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# Large normally hyperbolic cylinders in *a priori* stable Hamiltonian systems

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A major problem in dynamical systems consists in studying the Hamiltonian systems on  $\mathbb{T}^n \times \mathbb{R}^n$  of the form

$$H(q, p) = h(p) - \epsilon^2 G(t, q, p), \quad (t, q, p) \in \mathbb{T} \times \mathbb{T}^n \times \mathbb{R}^n. \quad (H)$$

Here  $\epsilon$  should be considered as a small perturbation parameter, we put a square because the sign of the perturbation will play a role in our discussion. In the unperturbed system ( $\epsilon = 0$ ) the momentum variable  $p$  is constant.

We want to study the dynamics of the perturbed system in the neighborhood of a torus  $\{p = p_0\}$ , corresponding to a resonant frequency. There is no loss of generality in assuming that the frequency is of the form

$$\partial h(p_0) = (\omega, 0) \in \mathbb{R}^m \times \mathbb{R}^r.$$

If the restricted frequency  $\omega$  is non-resonant in  $\mathbb{R}^m$ , then it is expected that the averaged system

$$H_a(q, p) = H_a(q_1, q_2, p_1, p_2) = h(p) - \epsilon^2 V(q_2) \quad (H_a)$$

should locally approximate the dynamics of  $(H)$  near  $p = p_0$ , where  $q = (q_1, q_2) \in \mathbb{T}^r \times \mathbb{T}^m$  and  $p = (p_1, p_2) \in \mathbb{R}^r \times \mathbb{R}^m$ , and where

$$V(q_2) = \int G(t, q_1, q_2, p_0) dt dq_1.$$

We make the following hypothesis on the averaged system:

**Hypotheses 1.** *The function  $h$  is convex with positive definite Hessian and the averaged potential  $V$  has a non-degenerate local maximum at  $q_2 = 0$ .*

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Under Hypothesis 1, the averaged system has an invariant manifold of equations

$$(\partial_{p_2} h = 0, q_2 = 0) \in \mathbb{T}^m \times \mathbb{R}^n.$$

Because  $h$  has positive definite Hessian, the equation  $\partial_{p_2} h(p_1, p_2) = 0$  is equivalent to  $p_2 = P_2(p_1)$  for some function  $P_2 : \mathbb{R}^m \rightarrow \mathbb{R}^r$ . Therefore, the invariant manifold can be written in a parametric form as

$$(q_1, 0, p_1, P_2(q_1)); (q_1, p_1) \in \mathbb{T}^m \times \mathbb{R}^m$$

so it is a cylinder. Moreover, this manifold is Normally hyperbolic in the sense of [8]. It is then pretty well understood that some small pieces of this manifold persist in the initial system, meaning that some normally hyperbolic invariant manifold close to that cylinder exist. It can be seen as the center manifolds of a "Whiskered" (or partially hyperbolic) torus. The existence of such a torus was proved in [13], and it is well understood, see for example [3] that such a torus must be contained in an invariant cylinder which is normally hyperbolic. However, the approaches used so far to study the Whiskered torus rely on a rescaling of the momentum  $p$ , and produce an invariant annulus of size  $O(\epsilon)$ , although a cylinder of size  $O(1)$  in fact exists. This is what we will prove in the present paper:

**Theorem 1.** *Assume that  $H$  is smooth (or at least  $C^r$  for a sufficiently large  $r$ ) and satisfies Hypothesis 1. Assume that  $\omega$  is Diophantine, and fix  $\kappa > 0$ . Then there exists an open ball  $B \subset \mathbb{R}^m$  containing  $p_0$ , a neighborhood  $U$  of 0 in  $\mathbb{T}^r$ , a positive number  $\epsilon_0$  and, for  $\epsilon < \epsilon_0$  two  $C^1$  functions*

$$Q_2^\epsilon : \mathbb{T} \times \mathbb{T}^m \times B \rightarrow U \subset \mathbb{T}^r \quad \text{and} \quad P_2^\epsilon : \mathbb{T} \times \mathbb{T}^m \times B \rightarrow \mathbb{R}^r$$

such that the annulus

$$A^\epsilon = (t, q_1, Q_2^\epsilon(t, q_1, p_1), p_1, P_2^\epsilon(t, q_1, p_1))$$

is invariant for  $(H)$  (in the sense that the Hamiltonian vectorfield is tangent to it) and normally hyperbolic. We have  $P_2^\epsilon \rightarrow P_2^0$  uniformly as  $\epsilon \rightarrow 0$ , where  $P_2^0$  is the function  $(t, q_1, p_1) \mapsto P_2(q_1, p_1)$ . Moreover, we have  $\|P_2^\epsilon - P_2^0\|_{C^1} \leq \kappa$ , and  $\|Q_2^\epsilon\|_{C^1} \leq \kappa/\epsilon$ . Each invariant set of  $(H)$  (in the sense that it contains the full orbit of each of its points) contained in the domain

$$\mathcal{D}^\epsilon := \mathbb{T} \times \mathbb{T}^m \times U \times B \times \{p_2 \in \mathbb{R}^r : \|p_2\| \leq \epsilon\}$$

is contained in  $A^\epsilon$  for  $\epsilon < \epsilon_0$ .

The novelty here is that the ball  $B$  does not depend on  $\epsilon$ . Easy examples show that we can't expect a control of the asymptotic behaviour of  $Q_2^\epsilon$  in terms of the averaged system only except if we restrict to smaller domains depending on  $\epsilon$ . This asymptotic behaviour will also depend on the averaged systems at other frequencies.

Combining this result in the case  $m = 1$  (but for any dimension  $n$ ) with existing techniques on the *a priori* unstable situation (see *e. g.* [2, 5, 6, 13]), one can hope to obtain, under additional non-degeneracy assumptions, the existence of restricted Arnold diffusion in the following sense : There exists  $\delta > 0$  and  $\epsilon_0$  such that, for each  $\epsilon \in ]0, \epsilon_0[$  there exists an orbit  $(q_\epsilon(t), p_\epsilon(t))$  such that the image  $p_\epsilon(\mathbb{R})$  is not contained in any ball of radius  $\delta$  in  $\mathbb{R}^n$ . Once again, the key point here is that  $\delta$  can be chosen independant of  $\epsilon$ . Specifying the needed "non-degeneracy assumptions" will require some further work. The lack of control on the normally hyperbolic invariant cylinder may create substantial difficulties.

Of course, finding "global" Arnold diffusion, as announced in [11], that is orbits wandering in the whole phase space along different resonant lines (or far away along a given resonant line) requires a specific study of double resonances, where the existence of normally hyperbolic invariant cylinders can't be obtained by the method used in the present paper.

# 1 Averaging

In order to apply averaging methods, it is easier to consider the extended phase space

$$(t, e, q, p) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}^n \times \mathbb{R}^n$$

where the Hamiltonian flow can be seen as the Hamiltonian flow of the autonomous Hamiltonian function

$$\tilde{H}(t, e, q, p) = h(p) + e - \epsilon^2 G(t, q, p)$$

on one of its energy surfaces, for example  $\tilde{H} = 0$ . Then, we consider a smooth solution  $f(t, q)$  of the Homological equation

$$\partial_t f + \partial_q f \cdot (\omega, 0) = G(t, q, p_0) - V(q_2).$$

Such a solution exists because  $\omega$  is Diophantine, as can be checked easily by power series expansion. It is unique up to an additive constant. We consider the smooth symplectic diffeomorphism

$$\psi^\epsilon : (t, e, q, p) \mapsto (t, e + \epsilon^2 \partial_t f(t, q), q, p + \epsilon^2 \partial_q f(t, q))$$

and use the same notation for the diffeomorphism  $(t, q, p) \mapsto (t, q, p + \epsilon^2 \partial_q f(t, q))$ . We have

$$\tilde{H} \circ \psi^\epsilon = h(p) + e - \epsilon^2 V(q_2) - \epsilon^2 R(t, q, p) + O(\epsilon^4).$$

In other words, by the time-dependant symplectic change of coordinates  $\psi^\epsilon$ , we have reduced the study of  $H$  to the study of the time-dependant Hamiltonian

$$H_1(t, q, p) = h(p) - \epsilon^2 V(q_2) - \epsilon^2 R(t, q, p) + O(\epsilon^4)$$

where  $R = O(p)$ . As a consequence, Theorem 1 holds for  $H$  if it holds for  $H_1$ . More precisely, assume that there exists an invariant cylinder

$$\tilde{A}^\epsilon = (t, q_1, \tilde{Q}_2^\epsilon(t, q_1, p_1), p_1, \tilde{P}_2^\epsilon(t, q_1, p_1))$$

for  $H_1$ , with  $\|\tilde{Q}_2^\epsilon\|_{C^1} \leq \kappa/2\epsilon$  and  $\|P_2^\epsilon - P_2^0\|_{C^1} \leq \kappa/2$ . Then the annulus  $A^\epsilon := \psi^\epsilon(\tilde{A}^\epsilon)$  is invariant for  $H$ . Since  $\psi^\epsilon$  is  $\epsilon^2$ -close to the identity, while  $\|Q_2^\epsilon\|_{C^1} \leq \kappa/2\epsilon$ , the annulus  $A^\epsilon$  has the form

$$A^\epsilon = (t, q_1, Q_2^\epsilon(t, q_1, p_1), p_1, P_2^\epsilon(t, q_1, p_1))$$

for  $C^1$  functions  $Q_2^\epsilon, P_2^\epsilon$  which satisfy  $\|Q_2^\epsilon\|_{C^1} \leq \kappa/\epsilon$  and  $\|P_2^\epsilon - P_2^0\|_{C^1} \leq \kappa$ . We will prove that Theorem 1 holds for  $H_1$  in section 4. We first expose some useful tools.

## 2 Normally hyperbolic manifolds

We shall now present a version of the classical theory of Normally hyperbolic manifolds adapted for our purpose. On  $\mathbb{R}^{n_z} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ , let us consider the time dependant vectorfield

$$\begin{aligned} \dot{z} &= Z(t, z, x, y) \\ \dot{x} &= A(z)x \\ \dot{y} &= -B(z)y. \end{aligned}$$

We assume that the function

$$Z : \mathbb{R} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \longrightarrow \mathbb{R}^{n_z}$$

is bounded in  $C^1$ , and that the matrices  $A$  and  $B$  are  $C^1$ -bounded functions of  $z$ . Moreover, we assume that there exists constants  $a > b > 0$  such that

$$A(z)x \cdot x \geq a\|x\|^2 \quad , \quad B(z)y \cdot y \geq a\|y\|^2$$

for each  $x, y, z$ , and such that

$$\|\partial_{(t,z)}Z(t, z, x, y)\| \leq b$$

for all  $(t, z, x, y)$ . We consider the perturbed vectorfield

$$\begin{aligned} \dot{z} &= Z(t, z, x, y) + R_z(t, z, x, y) \\ \dot{x} &= A(z)x \quad + R_x(t, z, x, y) \\ \dot{y} &= -B(z)y \quad + R_y(t, z, x, y). \end{aligned}$$

where  $R = (R_z, R_x, R_y)$  is seen as a small perturbation.

**Theorem 2.** *There exists  $\epsilon > 0$  such that, when  $\|R\|_{C^1} < \epsilon$ , the maximal invariant set of the perturbed vectorfield contained in the domain*

$$\mathbb{R}^{n_z} \times \{x \in \mathbb{R}^{n_x} : \|x\| \leq 1\} \times \{y \in \mathbb{R}^{n_y} : \|y\| \leq 1\}$$

is a graph of the form

$$(t, z, X(t, z), Y(t, z))$$

where  $X$  and  $Y$  are  $C^1$  maps. This graph is normally hyperbolic. We have

$$\|(X, Y)\|_{C^0} \leq (2/a)\|R\|_{C^0}.$$

where  $C$  is a constant independent of  $R$ . If the data are  $C^r$ ,  $r \in \{1, 2, \dots, \infty\}$  and if  $rb < a$ , then the functions  $X$  and  $Y$  are  $C^r$ .

PROOF. The invariant space  $\mathbb{R}^{n_z}$  is Normally hyperbolic in the sense of [7, 8]. As a consequence, the standard theory applies and implies the existence of functions  $X$  and  $Y$  such that the graph  $(t, z, X(t, z), Y(t, z))$  is invariant, normally hyperbolic, and contained in the prescribed domain. Note that we are slightly outside of the hypotheses of the statements in [8] because our unperturbed manifold is not compact. However, it is easy to see that the results actually depend on uniform estimates rather than on compactness (see [9], Appendix B, for example, see also [4]), and we assumed such uniform estimates.

Let us now prove the estimate on  $(X, Y)$ . We have the inequality

$$\dot{x} \cdot x \geq a\|x\|^2 + x \cdot R_x \geq a\|x\|(\|x\| - \|R_x\|_{C^0}/a)$$

which implies that

$$\dot{x} \cdot x \geq \|x\|\|R_x\|_{C^0}$$

if

$$2\|R_x\|_{C^0}/a \leq \|x\| \leq 1,$$

hence this domain can't intersect the invariant graph. Similar considerations show that the domain  $2\|R_y\|_{C^0}/a \leq \|y\| \leq 1$  can't intersect the graph.  $\square$

### 3 Hyperbolic Linear System

Let us consider the linear Hamiltonian system on  $\mathbb{R}^n \times \mathbb{R}^n$  generated by the Hamiltonian

$$H(q, p) = \frac{1}{2}\langle Bp, p \rangle - \frac{1}{2}\langle Aq, q \rangle,$$

where both  $A$  and  $B$  are positive definite symmetric matrices. We recall that this system can be reduced to

$$G(x, y) = \langle Dx, y \rangle,$$

where  $D$  is a positive definite symmetric matrix by a linear symplectic change of variables  $(q, p) \longrightarrow (x, y)$ . In order to do so, we consider the symmetric positive definite matrix

$$L := (A^{-1/2}(A^{1/2}BA^{1/2})^{1/2}A^{-1/2})^{1/2},$$

which is the only symmetric and positive definite solution of the equation  $L^2AL^2 = B$ . Considering the change of variables

$$x = \frac{1}{\sqrt{2}}(Lp + L^{-1}q) \quad ; \quad y = \frac{1}{\sqrt{2}}(Lp - L^{-1}q)$$

or equivalently

$$q = \frac{1}{\sqrt{2}}L(x - y) \quad ; \quad p = \frac{1}{\sqrt{2}}L^{-1}(x + y),$$

an elementary calculation shows that we obtain the desired form for the Hamiltonian in coordinates  $(x, y)$ , with

$$D = LAL.$$

As a consequence, the equations of motions in the new variables take the block-diagonal form

$$\dot{x} = Dx \quad ; \quad \dot{y} = -Dy.$$

In the original coordinates  $(q, p)$  the stable space (which is the space  $x = 0$ ) is the space  $\{(q, -L^2q), q \in \mathbb{R}^n\}$  while the unstable space is  $\{(q, L^2q), q \in \mathbb{R}^n\}$ .

### 4 Proof of Theorem 1

We now prove Theorem 1 for the Hamiltonian

$$H_1(t, q, p) = h(p) - \epsilon^2 V(q_2) - \epsilon^2 R(t, q, p) + O(\epsilon^{2+\gamma}),$$

where  $R = O(p - p_0)$  and  $\gamma > 0$  ( $\gamma = 2$  in our situation). We assume that Hypothesis 1 holds. We lift all the angular variable to the universal covering, and see  $H_1$  as a Hamiltonian of the variables

$$(t, q, p) = (t, q_1, q_2, p_1, p_2) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^r$$

which is one-periodic in  $t, q$ . We assume that  $p_0 = 0$ . It is useful to introduce two new positive parameters  $\alpha$  and  $\delta$ . We always assume that

$$0 < \epsilon < \delta < \alpha < 1.$$

In the sequel, we shall chose  $\alpha$  small, then  $\delta$  small with respect to  $\alpha$ , and work with  $\epsilon$  small enough with respect to  $\alpha$  and  $\delta$ . Since we are only interested in the dynamics near  $p = 0, q_2 = 0$ , we define

$$R_\delta(t, q, p) = \rho(\|p\|/\delta)R(t, q, p)$$

where  $\rho : [0, \infty) \rightarrow [0, 1]$  is a smooth function which is equal to 1 on  $[0, 1]$  and to 0 on  $[2, \infty)$ . We set  $A = \partial^2 V(0)$ , it is a positive definite symmetric matrix. We now define

$$F_\delta(q_2) = \rho(\|q_2\|/\sqrt{\delta})(\partial_{q_2} V(q_2) - Aq_2),$$

so that the equality

$$\partial_2 V(q_2) = Aq_2 + F_\delta(q_2)$$

holds when  $q_2 \leq \sqrt{\delta}$ . Finally, we define

$$B(p_1) := \partial_{p_2}^2 h(P_2(p_1)).$$

Recall that  $P_2(p_1)$  is the local solution of  $\partial_2 h(p_1, \cdot) = 0$ . We will then truncate once more the higher order terms by setting

$$Y_\epsilon(p) = \rho(\|p_2 - P_2(p_1)\|/\epsilon)(\partial_{p_2} h(p) - B(p_1)(p_2 - P_2(p_1))).$$

The following vectorfield is equal to the Hamiltonian vectorfield of  $H_1$  on the domain  $\{\|p_1\| \leq \delta, \|p_2 - P_2(p_1)\| \leq \epsilon, \|q_2\| \leq \sqrt{\delta}\}$ :

$$\begin{aligned} \dot{q}_1 &= \partial_{p_1} h(p) & -\epsilon^2 \partial_{p_1} R(t, q, p) & + O(\epsilon^{2+\gamma}) \\ \dot{p}_1 &= & \epsilon^2 \partial_{q_1} R_\delta(t, q, p) & + O(\epsilon^{2+\gamma}) \\ \dot{q}_2 &= B(p_1)(p_2 - P_2(p_1)) + Y_\epsilon(p) - \epsilon^2 \partial_{p_2} R(t, q, p) & + O(\epsilon^{2+\gamma}) \\ \dot{p}_2 &= \epsilon^2 Aq_2 & + \epsilon^2 F_\delta(q_2) + \epsilon^2 \partial_{q_2} R_\delta(t, q, p) & + O(\epsilon^{2+\gamma}) \end{aligned}$$

We write this vectorfield in the form

$$\begin{aligned} \dot{q}_1 &= \partial_{p_1} h(p) & + \epsilon^2 \chi(t, q, p) \\ \dot{p}_1 &= 0 & + \epsilon^2 \delta \chi(t, q, p/\delta) + \epsilon^{2+\gamma} \chi(t, q, p) \\ \dot{q}_2 &= B(p_1)(p_2 - P_2(p_1)) + \epsilon^2 \chi(p_1, (p_2 - P_2(p_1))/\epsilon) + \epsilon^2 \chi(t, q, p) \\ \dot{p}_2 &= \epsilon^2 Aq_2 & + \epsilon^2 \delta \chi(q_2/\sqrt{\delta}) + \epsilon^2 \delta \chi(t, q_1, q_2, p/\delta) + \epsilon^{2+\gamma} \chi(t, q, p) \end{aligned}$$

with the convention that  $\chi(\cdot)$  always denotes a  $C^1$  function of its arguments, depending on  $\epsilon$  and  $\delta$ , but bounded in  $C^1$  independantly of  $\delta$  and  $\epsilon$ . Motivated by section 2, we set

$$L(p_1) = (A^{-1/2}(A^{1/2}B(p_1)A^{1/2})^{1/2}A^{-1/2})^{1/2},$$

and perform the change of variables  $(t, q_1, p_1, q_2, p_2) \rightarrow (\tau, \theta, r, x, y)$  given by:

$$\begin{aligned} \tau &= \epsilon t, & \theta &= \epsilon \alpha q_1, & r &= p_1, \\ x &= L(p_1)(p_2 - P_2(p_1)) + \epsilon L^{-1}(p_1)q_2, & y &= L(p_1)(p_2 - P_2(p_1)) - \epsilon L^{-1}(p_1)q_2, \end{aligned}$$

recalling that  $\alpha$  is a fixed positive parameter. Equivalently, this can be written

$$t = \tau/\epsilon, \quad q_1 = \theta/\epsilon\alpha, \quad p_1 = r, \quad q_2 = L(r)(x - y)/2\epsilon, \quad p_2 = P_2(r) + L^{-1}(r)(x + y)/2.$$

In the new coordinates, the principal part of the vectorfield takes the form

$$\dot{\theta} = \alpha \Omega(r, x, y), \quad \dot{r} = 0, \quad \dot{x} = D(r)x, \quad \dot{y} = -D(r)y,$$

with

$$\Omega(r, x, y) := \partial_{p_1} h(r, P_2(r) + L^{-1}(r)(x + y)/2)$$

and

$$D(r) := L(r)AL(r) = L^{-1}(r)B(r)L^{-1}(r).$$

Recall that the equality above holds because  $L(r)$  solves the equation  $L^2(r)AL^2(r) = B(r)$ . Let us detail the calculations leading to the expressions of  $\dot{x} := dx/d\tau$  (the calculation for  $\dot{y}$  is similar):

$$\begin{aligned} \dot{x} &= \epsilon \dot{x} = L(p_1)(\dot{p}_2 - \partial_{p_1} P_2 \cdot \dot{p}_1) + \epsilon L^{-1}(p_1)\dot{q}_2 + (\partial_{p_1} L \cdot \dot{p}_1)(p_2 - P_2(p_1)) + \epsilon(\partial_{p_1}(L^{-1}) \cdot \dot{p}_1)q_2 \\ &= \epsilon^2 L(p_1)Aq_2 + \epsilon L^{-1}(p_1)B(p_1)(p_2 - P_2(p_1)) \\ &\quad + \epsilon^2 \delta \chi(t, q, p/\delta, x, y) + \epsilon^3 \chi(p_1, (x+y)/\epsilon) + \epsilon^2 \delta \chi(q_2/\sqrt{\delta}) + \epsilon^{2+\gamma} \chi(t, q, p, x, y) \\ &= \epsilon L(r)AL(r)(x-y)/2 + \epsilon L^{-1}(r)B(r)L^{-1}(r)(x+y)/2 \\ &\quad + \epsilon^2 \delta \chi(\tau/\epsilon, \theta/\epsilon, r/\delta, x/\delta, y/\delta, x/\epsilon, y/\epsilon) + \epsilon^2 \delta \chi(r/\sqrt{\delta}, x/\sqrt{\delta}\epsilon, y/\sqrt{\delta}\epsilon) \\ &= \epsilon D(r)x + \epsilon^2 \delta \chi(\tau/\epsilon, \theta/\epsilon, r/\delta, x/\delta, y/\delta, x/\epsilon, y/\epsilon) + \epsilon^2 \delta \chi(r/\sqrt{\delta}, x/\sqrt{\delta}\epsilon, y/\sqrt{\delta}\epsilon). \end{aligned}$$

We are only interested in the dynamics inside the domain  $\{\|p_1\| \leq \delta, \|p_2 - P_2(p_1)\| \leq \epsilon, \|q_2\| \leq \sqrt{\delta}\}$  which in the new coordinates is contained in  $\{\|r\| \leq \delta, \|(x, y)\| \leq \epsilon\}$ , so that we can modify the functions  $\Omega$  and  $D$  outside of a fixed domain and assume that they are bounded in  $C^1$ . Then we can choose  $\alpha < 1$  once and for all in order that the principal part of the vectorfield satisfies the hypotheses of Theorem 2. The full vectorfield can be written in the new coordinates, where  $\dot{f} := df/d\tau$ :

$$\begin{aligned} \dot{\theta} &= \alpha \Omega(r, x, y) + \epsilon^2 \chi(\tau/\epsilon, \theta/\epsilon, r, x/\epsilon, y/\epsilon) \\ \dot{r} &= 0 \quad + \epsilon \delta \chi(\tau/\epsilon, \theta/\epsilon, r/\delta, x/\delta, y/\delta, x/\epsilon, y/\epsilon) \\ \dot{x} &= D(r)x \quad + \epsilon \delta \chi(\tau/\epsilon, \theta/\epsilon, r/\delta, x/\delta, y/\delta, x/\epsilon, y/\epsilon) + \epsilon \delta \chi(r/\sqrt{\delta}, x/\sqrt{\delta}\epsilon, y/\sqrt{\delta}\epsilon) \\ \dot{y} &= -D(r)y \quad + \epsilon \delta \chi(\tau/\epsilon, \theta/\epsilon, r/\delta, x/\delta, y/\delta, x/\epsilon, y/\epsilon) + \epsilon \delta \chi(r/\sqrt{\delta}, x/\sqrt{\delta}\epsilon, y/\sqrt{\delta}\epsilon). \end{aligned}$$

In this expression, we observe that the uniform norm of the perturbation is  $O(\epsilon\delta)$  while the  $C^1$  norm is  $O(\sqrt{\delta})$  (recall that  $0 < \epsilon < \delta < 1$ ). We can apply Theorem 2 and find a unique bounded normally hyperbolic invariant graph

$$(\tau, \theta, X(\tau, \theta, r), r, Y(\tau, \theta, r)).$$

Moreover, we have that

$$\|(X, Y)\|_{C^0} \leq C\epsilon\delta.$$

In the initial coordinates, we have an invariant graph

$$(t, q_1, Q_2^\epsilon(t, q_1, p_1), p_1, P_2^\epsilon(t, q_1, p_1))$$

with

$$Q_2^\epsilon(t, q_1, p_1) = L(p_1)(X(\epsilon t, \epsilon q_1, p_1) - Y(\epsilon t, \epsilon q_1, p_1))/2\epsilon$$

and

$$P_2^\epsilon(t, q_1, p_1) = P_2(p_1) + L^{-1}(p_1)(X(\epsilon t, \epsilon q_1, p_1) + Y(\epsilon t, \epsilon q_1, p_1))/2.$$

Because this graph is the unique bounded invariant graph, the functions  $Q_2^\epsilon$  and  $P_2^\epsilon$  are periodic in  $(t, q_1)$ . Observe that

$$\|Q_2^\epsilon\|_{C^0} \leq C\delta, \quad \|P_2^\epsilon\|_{C^0} \leq C\epsilon\delta.$$

We conclude that that the annulus

$$\{(t, q_1, Q_2(t, q_1, p_1), p_1, P_2(t, q_1, p_1)) : t \in \mathbb{T}, q_1 \in \mathbb{T}^m, p_1 \in \mathbb{R}^m, \|p_1\| \leq \delta\} \subset \mathbb{T} \times \mathbb{T}^n \times \mathbb{R}^n$$



is contained in the domain

$$\{\|q_2\| \leq \sqrt{\delta}, \|p_1\| \leq \delta, \|p_2\| \leq \epsilon\}$$

where our vectorfield coincides with the Hamiltonian vectorfield of  $H_1$ . It is thus an invariant annulus for  $H_1$ . We have proved Theorem 1 for  $H_1$ , we conclude from Section 1 that Theorem 1 holds for  $H$ .

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