associated interior Maxwell boundary value problem. We obtain four different families of integral equations for which we can show that by choosing some parameters in an appropriate way, they become uniquely solvable for all real frequencies.

Keywords : scattering problems, Maxwell equations, boundary integral equations, Helmholtz decomposition.

1 Introduction

We consider the scattering of time-harmonic electromagnetic waves in $\mathbb{R}^3$ by a bounded obstacle with a smooth and simply connected boundary. We assume that the dielectric permittivity and the magnetic permeability take constant, in general different, values in the interior and in the exterior of the domain. This problem is described by the system of Maxwell's equations, valid in the sense of distributions in $\mathbb{R}^3$, which implies two transmission conditions expressing the continuity of the tangential components of the fields across the interface. The transmission problem is completed by the Silver-Müller radiation condition at infinity (see [21] and [22]).

It is well known that this problem can be reduced in several different ways to systems of two boundary integral equations for two unknown tangential vector fields on the interface (see [7, 22]). These pairs of integral equations are not always uniquely solvable for all values of the exterior wave number, although the underlying Maxwell interface problem is uniquely solvable under standard assumptions on the material coefficients. It can be desirable to solve a single integral equation for a single unknown, rather than a system of two equations of two unknowns (see [1], [8] and [23]).

In this paper, we study methods for solving the transmission problem using a single boundary integral equation for a single unknown tangential vector field on the interface. We follow ideas of [18] where R. E. Kleinman and P. A. Martin considered the analogous question for the acoustic interface scattering problem. The method consists of representing the solution in one domain by some combination of a single layer potential and a double layer potential, and inserting this representation into the transmission conditions and the Calderón relations of the other domain. Several different integral equations of the first kind or of the second kind, containing two arbitrary parameters, can be obtained in this way, and in the scalar case, the parameters can be chosen in such a way that no spurious real frequencies are introduced. Following the same procedure in the electromagnetic case, one encounters two main difficulties:

The first problem is that some boundary integral operators that are compact in the scalar case are no longer compact, and therefore arguments based on the theory of Fredholm integral equations of the second kind have to be refined in order to show well-posedness of the corresponding integral equations.

The second problem comes from a lack of ellipticity. The spurious frequencies are associated with the spectrum of a certain interior boundary value problem of the third kind, and whereas in the scalar case this is an elliptic boundary value problem whose spectrum can be moved off the real line by the right choice of parameters, in the Maxwell case this boundary value problem is not elliptic, in general. Thus an additional idea is needed to avoid real irregular frequences. Since the Kleinman-Martin method has similarities to the combined field integral equation methods, we use a regularizer introduced by M. Windisch and O. Steinbach in [23] in the context of combined field integral equations for the time-harmonic Maxwell equations. This regularizer is a positive definite boundary integral operator with a similar structure as the operator of the electrical field integral equation, but it is not a compact operator like those used in [10] and [6] for regularizing the exterior electromagnetic scattering problem. Its introduction changes the boundary condition in the associated interior boundary value problem from a non-elliptic local impedance-like condition to a non-local, but elliptic, boundary condition.

This work contains results from the thesis [20] where this integral formulation of the transmission problem is used to study the shape derivatives of the solution of the dielectric scattering problem, in the context of a problem of optimising the shape of a dielectric lens in order to obtain a prescribed radiation pattern.

In Section 3 we recall briefly some results about traces and potentials for Maxwell's equations in Sobolev spaces, following the notation of [7] and [6]. We also recall some properties of the Laplace-Beltrami operator which will then be used for a Helmholtz decomposition of the boundary energy space shown by A. De La Bourdonnaie in [16]. In the following sections, this allows us to prove the invertibility of boundary integral operators by using a new integral representation of the potential's traces. Sections 4 and 5 contain the details of the method for solving the transmission problem using single boundary integral equations. In Section 4, we start from a layer representation for the exterior field whereas in Section 5, we use a layer representation for the interior field. In either case, we derive two boundary integral equations of the second kind and we show their unique solvability under suitable conditions on an associated interior boundary value

problem. Moreover, we show that the integral operators in each integral equation are Fredholm of index zero. We also construct the solution of the transmission problem using the solution of any of the four integral equations. We finally show how to choose the free parameters so that the associated interior boundary value problem is uniquely solvable, and as a consequence, we can construct an integral representation of the solution which yields uniquely solvable boundary integral equations for all real frequencies.

2 The dielectric scattering problem

Let Ω denote a bounded domain in $\mathbb{R}^3$ and let Ω^c denote the exterior domain $\mathbb{R}^3\backslash\overline{\Omega}$. In this paper, we will assume that the boundary Γ of Ω is a smooth and simply connected closed surface, so that Ω is diffeomorphic to a ball. Let $\mathbf{n}$ denote the outer unit normal vector on the boundary Γ.

In Ω (resp. Ω^c) the electric permittivity ϵ_i (resp. ϵ_e) and the magnetic permeability μ_i (resp. μ_e) are positive constants. The frequency ω is the same in Ω and in Ω^c. The interior wave number κ_i and the exterior wave number κ_e are complex constants of non negative imaginary part.

Notation: For a domain $G \subset \mathbb{R}^3$ we denote by $H^s(G)$ the usual L^2-based Sobolev space of order $s \in \mathbb{R}$, and by $H^s_{\rm loc}(\overline{G})$ the space of functions whose restrictions to any bounded subdomain B of G belong to $H^s(B)$. Spaces of vector functions will be denoted by boldface letters, thus

$$\mathbf{H}^s(G) = \left(H^s(G)\right)^3 .$$

If D is a differential operator, we write:

$$\begin{aligned} \mathbf{H}(\mathsf{D},\Omega) &= \{u \in \mathbf{L}^2(\Omega) : \mathsf{D}u \in \mathbf{L}^2(\Omega)\} \\ \mathbf{H}_{\rm loc}(\mathsf{D},\overline{\Omega^c}) &= \{u \in \mathbf{L}^2_{\rm loc}(\overline{\Omega^c}) : \mathsf{D}u \in \mathbf{L}^2_{\rm loc}(\overline{\Omega^c})\} \end{aligned}$$

The space $\mathbf{H}(\mathsf{D},\Omega)$ is endowed with the natural graph norm. This defines in particular the Hilbert spaces $\mathbf{H}(\mathbf{curl},\Omega)$ and $\mathbf{H}(\mathbf{curl}\,\mathbf{curl},\Omega)$.

The time-harmonic dielectric scattering problem is formulated as follows.

The dielectric scattering problem : Given an incident field $\mathbf{E}^{inc} \in \mathbf{H}_{\rm loc}(\mathbf{curl},\mathbb{R}^3)$ that satisfies $\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}^{inc} - \kappa_e^2\mathbf{E}^{inc} = 0$ in a neighborhood of $\overline{\Omega}$, we seek two fields $\mathbf{E}^i \in \mathbf{H}(\mathbf{curl},\Omega)$ and $\mathbf{E}^s \in \mathbf{H}_{\rm loc}(\mathbf{curl},\overline{\Omega^c})$ satisfying the time-harmonic Maxwell equations

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}^i - \kappa_i^2\mathbf{E}^i = 0 \quad \text{in } \Omega, \tag{2.1}$$

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}^s - \kappa_e^2\mathbf{E}^s = 0 \quad \text{in } \Omega^c, \tag{2.2}$$

the two transmission conditions,

$$\mathbf{n}\times\mathbf{E}^i = \mathbf{n}\times(\mathbf{E}^s + \mathbf{E}^{inc}) \qquad \text{on } \Gamma \tag{2.3}$$

$$\mu_i^{-1}(\mathbf{n}\times\mathbf{curl}\,\mathbf{E}^i) = \mu_e^{-1}\mathbf{n}\times\mathbf{curl}(\mathbf{E}^s + \mathbf{E}^{inc}) \qquad \text{on } \Gamma \tag{2.4}$$

and the Silver-Müller radiation condition:

$$\lim_{|x|\to+\infty} |x| \left| \mathbf{curl}\, \mathbf{E}^s(x) \times \frac{x}{|x|} - i\kappa_e \mathbf{E}^s(x) \right| = 0. \tag{2.5}$$

It is well known that this problem has at most one solution under some mild restrictions on the dielectric constants. We give sufficient conditions in the next theorem, and for completeness we give its simple proof.

Theorem 2.1 *Assume that the constants μ_i, κ_i, μ_e and κ_e satisfy:*

(i) κ_e is real and positive or $\mathrm{Im}(\kappa_e) > 0$,

(ii) $\mathrm{Im}\left(\overline{\kappa_e}\dfrac{\mu_e}{\mu_i}\right) \le 0$ and $\mathrm{Im}\left(\overline{\kappa_e}\dfrac{\mu_e}{\mu_i}\kappa_i^2\right) \ge 0$.

Then the dielectric scattering problem has at most one solution.

Proof. We use similar arguments as in the acoustic case [18]. Assume that $\mathbf{E}^{inc} = 0$. Let $(\mathbf{E}^i, \mathbf{E}^s)$ be a solution of the homogeneous scattering problem. Let B_R be a ball of radius R large enough such that $\Omega \subset B_R$ and let $\mathbf{n}_R$ the unit outer normal vector to B_R. Integration by parts using the Maxwell equations (2.1) and (2.2) and the transmission conditions (2.3) and (2.4) gives :

$$\int_{\partial B_R} (\mathbf{curl}\, \mathbf{E}^s \times \mathbf{n}_R) \cdot \overline{\mathbf{E}^s} = \int_{B_R\setminus\Omega} \{|\,\mathbf{curl}\, \mathbf{E}^s|^2 - \kappa_e^2 |\mathbf{E}^s|^2\} + \frac{\mu_e}{\mu_i} \int_\Omega \{|\,\mathbf{curl}\, \mathbf{E}^i|^2 - \kappa_i^2 |\mathbf{E}^i|^2\}$$

We multiply this by $\overline{\kappa_e}$ and take the imaginary part:

$$\begin{aligned}\mathrm{Im}\left(\overline{\kappa_e}\int_{\partial B_R} (\mathbf{curl}\, \mathbf{E}^s \times \mathbf{n}_R) \cdot \overline{\mathbf{E}^s}\right) = &\mathrm{Im}(\overline{\kappa_e}) \left(\int_{B_R\setminus\Omega} \{|\,\mathbf{curl}\, \mathbf{E}^s|^2 + |\kappa_e \mathbf{E}^s|^2\}\right) \\ &+ \mathrm{Im}\left(\overline{\kappa_e}\frac{\mu_e}{\mu_i}\right) \int_\Omega |\,\mathbf{curl}\, \mathbf{E}^i|^2 - \mathrm{Im}\left(\overline{\kappa_e}\frac{\mu_e}{\mu_i}\kappa_i^2\right) \int_\Omega |\mathbf{E}^i|^2.\end{aligned}$$

Under the hypotheses (i) and (ii), all terms on the right hand side are non-positive.

Thanks to the Silver-Müller condition, we have

$$\lim_{R\to+\infty} \int_{\partial B_R} |\,\mathbf{curl}\, \mathbf{E}^s \times \mathbf{n}_R - i\kappa_e \mathbf{E}^s|^2 = 0.$$

Developping this expression, we get

$$\lim_{R\to+\infty} \int_{\partial B_R} |\,\mathbf{curl}\, \mathbf{E}^s \times \mathbf{n}_R|^2 + |\kappa_e \mathbf{E}^s|^2 - 2\,\mathrm{Re}\left(\mathbf{curl}\, \mathbf{E}^s \times \mathbf{n}_R \cdot \overline{i\kappa_e \mathbf{E}^s}\right) = 0.$$

As we have seen, we have

$$\int_{\partial B_R} \mathrm{Re}\left(\mathbf{curl}\, \mathbf{E}^s \times \mathbf{n}_R \cdot \overline{i\kappa_e \mathbf{E}}\right) = \mathrm{Im} \int_{\partial B_R} \left(\overline{\kappa_e}\, \mathbf{curl}\, \mathbf{E}^s \times \mathbf{n}_R \cdot \overline{\mathbf{E}}\right) \le 0.$$

It follows that

$$\lim_{R\to+\infty}\int_{\partial B_R}|\mathbf{E}^s|^2=0.$$

Thus, by Rellich's lemma [10], $\mathbf{E}^s=0$ in Ω^c. Using the transmission conditions, we obtain $\gamma_D\mathbf{E}^i=\gamma_{N_{\kappa_i}}\mathbf{E}^i=0$. It follows that $\mathbf{E}^i=0$ in Ω. ■

3 Traces and electromagnetic potentials

We use some well known results about traces of vector fields and integral representations of time-harmonic electromagnetic fields. Details can be found in [3, 4, 5, 7, 12, 22].

Definition 3.1 *For a vector function* $\mathbf{u}\in(\mathscr{C}^\infty(\overline{\Omega}))^3$ *and a scalar function* $v\in\mathscr{C}^\infty(\overline{\Omega})$ *we define the traces :*

$$\gamma v=v_{|\Gamma},$$
$$\gamma_D\mathbf{u}:=(\mathbf{n}\times\mathbf{u})_{|\Gamma}\ \textit{(Dirichlet) and}$$
$$\gamma_{N_\kappa}\mathbf{u}:=\kappa^{-1}(\mathbf{n}\times\mathbf{curl}\,\mathbf{u})_{|\Gamma}\ \textit{(Neumann).}$$

We introduce the Hilbert spaces $H^s(\Gamma)=\gamma\left(H^{s+\frac12}(\Omega)\right)$, and $\mathbf{H}^s_\times(\Gamma)=\gamma_D\left(\mathbf{H}^{s+\frac12}(\Omega)\right)$ For $s>0$, the traces

$$\gamma: H^{s+\frac12}(\Omega)\to H^s(\Gamma),$$
$$\gamma_D: \mathbf{H}^{s+\frac12}(\Omega)\to\mathbf{H}^s_\times(\Gamma)$$

are then continuous. The dual of $H^s(\Gamma)$ and $\mathbf{H}^s_\times(\Gamma)$ with respect to the L^2 (or $\mathbf{L}^2$) scalar product is denoted by $H^{-s}(\Gamma)$ and $\mathbf{H}^{-s}_\times(\Gamma)$, respectively.

We use the surface differential operators: The tangential gradient denoted by ∇_Γ, the surface divergence denoted by div_Γ, the tangential vector curl denoted by $\mathbf{curl}_\Gamma$ and the surface scalar curl denoted by curl_Γ. For their definitions we refer to [5], [12] and [22].

Definition 3.2 *We define the Hilbert space*

$$\mathbf{H}^{-\frac12}_\times(\mathrm{div}_\Gamma,\Gamma)=\left\{\mathbf{j}\in\mathbf{H}^{-\frac12}_\times(\Gamma),\mathrm{div}_\Gamma\,\mathbf{j}\in H^{-\frac12}(\Gamma)\right\}$$

endowed with the norm

$$||\cdot||_{\mathbf{H}^{-\frac12}_\times(\mathrm{div}_\Gamma,\Gamma)}=||\cdot||_{\mathbf{H}^{-\frac12}_\times(\Gamma)}+||\,\mathrm{div}_\Gamma\cdot||_{H^{-\frac12}(\Gamma)}.$$

The skew-symmetric bilinear form

$$\begin{aligned}\mathcal{B}:\ \mathbf{H}^{-\frac12}_\times(\mathrm{div}_\Gamma,\Gamma)\times\mathbf{H}^{-\frac12}_\times(\mathrm{div}_\Gamma,\Gamma)&\to\mathbb{C}\\(\mathbf{j},\mathbf{m})&\to\mathcal{B}(\mathbf{j},\mathbf{m})=\int_\Gamma(\mathbf{n}\times\mathbf{j})\cdot\mathbf{m}\,d\sigma\end{aligned}$$

defines a non-degenerate duality product on $\mathbf{H}^{-\frac12}_\times(\mathrm{div}_\Gamma,\Gamma)$.

Lemma 3.3 *The operators γ_D and γ_N are linear and continuous from $(\mathscr{C}^\infty(\overline{\Omega}))^3$ to $\mathbf{L}^2_\times(\Gamma)$ and they can be extended to continuous linear operators from $\mathbf{H}(\mathbf{curl},\Omega)$ and $\mathbf{H}(\mathbf{curl},\Omega)\cap\mathbf{H}(\mathbf{curl}\,\mathbf{curl},\Omega)$, respectively, to $\mathbf{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$. Moreover, for all $\mathbf{u},\ \mathbf{v}\in\mathbf{H}(\mathbf{curl},\Omega)$, we have:*

$$\int_\Omega \left[(\mathbf{curl}\,\mathbf{u}\cdot\mathbf{v})-(\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v})\right]dx=\mathcal{B}(\gamma_D\mathbf{u},\gamma_D\mathbf{v}). \tag{3.1}$$

For $\mathbf{u}\in\mathbf{H}_{\mathrm{loc}}(\mathbf{curl},\overline{\Omega^c})$ and $\mathbf{v}\in\mathbf{H}_{\mathrm{loc}}(\mathbf{curl}\,\mathbf{curl},\overline{\Omega^c}))$ we define $\gamma_D^c\mathbf{u}$ and $\gamma_N^c\mathbf{v}$ in the same way and the same mapping properties hold true.

Recall that we assume that the boundary Γ is smooth and topologically trivial. For a proof of the following result, we refer to [3, 12, 22].

Lemma 3.4 *Let $t\in\mathbb{R}$. The Laplace-Beltrami operator*

$$\Delta_\Gamma=\mathrm{div}_\Gamma\nabla_\Gamma=-\,\mathrm{curl}_\Gamma\,\mathbf{curl}_\Gamma \tag{3.2}$$

is linear and continuous from $H^{t+2}(\Gamma)$ to $H^t(\Gamma)$.
It is an isomorphism from $H^{t+2}(\Gamma)/\mathbb{R}$ to the space $H^t_*(\Gamma)$ defined by

$$u\in H^t_*(\Gamma)\quad\Longleftrightarrow\quad u\in H^t(\Gamma)\ \textit{and}\ \int_\Gamma u=0.$$

We note the following equalities:

$$\mathrm{curl}_\Gamma\nabla_\Gamma=0\ \text{and}\ \ \mathrm{div}_\Gamma\,\mathbf{curl}_\Gamma=0 \tag{3.3}$$

$$\mathrm{div}_\Gamma(\mathbf{n}\times\mathbf{j})=-\,\mathrm{curl}_\Gamma\,\mathbf{j}\ \text{and}\ \ \mathrm{curl}_\Gamma(\mathbf{n}\times\mathbf{j})=\mathrm{div}_\Gamma\,\mathbf{j} \tag{3.4}$$

Let κ be a complex number such that $\mathrm{Im}(\kappa)\geq 0$ and let

$$G(\kappa,|x-y|)=\frac{e^{i\kappa|x-y|}}{4\pi|x-y|}$$

be the fundamental solution of the Helmholtz equation

$$\Delta u+\kappa^2u=0.$$

The single layer potential ψ_κ is given by :

$$(\psi_\kappa u)(x)=\int_\Gamma G(\kappa,|x-y|)u(y)d\sigma(y)\qquad x\in\mathbb{R}^3\backslash\Gamma,$$

and its trace by

$$V_\kappa u(x)=\int_\Gamma G(\kappa,|x-y|)u(y)d\sigma(y)\qquad x\in\Gamma.$$

For a proof of the following well-known result, see [17, 22].

Lemma 3.5 *The operators*

$$\begin{aligned}\psi_\kappa &: H^{-\frac{1}{2}}(\Gamma) \to H^1_{\rm loc}(\mathbb{R}^3)\\ V_\kappa &: H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)\end{aligned}$$

are continuous.

We define the electric potential Ψ_{E_κ} generated by $\mathbf{j} \in \mathbf{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ by

$$\Psi_{E_\kappa}\mathbf{j} := \kappa\psi_\kappa\,\mathbf{j} + \kappa^{-1}\nabla\psi_\kappa\,\mathrm{div}_\Gamma\,\mathbf{j}$$

This can be written as $\Psi_{E_\kappa}\mathbf{j} := \kappa^{-1}\,\mathbf{curl}\,\mathbf{curl}\,\psi_\kappa\,\mathbf{j}$ because of the Helmholtz equation and the identity $\mathbf{curl}\,\mathbf{curl} = -\Delta + \nabla\,\mathrm{div}$ (cf [3]).

We define the magnetic potential Ψ_{M_κ} generated by $\mathbf{m} \in \mathbf{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ by

$$\Psi_{M_\kappa}\mathbf{m} := \mathbf{curl}\,\psi_\kappa\mathbf{m}.$$

These potentials satisfy

$$\kappa^{-1}\,\mathbf{curl}\,\Psi_{E_\kappa} = \Psi_{M_\kappa} \quad \text{and} \quad \kappa^{-1}\,\mathbf{curl}\,\Psi_{M_\kappa} = \Psi_{E_\kappa}.$$

We denote the identity operator by I.

Lemma 3.6 *The potentials Ψ_{E_κ} et Ψ_{M_κ} are continuous from $\mathbf{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ to $\mathbf{H}_{\rm loc}(\mathbf{curl}, \mathbb{R}^3)$. For $\mathbf{j} \in \mathbf{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ we have*

$$(\mathbf{curl}\,\mathbf{curl} - \kappa^2\mathrm{I})\Psi_{E_\kappa}\mathbf{j} = 0 \text{ and } (\mathbf{curl}\,\mathbf{curl} - \kappa^2\mathrm{I})\Psi_{M_\kappa}\mathbf{m} = 0 \text{ in } \mathbb{R}^3\backslash\Gamma$$

and $\Psi_{E_\kappa}\mathbf{j}$ and $\Psi_{M_\kappa}\mathbf{m}$ satisfy the Silver-Müller condition.

It follows that the traces γ_D, γ_{N_κ}, γ_D^c and $\gamma_{N_\kappa}^c$ can be applied to Ψ_{E_κ} and Ψ_{M_κ}, resulting in continuous mappings from $\mathbf{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ to itself satisfying

$$\gamma_{N_\kappa}\Psi_{E_\kappa} = \gamma_D\Psi_{M_\kappa} \quad \text{and} \quad \gamma_{N_\kappa}\Psi_{M_\kappa} = \gamma_D\Psi_{E_\kappa}\,.$$

Defining

$$\begin{aligned}[\gamma_D] &= \gamma_D - \gamma_D^c, \quad \{\gamma_D\} &= -\frac{1}{2}\left(\gamma_D + \gamma_D^c\right),\\ [\gamma_{N_\kappa}] &= \gamma_{N_\kappa} - \gamma_{N_\kappa}^c, \quad \{\gamma_{N_\kappa}\} &= -\frac{1}{2}\left(\gamma_{N_\kappa} + \gamma_{N_\kappa}^c\right).\end{aligned}$$

we have the following *jump relations* (see [4]):

$$\begin{aligned}[\gamma_D]\,\Psi_{E_\kappa} &= 0, \quad [\gamma_{N_\kappa}]\,\Psi_{E_\kappa} &= -\mathrm{I},\\ [\gamma_{N_\kappa}]\,\Psi_{M_\kappa} &= -\mathrm{I}, \quad [\gamma_D]\,\Psi_{M_\kappa} &= 0.\end{aligned}$$

Now assume that $\mathbf{E} \in \mathbf{L}^2_{\rm loc}(\mathbb{R}^3)$ belongs to $\mathbf{H}(\mathbf{curl}, \Omega)$ in the interior domain and to $\mathbf{H}_{\rm loc}(\mathbf{curl}, \overline{\Omega^c})$ in the exterior domain and satisfies the equation

$$(\mathbf{curl}\,\mathbf{curl} - \kappa^2 \mathrm{I})\,\mathbf{E} = 0 \tag{3.5}$$

in $\mathbb{R}^3 \setminus \Gamma$ and the Silver-Müller condition. Then if we set $\mathbf{j} = [\gamma_{N_\kappa}]\mathbf{E}$, $\mathbf{m} = [\gamma_D]\mathbf{E}$, we have on $\mathbb{R}^3 \setminus \Gamma$ the Stratton-Chu **integral representation**

$$\mathbf{E} = -\Psi_{E_\kappa}\,\mathbf{j} - \Psi_{M_\kappa}\mathbf{m}. \tag{3.6}$$

Special cases of (3.6) are: If $(\mathbf{E}^i, \mathbf{E}^s)$ solves the dielectric scattering problem, then

$$-\Psi_{E_{\kappa_e}}\gamma^c_{N_{\kappa e}}\mathbf{E}^s - \Psi_{M_{\kappa_e}}\gamma^c_D\mathbf{E}^s = \begin{cases} -\mathbf{E}^s & x \in \Omega^c \\ 0 & x \in \Omega \end{cases} \tag{3.7}$$

$$\Psi_{E_{\kappa_e}}\gamma^c_{N_{\kappa e}}\left(\mathbf{E}^s + \mathbf{E}^{inc}\right) + \Psi_{M_{\kappa_e}}\gamma^c_D\left(\mathbf{E}^s + \mathbf{E}^{inc}\right) = \begin{cases} \mathbf{E}^s & x \in \Omega^c \\ -\mathbf{E}^{inc} & x \in \Omega \end{cases} \tag{3.8}$$

$$-\Psi_{E_{\kappa_i}}\gamma_{N_{\kappa_i}}\mathbf{E}^i - \Psi_{M_{\kappa_i}}\gamma_D\mathbf{E}^i = \begin{cases} 0 & x \in \Omega^c \\ \mathbf{E}^i & x \in \Omega \end{cases} \tag{3.9}$$

We can now define the main boundary integral operators:

$$C_\kappa = \{\gamma_D\}\Psi_{E_\kappa} = \{\gamma_{N_\kappa}\}\Psi_{M_\kappa},$$

$$M_\kappa = \{\gamma_D\}\Psi_{M_\kappa} = \{\gamma_{N_\kappa}\}\Psi_{E_\kappa}.$$

These are bounded operators in $\mathbf{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$.

As tools, we will need variants of these operators:

Definition 3.7 *Define the operators $C_{\kappa,0}$ and C_0^* for $\mathbf{j} \in \mathbf{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ by :*

$$C_{\kappa,0}\,\mathbf{j} = -\kappa\,\mathbf{n} \times V_0\,\mathbf{j} + \kappa^{-1}\,\mathbf{curl}_\Gamma\,V_0\,\mathrm{div}_\Gamma\,\mathbf{j}$$

$$C_0^*\,\mathbf{j} = \mathbf{n} \times V_0\,\mathbf{j} + \mathbf{curl}_\Gamma\,V_0\,\mathrm{div}_\Gamma\,\mathbf{j}$$

Note that C_0^* differs from $C_{1,0}$ by the relative sign of the two terms.

The following theorem is proved in [22] section 5.5.

Theorem 3.8 *The operators $C_\kappa - C_{\kappa,0}$ and M_κ are compact operators from $\mathbf{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ to itself.*

The Calderón projectors for the time-harmonic Maxwell system (3.5) are $P = \frac{1}{2}\mathrm{I} + A_\kappa$ and $P^c = \frac{1}{2}\mathrm{I} - A_\kappa$ where

$$A = \begin{pmatrix} M_\kappa & C_\kappa \\ C_\kappa & M_\kappa \end{pmatrix}.$$

We have $P \circ P^c \equiv 0$ and therefore

$$C_\kappa^2 = \frac{1}{4}\mathrm{I} - M_\kappa^2. \tag{3.10}$$

It follows that the operator C_κ is Fredholm of index zero.

The following theorem was proved in [23].

Lemma 3.9 *The operator C_0^* is self-adjoint and elliptic for the bilinear form $\mathcal{B}$ and invertible on $\mathsf{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$.*

Indeed, for $\mathbf{j} \in \mathsf{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ we have

$$\mathcal{B}(\mathbf{j}, C_0^* \bar{\mathbf{j}}) = \int_\Gamma \left\{ \mathbf{j} \cdot V_0 \bar{\mathbf{j}} + \mathrm{div}_\Gamma \mathbf{j} \; V_0 \, \mathrm{div}_\Gamma \bar{\mathbf{j}} \right\}$$

and the result follows from the $H^{-\frac{1}{2}}(\Gamma)$-ellipticity of the scalar single layer potential operator V_0.

4 Integral equations 1

In this section, we present the first method for solving the dielectric problem, following the procedure of R. E. Kleinman and P. A. Martin [18]: We use a layer ansatz on the exterior field to construct two alternative boundary integral equations.
In the scalar case, one represents the exterior field as a linear combination of a single layer potential and a double layer potential, both generated by the same density. It turns out that this simple idea does not suffice in the electromagnetic case if one wants to avoid irregular frequencies. Our approach is related to the idea of "modified combined field integral equations": We compose one of the electromagnetic potential operators with an elliptic and invertible boundary integral operator, namely C_0^*. More precisely, we assume that $\mathbf{E}^s$ admits the following integral representation :

$$\mathbf{E}^s(x) = -a(\Psi_{E_{\kappa_e}} \mathbf{j})(x) - b(\Psi_{M_{\kappa_e}} C_0^* \, \mathbf{j})(x) \qquad \text{for } x \in \Omega^c \,. \tag{4.1}$$

Here $\mathbf{j} \in \mathsf{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ is the unknown density and a and b are arbitrary complex constants.

We set $\rho = \dfrac{\mu_e \kappa_i}{\mu_i \kappa_e}$. The transmission conditions can be rewritten :

$$\gamma_D \mathbf{E}^i = \gamma_D^c \mathbf{E}^s + \gamma_D \mathbf{E}^{inc} \text{ and } \gamma_{N_{\kappa_i}} \mathbf{E}^i = \rho^{-1} \left(\gamma_{N_{\kappa_e}}^c \mathbf{E}^s + \gamma_{N_{\kappa_e}} \mathbf{E}^{inc} \right).$$

Using this in the integral representation formula (3.9) in Ω, we get:

$$\mathbf{E}^i = -\frac{1}{\rho} \Psi_{E_{\kappa_i}} \left(\gamma_{N_{\kappa_e}}^c \mathbf{E}^s + \gamma_{N_{\kappa_e}} \mathbf{E}^{inc} \right) - \Psi_{M_{\kappa_i}} \left(\gamma_D^c \mathbf{E}^s + \gamma_D \mathbf{E}^{inc} \right) \text{ in } \Omega. \tag{4.2}$$

We take traces in (4.1) and obtain the Calderón relations

$$\gamma_D^c \mathbf{E}^s = \left\{ aC_{\kappa_e} - b\left(\frac{1}{2}\mathrm{I} - M_{\kappa_e}\right) C_0^* \right\} \mathbf{j} \quad = L_e \mathbf{j} \text{ on } \Gamma, \tag{4.3}$$

$$\gamma_{N_{\kappa_e}}^c \mathbf{E}^s = \left\{ -a\left(\frac{1}{2}\mathrm{I} - M_{\kappa_e}\right) + bC_{\kappa_e} C_0^* \right\} \mathbf{j} \quad = N_e \mathbf{j} \text{ on } \Gamma. \tag{4.4}$$

On the other hand, taking traces in (4.2) gives:

$$\rho\left(-\frac{1}{2}\mathrm{I} + M_{\kappa_i}\right)\left(\gamma_D^c \mathbf{E}^s + \gamma_D \mathbf{E}^{inc}\right) + C_{\kappa_i}\left(\gamma_{N_{\kappa_e}}^c \mathbf{E}^s + \gamma_{N_{\kappa_e}} \mathbf{E}^{inc}\right) = 0 \text{ on } \Gamma, \tag{4.5}$$

$$\left(-\frac{1}{2}\mathrm{I} + M_{\kappa_i}\right)\left(\gamma_{N_{\kappa_e}}^c \mathbf{E}^s + \gamma_{N_{\kappa_e}} \mathbf{E}^{inc}\right) + \rho C_{\kappa_i}\left(\gamma_D^c \mathbf{E}^s + \gamma_D \mathbf{E}^{inc}\right) = 0 \text{ on } \Gamma. \tag{4.6}$$

We can now substitute (4.3) and (4.4) into (4.5) and get our first integral equation:

$$\mathbf{S}\,\mathbf{j} = \rho\left(-\frac{1}{2}\mathrm{I} + M_{\kappa_i}\right) L_e \mathbf{j} + C_{\kappa_i} N_e \mathbf{j} = f \tag{4.7}$$

$$\text{where } f = -\rho\left(-\frac{1}{2}\mathrm{I} + M_{\kappa_i}\right)\gamma_D \mathbf{E}^{inc} - C_{\kappa_i}\gamma_{N_{\kappa_e}} \mathbf{E}^{inc}. \tag{4.8}$$

If we substitute (4.3) and (4.4) into (4.6), we get our second integral equation:

$$\mathbf{T}\,\mathbf{j} = \rho C_{\kappa_i} L_e \mathbf{j} + \left(-\frac{1}{2}\mathrm{I} + M_{\kappa_i}\right) N_e \mathbf{j} = g \tag{4.9}$$

$$\text{where } g = -\rho C_{\kappa_i}\gamma_D \mathbf{E}^{inc} - \left(-\frac{1}{2}\mathrm{I} + M_{\kappa_i}\right)\gamma_{N_{\kappa_e}} \mathbf{E}^{inc}. \tag{4.10}$$

Thus we obtain two boundary integral equations for the unknown $\mathbf{j}$. Having solved either one, we construct $\mathbf{E}^s$ using (4.1) and $\mathbf{E}^i$ using (4.2), (4.3), (4.4):

$$\mathbf{E}^i = -\frac{1}{\rho}\left(\Psi_{E_{\kappa_i}}\{\gamma_{N_{\kappa_e}} \mathbf{E}^{inc} + N_e \mathbf{j}\}\right) - \left(\Psi_{M_{\kappa_i}}\{\gamma_D \mathbf{E}^{inc} + L_e \mathbf{j}\}\right). \tag{4.11}$$

Theorem 4.1 *If* $\mathbf{j} \in \mathbf{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ *solves* (4.7) *or* (4.9)*, then* $\mathbf{E}^s$ *and* $\mathbf{E}^i$ *given by* (4.1) *and* (4.11) *solve the transmission problem.*

Proof. We know that $\mathbf{E}^i$ and $\mathbf{E}^s$ satisfy the Maxwell equations and the Silver-Müller condition. It remains to verify that $\mathbf{E}^s$ and $\mathbf{E}^i$ satisfy the transmission conditions (2.3) and (2.4). Using the integral representation (4.1) and (4.2) of $\mathbf{E}^s$ and $\mathbf{E}^i$, a simple computation gives:

$$\rho(\gamma_D^c \mathbf{E}^s + \gamma_D \mathbf{E}^{inc} - \gamma_D \mathbf{E}^i) = \mathbf{S}\,\mathbf{j} - f \tag{4.12}$$

and

$$\gamma_{N_{\kappa_e}}^c \mathbf{E}^s + \gamma_{N_{\kappa_e}} \mathbf{E}^{inc} - \rho\gamma_{N_{\kappa_i}} \mathbf{E}^i = \mathbf{T}\,\mathbf{j} - g \tag{4.13}$$

We deduce that

- if $\mathbf{j}$ solves (4.7), then relation (4.12) proves that the condition (2.3) is satisfied,
- if $\mathbf{j}$ solves (4.9), then relation (4.13) proves that the condition (2.4) is satisfied.

Now we show that (4.7) and (4.9) are in fact equivalent. Define :

$$\mathbf{u}(x) = -\Psi_{E_{\kappa_i}}\{\gamma_{N_{\kappa_e}}\mathbf{E}^{inc} + N_e\,\mathbf{j}\}(x) - \rho\Psi_{M_{\kappa_i}}\{\gamma_D\mathbf{E}^{inc} + L_e\,\mathbf{j}\}(x) \quad \text{for } x \in \Omega^c.$$

This field $\mathbf{u}$ is in $\mathbf{H}_{\rm loc}(\mathbf{curl}, \overline{\Omega^c})$ and satisfies the Maxwell system

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - \kappa_i^2\mathbf{u} = 0 \tag{4.14}$$

in Ω^c. On the boundary Γ we have:

$$\gamma_D^c\mathbf{u} = \mathsf{S}\,\mathbf{j} - f \quad \text{and} \quad \gamma_{N_{\kappa_i}}^c\mathbf{u} = \mathsf{T}\,\mathbf{j} - g.$$

Since $\mathbf{u}$ solves (4.14) in Ω^c and satisfies the Silver-Müller condition, it follows:

$$\mathbf{j} \text{ satisfies (4.7)} \Rightarrow \gamma_D^c\mathbf{u} = 0 \Rightarrow \mathbf{u} \equiv 0 \text{ in } \overline{\Omega}^c \Rightarrow \gamma_{N_{\kappa_i}}^c\mathbf{u} = 0 \Rightarrow \mathbf{j} \text{ satisfies (4.9)}.$$

$$\mathbf{j} \text{ satisfies (4.9)} \Rightarrow \gamma_{N_{\kappa_i}}^c\mathbf{u} = 0 \Rightarrow \mathbf{u} \equiv 0 \text{ in } \Omega^c \Rightarrow \gamma_D^c\mathbf{u} = 0 \Rightarrow \mathbf{j} \text{ satisfies (4.7)}.$$

As a consequence, if $\mathbf{j}$ solves one of the two integral equations, it solves both, and then both transmission conditions (2.3) and (2.4) are satisfied. ■

The next theorem is concerned with the uniqueness of the solutions of the boundary integral equations (4.7) and (4.9), i.e., with the existence of nontrivial solutions of the following homogeneous forms of (4.7) and (4.9):

$$\rho\left(-\frac{1}{2}\mathrm{I} + M_{\kappa_i}\right)L_e\,\mathbf{j}_0 + C_{\kappa_i}N_e\,\mathbf{j}_0 = 0, \tag{4.15}$$

$$\rho C_{\kappa_i}L_e\,\mathbf{j}_0 + \left(-\frac{1}{2}\mathrm{I} + M_{\kappa_i}\right)N_e\,\mathbf{j}_0 = 0. \tag{4.16}$$

As in the scalar case [18], we associate with the dielectric scattering problem a new interior boundary value problem, the eigenvalues of which determine uniqueness for the integral equations.

Associated interior problem: For $a, b \in \mathbb{C}$, consider the boundary value problem

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - \kappa_e^2\mathbf{u} = 0 \quad \text{in } \Omega, \qquad a\gamma_D\mathbf{u} - bC_0^*\gamma_{N_e}\mathbf{u} = 0 \quad \text{on } \Gamma. \tag{4.17}$$

Lemma 4.2 . *Let $a, b \in \mathbb{C}\setminus 0$. If $\mathrm{Im}\,\dfrac{b}{a} \neq 0$, then the associated interior problem* (4.17) *does not admit any real eigenvalue.*

Proof. Let κ_e^2 be an eigenvalue of the interior problem such that $\kappa_e \in \mathbb{R}$ and let $\mathbf{u} \neq 0$ be an eigenfunction. Using Green's theorem we have:

$$\int_\Omega |\,\mathbf{curl}\,\mathbf{u}|^2 - \kappa_e^2\int_\Omega |\mathbf{u}|^2 = \kappa_e\mathcal{B}(\gamma_{N_{\kappa_e}}\mathbf{u}, \gamma_D\overline{\mathbf{u}}) = \frac{a\kappa_e}{b}\,\mathcal{B}((C_0^*)^{-1}(\gamma_D\mathbf{u}), \gamma_D\overline{\mathbf{u}}) \text{ if b} \neq 0$$
$$= \kappa_e\overline{\left(\frac{b}{a}\right)}\mathcal{B}(\gamma_{N_{\kappa_e}}\mathbf{u}, C_0^*(\gamma_{N_{\kappa_e}}\overline{\mathbf{u}})) \text{ if a} \neq 0$$

Taking the imaginary part, under the hypothesis of the lemma if follows

$$\mathcal{B}((C_0^*)^{-1}(\gamma_D \mathbf{u}), \gamma_D \overline{\mathbf{u}}) = 0 \text{ and } \mathcal{B}(\gamma_{N_{\kappa_e}} \mathbf{u}, C_0^*(\gamma_{N_{\kappa_e}} \overline{\mathbf{u}})) = 0.$$

As C_0^* is elliptic for the bilinear form $\mathcal{B}$, the traces $\gamma_D \mathbf{u}$ and $\gamma_{N_{\kappa_e}} \mathbf{u}$ then vanish. Thanks to the Stratton-Chu representation formula (3.6) in Ω, we deduce that $\mathbf{u} = 0$, which contradicts the initial assumption. ■

Remark 4.3 *Note that this associated interior problem is not an impedance problem (or Robin problem) as in the scalar case [18]. If we replace in* (4.17) *the operator C_0^* by the identity, we obtain a "pseudo-impedance" type problem. This is a non-elliptic problem, about whose spectrum we have no information. That the problem is non-elliptic can be seen as follows: If it were elliptic, its principal part would be elliptic, too. This would be the vector Laplace operator with the "Neumann" condition $\gamma_{N_{\kappa_e}} \mathbf{u} = 0$. Any gradient of a harmonic function in $H^1(\Omega)$ will satisfy the homogeneous problem, which therefore has an infinite-dimensional nullspace, contradiciting ellipticity. Note that the issue here is not the apparent non-elliptic nature of the interior Maxwell* $\mathbf{curl}\,\mathbf{curl}$ *operator, which can easily be remedied by the usual regularization that adds $-\nabla \operatorname{div}$, but the manifestly non-elliptic nature of the Maxwell "Neumann" boundary operator. For a "true" impedance problem, the operator C_0^* would have to be replaced not by the identity, but by the rotation operator $\mathbf{j} \mapsto \mathbf{n} \times \mathbf{j}$. This operator leads out of the space $\mathbf{H}_{\times}^{-\frac{1}{2}}(\operatorname{div}_\Gamma)$, however, which rules it out for our purposes.*

For our integral equations, the problem (4.17) plays the same role as the Robin problem for the scalar case in [18].

Theorem 4.4 *Assume that the hypotheses of Theorem 2.3 are satisfied. Then for $(a, b) \neq (0, 0)$, the homogeneous integral equations* (4.15) *and* (4.16) *admit nontrivial solutions if and only if κ_e^2 is an eigenvalue of the associated interior problem.*

Proof. Assume that $\mathbf{j}_0 \neq 0$ solves (4.15) or (4.16).
We construct $\mathbf{u}_2$ and $\mathbf{u}_1$ as follows:

$$\mathbf{u}_2(x) = -a\Psi_{E_{\kappa_e}} \mathbf{j}_0(x) - b\Psi_{M_{\kappa_e}} C_0^* \mathbf{j}_0(x) \text{ for } x \in \Omega^c$$
$$\mathbf{u}_1(x) = -\frac{1}{\rho}\Psi_{E_{\kappa_i}}(N_e \mathbf{j}_0)(x) - \Psi_{M_{\kappa_i}}(L_e \mathbf{j}_0)(x) \text{ for } x \in \Omega$$

By Theorem 4.1, $\mathbf{u}_1$ and $\mathbf{u}_2$ together solve the transmission problem with $\mathbf{E}^{inc} = 0$.
Since this problem admits at most one solution, we have $\mathbf{u}_2 \equiv 0$ in Ω^c and $\mathbf{u}_1 \equiv 0$ in Ω.
Now we set $\mathbf{u}(x) = -a\Psi_{E_{\kappa_e}} \mathbf{j}_0(x) - b\Psi_{M_{\kappa_e}} C_0^* \mathbf{j}_0(x)$ for $x \in \Omega$.
We have on Γ :

$$\gamma_D^c \mathbf{u}_2 - \gamma_D \mathbf{u} = b\, C_0^* \mathbf{j}_0, \tag{4.18}$$
$$\gamma_{N_{\kappa_e}}^c \mathbf{u}_2 - \gamma_{N_{\kappa_e}} \mathbf{u} = a\, \mathbf{j}_0\,. \tag{4.19}$$

Since $\gamma_D^c \mathbf{u}_2 = \gamma_{N_e}^c \mathbf{u}_2 = 0$ on Γ, we find

$$a\gamma_D \mathbf{u} - bC_0^* \gamma_{N_{\kappa_e}} \mathbf{u} = 0 \text{ on } \Gamma.$$

Thus $\mathbf{u}$ is an eigenfunction associated with the eigenvalue κ_e^2 of the interior problem or $\mathbf{u} \equiv 0$. But this latter possibility can be eliminated since it implies that $\gamma_D \mathbf{u} = \gamma_{N_{\kappa_e}} \mathbf{u} = 0$, whence $\mathbf{j}_0 = 0$ by (4.18) and (4.19), which is contrary to the assumption.
Conversely, assume that κ_e^2 is an eigenvalue of the associated interior problem. Let $v_0 \not\equiv 0$ be a corresponding eigenfunction. The Calderón relations in Ω imply that :

$$\begin{aligned} -C_{\kappa_e} \gamma_{N_{\kappa_e}} v_0 + \left(\frac{1}{2} \mathrm{I} - M_{\kappa_e} \right) \gamma_D v_0 = 0, \\ \left(\frac{1}{2} \mathrm{I} - M_{\kappa_e} \right) \gamma_{N_{\kappa_e}} v_0 - C_{\kappa_e} \gamma_D v_0 = 0. \end{aligned}$$

Using the equality $a\gamma_D v_0 - b\, C_0^* \gamma_{N_{\kappa_e}} v_0 = 0$, we obtain

$$L_e (C_0^*)^{-1} \gamma_D v_0 = 0, \quad N_e (C_0^*)^{-1} \gamma_D v_0 = 0, \quad L_e \gamma_{N_{\kappa_e}} v_0 = 0, \quad N_e \gamma_{N_{\kappa_e}} v_0 = 0.$$

If $b \neq 0$ then $\gamma_D v_0 \neq 0$, and $\mathbf{j}_0 = (C_0^*)^{-1} \gamma_D v_0$ is a nontrivial solution of (4.15) and (4.16).
If $b = 0$ then $\gamma_{N_{\kappa_e}} v_0 \neq 0$, and $\mathbf{j}_0 = \gamma_{N_{\kappa_e}} v_0$ is a nontrivial solution of (4.15) and (4.16). ■

Theorem 4.5 *Assume that the constants a, b, μ_e, μ_i, κ_e and κ_i satisfy:*

$$(b\kappa_e + 2a) \neq 0, \ \left(1 + \frac{\mu_e}{\mu_i} \right) \neq 0, \ (b - 2a\kappa_e) \neq 0 \text{ and } \left(1 + \frac{\mu_e \kappa_i^2}{\mu_i \kappa_e^2} \right) \neq 0.$$

Then $\mathbf{S}$ is a Fredholm operator of index zero on $\mathbf{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$.

Proof. We can rewrite $\mathbf{S}$ as follows:

$$\begin{aligned} \mathbf{S} = \ & \frac{1}{4} b\rho C_0^* - \frac{1}{2} b\rho (M_{\kappa_i} + M_{\kappa_e}) C_0^* + b\rho M_{\kappa_i} M_{\kappa_e} C_0^* - \frac{1}{2} a\rho (C_{\kappa_e} - C_{\kappa_e,0}) \\ & - \frac{1}{2} a (C_{\kappa_i} - C_{\kappa_i,0}) - \frac{1}{2} a (\rho C_{\kappa_e,0} + C_{\kappa_i,0}) + a\rho M_{\kappa_i} C_{\kappa_e} + a C_{\kappa_i} M_{\kappa_e} \\ & + b(C_{\kappa_i} - C_{\kappa_i,0}) C_{\kappa_e} C_0^* + b C_{\kappa_i,0} (C_{\kappa_e} - C_{\kappa_e,0}) C_0^* + b C_{\kappa_i,0} C_{\kappa_e,0} C_0^*. \end{aligned}$$

Thus $\mathbf{S}$ is a compact perturbation of the operator

$$\mathbf{S}_1 = b \left(\frac{1}{4} \rho \mathrm{I} + C_{\kappa_i,0} C_{\kappa_e,0} \right) C_0^* - \frac{1}{2} a \left(\rho C_{\kappa_e,0} + C_{\kappa_i,0} \right).$$

We have to show that the operator $\mathbf{S}_1$ is Fredholm of index zero. For this we use the *Helmholtz decomposition* of $\mathbf{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$:

$$\mathbf{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) = \nabla_\Gamma H^{\frac{3}{2}}(\Gamma) \oplus \mathbf{curl}_\Gamma H^{\frac{1}{2}}(\Gamma). \tag{4.20}$$

For a detailed proof of (4.20) see [16]. Note that we are assuming that Ω has trivial topology.

The terms in the decomposition $\mathbf{j} = \nabla_\Gamma p + \mathbf{curl}_\Gamma\, q$ for $\mathbf{j} \in \mathbf{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ are obtained by solving the Laplace-Beltrami equation (see Lemma 3.4):

$$p = \Delta_\Gamma^{-1}\,\mathrm{div}_\Gamma\,\mathbf{j}\,, \qquad q = -\Delta_\Gamma^{-1}\,\mathrm{curl}_\Gamma\,\mathbf{j}\,.$$

The mapping

$$\begin{array}{rcl} \mathbf{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) & \to & H^{\frac{3}{2}}(\Gamma)/\mathbb{R} \times H^{\frac{1}{2}}(\Gamma)/\mathbb{R} \\ \mathbf{j} = \nabla_\Gamma p + \mathbf{curl}_\Gamma\, q & \mapsto & \begin{pmatrix} p \\ q \end{pmatrix} \end{array} \tag{4.21}$$

is an isomorphism. Using this isomorphism, we can rewrite the operator $\mathbf{S}_1$ as an operator $\mathcal{S}_1$ defined from $H^{\frac{3}{2}}(\Gamma)/\mathbb{R} \times H^{\frac{1}{2}}(\Gamma)/\mathbb{R}$ into itself. Then to show that $\mathbf{S}_1$ it is Fredholm of index zero it suffices to show that $\mathcal{S}_1$ has this property. Let us begin by rewriting C_0^* and $C_{\kappa,0}$ as operators $\mathcal{C}_0^*$ and $\mathcal{C}_{\kappa,0}$ defined on $H^{\frac{3}{2}}(\Gamma)/\mathbb{R} \times H^{\frac{1}{2}}(\Gamma)/\mathbb{R}$. We have to determine $P_0 \in H^{\frac{3}{2}}(\Gamma)/\mathbb{R}$ and $Q_0 \in H^{\frac{1}{2}}(\Gamma)/\mathbb{R}$ such that $C_0^*(\nabla_\Gamma p + \mathbf{curl}_\Gamma\, q) = \nabla_\Gamma P_0 + \mathbf{curl}_\Gamma\, Q_0$, and this defines $\mathcal{C}_0^*$ by:

$$\mathcal{C}_0^* \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} P_0 \\ Q_0 \end{pmatrix}.$$

We have

$$P_0 = \Delta_\Gamma^{-1}\,\mathrm{div}_\Gamma\, C_0^*(\nabla_\Gamma p + \mathbf{curl}_\Gamma\, q)$$

and

$$Q_0 = -\Delta_\Gamma^{-1}\,\mathrm{curl}_\Gamma\, C_0^*(\nabla_\Gamma p + \mathbf{curl}_\Gamma\, q).$$

Using the integral representation of C_0^* and the equalities (3.3) and (3.4) we obtain:

$$\mathcal{C}_0^* = \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} \\ \mathcal{C}_{21,1} + \mathcal{C}_{21,2} & \mathcal{C}_{22} \end{pmatrix},$$

where

$$\begin{array}{lcllcl} C_{11} & = & -\Delta_\Gamma^{-1}\,\mathrm{curl}_\Gamma\, V_0 \nabla_\Gamma, & C_{12} & = & -\Delta_\Gamma^{-1}\,\mathrm{curl}_\Gamma\, V_0\, \mathbf{curl}_\Gamma, \\ C_{21,1} & = & -\Delta_\Gamma^{-1}\,\mathrm{div}_\Gamma\, V_0 \nabla_\Gamma, & C_{22} & = & -\Delta_\Gamma^{-1}\,\mathrm{div}_\Gamma\, V_0\, \mathbf{curl}_\Gamma, \\ C_{21,2} & = & V_0 \Delta_\Gamma. & & & \end{array}$$

Some of these operators are of lower order than what a simple counting of orders (with -1 for the order of V_0) would give:

Lemma 4.6 *The operators* $\mathrm{curl}_\Gamma\, V_0 \nabla_\Gamma$ *and* $\mathrm{div}_\Gamma\, V_0\, \mathbf{curl}_\Gamma$ *are linear and continuous from* $H^t(\Gamma)$ *into itself.*

Proof. These results are due to the equalites (3.3). One can write (see [22, page 240]):

$$\begin{array}{rcl} \mathrm{curl}_\Gamma\, V_0 \nabla_\Gamma u(x) & = & \displaystyle\int_\Gamma \mathbf{n}(x) \cdot \mathbf{curl}^x \left\{ G(0, |x-y|) \nabla_\Gamma u(y) \right\} d\sigma(y) \\ & = & \displaystyle\int_\Gamma \left\{ (\mathbf{n}(x) - \mathbf{n}(y)) \times \nabla^x G(0, |x-y|) \right\} \cdot \nabla_\Gamma u(y) d\sigma(y) \\ & & -V_0\,\mathrm{curl}_\Gamma\, \nabla_\Gamma u. \end{array}$$

The second term on the right hand side vanishes, and the kernel

$$(\mathbf{n}(x) - \mathbf{n}(y)) \times \nabla^x G(0, |x-y|)$$

has the same weak singularity as the fundamental solution $G(0, |x-y|)$. We deduce the lemma using similar arguments for the other operator. ■

As a consequence, the operators $\mathcal{C}_{11}$ and $\mathcal{C}_{22}$ are of order -2, the operators $\mathcal{C}_{12}$ and $\mathcal{C}_{21,1}$ are of order -1 and the operator $C_{21,2}$ is of order 1. Therefore, $\mathcal{C}_0^*$ is a compact perturbation of

$$\begin{pmatrix} 0 & \mathcal{C}_{12} \\ \mathcal{C}_{21,2} & 0 \end{pmatrix}$$

By definition of $C_{\kappa,0}$, the operator $\mathcal{C}_{\kappa,0}$ can be written as:

$$\begin{aligned} \mathcal{C}_{\kappa,0} &= \begin{pmatrix} -\kappa\mathcal{C}_{11} & -\kappa\mathcal{C}_{12} \\ -\kappa\mathcal{C}_{21,1} + \kappa^{-1}\mathcal{C}_{21,2} & -\kappa\mathcal{C}_{22} \end{pmatrix} \\ &= \begin{pmatrix} -\kappa & 0 \\ 0 & \kappa^{-1} \end{pmatrix} \mathcal{C}_0^* - (\kappa + \kappa^{-1}) \begin{pmatrix} 0 & 0 \\ \mathcal{C}_{21,1} & \mathcal{C}_{22} \end{pmatrix}. \end{aligned}$$

The second term on the right hand side is compact on $H^{\frac{3}{2}}(\Gamma)/\mathbb{R} \times H^{\frac{1}{2}}(\Gamma)/\mathbb{R}$.

Since $\mathcal{C}_{\kappa,0}$ is a compact perturbation of

$$\begin{pmatrix} -\kappa & 0 \\ 0 & \kappa^{-1} \end{pmatrix} \mathcal{C}_0^*,$$

the sum $\mathcal{C}_{\kappa_i,0} + \rho\mathcal{C}_{\kappa_e,0}$ is a compact perturbation of

$$\begin{pmatrix} -(\kappa_i + \rho\kappa_e) & 0 \\ 0 & (\kappa_i^{-1} + \rho\kappa_e^{-1}) \end{pmatrix} \mathcal{C}_0^*.$$

Lemma 4.7 *The operator C_0^{*2} is a compact perturbation of $-\frac{1}{4}\mathrm{I}$.*

Proof. It suffices to consider the principal part of (3.10). ■

Thus the operator $\mathcal{C}_{\kappa_i,0}\mathcal{C}_{\kappa_e,0}$ is a compact perturbation of

$$\frac{1}{4}\begin{pmatrix} \kappa_i\kappa_e^{-1} & 0 \\ 0 & \kappa_i^{-1}\kappa_e \end{pmatrix}.$$

Collecting all the results, we found that $\mathcal{S}_1$ is a compact perturbation of

$$\begin{pmatrix} \frac{1}{4}b(\rho + \kappa_i\kappa_e^{-1}) - \frac{1}{2}a(\kappa_i + \rho\kappa_e) & 0 \\ 0 & \frac{1}{4}b(\rho + \kappa_i^{-1}\kappa_e) + \frac{1}{2}a(\kappa_i^{-1} + \rho\kappa_e^{-1}) \end{pmatrix} \mathcal{C}_0^*.$$

We recall that $\rho = \dfrac{\mu_e \kappa_i}{\mu_i \kappa_e}$. The matrix written above is invertible if:

$$\frac{1}{4} b(\rho + \kappa_i \kappa_e^{-1}) - \frac{1}{2} a(\kappa_i + \rho \kappa_e) \neq 0 \quad \Leftrightarrow \quad \frac{1}{4}(b - 2a\kappa_e)\left(1 + \frac{\mu_i}{\mu_e}\right) \neq 0$$

and

$$\frac{1}{4} b(\rho + \kappa_i^{-1} \kappa_e) + \frac{1}{2} a(\kappa_i^{-1} + \rho \kappa_e^{-1}) \neq 0 \quad \Leftrightarrow \quad \frac{1}{4}(b\kappa_e + 2a)\left(1 + \frac{\mu_i \kappa_e^2}{\mu_e \kappa_i^2}\right) \neq 0.$$

Since the operator C_0^* is invertible, we conclude that under the conditions of the theorem the operator $\mathbf{S}_1$ is Fredholm of index zero and therefore $\mathbf{S}$ too. ■

Using similar arguments we obtain the following theorem.

Theorem 4.8 *Assume that the constants a, b, μ_e, μ_i, κ_e et κ_i satisfy:*

$$\left(a\big(1 + \frac{\mu_e \kappa_i^2}{\mu_i \kappa_e^2}\big) + \frac{b}{2\kappa_e}\big(1 + \frac{\mu_e}{\mu_i}\big)\right) \cdot \left(a\big(1 + \frac{\mu_e}{\mu_i}\big) - \frac{b\kappa_e}{2}\big(1 + \frac{\mu_e \kappa_i^2}{\mu_i \kappa_e^2}\big)\right) \neq 0$$

Then $\mathbf{T}$ is a Fredholm operator of index zero on $\mathbf{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$.

Note that under standard hypotheses on the materials and for real frequencies, the material factors such as $\big(1 + \dfrac{\mu_e}{\mu_i}\big)$ and $\big(1 + \dfrac{\mu_e \kappa_i^2}{\mu_i \kappa_e^2}\big)$ are always non-zero.

5 Integral equations 2

The second method is based on a layer ansatz for the interior field: We assume that the interior electric field $\mathbf{E}^i$ can be represented either by $\Psi_{E_{\kappa_i}} \mathbf{j}$ or by $\Psi_{M_{\kappa_i}} \mathbf{j}$ where the density $\mathbf{j} \in \mathbf{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ is the unknown function we have to determine. We begin with the Stratton-Chu representation formula (3.8) in Ω^c:

$$\mathbf{E}^s(x) = \Psi_{E^s} \gamma_{N_{\kappa_e}}^c \left(\mathbf{E}^s + \mathbf{E}^{inc}\right)(x) + \Psi_{M_{\kappa_e}} \gamma_D^c \left(\mathbf{E}^s + \mathbf{E}^{inc}\right)(x) \qquad x \in \Omega^c \tag{5.1}$$

We then apply the exterior traces γ_D^c and $\gamma_{N_{\kappa_e}}^c$ and use on both sides of (5.1) the transmission conditions. The result is a relation between the traces of $\mathbf{E}^i$ on Γ:

$$\gamma_D \mathbf{E}^i - \gamma_D^c \mathbf{E}^{inc} = -\rho C_{\kappa_e} \gamma_{N_{\kappa_i}} \mathbf{E}^i + \left(-\frac{1}{2}\mathrm{I} + M_{\kappa_e}\right) \gamma_D \mathbf{E}^i, \tag{5.2}$$

$$\rho \gamma_{N_{\kappa_i}} \mathbf{E}^i - \gamma_{N_{\kappa_e}}^c \mathbf{E}^{inc} = -C_{\kappa_e} \gamma_D \mathbf{E}^i + \rho \left(-\frac{1}{2}\mathrm{I} + M_{\kappa_e}\right) \gamma_{N_{\kappa_i}} \mathbf{E}^i. \tag{5.3}$$

In the scalar case, to construct the integral equations one would simply take a linear combination of (5.2) and (5.3). Here we multiply (5.2) by a and (5.3) by bC_0^* and subtract to obtain:

$$\rho L_e' \gamma_{N_{\kappa_i}} \mathbf{E}^i - N_e' \gamma_D \mathbf{E}^i = h \qquad \text{sur } \Gamma \tag{5.4}$$

where the operators L_e' and N_e' are defined for all $\mathbf{j} \in \mathbf{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ by :

$$L'_e \mathbf{j} = \left\{ aC_{\kappa_e} - bC_0^* \left(\frac{1}{2}\mathrm{I} + M_{\kappa_e} \right) \right\} \mathbf{j},$$

$$N'_e \mathbf{j} = \left\{ -a \left(\frac{1}{2}\mathrm{I} + M_{\kappa_e} \right) + bC_0^* C_{\kappa_e} \right\} \mathbf{j},$$

and

$$h = a\gamma_D \mathbf{E}^{inc} - bC_0^* \gamma_{N_{\kappa_e}} \mathbf{E}^{inc} \in \mathbf{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma). \tag{5.5}$$

If $\mathbf{E}^i$ is represented by the potential $\Psi_{E_{\kappa_i}}$ applied to a density $\mathbf{j} \in \mathbf{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$:

$$\mathbf{E}^i(x) = -(\Psi_{E_{\kappa_i}} \mathbf{j})(x), \qquad x \in \Omega, \tag{5.6}$$

we obtain :

$$\gamma_D \mathbf{E}^i = C_{\kappa_i} \mathbf{j} \qquad \text{and} \qquad \gamma_{N_{\kappa_i}} \mathbf{E}^i = \left(\frac{1}{2}\mathrm{I} + M_{\kappa_i} \right) \mathbf{j} \qquad \text{on } \Gamma \tag{5.7}$$

Substituting (5.7) in (5.4), we obtain a first integral equation:

$$\mathbf{S}' \mathbf{j} = \left\{ \rho L'_e \left(\frac{1}{2}\mathrm{I} + M_{\kappa_i} \right) - N'_e C_{\kappa_i} \right\} \mathbf{j} = h \qquad \text{on } \Gamma \tag{5.8}$$

This is an integral equation for the unknown $\mathbf{j} \in \mathbf{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$. Having solved this equation, we construct $\mathbf{E}^i$ and $\mathbf{E}^s$ by the representations (5.6) in Ω and :

$$\mathbf{E}^s = \rho \left(\Psi_{E_{\kappa_e}} \left(\frac{1}{2}\mathrm{I} + M_{\kappa_i} \right) \mathbf{j} \right)(x) + \left(\Psi_{M_{\kappa_e}} C_{\kappa_i} \mathbf{j} \right)(x) \qquad x \in \Omega^c. \tag{5.9}$$

If $\mathbf{E}^i$ is represented by the potential $\Psi_{M_{\kappa_i}}$ applied to a density $\mathbf{m} \in \mathbf{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$:

$$\mathbf{E}^i(x) = -(\Psi_{M_{\kappa_i}} \mathbf{m})(x), \qquad x \in \Omega, \tag{5.10}$$

we obtain:

$$\gamma_D \mathbf{E}^i = \left(\frac{1}{2} I + M_{\kappa_i} \right) \mathbf{m} \qquad \textit{and} \qquad \gamma_{N_{\kappa_i}} \mathbf{E}^i = C_{\kappa_i} \mathbf{m} \qquad \text{on } \Gamma. \tag{5.11}$$

Substituting (5.11) in (5.4), we obtain a second integral equation:

$$\mathbf{T}' \mathbf{m} = \left\{ \rho L'_e C_{\kappa_i} - N'_e \left(\frac{1}{2}\mathrm{I} + M_{\kappa_i} \right) \right\} \mathbf{m} = h \qquad \text{on } \Gamma. \tag{5.12}$$

This is an integral equation for the unknown $\mathbf{m} \in \mathbf{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$. Having solved this equation, we construct $\mathbf{E}^i$ and $\mathbf{E}^s$ by the representations (5.10) in Ω and:

$$\mathbf{E}^s(x) = \rho \left(\Psi_{E_{\kappa_e}} C_{\kappa_i} \mathbf{m} \right)(x) + \left(\Psi_{M_{\kappa_e}} \left(\frac{1}{2}\mathrm{I} + M_{\kappa_i} \right) \mathbf{m} \right)(x), \qquad x \in \Omega^c. \tag{5.13}$$

Contrary to the preceding method from the previous section, the two integral equations are not equivalent, in general. The following theorem corresponds to Theorem 4.1. The proof is similar to the scalar case.

Theorem 5.1 *We assume that κ_e^2 is not an eigenvalue of the associated interior problem* (4.17).
If $\mathbf{j} \in \mathbf{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ *solves* (5.8), $\mathbf{E}^i$ *and* $\mathbf{E}^s$, *given by* (5.6) *et* (5.9) *respectively, solve the dielectric scattering problem.*
If $\mathbf{m} \in \mathbf{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ *solves* (5.12), $\mathbf{E}^i$ *and* $\mathbf{E}^s$, *given by* (5.10) *and* (5.13) *respectively, solve the dielectric scattering problem.*

PROOF. In each case the integral representations of $\mathbf{E}^i$ and $\mathbf{E}^s$ satisfy the Maxwell equations and the Silver-Müller condition. It remains to prove that the tranmission conditions are satisfied. We prove it for the equation (5.12), the arguments being similar for (5.8).
Assume that $\mathbf{m}$ solves (5.12) which we rewrite as :

$$\begin{aligned} a\left\{\rho C_{\kappa_e} C_{\kappa_i}\mathbf{m} + \left(\frac{1}{2}\mathrm{I} + M_{\kappa_e}\right)\left(\frac{1}{2}\mathrm{I} + M_{\kappa_i}\right)\mathbf{m} - \gamma_D \mathbf{E}^{inc}\right\} \\ -bC_0^*\left\{\rho\left(\frac{1}{2}\mathrm{I} + M_{\kappa_e}\right) C_{\kappa_i}\mathbf{m} + C_{\kappa_e}\left(\frac{1}{2}\mathrm{I} + M_{\kappa_i}\right)\mathbf{m} - \gamma_{N_{\kappa_e}}\mathbf{E}^{inc}\right\} \quad = 0. \end{aligned} \tag{5.14}$$

Then, using the integral representation (5.13) of $\mathbf{E}^s$, we obtain :

$$(\gamma_D^c \mathbf{E}^s + \gamma_D^c \mathbf{E}^{inc} - \gamma_D \mathbf{E}^i) = -\rho C_{\kappa_e} C_{\kappa_i}\mathbf{m} - \left(\frac{1}{2}\mathrm{I} + M_{\kappa_e}\right)\left(\frac{1}{2}\mathrm{I} + M_{\kappa_i}\right)\mathbf{m} + \gamma_D \mathbf{E}^{inc},$$

$$(\gamma_{N_{\kappa_e}}^c \mathbf{E}^s + \gamma_{N_{\kappa_e}}^c - \rho\gamma_{N_{\kappa_i}}\mathbf{E}^i) = -\rho\left(\frac{1}{2}\mathrm{I} + M_{\kappa_e}\right) C_{\kappa_i}\mathbf{m} - C_{\kappa_e}\left(\frac{1}{2}\mathrm{I} + M_{\kappa_i}\right)\mathbf{m} + \gamma_{N_{\kappa_e}}^c \mathbf{E}^{inc}.$$

We have to show that the right hand sides of these equalities vanish.
We introduce the function $\mathbf{v}$ defined on Ω by:

$$\mathbf{v}(x) = -\rho\Psi_{E_{\kappa_e}} C_{\kappa_i}\mathbf{m} - \Psi_{M_{\kappa_e}}\left(\frac{1}{2}\mathrm{I} + M_{\kappa_i}\right)\mathbf{m} - \mathbf{E}^{inc}.$$

By equation (5.14) we have $a\gamma_D\mathbf{v} - bC_0^*\gamma_{N_{\kappa_e}}\mathbf{v} = 0$. Since $\mathbf{E}^{inc}$ satisfies the Maxwell system $\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} - \kappa_e^2\mathbf{v} = 0$ in Ω, $\mathbf{v}$ satisfies it, too. By hypothesis, κ_e^2 is not an eigenvalue of the associated interior problem, which implies $\mathbf{v} \equiv 0$ in Ω. In particular, $\gamma_D\mathbf{v}$ et $\gamma_{N_{\kappa_e}}\mathbf{v}$ vanish, which shows that the above right hand sides are indeed zero and that the transmission conditions are satisfied. ■

Theorem 5.2 *Assume that the hypotheses of Theorem 2.3 are satisfied and that κ_e^2 is not an eigenvalue of the associated interior problem* (4.17). *Then the operators $\mathbf{S}'$ and $\mathbf{T}'$ are injective.*

PROOF. We prove the result for the operator $\mathbf{T}'$, similar arguments being valid for $\mathbf{S}'$.
Assume that $\mathbf{m}_0 \in \mathbf{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ solves the homogeeous equation :

$$\mathbf{T}'\mathbf{m}_0 = \rho L_e' C_{\kappa_i}\mathbf{m}_0 - N_e'\left(\frac{1}{2}\mathrm{I} + M_{\kappa_i}\right)\mathbf{m}_0 = 0. \tag{5.15}$$

We want to show that $\mathbf{m}_0 = 0$.
We construct $\mathbf{v}_1$ and $\mathbf{v}_2$ as follows:

$$\mathbf{v}_2(x) = \rho(\Psi_{E_{\kappa_e}} C_{\kappa_i} \mathbf{m}_0)(x) + \left(\Psi_{M_{\kappa_e}} \left\{ \frac{1}{2} \mathrm{I} + M_{\kappa_i} \right\} \mathbf{m}_0 \right)(x), \qquad x \in \Omega^c,$$

and

$$\mathbf{v}_1(x) = -(\Psi_{M_{\kappa_i}} \mathbf{m}_0)(x), \qquad x \in \Omega.$$

By Theorem 5.1, these functions solve the homogeneous scattering problem (i.e. when $\mathbf{E}^{inc} \equiv 0$), and therefore $\mathbf{v}_1 \equiv 0$ in Ω and $\mathbf{v}_2 \equiv 0$ in Ω^c. Now we define

$$\mathbf{v}(x) = -(\Psi_{M_{\kappa_i}} \mathbf{m}_0)(x) \qquad x \in \Omega^c$$

We have $\gamma^c_{N_{\kappa_i}} \mathbf{v} = C_{\kappa_i} \mathbf{m}_0 = \gamma_{N_{\kappa_i}} \mathbf{v}_1 = 0$. Since $\mathbf{v}$ satisfies the Silver-Müller condition, we have $\mathbf{v} \equiv 0$ in Ω^c. Thus $\mathbf{v} \equiv 0$ is $\mathbb{R}^3$ and $[\gamma_D]\mathbf{v} = \mathbf{m}_0 = 0$. ■

Remark 5.3 *The operators $\mathbf{S}'$ and $\mathbf{T}'$ are the dual operators of $\mathbf{S}$ and $\mathbf{T}$, respectively, for the bilinear form $\mathcal{B}$. Therefore they are Fredholm of index zero under the same hypotheses as those given in Theorems 4.5 and 4.8.*

In order that each of the four integral equations admit a unique solution for all positive real values of κ_e, we will now give an example of how to choose the constants a and b such that the associated interior problem does not admit any real eigenvalue.

We sumarize all the previous results by the final theorem.

Theorem 5.4 *Assume that:*

(i) κ_e *is a positive real number,*

(ii) $a = 1$ *and* $b = i\eta$ *with* $\eta \in \mathbb{R}\backslash\{0\}$,

(iii) $\dfrac{\mu_i}{\mu_e} \neq -1$, $\dfrac{\mu_e \kappa_i^2}{\mu_i \kappa_e^2} \neq -1$.

Then the operators $\mathbf{S}$, $\mathbf{T}$, $\mathbf{S}'$ and $\mathbf{T}'$ are invertible. Moreover, given $\mathbf{E}^{inc} \in \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}, \mathbb{R}^3)$, the integral representations {(4.1), (4.11)}, {(5.6), (5.9)} and {(5.10), (5.13)} of $\mathbf{E}^i$ et $\mathbf{E}^s$ give the solution of the dielectric scattering problem for all positive real values of κ_e.

6 Conclusion

In this paper we have described and analyzed modified boundary integral equations to solve a radiation problem for the Maxwell system that are stable for all wave numbers. In Section 4 we have derived two boundary integral equations using an ansatz for the exterior field and in Section 5 we have derived two integral equations using an ansatz for the interior field. Note that if it is only the exterior field that is of interest, one can choose an integral equation which gives a simple representation for $\mathbf{E}^s$, e.g. (4.7) or (4.9). This choice was used in the PhD thesis [20] for an application in an optimization problem concerning the far field pattern. For numerical results using this method, we refer to [20].

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