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Test on the components of mixture densities

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Abstract

This paper deals with statistical tests on the components of mixture densities. We propose to test whether the densities of two independent samples of independent random variables Y_1, \dots, Y_n and Z_1, \dots, Z_n result from the same mixture of M components or not. We provide a test procedure which is proved to be asymptotically optimal according to the minimax setting. We extensively discuss the connection between the mixing weights and the performance of the testing procedure and illustrate it with numerical examples.

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1 Introduction

Since more than 20 years, the mixture model has gained a lot of attention. This is due to its ease of interpretation by viewing each component as a distinct group in the data. This model has been widely applied in several areas such as finance, economy, biology, astronomy, survey methods,...

Most of the theoretical results in the literature deal with the estimation of the components or of the mixing weights. There are two types of mixture models : the most popular one has fixed mixing weights and the other one has varying mixing weights.

On the one hand, many statisticians have been interested in estimating the mixing weights. For example, Hall [12], Titterington [24] and Hall and Titterington [13] have considered nonparametric estimation of the mixing weights. Two other examples about the mixing weights are the estimation of a functional of the weights by van de Geer [25] and the computation of confidence intervals by Qin [22]. On the other hand, one can be interested in estimating the components of the mixture. This can be easily done with varying mixture weights by applying several well-known methods such as histograms in Lodakto and Maiboroda [16], empirical distribution in Maiboroda [19] or wavelet thresholding methods in Pokhyl'ko [20]. Finally, the mixing weights and the mixture components can also be estimated both and at the same time, this result holds in a particular setting for k-variate data introduced by Hall and Zhou [14].

More recently, the mixture model has also been studied in the testing problem framework. The usual addressed question is whether the observations come from a non-trivial mixture model or from a trivial one (i.e. with only one component). This has been done for example by Garel [10] and [11] and Delmas [8] in the case of fixed mixing weights and by Maiboroda [18] in the case of varying mixing weights. Their homogeneity tests which rely respectively on the likelihood ratio test and on a Kolmogorov-Smirnov type test are proved to be consistent. Here we propose to study a testing problem with two samples in a mixture model with varying mixing weights.

Although the varying mixing weights model does not seem natural at first sight, one can think of several situations where it can be useful. Let us give three examples that will help the reader to recognize its usefulness.

Social science

This first example is the closest to the varying mixing weights model that is studied here. Let us consider an organization divided into several departments such as an enterprise. Aggregated informations are only known at the department level, e.g. proportion of men and women, proportion of graduates and undergraduates, proportion of married and unmarried people, etc... The researcher is interested in a variable for these subgroups such as salary. For each person, the researcher has only recorded salary and department. The information of interest which allows to divide the sample into subgroups is unavailable at the individual level. This can happen if the researcher has forgotten to record this

information when collecting the data; this frequently happens when a new question arises during the study of the data. Another reason can be that the law forbids to record such information at the individual level; for example this is the case of origins or races in many countries. There is a wealth of works on partially missing data (see McKnight et al. [17] for example) but the case of entirely missing data has never been really considered. From our point of view, a varying mixing weights model is a way to cope with this lack of information at the individual level and to allow the researcher to reconstruct information for each subgroup. Although we are aware of methodological problems, we want to emphasize that in this case the varying mixing weights are exactly known to the researcher; indeed, aggregated information often exists and is much easier to collect than individual information.

Image analysis

Let us assume a simple picture taken at a party and consisting of people and background. There is usually no way to distinguish at the pixel level whether it comes from people or background. Nevertheless one can think of some kind of aggregated information to roughly divide the image into several areas. In the center of the image, there are usually mainly people and only little background. In the area surrounding the center, there is mainly background although few people can be scattered here and there. Therefore the image can be divided into two areas. This written description of the image can be translated into a mathematical description namely the varying mixing weights model. In this model, the statistician will be able to extract distinctive features of the picture concerning people or background. We are aware that methodological problems can appear in this setup. For example, spatial structure are not taken into account. One can consider that the weights are only roughly known which can be a problem. Nevertheless for some types of images, such as satellite images, one can assume that the weights are accurately known. Indeed, as the area under scrutiny is exactly known from a geographical point of view, one can use aggregated information about surfaces such as proportion of forest, land, city, water, etc...

Finance

Mixture densities have been proved to be useful in volatility modeling (see Bernhard and Leblang [3], Avellaneda [2] for example). If one consider the volatility clustering effect (see Cont [6] for example), one can roughly divide time into periods where the proportions of high and low volatility are estimated. Indeed during each period it might be hard to exactly label observations corresponding to low or high volatility. Therefore the varying mixing weights model can be considered and help to extract useful features of the mixture components. This case with estimated proportions is not solved here. Although it is beyond the scope of this paper, we briefly discuss it in Section 4.

Let us now come back to our testing problem with two samples in a mixture model with varying mixing weights: let Y_1, \dots, Y_n and Z_1, \dots, Z_n be two inde-

pendent n -samples of independent random variables. We propose to study in this paper whether these two samples of random variables come from the same mixture of M unknown densities p_u ($1 \leq u \leq M$) or not. We assume that the mixing weights associated with each observation are available to the statistician. In Butucea and Tribouley [4] some procedures are proposed to test if two n -samples of i.i.d. variables have common probability density. Their setting is equivalent to the case $M = 1$ in our mixture problem. Here the problem appears more complex since the two samples are not based on random variables with the same marginal densities. Our results show that there is no loss in the minimax rate compared to the simpler case studied by Butucea and Tribouley [4]. In Section 2 we provide an asymptotically minimax test which is based on wavelet methods and we prove the dependence between the mixing weights and the constants appearing in the definition of the minimax rate of testing. Until now this phenomenon has never been studied and is extensively discussed in this paper. In addition to our theoretical result some numerical experiments are given in Section 3 in order to illustrate the strong connection between the mixing weights and the performance of the test. As expected, our test performs very well for various mixture models. Sections 4 and 5 are respectively devoted to possible extensions of work and to proofs of main results.

Before describing our testing problem we introduce the wavelet framework that will be used.

Wavelet framework

We first recall that wavelets have been often applied in different mathematical fields such as in approximation theory, in signal analysis and in statistics for instance. In particular, many recent statistical works on estimation (see among others Autin [1], Donoho et al [9], Cohen et al [5]) and on hypothesis testing (see Spokoiny [23]) use the wavelet setting to provide efficient estimators and tests. There are many explanations for the huge interest of the wavelet setting. One of them is that wavelets bases are localized both in frequency and in time, contrary to the classical Fourier basis which is only localized in frequency. As a consequence, the wavelet setting appears to be well adapted to describe local characteristics of a signal to be reconstructed.

Let ϕ and ψ be two compactly supported functions of $L_2(\mathbb{R})$ and denote for all j in \mathbb{N} and all k in \mathbb{Z} and all x in \mathbb{R} , $\phi_{jk}(x) = 2^{j/2} \phi(2^j x - k)$ and $\psi_{jk}(x) = 2^{j/2} \psi(2^j x - k)$.

Suppose that for any j in \mathbb{N} :

- $\{\phi_{jk}, \psi_{j'k}; j' \geq j; k \in \mathbb{Z}\}$ constitutes an orthonormal basis of $L_2(\mathbb{R})$,
- $support(\phi) \cup support(\psi) \subset [-L, L[$ for some $L > 0$.

Some most popular examples of such bases, called compactly supported orthonormal wavelet bases, are given in Daubechies [7]. The function ϕ is called

the scaling function and ψ the associated wavelet.

Any function h in $L_2(\mathbb{R})$ can be represented as:

$$h(t) = \sum_{k \in \mathbb{Z}} \alpha_{jk} \phi_{jk}(t) + \sum_{j' \geq j} \sum_{k \in \mathbb{Z}} \beta_{jk} \psi_{j'k}(t)$$

where $\forall j \in \mathbb{N}, \forall j' \geq j, \forall k \in \mathbb{Z}$:

- $\alpha_{jk} = \int_{I_{jk}} h(t) \phi_{jk}(t) dt$ and $\beta_{j'k} = \int_{I_{j'k}} h(t) \psi_{j'k}(t) dt$,
- $I_{jk} = \{x \in \mathbb{R}; -L \leq 2^j x - k < L\} = [\frac{k-L}{2^j}, \frac{k+L}{2^j}[$.

Let us now describe the testing problem we focus on.

Description of the testing problem

Let Y_1, \dots, Y_n be a sample of independent random variables with unknown marginal densities

$$f_i(\cdot) = \sum_{u=1}^M \omega_u(i) p_u(\cdot), \quad 1 \leq i \leq n.$$

and let Z_1, \dots, Z_n be another sample of independent random variables with unknown marginal densities

$$g_i(\cdot) = \sum_{u=1}^M \sigma_u(i) q_u(\cdot), \quad 1 \leq i \leq n.$$

We also assume that the two samples are independent.

Here and in what follows, we suppose that the mixing weights $(\omega_u(i), 1 \leq u \leq M, 1 \leq i \leq n)$ and $(\sigma_u(i), 1 \leq u \leq M, 1 \leq i \leq n)$ are known to the statistician and satisfy

- $\forall (u, i) \in \{1, \dots, M\} \times \{1, \dots, n\}, \min(\omega_u(i), \sigma_u(i)) \geq 0$,
- $\forall i \in \{1, \dots, n\}, \sum_{u=1}^M \omega_u(i) = \sum_{u=1}^M \sigma_u(i) = 1$,

and are known by the statistician whereas the densities p_u and q_u ($1 \leq u \leq M$) are unknown.

Let us denote $\vec{p} = (p_1, \dots, p_M)$ and $\vec{q} = (q_1, \dots, q_M)$.

We study in this paper a nonparametric procedure to test whether the samples result from the same mixture of densities. Let \mathcal{D} denote the set of all probability

densities with respect to the Lebesgue measure on \mathbb{R} . For any real number $R > 0$, we define

$$\Theta_0(R) = \{(\vec{p}, \vec{q}) : \forall u \in \{1, \dots, M\}, p_u = q_u \in \mathcal{S}(R) = \mathcal{D} \cap \mathbb{L}_\infty(R) \cap \mathbb{L}_2(R)\},$$

and we consider the following null hypothesis

$$\mathcal{H}_0 : (\vec{p}, \vec{q}) \in \Theta_0(R).$$

For a given $C > 0$, we define

$$\Theta_1(R, C, n, s) = \left\{ (\vec{p}, \vec{q}) : \forall u \in \{1, \dots, M\}, p_u - q_u \in \mathcal{B}_{2, \infty}^s(R), \right. \\ \left. \exists u \in \{1, \dots, M\}, (p_u, q_u) \in \Lambda_n(R, C) \right\},$$

where $\Lambda_n(R, C) = \{(p, q) \in (\mathcal{D} \cap \mathbb{L}_\infty(R))^2, \|p - q\|_2 \geq Cr_n\}$, for a sequence r_n tending to 0 when n goes to infinity and $\mathcal{B}_{2, \infty}^s(R)$ is the R -ball of a functional space defined below. We consider the following alternative

$$\mathcal{H}_1 : (\vec{p}, \vec{q}) \in \Theta_1(R, C, n, s).$$

As usual in the nonparametric setting, we focus on a large class of functions having some regularity so as to derive optimal properties. For the chosen wavelet basis, the space $\mathcal{B}_{2, \infty}^s(R)$ represents the R -ball of the so-called Besov body which is composed of all the functions $h \in L_2(\mathbb{R})$ for which the sequence of wavelet coefficients $(\alpha_{jk}, \beta_{j'k}, j \in \mathbb{N}, j' \geq j, k \in \mathbb{Z})$ satisfies:

$$\sup_{j \in \mathbb{N}} 2^{2js} \sum_{j' \geq j} \sum_{k \in \mathbb{Z}} \beta_{j'k}^2 \leq R.$$

The minimax setting

In this paragraph we recall the minimax approach which is often used to evaluate the performances of testing procedures. Given the sum of the probability errors, say $\gamma \in [0, 1]$, we study the optimal separation rate r_n between the null hypothesis and the alternative. This rate r_n is the best possible rate separating at least one of the M couples of density components p_u and q_u . It is usually called *the minimax rate*. Let us recall the classical definition for the separation rate.

Definition 1.1 *Let $0 < \gamma < 1$. We say that r_n is the minimax rate separating \mathcal{H}_0 and \mathcal{H}_1 of our testing problem at level γ if the two following statements are satisfied:*

1. *there exist a sequence of test procedures Δ_n^* and a constant C_γ such that*

$$\lim_{n \rightarrow \infty} \sup \left(\sup_{(\vec{p}, \vec{q}) \in \Theta_0(R)} \mathbb{P}_{\vec{p}, \vec{q}}(\Delta_n^* = 1) + \sup_{(\vec{p}, \vec{q}) \in \Theta_1(R, C, n, s)} \mathbb{P}_{\vec{p}, \vec{q}}(\Delta_n^* = 0) \right) \leq \gamma \quad (1)$$

for all $C > C_\gamma$;

2. there exists a constant c_γ such that

$$\lim_{n \rightarrow \infty} \inf_{\Delta} \inf_{(\vec{p}, \vec{q}) \in \Theta_0(R)} \mathbb{P}_{\vec{p}, \vec{q}}(\Delta = 1) + \sup_{(\vec{p}, \vec{q}) \in \Theta_1(R, C, n, s)} \mathbb{P}_{\vec{p}, \vec{q}}(\Delta = 0) > \gamma \quad (2)$$

for all $C < c_\gamma$, where the infimum is taken over all test procedures Δ .

Hypotheses on the model

In our study we suppose that the mixing weights $(\omega_u(i), 1 \leq u \leq M, 1 \leq i \leq n)$ and $(\sigma_u(i), 1 \leq u \leq M, 1 \leq i \leq n)$ satisfy an added hypothesis. Let us denote by $\Omega = (\Omega)_{u,i}$ the matrix with coefficients $\Omega_{u,i} = \omega_u(i)$ and $\Sigma = (\Sigma)_{u,i}$ the matrix with coefficients $\Sigma_{u,i} = \sigma_u(i)$.

HYP-1 The smallest eigenvalues of the $(M \times M)$ -matrices $\Gamma_n = \Omega\Omega^*$ and $\Gamma'_n = \Sigma\Sigma^*$ are both larger than or equal to Kn , with $0 < K < 1$.

We recall the following proposition due to Maiboroda [19].

Proposition 1.1 *Suppose that the previous conditions are satisfied by the mixing weights $(\omega_u(i), 1 \leq u \leq M, 1 \leq i \leq n)$ and $(\sigma_u(i), 1 \leq u \leq M, 1 \leq i \leq n)$ associated with the model. Then, there exists a solution of the two problems*

$$[\text{find } a_l = \{a_l(i), i = 1, \dots, n\} \text{ such that } \langle \omega_k, a_l \rangle_n := \frac{1}{n} \sum_{i=1}^n \omega_k(i) a_l(i) = \delta_{kl}],$$

$$[\text{find } b_l = \{b_l(i), i = 1, \dots, n\} \text{ such that } \langle \sigma_k, b_l \rangle_n := \frac{1}{n} \sum_{i=1}^n \sigma_k(i) b_l(i) = \delta_{kl}],$$

where δ_{kl} is the Kronecker delta. According to HYP-1 this solution satisfies

$$\sum_{l=1}^M \langle a_l, a_l \rangle_n := \frac{1}{n} \sum_{l=1}^M \sum_{i=1}^n a_l^2(i) \leq \frac{M}{K}, \quad (3)$$

$$\sum_{l=1}^M \langle b_l, b_l \rangle_n := \frac{1}{n} \sum_{l=1}^M \sum_{i=1}^n b_l^2(i) \leq \frac{M}{K}. \quad (4)$$

2 Nonparametric test procedure

This paragraph deals with the case where the regularity s of the Besov body that appears in \mathcal{H}_1 is known. From now on we denote by a_l and b_l the n -vectors which are the solutions of the two optimization problems appearing in Proposition 1.1. Let us describe the asymptotically minimax decision rule.

2.1 Definition of the test procedure

For each level parameter j , we define the test procedure Δ_j comparing the test statistic

$$T_j = \frac{1}{n^2} \sum_{l=1}^M \sum_k \sum_{i_1 \neq i_2} [a_l(i_1) \phi_{jk}(Y_{i_1}) - b_l(i_1) \phi_{jk}(Z_{i_1})] [a_l(i_2) \phi_{jk}(Y_{i_2}) - b_l(i_2) \phi_{jk}(Z_{i_2})]$$

with a threshold value $t_n = t r_n^2$ where t is a constant chosen later. We define

$$\Delta_j = \begin{cases} 1 & \text{if } T_j > t_n, \\ 0 & \text{if } T_j \leq t_n. \end{cases}$$

2.2 Properties of the test statistic

In this section, we provide two propositions which will be crucial when evaluating the performance of our test procedure. They deal with the behaviors of its expectation and its variance.

Proposition 2.1 *Let j be any given level parameter. Then,*

$$\mathbb{E}_{\bar{p}, \bar{q}}(T_j) = \sum_{l=1}^M \sum_k \left(\int_{\mathbb{R}} (p_l - q_l) \phi_{jk} \right)^2 - \frac{1}{n^2} \sum_{l=1}^M \sum_k \sum_{i=1}^n \left(\int_{\mathbb{R}} (a_l(i) f_i - b_l(i) g_i) \phi_{jk} \right)^2.$$

Remark 2.1 *For the particular case where the sequences of the mixing weights $(\omega_u(i), 1 \leq u \leq M, 1 \leq i \leq n)$ and $(\sigma_u(i), 1 \leq u \leq M, 1 \leq i \leq n)$ are identical, the test statistic T_j is centered under the null hypothesis.*

Corollary 2.1 *For any $j \in \mathbb{N}$,*

$$\left| \mathbb{E}_{\bar{p}, \bar{q}}(T_j) - \sum_{l=1}^M \sum_k \left(\int_{\mathbb{R}} (p_l - q_l) \phi_{jk} \right)^2 \right| \leq \frac{8LMR^2}{Kn}.$$

Proposition 2.2 *There exists a constant $C_T = C_T(R, L, \|\phi\|_{\infty}) > 0$ such that*

$$\text{Var}_{\bar{p}, \bar{q}}(T_j) \leq C_T \left(\frac{2^j}{n^2} + \frac{1}{n} \sum_l \|p_l - q_l\|_2^2 + \sqrt{\frac{2^j}{n^3}} \sum_l \|p_l - q_l\|_2 \right) \frac{M^2}{K^2}.$$

Remark 2.2 *Under the null hypothesis the variance of the test statistic T_j is less than or equal to $C_T M^2 K^{-2} 2^j n^{-2}$.*

2.3 Minimax performance of the test procedure

For any $s > 0$, let $(r_n)_{n \in \mathbb{N}}$ be the sequence such that

$$r_n = n^{-\frac{2s}{1+4s}} \quad \forall n \in \mathbb{N}^*.$$

The following theorem shows that the test procedure defined in section 2 provides an accurate upper bound when it is well calibrated.

Theorem 2.1 (Upper bound) Fix $0 < \gamma < 1$ and consider the test procedure $\Delta_s^* = \Delta_{j_n}$ where j_n is the smallest integer such that $2^{-j_n} \leq n^{-\frac{2}{1+4s}}$. Let t and C_γ be two positive real numbers defined as follows :

$$t = \left(2\sqrt{\frac{C_T}{\gamma}} + 8LR^2 \right) \frac{M}{K},$$

$$C_\gamma^2 = 2 \left(\frac{1}{K} \sqrt{\frac{6 C_T}{\gamma}} + R + \frac{t}{M} \right).$$

Then

$$\limsup_{n \rightarrow \infty} \left(\sup_{(\bar{p}, \bar{q}) \in \Theta_0(R)} \mathbb{P}_{\bar{p}, \bar{q}}(\Delta_s^* = 1) + \sup_{(\bar{p}, \bar{q}) \in \Theta_1(R, C, n, s)} \mathbb{P}_{\bar{p}, \bar{q}}(\Delta_s^* = 0) \right) \leq \gamma \quad (5)$$

for all $C > C_\gamma$.

Although the exact value of the constant C_T is very complicated, it can be exactly calculated by following the proofs.

Now, let us focus on the lower bound associated with our nonparametric testing problem \mathcal{H}_0 versus \mathcal{H}_1 .

We aim at providing a constant c_γ such that we ensure that no test procedure is able to choose \mathcal{H}_0 or \mathcal{H}_1 with a sum of the probability errors less than γ ($0 < \gamma < 1$). Obviously, the smaller the distance between c_γ and C_γ the more accurate our results. The next theorem proves that our test procedure is asymptotically minimax.

Similarly to the classical methods for providing lower bounds (see for instance Gayraud and Pouet [21] or Butucea and Tribouley [4]) we shall consider a subspace of $\Lambda_n(R, C)$ that is, for any chosen $C_1 > 0$,

$$\tilde{\Lambda}_n(R, C, C_1) = \left\{ (p, q) \in \Lambda_n(R, C); \sup_{z \in [0, 1]} \min(p(z), q(z)) \geq C_1 \right\}. \quad (6)$$

Theorem 2.2 (Lower bound) Let $0 < \gamma < 1$, $s > 0$ and let $c_\gamma > 0$ satisfy the following equation

$$c_\gamma^4 = \left(\frac{C_1^2}{L K^2} \ln[4(1 - \gamma)^2 + 1] \wedge 2R^2 \right) \frac{2^{-4s}}{4M^2}.$$

Then for all $C < c_\gamma$

$$\liminf_{n \rightarrow \infty} \inf_{\Delta} \left(\sup_{(\bar{p}, \bar{q}) \in \Theta_0(R)} \mathbb{P}_{\bar{p}, \bar{q}}(\Delta = 1) + \sup_{(\bar{p}, \bar{q}) \in \Theta_1(R, C, n, s)} \mathbb{P}_{\bar{p}, \bar{q}}(\Delta = 0) \right) > \gamma \quad (7)$$

where the infimum is taken over all test procedure Δ .

From Theorems 2.1 and 2.2 we deduce

Corollary 2.2 For any $s > 0$, the test procedure Δ_s^* is asymptotically minimax and the minimax rate separating \mathcal{H}_0 and \mathcal{H}_1 is $r_n = n^{-\frac{2s}{1+2s}}$.

2.4 Discussion about the constants c_γ and C_γ

In the two previous theorems we exhibited two constants appearing in the upper and the lower bounds. We think that the connection between these constants and the model's parameters M and K is a novelty and really deserves a discussion. Indeed, we keep in mind that

- C_γ is the minimal value for C such that our test statistic is able to detect if all the mixture components are identical in the two populations with the sum of the probability errors not exceeding γ ;
- c_γ is the maximal value for C such that no test statistic is able to detect if all the mixture components are identical in the two populations with the sum of probability errors not exceeding γ .

As a consequence we proved that our test statistic is optimal in the minimax sense since it attains the minimax rate of convergence separating \mathcal{H}_0 and \mathcal{H}_1 .

According to the definitions of c_γ and C_γ we let the reader be aware that:

- the smaller the constant K , the larger the family of the mixing weights satisfying HYP-1;
- the smaller the constant M , the bigger (= the worse) the constant C_γ and the bigger the constant c_γ ;
- the smaller the constant K , the bigger (= the worse) the constant C_γ and the bigger the constant c_γ .

Although the exact separation constant is not established in this study (since $c_\gamma \neq C_\gamma$), we prove that c_γ and C_γ strongly depend on the smallest eigenvalue of the matrices $\Omega\Omega^*$ and $\Sigma\Sigma^*$.

3 Numerical experiments

The aim of this section is to illustrate by numerical experiments on Matlab the good performance of the test procedures based on the statistics T_{j_n} . First, we give two examples of mixture models to show the interest of the problem we have considered. Next we illustrate the behavior of the test statistics T_{j_n} .

3.1 Examples of mixture models

Figures 1 and 2: [Mixture with two components]

Consider two populations sampled from the same mixture densities such that

- the size of the two populations (Y, Z) is $n = 500$,
- the ranks of the matrices of the mixing weights Ω^* and Σ^* are 2,

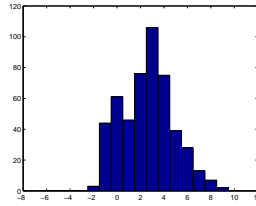


Figure 1: Histogram of population Y .

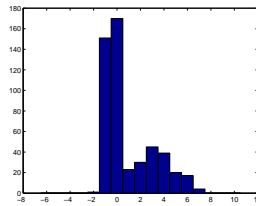


Figure 2: Histogram of population Z .

- the two components of the mixtures are the uniform density $\mathcal{U}([-1, 0])$ and the normal density $\mathcal{N}(3, 4)$.

Figures 3 and 4: [Mixture with three components]

Consider two populations sampled from the same mixture densities such that

- the size of the two populations (Y, Z) is $n = 500$,
- the ranks of the matrices of the mixing weights Ω^* and Σ^* are 3,
- the three components of the mixtures are the normal densities $\mathcal{N}(-2, 1)$, $\mathcal{N}(0, 1)$ and $\mathcal{N}(2, 1)$.

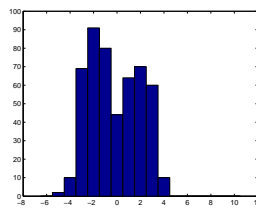


Figure 3: Histogram of population Y .

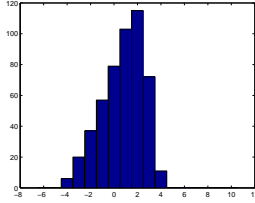


Figure 4: Histogram of population Z .

We remark that in all cases the histograms of the observations are quite different, although they correspond to mixture models with the same components. So the previous schemes show how hard it is to guess whether the mixture components of the two populations (Y, Z) are exactly the same or not. Hence, it justifies that the statistician needs a judicious test statistic to decide whether the populations (Y, Z) have the same mixture components or not.

3.2 Construction of the test procedure: calibration of t_n

In the theoretical part of this paper we provide a decision rule to test \mathcal{H}_0 against \mathcal{H}_1 . This decision rule Δ_{j_n} relies on the sign of $T_{j_n} - t_n$, where t_n is the threshold value depending on the sum of the errors γ and T_{j_n} is the test statistic. In the positive case (resp. in the negative case) Δ_{j_n} proposes to accept \mathcal{H}_1 (resp. \mathcal{H}_0). From the practical point of view, we give some hints to adjust the threshold value t_n . Here we use the Haar basis and we set $s = 4$. For this, we decide to consider two different approaches.

The *first* approach consists in fixing the first type error, $0 < \gamma_1 < 1$, and in choosing t_n as the quantile of order $1 - \gamma_1$ of the test statistic obtained after 1000 replications of the chosen mixture model.

The *second* approach consists in choosing t_n as the value for which the sum of the two errors is the minimal one according of the statistic of test obtained after 1000 replications of the mixture model chosen.

3.3 Connection between K and the performance of the test procedure.

The aim of this paragraph is to illustrate the connection between the value of K and the performance of our test procedure. We provide simulations of some Gaussian mixture models and we give for different values of n

- the value of t_n associated with a first type error equal to 10%,

- the power \mathcal{P} of the test procedure based on the threshold value t_n ,
- the minimum of the global error γ_{opt} - the sum of the first type and the second type errors - reachable by the test procedure,
- the value t_{opt} which corresponds to the global error γ_{opt} .

We consider two samples : Y_1, \dots, Y_n and Z_1, \dots, Z_n . There are two mixture components such that

- under H_0 , we have $p_1(\cdot) = q_1(\cdot) \sim \mathcal{N}(-2, 1)$ and $p_2(\cdot) = q_2(\cdot) \sim \mathcal{N}(3, 4)$,
- under H_1 , we have $p_1(\cdot) \sim \mathcal{N}(-2, 1)$, $p_2(\cdot) \sim \mathcal{N}(3, 4)$, $q_1(\cdot) \sim \mathcal{N}(0, 1)$ and $q_2(\cdot) \sim \mathcal{N}(1, 1)$.

The weights of the samples Y and Z for Gaussian Model 1 are described in the following table :

Sample	Range of i	$\sigma_1(i)$ or $\omega_1(i)$	$\sigma_2(i)$ or $\omega_2(i)$	
Y	$i = 1, \dots, 0.8 n$	0.6	0.4	Model 1
	$i = 0.8 n + 1, \dots, n$	0.4	0.6	
Z	$i = 1, \dots, 0.3 n$	0.2	0.8	
	$i = 0.3 n + 1, \dots, n$	0.5	0.5	

The results are gathered in the following table. We point out that the constant K related to the smallest eigenvalue is very small in this case. Therefore we must expect poor results.

Gaussian Model 1	$n = 200$	$n = 500$	$n = 1000$	
t_n	0.289	0.135	0.080	$K = 0.013$
\mathcal{P}	36.7%	68.1%	85.7%	
γ_{opt}	52.6%	38.2%	23.5%	
t_{opt}	0.022	0.080	0.092	

The weights of the samples Y and Z for Gaussian Model 2 are described in the following table :

Sample	Range of i	$\sigma_1(i)$ or $\omega_1(i)$	$\sigma_2(i)$ or $\omega_2(i)$	
Y	$i = 1, \dots, 0.8 n$	0.8	0.2	Model 2
	$i = 0.8 n + 1, \dots, n$	0.3	0.7	
Z	$i = 1, \dots, 0.3 n$	0.1	0.9	
	$i = 0.3 n + 1, \dots, n$	0.4	0.6	

For this setup, the constant K is almost three times the one appearing in Gaussian Model 1. Therefore we expect to see improved results.

Gaussian Model 2	$n = 200$	$n = 500$	$n = 1000$
t_n	0.994	0.061	0.027
\mathcal{P}	85.2%	91.5%	96.8%
γ_{opt}	24.6%	16.3%	9.5%
t_{opt}	0.078	0.103	0.047

$K = 0.033$

The weights of the samples Y and Z for Gaussian Model 3 are described in the following table :

Sample	Range of i	$\sigma_1(i)$ or $\omega_1(i)$	$\sigma_2(i)$ or $\omega_2(i)$
Y	$i = 1, \dots, 0.8 n$	0.8	0.2
	$i = 0.8 n + 1, \dots, n$	0.3	0.7
Z	$i = 1, \dots, 0.3 n$	0.9	0.1
	$i = 0.3 n + 1, \dots, n$	0.3	0.7

Model 3

In this setup, the constant K is more than five times the one appearing in Gaussian Model 1 and more than twice the one appearing in Gaussian Model 2. Therefore we expect better results.

Gaussian Model 3	$n = 200$	$n = 500$	$n = 1000$
t_n	0.054	0.030	0.015
\mathcal{P}	97.1%	96.7%	98.1%
γ_{opt}	10.5%	9.6%	6.5%
t_{opt}	0.066	0.064	0.034

$K = 0.068$

According to the numerical results above, it is clear that for a fixed n , the larger the value of K , the better the performance of the test procedure. Indeed, when the error of the first type is 10%, we see that increasing values of K increases the power of the test procedure. Moreover, we remark that the optimal global error γ_{opt} increases when the value of K decreases. In fact, this is not surprising as this behavior was predicted by our theoretical results :the smaller the value of K the larger the constant C_γ (see theorem 2.1). In other words, in a mixture model with a small value of K one needs a lot of observations n to hope good performance of our test procedure.

4 Open questions

As a conclusion, we have provided a statistical procedure for a testing problem on the mixture components of two populations (Y, Z). This one was proved to be optimal in the minimax sense (Theorems 2.1 and 2.2). In addition, we explained how the weights of the mixture model influence the performance of

the statistical rule. All these theoretical results are illustrated by our numerical experiments.

It seems to us important to give some hints about possible extensions of this work. From the theoretical and practical points of view, it would be interesting to study the same problem without assuming that the mixing weights are exactly known to the statistician. Several explanations can be given

- the statistician can estimate the mixing weights for an observation by using covariates and an appropriate predictive model such as the logistic one,
- a Bayesian approach is chosen for the mixing weights,
- exogenous information allows the statistician to roughly estimate the mixing weights.

In this case several natural questions arise

- What statistical rule should be considered?
- What kind of performance can be expected for such a rule?
- How much do random mixing weights deteriorate the performance?

Such questions are beyond the scope of this article and their answers certainly involve random matrices theory.

Finally, it would be nice to show how to choose the adequate value of t_n in a better way than the complicated one given in Theorem 2.2.

5 Proofs of main results

This section is devoted to the proofs of our results. The proofs often need technical lemmas which shall be proved in Appendix. For the sake of simplicity we sometimes omit \vec{p} and \vec{q} in the indices when there is no ambiguity.

5.1 Proofs of Propositions and Corollaries

Proof of Proposition 1.1: We refer to Maiboroda [19]. A solution of the two problems is given for any $(l, i) \in \{1, \dots, M\} \times \{1, \dots, n\}$ by

$$a_l(i) = \frac{1}{\det(\Gamma_n)} \sum_{u=1}^M (-1)^{l+u} \gamma_{lu} \omega_u(i)$$

$$b_l(i) = \frac{1}{\det(\Gamma'_n)} \sum_{u=1}^M (-1)^{l+u} \gamma'_{lu} \sigma_u(i)$$

where γ_{lu} and γ'_{lu} are respectively the minor (l, u) of the matrix Γ_n and the minor (l, u) of the matrix Γ'_n . Inequalities (3) and (4) are obtained by using lemma 6.1.

□

Proof of Proposition 2.1: Let us evaluate the expectation of T_j .

$$\begin{aligned} \mathbb{E}_{\vec{p}, \vec{q}}(T_j) &= \mathbb{E}_{\vec{p}, \vec{q}} \left(\frac{1}{n^2} \sum_{l=1}^M \sum_k \sum_{i_1 \neq i_2} (a_l(i_1)\phi_{jk}(Y_{i_1}) - b_l(i_1)\phi_{jk}(Z_{i_1}))(a_l(i_2)\phi_{jk}(Y_{i_2}) - b_l(i_2)\phi_{jk}(Z_{i_2})) \right) \\ &= \frac{1}{n^2} \sum_{l=1}^M \sum_k \sum_{i_1 \neq i_2} \mathbb{E}_{\vec{p}, \vec{q}} [a_l(i_1)\phi_{jk}(Y_{i_1}) - b_l(i_1)\phi_{jk}(Z_{i_1})] \mathbb{E}_{\vec{p}, \vec{q}} [a_l(i_2)\phi_{jk}(Y_{i_2}) - b_l(i_2)\phi_{jk}(Z_{i_2})], \end{aligned}$$

since the random variables (Y_{i_1}, Z_{i_1}) and (Y_{i_2}, Z_{i_2}) are independent. We have for all $1 \leq i \leq n$,

$$\mathbb{E}_{\vec{p}, \vec{q}} [a_l(i)\phi_{jk}(Y_i) - b_l(i)\phi_{jk}(Z_i)] = \int_{\mathbb{R}} \left(\sum_{u=1}^M (a_l(i)\omega_u(i)p_u - b_l(i)\sigma_u(i)q_u) \right) \phi_{jk}.$$

By introducing the diagonal term $i_1 = i_2$ in the sum, we get

$$\begin{aligned} \mathbb{E}_{\vec{p}, \vec{q}}(T_j) &= \frac{1}{n^2} \sum_{l=1}^M \sum_k \left(\int_{\mathbb{R}} \phi_{jk} \left(\sum_{i=1}^n \sum_{u=1}^M a_l(i)\omega_u(i)p_u - \sum_{i=1}^n \sum_{u=1}^M b_l(i)\sigma_u(i)q_u \right) \right)^2 \\ &\quad - \frac{1}{n^2} \sum_{l=1}^M \sum_k \sum_{i=1}^n \left(\int_{\mathbb{R}} (a_l(i)f_i - b_l(i)g_i) \phi_{jk} \right)^2 \\ &= \sum_{l=1}^M \sum_k \left(\int_{\mathbb{R}} (p_l - q_l) \phi_{jk} \right)^2 - \frac{1}{n^2} \sum_{l=1}^M \sum_k \sum_{i=1}^n \left(\int_{\mathbb{R}} (a_l(i)f_i - b_l(i)g_i) \phi_{jk} \right)^2, \end{aligned}$$

because of the two properties $\frac{1}{n} \sum_{i=1}^n a_l(i)\omega_u(i) = \delta_{lu}$ and $\frac{1}{n} \sum_{i=1}^n b_l(i)\sigma_u(i) = \delta_{lu}$.

Thus the result for the expectation is proved. □

Proof of Corollary 2.1:

According to proposition 2.1 we only have to bound the quantity

$$\frac{1}{n^2} \sum_{l=1}^M \sum_k \sum_{i=1}^n \left(\int_{\mathbb{R}} (a_l(i)f_i - b_l(i)g_i) \phi_{jk} \right)^2.$$

Using the Cauchy-Schwarz inequality and lemma 6.3, we have

$$\begin{aligned}
\sum_{l=1}^M \sum_k \sum_{i=1}^n \left(\int_{\mathbb{R}} (a_l(i)f_i - b_l(i)g_i) \phi_{jk} \right)^2 &\leq \sum_{l=1}^M \sum_k \sum_{i=1}^n \int_{I_{jk}} (a_l(i)f_i - b_l(i)g_i)^2 \int \phi_{jk}^2 \\
&= \sum_{l=1}^M \sum_{i=1}^n \left[\sum_k \int_{I_{jk}} (a_l(i)f_i - b_l(i)g_i)^2 \right] \\
&\leq 2 \sum_{i=1}^n \sum_{l=1}^M \left[\sum_k \int_{I_{jk}} (a_l(i)f_i)^2 + \int_{I_{jk}} (b_l(i)g_i)^2 \right] \\
&\leq 4L \left(\sum_{i=1}^n \sum_{l=1}^M a_l^2(i) \|f_i\|_2^2 + \sum_{i=1}^n \sum_{l=1}^M b_l^2(i) \|g_i\|_2^2 \right) \\
&\leq \frac{8LMR^2n}{K}.
\end{aligned}$$

Last inequality is due to proposition 1.1 and the fact that for all $1 \leq i \leq n$ the density functions f_i and g_i belong to $\mathbb{L}_2(R)$. \square

Proof of Proposition 2.2: Let us consider the variance of T_j . For all (i_1, i_2) , let $h_j(i_1, i_2)$ denote the quantity

$$h_j(i_1, i_2) = \sum_k \sum_{l=1}^M (a_l(i_1)\phi_{jk}(Y_{i_1}) - b_l(i_1)\phi_{jk}(Z_{i_1})) (a_l(i_2)\phi_{jk}(Y_{i_2}) - b_l(i_2)\phi_{jk}(Z_{i_2})).$$

The variance of T_j satisfies

$$\begin{aligned}
n^4 \mathbb{V}ar_{\vec{p}, \vec{q}}(T_j) &= \mathbb{V}ar_{\vec{p}, \vec{q}} \left(\sum_{i_1 \neq i_2} h_j(i_1, i_2) \right) \\
&= \sum_{i_1 \neq i_2, i_3 \neq i_4} \mathbb{C}ov(h_j(i_1, i_2), h_j(i_3, i_4)) \\
&= \sum_{i_1 \neq i_2} \mathbb{V}ar(h_j(i_1, i_2)) + \sum_{i_1 \neq i_2} \mathbb{C}ov(h_j(i_1, i_2), h_j(i_2, i_1)) \\
&\quad + \sum_{i_1 \neq i_2 \neq i_3} \mathbb{C}ov(h_j(i_1, i_2), h_j(i_1, i_3)) + \sum_{i_1 \neq i_2 \neq i_3} \mathbb{C}ov(h_j(i_1, i_2), h_j(i_2, i_3)) \\
&\quad + \sum_{i_1 \neq i_2 \neq i_3} \mathbb{C}ov(h_j(i_1, i_2), h_j(i_3, i_1)) + \sum_{i_1 \neq i_2 \neq i_3} \mathbb{C}ov(h_j(i_1, i_2), h_j(i_3, i_2)) \\
&\quad + \sum_{i_1 \neq i_2 \neq i_3 \neq i_4} \mathbb{C}ov(h_j(i_1, i_2), h_j(i_3, i_4)) \\
&= \sum_{u=1}^7 A_u.
\end{aligned}$$

Using independence arguments,

$$A_7 = \sum_{i_1 \neq i_2 \neq i_3 \neq i_4} \mathbb{C}ov(h_j(i_1, i_2), h_j(i_3, i_4)) = 0.$$

We are still required to bound for the quantities A_i ($1 \leq i \leq 6$). Since the ways to bound A_1 and A_2 (resp. A_3, A_4, A_5 and A_6) are similar, we will only bound A_1 and A_3 . Such bounds are given in lemmas 6.7 and 6.8. The proof of proposition 2.2 is a direct consequence of lemmas 6.7 and 6.8 by taking $C_T = 2 \bar{C}_T \vee 4 \tilde{C}_T$. \square

5.2 Proofs of Theorems

Proof of Theorem 2.1.

Let us fix $0 < \gamma < 1$ and $s > 0$. Under the null hypothesis, we use directly the well-known Bienayme-Chebyshev inequality.

$$\begin{aligned} \mathbb{P}_{\vec{p}, \vec{p}} (\Delta_s^* = 1) &= \mathbb{P}_{\vec{p}, \vec{p}} (T_{j_n} > t_n) \\ &\leq \mathbb{P}_{\vec{p}, \vec{p}} \left(T_{j_n} - \mathbb{E}(T_{j_n}) > t_n - \frac{8LMR^2}{Kn} \right) \\ &\leq \frac{\text{Var}_{\vec{p}, \vec{p}}(T_{j_n})}{\left(t_n - \frac{8LMR^2}{Kn}\right)^2} \\ &\leq \frac{C_T M^2 2^{j_n}}{n^2 K^2 \left(t - \frac{8LMR^2}{K}\right)^2 r_n^4}. \end{aligned}$$

The last inequality is obtained using remark 2.2. According to the choices of the level j_n and the threshold t_n , we have

$$\frac{C_T M^2 2^{j_n}}{n^2 K^2 \left(t - \frac{8LMR^2}{K}\right)^2 r_n^4} \leq \frac{2C_T M^2}{K^2 \left(t - \frac{8LMR^2}{K}\right)^2}.$$

Then

$$\mathbb{P}_{\vec{p}, \vec{p}} (\Delta_s^* = 1) \leq \frac{\gamma}{2}.$$

Under the alternative, we use the expectation of the test statistic and some approximation argument. The second type error is

$$\mathbb{P}_{\vec{p}, \vec{q}} (\Delta_s^* = 0) = \mathbb{P}_{\vec{p}, \vec{q}} \left(-T_{j_n} + \mathbb{E}_{\vec{p}, \vec{q}}(T_{j_n}) \geq -t_n + \mathbb{E}_{\vec{p}, \vec{q}}(T_{j_n}) \right).$$

The wavelet expansion in the Besov body $\mathcal{B}_{2,\infty}^s$ leads to

$$\begin{aligned} \mathbb{E}_{\vec{p}, \vec{q}}(T_{j_n}) - t_n &= \sum_{l=1}^M \|p_l - q_l\|_2^2 - \sum_{l=1}^M \sum_{j \geq j_n} \sum_k \left(\int_{\mathbb{R}} (p_l - q_l) \psi_{jk} \right)^2 \\ &\quad - \frac{1}{n^2} \sum_{l=1}^M \sum_k \sum_{i=1}^n \left(\int_{\mathbb{R}} (a_l(i) f_i - b_l(i) g_i) \phi_{j_n k} \right)^2 - t_n \\ &\geq \sum_{l=1}^M \|p_l - q_l\|_2^2 - M R 2^{-2j_n s} - \frac{8LMR^2}{Kn} - t_n. \\ &\geq \frac{1}{2} \sum_{l=1}^M \|p_l - q_l\|_2^2 - M R 2^{-2j_n s} - t_n, \end{aligned}$$

for any n large enough.

As a consequence, applying the Bienayme-Chebychev inequality leads to

$$\begin{aligned} & \mathbb{P}_{\vec{p}, \vec{q}}(-T_{j_n} + \mathbb{E}_{f,g}(T_{j_n}) \geq -t_n + \mathbb{E}_{f,g}(T_{j_n})) \\ & \leq \frac{C_T M^2 \left(2^{j_n} + n \sum_l \|p_l - q_l\|_2^2 + \sqrt{2^{j_n} n} \sum_l \|p_l - q_l\|_2 \right)}{n^2 K^2 \left(\frac{1}{2} \sum_{l=1}^M \|p_l - q_l\|_2^2 - M R 2^{-2j_n s} - t_n \right)^2}. \end{aligned}$$

The choice of j_n and the fact that the functions are in the alternative entail the following upper bound

$$\mathbb{P}_{\vec{p}, \vec{q}}(\Delta_s^* = 0) \leq \frac{C_T M^2 \left(2^{j_n} + n \sum_l \|p_l - q_l\|_2^2 + \sqrt{2^{j_n} n} \sum_l \|p_l - q_l\|_2 \right)}{K^2 n^2 \left(\frac{1}{2} \sum_{l=1}^M \|p_l - q_l\|_2^2 - M R 2^{-2j_n s} - t r_n^2 \right)^2}.$$

According to the choices of j_n , and r_n , one gets for n large enough:

$$\begin{aligned} \mathbb{P}_{\vec{p}, \vec{q}}(\Delta_s^* = 0) & \leq \frac{C_T M^2 \left(2^{j_n} + n \sum_l \|p_l - q_l\|_2^2 + \sqrt{2^{j_n} n} \sum_l \|p_l - q_l\|_2 \right)}{n^2 K^2 \left(\frac{1}{2} - \frac{R}{C^2} - \frac{t}{MC^2} \right)^2 \left(\sum_{l=1}^M \|p_l - q_l\|_2^2 \right)^2} \\ & \leq \frac{3C_T}{\left(\frac{1}{2} - \frac{R}{C^2} - \frac{t}{MC^2} \right)^2 K^2 C^4}. \end{aligned}$$

For all $C > C_\gamma$, we finally obtain

$$\mathbb{P}_{\vec{p}, \vec{q}}(\Delta_s^* = 0) \leq \frac{\gamma}{2}.$$

The results on the first-type and second-type errors show that if $C > C_\gamma$ the sum of the errors is less than γ . Therefore the upper bound is proved. \square

Proof of Theorem 2.2.

Let $\gamma \in]0, 1[$, $C > 0$ and $C_1 > 0$. We define

$$\begin{aligned} \tilde{\Theta}_1(R, C, C_1, n, s) & = \left\{ (\vec{p}, \vec{q}) : \forall u \in \{1, \dots, M\}, p_u - q_u \in \mathcal{B}_{2, \infty}^s(R), \right. \\ & \quad \left. \exists u \in \{1, \dots, M\}, (p_u, q_u) \in \tilde{\Lambda}_n(R, C, C_1) \right\}, \end{aligned}$$

where $\tilde{\Lambda}_n(R, C, C_1)$ is defined in (6). It is well-known that

$$\begin{aligned} & \inf_{\Delta} \left(\sup_{(\bar{p}, \bar{q}) \in \Theta_0(R)} \mathbb{P}_{\bar{p}, \bar{q}}(\Delta = 1) + \sup_{(\bar{p}, \bar{q}) \in \Theta_1(R, C, n, s)} \mathbb{P}_{\bar{p}, \bar{q}}(\Delta = 0) \right) \\ & \geq \inf_{\Delta} \left(\sup_{(\bar{p}, \bar{q}) \in \Theta_0(R)} \mathbb{P}_{\bar{p}, \bar{q}}(\Delta = 1) + \sup_{(\bar{p}, \bar{q}) \in \tilde{\Theta}_1(R, C, C_1, n, s)} \mathbb{P}_{\bar{p}, \bar{q}}(\Delta = 0) \right) \\ & \geq 1 - \frac{1}{2} \|\mathbb{P}_{\bar{p}, \bar{p}} - \mathbb{P}_{\pi}\|, \end{aligned}$$

where $\|\cdot\|$ is the \mathbb{L}_1 - distance and π is an a priori probability measure on the set $\Lambda_n(R, C)$. First we define the probability measure π and its support. Let $\theta = (\theta_1, \dots, \theta_M)$ denote an eigenvector associated with the smallest eigenvalue of $\Sigma\Sigma^*$ - which is Kn according to HYP-2 - such that $\|\theta\|_2 = 1$.

Recall that here j_n is the same as the one defined in theorem 2.1. Let \mathcal{T} be the subset of \mathbb{Z} containing every integer k satisfying the following properties

- $k \in \mathcal{T} \implies \left[\frac{k-L}{2^{j_n}}, \frac{k+L}{2^{j_n}} \right] \subset [0, 1[;$
- $(k, k') \in \mathcal{T} \times \mathcal{T}$ with $k \neq k' \implies \left[\frac{k-L}{2^{j_n}}, \frac{k+L}{2^{j_n}} \right] \cap \left[\frac{k'-L}{2^{j_n}}, \frac{k'+L}{2^{j_n}} \right] = \emptyset$.

The cardinal of \mathcal{T} is clearly equal to $T = \lfloor \frac{2^{j_n-1}}{L} \rfloor$ and we denote its elements k_1, \dots, k_T . The following parametric family of functions is considered

$$q_{l, \zeta}(z) = p_l(z) + 2^{s+1} C \sqrt{ML} \theta_l \sum_{k \in \mathcal{T}} \zeta_k 2^{-j_n s - \frac{j_n}{2}} \psi_{j_n k}(z),$$

where $\zeta_k = +1$ or -1 .

Remark that ζ_k does not depend on the index l . Therefore the density of Z_i is

$$g_{i, \zeta}(z) = \sum_{l=1}^M \sigma_l(i) \sqrt{ML} \theta_l 2^{s+1} C \sum_{k \in \mathcal{T}} \zeta_k 2^{-j_n s - \frac{j_n}{2}} \psi_{j_n k}(z) + \sum_{l=1}^M \sigma_l(i) p_l(z).$$

The probability measure π is such that the ζ_k 's are independent Rademacher random variables with parameter $\frac{1}{2}$.

The function $q_{l, \zeta}$ is a density. Indeed, for n large, $q_{l, \zeta}$ is non-negative. Moreover, as $\psi_{j_n k}$ is a wavelet, we have $\int \psi_{j_n k} = 0$ and therefore $\int q_{l, \zeta} = 1$. If $C < \sqrt{R/M} 2^{2s+2}$, then $q_{l, \zeta} - p_l$ belongs to the ball of the Besov body $\mathcal{B}_{2, \infty}^s(R)$. There exists l such that

$$M\theta_l^2 \geq 1 \quad \text{and} \quad \|p_l - q_{l, \zeta}\|_2^2 = TLMC^2 2^{2+2s-2j_n s - j_n} \theta_l^2 \geq C^2 n^{-\frac{4s}{4s+1}}.$$

Therefore the probability measure π is solely concentrated on the alternative.

It is well-known that the \mathbb{L}_1 distance can be bounded by the \mathbb{L}_2 distance. We have

$$\begin{aligned} \|\mathbb{P}_{\vec{p}, \vec{p}} - \mathbb{P}_\pi\| &\leq \sqrt{\mathbb{E}_{\vec{p}, \vec{p}} \left[\left(\frac{d\mathbb{P}_\pi}{d\mathbb{P}_{\vec{p}, \vec{p}}} \right)^2 \right] - 1} \\ &= \sqrt{\mathbb{E}_{\vec{p}, \vec{p}} \left[\left(\mathbb{E}_\pi \left(\prod_{i=1}^n \frac{g_{i,\zeta}(Z_i)}{g_i(Z_i)} \right) \right)^2 \right] - 1}. \end{aligned} \quad (8)$$

Therefore it suffices to evaluate the second-order moment of the likelihood ratio:

$$\begin{aligned} &\mathbb{E}_{\vec{p}, \vec{p}} \left[\left(\mathbb{E}_\pi \left(\prod_{i=1}^n \frac{g_{i,\zeta}(Z_i)}{g_i(Z_i)} \right) \right)^2 \right] \\ &= \mathbb{E}_{\vec{p}, \vec{p}} \left[\left(\prod_{k \in \mathcal{T}} \int \prod_{i=1}^n \left(1 + 2^{s+1} C \sqrt{ML} \zeta_k 2^{-j_n s - \frac{j_n}{2}} \frac{\psi_{j_n k}(Z_i)}{g_i(Z_i)} \sum_{l=1}^M \theta_l \sigma_l(i) \right) d\pi(\zeta_1, \dots, \zeta_T) \right)^2 \right]. \end{aligned}$$

Let us introduce the following random variables

$$\tilde{Z}_{ik} = 2^{s+1} C \sqrt{ML} 2^{-j_n s - \frac{j_n}{2}} \frac{\psi_{j_n k}(Z_i)}{g_i(Z_i)} \sum_{l=1}^M \theta_l \sigma_l(i).$$

We have

$$\begin{aligned} &\mathbb{E}_{\vec{p}, \vec{p}} \left[\left(\prod_{k \in \mathcal{T}} \int \prod_{i=1}^n \left(1 + 2^{s+1} C \sqrt{ML} \zeta_k 2^{-j_n s - \frac{j_n}{2}} \frac{\psi_{j_n k}(Z_i)}{g_i(Z_i)} \sum_{l=1}^M \theta_l \sigma_l(i) \right) d\pi(\zeta_1, \dots, \zeta_T) \right)^2 \right] \\ &= \mathbb{E}_{\vec{p}, \vec{p}} \left[\prod_{k \in \mathcal{T}} \frac{1}{4} \left[\prod_{i=1}^n (1 + \tilde{Z}_{ik}) + \prod_{i=1}^n (1 - \tilde{Z}_{ik}) \right]^2 \right] \\ &= \mathbb{E}_{\vec{p}, \vec{p}} \left[\prod_{k \in \mathcal{T}} \frac{1}{4} \left(\prod_{i=1}^n (1 + 2\tilde{Z}_{ik} + \tilde{Z}_{ik}^2) + \prod_{i=1}^n (1 - 2\tilde{Z}_{ik} + \tilde{Z}_{ik}^2) + 2 \prod_{i=1}^n (1 - \tilde{Z}_{ik}^2) \right) \right] \\ &= \mathbb{E}_{\vec{p}, \vec{p}} \left[\prod_{k \in \mathcal{T}} \frac{1}{4} \left\{ 2 \prod_{i=1}^n (1 + \tilde{Z}_{ik}^2) + 2 \prod_{i=1}^n (1 - \tilde{Z}_{ik}^2) \right. \right. \\ &\quad \left. \left. + \sum_{i=1}^n \tilde{Z}_{ik} h_i(\tilde{Z}_{1k}, \dots, \tilde{Z}_{i-1,k}, \tilde{Z}_{i+1,k}, \dots, \tilde{Z}_{nk}) \right\} \right] \\ &= \mathbb{E}_{\vec{p}, \vec{p}} \left[\prod_{k \in \mathcal{T}} \frac{1}{2} \left(\prod_{i=1}^n (1 + \tilde{Z}_{ik}^2) + \prod_{i=1}^n (1 - \tilde{Z}_{ik}^2) \right) \right. \\ &\quad \left. + \sum_{r=1}^T \sum_{i=1}^n \tilde{Z}_{ik_r} \tilde{h}(\tilde{Z}_{1k_1}, \dots, \tilde{Z}_{n,k_r-1}, \tilde{Z}_{1k_r}, \dots, \tilde{Z}_{i-1,k_r}, \tilde{Z}_{i+1,k_r}, \dots, \tilde{Z}_{nk_r}, \tilde{Z}_{1,k_r+1}, \dots, \tilde{Z}_{nk_T}) \right], \end{aligned}$$

where the functions h_i and \tilde{h}_i are sums of products of their arguments. As $\mathbb{E}_{\vec{p}, \vec{p}}(\tilde{Z}_{ik}) = 0$ and $\tilde{Z}_{ik}\tilde{Z}_{ik'} = 0$ for $k \neq k'$, the last term vanishes. Thus we are only interested in the first term.

Define for all $k \in \mathcal{T}$:

$$\begin{aligned} h_l(k) &= \sum_{1 \leq i_1 < i_2 < \dots < i_l \leq n} \tilde{Z}_{i_1 k}^2 \tilde{Z}_{i_2 k}^2 \dots \tilde{Z}_{i_l k}^2, \\ h_0(k) &= 2. \end{aligned}$$

Then, we have

$$\begin{aligned} \mathbb{E}_{\vec{p}, \vec{p}} \left[\prod_{k \in \mathcal{T}} \frac{1}{2} \left(\prod_{i=1}^n (1 + \tilde{Z}_{ik}^2) + \prod_{i=1}^n (1 - \tilde{Z}_{ik}^2) \right) \right] &= \mathbb{E}_{\vec{p}, \vec{p}} \left[\left(\frac{1}{2} \right)^T \prod_{k \in \mathcal{T}} \left(\sum_{\substack{l=0 \\ l \text{ even}}}^n h_l(k) \right) \right] \\ &= \sum_{\substack{l_1, \dots, l_T=0 \\ l_1, \dots, l_T \text{ even}}}^n \left(\frac{1}{2} \right)^T \mathbb{E}_{\vec{p}, \vec{p}} \left(\prod_{r=1}^T h_{l_r}(k_r) \right) \\ &\leq \sum_{\substack{l_1, \dots, l_T=0 \\ l_1, \dots, l_T \text{ even}}}^n \left(\frac{1}{2} \right)^T \prod_{r=1}^T \mathbb{E}_{\vec{p}, \vec{p}}(h_{l_r}(k_r)) \\ &\leq \prod_{k \in \mathcal{T}} \frac{1}{2} \left(\sum_{\substack{l=0 \\ l \text{ even}}}^n \mathbb{E}_{\vec{p}, \vec{p}}[h_l(k)] \right) \\ &\leq \prod_{k \in \mathcal{T}} \frac{1}{2} \left(\sum_{\substack{l=0 \\ l \text{ even}}}^n \sum_{1 \leq i_1 < \dots < i_l \leq n} \mathbb{E}_{\vec{p}, \vec{p}}[\tilde{Z}_{i_1 k}^2] \dots \mathbb{E}_{\vec{p}, \vec{p}}[\tilde{Z}_{i_l k}^2] \right) \\ &\leq \prod_{k \in \mathcal{T}} \frac{1}{2} \left(\prod_{i=1}^n (1 + \mathbb{E}_{\vec{p}, \vec{p}}[\tilde{Z}_{ik}^2]) + \prod_{i=1}^n (1 - \mathbb{E}_{\vec{p}, \vec{p}}[\tilde{Z}_{ik}^2]) \right) \\ &\leq \prod_{k \in \mathcal{T}} \cosh \left(\sum_{i=1}^n \mathbb{E}_{\vec{p}, \vec{p}}(\tilde{Z}_{ik}^2) \right) \\ &\leq \exp \left(\frac{1}{2} \sum_{k \in \mathcal{T}} \left(\sum_{i=1}^n \mathbb{E}_{\vec{p}, \vec{p}}(\tilde{Z}_{ik}^2) \right)^2 \right). \end{aligned}$$

Each $\mathbb{E}_{\vec{p}, \vec{p}}(\tilde{Z}_{ik}^2)$ is bounded as follows,

$$\mathbb{E}_{\vec{p}, \vec{p}}(\tilde{Z}_{ik}^2) \leq 2^{2s+2-2j_n s-j_n} \frac{C^2}{C_1} M L \left(\sum_{l=1}^M \theta_l \sigma_l(i) \right)^2.$$

Therefore this bound entails

$$\begin{aligned}
\exp\left(\frac{1}{2}\sum_{k\in\mathcal{T}}\left(\sum_{i=1}^n\mathbb{E}_{\vec{p},\bar{p}}\left(\tilde{Z}_{ik}^2\right)\right)^2\right) &\leq \exp\left(\frac{1}{2}\sum_{k\in\mathcal{T}}2^{4s+4}C^42^{-4j_n s-2j_n}\frac{L^2M^2}{C_1^2}\left(\sum_{i=1}^n\sum_{l,m=1}^M\theta_l\theta_m\sigma_l(i)\sigma_m(i)\right)^2\right) \\
&\leq \exp\left(\frac{1}{2}\sum_{k\in\mathcal{T}}2^{4s+4}C^42^{-4j_n s-2j_n}\frac{L^2M^2}{C_1^2}(\theta^*\Gamma'_n\theta)^2\right) \\
&= \exp\left(\sum_{k\in\mathcal{T}}2^{4s+3}C^42^{-4j_n s-2j_n}\frac{L^2M^2}{C_1^2}(Kn)^2\right) \\
&\leq \exp\left(2^{4s+2}M^2K^2\frac{LC^4}{C_1^2}\right). \tag{9}
\end{aligned}$$

Inequalities (8) and (9) lead to

$$\|\mathbb{P}_{\vec{p},\bar{p}} - \mathbb{P}_\pi\| \leq \sqrt{\exp\left(2^{4s+2}M^2K^2\frac{LC^4}{C_1^2}\right) - 1}.$$

The choice of any constant C such that $C < c_\gamma$ entails that the left-hand side of (9) is strictly smaller than $2(1 - \gamma)$. \square

6 Appendix

This section contains the technical lemmas used in the proofs of the main results.

Lemma 6.1

$$\sum_{l=1}^M\sum_{i=1}^na_l^2(i) \leq \frac{Mn}{K}, \tag{10}$$

$$\sum_{l=1}^M\sum_{i=1}^nb_l^2(i) \leq \frac{Mn}{K}. \tag{11}$$

Proof of Lemma 6.1:

The proofs of (10) and (11) are identical, that's why we only prove (10). Let $\lambda_{\min}(\Gamma_n)$ be the smallest non negative eigenvalue of the matrix Γ_n . Let $A = (A)_{1\leq j\leq n, 1\leq l\leq M}$ denote the $(n \times M)$ matrix with coefficients $A_{j,l} = a_l(j)$. Since the matrix AA^* has at most M non negative eigenvalues, we have

$$\sum_{l=1}^M\sum_{i=1}^na_l^2(i) = \text{trace}(AA^*) \leq M\lambda_{\max}(AA^*). \tag{12}$$

Clearly, the following implication holds

λ is a non negative eigenvalue of $AA^* \implies n^2\lambda^{-1}$ is an eigenvalue of Γ_n .

So

$$\lambda_{max}(AA^*) \leq \frac{n^2}{\lambda_{min}(\Gamma_n)}. \quad (13)$$

Lemma 6.1 is proved by inequalities (12) and (13) and under hypotheses HYP-1 and HYP-2. \square

Lemma 6.2 For all $(j, k) \in \mathbb{Z} \times \mathbb{Z}$, let us put

$$I_{jk} = \left[\frac{k-L}{2^j}, \frac{k+L}{2^j} \right[.$$

Then for any fixed (j, k)

$$\text{Card}\{k' \in \mathbb{Z} : I_{jk} \cap I_{jk'} \neq \emptyset\} \leq 4L.$$

Proof of Lemma 6.2:

Clearly, $I_{jk} \cap I_{jk'} = \emptyset \iff k' - L \geq k + L$ or $k' + L \leq k - L$.

Hence, $I_{jk} \cap I_{jk'} \neq \emptyset \iff k - 2L < k' < k + 2L$.

As a consequence, we have

$$\text{Card}\{k' \in \mathbb{Z} : I_{jk} \cap I_{jk'} \neq \emptyset\} \leq 4L.$$

\square

Lemma 6.3 For any function $h \in L_1(\mathbb{R})$

$$\sum_k \int_{I_{jk}} |h(x)| dx \leq 2L \|h\|_1.$$

Proof of Lemma 6.3: Let us define for any $h \in L_1(\mathbb{R})$:

$$p_{jk}(h) = \int_{I_{jk}} |h(x)| dx, \quad \forall j \in \mathbb{N}, \quad \forall k \in \mathbb{Z}.$$

Judging from the definition of the intervals I_{jk} , we easily prove that for any $j \in \mathbb{N}$,

$$\sum_k p_{jk}(h) = \sum_{u=1}^{2L} \sum_{i \in \mathbb{Z}} p_{j, 2Li+u}(h) \leq \sum_{u=1}^{2L} \int_{\mathbb{R}} |h(x)| dx = 2L \|h\|_1.$$

\square

Lemma 6.4 *Let W be either Y or Z . For any $1 \leq i \leq n$ and any (j, k) , we have*

$$|\mathbb{E}(\phi_{jk}(W_i))| \leq \left(2L \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty)\right)^{\frac{1}{2}} 2^{-\frac{j}{2}}.$$

Proof of Lemma 6.4:

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |\mathbb{E}(\phi_{jk}(W_i))| &\leq \left| \int \phi_{jk} f_i \right| \vee \left| \int \phi_{jk} g_i \right| \\ &\leq \int |\phi_{jk}| \sup_l \|p_l\|_\infty \vee \int |\phi_{jk}| \sup_l \|q_l\|_\infty \\ &\leq \left(2L \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty)\right)^{\frac{1}{2}} 2^{-\frac{j}{2}}. \end{aligned}$$

□

Lemma 6.5 *Let W be either Y or Z and c be either a or b . For any $1 \leq i \leq n$ and any (j, k) , the following inequalities hold*

$$\begin{aligned} \sum_{k'} |\mathbb{E}(\phi_{jk}(W_i)\phi_{jk'}(W_i))| &\leq 4L \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty), \\ \sup_l \left| \sum_k \int \phi_{jk}(p_l - q_l) \right| &\leq 4L \|\phi\|_\infty 2^{\frac{j}{2}}, \\ \sup_l |c_l(i)| &\leq \sqrt{n \sum_l \langle c_l, c_l \rangle_n}. \end{aligned}$$

Proof of Lemma 6.5:

Since the wavelets are compactly supported, for any fixed k the sum over k' has at most $4L$ terms which are non zeros (see lemma 6.2). So, the Cauchy-Schwarz inequality entails that

$$\begin{aligned} \sum_{k'} |\mathbb{E}(\phi_{jk}(W_i)\phi_{jk'}(W_i))| &\leq \sum_{k'} \int |\phi_{jk}| |\phi_{jk'}| f_i \vee \sum_{k'} \int |\phi_{jk}| |\phi_{jk'}| g_i \\ &\leq \sum_{k'} \left(\|f_i\|_\infty \int |\phi_{jk}| |\phi_{jk'}| \right) \vee \sum_{k'} \left(\|g_i\|_\infty \int |\phi_{jk}| |\phi_{jk'}| \right) \\ &\leq \left(\sup_l \|p_l\|_\infty \sum_{k'} \int |\phi_{jk}| |\phi_{jk'}| \right) \vee \left(\sup_l \|q_l\|_\infty \sum_{k'} \int |\phi_{jk}| |\phi_{jk'}| \right) \\ &\leq 4L \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty). \end{aligned}$$

We also have

$$\begin{aligned}
\sup_l \left| \sum_k \int \phi_{jk}(p_l - q_l) \right| &\leq 2^{\frac{j}{2}} \|\phi\|_\infty \sup_l \sum_k \int_{I_{jk}} |p_l - q_l| \\
&\leq 2L \left(\int p_l + \int q_l \right) \|\phi\|_\infty 2^{\frac{j}{2}} \\
&= 4L \|\phi\|_\infty 2^{\frac{j}{2}}.
\end{aligned}$$

Clearly, for any $1 \leq i \leq n$,

$$\begin{aligned}
\sup_l |c_l(i)| &\leq \sup_l \sqrt{\sum_i c_l^2(i)} \\
&\leq \sqrt{n \sum_l \langle c_l, c_l \rangle_n}.
\end{aligned}$$

□

Lemma 6.6 *Let $p_l, q_l, p_{l'}$ and $q_{l'}$ be four probability densities in \mathbb{L}_2 . Then, for any $j \in \mathbb{N}$*

$$\begin{aligned}
\sum_k \left(\int \phi_{jk} p_l - \int \phi_{jk} q_l \right)^2 &\leq 2L \|p_l - q_l\|_2^2; \\
\sum_k \sum_{k': I_{jk} \cap I_{jk'} \neq \emptyset} \left| \left(\int \phi_{jk} p_l - \int \phi_{jk} q_l \right) \left(\int \phi_{jk'} p_{l'} - \int \phi_{jk'} q_{l'} \right) \right| &\leq 4L^2 (\|p_l - q_l\|_2^2 + \|p_{l'} - q_{l'}\|_2^2).
\end{aligned}$$

Proof of Lemma 6.6:

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\sum_k \left(\int \phi_{jk} p_l - \int \phi_{jk} q_l \right)^2 &\leq \sum_k \int_{I_{jk}} (p_l - q_l)^2 \\
&\leq 2L \|p_l - q_l\|_2^2.
\end{aligned}$$

Lemma 6.3 entails that

$$\begin{aligned}
& \sum_k \sum_{k': I_{jk} \cap I_{jk'} \neq \emptyset} \left| \left(\int \phi_{jk} p_l - \int \phi_{jk} q_l \right) \left(\int \phi_{jk'} p_{l'} - \int \phi_{jk'} q_{l'} \right) \right| \\
& \leq \frac{1}{2} \left[\sum_k \sum_{k': I_{jk} \cap I_{jk'} \neq \emptyset} \left(\int \phi_{jk} p_l - \int \phi_{jk} q_l \right)^2 + \sum_k \sum_{k': I_{jk} \cap I_{jk'} \neq \emptyset} \left(\int \phi_{jk} p_{l'} - \int \phi_{jk} q_{l'} \right)^2 \right] \\
& \leq \frac{1}{2} \left[4L \sum_k \int \phi_{jk}^2 \int_{I_{jk}} (p_l - q_l)^2 + 4L \sum_k \int \phi_{jk}^2 \int_{I_{jk}} (p_{l'} - q_{l'})^2 \right] \\
& \leq \frac{1}{2} (8L^2 \|p_l - q_l\|_2^2 + 8L^2 \|p_{l'} - q_{l'}\|_2^2) \\
& \leq 4L^2 (\|p_l - q_l\|_2^2 + \|p_{l'} - q_{l'}\|_2^2).
\end{aligned}$$

□

Lemma 6.7 *There exists a constant $\bar{C}_T = \bar{C}_T(R, L, \|\phi\|_\infty) > 0$ such that*

$$A_1 := \sum_{i_1 \neq i_2} \text{Var}_{\bar{p}, \bar{q}}(h_j(i_1, i_2)) \leq \bar{C}_T \frac{M^2}{K^2} 2^j n^2.$$

Proof of Lemma 6.7:

Let us evaluate each variance

$$\text{Var}_{\bar{p}, \bar{q}}(h_j(i_1, i_2)) = \text{Cov}(h_j(i_1, i_2), h_j(i_1, i_2)).$$

We expand the covariance

$$\begin{aligned}
& \mathbb{C}ov((a_l(i_1)\phi_{jk}(Y_{i_1}) - b_l(i_1)\phi_{jk}(Z_{i_1}))(a_l(i_2)\phi_{jk}(Y_{i_2}) - b_l(i_2)\phi_{jk}(Z_{i_2})), \\
& (a_{l'}(i_1)\phi_{jk'}(Y_{i_1}) - b_{l'}(i_1)\phi_{jk'}(Z_{i_1}))(a_{l'}(i_2)\phi_{jk'}(Y_{i_2}) - b_{l'}(i_2)\phi_{jk'}(Z_{i_2}))) \\
&= \mathbb{C}ov(a_l(i_1)\phi_{jk}(Y_{i_1})a_l(i_2)\phi_{jk}(Y_{i_2}), a_{l'}(i_1)\phi_{jk'}(Y_{i_1})a_{l'}(i_2)\phi_{jk'}(Y_{i_2})) \\
&\quad - \mathbb{C}ov(a_l(i_1)\phi_{jk}(Y_{i_1})a_l(i_2)\phi_{jk}(Y_{i_2}), a_{l'}(i_1)\phi_{jk'}(Y_{i_1})b_{l'}(i_2)\phi_{jk'}(Z_{i_2})) \\
&\quad - \mathbb{C}ov(a_l(i_1)\phi_{jk}(Y_{i_1})a_l(i_2)\phi_{jk}(Y_{i_2}), b_{l'}(i_1)\phi_{jk'}(Z_{i_1})a_{l'}(i_2)\phi_{jk'}(Y_{i_2})) \\
&\quad + \mathbb{C}ov(a_l(i_1)\phi_{jk}(Y_{i_1})a_l(i_2)\phi_{jk}(Y_{i_2}), b_{l'}(i_1)\phi_{jk'}(Z_{i_1})b_{l'}(i_2)\phi_{jk'}(Z_{i_2})) \\
&\quad - \mathbb{C}ov(a_l(i_1)\phi_{jk}(Y_{i_1})b_l(i_2)\phi_{jk}(Z_{i_2}), a_{l'}(i_1)\phi_{jk'}(Y_{i_1})a_{l'}(i_2)\phi_{jk'}(Y_{i_2})) \\
&\quad + \mathbb{C}ov(a_l(i_1)\phi_{jk}(Y_{i_1})b_l(i_2)\phi_{jk}(Z_{i_2}), a_{l'}(i_1)\phi_{jk'}(Y_{i_1})b_{l'}(i_2)\phi_{jk'}(Z_{i_2})) \\
&\quad + \mathbb{C}ov(a_l(i_1)\phi_{jk}(Y_{i_1})b_l(i_2)\phi_{jk}(Z_{i_2}), b_{l'}(i_1)\phi_{jk'}(Z_{i_1})a_{l'}(i_2)\phi_{jk'}(Y_{i_2})) \\
&\quad - \mathbb{C}ov(a_l(i_1)\phi_{jk}(Y_{i_1})b_l(i_2)\phi_{jk}(Z_{i_2}), b_{l'}(i_1)\phi_{jk'}(Z_{i_1})b_{l'}(i_2)\phi_{jk'}(Z_{i_2})) \\
&\quad - \mathbb{C}ov(b_l(i_1)\phi_{jk}(Z_{i_1})a_l(i_2)\phi_{jk}(Y_{i_2}), a_{l'}(i_1)\phi_{jk'}(Y_{i_1})a_{l'}(i_2)\phi_{jk'}(Y_{i_2})) \\
&\quad + \mathbb{C}ov(b_l(i_1)\phi_{jk}(Z_{i_1})a_l(i_2)\phi_{jk}(Y_{i_2}), a_{l'}(i_1)\phi_{jk'}(Y_{i_1})b_{l'}(i_2)\phi_{jk'}(Z_{i_2})) \\
&\quad + \mathbb{C}ov(b_l(i_1)\phi_{jk}(Z_{i_1})a_l(i_2)\phi_{jk}(Y_{i_2}), b_{l'}(i_1)\phi_{jk'}(Z_{i_1})a_{l'}(i_2)\phi_{jk'}(Y_{i_2})) \\
&\quad - \mathbb{C}ov(b_l(i_1)\phi_{jk}(Z_{i_1})a_l(i_2)\phi_{jk}(Y_{i_2}), b_{l'}(i_1)\phi_{jk'}(Z_{i_1})b_{l'}(i_2)\phi_{jk'}(Z_{i_2})) \\
&\quad + \mathbb{C}ov(b_l(i_1)\phi_{jk}(Z_{i_1})b_l(i_2)\phi_{jk}(Z_{i_2}), a_{l'}(i_1)\phi_{jk'}(Y_{i_1})a_{l'}(i_2)\phi_{jk'}(Y_{i_2})) \\
&\quad - \mathbb{C}ov(b_l(i_1)\phi_{jk}(Z_{i_1})b_l(i_2)\phi_{jk}(Z_{i_2}), a_{l'}(i_1)\phi_{jk'}(Y_{i_1})b_{l'}(i_2)\phi_{jk'}(Z_{i_2})) \\
&\quad - \mathbb{C}ov(b_l(i_1)\phi_{jk}(Z_{i_1})b_l(i_2)\phi_{jk}(Z_{i_2}), b_{l'}(i_1)\phi_{jk'}(Z_{i_1})a_{l'}(i_2)\phi_{jk'}(Y_{i_2})) \\
&\quad + \mathbb{C}ov(b_l(i_1)\phi_{jk}(Z_{i_1})b_l(i_2)\phi_{jk}(Z_{i_2}), b_{l'}(i_1)\phi_{jk'}(Z_{i_1})b_{l'}(i_2)\phi_{jk'}(Z_{i_2})).
\end{aligned}$$

According to independence arguments, the following terms are clearly equal to zero:

$$\begin{aligned}
& \mathbb{C}ov(a_l(i_1)\phi_{jk}(Y_{i_1})a_l(i_2)\phi_{jk}(Y_{i_2}), b_{l'}(i_1)\phi_{jk'}(Z_{i_1})b_{l'}(i_2)\phi_{jk'}(Z_{i_2})), \\
& \mathbb{C}ov(a_l(i_1)\phi_{jk}(Y_{i_1})b_l(i_2)\phi_{jk}(Z_{i_2}), b_{l'}(i_1)\phi_{jk'}(Z_{i_1})a_{l'}(i_2)\phi_{jk'}(Y_{i_2})), \\
& \mathbb{C}ov(b_l(i_1)\phi_{jk}(Z_{i_1})a_l(i_2)\phi_{jk}(Y_{i_2}), a_{l'}(i_1)\phi_{jk'}(Y_{i_1})b_{l'}(i_2)\phi_{jk'}(Z_{i_2})), \\
& \mathbb{C}ov(b_l(i_1)\phi_{jk}(Z_{i_1})b_l(i_2)\phi_{jk}(Z_{i_2}), a_{l'}(i_1)\phi_{jk'}(Y_{i_1})a_{l'}(i_2)\phi_{jk'}(Y_{i_2})).
\end{aligned}$$

The remaining terms can be split into two types: those involving two different random variables and those involving three different random variables. Let us handle these two cases separately. First, we consider the case with two different random variables. We need to bound terms such as

$$\begin{aligned}
& \sum_{i_1 \neq i_2} \sum_{k, k'} \mathbb{C}ov(a_l(i_1)\phi_{jk}(Y_{i_1})a_l(i_2)\phi_{jk}(Y_{i_2}), a_{l'}(i_1)\phi_{jk'}(Y_{i_1})a_{l'}(i_2)\phi_{jk'}(Y_{i_2})) \\
&= \sum_{i_1 \neq i_2} \sum_{k, k'} a_l(i_1)a_l(i_2)a_{l'}(i_1)a_{l'}(i_2) \mathbb{E}(\phi_{jk}(Y_{i_1})\phi_{jk'}(Y_{i_1})) \mathbb{E}(\phi_{jk}(Y_{i_2})\phi_{jk'}(Y_{i_2})) \\
&\quad - \sum_{i_1 \neq i_2} \sum_{k, k'} a_l(i_1)a_l(i_2)a_{l'}(i_1)a_{l'}(i_2) \mathbb{E}(\phi_{jk}(Y_{i_1})) \mathbb{E}(\phi_{jk'}(Y_{i_1})) \mathbb{E}(\phi_{jk}(Y_{i_2})) \mathbb{E}(\phi_{jk'}(Y_{i_2})).
\end{aligned}$$

As the wavelets are compactly supported, we get for any (i_1, i_2) ,

$$\begin{aligned}
& \left| \sum_{k,k'} a_l(i_1) a_l(i_2) a_{l'}(i_1) a_{l'}(i_2) \mathbb{E}(\phi_{jk}(Y_{i_1}) \phi_{jk'}(Y_{i_1})) \mathbb{E}(\phi_{jk}(Y_{i_2}) \phi_{jk'}(Y_{i_2})) \right| \\
& \leq \|f_{i_1}\|_\infty \sum_{k,k'} |a_l(i_1) a_l(i_2) a_{l'}(i_1) a_{l'}(i_2)| \int |\phi_{jk} \phi_{jk'}| f_{i_2} \\
& \leq 2^{j+3} L^2 \|\phi\|_\infty^2 \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty) |a_l(i_1) a_l(i_2) a_{l'}(i_1) a_{l'}(i_2)|.
\end{aligned}$$

The second sum is much simpler to bound. According to lemma 6.4 it can be bounded as follows

$$\begin{aligned}
& \left| \sum_{k,k'} a_l(i_1) a_l(i_2) a_{l'}(i_1) a_{l'}(i_2) \mathbb{E}(\phi_{jk}(Y_{i_1})) \mathbb{E}(\phi_{jk'}(Y_{i_1})) \mathbb{E}(\phi_{jk}(Y_{i_2})) \mathbb{E}(\phi_{jk'}(Y_{i_2})) \right| \\
& \leq \sum_{k,k'} |a_l(i_1) a_l(i_2) a_{l'}(i_1) a_{l'}(i_2)| \mathbb{E}(|\phi_{jk}(Y_{i_1})|) \mathbb{E}(|\phi_{jk'}(Y_{i_1})|) \left(\sqrt{2L} 2^{-\frac{j}{2}}\right)^2 \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty) \\
& = L 2^{1-j} \sum_{k,k'} |a_l(i_1) a_l(i_2) a_{l'}(i_1) a_{l'}(i_2)| \int_{I_{jk}} |\phi_{jk}| f_{i_1} \int_{I_{jk'}} |\phi_{jk'}| f_{i_1} \left(\sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty)\right) \\
& \leq 8L^3 |a_l(i_1) a_l(i_2) a_{l'}(i_1) a_{l'}(i_2)| \|\phi\|_\infty^2 \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty).
\end{aligned}$$

Let us now focus on the sums over i_1, i_2, l and l' .

$$\begin{aligned}
\sum_{i_1 \neq i_2} \sum_{l, l'} |a_l(i_1) a_l(i_2) a_{l'}(i_1) a_{l'}(i_2)| & \leq \sum_{i_1, i_2} \sum_{l, l'} \frac{1}{2} (a_l(i_1)^2 a_{l'}(i_2)^2 + a_{l'}(i_1)^2 a_l(i_2)^2) \\
& \leq n^2 \sum_{l, l'} \langle a_l, a_l \rangle_n \langle a_{l'}, a_{l'} \rangle_n \\
& \leq \frac{M^2 n^2}{K^2}.
\end{aligned}$$

We see that this term behaves like n^2 . The three other terms featuring only two different random variables are handled in the same way.

Therefore it remains to evaluate the eight terms with three different random variables. For example, let us consider

$$\mathbb{Cov}(a_l(i_1) \phi_{jk}(Y_{i_1}) a_l(i_2) \phi_{jk}(Y_{i_2}), a_{l'}(i_1) \phi_{jk'}(Y_{i_1}) a_{l'}(i_2) \phi_{jk'}(Z_{i_2})),$$

and let us omit for a moment the sums over i_1, i_2, k, k', l and l' . The covariance can be expanded as

$$\mathbb{Cov}(\phi_{jk}(Y_{i_1}) \phi_{jk}(Y_{i_2}), \phi_{jk'}(Y_{i_1}) \phi_{jk'}(Z_{i_2})) = \mathbb{E}(\phi_{jk'}(Z_{i_2})) \mathbb{E}(\phi_{jk}(Y_{i_2})) \mathbb{Cov}(\phi_{jk}(Y_{i_1}), \phi_{jk'}(Y_{i_1})).$$

When we add the sums over k and k' , the second term is exactly handled as the second term above in the case of two different random variables. Thus,

it remains to consider the first summand. As above, the compactness of the wavelet entails that

$$\begin{aligned}
& \left| \sum_{k,k'} \mathbb{E}(\phi_{jk'}(Z_{i_2})) \mathbb{E}(\phi_{jk}(Y_{i_2})) \text{Cov}(\phi_{jk}(Y_{i_1}), \phi_{jk'}(Y_{i_1})) \right| \\
& \leq \sum_{k,k'} |\mathbb{E}(\phi_{jk'}(Z_{i_2})) \mathbb{E}(\phi_{jk}(Y_{i_2})) \mathbb{E}(\phi_{jk}(Y_{i_1})) \mathbb{E}(\phi_{jk'}(Y_{i_1}))| \\
& \quad + \sum_{k,k'} |\mathbb{E}(\phi_{jk'}(Z_{i_2})) \mathbb{E}(\phi_{jk}(Y_{i_2})) \mathbb{E}(\phi_{jk}(Y_{i_1})\phi_{jk'}(Y_{i_1}))| \\
& = A_{11} + A_{12},
\end{aligned}$$

According to lemmas 6.4 and 6.5, we have

$$\begin{aligned}
A_{11} &= \sum_{k,k'} |\mathbb{E}(\phi_{jk'}(Z_{i_2})) \mathbb{E}(\phi_{jk}(Y_{i_2})) \mathbb{E}(\phi_{jk}(Y_{i_1})) \mathbb{E}(\phi_{jk'}(Y_{i_1}))| \\
&\leq \left(2^{1-j} L \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty) \right) \sum_k \int |\phi_{jk} g_{i_2}| \sum_{k'} \int |\phi_{jk'} f_{i_2}| \\
&\leq 8L^3 \|\phi\|_\infty^2 \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty)
\end{aligned}$$

and

$$\begin{aligned}
A_{12} &= \sum_{k,k'} |\mathbb{E}(\phi_{jk'}(Z_{i_2})) \mathbb{E}(\phi_{jk}(Y_{i_2})) \mathbb{E}(\phi_{jk}(Y_{i_1})\phi_{jk'}(Y_{i_1}))| \\
&\leq \left(2^{1-j} L \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty) \right) \sum_{k,k'} |\mathbb{E}(\phi_{jk}(Y_{i_1})\phi_{jk'}(Y_{i_1}))| \\
&\leq 4L \left(2^{1-j} L \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty) \right) 2^{\frac{j}{2}} \|\phi\|_\infty \sum_k \int |\phi_{jk} f_{i_1}| \\
&\leq 8L^2 \left(2L \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty) \right) \|\phi\|_\infty^2 \int f_{i_1} \\
&\leq 16L^3 \|\phi\|_\infty^2 \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty).
\end{aligned}$$

It remains to sum over i_1 and i_2 as the sums over l and l' are not important (they only change the constant). We have

$$\begin{aligned}
\sum_{i_1 \neq i_2} \sum_{l,l'} |a_l(i_1) a_l(i_2) a_{l'}(i_1) b_{l'}(i_2)| &\leq \sum_{i_1, i_2} \sum_{l, l'} |a_l(i_1) a_l(i_2) a_{l'}(i_1) b_{l'}(i_2)| \\
&\leq \frac{1}{2} \sum_{l, l'} \left(\sum_{i_1, i_2} a_l(i_1)^2 b_{l'}(i_2)^2 + \sum_{i_1, i_2} a_{l'}(i_1)^2 a_l(i_2)^2 \right) \\
&= \frac{n^2}{2} \sum_{l, l'} (\langle a_l, a_l \rangle_n \langle b_{l'}, b_{l'} \rangle_n + \langle a_{l'}, a_{l'} \rangle_n \langle a_l, a_l \rangle_n) \\
&\leq \frac{M^2 n^2}{K^2}.
\end{aligned}$$

Clearly, this term behaves like n^2 . The other covariances involving three random variables are handled exactly in the same way.

By combining all the previous bounds, we conclude that

$$A_1 \leq (k_1 + k_2 2^j) \frac{M^2 n^2}{K^2}, \quad \text{with } k_1 = 224RL^3 \|\phi\|_\infty^2, \quad k_2 = 32RL^2 \|\phi\|_\infty^2.$$

As a consequence if we write $\tilde{C}_T = k_1 + k_2$ one gets

$$A_1 \leq \tilde{C}_T \frac{M^2}{K^2} 2^j n^2.$$

□

Lemma 6.8 *There exists a constant $\tilde{C}_T = \tilde{C}_T(R, L, \|\phi\|_\infty) > 0$ such that for any $j \in \mathbb{N}$*

$$A_3 := \sum_{i_1 \neq i_2 \neq i_3} \text{Cov}(h_j(i_1, i_2), h_j(i_1, i_3)) \leq \tilde{C}_T \frac{M^2}{K^2} \left[n^3 \sum_l \|p_l - q_l\|_2^2 + 2^{\frac{j}{2}} n^{\frac{5}{2}} \sum_l \|p_l - q_l\|_2 \right].$$

Proof of Lemma 6.8:

Clearly, the term A_3 can be bounded as follows

$$\begin{aligned} A_3 &= \sum_{i_1 \neq i_2 \neq i_3} \text{Cov}(h_j(i_1, i_2), h_j(i_1, i_3)) \\ &= \sum_{i_1 \neq i_2 \neq i_3} \sum_{k, k'} \sum_{l, l'} \text{Cov}((a_l(i_1)\phi_{jk}(Y_{i_1}) - b_l(i_1)\phi_{jk}(Z_{i_1}))(a_l(i_2)\phi_{jk}(Y_{i_2}) - b_l(i_2)\phi_{jk}(Z_{i_2})), \\ &\quad (a_{l'}(i_1)\phi_{jk'}(Y_{i_1}) - b_{l'}(i_1)\phi_{jk'}(Z_{i_1}))(a_{l'}(i_3)\phi_{jk'}(Y_{i_3}) - b_{l'}(i_3)\phi_{jk'}(Z_{i_3}))) \\ &= \sum_{i_1 \neq i_2 \neq i_3} \sum_{k, k'} \sum_{l, l'} \text{Cov}(a_l(i_1)\phi_{jk}(Y_{i_1}) - b_l(i_1)\phi_{jk}(Z_{i_1}), a_{l'}(i_1)\phi_{jk'}(Y_{i_1}) - b_{l'}(i_1)\phi_{jk'}(Z_{i_1})) \\ &\quad \times \mathbb{E}(a_l(i_2)\phi_{jk}(Y_{i_2}) - b_l(i_2)\phi_{jk}(Z_{i_2})) \mathbb{E}(a_{l'}(i_3)\phi_{jk'}(Y_{i_3}) - b_{l'}(i_3)\phi_{jk'}(Z_{i_3})) \\ &= \sum_{i_1, i_2, i_3} \sum_{k, k'} \sum_{l, l'} \text{Cov}(a_l(i_1)\phi_{jk}(Y_{i_1}) - b_l(i_1)\phi_{jk}(Z_{i_1}), a_{l'}(i_1)\phi_{jk'}(Y_{i_1}) - b_{l'}(i_1)\phi_{jk'}(Z_{i_1})) \\ &\quad \mathbb{E}(a_l(i_2)\phi_{jk}(Y_{i_2}) - b_l(i_2)\phi_{jk}(Z_{i_2})) \mathbb{E}(a_{l'}(i_3)\phi_{jk'}(Y_{i_3}) - b_{l'}(i_3)\phi_{jk'}(Z_{i_3})) \\ &\quad - \sum_{i_1=i_2, i_3} \sum_{k, k'} \sum_{l, l'} \text{Cov}(a_l(i_1)\phi_{jk}(Y_{i_1}) - b_l(i_1)\phi_{jk}(Z_{i_1}), a_{l'}(i_1)\phi_{jk'}(Y_{i_1}) - b_{l'}(i_1)\phi_{jk'}(Z_{i_1})) \\ &\quad \mathbb{E}(a_l(i_2)\phi_{jk}(Y_{i_2}) - b_l(i_2)\phi_{jk}(Z_{i_2})) \mathbb{E}(a_{l'}(i_3)\phi_{jk'}(Y_{i_3}) - b_{l'}(i_3)\phi_{jk'}(Z_{i_3})) \\ &\quad - \sum_{i_1=i_3, i_2} \sum_{k, k'} \sum_{l, l'} \text{Cov}(a_l(i_1)\phi_{jk}(Y_{i_1}) - b_l(i_1)\phi_{jk}(Z_{i_1}), a_{l'}(i_1)\phi_{jk'}(Y_{i_1}) - b_{l'}(i_1)\phi_{jk'}(Z_{i_1})) \\ &\quad \mathbb{E}(a_l(i_2)\phi_{jk}(Y_{i_2}) - b_l(i_2)\phi_{jk}(Z_{i_2})) \mathbb{E}(a_{l'}(i_3)\phi_{jk'}(Y_{i_3}) - b_{l'}(i_3)\phi_{jk'}(Z_{i_3})) \\ &\quad + \sum_{i_1=i_2=i_3} \sum_{k, k'} \sum_{l, l'} \text{Cov}(a_l(i_1)\phi_{jk}(Y_{i_1}) - b_l(i_1)\phi_{jk}(Z_{i_1}), a_{l'}(i_1)\phi_{jk'}(Y_{i_1}) - b_{l'}(i_1)\phi_{jk'}(Z_{i_1})) \\ &\quad \mathbb{E}(a_l(i_2)\phi_{jk}(Y_{i_2}) - b_l(i_2)\phi_{jk}(Z_{i_2})) \mathbb{E}(a_{l'}(i_3)\phi_{jk'}(Y_{i_3}) - b_{l'}(i_3)\phi_{jk'}(Z_{i_3})) \\ &= A_{31} - A_{32} - A_{33} + A_{34} \\ &\leq |A_{31}| + |A_{32}| + |A_{33}| + |A_{34}|. \end{aligned}$$

We will separately bound each term.

Let us start with $|A_{31}|$. The first step is to expand the covariance.

$$\begin{aligned}
|A_{31}| &= \left| \sum_{i_1, i_2, i_3} \sum_{k, k'} \sum_{l, l'} \text{Cov}(a_l(i_1)\phi_{jk}(Y_{i_1}) - b_l(i_1)\phi_{jk}(Z_{i_1}), a_{l'}(i_1)\phi_{jk'}(Y_{i_1}) - b_{l'}(i_1)\phi_{jk'}(Z_{i_1})) \right. \\
&\quad \left. \mathbb{E}(a_l(i_2)\phi_{jk}(Y_{i_2}) - b_l(i_2)\phi_{jk}(Z_{i_2})) \right| \\
&= n^2 \left| \sum_{i_1} \sum_{k, k'} \sum_{l, l'} \text{Cov}((a_l(i_1)\phi_{jk}(Y_{i_1}) - b_l(i_1)\phi_{jk}(Z_{i_1})), (a_{l'}(i_1)\phi_{jk'}(Y_{i_1}) - b_{l'}(i_1)\phi_{jk'}(Z_{i_1}))) \right. \\
&\quad \left. \left(\int \phi_{jk} p_l - \int \phi_{jk} q_l \right) \left(\int \phi_{jk'} p_{l'} - \int \phi_{jk'} q_{l'} \right) \right| \\
&= n^2 \left| \sum_{i_1} \sum_{k, k'} \sum_{l, l'} [\mathbb{E}(a_l(i_1)\phi_{jk}(Y_{i_1})a_{l'}(i_1)\phi_{jk'}(Y_{i_1})) + \mathbb{E}(b_l(i_1)\phi_{jk}(Z_{i_1})b_{l'}(i_1)\phi_{jk'}(Z_{i_1})) \right. \\
&\quad \left. - \mathbb{E}(a_l(i_1)\phi_{jk}(Y_{i_1}))\mathbb{E}(a_{l'}(i_1)\phi_{jk'}(Y_{i_1})) - \mathbb{E}(b_l(i_1)\phi_{jk}(Z_{i_1}))\mathbb{E}(b_{l'}(i_1)\phi_{jk'}(Z_{i_1}))] \right. \\
&\quad \left. \left(\int \phi_{jk} p_l - \int \phi_{jk} q_l \right) \left(\int \phi_{jk'} p_{l'} - \int \phi_{jk'} q_{l'} \right) \right|.
\end{aligned}$$

The first two terms involve only one expectation and can be bounded in the same way. Therefore let us bound the quantity

$$\left| \sum_{i_1} \sum_{k, k'} \sum_{l, l'} \mathbb{E}(a_l(i_1)\phi_{jk}(Y_{i_1})a_{l'}(i_1)\phi_{jk'}(Y_{i_1})) \left(\int \phi_{jk} p_l - \int \phi_{jk} q_l \right) \left(\int \phi_{jk'} p_{l'} - \int \phi_{jk'} q_{l'} \right) \right|.$$

$$\text{Clearly } \sum_{i_1} |a_l(i_1)a_{l'}(i_1)| \leq n\sqrt{\langle a_l, a_l \rangle_n \langle a_{l'}, a_{l'} \rangle_n} \leq \frac{M}{K} n.$$

Since $|\mathbb{E}(\phi_{jk}(Y_{i_1})\phi_{jk'}(Y_{i_1}))| \leq \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty)$, lemma 6.6 entails that

$$\sum_k \sum_{k': I_{jk} \cap I_{jk'} \neq \emptyset} \left| \left(\int \phi_{jk} p_l - \int \phi_{jk} q_l \right) \left(\int \phi_{jk'} p_{l'} - \int \phi_{jk'} q_{l'} \right) \right| \leq 4L^2 (\|p_l - q_l\|_2^2 + \|p_{l'} - q_{l'}\|_2^2).$$

Then one deduces that for any $1 \leq i_1 \leq n$

$$\begin{aligned}
&\sum_{k, k'} \sum_{l, l'} |\mathbb{E}(\phi_{jk}(Y_{i_1})\phi_{jk'}(Y_{i_1}))| \left| \left(\int \phi_{jk} p_l - \int \phi_{jk} q_l \right) \left(\int \phi_{jk'} p_{l'} - \int \phi_{jk'} q_{l'} \right) \right| \\
&\leq 8L^2 \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty) \sum_l \|p_l - q_l\|_2^2.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left| \sum_{i_1} \sum_{k, k'} \sum_{l, l'} \mathbb{E}(a_l(i_1)\phi_{jk}(Y_{i_1})a_{l'}(i_1)\phi_{jk'}(Y_{i_1})) \left(\int \phi_{jk} p_l - \int \phi_{jk} q_l \right) \left(\int \phi_{jk'} p_{l'} - \int \phi_{jk'} q_{l'} \right) \right| \\
&\leq \frac{8ML^2}{K} \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty) \sum_l \|p_l - q_l\|_2^2 n.
\end{aligned}$$

Now we come to the last two terms which involve two expectations. Let us consider for example the quantity

$$\begin{aligned}
& \left| \sum_{k,k'} \mathbb{E}(\phi_{jk}(Y_{i_1})) \mathbb{E}(\phi_{jk'}(Y_{i_1})) \left(\int \phi_{jk} p_l - \int \phi_{jk} q_l \right) \left(\int \phi_{jk'} p_{l'} - \int \phi_{jk'} q_{l'} \right) \right| \\
& \leq \sum_{k,k'} |\mathbb{E}(\phi_{jk}(Y_{i_1})) \mathbb{E}(\phi_{jk'}(Y_{i_1}))| \frac{1}{2} \left\{ \left(\int \phi_{jk} p_l - \int \phi_{jk} q_l \right)^2 + \left(\int \phi_{jk'} p_{l'} - \int \phi_{jk'} q_{l'} \right)^2 \right\} \\
& \leq \sup_{l,l'} \left[\sqrt{2L} (\|p_l\|_\infty \vee \|q_l\|_\infty)^{\frac{1}{2}} 2^{-\frac{j}{2}} \sum_k |\mathbb{E}(\phi_{jk}(Y_{i_1}))| \sum_{k'} \left(\int \phi_{jk'} p_{l'} - \int \phi_{jk'} q_{l'} \right)^2 \right] \\
& \leq 4\sqrt{2L}^{\frac{5}{2}} \|\phi\|_\infty \left(\sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty) \right)^{\frac{1}{2}} \sup_{l'} \|p_{l'} - q_{l'}\|_2^2.
\end{aligned}$$

Last inequalities are obtained by using lemma 6.6 for any $1 \leq i_1 \leq n$. Hence

$$\begin{aligned}
& \left| \sum_{i_1} \sum_{k,k'} \sum_{l,l'} \mathbb{E}(a_l(i_1) \phi_{jk}(Y_{i_1})) \mathbb{E}(a_{l'}(i_1) \phi_{jk'}(Y_{i_1})) \left(\int \phi_{jk} p_l - \int \phi_{jk} q_l \right) \left(\int \phi_{jk'} p_{l'} - \int \phi_{jk'} q_{l'} \right) \right| \\
& \leq \frac{4\sqrt{2}M}{K} L^{\frac{5}{2}} \|\phi\|_\infty \left(\sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty) \right)^{\frac{1}{2}} \sup_{l'} \|p_{l'} - q_{l'}\|_2^2 n.
\end{aligned}$$

Therefore the two last bounds entail that

$$|A_{31}| \leq c_{31} \frac{M^2 n^3}{K} \sum_l \|p_l - q_l\|_2^2, \quad \text{where } c_{31} = 4L^2 \sqrt{R} \left(2\sqrt{R} + \sqrt{2L} \|\phi\|_\infty \right).$$

The way to bound A_{32} and A_{33} is trickier. We have

$$\begin{aligned}
|A_{32}| & \leq \left| \sum_{l,l'} \sum_{i_1, i_2} \sum_{k,k'} [a_l(i_1) a_{l'}(i_1) \text{Cov}(\phi_{jk}(Y_{i_1}), \phi_{jk'}(Y_{i_1})) + b_l(i_1) b_{l'}(i_1) \text{Cov}(\phi_{jk}(Z_{i_1}), \phi_{jk'}(Z_{i_1}))] \right. \\
& \quad \left. [a_l(i_2) \mathbb{E}(\phi_{jk}(Y_{i_2})) - b_l(i_2) \mathbb{E}(\phi_{jk}(Z_{i_2}))] [a_{l'}(i_2) \mathbb{E}(\phi_{jk'}(Y_{i_2})) - b_{l'}(i_2) \mathbb{E}(\phi_{jk'}(Z_{i_2}))] \right| \\
& \leq \left| \sum_{l,l'} \sum_{i_1} \sum_{k,k'} [a_l(i_1) a_{l'}(i_1) \text{Cov}(\phi_{jk}(Y_{i_1}), \phi_{jk'}(Y_{i_1})) + b_l(i_1) b_{l'}(i_1) \text{Cov}(\phi_{jk}(Z_{i_1}), \phi_{jk'}(Z_{i_1}))] \right. \\
& \quad \left. [a_l(i_1) \mathbb{E}(\phi_{jk}(Y_{i_1})) - b_l(i_1) \mathbb{E}(\phi_{jk}(Z_{i_1}))] \left(n \int \phi_{jk'} p_{l'} - n \int \phi_{jk'} q_{l'} \right) \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \left| \sum_{l,l'} \sum_{i_1} \sum_{k,k'} a_l(i_1) a_{l'}(i_1) \mathbb{E}(\phi_{jk}(Y_{i_1}) \phi_{jk'}(Y_{i_1})) \right. \\
&\quad \left. (a_l(i_1) \mathbb{E}(\phi_{jk}(Y_{i_1})) - b_l(i_1) \mathbb{E}(\phi_{jk}(Z_{i_1}))) \left(n \int \phi_{jk'} p_{l'} - n \int \phi_{jk'} q_{l'} \right) \right| \\
&\quad + \left| \sum_{l,l'} \sum_{i_1} \sum_{k,k'} a_l(i_1) a_{l'}(i_1) \mathbb{E}(\phi_{jk}(Y_{i_1})) \mathbb{E}(\phi_{jk'}(Y_{i_1})) \right. \\
&\quad \left. (a_l(i_1) \mathbb{E}(\phi_{jk}(Y_{i_1})) - b_l(i_1) \mathbb{E}(\phi_{jk}(Z_{i_1}))) \left(n \int \phi_{jk'} p_{l'} - n \int \phi_{jk'} q_{l'} \right) \right| \\
&\quad + \left| \sum_{l,l'} \sum_{i_1} \sum_{k,k'} b_l(i_1) b_{l'}(i_1) \mathbb{E}(\phi_{jk}(Z_{i_1}) \phi_{jk'}(Z_{i_1})) \right. \\
&\quad \left. (a_l(i_1) \mathbb{E}(\phi_{jk}(Y_{i_1})) - b_l(i_1) \mathbb{E}(\phi_{jk}(Z_{i_1}))) \left(n \int \phi_{jk'} p_{l'} - n \int \phi_{jk'} q_{l'} \right) \right| \\
&\quad + \left| \sum_{l,l'} \sum_{i_1} \sum_{k,k'} b_l(i_1) b_{l'}(i_1) \mathbb{E}(\phi_{jk}(Z_{i_1})) \mathbb{E}(\phi_{jk'}(Z_{i_1})) \right. \\
&\quad \left. (a_l(i_1) \mathbb{E}(\phi_{jk}(Y_{i_1})) - b_l(i_1) \mathbb{E}(\phi_{jk}(Z_{i_1}))) \left(n \int \phi_{jk'} p_{l'} - n \int \phi_{jk'} q_{l'} \right) \right|.
\end{aligned}$$

The calculations are rather lengthy and involve eight terms. But the bright side is that the terms can be split into two groups. There are terms involving two expectations such as

$$\sum_{l,l'} \sum_{i_1} \sum_{k,k'} a_l(i_1) a_{l'}(i_1) \mathbb{E}(\phi_{jk}(Y_{i_1}) \phi_{jk'}(Y_{i_1})) a_l(i_1) \mathbb{E}(\phi_{jk}(Y_{i_1})) \left(n \int \phi_{jk'} p_{l'} - n \int \phi_{jk'} q_{l'} \right),$$

and terms involving three expectations such as

$$\sum_{l,l'} \sum_{i_1} \sum_{k,k'} a_l(i_1) a_{l'}(i_1) \mathbb{E}(\phi_{jk}(Y_{i_1})) \mathbb{E}(\phi_{jk'}(Y_{i_1})) a_l(i_1) \mathbb{E}(\phi_{jk}(Y_{i_1})) \left(n \int \phi_{jk'} p_{l'} - n \int \phi_{jk'} q_{l'} \right).$$

Still using lemmas 6.4 and 6.5, we have

$$\begin{aligned}
&\left| \sum_{l,l'} \sum_{i_1} \sum_{k,k'} a_l(i_1) a_{l'}(i_1) \mathbb{E}(\phi_{jk}(Y_{i_1}) \phi_{jk'}(Y_{i_1})) a_l(i_1) \mathbb{E}(\phi_{jk}(Y_{i_1})) \left(n \int \phi_{jk'} p_{l'} - n \int \phi_{jk'} q_{l'} \right) \right| \\
&\leq 8\sqrt{2} L^{\frac{5}{2}} \|\phi\|_{\infty}^2 \sup_l \|p_l\|_{\infty}^{\frac{1}{2}} \sum_{l=1}^M \|p_l - q_l\|_2 \left(\sum_{l=1}^M \langle a_l, a_l \rangle_n \right)^{\frac{3}{2}} 2^{\frac{j}{2}} n^{\frac{5}{2}} \\
&\leq 8\sqrt{\frac{2M^3}{K^3}} L^{\frac{5}{2}} \|\phi\|_{\infty}^2 \sup_l \|p_l\|_{\infty}^{\frac{1}{2}} \sum_{l=1}^M \|p_l - q_l\|_2 2^{\frac{j}{2}} n^{\frac{5}{2}};
\end{aligned}$$

$$\begin{aligned}
& \left| \sum_{l,l'} \sum_{i_1} \sum_{k,k'} a_l(i_1) a_{l'}(i_1) \mathbb{E}(\phi_{jk}(Y_{i_1}) \phi_{jk'}(Y_{i_1})) b_l(i_1) \mathbb{E}(\phi_{jk}(Z_{i_1})) \left(n \int \phi_{jk'} p_{l'} - n \int \phi_{jk'} q_{l'} \right) \right| \\
& \leq 4\sqrt{2} L^{\frac{5}{2}} \|\phi\|_\infty^2 \sup_l \|q_l\|_\infty^{\frac{1}{2}} \sum_{l=1}^M \|p_l - q_l\|_2 \left(\sum_{l=1}^M \langle a_l, a_l \rangle_n \right)^{\frac{1}{2}} \sum_{l=1}^M (\langle a_l, a_l \rangle_n + \langle b_l, b_l \rangle_n) 2^{\frac{j}{2}} n^{\frac{5}{2}} \\
& \leq 8\sqrt{\frac{2M^3}{K^3}} L^{\frac{5}{2}} \|\phi\|_\infty^2 \sup_l \|q_l\|_\infty^{\frac{1}{2}} \sum_{l=1}^M \|p_l - q_l\|_2 2^{\frac{j}{2}} n^{\frac{5}{2}}; \\
& \left| \sum_{l,l'} \sum_{i_1} \sum_{k,k'} b_l(i_1) b_{l'}(i_1) \mathbb{E}(\phi_{jk}(Z_{i_1}) \phi_{jk'}(Z_{i_1})) b_l(i_1) \mathbb{E}(\phi_{jk}(Z_{i_1})) \left(n \int \phi_{jk'} p_{l'} - n \int \phi_{jk'} q_{l'} \right) \right| \\
& \leq 8\sqrt{2} L^{\frac{5}{2}} \|\phi\|_\infty^2 \sup_l \|q_l\|_\infty^{\frac{1}{2}} \sum_{l=1}^M \|p_l - q_l\|_2 \left(\sum_{l=1}^M \langle b_l, b_l \rangle_n \right)^{\frac{3}{2}} 2^{\frac{j}{2}} n^{\frac{5}{2}} \\
& \leq 8\sqrt{\frac{2M^3}{K^3}} L^{\frac{5}{2}} \|\phi\|_\infty^2 \sup_l \|q_l\|_\infty^{\frac{1}{2}} \sum_{l=1}^M \|p_l - q_l\|_2 2^{\frac{j}{2}} n^{\frac{5}{2}}; \\
& \left| \sum_{l,l'} \sum_{i_1} \sum_{k,k'} b_l(i_1) b_{l'}(i_1) \mathbb{E}(\phi_{jk}(Z_{i_1}) \phi_{jk'}(Z_{i_1})) a_l(i_1) \mathbb{E}(\phi_{jk}(Y_{i_1})) \left(n \int \phi_{jk'} p_{l'} - n \int \phi_{jk'} q_{l'} \right) \right| \\
& \leq 4\sqrt{2} L^{\frac{5}{2}} \|\phi\|_\infty^2 \sup_l \|p_l\|_\infty^{\frac{1}{2}} \sum_{l=1}^M \|p_l - q_l\|_2 \left(\sum_{l=1}^M \langle b_l, b_l \rangle_n \right)^{\frac{1}{2}} \sum_{l=1}^M (\langle a_l, a_l \rangle_n + \langle b_l, b_l \rangle_n) 2^{\frac{j}{2}} n^{\frac{5}{2}} \\
& \leq 8\sqrt{\frac{2M^3}{K^3}} L^{\frac{5}{2}} \|\phi\|_\infty^2 \sup_l \|p_l\|_\infty^{\frac{1}{2}} \sum_{l=1}^M \|p_l - q_l\|_2 2^{\frac{j}{2}} n^{\frac{5}{2}}.
\end{aligned}$$

Next we come to the second term. We have

$$\begin{aligned}
& \left| \sum_{l,l'} \sum_{i_1} \sum_{k,k'} a_l(i_1) a_{l'}(i_1) \mathbb{E}(\phi_{jk}(Y_{i_1})) \mathbb{E}(\phi_{jk'}(Y_{i_1})) a_l(i_1) \mathbb{E}(\phi_{jk}(Y_{i_1})) \left(n \int \phi_{jk'} p_{l'} - n \int \phi_{jk'} q_{l'} \right) \right| \\
& \leq 4\sqrt{2} \|\phi\|_\infty^2 L^{\frac{5}{2}} \sup_l \|p_l\|_\infty^{\frac{1}{2}} \sum_{l=1}^M \|p_l - q_l\|_2 \left(\sum_{l=1}^M \langle a_l, a_l \rangle_n \right)^{\frac{3}{2}} 2^{\frac{j}{2}} n^{\frac{5}{2}} \\
& \leq 4\sqrt{2\frac{M^3}{K^3}} \|\phi\|_\infty^2 L^{\frac{5}{2}} \sup_l \|p_l\|_\infty^{\frac{1}{2}} \sum_{l=1}^M \|p_l - q_l\|_2 2^{\frac{j}{2}} n^{\frac{5}{2}},
\end{aligned}$$

$$\begin{aligned}
& \left| \sum_{l,l'} \sum_{i_1} \sum_{k,k'} a_l(i_1) a_{l'}(i_1) \mathbb{E}(\phi_{jk}(Y_{i_1})) \mathbb{E}(\phi_{jk'}(Y_{i_1})) b_l(i_1) \mathbb{E}(\phi_{jk}(Z_{i_1})) \left(n \int \phi_{jk'} p_{l'} - n \int \phi_{jk'} q_{l'} \right) \right| \\
& \leq 2\sqrt{2} \|\phi\|_\infty^2 L^{\frac{5}{2}} \sup_l \|q_l\|_\infty^{\frac{1}{2}} \sum_{l=1}^M \|p_l - q_l\|_2 \left(\sum_{l=1}^M \langle a_l, a_l \rangle_n \right)^{\frac{1}{2}} \sum_{l=1}^M (\langle a_l, a_l \rangle_n + \langle b_l, b_l \rangle_n) 2^{\frac{j}{2}} n^{\frac{5}{2}} \\
& \leq 4\sqrt{2} \frac{M^3}{K^3} \|\phi\|_\infty^2 L^{\frac{5}{2}} \sup_l \|q_l\|_\infty^{\frac{1}{2}} \sum_{l=1}^M \|p_l - q_l\|_2 2^{\frac{j}{2}} n^{\frac{5}{2}}, \\
& \left| \sum_{l,l'} \sum_{i_1} \sum_{k,k'} b_l(i_1) b_{l'}(i_1) \mathbb{E}(\phi_{jk}(Z_{i_1})) \mathbb{E}(\phi_{jk'}(Z_{i_1})) b_l(i_1) \mathbb{E}(\phi_{jk}(Z_{i_1})) \left(n \int \phi_{jk'} p_{l'} - n \int \phi_{jk'} q_{l'} \right) \right| \\
& \leq 4\sqrt{2} \|\phi\|_\infty^2 L^{\frac{5}{2}} \sup_l \|q_l\|_\infty^{\frac{1}{2}} \sum_{l=1}^M \|p_l - q_l\|_2 \left(\sum_{l=1}^M \langle b_l, b_l \rangle_n \right)^{\frac{3}{2}} 2^{\frac{j}{2}} n^{\frac{5}{2}} \\
& \leq 4\sqrt{2} \frac{M^3}{K^3} \|\phi\|_\infty^2 L^{\frac{5}{2}} \sup_l \|q_l\|_\infty^{\frac{1}{2}} \sum_{l=1}^M \|p_l - q_l\|_2 2^{\frac{j}{2}} n^{\frac{5}{2}}, \\
& \left| \sum_{l,l'} \sum_{i_1} \sum_{k,k'} b_l(i_1) b_{l'}(i_1) \mathbb{E}(\phi_{jk}(Z_{i_1})) \mathbb{E}(\phi_{jk'}(Z_{i_1})) a_l(i_1) \mathbb{E}(\phi_{jk}(Y_{i_1})) \left(n \int \phi_{jk'} p_{l'} - n \int \phi_{jk'} q_{l'} \right) \right| \\
& \leq 2\sqrt{2} \|\phi\|_\infty^2 L^{\frac{5}{2}} \sup_l \|p_l\|_\infty^{\frac{1}{2}} \sum_{l=1}^M \|p_l - q_l\|_2 \left(\sum_{l=1}^M \langle b_l, b_l \rangle_n \right)^{\frac{1}{2}} \sum_{l=1}^M (\langle a_l, a_l \rangle_n + \langle b_l, b_l \rangle_n) 2^{\frac{j}{2}} n^{\frac{5}{2}} \\
& \leq 4\sqrt{2} \frac{M^3}{K^3} \|\phi\|_\infty^2 L^{\frac{5}{2}} \sup_l \|p_l\|_\infty^{\frac{1}{2}} \sum_{l=1}^M \|p_l - q_l\|_2 2^{\frac{j}{2}} n^{\frac{5}{2}}.
\end{aligned}$$

All these bounds entail that

$$|A_{32}| \leq c_{32} \left(\frac{M}{K} \right)^{\frac{3}{2}} 2^{\frac{j}{2}} n^{\frac{5}{2}} \sum_{l=1}^M \|p_l - q_l\|_2,$$

with $c_{32} = 48\sqrt{2RL^5} \|\phi\|_\infty^2$. As a consequence, one similarly gets

$$|A_{33}| \leq c_{33} \left(\frac{M}{K} \right)^{\frac{3}{2}} 2^{\frac{j}{2}} n^{\frac{5}{2}} \sum_{l=1}^M \|p_l - q_l\|_2,$$

with $c_{33} = c_{32}$.

Let us now consider $|A_{34}|$.

$$\begin{aligned}
|A_{34}| & \leq \left| \sum_{i_1} \sum_{k,k'} \sum_{l,l'} \text{Cov}(a_l(i_1) \phi_{jk}(Y_{i_1}) - b_l(i_1) \phi_{jk}(Z_{i_1}), a_{l'}(i_1) \phi_{jk'}(Y_{i_1}) - b_{l'}(i_1) \phi_{jk'}(Z_{i_1})) \right. \\
& \quad \left. \mathbb{E}(a_l(i_1) \phi_{jk}(Y_{i_1}) - b_l(i_1) \phi_{jk}(Z_{i_1})) \mathbb{E}(a_{l'}(i_1) \phi_{jk'}(Y_{i_1}) - b_{l'}(i_1) \phi_{jk'}(Z_{i_1})) \right|.
\end{aligned}$$

Once again, we apply the Cauchy-Schwarz inequality,

$$\begin{aligned} \left| \sum_{i_1} \sum_{l'} \mathbb{E} (a_{l'}(i_1) \phi_{jk'}(Y_{i_1}) - b_{l'}(i_1) \phi_{jk'}(Z_{i_1})) \right| &= n \left| \sum_{l'} \int \phi_{jk'}(p_{l'} - q_{l'}) \right| \\ &\leq n \sum_{l'} \|p_{l'} - q_{l'}\|_2. \end{aligned}$$

According to lemma 6.5, we have for any $1 \leq i_1 \leq n$ and any l ,

$$\begin{aligned} \left| \sum_k \mathbb{E} (a_l(i_1) \phi_{jk}(Y_{i_1}) - b_l(i_1) \phi_{jk}(Z_{i_1})) \right| &\leq \left(|a_l(i_1)| \left| \sum_k \int \phi_{jk} f_{i_1} \right| \right) \vee \left(|b_l(i_1)| \left| \sum_k \int \phi_{jk} g_{i_1} \right| \right) \\ &\leq 2L(|a_l(i_1)| \vee |b_l(i_1)|) 2^{\frac{j}{2}} \|\phi\|_\infty \\ &\leq 2L \|\phi\|_\infty 2^{\frac{j}{2}} \left(\sum_{l=1}^M \langle a_l, a_l \rangle_n \vee \sum_{l=1}^M \langle b_l, b_l \rangle_n \right)^{\frac{1}{2}} \sqrt{n} \\ &\leq 2L \sqrt{\frac{M}{K}} \|\phi\|_\infty 2^{\frac{j}{2}} \sqrt{n}. \end{aligned}$$

According to lemmas 6.4 and 6.5, we have for any fixed k ,

$$\begin{aligned} &\sum_{i_1} \sum_{k'} |a_l(i_1) a_{l'}(i_1)| \left(\left| \int \phi_{jk} \phi_{jk'} f_{i_1} \right| + \left| \int \phi_{jk} f_{i_1} \int \phi_{jk'} f_{i_1} \right| \right) \\ &\leq n \sqrt{\langle a_l, a_l \rangle_n \langle a_{l'}, a_{l'} \rangle_n} \left(\sum_{k'} \left| \int \phi_{jk} \phi_{jk'} f_{i_1} \right| + \left| \int \phi_{jk} f_{i_1} \right| \sum_{k'} \left| \int \phi_{jk'} f_{i_1} \right| \right) \\ &\leq n \sqrt{\langle a_l, a_l \rangle_n \langle a_{l'}, a_{l'} \rangle_n} \left(4L \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty) + (2L)^{\frac{3}{2}} \|\phi\|_\infty \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty)^{\frac{1}{2}} \right) \\ &\leq \frac{M}{K} \left(4L \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty) + (2L)^{\frac{3}{2}} \|\phi\|_\infty \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty)^{\frac{1}{2}} \right) n \end{aligned}$$

and

$$\begin{aligned} &\sum_{i_1} \sum_{k'} |b_l(i_1) b_{l'}(i_1)| \left(\left| \int \phi_{jk} \phi_{jk'} g_{i_1} \right| + \left| \int \phi_{jk} g_{i_1} \int \phi_{jk'} g_{i_1} \right| \right) \\ &\leq n \sqrt{\langle b_l, b_l \rangle_n \langle b_{l'}, b_{l'} \rangle_n} \left(\sum_{k'} \left| \int \phi_{jk} \phi_{jk'} g_{i_1} \right| + \left| \int \phi_{jk} g_{i_1} \right| \sum_{k'} \left| \int \phi_{jk'} g_{i_1} \right| \right) \\ &\leq n \sqrt{\langle b_l, b_l \rangle_n \langle b_{l'}, b_{l'} \rangle_n} \left(4L \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty) + (2L)^{\frac{3}{2}} \|\phi\|_\infty \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty)^{\frac{1}{2}} \right) \\ &\leq \frac{M}{K} \left(4L \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty) + (2L)^{\frac{3}{2}} \|\phi\|_\infty \sup_l (\|p_l\|_\infty \vee \|q_l\|_\infty)^{\frac{1}{2}} \right) n. \end{aligned}$$

Hence,

$$|A_{34}| \leq c_{34} 2^{\frac{j}{2}} \left(\frac{M}{K} \right)^{\frac{3}{2}} n^{\frac{5}{2}} \sum_l \|p_l - q_l\|_2,$$

with $c_{34} = 4L\|\phi\|_\infty\sqrt{R}\left(4L\sqrt{R} + (2L)^{\frac{3}{2}}\|\phi\|_\infty\right)$.

When we carefully look at the bounds of A_{3i} for $i \in \{1, 2, 3, 4\}$, we deduce that there exists a $Cste > 0$ such that

$$A_3 \leq \tilde{C}_T \frac{M^2}{K^2} \left[n^3 \sum_l \|p_l - q_l\|_2^2 + 2^{\frac{j}{2}} n^{\frac{5}{2}} \sum_l \|p_l - q_l\|_2 \right],$$

with $\tilde{C}_T = \sum_{i=1}^4 c_{3i}$. □

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