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# Explicit expression of the counting generating function for Gessel's walk

Irina Kurkova\*

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December 2, 2009

## Abstract

We consider the so-called Gessel's walk, that is the planar random walk that is confined to the first quadrant and that can move in unit steps to the West, North-East, East and South-West. For this walk we make explicit the generating function of the number of paths starting at  $(0, 0)$  and ending at  $(i, j)$  in time  $k$ .

*Keywords* : lattice walks, generating function, Riemann-Hilbert boundary value problem, conformal gluing function, Weierstrass elliptic function, Riemann surface, uniformization.

*AMS 2000 Subject Classification* : primary 05A15 ; secondary 30F10, 30D05.

## 0 Introduction and main results

The enumeration of lattice walks is a classical problem in combinatorics. The one of Gessel's walk seems to puzzle the mathematics community already for several years [Ges86, PW08, KKZ09, Ayy09, Pin09, BK09]. This is a planar random walk that is confined to the first quadrant and that can move in the interior in unit steps to the West, North-East, East and South-West, see Figure 1. For  $(i, j) \in \mathbb{Z}_+^2$  and  $k \in \mathbb{Z}_+$ , set

$$q(i, j, k) = \#\{\text{walks starting at } (0, 0) \text{ and ending at } (i, j) \text{ in time } k\}.$$

I. Gessel conjectured around 2001 that  $q(2k, 0, 0) = 16^k [(5/6)_k (1/2)_k] / [(2)_k (5/3)_k]$ , where  $(a)_k = a(a+1) \cdots (a+k-1)$ . In 2008, M. Kauers, C. Koutschan and D. Zeilberger yielded a remarkable although heavily computer-aided proof of this conjecture, see [KKZ09].

The articles [Ayy09, Pin09] give connections between Gessel's walk and other interesting models. Namely, S. Ping in [Pin09] establishes a probabilistic model for Gessel's

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walk concerned with vicious walkers, and A. Ayyer in [Ayy09] interprets such walks as Dick words with two sets of letters and gives explicit formulas for a restricted class of such words. Both of these approaches are yet in some way of providing “human” proof of Gessel conjecture but may certainly help for a better understanding of Gessel’s walk.

M. Petkovsek and H. Wilf in [PW08] state some similar conjectures for the number of walks ending at other points. Two of them have been proved by S. Ping in [Pin09]. M. Petkovsek and H. Wilf in [PW08] obtain also an infinite lower-triangular system of linear equations satisfied by the values of  $f(k, i, 0)$  and  $f(k, 0, j) + f(k, 0, j - 1)$  and express these values as determinants of lower-Hessenberg matrices with unit superdiagonals whose non-zero entries are products of two binomial coefficients.

Finally, A. Bostan and M. Kauers in [BK09] show that the complete generating function for Gessel’s walk

$$Q(x, y, z) = \sum_{i, j, k \geq 0} q(i, j, k) x^i y^j z^k$$

is algebraic. Their proof involves, among other tools, computer calculations using a powerful computer algebra system Magma, it required immense computational effort.

Curiously, in spite of this vivid interest to Gessel’s walk, the complete generating function  $Q(x, y, z)$  or even  $Q(0, y, z)$  or  $Q(x, 0, z)$  have *not* been yet make explicit in a closed form. Furthermore, recently M. Bousquet-Mélou and M. Mishna have undertaken the systematic analysis of enumeration of the walks confined to the quarter plane  $\mathbb{Z}_+^2$  starting from the origin and making steps at any point of  $\mathbb{Z}_+^2$  from a given subset of  $\{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ . There are  $2^8$  such models. Moreover they show that, after eliminating trivial models and those that are equivalent to models of walks confined to a half-plane and solved by known methods, it remains 79 inherently different problems to study. Following the idea of Book [FIM99], they associate to each model a group  $G$  of birational transformations (for details on this group, see Subsection 1.1 below). This group is finite in 23 cases and infinite in the 56 other cases. They are able to solve 22 models associated with a finite group. The only case with finite group that remained unsolved is the model of Gessel’s walk.

The aim of this paper is to solve Gessel’s walk model, *i.e.* to represent in a closed form the generating function  $Q(x, y, z)$ .

Let us observe that for any  $i$  and  $j$ ,  $q(k, i, j) \leq 4^k$ , so that  $Q(x, y, z)$  is holomorphic in  $\{|x| < 1, |y| < 1, |z| < 1/4\}$  and continuous up to  $\{|x| \leq 1, |y| \leq 1, |z| < 1/4\}$ . Our starting point is the functional equation already stated in [BMM08] and exploited in [PW08], valid *a priori* on  $\{|x| \leq 1, |y| \leq 1, |z| < 1/4\}$  :

$$L(x, y, z)Q(x, y, z) = zQ(x, 0, z) + z(y + 1)Q(0, y, z) - zQ(0, 0, z) - xy, \quad (1)$$

where  $L(x, y, z) = xyz[1/x + 1/(xy) + x + xy - 1/z]$ .

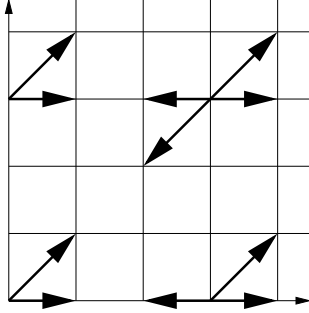


Figure 1: Gessel's walk

Our method heavily relies on the profound analytic approach developed in Book [FIM99] by G. Fayolle, R. Iasnogorodski and V. Malyshev. There the authors compute the generating functions of stationary probabilities for some ergodic random walks in the quarter plane. These random walks have four domains of spatial homogeneity : the interior  $\{(i, j) : i > 0, j > 0\}$ , the  $x$ -axis  $\{(i, 0) : i > 0\}$ , the  $y$ -axis  $\{(0, j) : j > 0\}$  and the origin  $\{0, 0\}$  ; in the interior the only (at most eight) possible non-zero jump probabilities are at distance one. They reduce the problem to the solution of the following functional equation on  $\{|x| \leq 1, |y| \leq 1\}$ ,

$$K(x, y)\Pi(x, y) = k(x, y)\pi(x) + \tilde{k}(x, y)\tilde{\pi}(y) + k_0(x, y)\pi_{00}, \quad (2)$$

with known polynomials  $K(x, y)$ ,  $k_0(x, y)$ ,  $k(x, y)$ ,  $\tilde{k}(x, y)$  and with functions  $\Pi(x, y)$ ,  $\pi(x)$ ,  $\tilde{\pi}(y)$  unknown but holomorphic in unit discs, continuous up to the boundary. First, they continue  $\pi(x)$  and  $\tilde{\pi}(y)$  as meromorphic (with poles that can be identified) to the whole complex plane cut along some segment. This ingenious continuation procedure is the crucial step of Book [FIM99]. After that, they show that  $\pi(x)$  and  $\tilde{\pi}(y)$  verify a boundary value problem of Riemann-Carleman type, and they solve it by converting it into a boundary value problem of Riemann-Hilbert type.

Compared to (2), our equation (1) seems a bit more difficult to analyze, as it involves a complementary parameter  $z$ , that we will suppose to be fixed in  $]0, 1/4[$ , so that  $zQ(x, 0, z)$  is an unknown function of  $x \in \{|x| \leq 1\}$  and  $z(y+1)Q(x, y, z)$  is an unknown function of  $y \in \{|y| \leq 1\}$ . From the other point of view, the coefficients  $k_0(x, y)$ ,  $k(x, y)$  and  $\tilde{k}(x, y)$  in front of unknowns  $zQ(x, 0, z)$ ,  $z(y+1)Q(0, y, z)$  and  $zQ(0, 0, z)$  are absent. This will allow us to continue  $zQ(x, 0, z)$  and  $z(y+1)Q(0, y, z)$  as holomorphic and not only meromorphic functions and, consequently, to simplify substantially the solution.

We are now going to state the main results of this paper. To begin with, let us have a closer look to the kernel  $L(x, y, z)$  that appears in (1) and let us take some notations.

The polynomial  $L(x, y, z)$  can be written as  $L(x, y, z) = \tilde{a}(y, z)x^2 + \tilde{b}(y, z)x + \tilde{c}(y, z) = a(x, z)y^2 + b(x, z)y + c(x, z)$ , with  $\tilde{a}(y, z) = zy(y+1)$ ,  $\tilde{b}(y, z) = -y$ ,  $\tilde{c}(y, z) = z(y+1)$

and  $a(x, z) = zx^2$ ,  $b(x, z) = zx^2 - x + z$ ,  $c(x, z) = z$ . Define also  $\tilde{d}(y, z) = \tilde{b}(y, z)^2 - 4\tilde{a}(y, z)\tilde{c}(y, z)$  and  $d(x, z) = b(x, z)^2 - 4a(x, z)c(x, z)$ .

For any  $z \in ]0, 1/4[$ ,  $\tilde{d}$  has one root equal to zero and two real positive roots, that we denote by  $y_2(z) < 1 < y_3(z)$ . We have  $y_2(z) = [1 - 8z^2 - (1 - 16z^2)^{1/2}]/[8z^2]$  and  $y_3(z) = [1 - 8z^2 + (1 - 16z^2)^{1/2}]/[8z^2]$ ; we will also note  $y_1(z) = 0$  and  $y_4(z) = \infty$ .

Likewise, for all  $z \in ]0, 1/4[$ ,  $d$  has four real positive roots, that we denote by  $x_1(z) < x_2(z) < 1 < x_3(z) < x_4(z)$ . Their explicit expression is  $x_1(z) = [1 + 2z - (1 + 4z)^{1/2}]/[2z]$ ,  $x_2(z) = [1 - 2z - (1 - 4z)^{1/2}]/[2z]$ ,  $x_3(z) = [1 - 2z + (1 - 4z)^{1/2}]/[2z]$  and  $x_4(z) = [1 + 2z + (1 + 4z)^{1/2}]/[2z]$ .

With these notations we have  $L(x, y, z) = 0$  if and only if  $(\tilde{b}(y, z) + 2\tilde{a}(y, z)x)^2 = \tilde{d}(y, z)$  or  $(b(x, z) + 2a(x, z)y)^2 = d(x, z)$ . In particular, the algebraic functions  $X(y, z)$  and  $Y(x, z)$  defined by  $L(X(y, z), y, z) = 0$  and  $L(x, Y(x, z), z) = 0$  have two branches, meromorphic on respectively  $\mathbb{C} \setminus ([y_1(z), y_2(z)] \cup [y_3(z), y_4(z)])$  and  $\mathbb{C} \setminus ([x_1(z), x_2(z)] \cup [x_3(z), x_4(z)])$ .

The following straightforward results give some properties of the two branches of the algebraic functions  $X(y, z)$  and  $Y(x, z)$ .

**Lemma 1.** *Call  $X_0(y, z) = [-\tilde{b}(y, z) + \tilde{d}(y, z)^{1/2}]/[2\tilde{a}(y, z)]$  and  $X_1(y, z) = [-\tilde{b}(y, z) - \tilde{d}(y, z)^{1/2}]/[2\tilde{a}(y, z)]$  the branches of  $X(y, z)$ . For all  $y \in \mathbb{C}$ , we have  $|X_0(y, z)| \leq |X_1(y, z)|$ .*

*On  $\mathbb{C} \setminus ([y_1(z), y_2(z)] \cup [y_3(z), y_4(z)])$ ,  $X_0$  has a simple zero at  $-1$ , no other zero and no pole;  $X_1$  has a simple pole at  $-1$ , no other pole and no zero. Finally, both  $X_0$  and  $X_1$  become infinite at  $y_1(z) = 0$  and zero at  $y_4(z) = \infty$ .*

*Now we call  $Y_0(x, z) = [-b(x, z) + d(x, z)^{1/2}]/[2a(x, z)]$  and  $Y_1(x, z) = [-b(x, z) - d(x, z)^{1/2}]/[2a(x, z)]$  the branches of  $Y(x, z)$ . For all  $x \in \mathbb{C}$ , we have  $|Y_0(x, z)| \leq |Y_1(x, z)|$ .*

*On  $\mathbb{C} \setminus ([x_1(z), x_2(z)] \cup [x_3(z), x_4(z)])$ ,  $Y_0$  has a double zero at  $\infty$ , no other zero and no pole;  $Y_1$  has a double pole at  $0$ , no other pole and no zero.*

Both  $X_i(y, z)$ ,  $i = 0, 1$ , are not defined for  $y$  in a branch cut, in other words for  $y \in [y_1(z), y_2(z)] \cup [y_3(z), y_4(z)]$ . However, the limits  $X_i^\pm(y, z)$  defined by  $X_i^+(y, z) = \lim X_i(\hat{y}, z)$  as  $\hat{y} \rightarrow y$  from the *upper* side of the cut and  $X_i^-(y, z) = \lim X_i(\hat{y}, z)$  as  $\hat{y} \rightarrow y$  from the *lower* side of the cut are well defined. Since for  $y$  in a branch cut,  $\tilde{d}(y, z) < 0$ , these two quantities are complex conjugate the one from the other.

A similar remark holds for  $Y_i(x, z)$ ,  $i = 0, 1$ , for  $x \in [x_1(z), x_2(z)] \cup [x_3(z), x_4(z)]$ .

In fact we have :

$$X_0^\pm(y, z) = \frac{-\tilde{b}(y, z) \mp i[-\tilde{d}(y, z)]^{1/2}}{2\tilde{a}(y, z)}, \quad Y_0^\pm(x, z) = \frac{-b(x, z) \mp i[-d(x, z)]^{1/2}}{2a(x, z)}, \quad (3)$$

$$X_1^\pm(y, z) = X_0^\mp(y, z) \text{ and } Y_1^\pm(x, z) = Y_0^\mp(x, z).$$

**Lemma 2.** *Consider  $X([y_1(z), y_2(z)], z)$  and  $Y([x_1(z), x_2(z)], z)$ . (i) These two curves are symmetrical w.r.t. the real axis and not included in the unit disc. (ii)  $X([y_1(z), y_2(z)], z)$  contains  $\infty$ ,  $Y([x_1(z), x_2(z)], z)$  is closed. (iii) Both of them split the plane into*

two connected components, we call  $\mathcal{G}X([y_1(z), y_2(z)], z)$  and  $\mathcal{G}Y([x_1(z), x_2(z)], z)$  the connected components of 0. They verify  $\mathcal{G}X([y_1(z), y_2(z)], z) \subset \mathbb{C} \setminus [x_3(z), x_4(z)]$  and  $\mathcal{G}Y([x_1(z), x_2(z)], z) \subset \mathbb{C} \setminus [y_3(z), y_4(z)]$ .

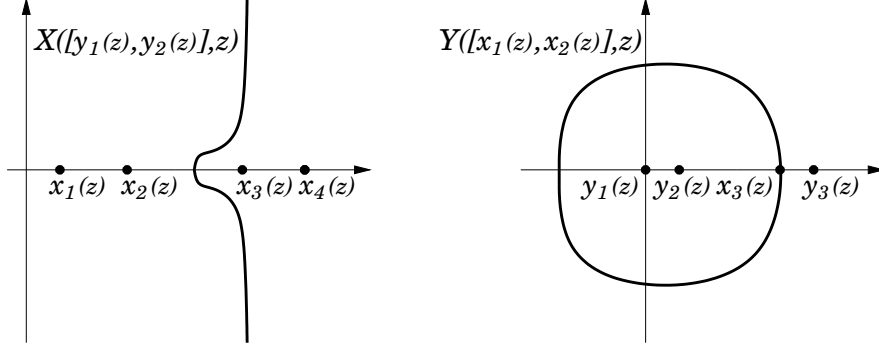


Figure 2: The curves  $X([y_1(z), y_2(z)], z)$  and  $Y([x_1(z), x_2(z)], z)$

Note that complete proofs of Lemmas 1 and 2 can be found in Part 5.3 of [FIM99]. These notations and results on the kernel  $L(x, y, z)$  are enough in order to state our results.

First of all, we would like to show that  $zQ(x, 0, z)$  and  $z(y + 1)Q(0, y, z)$  verify some boundary value problem of Riemann-Carleman type. It turns out that the associated boundary conditions verified by  $zQ(x, 0, z)$  and  $z(y + 1)Q(0, y, z)$  hold respectively on the curves  $X([y_1(z), y_2(z)], z)$  and  $Y([x_1(z), x_2(z)], z)$ , which are not included in unit disc, where the functions  $zQ(x, 0, z)$  and  $z(y + 1)Q(0, y, z)$  are thus *a priori* not defined. For this reason, we first need to continue the generating functions up to these curves. In fact we will prove the following – the proof of which being the central subject of Section 2.

**Theorem 3.** *The functions  $zQ(x, 0, z)$  and  $z(y + 1)Q(0, y, z)$  can be holomorphically continued from their unit disc up to  $\mathbb{C} \setminus [x_3(z), x_4(z)]$  and  $\mathbb{C} \setminus [y_3(z), y_4(z)]$  respectively. Furthermore for any  $y \in \mathbb{C} \setminus [y_1(z), y_2(z)] \cup [y_3(z), y_4(z)]$ ,*

$$zQ(X_0(y, z), 0, z) + z(y + 1)Q(0, y, z) - zQ(0, 0, z) - X_0(y, z)y = 0, \quad (4)$$

and for any  $x \in \mathbb{C} \setminus [x_1(z), x_2(z)] \cup [x_3(z), x_4(z)]$ ,

$$zQ(x, 0, z) + z(Y_0(x, z) + 1)Q(0, Y_0(x, z), z) - zQ(0, 0, z) - xY_0(x, z) = 0. \quad (5)$$

**Remark 4.** *For  $y \in \{|y| \leq 1\}$  such that  $|X_0(y, z)| \leq 1$ , (4) follows immediately from (1). Likewise, (5) is a straightforward consequence of (1) for any  $x \in \{|x| \leq 1\}$  such that  $|Y_0(x, z)| \leq 1$ . The fact that equations (4) and (5) are verified not only for these values of  $y$  and  $x$  but actually on  $\mathbb{C} \setminus [y_1(z), y_2(z)] \cup [y_3(z), y_4(z)]$  and  $\mathbb{C} \setminus [x_1(z), x_2(z)] \cup [x_3(z), x_4(z)]$  respectively will be shown in Section 2.*

**Remark 5.** *In the proof of Theorem 3, we will see that the function  $zQ(0, y, z)$  can also be holomorphically continued from the unit disc up to  $\mathbb{C} \setminus [y_3(z), y_4(z)]$ .*

Now we explain how to obtain the above mentioned boundary conditions verified by the functions  $zQ(x, 0, z)$  and  $z(y+1)Q(0, y, z)$ .

Let  $y \in [y_1(z), y_2(z)]$ , and let  $\hat{y}^+$  and  $\hat{y}^-$  be close to  $y$ , such that  $\hat{y}^+$  is in the *upper* half-plane and  $\hat{y}^-$  in the *lower* half-plane. Then we have (4) for both  $\hat{y}^+$  and  $\hat{y}^-$ . If now  $\hat{y}^+ \rightarrow y$  and  $\hat{y}^- \rightarrow y$ , then we obtain  $X_0(\hat{y}^+, z) \rightarrow X_0^+(y, z)$  and  $X_0(\hat{y}^-, z) \rightarrow X_0^-(y, z) = X_1^+(y, z)$ . So we have proved that for any  $y \in [y_1(z), y_2(z)]$ ,

$$zQ(X_0^+(y, z), 0, z) + z(y+1)Q(0, y, z) - zQ(0, 0, z) - X_0^+(y, z)y = 0, \quad (6)$$

$$zQ(X_1^+(y, z), 0, z) + z(y+1)Q(0, y, z) - zQ(0, 0, z) - X_1^+(y, z)y = 0. \quad (7)$$

Subtracting (7) from (6) we get that for any  $y \in [y_1(z), y_2(z)]$ ,

$$z[Q(X_0^+(y, z), 0, z) - Q(X_1^+(y, z), 0, z)] = X_0^+(y, z)y - X_1^+(y, z)y. \quad (8)$$

Then, using the fact that for  $i = 0, 1$ ,  $y \in [y_1(z), y_2(z)]$  and  $z \in ]0, 1/4[$ ,  $Y_0(X_i^\pm(y, z), z) = y$  – which can be proved by elementary considerations starting from Lemma 1, or by the use of Lemma 17 –, we get the first part of (9) below :

$$\begin{aligned} \forall t \in X([y_1(z), y_2(z)], z) : \quad & z[Q(t, 0, z) - Q(\bar{t}, 0, z)] = tY_0(t, z) - \bar{t}Y_0(\bar{t}, z), \\ \forall t \in Y([x_1(z), x_2(z)], z) : & z[(t+1)Q(0, t, z) - (\bar{t}+1)Q(0, \bar{t}, z)] = X_0(t, z)t - X_0(\bar{t}, z)\bar{t}. \end{aligned} \quad (9)$$

Likewise, we could prove the second part of (9).

Note that as a consequence of (6) and (7), (4) is in some sense also verified for  $y \in [y_1(z), y_2(z)]$  – the same is true for (5) and  $x \in [x_1(z), x_2(z)]$ .

With Lemma 2, Theorem 3 and (9), we get that  $zQ(x, 0, z)$  and  $z(y+1)Q(0, y, z)$  can be found among the functions holomorphic in  $\mathcal{G}X([y_1(z), y_2(z)], z)$  and  $\mathcal{G}Y([x_1(z), x_2(z)], z)$ , continuous up to the boundary and verifying the boundary conditions (9).

A such problem is called a boundary value problem of Riemann-Carleman type. A standard way to solve it consists in converting it to a boundary value problem of Riemann-Hilbert type by use of a *conformal gluing function* (CGF).

For any detail about boundary value problems and conformal gluing, we refer to [Lit00].

**Definition 6.** *Let  $C \subset \mathbb{C} \setminus \{0\}$  be a curve symmetrical w.r.t. the real axis and splitting the complex plane into two connected components, and let  $\mathcal{G}C$  be the connected component of 0. A function  $u$  is said to be a CGF for the curve  $C$  if (i)  $u$  is meromorphic in  $\mathcal{G}C$  (ii)  $u$  establishes a conformal mapping of  $\mathcal{G}C$  onto the complex plane cut along some arc (iii) for all  $t \in C$ ,  $u(t) = u(\bar{t})$ .*

Let  $w(t, z)$  and  $\tilde{w}(t, z)$  be CGF for  $X([y_1(z), y_2(z)], z)$  and  $Y([x_1(z), x_2(z)], z)$  – the existence (but *no* explicit expression) of  $w$  and  $\tilde{w}$  is ensured by general results on conformal gluing.

Transforming the boundary value problems of Riemann-Carleman type into boundary value problems of Riemann-Hilbert type thanks to  $w$  and  $\tilde{w}$ , solving them and working out the solutions we will prove the following.

**Theorem 7.** *The function  $z[Q(x, 0, z) - Q(0, 0, z)]$  has the following explicit expression for  $z \in ]0, 1/4[$  and  $x \in \mathbb{C} \setminus [x_3(z), x_4(z)]$  :*

$$z[Q(x, 0, z) - Q(0, 0, z)] = xY_0(x, z) + \frac{1}{\pi} \int_{x_1(z)}^{x_2(z)} \frac{t[-d(t, z)]^{1/2}}{2a(t, z)} \left[ \frac{\partial_t w(t, z)}{w(t, z) - w(x, z)} - \frac{\partial_t w(t, z)}{w(t, z) - w(0, z)} \right] dt,$$

$w$  being a CGF for the curve  $X([y_1(z), y_2(z)], z)$ .

The function  $z[(y+1)Q(0, y, z) - Q(0, 0, z)]$  has the following explicit expression for  $z \in ]0, 1/4[$  and  $y \in \mathbb{C} \setminus [y_3(z), y_4(z)]$  :

$$z[(y+1)Q(0, y, z) - Q(0, 0, z)] = X_0(y, z)y + \frac{1}{\pi} \int_{y_1(z)}^{y_2(z)} \frac{t[-\tilde{d}(t, z)]^{1/2}}{2\tilde{a}(t, z)} \left[ \frac{\partial_t \tilde{w}(t, z)}{\tilde{w}(t, z) - \tilde{w}(y, z)} - \frac{\partial_t \tilde{w}(t, z)}{\tilde{w}(t, z) - \tilde{w}(0, z)} \right] dt,$$

$\tilde{w}$  being a CGF for the curve  $Y([x_1(z), x_2(z)], z)$ .

The function  $Q(0, 0, z)$  has the following explicit expression for  $z \in ]0, 1/4[$  :

$$Q(0, 0, z) = -\frac{1}{\pi} \int_{y_1(z)}^{y_2(z)} \frac{t[-\tilde{d}(t, z)]^{1/2}}{2\tilde{a}(t, z)} \left[ \frac{\partial_t \tilde{w}(t, z)}{\tilde{w}(t, z) - \tilde{w}(-1, z)} - \frac{\partial_t \tilde{w}(t, z)}{\tilde{w}(t, z) - \tilde{w}(0, z)} \right] dt,$$

$\tilde{w}$  being a CGF for the curve  $Y([x_1(z), x_2(z)], z)$ .

The function  $Q(x, y, z)$  has the explicit expression obtained by using the ones of  $Q(x, 0, z)$ ,  $Q(0, y, z)$  and  $Q(0, 0, z)$  in (1).

All functions in the integrands above are explicit, except for the CGF  $w$  and  $\tilde{w}$ . In [FIM99] suitable CGF are computed *implicitly* by means of the reciprocal of some known functions (see the formulas (24) and (25) below for the details). Starting from this representation, we are able to make *explicit* these functions for Gessel's walk.

In order to state the result we need to define  $G_2(z) = (4/27)(1 + 224z^2 + 256z^4)$ ,  $G_3(z) = (8/729)(1+16z^2)(1-24z+16z^2)(1+24z+16z^2)$ ,  $K(z)$  as the only positive root of  $K^4 - G_2(z)K^2/2 - G_3(z)K - G_2(z)^2/48 = 0$  - noting  $r_k(z) = [G_2(z) - \exp(2k\pi/3)(G_2(z))^3 - 27G_3(z)^2]^{1/3}/3$  we have  $K(z) = [-r_0(z)^{1/2} + r_1(z)^{1/2} + r_2(z)^{1/2}]/2$  - and

$$\begin{aligned} F(t, z) &= \frac{1 - 24z + 16z^2}{3} - \frac{4(1 - 4z)^2}{z} \frac{t^2}{(t - x_2(z))(t - 1)^2(t - x_3(z))}, \\ \tilde{F}(t, z) &= \frac{1 - 24z + 16z^2}{3} + \frac{4(1 - 4z)^2}{z} \frac{t(t + 1)^2}{[(t - x_2(z))(t - x_3(z))]^2}. \end{aligned} \quad (10)$$



**Theorem 8.** *A suitable CGF for the curve  $X([y_1(z), y_2(z)], z)$  is the only function having a pole at  $x_2(z)$  and solution of*

$$w^3 - w^2[F(t, z) + 2K(z)] + w[2K(z)F(t, z) + K(z)^2/3 + G_2(z)/2] - [K(z)^2F(t, z) + 19G_2(z)K(z)/18 + G_3(z) - 46K(z)^3/27] = 0. \quad (11)$$

*Likewise, a suitable CGF for the curve  $Y([x_1(z), x_2(z)], z)$  is the only function having a pole at  $x_3(z)$  and solution of the equation obtained from (11) by replacing  $F$  by  $\tilde{F}$ , see (10).*

Let us now outline some facts around Theorems 3, 7 and 8.

**Remark 9.** *Since (1) is valid at least on  $\{|x| \leq 1, |y| \leq 1, |z| < 1/4\}$ , then for any such  $(\hat{x}, \hat{y}, z)$  with  $L(\hat{x}, \hat{y}, z) = 0$ , the right-hand side of (1) equals zero, so that*

$$z[Q(\hat{x}, 0, z) - Q(0, 0, z)] + z[(\hat{y} + 1)Q(0, \hat{y}, z) - Q(0, 0, z)] + zQ(0, 0, z) - \hat{x}\hat{y} = 0. \quad (12)$$

*We deduce that*

$$zQ(0, 0, z) = -z[Q(\hat{x}, 0, z) - Q(0, 0, z)] - z[(\hat{y} + 1)Q(0, \hat{y}, z) - Q(0, 0, z)] + \hat{x}\hat{y}. \quad (13)$$

*In Theorem 7 we have chosen to substitute  $(\hat{x}, \hat{y}, z) = (0, -1, z)$ , which is such that  $L(\hat{x}, \hat{y}, z) = 0$  since  $X_0(-1, z) = 0$ , see Lemma 1. Moreover, we show in Theorem 3 that for any  $z \in ]0, 1/4[$ , the equation (12) is valid not only on  $\{(x, y) \in \mathbb{C}^2 : L(x, y, z) = 0\} \cap \{|x| \leq 1, |y| \leq 1\}$  but in a much larger domain of the algebraic curve  $\{(x, y) \in \mathbb{C}^2 : L(x, y, z) = 0\}$ . Namely, if  $(\hat{x}, \hat{y})$  is such that  $x \notin [x_3(z), x_4(z)]$  and  $\hat{y} = Y_0(\hat{x}, z)$  or  $y \notin [y_3(z), y_4(z)]$  and  $\hat{x} = X_0(\hat{y}, z)$ , then (12) is still valid. Substituting any  $(\hat{x}, \hat{y})$  from this domain into (12) yields us  $zQ(0, 0, z)$  as in (13).*

**Remark 10.** *With the analytical approach proposed here, it would be possible, without additional difficulty, to obtain explicitly the generating function of the number of walks beginning at an arbitrary initial state  $(i_0, j_0)$  and ending at  $(i, j)$  in time  $k$ . Indeed, the only significant difference is that the product  $xy$  in (1) would be then replaced by  $x^{i_0+1}y^{j_0+1}$ .*

**Remark 11.** *Making in Theorem 7 the changes of variable  $w = w(t, z)$  and  $\tilde{w} = \tilde{w}(t, z)$ , we obtain that the generating functions  $zQ(x, 0, z)$  and  $z(y + 1)Q(0, y, z)$  are essentially Cauchy-type integrals of algebraic functions.*

*In particular, it could be deduced from the work [PRY04] – which gives criteria for a Cauchy-type integral of an algebraic function to be algebraic – that as functions of  $x$  and  $y$  respectively,  $zQ(x, 0, z)$  and  $z(y + 1)Q(0, y, z)$  are algebraic functions, what would give an other proof to some results contained in [BK09].*

**Remark 12.** *In Theorem 7, the functions  $z[Q(x, 0, z) - Q(0, 0, z)]$  and  $z[(y + 1)Q(0, y, z) - Q(0, 0, z)]$  are written as the sum of two functions not holomorphic but algebraic near respectively  $[x_1(z), x_2(z)]$  and  $[y_1(z), y_2(z)]$ . The sum of these two algebraic functions is*

of course holomorphic near these segments. By an application of the residue theorem as in Section 4 of [KR09], we could write both generating functions as functions manifestly holomorphic near these segments and having in fact their first singularity at respectively  $x_3(z)$  and  $y_3(z)$ .

The rest of the paper is organized as follows.

In Section 1 we prove Theorem 8. There the implicit representation of the CGF given in [FIM99] (recalled here in Subsections 1.1 and 1.2) in a general setting is developed in Subsection 1.3 to the case of Gessel's walk.

The proof of Theorem 3 is postponed to the last Section 2. The main idea of the holomorphic continuation procedure is borrowed again from [FIM99], we show how it works with the parameter  $z \in ]0, 1/4[$ . Finally we give the proof of Theorem 7.

*Proof of Theorem 7.* The proof is composed of two steps : the first one, inspired by [FIM99], will allow us to obtain integral representations of the functions  $zQ(x, 0, z)$  and  $z(y+1)Q(0, y, z)$  on the curves  $X([y_1(z), y_2(z)], z)$  and  $Y([x_1(z), x_2(z)], z)$  ; the second one will consist in transforming these formulations into the integrals on real segments written in the statement of Theorem 7.

Let us begin by solving the boundary value problems of Riemann-Carleman type with boundary conditions (9). The use of CGF allows us, as in [FIM99] or [Lit00], to transform them into boundary value problems of Riemann-Hilbert type. Following again [FIM99] or [Lit00] we solve them and in this way we obtain representations of the unknown functions  $zQ(x, 0, z)$  and  $z(y+1)Q(0, y, z)$  as integrals along the curves  $X([y_1(z), y_2(z)], z)$  and  $Y([x_1(z), x_2(z)], z)$ . For  $zQ(x, 0, z)$ , we get that up to some additive function of  $z$ ,

$$zQ(x, 0, z) = \frac{1}{2\pi i} \int_{X([y_1(z), y_2(z)], z)} tY_0(t, z) \frac{\partial_t w(t, z)}{w(t, z) - w(x, z)} dt, \quad (14)$$

where  $w$  is the CGF used for  $X([y_1(z), y_2(z)], z)$ . Similarly, we could write an integral representation of  $z(y+1)Q(0, y, z)$ .

We are now going to transform this integral representation of  $zQ(x, 0, z)$ . To begin with, let  $C(\epsilon, z)$  be any contour such that

- (i)  $C(\epsilon, z)$  is connected and contains  $\infty$ ,
- (ii)  $C(\epsilon, z) \subset (\mathcal{G}X([y_1(z), y_2(z)], z) \cup X([y_1(z), y_2(z)], z)) \setminus [x_1(z), x_2(z)]$ ,
- (iii)  $\lim_{\epsilon \rightarrow 0} C(\epsilon, z) = X([y_1(z), y_2(z)], z) \cup S(z)$ , where we have denoted by  $S(z)$  the segment  $[x_1(z), X(y_2(z), z)]$  traversed from  $X(y_2(z), z)$  to  $x_1(z)$  along the lower edge of the slit and then back to  $X(y_2(z), z)$  along the upper edge,

and let  $\mathcal{G}C(\epsilon, z)$  be the connected component of 0 of  $\mathbb{C} \setminus C(\epsilon, z)$ .

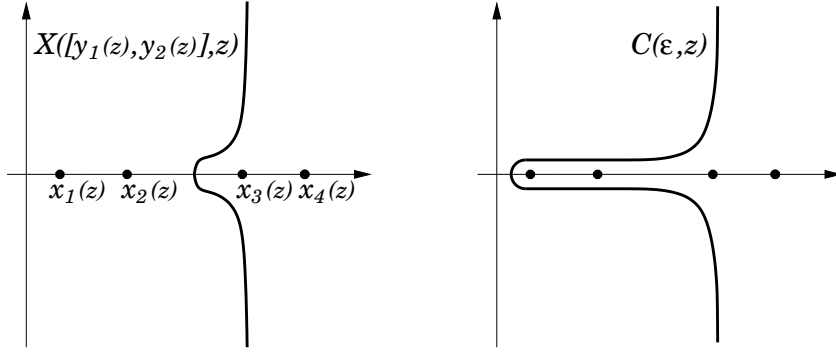


Figure 3: The curve  $X([y_1(z), y_2(z)], z)$  and the new contour of integration  $C(\epsilon, z)$

Now we apply the residue theorem to the integrand of (14) on the contour  $C(\epsilon, z)$ . As a function of  $t$ ,  $tY_0(t, z)$  is holomorphic  $\mathcal{G}C(\epsilon, z)$ , thanks to Lemma 1 and the property (ii) of the contour. Likewise, by using Definition 6 and the property (ii), we get that  $\partial_t w(t, z)/(w(t, z) - w(x, z))$  is meromorphic on  $\mathcal{G}C(\epsilon, z)$ , with a unique pole at  $t = x$ . Therefore we have :

$$\frac{1}{2\pi i} \int_{C(\epsilon, z)} tY_0(t, z) \frac{\partial_t w(t, z)}{w(t, z) - w(x, z)} dt = xY_0(x, z). \quad (15)$$

Then, making  $\epsilon$  go to 0, using (14), (15) and the property (iii) of the contour, we obtain that up to an additive function of  $z$ ,

$$zQ(x, 0, z) = xY_0(x, z) - \frac{1}{2\pi i} \int_{S(z)} tY_0(t, z) \frac{\partial_t w(t, z)}{w(t, z) - w(x, z)} dt. \quad (16)$$

Since the integrand in (16) is holomorphic at any point of  $]x_2(z), X(y_2(z), z)[$ , we have

$$\int_{S(z)} tY_0(t, z) \frac{\partial_t w(t, z)}{w(t, z) - w(x, z)} dt = \int_{x_1(z)}^{x_2(z)} [tY_0^+(t, z) - tY_0^-(t, z)] \frac{\partial_t w(t, z)}{w(t, z) - w(x, z)} dt, \quad (17)$$

so that with (3) we immediately obtain the expression of  $z[Q(x, 0, z) - Q(0, 0, z)]$  stated in Theorem 7.

Likewise, we could obtain the expression of  $z[(y + 1)Q(0, y, z) - Q(0, 0, z)]$  written in Theorem 7. The formula for the  $Q(0, 0, z)$  has been already proven in Remark 9.  $\square$

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# 1 Study of the conformal gluing function

**Notation.** For the sake of shortness we will, from now on, drop the dependence of the different quantities w.r.t.  $z \in ]0, 1/4[$ .

The main subject of Section 1 is to prove Theorem 8. In other words, we will prove that the CGF proposed in [FIM99] satisfy the conclusions of Theorem 8.

This definition of the CGF given in [FIM99] is recalled here in Subsection 1.2, see particularly (24) and (25). It uses some functions defined on a uniformization of the algebraic curve  $\{(x, y) \in \mathbb{C}^2 : L(x, y, z) = 0\}$ , so that we begin Section 1 by studying this uniformization ; this Subsection 1.1 is also necessary in Section 2, where we will prove Theorem 3.

## 1.1 Uniformization

We will note  $\mathcal{L}$  the algebraic curve  $\{(x, y) \in \mathbb{C}^2 : L(x, y, z) = 0\}$ ,  $L$  being defined in (1).

**Proposition 13.** *For any  $z \in ]0, 1/4[$ ,  $\mathcal{L}$  is a Riemann surface of genus one.*

*Proof.* We have shown in Section 0 that  $L(x, y, z) = 0$  if and only if  $(b(x) + 2a(x)y)^2 = d(x)$ . But the Riemann surface of the square root of a polynomial which has four distinct roots of order one has genus one, see *e.g.* [JS87], therefore the genus of  $\mathcal{L}$  is also one.  $\square$

With Proposition 13 it is immediate that  $\mathcal{L}$  is isomorphic to some torus ; in other words there exists a two-dimensional lattice  $\Omega$  such that  $\mathcal{L}$  is isomorphic to  $\mathbb{C}/\Omega$ . A suitable lattice  $\Omega$  (in fact the *only possible* lattice, up to a homothetic transformation) is made explicit in Parts 3.1 and 3.3 of [FIM99], namely  $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ , where

$$\omega_1 = \int_{x_1}^{x_2} \frac{dx}{[-d(x)]^{1/2}}, \quad \omega_2 = \int_{x_2}^{x_3} \frac{dx}{[d(x)]^{1/2}}. \quad (18)$$

We are now going to give a uniformization of the surface  $\mathcal{L}$ , in other words we will make explicit two functions  $x(\omega), y(\omega)$  elliptic w.r.t. the lattice  $\Omega$  such that  $\mathcal{L} = \{(x(\omega), y(\omega)), \omega \in \mathbb{C}\} = (\{(x(\omega), y(\omega)), \omega \in \mathbb{C}/\Omega\})$ . By using the same arguments as in Part 3.3 of [FIM99], we immediately obtain that we can take

$$x(\omega) = x_4 + \frac{d'(x_4)}{\wp(\omega) - d''(x_4)/6}, \quad y(\omega) = \frac{1}{2a(x(\omega))} \left[ -b(x(\omega)) + \frac{d'(x_4)\wp'(\omega)}{2(\wp(\omega) - d''(x_4)/6)^2} \right], \quad (19)$$

$\wp$  being the Weierstrass elliptic function with periods  $\omega_1, \omega_2$ .

By convenience, we will consider from now on that the coordinates of the uniformization  $x$  and  $y$  are defined on  $\mathbb{C}/\Omega$  rather than on  $\mathbb{C}$ .

It is well-known that  $\wp$  is characterized by its invariants  $g_2, g_3$  through

$$\wp'(\omega)^2 = 4\wp(\omega)^3 - g_2\wp(\omega) - g_3. \quad (20)$$

**Lemma 14.** *The invariants  $g_2, g_3$  of  $\wp$  are equal to :*

$$g_2 = (4/3)(1 - 16z^2 + 16z^4), \quad g_3 = -(8/27)(1 - 8z^2)(1 - 16z^2 - 8z^4).$$

*Proof.* It is well-known that  $4\wp(\omega)^3 - g_2\wp(\omega) - g_3 = 4(\wp(\omega) - \wp(\omega_1/2))(\wp(\omega) - \wp((\omega_1 + \omega_2)/2))(\wp(\omega) - \wp(\omega_2/2))$  ; in particular the invariants can be calculated in terms of the values of  $\wp$  at the half-periods. But it is proved in Part 3.3 of [FIM99] that setting  $f(t) = d'(x_4)/(t - x_4) + d''(x_4)/6$  we have  $\wp(\omega_1/2) = f(x_3)$ ,  $\wp((\omega_1 + \omega_2)/2) = f(x_2)$  and  $\wp(\omega_2/2) = f(x_1)$ , so that Lemma 14 follows from a direct calculation.  $\square$

Now that the uniformization (19) is completely and explicitly defined, it is natural to be interested in how it transforms the important cycles that are the branch cuts  $[x_1, x_2]$ ,  $[x_3, x_4]$ ,  $[y_1, y_2]$  and  $[y_3, y_4]$ . For this we need to define a new period, namely

$$\omega_3 = \int_{-\infty}^{x_1} \frac{dx}{[d(x)]^{1/2}}. \quad (21)$$

We will importantly use the fact that  $\omega_3 \in ]0, \omega_2[$  – this is proved in Part 3.3 of [FIM99].

**Proposition 15.** *We have  $x^{-1}([x_1, x_2]) = [0, \omega_1[ + \omega_2/2$  and  $x^{-1}([x_3, x_4]) = [0, \omega_1[$ ,  $y^{-1}([y_1, y_2]) = [0, \omega_1[ + (\omega_2 + \omega_3)/2$  and  $y^{-1}([y_3, y_4]) = [0, \omega_1[ + \omega_3/2$ .*

Proposition 15 follows from repeating some arguments in Part 5.5 of [FIM99], and is illustrated on Figure 4 below.

Now we define  $S(x, y) = 1/x + 1/(xy) + x + xy$ , the generating function of the jump probabilities of Gessel's walk, and we consider the following birational transformations :

$$\Psi(x, y) = (x, 1/(x^2y)), \quad \Phi(x, y) = (1/(xy), y). \quad (22)$$

They are such that  $S \circ \Psi = S \circ \Phi = S$ . Then, as in [FIM99], we define the *group of the random walk* as the group  $G$  generated by  $\Psi$  and  $\Phi$ . This is well known, see e.g. [BMM08], that  $G$  is of order eight for the process considered here : in other words  $\inf\{n \in \mathbb{N} : (\Phi \circ \Psi)^n = \text{id}\} = 4$ .

If  $(x, y) \in \mathbb{C}^2$  is such that  $L(x, y, z) = 0$  and if  $\theta$  is any element of  $G$ , then  $L(\theta(x, y), z) = 0$ . In other words the group  $G$  can also be understood as a group of automorphisms of the algebraic curve  $\mathcal{L}$ .

It is also shown in Part 3.1 of [FIM99] that the automorphisms  $\Psi$  and  $\Phi$  defined on  $\mathcal{L}$  become on  $\mathbb{C}/\Omega$  the automorphisms  $\psi$  and  $\phi$  with the following expressions :

$$\psi(\omega) = -\omega, \quad \phi(\omega) = -\omega + \omega_3. \quad (23)$$

They are such that  $\psi^2 = \phi^2 = \text{id}$ ,  $x \circ \psi = x$  and  $y \circ \phi = y$ . A crucial fact is the following.

**Proposition 16.** *For all  $z \in ]0, 1/4[$ , we have  $\omega_3 = 3\omega_2/4$ .*

*Proof.* Since the group generated by  $\Psi$  and  $\Phi$  is of order eight, so is the group generated by  $\psi$  and  $\phi$ , in other words  $\inf\{n \in \mathbb{N} : (\phi \circ \psi)^n = \text{id}\} = 4$ . With (23) this immediately implies that  $4\omega_3$  is some point of the lattice  $\Omega$ . But we already know that  $\omega_3 \in ]0, \omega_2[$  so that two possibilities remain : either  $\omega_3 = \omega_2/4$  or  $\omega_3 = 3\omega_2/4$ .

In addition, essentially because the covariance of Gessel's walk is positive, we can repeat the same arguments as in Section 4 of [KR09] (see page 14) and in this way we obtain that  $\omega_3$  is necessary larger than  $\omega_2/2$ , which entails Proposition 16.  $\square$

## 1.2 Implicit expression of the CGF

As said in Section 0, the *existence* of CGF for the curves  $X([y_1, y_2])$  and  $Y([x_1, x_2])$  follows from general results on conformal gluing ; actually finding *explicit* expressions for CGF is more problematic.

But by using the same arguments as in Part 5.5 of [FIM99], we obtain that the following function is a suitable CGF for  $X([y_1, y_2])$  :

$$w(t) = \wp_{1,3}(x^{-1}(t) - (\omega_1 + \omega_2)/2), \quad (24)$$

$\wp_{1,3}$  being the Weierstrass elliptic function with periods  $\omega_1, \omega_3$  and  $x^{-1}$  the reciprocal function of the first coordinate of the uniformization (19) ; the periods  $\omega_1, \omega_2, \omega_3$  are defined in (18) and (21).

In Section 4 of [KR09], we have studied some properties of the function (24), and we have shown that since  $\omega_3 > \omega_2/2$  (we recall from Proposition 16 that  $\omega_3 = 3\omega_2/4$ ), the function (24) is in fact meromorphic on  $\mathbb{C} \setminus [x_3, x_4]$  and has there a unique pole, at  $x_2$ .

In order to find explicitly a CGF for the curve  $Y([x_1, x_2])$ , we remark that

$$\tilde{w}(t) = w(X_0(t)) \quad (25)$$

is suitable – this is a consequence of the facts that  $w$  is a CGF for  $X([y_1, y_2])$  and that  $X_0 : \mathcal{G}Y([x_1, x_2]) \setminus [y_1, y_2] \rightarrow \mathcal{G}X([y_1, y_2]) \setminus [x_1, x_2]$  is conformal, as stated in Lemma 17.

More globally,  $\tilde{w}$  defined by (25) is meromorphic on  $\mathbb{C} \setminus [y_3, y_4]$ , and has there a unique pole, of order two and at  $Y(x_2) = x_3$  – this is a consequence of some properties of  $w$  already mentioned and of the fact that  $X_0(\mathbb{C}) \subset \mathbb{C} \setminus [x_3, x_4]$ , see Lemma 17 below.

**Lemma 17.**  $X_0 : \mathcal{G}Y([x_1, x_2]) \setminus [y_1, y_2] \rightarrow \mathcal{G}X([y_1, y_2]) \setminus [x_1, x_2]$  and  $Y_0 : \mathcal{G}X([y_1, y_2]) \setminus [x_1, x_2] \rightarrow \mathcal{G}Y([x_1, x_2]) \setminus [y_1, y_2]$  are conformal and reciprocal the one from the other. In addition,  $X_0(\mathbb{C}) \subset \mathbb{C} \setminus [x_3, x_4]$  and  $Y_0(\mathbb{C}) \subset \mathbb{C} \setminus [y_3, y_4]$

For the proof of Lemma 17, we refer to Part 5.3 of [FIM99].

## 1.3 Proof of Theorem 8

*Proof of Theorem 8.* In this proof, we are going here to note  $\omega_4 = \omega_2/4$  and  $\wp_{1,4}$  the Weierstrass elliptic function with periods  $\omega_1, \omega_4$ . Moreover, we recall that  $\wp$  and  $\wp_{1,3}$  are the Weierstrass elliptic functions with respective periods  $\omega_1, \omega_2$  and  $\omega_1, \omega_3 = 3\omega_2/4$ .

To begin with, let us mention the following fact. Let  $\wp$  be the Weierstrass elliptic function with periods noted  $\hat{\omega}, \tilde{\omega}$  and let  $n$  be some positive integer. Then the Weierstrass elliptic function with periods  $\hat{\omega}, \tilde{\omega}/n$  can be written in terms of  $\wp$  as follows (see *e.g.* <http://functions.wolfram.com/EllipticFunctions/WeierstrassP/16/06/03/>) :

$$\wp(\omega) + \sum_{k=1}^{n-1} [\wp(\omega + k\tilde{\omega}/n) - \wp(k\tilde{\omega}/n)]. \quad (26)$$

Then, by using *e.g.* the addition theorem (27) for the Weierstrass elliptic function  $\wp$  in (26) and next (20), we obtain that *the Weierstrass elliptic function with periods  $\hat{\omega}, \tilde{\omega}/n$  is a rational function of the Weierstrass elliptic function with periods  $\hat{\omega}, \tilde{\omega}$ .*

The proof of Theorem 8 will follow from applying this fact twice : (i) first, since  $\omega_4 = \omega_2/4$ , we will express  $\wp_{1,4}$  as a rational function of  $\wp$ , (ii) then, since  $\omega_4 = \omega_3/3$ , we will express  $\wp_{1,4}$  as a rational function of  $\wp_{1,3}$ .

Before making explicit the rational transformations that appear in (i) and (ii), we explain how to conclude the proof of Theorem 8. An immediate consequence of (i) and (ii) is the possibility of writing  $\wp_{1,3}$  as an algebraic function of  $\wp$ . In particular, it is immediate from that and from the addition theorem (27) for  $\wp$  that the formula  $w(t) = \wp_{1,3}(\wp^{-1}(f(t)) - (\omega_1 + \omega_2)/2)$ , with  $f(t) = d'(x_4)/(t - x_4) + d''(x_4)/6$  – which is the CGF under consideration, see (19) and (24) – defines an algebraic function of  $t$ .

*Explicit expression of the rational function for (i).* With (26) we can write

$$\wp_{1,4}(\omega) = \wp(\omega) + \wp(\omega + \omega_2/2) + \wp(\omega + \omega_2/4) + \wp(\omega + 3\omega_2/4) - \wp(\omega_2/2) - \wp(\omega_2/4) - \wp(3\omega_2/4).$$

Then, by using the addition theorem for  $\wp$ , namely the following formula, valid for all  $\omega, \tilde{\omega}$ , that can be found *e.g.* in [Law89],

$$\wp(\omega + \tilde{\omega}) = -\wp(\omega) - \wp(\tilde{\omega}) + \frac{1}{4} \left[ \frac{\wp'(\omega) - \wp'(\tilde{\omega})}{\wp(\omega) - \wp(\tilde{\omega})} \right]^2, \quad (27)$$

as well as the equalities  $\wp(\omega_2/4) = \wp(3\omega_2/4)$ ,  $\wp'(\omega_2/4) = -\wp'(3\omega_2/4)$  and  $\wp'(\omega_2/2) = 0$  – obtained from the facts that  $\wp(\omega_2/2 + \omega)$  is even and  $\wp'(\omega_2/2 + \omega)$  is odd –, we get

$$\wp_{1,4}(\omega) = -2\wp(\omega) + \frac{\wp'(\omega)^2 + \wp'(\omega_2/4)^2}{2[\wp(\omega) - \wp(\omega_2/4)]^2} + \frac{\wp'(\omega)^2}{4[\wp(\omega) - \wp(\omega_2/2)]^2} - \wp(\omega_2/2) - 2\wp(\omega_2/4). \quad (28)$$

We recall from the proof of Lemma 14 that  $\wp(\omega_2/2) = f(x_1)$ . In other words, for the right-hand side of (28) to be completely explicit, it remains to find the expression  $\wp(\omega_2/4)$  and  $\wp'(\omega_2/4)$  in terms of  $z$ . Starting from the known value of  $\wp(\omega_2/2)$ , it is easy to

obtain the value of  $\wp(\omega_2/4)$ , by using *e.g.* the formula below (a proof of which being given in [Law89]) :

$$\wp(\omega_2/4) = \wp(\omega_2/2) + [(\wp(\omega_2/2) - \wp(\omega_1/2))(\wp(\omega_2/2) - \wp((\omega_1 + \omega_2)/2))]^{1/2}. \quad (29)$$

Then we use that  $\wp(\omega_1/2) = f(x_3)$ ,  $\wp((\omega_1 + \omega_2)/2) = f(x_2)$ , and after simplification we obtain  $\wp(\omega_2/4) = (1 + 4z^2)/3$ . As a consequence and with (20) and Lemma 14, we get  $\wp'(\omega_2/4) = -8z^2$ . In conclusion, the right-hand side of (28) is completely known.

In particular, evaluating (28) at  $\omega = \wp^{-1}(f(t)) - (\omega_1 + \omega_2)/2$  and using again the addition formula (27) for  $\wp$ , we obtain that the right-hand side of (28) is a rational function of  $t$  that can be explicitly obtained in terms of  $t$  and  $z$  ; after a substantial but elementary calculation we get  $\wp_{1,4}(\wp^{-1}(f(t)) - (\omega_1 + \omega_2)/2) = F(t)$ ,  $F$  being defined in (10).

*Explicit expression of the rational function for (ii).* Using the same arguments that have allowed us to obtain (28) from (26), we obtain that  $\wp_{1,4}$  is the following rational function of  $\wp_{1,3}$  :

$$\wp_{1,4}(\omega) = -\wp_{1,3}(\omega) + \frac{\wp'_{1,3}(\omega)^2 + \wp'_{1,3}(\omega_3/3)^2}{2[\wp_{1,3}(\omega) - \wp_{1,3}(\omega_3/3)]^2} - 4\wp_{1,3}(\omega_3/3). \quad (30)$$

By using (30) and the equality  $\wp'_{1,3}(\omega)^2 = 4\wp_{1,3}(\omega)^3 - g_{2,1,3}\wp_{1,3}(\omega) - g_{3,1,3}$ , where  $g_{2,1,3}, g_{3,1,3}$  are the invariants associated with  $\wp_{1,3}$ , we get that  $\wp_{1,4}$  is a rational function of  $\wp_{1,3}$  ; moreover, with Lemma 18, the equality  $\wp'_{1,3}(\omega_3/3)^2 = 4\wp_{1,3}(\omega_3/3)^3 - g_{2,1,3}\wp_{1,3}(\omega_3/3) - g_{3,1,3}$  and Lemma 19, the coefficients of this rational function in terms of  $z$  are explicitly known.

*Proof of (11).* Now we remark that with Lemmas 18 and 19, (30) can be written as

$$\begin{aligned} \wp_{1,3}(\omega)^3 - \wp_{1,3}(\omega)^2[\wp_{1,4}(\omega) + 2K] + \wp_{1,3}(\omega)[2K\wp_{1,4}(\omega) + K^2/3 + G_2/2] \\ - [K^2\wp_{1,4}(\omega) + 19G_2K/18 + G_3 - 46K^3/27] = 0. \end{aligned}$$

In particular, evaluating this equality at  $\omega = \wp^{-1}(f(t)) - (\omega_1 + \omega_2)/2$ , using the fact already proved that  $\wp_{1,4}(\wp^{-1}(f(t)) - (\omega_1 + \omega_2)/2) = F(t)$  as well as the definition (24) of  $w$ , we obtain (11).

*End of the proof of Theorem 11.* If  $F$  is infinite at some point, then the equality (11) becomes  $(w - K)^2 = 0$ . In particular, at a such point at least two roots of (11) take finite values. In addition, by using the root-coefficient relationships, it is clear that at a point where  $F$  is infinite, at least one root of (11) is infinite. This proves that at any point where  $F$  is infinite, there is one and only one root of (11) which is infinite.

In particular, since  $F$  is infinite at  $x_2$ , see (10), and since  $w$  has a pole at  $x_2$ , see Subsection 1.2,  $w$  can be characterized as the only solution of (11) with a pole at  $x_2$ .

Likewise, we could prove the corresponding fact for  $\tilde{w}$ . Theorem 11 is proved.  $\square$

Let  $G_2, G_3, K$  be the quantities defined in Section 0, above Theorem 8.

**Lemma 18.**  $\wp_{1,3}(\omega_3/3) = K$ .



**Lemma 19.**  $g_{2,1,3}, g_{3,1,3}$ , the invariants of  $\wp_{1,3}$ , have the following explicit expressions :

$$g_{2,1,3} = 40K^2/3 - G_2, \quad g_{3,1,3} = -280K^3/27 + 14KG_2/9 + G_3.$$

*Proof of Lemmas 18 and 19.* Start by expanding  $\wp_{1,4}$  at 0 in two different ways. Firstly, by using (28) and by simplifying, we obtain :

$$\wp_{1,4}(\omega) = \omega^{-2} + [9G_2/20]\omega^2 - [27G_3/28]\omega^4 + O(\omega^6). \quad (31)$$

Secondly, we can also use (30) and after some calculation we get :

$$\wp_{1,4}(\omega) = \omega^{-2} + [6K^2 - 9g_{2,1,3}/20]\omega^2 + [10K^3 - 3Kg_{2,1,3}/2 - 27g_{3,1,3}/28]\omega^4 + O(\omega^6). \quad (32)$$

Lemma 19 follows immediately, by identifying the expansions (31) and (32).

As for Lemma 18, it will be a consequence of Lemma 19 and of the following result, proved *e.g.* in [Law89] : the quantity  $K = \wp_{1,3}(\omega_3/3)$  is the only positive solution of the following equation :  $K^4 - g_{2,1,3}K^2/2 - g_{3,1,3}K - g_{2,1,3}^2/48 = 0$ . But thanks to Lemma 19, we can replace  $g_{2,1,3}$  and  $g_{3,1,3}$  by their expression in terms of  $K$  ; in this way we obtain that  $K$  verifies the equation  $K^4 - G_2K^2/2 - G_3K - G_2^2/48 = 0$ .  $\square$

## 2 Holomorphic continuation of $zQ(x, 0, z)$ and $z(y+1)Q(0, y, z)$

In this part we are going to prove Theorem 3, in other words we are going to show that  $zQ(x, 0, z)$  and  $z(y+1)Q(0, y, z)$  can be holomorphically continued up to  $\mathbb{C} \setminus [x_3, x_4]$  and  $\mathbb{C} \setminus [y_3, y_4]$  respectively.

In fact, we are going to show that  $Q(x, 0, z)$  and  $Q(0, y, z)$  can be holomorphically continued up to  $\mathbb{C} \setminus [x_3, x_4]$  and  $\mathbb{C} \setminus [y_3, y_4]$  respectively, which is an equivalent assertion, since as said in Section 0,  $Q(0, y, z)$  is bounded at  $-1$ .

For this we will use the following procedure :

- (i) First, we will lift the functions  $Q(x, 0, z)$  and  $Q(0, y, z)$  up to  $\mathbb{C}/\Omega$  by setting  $q_x(\omega) = Q(x(\omega), 0, z)$  and  $q_y(\omega) = Q(0, y(\omega), z)$ . The functions  $q_x$  and  $q_y$  are *a priori* well defined on  $x^{-1}(\{|x| \leq 1\})$  and  $y^{-1}(\{|y| \leq 1\})$  respectively.
- (ii) Then, we will prove the following.

**Theorem 20.**  $q_x$  and  $q_y$ , initially well defined on  $x^{-1}(\{|x| \leq 1\})$  and  $y^{-1}(\{|y| \leq 1\})$  respectively, can be holomorphically continued up to the whole parallelogram  $\mathbb{C}/\Omega$  cut along respectively  $[0, \omega_1[$  and  $[0, \omega_1[ + \omega_3/2$ . Moreover, these continuations verify

$$\forall \omega \in \mathbb{C}/\Omega \setminus [0, \omega_1[ : q_x(\omega) = q_x(\psi(\omega)), \quad \forall \omega \in \mathbb{C}/\Omega \setminus ([0, \omega_1[ + \omega_3/2) : q_y(\omega) = q_y(\phi(\omega)), \quad (33)$$

and

$$\forall \omega \in ]3\omega_2/8, \omega_2[ \times ]0, \omega_1/\iota[ : zq_x(\omega) + z(y(\omega) + 1)q_y(\omega) - zQ(0, 0, z) - x(\omega)y(\omega) = 0. \quad (34)$$

(iii) Finally, we will set  $Q(x, 0, z) = q_x(\omega)$  if  $x(\omega) = x$  and  $Q(0, y, z) = q_y(\omega)$  if  $y(\omega) = y$ . Thanks to (33) and Proposition 15, these relations define  $Q(x, 0, z)$  and  $Q(0, y, z)$  on respectively  $\mathbb{C} \setminus [x_3, x_4]$  and  $\mathbb{C} \setminus [y_3, y_4]$  not ambiguously, as holomorphic functions.

Moreover, both (4) and (5) are immediate consequences of (34).

Items (i) and (iii) are straightforward. For the proof of (ii), it will be useful first to find the location of the cycles  $x^{-1}(\{|x| = 1\})$  and  $y^{-1}(\{|y| = 1\})$  on  $\mathbb{C}/\Omega$ , this is the subject of the following result, illustrated on Figure 4 below.

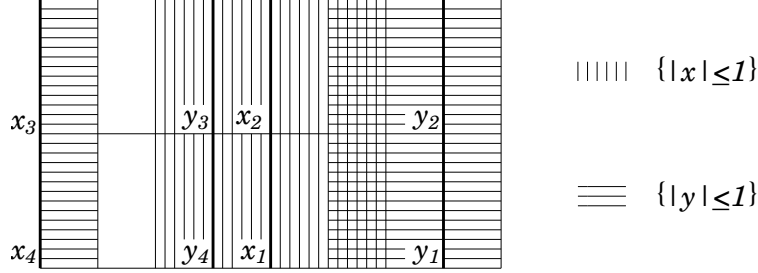


Figure 4: Location of the important cycles on the surface  $[0, \omega_2[ \times [0, \omega_1/\iota[$

**Proposition 21.** *We have  $x^{-1}(\{|x| = 1\}) = ([0, \omega_1[ + \omega_2/4) \cup ([0, \omega_1[ + 3\omega_2/4)$  and  $y^{-1}(\{|y| = 1\}) = ([0, \omega_1[ + \omega_2/8) \cup ([0, \omega_1[ + 5\omega_2/8)$ .*

*Proof.* The details are of course essentially the same for  $x$  and  $y$ , so that we are going to prove only the assertion concerning  $x$ . We are going to show it with three steps.

But first of all we note that because of the equality  $x \circ \psi = x$ , it is sufficient to prove that  $x^{-1}(\{|x| = 1\}) \cap ([0, \omega_2/2] \times [0, \omega_1/\iota]) = [0, \omega_1[ + \omega_2/4$  – the advantage of this being that  $\wp$ , and therefore also  $x$ , are one-to-one in  $[0, \omega_2/2] \times [0, \omega_1/\iota]$ .

*Firstly*, we prove that  $x(\omega_2/4 + \omega_1/2) = 1$ . For this we recall from the proof of Theorem 11 that  $\wp(\omega_2/4) = (1 + 4z^2)/3$ . Then with the addition theorem (27) we immediately obtain the explicit value of  $\wp(\omega_2/4 + \omega_1/2)$ , since the ones of  $\wp(\omega_2/4)$ ,  $\wp'(\omega_2/4) = -8z^2$  and  $\wp(\omega_1/2) = f(x_3)$  are known. Finally, after a simple calculation and by using (19), we get  $x(\omega_2/4 + \omega_1/2) = 1$ .

*Secondly*, we show that  $x^{-1}(\{|x| = 1\}) \cap ([0, \omega_2/2] \times [0, \omega_1/\iota]) \subset [0, \omega_1[ + \omega_2/4$ . For this let  $\theta \in [0, 2\pi[$ . With (19) we have  $x(\omega) = \exp(i\theta)$  if and only if  $\wp(\omega) = f(\exp(i\theta))$ . Since  $\omega \in [0, \omega_2/2] \times [0, \omega_1/\iota]$ , we can use the explicit expression of the reciprocal function of  $\wp$  and with the first step we obtain :

$$\omega = \omega_2/4 + \omega_1/2 + \int_{f(1)}^{f(\exp(i\theta))} \frac{dt}{[4t^3 - g_2t - g_3]^{1/2}} = \omega_2/4 + \omega_1/2 + \frac{1}{2} \int_{\exp(i\theta)}^1 \frac{dx}{[d(x)]^{1/2}}, \quad (35)$$

$d$  being the polynomial defined in Section 0. Note that the second equality above is got with the same calculations as in Part 3.3 of [FIM99].

Now we remark that  $d(x) = x^4 d(1/x)$ . In particular, the change of variable  $x \mapsto 1/x$  in the integral  $\int_{\exp(i\theta)}^1 dx/[d(x)]^{1/2}$  yields  $\int_{\exp(i\theta)}^1 dx/[d(x)]^{1/2} = -\int_{\exp(-i\theta)}^1 dx/[d(x)]^{1/2}$ . As a consequence, this integral belongs to  $i\mathbb{R}$ .

In conclusion, with (35) we have actually shown that  $x^{-1}(\{|x| = 1\}) \cap ([0, \omega_2/2] \times [0, \omega_1/i]) \subset [0, \omega_1[ + \omega_2/4$ .

*Thirdly*, we prove that the inclusion above has to be an equality. Indeed, if it was not the case the curve  $x^{-1}(\{|x| = 1\}) \cap ([0, \omega_2/2] \times [0, \omega_1/i])$  would be curve *not* closed, which is a manifest contradiction with the fact that its image through the function  $x$ , meromorphic and one-to-one in  $[0, \omega_2/2] \times [0, \omega_1/i]$ , namely the unit circle, is a closed curve.  $\square$

*Proof of Theorem 20.* The proof is composed of four steps. We will first, in three items, define the continuations of  $q_x$  and  $q_y$  on the whole parallelogram  $\mathbb{C}/\Omega$  appropriately cut. In conclusion we will verify that the functions so constructed actually verify the facts claimed in the statement of Theorem 20.

- We define  $q_x(\omega)$  on  $x^{-1}(\{|x| \leq 1\})$  by  $Q(x(\omega), 0, z)$  and  $q_y(\omega)$  on  $y^{-1}(\{|y| \leq 1\})$  by  $Q(0, y(\omega), z)$ . Note that as a consequence of Proposition 21 we have  $x^{-1}(\{|x| \leq 1\}) = [\omega_2/4, 3\omega_2/4] \times [0, \omega_1/i[$  and  $y^{-1}(\{|y| \leq 1\}) = [5\omega_2/8, 9\omega_2/8] \times [0, \omega_1/i[$ .
- Motivated by (1), on  $]3\omega_2/4, \omega_2[ \times [0, \omega_1/i[ \subset y^{-1}(\{|y| \leq 1\})$  we set  $q_x(\omega) = -(y(\omega) + 1)q_y(\omega) + Q(0, 0, z) + x(\omega)y(\omega)/z$  and on  $]3\omega_2/8, 5\omega_2/8[ \times [0, \omega_1/i[ \subset x^{-1}(\{|x| \leq 1\})$  we set  $(y(\omega) + 1)q_y(\omega) = -q_x(\omega) + Q(0, 0, z) + x(\omega)y(\omega)/z$ .
- On  $]0, \omega_2/4[ \times [0, \omega_1/i[$  we define  $q_x(\omega)$  by  $q_x(\phi(\omega))$  – note that with (23) we have  $\phi(]0, \omega_2/4[ \times [0, \omega_1/i[) = [3\omega_2/4, \omega_2[ \times [0, \omega_1/i[$  – and on  $[\omega_2/8, 3\omega_2/8[ \times [0, \omega_1/i[$  we define  $q_y(\omega)$  by  $q_y(\psi(\omega))$  – by using (23) we have  $\psi([\omega_2/8, 3\omega_2/8[ \times [0, \omega_1/i[) = ]3\omega_2/8, 5\omega_2/8[ \times [0, \omega_1/i[$ .

The functions  $q_x$  and  $q_y$  are now well defined on the whole parallelogram  $\mathbb{C}/\Omega$  cut along  $[0, \omega_1[$  and  $[0, \omega_1[ + \omega_3/2$  respectively.

Note that the definition given in the first item is quite natural. The one stated in the second item is also natural since on  $x^{-1}(\{|x| \leq 1\}) \cap y^{-1}(\{|y| \leq 1\}) = [5\omega_2/8, 3\omega_2/4] \times [0, \omega_1/i[$ , the equality  $zq_x(\omega) + z(y(\omega) + 1)q_y(\omega) - zQ(0, 0, z) - x(\omega)y(\omega) = 0$  holds, see (1). The definition set in the third item is to ensure that (33) is valid.

Note also that (34) is immediately true, by construction of the continuations.

We are now going to verify (33) for  $q_x$ . Since  $x \circ \psi = x$ , (33) is obviously verified on  $[\omega_2/4, 3\omega_2/4] \times [0, \omega_1/i[ = \psi([\omega_2/4, 3\omega_2/4] \times [0, \omega_1/i[)$ . Moreover, with the third item, (33) is verified for  $q_x$  on  $]0, \omega_2/4[ \times [0, \omega_1/i[$ , and since  $\psi^2 = \text{id}$ , (33) is also true for  $q_x$  on  $]3\omega_2/4, \omega_2[ \times [0, \omega_1/i[$ , and finally on the whole  $\mathbb{C}/\Omega \setminus [0, \omega_1[$ .

Likewise, we verify easily that (33) is valid for  $q_y$  on  $\mathbb{C}/\Omega \setminus ([0, \omega_1[ + 3\omega_2/8)$ .

It remains to prove that the continuations of  $q_x$  and  $q_y$  are holomorphic on  $\mathbb{C}/\Omega$  cut along  $[0, \omega_1[$  and  $[0, \omega_1[+3\omega_2/8$  respectively.

We show first that they are *meromorphic* on their respective cut parallelogram. For  $q_x$  the following cycles are *a priori* problematic :  $[0, \omega_1[$ ,  $[0, \omega_1[+\omega_2/4$  and  $[0, \omega_1[+3\omega_2/4$ .

In an open neighborhood of  $[0, \omega_1[+3\omega_2/4$ , we have  $q_x(\omega) = -(y(\omega) + 1)q_y(\omega) + Q(0, 0, z) + x(\omega)y(\omega)/z$ , so that  $q_x$  is in fact meromorphic in the neighborhood of the cycle  $[0, \omega_1[+3\omega_2/4$ . Since (33) holds,  $q_x$  is also meromorphic near  $[0, \omega_1[+\omega_2/4 = \psi([0, \omega_1[+3\omega_2/4)$ , so that only  $[0, \omega_1[$  remains *a priori* a singular cycle.

Similarly, we could show that  $q_y$  is meromorphic on  $\mathbb{C}/\Omega \setminus ([0, \omega_1[+3\omega_2/8)$ .

Let us now prove that these continuations are actually *holomorphic* on their respective cut parallelogram.

$q_x$  is obviously holomorphic on  $] \omega_2/4, 3\omega_2/4[ \times [0, \omega_1/\iota[$ , since it is there defined through its power series.

On  $]5\omega_2/8, \omega_2[ \times [0, \omega_1/\iota[$ , we have  $q_x(\omega) = -(y(\omega) + 1)q_y(\omega) + Q(0, 0, z) + x(\omega)y(\omega)/z$ , and all terms of the right-hand side of this equality are holomorphic on this domain – at  $7\omega_2/8$ ,  $x$  has a pole of order one and  $y$  has a zero of order two so that the product  $xy$  is holomorphic near  $7\omega_2/8$  (this is a consequence of Lemma 1 that the only poles of  $x$  are at  $\omega_2/8, 7\omega_2/8$ , its only zeros are at  $3\omega_2/8, 5\omega_2/8$  and that the only pole of  $y$  (of order two) is at  $3\omega_2/8$  and its only zero (of order two) is at  $7\omega_2/8$ ).

On  $]0, 3\omega_2/8[ \times [0, \omega_1/\iota[$ , we have  $q_x = q_x \circ \psi$ , so that  $q_x$  is holomorphic on this domain since it is on  $\psi([0, 3\omega_2/8[ \times [0, \omega_1/\iota[) = ]5\omega_2/8, \omega_2[ \times [0, \omega_1/\iota[$ .

Likewise, we could show that  $(y + 1)q_y$  is holomorphic on  $\mathbb{C}/\Omega \setminus ([0, \omega_1[+\omega_3/2)$ . This implies that  $q_y$  is holomorphic on the same set except at the points where  $y + 1 = 0$ . There are two possibilities in order to show that  $q_y$  is also holomorphic at these points, namely  $\omega_2/8$  and  $5\omega_2/8$ .

First, we can use the fact that the generating function  $Q(0, y, z)$  is bounded at  $-1$ , see Section 0, so that  $q_y(\omega) = Q(0, y(\omega), z)$ , being meromorphic and bounded near  $\omega_2/8$  and  $5\omega_2/8$ , is actually holomorphic at these points.

We can also remark that with (34),  $(y(5\omega_2/8) + 1)q_y(5\omega_2/8) = 0$ , since  $x(5\omega_2/8) = 0$ . Moreover, since  $\phi(5\omega_2/8) = \omega_2/8$ ,  $(y(\omega_2/8) + 1)q_y(\omega_2/8) = 0$ . In other words, at  $\omega = \omega_2/8$  and  $\omega = 5\omega_2/8$ , both holomorphic functions  $(y + 1)q_y$  and  $(y + 1)$  have a zero of order exactly one, it follows immediatly that  $q_y$  is holomorphic at  $\omega$ .  $\square$

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