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To cite this version:

Irina Kurkova, Kilian Raschel. Explicit expression of the counting generating function for Gessel's walk. 2009. hal-00438190v2

HAL Id: hal-00438190

https://hal.science/hal-00438190v2

Preprint submitted on 6 Mar 2010 (v2), last revised 18 Nov 2010 (v3)

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Explicit expression of the counting generating function for Gessel's walk

Irina Kurkova* Kilian Raschel*

December 8, 2009

Abstract

We consider the so-called Gessel's walk, that is the planar random walk that is confined to the first quadrant and that can move in unit steps to the West, North-East, East and South-West. For this walk we make explicit the generating function of the number of paths starting at $(0,0)$ and ending at (i,j) in time k.

Keywords : lattice walks, generating function, Riemann-Hilbert boundary value problem, conformal gluing function, Weierstrass elliptic function, Riemann surface, uniformization.

AMS 2000 *Subject Classification : primary 05A15 ; secondary 30F10, 30D05.*

0 Introduction and main results

The enumeration of lattice walks is a classical problem in combinatorics. The one of Gessel's walk seems to puzzle the mathematics community already for several years [Ges86, PW08, KKZ09, Ayy09, Pin09, BK09]. This is a planar random walk that is confined to the first quadrant and that can move in the interior in unit steps to the West, North-East, East and South-West, see Figure 1. For $(i,j) \in \mathbb{Z}_+^2$ and $k \in \mathbb{Z}_+$, set

$$q(i,j,k) = \#\big\{\text{walks starting at } (0,0) \text{ and ending at } (i,j) \text{ in time } k\big\}.$$

I. Gessel conjectured around 2001 that $q(0,0,2k) = 16^k\big[(5/6)_k(1/2)_k\big]\big/\big[(2)_k(5/3)_k\big]$, where $(a)_k = a(a+1)\cdots(a+k-1)$. In 2008, M. Kauers, C. Koutschan and D. Zeilberger yielded a remarkable although heavily computer-aided proof of this conjecture, see [KKZ09].

The articles [Ayy09, Pin09] give connections between Gessel's walk and other interesting models. Namely, S. Ping in [Pin09] establishes a probabilistic model for Gessel's walk concerned with vicious walkers, and A. Ayyer in [Ayy09] interprets such walks as Dick words with two sets of letters and gives explicit formulas for a restricted class of

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such words. Both of these approaches are yet in some way of providing "human" proof of Gessel conjecture but may certainly help for a better understanding of Gessel's walk.

M. Petkovsek and H. Wilf in [PW08] state some similar conjectures for the number of walks ending at other points. Two of them have been proved by S. Ping in [Pin09]. M. Petkovsek and H. Wilf in [PW08] obtain also an infinite lower-triangular system of linear equations satisfied by the values of $q(i,0,k)$ and $q(0,j,k)+q(0,j-1,k)$ and express these values as determinants of lower Hessenberg matrices with unit superdiagonals whose non-zero entries are products of two binomial coefficients.

Finally, A. Bostan and M. Kauers in [BK09] show that the complete generating function for Gessel's walk

$$Q(x,y,z) = \sum_{i,j,k\geq 0} q(i,j,k)x^i y^j z^k$$

is algebraic and make explicit minimal polynomials for $Q(x,0,z)$ and $Q(0,y,z)$. The proof of A. Bostan and M. Kauers given in [BK09] involves, among other tools, computer calculations using a powerful computer algebra system Magma, it required immense computational effort.

Curiously, in spite of this vivid interest to Gessel's walk, the complete generating function $Q(x,y,z)$ or even $Q(0,y,z)$ or $Q(x,0,z)$ have *not* yet been analyzed *without computer help*, up to our knowledge.

Furthermore, recently M. Bousquet-Mélou and M. Mishna in [BMM08] have undertaken the systematic analysis of enumeration of the walks confined to the quarter plane $\mathbb{Z}_+^2$ starting from the origin and making steps at any point of $\mathbb{Z}_+^2$ from a given subset of $\{-1,0,1\}^2 \setminus \{(0,0)\}$. There are 2^8 such models. Moreover they show that, after eliminating trivial models and those that are equivalent to models of walks confined to a half-plane and solved by known methods, it remains 79 inherently different problems to study. Following the idea of Book [FIM99], they associate to each model a group G of birational transformations (for details on this group, see Subsection 1.1 below). This group is finite in 23 cases and infinite in the 56 other cases. They are able to solve "mathematically", *i.e.* to make explicit the function $Q(x,y,z)$ *without computer help*, 22 models associated with a finite group. The only case with finite group that remained unsolved is the model of Gessel's walk.

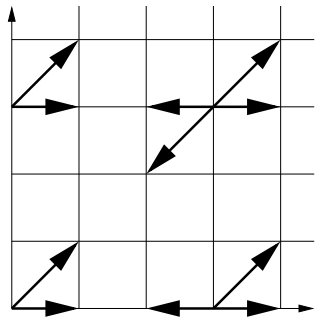

Figure 1: Gessel's walk

The aim of this paper is to solve Gessel's walk model, *i.e.* to represent in a closed form the generating function $Q(x,y,z)$ *without computer help.*

In addition of being not computer-aided, our method presents the advantage of being generalizable up to the case of all 2^8 walks described above with unit steps in the quarter plane and associated with a finite or infinite group.

Let us observe that for any i, j and k, $q(i,j,k) \leq 4^k$, so that $Q(x,y,z)$ is holomorphic in $\{|x|<1, |y|<1, |z|<1/4\}$ and continuous up to $\{|x|\leq 1, |y|\leq 1, |z|<1/4\}$. Our starting point is the functional equation already stated in [BMM08] and exploited in [PW08], valid *a priori* on $\{|x|\leq 1, |y|\leq 1, |z|<1/4\}$:

$$L(x,y,z)Q(x,y,z) = zQ(x,0,z) + z(y+1)Q(0,y,z) - zQ(0,0,z) - xy, \tag{1}$$

where $L(x,y,z) = xyz\big(1/x + 1/(xy) + x + xy - 1/z\big)$.

Our method heavily relies on the profound analytic approach developed in Book [FIM99] by G. Fayolle, R. Iasnogorodski and V. Malyshev. There the authors compute the generating functions of stationary probabilities for some ergodic random walks in the quarter plane. These random walks have four domains of spatial homogeneity : the interior $\{(i,j) : i>0, j>0\}$, the x-axis $\{(i,0) : i>0\}$, the y-axis $\{(0,j) : j>0\}$ and the origin $\{0,0\}$; in the interior the only (at most eight) possible non-zero jump probabilities are at distance one. They reduce the problem to the solution of the following functional equation on $\{|x|\leq 1, |y|\leq 1\}$,

$$K(x,y)\Pi(x,y) = k(x,y)\pi(x) + \widetilde{k}(x,y)\widetilde{\pi}(y) + k_0(x,y)\pi_{00}, \tag{2}$$

with known polynomials $K(x,y)$, $k(x,y)$, $\tilde{k}(x,y)$, $k_0(x,y)$ and with functions $\Pi(x,y)$, $\pi(x)$, $\tilde{\pi}(y)$ unknown but holomorphic in unit discs, continuous up to the boundary. First, they continue $\pi(x)$ and $\tilde{\pi}(y)$ meromorphically (with poles that can be identified) to the whole complex plane cut along some segment. This ingenious continuation procedure is the crucial step of Book [FIM99]. After that, they show that $\pi(x)$ and $\tilde{\pi}(y)$ verify a boundary value problem of Riemann-Carleman type and they solve it by converting it into a boundary value problem of Riemann-Hilbert type.

Compared to (2), our equation (1) seems a bit more difficult to analyze, as it involves a complementary parameter z. From the other point of view, the coefficients $k(x,y)$, $\tilde{k}(x,y)$ and $k_0(x,y)$ in front of unknowns $zQ(x,0,z)$, $z(y+1)Q(0,y,z)$ and $zQ(0,0,z)$ are absent. This will allow us to continue $zQ(x,0,z)$ and $z(y+1)Q(0,y,z)$ as holomorphic and not only meromorphic functions and, consequently, to simplify substantially the solutions.

In the sequel, we will suppose, for technical reasons, that z is fixed in $]0,1/4[$.

We are now going to state the main results of this paper. To begin with, let us have a closer look to the kernel $L(x,y,z)$ that appears in (1) and let us take some notations.

The polynomial $L(x,y,z)$ can be written as $L(x,y,z) = \tilde{a}(y,z)x^2 + \tilde{b}(y,z)x + \tilde{c}(y,z) = a(x,z)y^2 + b(x,z)y + c(x,z)$, with $\tilde{a}(y,z) = zy(y+1)$, $\tilde{b}(y,z) = -y$, $\tilde{c}(y,z) = z(y+1)$ and $a(x,z) = zx^2$, $b(x,z) = zx^2 - x + z$, $c(x,z) = z$. Define also $\tilde{d}(y,z) = \tilde{b}(y,z)^2 - 4\tilde{a}(y,z)\tilde{c}(y,z)$ and $d(x,z) = b(x,z)^2 - 4a(x,z)c(x,z)$.

For any $z \in]0,1/4[$, $\tilde{d}$ has one root equal to zero and two real positive roots, that we denote by $y_2(z) < 1 < y_3(z)$. We have $y_2(z) = [1-8z^2-(1-16z^2)^{1/2}]/[8z^2]$ and $y_3(z) = [1-8z^2+(1-16z^2)^{1/2}]/[8z^2]$; we will also note $y_1(z) = 0$ and $y_4(z) = \infty$.

Likewise, for all $z \in]0,1/4[$, d has four real positive roots, that we denote by $x_1(z) < x_2(z) < 1 < x_3(z) < x_4(z)$. Their explicit expression is $x_1(z) = [1+2z-(1+4z)^{1/2}]/[2z]$, $x_2(z) = [1-2z-(1-4z)^{1/2}]/[2z]$, $x_3(z) = [1-2z+(1-4z)^{1/2}]/[2z]$ and $x_4(z) = [1+2z+(1+4z)^{1/2}]/[2z]$.

With these notations we have $L(x,y,z) = 0$ if and only if $(\tilde{b}(y,z)+2\tilde{a}(y,z)x)^2 = \tilde{d}(y,z)$ or $(b(x,z)+2a(x,z)y)^2 = d(x,z)$. In particular, the algebraic functions $X(y,z)$ and $Y(x,z)$ defined by $L(X(y,z),y,z) = 0$ and $L(x,Y(x,z),z) = 0$ have two branches, meromorphic on respectively $\mathbb{C} \setminus \big([y_1(z),y_2(z)] \cup [y_3(z),y_4(z)]\big)$ and $\mathbb{C} \setminus \big([x_1(z),x_2(z)] \cup [x_3(z),x_4(z)]\big)$.

The following straightforward results give some properties of the two branches of the algebraic functions $X(y,z)$ and $Y(x,z)$.

Lemma 1. *Call $X_0(y,z) = [-\tilde{b}(y,z)+\tilde{d}(y,z)^{1/2}]/[2\tilde{a}(y,z)]$ and $X_1(y,z) = [-\tilde{b}(y,z)-\tilde{d}(y,z)^{1/2}]/[2\tilde{a}(y,z)]$ the branches of $X(y,z)$. For all $y \in \mathbb{C}$, we have $|X_0(y,z)| \le |X_1(y,z)|$.*

On $\mathbb{C} \setminus \big([y_1(z),y_2(z)] \cup [y_3(z),y_4(z)]\big)$, X_0 has a simple zero at -1, no other zero and no pole ; X_1 has a simple pole at -1, no other pole and no zero. Finally, both X_0 and X_1 become infinite at $y_1(z) = 0$ and zero at $y_4(z) = \infty$.

Now we call $Y_0(x,z) = [-b(x,z)+d(x,z)^{1/2}]/[2a(x,z)]$ and $Y_1(x,z) = [-b(x,z)-d(x,z)^{1/2}]/[2a(x,z)]$ the branches of $Y(x,z)$. For all $x \in \mathbb{C}$, we have $|Y_0(x,z)| \le |Y_1(x,z)|$.

On $\mathbb{C} \setminus \big([x_1(z),x_2(z)] \cup [x_3(z),x_4(z)]\big)$, Y_0 has a double zero at ∞, no other zero and no pole ; Y_1 has a double pole at 0, no other pole and no zero.

Both $X_i(y,z)$, $i = 0,1$, are not defined for y in a branch cut, in other words for $y \in [y_1(z),y_2(z)] \cup [y_3(z),y_4(z)]$. However, the limits $X_i^{\pm}(y,z)$ defined by $X_i^{+}(y,z) = \lim X_i(\hat{y},z)$ as $\hat{y} \to y$ from the *upper* side of the cut and $X_i^{-}(y,z) = \lim X_i(\hat{y},z)$ as $\hat{y} \to y$ from the *lower* side of the cut are well defined. Since for y in a branch cut, $\tilde{d}(y,z) < 0$, these two quantities are complex conjugate the one from the other.

A similar remark holds for $Y_i(x,z)$, $i = 0,1$, and $x \in [x_1(z),x_2(z)] \cup [x_3(z),x_4(z)]$.

In fact for respectively $y \in [y_1(z),y_2(z)]$ and $x \in [x_1(z),x_2(z)]$ we have :

$$X_0^{\pm}(y,z) = \frac{-\widetilde{b}(y,z) \mp \imath\big[-\widetilde{d}(y,z)\big]^{1/2}}{2\widetilde{a}(y,z)}, \qquad Y_0^{\pm}(x,z) = \frac{-b(x,z) \mp \imath\big[-d(x,z)\big]^{1/2}}{2a(x,z)}, \tag{3}$$

$X_1^{\pm}(y,z) = X_0^{\mp}(y,z)$ and $Y_1^{\pm}(x,z) = Y_0^{\mp}(x,z)$ – note that for (3) to be true for respectively $y \in [y_3(z),y_4(z)]$ and $x \in [x_3(z),x_4(z)]$, we have to exchange $X_0^{\pm}(y,z)$ and $Y_0^{\pm}(x,z)$ in $X_0^{\mp}(y,z)$ and $Y_0^{\mp}(x,z)$.

Lemma 2. *Consider $X([y_1(z),y_2(z)],z)$ and $Y([x_1(z),x_2(z)],z)$.* (i) *These two curves are symmetrical w.r.t. the real axis and not included in the unit disc.* (ii) *$X([y_1(z),y_2(z)],z)$ contains ∞ and $Y([x_1(z),x_2(z)],z)$ is closed.* (iii) *Both of them split the plane into two connected components, we call $\mathscr{G}X([y_1(z),y_2(z)],z)$ and $\mathscr{G}Y([x_1(z),x_2(z)],z)$ the connected components of 0. They verify $\mathscr{G}X([y_1(z),y_2(z)],z) \subset \mathbb{C} \setminus [x_3(z),x_4(z)]$ and $\mathscr{G}Y([x_1(z),x_2(z)],z) \subset \mathbb{C} \setminus [y_3(z),y_4(z)]$.*

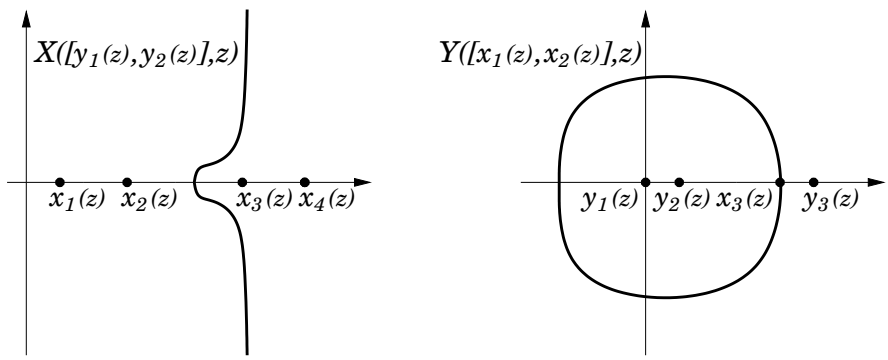

Figure 2: The curves $X([y_1(z), y_2(z)], z)$ and $Y([x_1(z), x_2(z)], z)$

Note that complete proofs of Lemmas 1 and 2 can be found in Part 5.3 of [FIM99]. These notations and results on the kernel $L(x, y, z)$ are enough in order to state our results.

First of all, we would like to show that $zQ(x, 0, z)$ and $z(y + 1)Q(0, y, z)$ verify some boundary value problems of Riemann-Carleman type. It turns out that the associated boundary conditions verified by $zQ(x, 0, z)$ and $z(y+1)Q(0, y, z)$ hold respectively on the curves $X([y_1(z), y_2(z)], z)$ and $Y([x_1(z), x_2(z)], z)$, which are *not* included in unit disc, see Lemma 2, and where therefore the functions $zQ(x, 0, z)$ and $z(y+1)Q(0, y, z)$ are *a priori* not defined. For this reason, we first need to continue the generating functions up to these curves. In fact we will show the following – the proof of which being the central subject of Section 2.

Theorem 3. *The functions $zQ(x, 0, z)$ and $z(y + 1)Q(0, y, z)$ can be holomorphically continued from their unit disc up to $\mathbb{C} \setminus [x_3(z), x_4(z)]$ and $\mathbb{C} \setminus [y_3(z), y_4(z)]$ respectively. Furthermore for any $y \in \mathbb{C} \setminus [y_1(z), y_2(z)] \cup [y_3(z), y_4(z)]$,*

$$zQ(X_0(y, z), 0, z) + z(y + 1)Q(0, y, z) - zQ(0, 0, z) - X_0(y, z)y = 0, \tag{4}$$

and for any $x \in \mathbb{C} \setminus [x_1(z), x_2(z)] \cup [x_3(z), x_4(z)]$,

$$zQ(x, 0, z) + z(Y_0(x, z) + 1)Q(0, Y_0(x, z), z) - zQ(0, 0, z) - xY_0(x, z) = 0. \tag{5}$$

Remark 4. *For $y \in \{|y| \leq 1\}$ such that $|X_0(y, z)| \leq 1$, (4) follows immediately from (1). Likewise, for $x \in \{|x| \leq 1\}$ such that $|Y_0(x, z)| \leq 1$, (5) is a straightforward consequence of (1). The fact that equations (4) and (5) are verified not only for these values of y and x but actually on $\mathbb{C} \setminus [y_1(z), y_2(z)] \cup [y_3(z), y_4(z)]$ and $\mathbb{C} \setminus [y_1(z), y_2(z)] \cup [x_3(z), x_4(z)]$ respectively will be shown in Section 2.*

Remark 5. *In the proof of Theorem 3, we will see that the function $Q(0, y, z)$ can also be holomorphically continued from the unit disc up to $\mathbb{C} \setminus [y_3(z), y_4(z)]$.*

Now we explain how to obtain the above mentioned boundary conditions verified by the functions $zQ(x,0,z)$ and $z(y+1)Q(0,y,z)$.

Let $y \in [y_1(z), y_2(z)]$, and let $\hat{y}^+$ and $\hat{y}^-$ be close to y, such that $\hat{y}^+$ is in the *upper* half-plane and $\hat{y}^-$ in the *lower* half-plane. Then we have (4) for both $\hat{y}^+$ and $\hat{y}^-$. If now $\hat{y}^+ \to y$ and $\hat{y}^- \to y$, then we obtain $X_0(\hat{y}^+, z) \to X_0^+(y,z)$ and $X_0(\hat{y}^-, z) \to X_0^-(y,z) = X_1^+(y,z)$. So we have proved that for any $y \in [y_1(z), y_2(z)]$,

$$zQ(X_0^+(y,z),0,z) + z(y+1)Q(0,y,z) - zQ(0,0,z) - X_0^+(y,z)y = 0, \tag{6}$$
$$zQ(X_1^+(y,z),0,z) + z(y+1)Q(0,y,z) - zQ(0,0,z) - X_1^+(y,z)y = 0. \tag{7}$$

Subtracting (7) from (6) we obtain that for any $y \in [y_1(z), y_2(z)]$,

$$z\big[Q(X_0^+(y,z),0,z) - Q(X_1^+(y,z),0,z)\big] = X_0^+(y,z)y - X_1^+(y,z)y. \tag{8}$$

Then, using the fact that for $i = 0,1$, $y \in [y_1(z), y_2(z)]$ and $z \in]0, 1/4[$, $Y_0(X_i^{\pm}(y,z),z) = y$ – which can be proved by elementary considerations starting from Lemma 1, or by the use of Lemma 17 –, we get the first part of (9) below :

$$\begin{aligned} &\forall t \in X([y_1(z), y_2(z)], z): && z\big[Q(t,0,z) - Q(\overline{t},0,z)\big] &&= tY_0(t,z) - \overline{t}Y_0(\overline{t},z), \\ &\forall t \in Y([x_1(z), x_2(z)], z): && z\big[(t+1)Q(0,t,z) - (\overline{t}+1)Q(0,\overline{t},z)\big] &&= X_0(t,z)t - X_0(\overline{t},z)\overline{t}. \end{aligned} \tag{9}$$

Likewise, we could prove the second part of (9).

Note that as a consequence of (6) and (7), (4) is in some sense also verified for $y \in [y_1(z), y_2(z)]$ – the same is true for (5) and $x \in [x_1(z), x_2(z)]$.

With Lemma 2, Theorem 3 and (9), we get that $zQ(x,0,z)$ *and* $z(y+1)Q(0,y,z)$ *can be found among the functions holomorphic in* $\mathscr{G}X([y_1(z), y_2(z)], z)$ *and* $\mathscr{G}Y([x_1(z), x_2(z)], z)$, *continuous up to the boundary and verifying the boundary conditions (9).*

Such problems are called boundary value problems of Riemann-Carleman type. A standard way to solve them consists in converting them into boundary value problems of Riemann-Hilbert type by use of *conformal gluing functions* (CGF).

For any detail about boundary value problems and conformal gluing, we refer to [Lit00].

Definition 6. *Let* $C \subset \mathbb{C} \setminus \{0\}$ *be a curve symmetrical w.r.t. the real axis and splitting the complex plane into two connected components, and let* $\mathscr{G}C$ *be the connected component of* 0. *A function* u *is said to be a CGF for the curve* C *if* (i) u *is meromorphic in* $\mathscr{G}C$ (ii) u *establishes a conformal mapping of* $\mathscr{G}C$ *onto the complex plane cut along some arc* (iii) *for all* $t \in C$, $u(t) = u(\overline{t})$.

Let $w(t,z)$ and $\tilde{w}(t,z)$ be CGF for $X([y_1(z), y_2(z)], z)$ and $Y([x_1(z), x_2(z)], z)$ – the existence (but *no* explicit expression) of w and $\tilde{w}$ is ensured by general results on conformal gluing, see *e.g.* [Lit00].

Transforming the boundary value problems of Riemann-Carleman type into boundary value problems of Riemann-Hilbert type thanks to w and $\tilde{w}$, solving them and working out the solutions we will prove the following.

Theorem 7. *The function* $z\big[Q(x,0,z)-Q(0,0,z)\big]$ *has the following explicit expression for* $z\in]0,1/4[$ *and* $x\in\mathbb{C}\setminus[x_3(z),x_4(z)]$ *:*

$$z\big[Q(x,0,z)-Q(0,0,z)\big]=$$
$$xY_0(x,z)+\frac{1}{\pi}\int_{x_1(z)}^{x_2(z)}\frac{t\big[-d(t,z)\big]^{1/2}}{2a(t,z)}\left[\frac{\partial_t w(t,z)}{w(t,z)-w(x,z)}-\frac{\partial_t w(t,z)}{w(t,z)-w(0,z)}\right]\mathrm{d}t,$$

w being a CGF for the curve $X([y_1(z),y_2(z)],z)$.

The function $z\big[(y+1)Q(0,y,z)-Q(0,0,z)\big]$ *has the following explicit expression for* $z\in]0,1/4[$ *and* $y\in\mathbb{C}\setminus[y_3(z),y_4(z)]$ *:*

$$z\big[(y+1)Q(0,y,z)-Q(0,0,z)\big]=$$
$$X_0(y,z)y+\frac{1}{\pi}\int_{y_1(z)}^{y_2(z)}\frac{t\big[-\widetilde{d}(t,z)\big]^{1/2}}{2\widetilde{a}(t,z)}\left[\frac{\partial_t \widetilde{w}(t,z)}{\widetilde{w}(t,z)-\widetilde{w}(y,z)}-\frac{\partial_t \widetilde{w}(t,z)}{\widetilde{w}(t,z)-\widetilde{w}(0,z)}\right]\mathrm{d}t,$$

$\tilde{w}$ being a CGF for the curve $Y([x_1(z),x_2(z)],z)$.

The function $Q(0,0,z)$ *has the following explicit expression for* $z\in]0,1/4[$ *:*

$$Q(0,0,z)=-\frac{1}{\pi}\int_{y_1(z)}^{y_2(z)}\frac{t\big[-\widetilde{d}(t,z)\big]^{1/2}}{2\widetilde{a}(t,z)}\left[\frac{\partial_t \widetilde{w}(t,z)}{\widetilde{w}(t,z)-\widetilde{w}(-1,z)}-\frac{\partial_t \widetilde{w}(t,z)}{\widetilde{w}(t,z)-\widetilde{w}(0,z)}\right]\mathrm{d}t,$$

$\tilde{w}$ being a CGF for the curve $Y([x_1(z),x_2(z)],z)$.

The function $Q(x,y,z)$ *has the explicit expression obtained by using the ones of* $Q(x,0,z)$*,* $Q(0,y,z)$ *and* $Q(0,0,z)$ *in (1).*

All functions in the integrands above are explicit, except for the CGF w and $\tilde{w}$. In [FIM99] suitable CGF are computed *implicitly* by means of the reciprocal of some known function (see the formulas (23) and (24) below for the details). Starting from this representation, we are able to make *explicit* these functions for Gessel's walk.

In order to state the result we need to define $G_2(z)=(4/27)(1+224z^2+256z^4)$, $G_3(z)=(8/729)(1+16z^2)(1-24z+16z^2)(1+24z+16z^2)$, $K(z)$ as the only positive root of $K^4-G_2(z)K^2/2-G_3(z)K-G_2(z)^2/48=0$ – noting $r_k(z)=[G_2(z)-\exp(2k\imath\pi/3)(G_2(z)^3-27G_3(z)^2)^{1/3}]/3$ we have $K(z)=[-r_0(z)^{1/2}+r_1(z)^{1/2}+r_2(z)^{1/2}]/2$ – and

$$\begin{aligned}F(t,z)&=\frac{1-24z+16z^2}{3}-\frac{4(1-4z)^2}{z}\frac{t^2}{(t-x_2(z))(t-1)^2(t-x_3(z))},\\ \widetilde{F}(t,z)&=\frac{1-24z+16z^2}{3}+\frac{4(1-4z)^2}{z}\frac{t(t+1)^2}{[(t-x_2(z))(t-x_3(z))]^2}.\end{aligned}\tag{10}$$

Theorem 8. *A suitable CGF for the curve* $X([y_1(z),y_2(z)],z)$ *is the only function having a pole at* $x_2(z)$ *and solution of*

$$\begin{aligned}w^3-w^2\big[F(t,z)+2K(z)\big]+w\big[2K(z)F(t,z)+K(z)^2/3+G_2(z)/2\big]\\ -\big[K(z)^2F(t,z)+19G_2(z)K(z)/18+G_3(z)-46K(z)^3/27\big]=0.\end{aligned}\tag{11}$$

Likewise, a suitable CGF for the curve $Y([x_1(z),x_2(z)],z)$ *is the only function having a pole at* $x_3(z)$ *and solution of the equation obtained from (11) by replacing* F *by* $\tilde{F}$*, see (10).*

Let us now outline some facts around Theorems 3, 7 and 8.

Remark 9. *Since (1) is valid at least on* $\{|x| \leq 1, |y| \leq 1, |z| < 1/4\}$*, then for any such* $(\hat{x}, \hat{y}, z)$ *with* $L(\hat{x}, \hat{y}, z) = 0$*, the right-hand side of (1) equals zero, so that*

$$z\big[Q(\hat{x}, 0, z) - Q(0, 0, z)\big] + z\big[(\hat{y} + 1)Q(0, \hat{y}, z) - Q(0, 0, z)\big] + zQ(0, 0, z) - \hat{x}\hat{y} = 0. \quad (12)$$

We deduce that

$$zQ(0, 0, z) = -z\big[Q(\hat{x}, 0, z) - Q(0, 0, z)\big] - z\big[(\hat{y} + 1)Q(0, \hat{y}, z) - Q(0, 0, z)\big] + \hat{x}\hat{y} \quad (13)$$

with the functions in the square brackets in right-hand side given by the first two formulas in Theorem 7. For the explicit expression of $zQ(0, 0, z)$ *given in Theorem 7, we have chosen to substitute* $(\hat{x}, \hat{y}, z) = (0, -1, z)$ *in (13), which is such that* $L(\hat{x}, \hat{y}, z) = 0$*, since with Lemma 1 we have* $X_0(-1, z) = 0$ *. Moreover, we show in Theorem 3 that for any* $z \in]0, 1/4[$*, the equation (12) is valid not only on* $\{(x, y) \in \mathbb{C}^2 : L(x, y, z) = 0\} \cap \{|x| \leq 1, |y| \leq 1\}$ *but in a much larger domain of the algebraic curve* $\{(x, y) \in \mathbb{C}^2 : L(x, y, z) = 0\}$*. Namely, if* $(\hat{x}, \hat{y}, z)$ *is such that* $z \in]0, 1/4[$*,* $\hat{x} \notin [x_3(z), x_4(z)]$ *and* $\hat{y} = Y_0(\hat{x}, z)$ *or* $\hat{y} \notin [y_3(z), y_4(z)]$ *and* $\hat{x} = X_0(\hat{y}, z)$*, then (12) is still valid. Substituting* any $(\hat{x}, \hat{y}, z)$ *from this domain into (12) yields us* $zQ(0, 0, z)$ *as in (13).*

Remark 10. *With the analytical approach proposed here, it would be possible, without additional difficulty, to obtain explicitly the generating function of the number of walks beginning at an arbitrary initial state* (i_0, j_0) *and ending at* (i, j) *in time* k*. Indeed, the only significant difference is that the product* xy *in (1) would be then replaced by* $x^{i_0+1}y^{j_0+1}$*.*

Remark 11. *Making in Theorem 7 the changes of variable* $w = w(t, z)$ *and* $\tilde{w} = \tilde{w}(t, z)$*, we obtain that the generating functions* $zQ(x, 0, z)$ *and* $z(y + 1)Q(0, y, z)$ *are essentially Cauchy-type integrals of algebraic functions.*

In particular, it could be deduced from the work [PRY04] – which gives criteria for a Cauchy-type integral of an algebraic function to be algebraic – that as functions of x *and* y *respectively,* $zQ(x, 0, z)$ *and* $z(y + 1)Q(0, y, z)$ *are algebraic functions, what would give an other proof to some results contained in [BK09].*

Remark 12. *In Theorem 7, the functions* $z\big[Q(x, 0, z) - Q(0, 0, z)\big]$ *and* $z\big[(y+1)Q(0, y, z) - Q(0, 0, z)\big]$ *are written as the sum of two functions not holomorphic but algebraic near respectively* $[x_1(z), x_2(z)]$ *and* $[y_1(z), y_2(z)]$*. The sum of these two algebraic functions is of course holomorphic near these segments, since they are included in the unit disc. By an application of the residue theorem as in Section 4 of [KR09], we could write both generating functions as functions* manifestly *holomorphic near these segments and having in fact their singularities near respectively* $[x_3(z), x_4(z)]$ *and* $[y_3(z), y_4(z)]$*.*

The rest of the paper is organized as follows.

In Section 1 we prove Theorem 8. There the implicit representation of the CGF given in [FIM99] (and recalled here in Subsections 1.1 and 1.2) in a general setting is developed in Subsection 1.3 to the case of Gessel's walk.

The proof of Theorem 3 is postponed to the last Section 2. The main idea of the holomorphic continuation procedure is borrowed again from [FIM99], we show how it works with the parameter $z \in]0, 1/4[$.

Finally, we give the proof of Theorem 7.

Proof of Theorem 7. The proof is composed of two steps : the first one, inspired by [FIM99], will allow us to obtain integral representations of the functions $zQ(x, 0, z)$ and $z(y+1)Q(0, y, z)$ on the curves $X([y_1(z), y_2(z)], z)$ and $Y([x_1(z), x_2(z)], z)$; the second one will consist in transforming these formulations into the integrals on real segments written in the statement of Theorem 7, which are more convenient, notably from a calculations point of view.

Let us begin by solving the boundary value problems of Riemann-Carleman type with boundary conditions (9). The use of CGF allows us, as in [FIM99] or [Lit00], to transform them into boundary value problems of Riemann-Hilbert type. Following again [FIM99] or [Lit00] we solve them and in this way we obtain representations of the unknown functions $zQ(x, 0, z)$ and $z(y + 1)Q(0, y, z)$ as integrals along the curves $X([y_1(z), y_2(z)], z)$ and $Y([x_1(z), x_2(z)], z)$. For $zQ(x, 0, z)$, we get that up to some additive function of z,

$$zQ(x, 0, z) = \frac{1}{2\pi\imath} \int_{X([y_1(z),y_2(z)],z)} tY_0(t, z) \frac{\partial_t w(t, z)}{w(t, z) - w(x, z)} \mathrm{d}t, \tag{14}$$

where w is the CGF used for $X([y_1(z), y_2(z)], z)$. Similarly, we could write an integral representation of $z(y + 1)Q(0, y, z)$, up to some additive function of z

We are now going to transform the integral representation (14) of $zQ(x, 0, z)$. To begin with, let $C(\epsilon, z)$ be any contour such that

(i) $C(\epsilon, z)$ is connected and contains ∞,

(ii) $C(\epsilon, z) \subset \big(\mathscr{G}X([y_1(z), y_2(z)], z) \cup X([y_1(z), y_2(z)], z)\big) \setminus [x_1(z), x_2(z)]$,

(iii) $\lim_{\epsilon\to 0} C(\epsilon, z) = X([y_1(z), y_2(z)], z) \cup S(z)$, where we have denoted by $S(z)$ the real segment $[x_1(z), X(y_2(z), z)]$ traversed from $X(y_2(z), z)$ to $x_1(z)$ along the lower edge of the slit and then back to $X(y_2(z), z)$ along the upper edge,

and let $\mathscr{G}C(\epsilon, z)$ be the connected component of 0 of $\mathbb{C} \setminus C(\epsilon, z)$.

Now we apply the residue theorem to the integrand of (14) on the contour $C(\epsilon, z)$. Thanks to Lemma 1 and the property (ii) of the contour $C(\epsilon, z)$, $tY_0(t, z)$ is, as a function of t, holomorphic in $\mathscr{G}C(\epsilon, z)$. Likewise, by using Definition 6 and the property (ii), we get that $\partial_t w(t, z)/(w(t, z) - w(t, x))$ is meromorphic on $\mathscr{G}C(\epsilon, z)$, with a unique pole at $t = x$. Therefore we have :

$$\frac{1}{2\pi\imath} \int_{C(\epsilon,z)} tY_0(t, z) \frac{\partial_t w(t, z)}{w(t, z) - w(x, z)} \mathrm{d}t = xY_0(x, z). \tag{15}$$

Then, making ϵ go to 0, using (14), (15) and the property (iii) of the contour, we obtain that up to an additive function of z,

$$zQ(x, 0, z) = xY_0(x, z) - \frac{1}{2\pi\imath} \int_{S(z)} tY_0(t, z) \frac{\partial_t w(t, z)}{w(t, z) - w(x, z)} \mathrm{d}t. \tag{16}$$

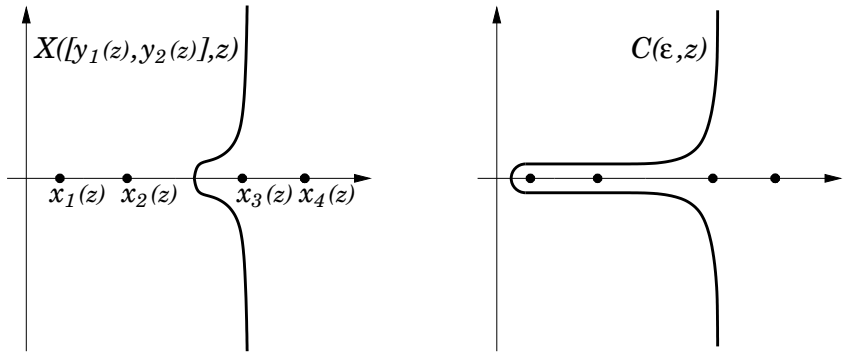

Figure 3: The curve $X([y_1(z), y_2(z)], z)$ and the new contour of integration $C(\epsilon, z)$

Since for any $x \in \mathscr{G}X([y_1(z), y_2(z)], z)$, the integrand in (16) is, as a function of t, holomorphic at any point of $]x_2(z), X(y_2(z), z)[$, we have

$$\int_{S(z)} tY_0(t,z)\frac{\partial_t w(t,z)}{w(t,z)-w(x,z)}\mathrm{d}t = \int_{x_1(z)}^{x_2(z)} \left[tY_0^+(t,z) - tY_0^-(t,z)\right]\frac{\partial_t w(t,z)}{w(t,z)-w(x,z)}\mathrm{d}t, \tag{17}$$

so that with (3) we immediately obtain, for $x \in \mathscr{G}X([y_1(z), y_2(z)], z)$, the expression of $z[Q(x,0,z) - Q(0,0,z)]$ stated in Theorem 7.

Likewise, we could obtain for $y \in \mathscr{G}Y([x_1(z), x_2(z)], z)$ the expression of $z[(y+1)Q(0,y,z) - Q(0,0,z)]$ written in Theorem 7.

The formula for $Q(0,0,z)$ has been already proved in Remark 9.

In fact, the integral representations of $z[Q(x,0,z) - Q(0,0,z)]$ and $z[(y+1)Q(0,y,z) - Q(0,0,z)]$ hold not only on $\mathscr{G}X([y_1(z), y_2(z)], z)$ and $\mathscr{G}Y([x_1(z), x_2(z)], z)$ but on $\mathbb{C} \setminus [x_3(z), x_4(z)]$ and $\mathbb{C} \setminus [y_3(z), y_4(z)]$; we will show this fact in Proposition 18, since the necessary tools will be naturally introduced in Subsections 1.1 and 1.2. □

Note. After the first version of this paper appeared on arXiv, we received an e-mail from M. van Hoeij who said us that he has, very recently, found explicitly $Q(x,y,z)$ by computing an explicit solution to the minimal polynomials for $Q(x,0,z)$ and $Q(0,y,z)$ given in [BK09] and by using (1). As already said, the results of A. Bostan and M. Kauers in [BK09], and consequently also the ones of M. van Hoeij, required the use of a powerful computer algebra system. However this computer-aided approach leads to another closed form of $Q(x,y,z)$ and gives a complementary interesting insight into Gessel's walk.

Acknowledgments. We thank Philippe Bougerol for putting our attention to the recent combinatorial problems and results. We thank Amaury Lambert for suggesting us to study Mireille Bousquet-Mélou papers.

1 Study of the conformal gluing functions

Notation. For the sake of shortness we will, from now on, drop the dependence of the different quantities w.r.t. $z \in]0, 1/4[$.

The main subject of Section 1 is to prove Theorem 8. For this we will define two functions, namely w and $\tilde{w}$, which thanks to Part 5.5 of [FIM99] are known to be suitable CGF for the curves $X([y_1, y_2])$ and $Y([x_1, x_2])$, and we will show that these functions verify the conclusions of Theorem 8.

These definitions of the CGF given in [FIM99] are recalled here in Subsection 1.2, see particularly (23) and (24). They require to define some functions on a uniformization of the algebraic curve $\{(x, y) \in \mathbb{C}^2 : L(x, y, z) = 0\}$, so that we begin Section 1 by studying a suitable uniformization of this curve – note that this Subsection 1.1 is also necessary in Section 2, where we will prove Theorem 3.

1.1 Uniformization

We will note $\mathscr{L}$ the algebraic curve $\{(x, y) \in \mathbb{C}^2 : L(x, y, z) = 0\}$, L being defined in (1).

Proposition 13. *For any $z \in]0, 1/4[$, $\mathscr{L}$ is a Riemann surface of genus one.*

Proof. We have shown in Section 0 that $L(x, y, z) = 0$ if and only if $(b(x)+2a(x)y)^2 = d(x)$. But the Riemann surface of the square root of a polynomial which has four distinct roots of order one has genus one, see *e.g.* [JS87], therefore the genus of $\mathscr{L}$ is also one. □

With Proposition 13 it is immediate that $\mathscr{L}$ is isomorphic to some torus ; in other words there exists a two-dimensional lattice Ω such that $\mathscr{L}$ is isomorphic to $\mathbb{C}/\Omega$. Such a suitable lattice Ω (in fact the *only possible* lattice, up to a homothetic transformation) is made explicit in Parts 3.1 and 3.3 of [FIM99], namely $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$, where

$$\omega_1 = \imath \int_{x_1}^{x_2} \frac{\mathrm{d}x}{[-d(x)]^{1/2}}, \qquad \omega_2 = \int_{x_2}^{x_3} \frac{\mathrm{d}x}{[d(x)]^{1/2}}. \tag{18}$$

We are now going to give a uniformization of the surface $\mathscr{L}$, in other words we are going to make explicit two functions $x(\omega), y(\omega)$ elliptic w.r.t. the lattice Ω such that $\mathscr{L} = \{(x(\omega), y(\omega)), \omega \in \mathbb{C}\} = (\{(x(\omega), y(\omega)), \omega \in \mathbb{C}/\Omega\})$. By using the same arguments as in Part 3.3 of [FIM99], we immediately obtain that we can take

$$x(\omega) = x_4 + \frac{d'(x_4)}{\wp(\omega) - d''(x_4)/6}, \quad y(\omega) = \frac{1}{2a(x(\omega))}\left[-b(x(\omega)) + \frac{d'(x_4)\wp'(\omega)}{2(\wp(\omega) - d''(x_4)/6)^2}\right], \tag{19}$$

$\wp$ being the Weierstrass elliptic function with periods ω_1, ω_2.

By convenience, we will consider, from now on, that the coordinates of the uniformization x and y are defined on $\mathbb{C}/\Omega$ rather than on $\mathbb{C}$.

It is well-known that $\wp$ is characterized by its invariants g_2, g_3 through

$$\wp'(\omega)^2 = 4\wp(\omega)^3 - g_2\wp(\omega) - g_3. \tag{20}$$

Lemma 14. *The invariants g_2, g_3 of $\wp$ are equal to :*

$$g_2 = (4/3)\big(1 - 16z^2 + 16z^4\big), \qquad g_3 = -(8/27)\big(1 - 8z^2\big)\big(1 - 16z^2 - 8z^4\big).$$

Proof. It is well-known that $4\wp(\omega)^3 - g_2\wp(\omega) - g_3 = 4\big(\wp(\omega) - \wp(\omega_1/2)\big)\big(\wp(\omega) - \wp((\omega_1 + \omega_2)/2)\big)\big(\wp(\omega) - \wp(\omega_2/2)\big)$; in particular the invariants can be calculated in terms of the values of $\wp$ at the half-periods. But it is clear (and proved in Part 3.3 of [FIM99]) that setting $f(t) = d'(x_4)/(t - x_4) + d''(x_4)/6$ we have $\wp(\omega_1/2) = f(x_3)$, $\wp((\omega_1 + \omega_2)/2) = f(x_2)$ and $\wp(\omega_2/2) = f(x_1)$, so that Lemma 14 follows from a direct calculation. □

Now that the uniformization (19) is completely and explicitly defined, it is natural to be interested in the reciprocal images through it of the important cycles that are the branch cuts $[x_1, x_2]$, $[x_3, x_4]$, $[y_1, y_2]$ and $[y_3, y_4]$. For this we need to define a new period, namely

$$\omega_3 = \int_{-\infty}^{x_1} \frac{\mathrm{d}x}{[d(x)]^{1/2}}. \tag{21}$$

We will importantly use the fact that $\omega_3 \in]0, \omega_2[$ – this is proved in Part 3.3 of [FIM99].

Proposition 15. *We have $x^{-1}([x_1, x_2]) = [0, \omega_1[+ \omega_2/2$ and $x^{-1}([x_3, x_4]) = [0, \omega_1[$, $y^{-1}([y_1, y_2]) = [0, \omega_1[+ (\omega_2 + \omega_3)/2$ and $y^{-1}([y_3, y_4]) = [0, \omega_1[+ \omega_3/2$.*

Proposition 15 follows from repeating the arguments of Part 5.5 of [FIM99], and is illustrated on Figure 4 below.

Now we define $S(x, y) = 1/x + 1/(xy) + x + xy$, the generating function of the jump probabilities of Gessel's walk, and we consider the following birational transformations :

$$\Psi(x, y) = \big(x, 1/(x^2 y)\big), \qquad \Phi(x, y) = \big(1/(xy), y\big).$$

They are such that $\Psi^2 = \Phi^2 = \mathrm{id}$ and $S \circ \Psi = S \circ \Phi = S$. Then, as in [FIM99], we define the *group of the random walk* as the group G generated by Ψ and Φ. This is well known, see *e.g.* [BMM08], that G is of order eight for the process considered here : in other words $\inf\{n \in \mathbb{N} : (\Phi \circ \Psi)^n = \mathrm{id}\} = 4$.

If $(x, y) \in \mathbb{C}^2$ is such that $L(x, y, z) = 0$ and if θ is any element of G, then obviously $L(\theta(x, y), z) = 0$. In other words the group G can also be understood as a group of automorphisms of the algebraic curve $\mathscr{L}$.

It is also shown in Part 3.1 of [FIM99] that these automorphisms Ψ and Φ defined on $\mathscr{L}$ become on $\mathbb{C}/\Omega$ the automorphisms ψ and ϕ with the following expressions :

$$\psi(\omega) = -\omega, \qquad \phi(\omega) = -\omega + \omega_3. \tag{22}$$

They are such that $\psi^2 = \phi^2 = \mathrm{id}$, $x \circ \psi = x$, $y \circ \psi = 1/(x^2 y)$, $x \circ \phi = 1/(xy)$ and $y \circ \phi = y$. A crucial fact is the following.

Proposition 16. *For all $z \in]0, 1/4[$, we have $\omega_3 = 3\omega_2/4$.*

Proof. Since the group generated by Ψ and Φ is of order eight, so is the group generated by ψ and ϕ, in other words $\inf\{n \in \mathbb{N} : (\phi \circ \psi)^n = \mathrm{id}\} = 4$. With (22) this immediately implies that $4\omega_3$ is some point of the lattice Ω, contrary to ω_3, $2\omega_3$ and $3\omega_3$. But we already know that $\omega_3 \in]0, \omega_2[$ so that two possibilities remain : either $\omega_3 = \omega_2/4$ or $\omega_3 = 3\omega_2/4$.

In addition, essentially because the covariance of Gessel's walk is positive, we can use the same arguments as in Section 4 of [KR09] (see page 14) and in this way we obtain that ω_3 is necessary larger than $\omega_2/2$, which entails Proposition 16. □

1.2 Implicit expression and global properties of the CGF

As said in Section 0, the *existence* of CGF for the curves $X([y_1, y_2])$ and $Y([x_1, x_2])$ follows from general results on conformal gluing, see *e.g.* [Lit00] ; actually finding *explicit* expressions for CGF is more problematic.

But by using the same analysis as in Part 5.5 of [FIM99], we obtain explicitly suitable CGF for these curves. Before writing the expression of these CGF, let us recall that $\wp$, and therefore also x, take each value of $\mathbb{C} \cup \{\infty\}$ twice on $[0, \omega_2[\times[0, \omega_1/\imath[$, but are one-to-one on $[0, \omega_2/2[\times[0, \omega_1/\imath[$. In particular, on this half-parallelogram x admits a reciprocal function, that we denote by x^{-1}.

Then with [FIM99] we state :

$$w(t) = \wp_{1,3}\big(x^{-1}(t) - (\omega_1 + \omega_2)/2\big), \tag{23}$$

$\wp_{1,3}$ being the Weierstrass elliptic function with periods ω_1, ω_3 and x^{-1} the reciprocal function of the first coordinate of the uniformization (19) ; the periods $\omega_1, \omega_2, \omega_3$ are defined in (18) and (21).

In Section 4 of [KR09], we have studied some properties of the function defined by (23), and we have shown that if $\omega_3 > \omega_2/2$ (which is actually the case here, see Proposition 16), then the function (23) is meromorphic on $\mathbb{C} \setminus [x_3, x_4]$ and has there a unique pole, at x_2.

In order to find explicitly a CGF for the curve $Y([x_1, x_2])$, we remark that

$$\widetilde{w}(t) = w(X_0(t)) \tag{24}$$

is suitable – this is a consequence of the facts that w is a CGF for $X([y_1, y_2])$ and that $X_0 : \mathscr{G}Y([x_1, x_2]) \setminus [y_1, y_2] \to \mathscr{G}X([y_1, y_2]) \setminus [x_1, x_2]$ is conformal, as stated in Lemma 17.

More globally, $\tilde{w}$ defined by (24) is meromorphic on $\mathbb{C} \setminus [y_3, y_4]$ and has there a unique pole, of order two and at $Y(x_2) = x_3$ – this is a consequence of some properties of w already mentioned and of the fact that $X_0(\mathbb{C}) \subset \mathbb{C} \setminus [x_3, x_4]$, see also Lemma 17 (for the proof, we refer to Part 5.3 of [FIM99]).

Lemma 17. *$X_0 : \mathscr{G}Y([x_1, x_2]) \setminus [y_1, y_2] \to \mathscr{G}X([y_1, y_2]) \setminus [x_1, x_2]$ and $Y_0 : \mathscr{G}X([y_1, y_2]) \setminus [x_1, x_2] \to \mathscr{G}Y([x_1, x_2]) \setminus [y_1, y_2]$ are conformal and reciprocal the one from the other. In addition, $X_0(\mathbb{C}) \subset \mathbb{C} \setminus [x_3, x_4]$ and $Y_0(\mathbb{C}) \subset \mathbb{C} \setminus [y_3, y_4]$*

Let us now complete the proof of Theorem 7, by showing the following.

Proposition 18. *The integral representations of* $z[Q(x,0,z)-Q(0,0,z)]$ *and* $z[(y+1)Q(0,y,z)-Q(0,0,z)]$ *given in Theorem 7 hold not only on* $\mathscr{G}X([y_1,y_2])$ *and* $\mathscr{G}X([x_1,x_2])$ *but on* $\mathbb{C}\setminus[x_3,x_4]$ *and* $\mathbb{C}\setminus[y_3,y_4]$.

Proof. It is clear from their explicit expression that these integral representations can be continued from $\mathscr{G}X([y_1,y_2])$ and $\mathscr{G}X([x_1,x_2])$ up to $\mathbb{C}\setminus\big([x_3,x_4]\cup(w^{-1}([x_1,x_2])\setminus[x_1,x_2])\big)$ and $\mathbb{C}\setminus\big([y_3,y_4]\cup(\tilde{w}^{-1}([y_1,y_2])\setminus[y_1,y_2])\big)$ respectively. In other words, in order to prove Proposition 18 it is enough to show that $w^{-1}([x_1,x_2])\setminus[x_1,x_2]=\emptyset$ and that $\tilde{w}^{-1}([y_1,y_2])\setminus[y_1,y_2]=\emptyset$.

To begin with, we explain why $w^{-1}([x_1,x_2])\setminus[x_1,x_2]=\emptyset$. By using the fact $\wp$ is one-to-one on $[0,\omega_2/2[\times[0,\omega_1/\imath[$, we easily obtain that $x^{-1}(\mathbb{C})=[0,\omega_2/2]\times[0,\omega_1/\imath[$. In particular, with Proposition 16, we have $x^{-1}(\mathbb{C})\subset]-\omega_3+\omega_2/2,\omega_2/2]\times[0,\omega_1/\imath[$. But $\wp_{1,3}$ takes each value of $\mathbb{C}\cup\{\infty\}$ twice on the parallelogram $]-\omega_3+\omega_2/2,\omega_2/2]\times[0,\omega_1/\imath[$, and with Lemma 14, $\wp_{1,3}([-\omega_1/2,0])=\wp_{1,3}([0,\omega_1/2])=w([x_1,x_2])$, so that we obtain $w^{-1}([x_1,x_2])\setminus[x_1,x_2]=\emptyset$

By using the same kind of arguments as above, as well as (24), we obtain that $\tilde{w}^{-1}([y_1,y_2])\setminus[y_1,y_2]=\emptyset$. □

1.3 Proof of Theorem 8

Proof of Theorem 8. We are going here to note $\omega_4=\omega_2/4$ and $\wp_{1,4}$ the Weierstrass elliptic function with periods ω_1,ω_4. Moreover, we recall that $\wp$ and $\wp_{1,3}$ are the Weierstrass elliptic functions with respective periods ω_1,ω_2 and $\omega_1,\omega_3=3\omega_2/4$.

To begin with, let us mention the following fact. Let $\breve{\wp}$ be the Weierstrass elliptic function with periods noted $\hat{\omega},\breve{\omega}$ and let n be some positive integer. Then the Weierstrass elliptic function with periods $\hat{\omega},\breve{\omega}/n$ can be written in terms of $\breve{\wp}$ as follows (see *e.g.* `http://functions.wolfram.com/EllipticFunctions/WeierstrassP/16/06/03/`) :

$$\breve{\wp}(\omega)+\sum_{k=1}^{n-1}\big[\breve{\wp}(\omega+k\breve{\omega}/n)-\breve{\wp}(k\breve{\omega}/n)\big]. \tag{25}$$

Then, *e.g.* by using the addition theorem (26) for the Weierstrass elliptic function $\breve{\wp}$ in (25) and next the identity (20), we obtain that *the Weierstrass elliptic function with periods* $\hat{\omega},\breve{\omega}/n$ *is a rational function of the Weierstrass elliptic function with periods* $\hat{\omega},\breve{\omega}$.

The proof of Theorem 8 will follow from applying this fact twice : (i) first, since $\omega_4=\omega_2/4$, we will express $\wp_{1,4}$ as a rational function of $\wp$, (ii) then, since $\omega_4=\omega_3/3$, we will express $\wp_{1,4}$ as a rational function of $\wp_{1,3}$.

Before making explicit the rational transformations that appear with (i) and (ii), we explain how to conclude the proof of Theorem 8. An immediate consequence of (i) and (ii) is the possibility of writing $\wp_{1,3}$ as an algebraic function of $\wp$. In particular, it is clear from that and from the addition theorem (26) for $\wp$ that the formula $w(t)=\wp_{1,3}\big(\wp^{-1}(f(t))-(\omega_1+\omega_2)/2\big)$, with $f(t)=d'(x_4)/(t-x_4)+d''(x_4)/6$ – which is the CGF under consideration, see (19) and (23) – defines an algebraic function of t.

Explicit expression of the rational function for (i). With (25) we can write

$$\wp_{1,4}(\omega) = \wp(\omega)+\wp(\omega+\omega_2/2)+\wp(\omega+\omega_2/4)+\wp(\omega+3\omega_2/4)-\wp(\omega_2/2)-\wp(\omega_2/4)-\wp(3\omega_2/4).$$

Then, by using the addition theorem for $\wp$, namely the following formula, valid for all $\omega, \tilde{\omega}$ – which can be found *e.g.* in [Law89] –,

$$\wp\big(\omega+\widetilde{\omega}\big) = -\wp\big(\omega\big) - \wp\big(\widetilde{\omega}\big) + \frac{1}{4}\left[\frac{\wp'\big(\omega\big) - \wp'\big(\widetilde{\omega}\big)}{\wp\big(\omega\big) - \wp\big(\widetilde{\omega}\big)}\right]^2, \tag{26}$$

as well as the equalities $\wp(\omega_2/4) = \wp(3\omega_2/4)$, $\wp'(\omega_2/4) = -\wp'(3\omega_2/4)$ and $\wp'(\omega_2/2) = 0$ – obtained from the facts that $\wp(\omega_2/2+\omega)$ is even and $\wp'(\omega_2/2+\omega)$ is odd –, we get

$$\wp_{1,4}(\omega) = -2\wp(\omega)+\frac{\wp'(\omega)^2+\wp'(\omega_2/4)^2}{2\big[\wp(\omega)-\wp(\omega_2/4)\big]^2}+\frac{\wp'(\omega)^2}{4\big[\wp(\omega)-\wp(\omega_2/2)\big]^2}-\wp(\omega_2/2)-2\wp(\omega_2/4). \tag{27}$$

Now we recall from the proof of Lemma 14 that $\wp(\omega_2/2) = f(x_1)$. In other words, for the right-hand side of (27) to be completely explicit, it remains to find the expressions of $\wp(\omega_2/4)$ and $\wp'(\omega_2/4)$ in terms of z.

But starting from the known value of $\wp(\omega_2/2)$, it is easy to obtain the expression of $\wp(\omega_2/4)$, by using *e.g.* the formula below (a proof of which being given in [Law89]) :

$$\wp\left(\omega_2/4\right) = \wp\left(\omega_2/2\right) + \big[\left(\wp\left(\omega_2/2\right) - \wp\left(\omega_1/2\right)\right)\left(\wp\left(\omega_2/2\right) - \wp\left(\left(\omega_1+\omega_2\right)/2\right)\right)\big]^{1/2}. \tag{28}$$

Then we use that $\wp(\omega_1/2) = f(x_3)$, $\wp((\omega_1+\omega_2)/2) = f(x_2)$, $\wp(\omega_2/2) = f(x_1)$ and after simplification we get $\wp(\omega_2/4) = (1+4z^2)/3$. As a consequence and with (20) and Lemma 14, we obtain $\wp'(\omega_2/4)^2 = 64z^4$. Since $\wp$ is decreasing on $]0,\omega_2/2[$, see [Law89], we have $\wp'(\omega_2/4) < 0$ and therefore $\wp'(\omega_2/4) = -8z^2$. In conclusion, the right-hand side of (27) is completely and explicitly known.

In particular, evaluating (27) at $\omega = \wp^{-1}(f(t)) - (\omega_1+\omega_2)/2$ and using again the addition formula (26) for $\wp$, we obtain that the right-hand side of (27) is a rational function of t that can be explicitly obtained in terms of t and z ; after a substantial but elementary calculation we get $\wp_{1,4}\big(\wp^{-1}(f(t)) - (\omega_1+\omega_2)/2\big) = F(t)$, F being defined in (10).

Explicit expression of the rational function for (ii). Using the same arguments that have allowed us to obtain (27) from (25), we obtain that $\wp_{1,4}$ is the following rational function of $\wp_{1,3}$:

$$\wp_{1,4}(\omega) = -\wp_{1,3}(\omega) + \frac{\wp'_{1,3}(\omega)^2+\wp'_{1,3}(\omega_3/3)^2}{2\big[\wp_{1,3}(\omega)-\wp_{1,3}(\omega_3/3)\big]^2} - 4\wp_{1,3}(\omega_3/3). \tag{29}$$

By using (29) and the equality $\wp'_{1,3}(\omega)^2 = 4\wp_{1,3}(\omega)^3 - g_{2,1,3}\wp_{1,3}(\omega) - g_{3,1,3}$, where $g_{2,1,3}, g_{3,1,3}$ are the invariants associated with $\wp_{1,3}$, we get that $\wp_{1,4}$ is a rational function of $\wp_{1,3}$; moreover, with Lemma 19, the equality $\wp'_{1,3}(\omega_3/3)^2 = 4\wp_{1,3}(\omega_3/3)^3 - g_{2,1,3}\wp_{1,3}(\omega_3/3) - g_{3,1,3}$ and Lemma 20, the coefficients of this rational function in terms of z are explicitly known.

Proof of (11). Now we remark that Lemmas 19 and 20 allow us to write (29) as

$$\wp_{1,3}(\omega)^3 - \wp_{1,3}(\omega)^2\big[\wp_{1,4}(\omega) + 2K\big] \;+\; \wp_{1,3}(\omega)\big[2K\wp_{1,4}(\omega) + K^2/3 + G_2/2\big] \\ - \;\big[K^2\wp_{1,4}(\omega) + 19G_2K/18 + G_3 - 46K^3/27\big] = 0.$$

In particular, evaluating this equality at $\omega = \wp^{-1}(f(t)) - (\omega_1 + \omega_2)/2$, using the fact already proved that $\wp_{1,4}\big(\wp^{-1}(f(t)) - (\omega_1 + \omega_2)/2\big) = F(t)$ as well as the definition (23) of w, we obtain (11).

End of the proof of Theorem 8. If F is infinite at some point, then the equality (11) becomes $(w - K)^2 = 0$. In particular, at a such point at least two roots of (11) take finite values. In addition, by using the root-coefficient relationships, it is clear that at a point where F is infinite, at least one root of (11) is infinite. This proves that at any point where F is infinite, there is one and only one root of (11) which is infinite.

In particular, since F is infinite at x_2, see (10), and since w has a pole at x_2, see Subsection 1.2, w can be characterized as the only solution of (11) with a pole at x_2.

Likewise, we could prove the corresponding fact for $\tilde{w}$. Theorem 8 is proved. □

The two following results have been used in the proof of Theorem 8. Let G_2, G_3, K be the quantities defined in Section 0 (above the statement of Theorem 8).

Lemma 19. $\wp_{1,3}(\omega_3/3) = K$.

Lemma 20. $g_{2,1,3}, g_{3,1,3}$, *the invariants of* $\wp_{1,3}$, *have the following explicit expressions :*

$$g_{2,1,3} = 40K^2/3 - G_2, \qquad g_{3,1,3} = -280K^3/27 + 14KG_2/9 + G_3.$$

Proof of Lemmas 19 and 20. Start by expanding $\wp_{1,4}$ at 0 in two different ways. Firstly, by using (27) and by simplifying we obtain :

$$\wp_{1,4}(\omega) = \omega^{-2} + \big[9G_2/20\big]\omega^2 - \big[27G_3/28\big]\omega^4 + O(\omega^6). \tag{30}$$

Secondly we can also use (29) in order to expand $\wp_{1,4}$ at 0 ; after some calculation we get :

$$\wp_{1,4}(\omega) = \omega^{-2} + \big[6K^2 - 9g_{2,1,3}/20\big]\omega^2 + \big[10K^3 - 3Kg_{2,1,3}/2 - 27g_{3,1,3}/28\big]\omega^4 + O(\omega^6). \tag{31}$$

Lemma 20 follows then immediately, by identifying the expansions (30) and (31).

As for Lemma 19, it will be a consequence of Lemma 20 and of the following result, proved *e.g.* in [Law89] : the quantity $K = \wp_{1,3}(\omega_3/3)$ is the only positive solution of the following equation : $K^4 - g_{2,1,3}K^2/2 - g_{3,1,3}K - g_{2,1,3}^2/48 = 0$. But thanks to Lemma 20, we can replace $g_{2,1,3}$ and $g_{3,1,3}$ by their expression in terms of G_2, G_3, K ; in this way we obtain that K verifies the equation $K^4 - G_2K^2/2 - G_3K - G_2^2/48 = 0$. □

2 Holomorphic continuation of $zQ(x, 0, z)$ and $z(y+1)Q(0, y, z)$

In this part we are going to prove Theorem 3, in other words we are going to show that $zQ(x, 0, z)$ and $z(y + 1)Q(0, y, z)$ can be holomorphically continued from their unit disc up to $\mathbb{C} \setminus [x_3, x_4]$ and $\mathbb{C} \setminus [y_3, y_4]$ respectively.

In fact, we are going to show that $Q(x,0,z)$ and $Q(0,y,z)$ can be holomorphically continued up to $\mathbb{C}\setminus[x_3,x_4]$ and $\mathbb{C}\setminus[y_3,y_4]$ respectively, which is an equivalent assertion, as shown at the end of the proof of Theorem 21.

For this we will use the following procedure :

(i) First, we will lift the functions $Q(x,0,z)$ and $Q(0,y,z)$ up to $\mathbb{C}/\Omega$ by setting $q_x(\omega)=Q(x(\omega),0,z)$ and $q_y(\omega)=Q(0,y(\omega),z)$. The functions q_x and q_y are *a priori* well defined on $x^{-1}(\{|x|\leq 1\})$ and $y^{-1}(\{|y|\leq 1\})$ respectively.

(ii) Then, we will prove the following.

Theorem 21. *q_x and q_y, initially well defined on $x^{-1}(\{|x|\leq 1\})$ and $y^{-1}(\{|y|\leq 1\})$ respectively, can be holomorphically continued up to the whole parallelogram $\mathbb{C}/\Omega$ cut along respectively $[0,\omega_1[$ and $[0,\omega_1[+\omega_3/2$. Moreover, these continuations verify*

$$\forall\omega\in\mathbb{C}/\Omega\setminus[0,\omega_1[\ :\ q_x(\omega)=q_x(\psi(\omega)),\quad \forall\omega\in\mathbb{C}/\Omega\setminus\big([0,\omega_1[+\omega_3/2\big)\ :\ q_y(\omega)=q_y(\phi(\omega)), \tag{32}$$

and

$$\forall\omega\in]3\omega_2/8,\omega_2[\times[0,\omega_1/\imath[\ :\ zq_x(\omega)+z(y(\omega)+1)q_y(\omega)-zQ(0,0,z)-x(\omega)y(\omega)=0. \tag{33}$$

Remark 22. *Moreover, both (4) and (5) are immediate consequences of (33).*

(iii) Finally, we will set $Q(x,0,z)=q_x(\omega)$ if $x(\omega)=x$ and $Q(0,y,z)=q_y(\omega)$ if $y(\omega)=y$. Thanks to (32) and Proposition 15, these equalities define $Q(x,0,z)$ and $Q(0,y,z)$ on respectively $\mathbb{C}\setminus[x_3,x_4]$ and $\mathbb{C}\setminus[y_3,y_4]$ not ambiguously, as holomorphic functions.

Items (i) and (iii) are straightforward. For the proof of (ii), it will be useful first to find the location of the cycles $x^{-1}(\{|x|=1\})$ and $y^{-1}(\{|y|=1\})$ on $\mathbb{C}/\Omega$, this is the subject of the following result, illustrated on Figure 4 below.

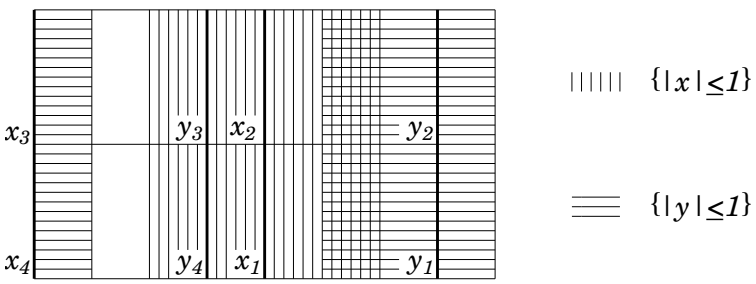

Figure 4: Location of the important cycles on the surface $[0,\omega_2[\times[0,\omega_1/\imath[$

Proposition 23. *We have $x^{-1}(\{|x|=1\})=\big([0,\omega_1[+\omega_2/4\big)\cup\big([0,\omega_1[+3\omega_2/4\big)$ and $y^{-1}(\{|y|=1\})=\big([0,\omega_1[+\omega_2/8\big)\cup\big([0,\omega_1[+5\omega_2/8\big)$.*

Proof. The details are of course essentially the same for x and y, so that we are going to prove only the assertion concerning x. The proof will be composed of three steps.

But first of all we note that because of the equality $x \circ \psi = x$, it is sufficient to prove that $x^{-1}(\{|x| = 1\}) \cap \big([0, \omega_2/2[\times[0, \omega_1/\imath[\big) = [0, \omega_1[+\omega_2/4$ – the advantage of this being that $\wp$, and therefore also x, are one-to-one in the half-parallelogram $[0, \omega_2/2[\times[0, \omega_1/\imath[$.

Firstly, we prove that $x(\omega_2/4+\omega_1/2) = 1$. For this we recall that $\wp(\omega_2/4) = (1+4z^2)/3$, $\wp'(\omega_2/4) = -8z^2$, $\wp(\omega_1/2) = f(x_3)$ and $\wp'(\omega_2/2) = 0$, see the proofs of Theorem 8 and Lemma 14. Then with the addition theorem (26) we immediately obtain the explicit value of $\wp(\omega_2/4 + \omega_1/2)$. Finally, after a simple calculation and by using (19), we get $x(\omega_2/4 + \omega_1/2) = 1$.

Secondly, we show that $x^{-1}(\{|x| = 1\}) \cap \big([0, \omega_2/2[\times[0, \omega_1/\imath[\big) \subset [0, \omega_1[+\omega_2/4$. For this let $\theta \in [0, 2\pi[$. With (19) we have $x(\omega) = \exp(\imath\theta)$ if and only if $\wp(\omega) = f(\exp(\imath\theta))$. Since $\omega \in [0, \omega_2/2] \times [0, \omega_1/\imath[$, we can use the well-known explicit expression of the reciprocal function of $\wp$ and with the first step we obtain :

$$\omega = \omega_2/4+\omega_1/2+\int_{f(1)}^{f(\exp(\imath\theta))} \frac{\mathrm{d}t}{[4t^3 - g_2 t - g_3]^{1/2}} = \omega_2/4+\omega_1/2+\frac{1}{2}\int_{\exp(\imath\theta)}^{1} \frac{\mathrm{d}x}{[d(x)]^{1/2}}, \quad (34)$$

d being defined in Section 0 and g_2, g_3 in Lemma 14. Note that the second equality above is got with the same calculations as in Part 3.3 of [FIM99].

Now we remark that $d(x) = x^4 d(1/x)$. In particular, the change of variable $x \mapsto 1/x$ in the integral $\int_{\exp(\imath\theta)}^{1} \mathrm{d}x/[d(x)]^{1/2}$ yields $\int_{\exp(\imath\theta)}^{1} \mathrm{d}x/[d(x)]^{1/2} = -\int_{\exp(-\imath\theta)}^{1} \mathrm{d}x/[d(x)]^{1/2}$. As a consequence, this integral belongs to $\imath\mathbb{R}$.

In conclusion, with (34) we have actually shown that $x^{-1}(\{|x| = 1\}) \cap \big([0, \omega_2/2[\times [0, \omega_1/\imath[\big) \subset [0, \omega_1[+\omega_2/4$.

Thirdly, we prove that the inclusion above has to be an equality. Indeed, if it was not the case the curve $x^{-1}(\{|x| = 1\}) \cap \big([0, \omega_2/2[\times[0, \omega_1/\imath[\big)$ would be curve *not* closed, which is a manifest contradiction with the facts that $\{|x| = 1\}$ is closed and that x is meromorphic and one-to-one in the half-parallelogram $[0, \omega_2/2[\times[0, \omega_1/\imath[$. □

Proof of Theorem 21. The proof is composed of two steps. We will first define the continuations of q_x and q_y on the whole parallelogram $\mathbb{C}/\Omega$ appropriately cut, and then we will verify that the functions so constructed actually verify the conclusions of Theorem 21.

- We define $q_x(\omega)$ on $x^{-1}(\{|x| \leq 1\})$ by $Q(x(\omega), 0, z)$ and $q_y(\omega)$ on $y^{-1}(\{|y| \leq 1\})$ by $Q(0, y(\omega), z)$ – note that as a consequence of Proposition 23 we have $x^{-1}(\{|x| \leq 1\}) = [\omega_2/4, 3\omega_2/4] \times [0, \omega_1/\imath[$ and $y^{-1}(\{|y| \leq 1\}) = [5\omega_2/8, 9\omega_2/8] \times [0, \omega_1/\imath[$.

- Motivated by (1), on $[3\omega_2/4, \omega_2[\times[0, \omega_1/\imath[\subset y^{-1}(\{|y| \leq 1\})$ we set $q_x(\omega) = -(y(\omega) + 1)q_y(\omega) + Q(0, 0, z) + x(\omega)y(\omega)/z$ and on $]3\omega_2/8, 5\omega_2/8] \times [0, \omega_1/\imath[\subset x^{-1}(\{|x| \leq 1\})$ we set $(y(\omega) + 1)q_y(\omega) = -q_x(\omega) + Q(0, 0, z) + x(\omega)y(\omega)/z$.

- On $]0, \omega_2/4] \times [0, \omega_1/\imath[$ we define $q_x(\omega)$ by $q_x(\phi(\omega))$ – note that with (22) we have $\phi(]0, \omega_2/4] \times [0, \omega_1/\imath[) = [3\omega_2/4, \omega_2[\times[0, \omega_1/\imath[$. On $[\omega_2/8, 3\omega_2/8[\times[0, \omega_1/\imath[$ we define $q_y(\omega)$ by $q_y(\psi(\omega))$ – by using (22) we have $\psi([\omega_2/8, 3\omega_2/8[\times[0, \omega_1/\imath[) =]3\omega_2/8, 5\omega_2/8] \times [0, \omega_1/\imath[$.

The functions q_x and q_y are now well defined on the whole parallelogram $\mathbb{C}/\Omega$ cut along $[0,\omega_1[$ and $[0,\omega_1[+\omega_3/2$ respectively.

Note that the definition given in the first item above is quite natural. The one stated in the second item is also natural since on $x^{-1}(\{|x| \leq 1\}) \cap y^{-1}(\{|y| \leq 1\}) = [5\omega_2/8, 3\omega_2/4] \times [0,\omega_1/\imath[$, the equality $q_x(\omega)+(y(\omega)+1)q_y(\omega)-Q(0,0,z)-x(\omega)y(\omega)/z = 0$ holds, see (1). The definition set in the third item is to ensure that (32) is valid.

Let us now prove that the functions q_x and q_y so continued verify the different assertions of Theorem 21.

Note first that (33) is immediately true, by construction of the continuations.

We are now going to verify (32) for q_x. By using the first item above as well as the equality $x \circ \psi = x$, (32) is obviously verified on $[\omega_2/4, 3\omega_2/4] \times [0,\omega_1/\imath[= \psi([\omega_2/4, 3\omega_2/4] \times [0,\omega_1/\imath[)$. Moreover, with the third item, (32) is verified for q_x on $]0,\omega_2/4] \times [0,\omega_1/\imath[$, and since $\psi^2 = \mathrm{id}$, (32) is also true for q_x on $[3\omega_2/4,\omega_2[\times[0,\omega_1/\imath[$, and finally on the whole $\mathbb{C}/\Omega \setminus [0,\omega_1[$.

Likewise, we verify easily that (32) is valid for q_y on $\mathbb{C}/\Omega \setminus \big([0,\omega_1[+3\omega_2/8\big)$.

It remains to prove that the continuations of q_x and q_y are holomorphic on $\mathbb{C}/\Omega$ cut along $[0,\omega_1[$ and $[0,\omega_1[+3\omega_2/8$ respectively.

We show first that they are *meromorphic* on their respective cut parallelogram. For q_x the following cycles are *a priori* problematic : $[0,\omega_1[$, $[0,\omega_1[+\omega_2/4$ and $[0,\omega_1[+3\omega_2/4$.

In an open neighborhood of $[0,\omega_1[+3\omega_2/4$, we have $q_x(\omega) = -(y(\omega)+1)q_y(\omega) + Q(0,0,z) + x(\omega)y(\omega)/z$, so that q_x is in fact meromorphic in the neighborhood of the cycle $[0,\omega_1[+3\omega_2/4$. Since (32) holds, q_x is also meromorphic near $[0,\omega_1[+\omega_2/4 = \psi([0,\omega_1[+3\omega_2/4)$, so that only $[0,\omega_1[$ remains *a priori* (and actually is) a singular cycle.

Similarly, we could show that q_y is meromorphic on $\mathbb{C}/\Omega \setminus \big([0,\omega_1[+3\omega_2/8\big)$.

Let us now prove that these continuations are actually *holomorphic* on their respective cut parallelogram.

q_x is obviously holomorphic on $]\omega_2/4, 3\omega_2/4] \times [0,\omega_1/\imath[$, since it is there defined through the power series $Q(x,0,z)$.

On $]5\omega_2/8,\omega_2[\times[0,\omega_1/\imath[$, we have $q_x(\omega) = -(y(\omega)+1)q_y(\omega) + Q(0,0,z) + x(\omega)y(\omega)/z$, and all terms of the right-hand side of this equality are holomorphic on this domain – at $7\omega_2/8$, x has a pole of order one and y has a zero of order two, see Lemma 24 below, so that the product xy is holomorphic near $7\omega_2/8$.

On $]0,3\omega_2/8[\times[0,\omega_1/\imath[$, we have $q_x = q_x \circ \psi$, so that q_x is holomorphic on this domain since it is on $\psi(]0,3\omega_2/8[\times[0,\omega_1/\imath[) =]5\omega_2/8,\omega_2[\times[0,\omega_1/\imath[$.

Likewise, we could show that $(y+1)q_y$ is holomorphic on $\mathbb{C}/\Omega \setminus \big([0,\omega_1[+\omega_3/2\big)$. This implies that q_y is holomorphic on the same set except at the points where $y+1=0$. There are two possibilities in order to show that q_y is also holomorphic at the points where $y+1=0$, namely $\omega_2/8$ and $5\omega_2/8$, in accordance with Lemma 24.

First, we can use the fact that the generating function $Q(0,y,z)$ is bounded at -1, see Section 0, so that $q_y(\omega) = Q(0,y(\omega),z)$, being meromorphic and bounded near $\omega_2/8$ and $5\omega_2/8$, is actually holomorphic at these points.

We can also remark that with (33), $(y(5\omega_2/8)+1)q_y(5\omega_2/8) = 0$, since with Lemma 24, $x(5\omega_2/8) = 0$. Moreover, since $\phi(5\omega_2/8) = \omega_2/8$, $(y(\omega_2/8) + 1)q_y(\omega_2/8) = 0$. In other words, at $\omega = \omega_2/8$ and $\omega = 5\omega_2/8$, both holomorphic functions $(y+1)q_y$ and $(y+1)$ have a zero, the first one of order equal or larger than one, the second one of order exactly one ; it follows immediately that q_y is holomorphic at ω. $\square$

The following result, which has been used in the proof of Theorem 21, follows easily from Lemma 1 and from the fact that the Weiertrass elliptic function $\wp$ takes on the parallelogram $[0, \omega_2[\times[0, \omega_1/\imath[$ each value of $\mathbb{C} \cup \{\infty\}$ twice.

Lemma 24. *The only poles of x are at $\omega_2/8, 7\omega_2/8$ and its only zeros are at $3\omega_2/8, 5\omega_2/8$. The only pole of y (of order two) is at $3\omega_2/8$ and its only zero (of order two) is at $7\omega_2/8$. The only zeros of $y + 1$ are at $\omega_2/8, 5\omega_2/8$.*

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