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Abstract

In this article we propose a new macroscopic model derived from a time-continuous version of the Vicsek Algorithm [1, 6, 10, 16], where we introduce the dependence of some local density and angle of vision in the orientation interaction.

With some adaptations to the concept of generalized collisional invariants, we prove that the methodology developped in [8] works to derive a macroscopic limit of this dynamical system. The system of PDE may present some zones of non-hyperbolicity, which is a new feature compared to the macroscopic model of [8].

We perform an asymptotic study of the coefficients obtained in this system of PDE's and show that there are indeed some cases where the system is not hyperbolic. This study is also useful to measure the influence of the angle of vision in the final continuum model.

Key words : Individual based model, Vicsek algorithm, orientation interaction, anisotropy, collisional invariants, non-hyperbolicity.

AMS Subject classification : 35M30, 35Q70, 35Q80, 82C22, 82C70, 92D50

1 Introduction

The study of complex particle systems is challenging, on the mathematical point of view, since they can exhibit some unusual behavior. The Vicsek algorithm [1, 6, 10, 16] has been proposed as a model for describing the behavior of individuals inside animal societies such as fish schools or flocks of birds. In this model, all the particles have constant speed and change their direction according to their neighbors. There is no obvious conservation relation except for the mass. If we want to derive some macroscopic model from the particular description, we can expect to find some non-conservative system of PDE's. This is the result of [8], where the kinetic version of the Vicsek algorithm they propose is shown to have a continuum limit at large scale, which involves a non-conservative system of PDE's. This system is shown to be hyperbolic.

In numerical experiments (see [5]), there is formation of bands of high density, strongly ordered, travelling through a disordered area of low density, at large time. This is obtained with the individual based model (IBM), but the macroscopic system of [8] seems to be unable to reproduce this dynamic. One of the possibility to obtain such behavior with a macroscopic model is to have a certain bistability in the equilibria (see for example [9, 15]), which is not the case. Another possibility could be the lack of hyperbolicity in some cases, leading to concentrations.

Another observation is that the order parameter, in these numerical experiments, is not constant. Numerically, it depends on the local density ρ. And this is not coherent with the local equilibria of the macroscopic model, since they all have the same order parameter. The idea of the present paper is to study some refinements to the kinetic model in order to reproduce properties observed on the individual based computations.

The first step in [8] to derive the kinetic model is to introduce a time-continuous version of the algorithm, since the original version is discrete in time. This imposes to add a new parameter ν, which can be viewed as a frequency of interaction between a particle and its neighbors. The mean-field kinetic model associated to the particle system consists of the following Kolmogorov-Fokker-Planck equation :

$$\partial_t f + \omega \cdot \nabla_x f + \nabla_\omega (Ff) = d\Delta_w f, \tag{1.1}$$

where Δ_ω is the Laplace-Belltrami operator on the unit sphere and f is the probability density function. Here ω is the angle variable, belonging to the unit sphere $\mathbb{S}_2$ (or $\mathbb{S}_1$, depending if we work in dimension 3 or 2). The force F is the alignment interaction, which depends on f and on ν. It tends to align the particles with the direction of the flux of f. The constant coefficient d represent the intensity of the noise. But there is no rigorous proof of the convergence of the dynamic system of particles to this kinetic model when the number of particles increase. We can reasonably expect that a limit should be of the previous form. The fact that ν and the noise parameter d are constant implies then that the local equilibrium obtained in the macroscopic limit is always of the same shape, independently of the local density. One of the hypothesis is that some extra noise could appear in the limit of a large number of particles, due to fluctuations in the process of computing the mean velocity. Indeed, if there are particles in the area of vision distributed according to some angular law, the central limit theorem tells us that the fluctuation between the mean velocity and the expected velocity for the law depends on the number of particles. So in this paper we want to see what happens when we add a dependence to some local density $\overline{\rho}$ for the coefficient d.

We will also add this dependence to the parameter ν (the rate of changing direction for one particle). We could indeed imagine that it is more likely to move when there is a large number of particles around, due to some social pressure. In [8], this parameter ν does also depend of the difference between one particle direction and its target direction. Most results will be given here without this dependence for convenience, but some computations have been done with, which will be given in annex.

The other refinement is to add some "angle of vision", that is to say that the kernel used to compute a mean around each particle will not be invariant under rotation, as in [8]. The procedure to add these features in the model is described in section 2. Finally we will detail a little bit more what are the differences if we work in 2 or 3 dimensions. Indeed some computations at the end lead to explicit expressions or not, according to the dimension, and all the results were on a 3-dimensional framework in [8].

The goal of this article is to study the features of the macroscopic limit of this new model. Indeed if we introduce a small parameter ε for the time and space scaling, we obtain a continuum model, as $\varepsilon \to 0$. The derivation of this model is given in section 3. The limit of the distribution function is given by $f^0 = M_{\rho,\Omega}$ where the function $M_{\rho,\Omega}$ plays here the same role as the maxwellian, in classical theory of rarefied gases. The total mass $\rho = \rho(x,t) \geq 0$ of f^0 and the director of its flux $\Omega = \Omega(x,t) \in \mathbb{S}_2$ (or $\mathbb{S}_1$) satisfy the following system of first order partial differential equations :

$$\partial_t \rho + \nabla_x \cdot (c_1 \rho \Omega) = 0. \tag{1.2}$$

$$\rho \left(\partial_t \Omega + c_2 (\Omega \cdot \nabla)\Omega\right) + \lambda \left(\mathrm{Id} - \Omega \otimes \Omega\right) \nabla_x \rho = 0, \tag{1.3}$$

where the convection speeds c_1, c_2 and the interaction parameter λ depend on ρ. Their expressions will be specified in the course of the paper (see (3.55) and (3.58)-(3.59)).

The purpose of this article is to study how the features introduced at the microscopic level influence the properties of the macroscopic model.

The first particularity here is that the shape of distribution in angle is dependent of the density ρ, and we cannot write the equilibrium of the form ρM_Ω, as in [8]. But if we are only interested in the system in ρ and Ω, it seems to be the same, except that the coefficients here are not constant. However, at section 4, we will see that the negativity of the coefficient λ can lead to a loss of hyperbolicity in some regimes, and that indeed, the dependence on ρ now allows λ to be negative. The other particularity of is that the difference between velocities c_1 and c_2 can be enhanced by the introduction of a restricted angle of vision.

2 A continuous Vicsek model with anisotropy and density dependent collision frequency

We start by presenting the individual-based model and the continuum limit we derived from it. The elements of the derivation of this macroscopic model will be given in section 3.

2.1 Starting point : particle dynamics

Here, we briefly recall the time-continuous version of the Vicsek algorithm, and introduce how we take in account the anisotropy of observation. The collision frequency, which is

a parameter specific to the time-continuous version (there is no such parameter in the original discrete model), is chosen dependent of a certain local density, in order to match some large-scale properties of the discrete model. Indeed this parameter is related to some order parameter for the equilibrium, and in simulations some links between local density and order parameter are observed (see [5]). In a modelling point of view, we can argue that the particles could tend to change there orientation more rapidly if the density is larger, due to some social pressure.

We consider a system of N particles with positions X_k in $\mathbb{R}^n$ (with $k \in [\![1, N]\!]$ and $n = 2$ or 3) and orientations ω_k in the unit sphere $\mathbb{S}_{n-1}$, which we will simply write $\mathbb{S}$ when there is no ambiguity.

The k^{th} particle moves at constant speed (say 1, since we can do a change of variable) following its orientation. This last one relax towards the mean orientation $\overline{\omega}_k$ of its neighbors, with rate ν (depending in some mean density $\overline{\rho}_k$ around it), under the constraint that ω_k is of norm 1 :

$$\frac{dX_k}{dt} = \omega_k, \tag{2.4}$$

$$\frac{d\omega_k}{dt} = \nu(\overline{\rho}_k)\,(\mathrm{Id} - \omega_k \otimes \omega_k)\overline{\omega}_k, \tag{2.5}$$

where we denote by $\mathrm{Id} - \omega_k \otimes \omega_k$ the projector on the plane orthogonal to ω_k, that is to say $(\mathrm{Id} - \omega \otimes \omega)\upsilon = \upsilon - (\upsilon \cdot \omega)\omega$. This projection is necessary in order to keep ω_k on the unit sphere.

It remains to define the way $\overline{\omega}_k$ and $\overline{\rho}_k$ are computed. In the original model the mean $\overline{\omega}_k$ is computed on all the neighbors in a ball of fixed radius R centered in the given particle. Here we take the mean according to a kernel K taking in account the distance between the given particle and a given neighbor, and the cosine of the angle between the orientation of the first particle and the right line joining the two particles :

$$\overline{\omega}_k = \frac{\overline{J}_k}{|\overline{J}_k|}, \quad \overline{J}_k = \sum_{j=1}^{N} K\left(|X_j - X_k|, \frac{X_j - X_k}{|X_j - X_k|} \cdot \omega_k\right)\omega_j. \tag{2.6}$$

We proceed in an analogous way to compute the local density $\overline{\rho}_k$, with eventually an other kernel $\widetilde{K}$:

$$\overline{\rho}_k = \sum_{j=1}^{N} \widetilde{K}\left(|X_j - X_k|, \frac{X_j - X_k}{|X_j - X_k|} \cdot \omega_k\right)\omega_j, \text{ with} \tag{2.7}$$

$$\int_{\xi \in \mathbb{R}^n} \widetilde{K}(|\xi|, |\xi|^{-1}\,\xi \cdot \omega)\mathrm{d}\xi = 1. \tag{2.8}$$

This last normalisation condition (which is not depending on $\omega \in \mathbb{S}$) is chosen so that the density is 1 if in average there is one particle per unit volume.

We finally have to add noise to this model to reach some interesting equilibrium, coming from the balance between the force of alignment and the diffusion caused by the noise. The idea is to add a brownian motion on the sphere (see [12] for more details on how to define such an object) for the orientation, and we obtain the following system of

stochastic differential equations :

$$\frac{dX_k}{dt} = \omega_k, \tag{2.9}$$

$$d\omega_k = (\mathrm{Id} - \omega_k \otimes \omega_k)(\nu(\overline{\rho}_k)\,\overline{\omega}_k\,dt + \sqrt{2D(\overline{\rho}_k)}\,dB_t^k), \tag{2.10}$$

where (B_t^k) are independent standard brownian motions on $\mathbb{R}^n$, and D is the intensity of the noise. Again here we let this parameter depend on the local density $\overline{\rho}_k$.

In [8], the relaxation coefficient ν depends on the (cosine of the) angle between the orientation of one particle and the mean orientation : in (2.10), we could replace $\nu(\overline{\rho}_k)$ by $\nu(\overline{\rho}_k, \omega_k \cdot \overline{\omega}_k)$, in order to take in account some "ability to turn". With our new features here, this would involve many more computations. For simplicity here, we will work without this dependence, but this can be done with (and leads to the same conclusion), following exactly the same method. I will only present the final results with this dependence in some cases, and add some remarks to explain the difference in some steps of the derivation.

Some numerical simulations tends to show that this time-continuous system present the same behavior at large scale as the discrete one (for example the formation of bands, as in [5]). More precise investigations on the numerical comparison between the discrete and continuous in time dynamical system are in progress.

2.2 The continuum model

In this paper, following the approach of [8], we derive the following continuum model from the particle dynamics described above :

$$\partial_t \rho + \nabla_x \cdot (c_1(\rho)\rho\Omega) = 0, \tag{2.11}$$

$$\rho\,(\partial_t \Omega + c_2(\rho)(\Omega \cdot \nabla_x)\Omega) + \lambda(\rho)\,(\mathrm{Id} - \Omega \otimes \Omega)\nabla_x \rho = 0. \tag{2.12}$$

The functions $\rho(x,t) \geqslant 0$ and $\Omega(x,t) \in \mathbb{S}$ describe the average density and particle direction at some point $x \in \mathbb{R}^n$.

The first equation is the conservation of mass : the density ρ moves through direction Ω with velocity $c_1(\rho)$. The second one describes the evolution of the direction Ω. The perturbations of this vector travel with velocity $c_2(\rho)$, influenced by a kind of pressure due to the density, of intensity $\lambda(\rho)$. The operator $\mathrm{Id} - \Omega \otimes \Omega$ (projecting orthogonally to Ω) ensures the direction Ω to be of unit norm.

The difference with the version in [8] is only in the definitions of these speeds c_1 and c_2, and of the parameter λ (which will be specified in the next section, see (3.55) and (3.58)-(3.59)). Here, they depend on the density ρ, and their expressions are slightly different (due to this dependency and the anisotropy of the observation).

In this article we will investigate the properties of (2.11)-(2.12) and show that, contrary to the result of [8], the model can be non hyperbolic in certain regimes. This question and the importance of the non-hyperbolicity will be discussed in section 4.

Again it is important to see that we have two different speeds, which means that the perturbations on the mean orientation do not travel at the same velocity than the "fluid". We will also see that the fact that there is an non isotropic observation kernel enhances the difference between c_1 and c_2.

3 Elements of the derivation of the continuum model

This derivation proceeds like in [8] but there are significant differences due to the additional complexity. In this section, we will recall briefly the method of [8] and will focus on the points which are specific to the present study.

The derivation proceeds in several steps. The first one consists in writing a kinetic version of the particle dynamics.

3.1 Step 1 : mean-field model

Let $f(x,\omega,t)$ be the probability density of finding one particle at time t, position $x \in \mathbb{R}^n$ and orientation $\omega \in \mathbb{S}$. The mean-field version of (2.9)-(2.10) is given by

$$\partial_t f + \omega \cdot \nabla_x f + \nabla_\omega \cdot (Ff) = \nabla_\omega \cdot (d(\overline{\rho})\nabla_\omega f), \tag{3.13}$$

with

$$F(x,\omega,t) = \nu(\overline{\rho})\,(\mathrm{Id} - \omega \otimes \omega)\overline{\Omega}(x,\omega,t), \tag{3.14}$$

$$\overline{\rho}(x,\omega,t) = \int_{y\in\mathbb{R}^n,\, v\in\mathbb{S}} \widetilde{K}\left(|x-y|, \frac{y-x}{|x-y|}\cdot\omega\right) f(y,v,t)\,\mathrm{d}y\,\mathrm{d}v \quad , \tag{3.15}$$

$$\overline{\Omega}(x,\omega,t) = \frac{J(x,\omega,t)}{|J(x,\omega,t)|}, \tag{3.16}$$

$$J(x,\omega,t) = \int_{y\in\mathbb{R}^n,\, v\in\mathbb{S}} K\left(|x-y|, \frac{y-x}{|x-y|}\cdot\omega\right) v\, f(y,v,t)\,\mathrm{d}y\,\mathrm{d}v\,. \tag{3.17}$$

The first equation (3.13) is the so called Kolmogorov-Focker-Planck equation. The force term $F(x,\omega,t)$ corresponds to the orientation interaction.

If there is no noise, as in (2.4)-(2.5), this is easy to derive formally this system, since the empirical distribution satisfy this same equation (with $d(\overline{\rho}) = 0$).

When the noise is added, some difficulties appear. We could try for example to use a method like the BBGKY hierarchy (see [11] for an example on an other model for self-propelled particles, the Cucker-Smale model). But since the interaction is not a sum of binary interactions, the distribution of one particle directly depends on the distribution of N particles, and we cannot reduce to a system involving only the one-particle and the two-particles distributions. The other difficulty is to deal with the classical assumption of propagation of chaos.

So we start with this model as a base for the derivation of the continuum model. A rigorous proof of how to obtain such a mean-field model from the individual dynamic system (2.9)-(2.10) is left to future work.

Remark 3.1.1 *If we want to take in account some "ability to turn", we just replace $\nu(\overline{\rho})$ by $\nu(\overline{\rho}, \omega\cdot\overline{\Omega})$.*

The following step consists in observing this system at large scale, in both space in time.

3.2 Step 2 : hydrodynamic scaling

The hydrodynamic scaling consists in the same rescaling for the time and space variable. We introduce a small parameter ε and we set $x' = \varepsilon x$, and $t' = \varepsilon t$. We define $f^\varepsilon(x', \omega, t') = f(x, \omega, t)$, and we rewrite the system (3.13)-(3.15) in this new coordinates.

The system is the same, except that there is a factor ε in front of each of the terms with space or time derivative. But now the expressions of $\overline{\rho}$ and $\overline{\Omega}$ depend on ε : for instance the flux $\overline{J}^\varepsilon$ is defined by

$$\overline{J}^\varepsilon(x,\omega,t) = \int_{y\in\mathbb{R}^n,\, \upsilon\in\mathbb{S}} K\left(\frac{|x-y|}{\varepsilon}, \frac{y-x}{|x-y|}\cdot\omega\right) \upsilon\, f^\varepsilon(y,\upsilon,t)\,\frac{\mathrm{d}y}{\varepsilon^n}\,\mathrm{d}\upsilon\,. \tag{3.18}$$

We have a similar expansion for $\overline{\rho}^\varepsilon$. The important point is to realize that $\overline{\rho}^\varepsilon$ and $\overline{\Omega}^\varepsilon$ can be easily expanded in terms of ε, and the non-locality only appears at high order.

We have the following expansions, the proof of which is given in lemma 1 of annex A.1.1 :

$$\overline{\Omega}^\varepsilon(x,\omega,t) = \Omega^\varepsilon(x,t) + \varepsilon\kappa\,(\omega\cdot\nabla_x)\,\Omega^\varepsilon(x,t) + O(\varepsilon^2)\,, \tag{3.19}$$

$$\overline{\rho}^\varepsilon(x,\omega,t) = \rho^\varepsilon(x,t) + \varepsilon\widetilde{\kappa}\,\omega\cdot\nabla_x\rho^\varepsilon(x,t) + O(\varepsilon^2)\,, \tag{3.20}$$

where the local density ρ^ε and orientation Ω^ε only depend on the space and time variables and are given by

$$\Omega^\varepsilon(x,t) = \frac{j^\varepsilon(x,t)}{|j^\varepsilon(x,t)|}, \quad \text{with} \quad j^\varepsilon(x,t) = \int_{\upsilon\in\mathbb{S}} \upsilon\, f^\varepsilon(x,\upsilon,t)\,\mathrm{d}\upsilon\,, \tag{3.21}$$

$$\rho^\varepsilon(x,t) = \int_{\upsilon\in\mathbb{S}} f^\varepsilon(x,\upsilon,t)\,\mathrm{d}\upsilon\,, \tag{3.22}$$

and κ and $\widetilde{\kappa}$ are constants, only depending on the observation kernels K and $\widetilde{K}$. We have that these constants are positive if the observation kernel is directed forward, and the more acute the angle of vision, the bigger the constant related to the kernel.

Now we can introduce these expansions in the mean-field model, and after some easy algebra, the rescaled model from (3.13)-(3.15) can be written in the form

$$\varepsilon(\partial_t f^\varepsilon + \omega\cdot\nabla_x f^\varepsilon + \kappa\,P(f^\varepsilon) + \widetilde{\kappa}\,R(f^\varepsilon)) = Q(f^\varepsilon) + O(\varepsilon^2)\,, \tag{3.23}$$

where Q, P and R are the operators given by the equations below. Notice that Q (the only term of order 0 in ε) only acts on the variable ω and the study of its properties will be important for the following.

$$Q(f) = -\nu(\rho[f])\nabla_\omega\cdot(F_{\Omega[f]}f) + d(\rho[f])\Delta_\omega f, \tag{3.24}$$

$$F_\Omega(\omega) = (\mathrm{Id} - \omega\otimes\omega)\Omega, \tag{3.25}$$

$$P(f) = \nu(\rho[f])\nabla_\omega\cdot(G_{\Omega[f]}f), \tag{3.26}$$

$$G_\Omega(\omega) = (\nabla_x\Omega)^T\omega - (\nabla_x\Omega : \omega\otimes\omega)\,\omega, \tag{3.27}$$

$$R(f) = \nabla_\omega\cdot\Big((\omega\cdot\nabla_x\rho)\,(\dot{\nu}(\rho[f])\,F_{\Omega[f]}f - \dot{d}(\rho[f])\,\nabla_\omega f)\Big)\,, \tag{3.28}$$

$$\rho[f] = \int_{\omega\in\mathbb{S}} f(.,\omega)\,\mathrm{d}\omega\,, \tag{3.29}$$

$$\Omega[f] = \frac{j[f]}{|\ j[f]\ |}, \quad \text{with} \quad j[f] = \int_{\omega\in\mathbb{S}} \omega\, f(.,\omega)\,\mathrm{d}\omega\,. \tag{3.30}$$

Here are the notations : $\dot{\nu}$ and $\dot{d}$ are the derivatives of ν (with respect to ρ), and $\nabla_x \Omega$ is the gradient tensor of Ω that is to say $(\nabla_x \Omega)_{ij} = \partial_{x_i} \Omega_j$. We write ^{T} the transpose of operators. And we use ":" to denote the "contraction" of two operators (if $A = (A_{ij})$ and $B = (B_{ij})$ then $A : B = \sum_{i,j=1,\dots,n} A_{ij} B_{ij}$, this is the trace of AB^T).

Remark 3.2.1 *If ν also depends on $\omega \cdot \overline{\Omega}$, the expression of the operator Q is the same with $\nu(\rho[f], \omega \cdot \Omega[f])$ instead of $\nu(\rho[f])$. But then the expressions of P and R complicate in a significant way, since there are also terms with the derivative of ν with respect to this second variable.*

Now we are ready to study this system when $\varepsilon \to 0$.

3.3 Step 3 : limit as $\varepsilon \to 0$

This is the main step, here we can express the fact that the continuum limit (2.11)-(2.12) is derived form the mean-field kinetic version of the particle dynamics in a more precise way.

Theorem 1 *The limit $\varepsilon \to 0$ of f^ε is given (formally) by $f^0 = M_{\rho,\Omega}$ where $\rho = \rho(x,t) \geq 0$ is the total mass of f^0 and $\Omega = \Omega(x,t) \in \mathbb{S}$ is the director of its flux :*

$$\rho(x,t) = \int_{\omega \in \mathbb{S}} f^0(x,\omega,t)\, \mathrm{d}\omega, \tag{3.31}$$

$$\Omega = \frac{j}{|j|}, \quad j(x,t) = \int_{\omega \in \mathbb{S}} f^0(x,\omega,t)\, \omega\, \mathrm{d}\omega, \tag{3.32}$$

and $M_{\rho,\Omega}$ is a given function of $\omega \cdot \Omega$ and $\dfrac{\nu}{d}$ which will be specified later on (see (3.36)). Furthermore, $\rho(x,t)$ and $\Omega(x,t)$ satisfy the following system of first order partial differential equations:

$$\partial_t \rho + \nabla_x \cdot (c_1 \rho \Omega) = 0. \tag{3.33}$$

$$\rho\, (\partial_t \Omega + c_2 (\Omega \cdot \nabla)\Omega) + \lambda\, (Id - \Omega \otimes \Omega) \nabla_x \rho = 0, \tag{3.34}$$

where the convection speeds c_1, c_2 and the interaction parameter λ depend on ρ. Their expressions will be given in this section (see (3.55) and (3.58)-(3.59)).

The method to obtain this result follows closely [8], and is only summarized here. We will focus on the details which are specific to this study.

3.3.1 Equilibria

The first important point is to determine the null space $\mathcal{E}$ of Q, since it contains the limits of (3.23). We find, like [8], that it is a n-dimensional manifold consisting of functions analogous to maxwellian distributions in the classical Boltzmann theory :

$$\mathcal{E} = \{ M_{\rho,\Omega}(\omega) \quad | \quad \rho \in \mathbb{R}_+, \quad \Omega \in \mathbb{S} \}, \tag{3.35}$$

where

$$M_{\rho,\Omega}(\omega) = C(\rho) e^{\alpha(\rho)\, \omega \cdot \Omega}. \tag{3.36}$$

Here we have written $\alpha(\rho) = \dfrac{\nu(\rho)}{d(\rho)}$ to simplify the notations. The main difference with [8] is the dependence on ρ for this equilibrium on a nonlinear way, coming from the dependence of ν and d. This will result in additional terms in the computations, and so in additional terms in the expressions of the constants in the limit model.

The normalization constant $C(\rho)$ is defined by $C(\rho) = \dfrac{\rho}{\int_{\mathbb{S}} e^{\alpha(\rho)\,\omega\cdot\Omega}\,\mathrm{d}\omega}$ (depending only on ρ and α, not on Ω), so that ρ is the total mass $M_{\rho,\Omega}(\omega)$, and we have that Ω is the director of its flux, that is to say, using the notations (3.29), (3.30) :

$$\rho[M_{\rho,\Omega}] = \rho\,, \quad \Omega[M_{\rho,\Omega}] = \Omega. \tag{3.37}$$

We also note, that, like in [8], the operator $Q(f)$ can be written as

$$Q(f) = d(\rho[f])\nabla_\omega \cdot \left[M_{\rho[f],\Omega[f]} \nabla_\omega \left(\frac{f}{M_{\rho[f],\Omega[f]}} \right) \right] \tag{3.38}$$

and satisfies

$$H(f) := \int_{\omega\in\mathbb{S}} Q(f) \frac{f}{M_{\rho[f],\Omega[f]}}\,\mathrm{d}\omega = -d \int_{\omega\in\mathbb{S}} M_{\rho[f],\Omega[f]} \left| \nabla_\omega \left(\frac{f}{M_{\rho[f],\Omega[f]}} \right) \right|^2 \mathrm{d}\omega \le 0. \tag{3.39}$$

We can easily compute the flux of this equilibrium :

$$j[M_{\rho,\Omega}] = \langle \cos\theta \rangle_{M_\alpha}\, \rho\Omega, \tag{3.40}$$

where for any function $\gamma(\cos\theta)$, the symbol $\langle \gamma(\cos\theta)\rangle_{M_\alpha}$ denotes the mean of γ following the density $\dfrac{1}{\rho} M_{\rho,\Omega}$, i.e.

$$\langle \gamma(\cos\theta)\rangle_{M_\alpha} = \frac{1}{\rho} \int_{\omega\in\mathbb{S}} M_{\rho,\Omega}(\omega)\gamma(\omega\cdot\Omega)\,\mathrm{d}\omega = \frac{\int_{\mathbb{S}} \gamma(\omega\cdot\Omega) e^{\alpha(\rho)\,\omega\cdot\Omega}\,\mathrm{d}\omega}{\int_{\mathbb{S}} e^{\alpha(\rho)\,\omega\cdot\Omega}\,\mathrm{d}\omega}. \tag{3.41}$$

Notice that $\langle \gamma(\cos\theta)\rangle_M$ only depends on $\alpha(\rho)$, not on Ω, we have the expression in dimension n (for $n = 2$ or $n = 3$) :

$$\langle \gamma(\cos\theta)\rangle_{M_\alpha} = \frac{\int_0^\pi \gamma(\cos\theta) e^{\alpha(\rho)\cos\theta}\, \sin^{n-2}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{\alpha(\rho)\cos\theta}\, \sin^{n-2}\theta\,\mathrm{d}\theta}. \tag{3.42}$$

Remark 3.3.1 *In the case where ν depends on ρ and $\omega\cdot\Omega$, we have to replace in all this point $\alpha(\rho)\,\omega\cdot\Omega$ by $\widehat{\alpha}(\rho,\omega\cdot\Omega)$, where $\widehat{\alpha}(\rho,\mu) = \displaystyle\int_0^\mu \frac{\nu(\rho,\tau)}{d(\rho)}\mathrm{d}\tau$.*

3.3.2 Collisional invariants

The second important point is the determination of generalized collisional invariants. Indeed, since there is no other conservation relation than the conservation of mass, the collision invariants reduce to the constants, and the integration of the equation against these invariants only gives one equation, which is not sufficient to describe the behavior

of the equilibrium (which lives on a n-dimensional manifold). The main idea in [8] was to overcome this problem with a generalization of the concept of the collisional invariants.

A collision invariant is a function such that the integration of $Q(f)$ against it gives zero for all function f of ω. So we ask for a generalized invariant to satisfy this definition only for a restricted subset of functions f. In the case where the dependence on ρ in the equilibria was linear, restricting to all functions with a given orientation Ω was sufficient to obtain the remaining equation. Here we have to restrict also to functions with a given density too.

We define the generalized collisional invariants $\mathcal{C}_{\rho,\Omega}$ of Q (associated with some $\rho > 0$ and some $\Omega \in \mathbb{S}$) as the functions ψ such that

$$\int_{\omega\in\mathbb{S}} Q(f)\,\psi\,\mathrm{d}\omega = 0, \quad \forall f \text{ such that } \Omega[f] = \Omega \text{ and } \rho[f] = \rho. \tag{3.43}$$

If we define the linear operator $L_{\rho,\Omega}$ by $L_{\rho,\Omega}(f) = d(\rho)\nabla_\omega \cdot \left[M_{\rho,\Omega}\nabla_\omega\left(\frac{f}{M_{\rho,\Omega}}\right)\right]$, we have that

$$\mathcal{C}_{\rho,\Omega} = \left\{\psi | \int_{\omega\in\mathbb{S}} L_{\rho,\Omega}(f)\,\psi\,\mathrm{d}\omega = 0, \quad \forall f \text{ such that } \Omega[f] = \Omega \text{ and } \rho[f] = \rho\right\}. \tag{3.44}$$

Using the linearity of the operators $j[.]$, $\rho[.]$, and $L_{\rho,\Omega}$, we get :

$$\mathcal{C}_{\rho,\Omega} = \left\{\psi | \int_{\omega\in\mathbb{S}} L_{\rho,\Omega}(f)\,\psi\,\mathrm{d}\omega = 0, \quad \forall f \text{ such that } j[f]\times\Omega = 0\right\}. \tag{3.45}$$

This computation has been done in [8], in dimension 3, so the result is that $\mathcal{C}_{\rho,\Omega}$ is of the form $Span\{1, \psi_{1,\alpha(\rho)}, \psi_{2,\alpha(\rho)}\}$. The expression of these functions can be done with an auxiliary function $g_\alpha(\mu)$, which is the unique regular solution on the following elliptic problem on $[-1,1]$ (in dimension 3) :

$$-(1-\mu^2)\partial_\mu(e^{\alpha\mu}(1-\mu^2)\partial_\mu g) + e^{\alpha\mu} g = -(1-\mu^2)^{3/2}e^{\alpha\mu}. \tag{3.46}$$

Indeed, this problem has a unique solution, in the space V (a " weighted H^1") defined by $V = \{g \,|\, (1-\mu^2)^{-1/2} g \in L^2(-1,1), \quad (1-\mu^2)^{1/2}\partial_\mu g \in L^2(-1,1)\}$, and this solution is negative (by the maximum principle).

Then, with the spherical coordinates (θ,ϕ) in a cartesian basis (e_1, e_2, Ω), we have the following expressions for the general collisional invariants : $\psi_{1,\alpha} = -g_\alpha(\cos\theta)\sin\phi$, and $\psi_{2,\alpha} = g_\alpha(\cos\theta)\cos\phi$.

We introduce to simplify notations the function $h_\rho(\mu) = (1-\mu^2)^{-1/2} g_{\alpha(\rho)} \in L^2(-1,1)$ that is to say $h_\rho(\cos\theta) = \dfrac{g_{\alpha(\rho)}(\cos\theta)}{\sin\theta}$ and we have

$$\vec{\psi}_{\rho,\Omega}(\omega) = (\Omega\times\omega)\,h_\rho(\Omega\cdot\omega) = \psi_{1,\alpha(\rho)}e_1 + \psi_{2,\alpha(\rho)}e_2\,. \tag{3.47}$$

We call $\vec{\psi}$ the generalized collisional invariant vector associated to ρ and Ω.

In dimension 2, we also have a two-dimensional vector space of generalized collisional invariants (associated to some density ρ and orientation Ω). It is spanned by the constants and a function $\psi_{\rho,\Omega}$, which is the unique solution with zero average of the following elliptic

equation (here we identify functions of $\mathbb{S}_1$ and 2π-periodic functions of $\mathbb{R}$, with $\theta = 0$ corresponding to Ω) :

$$\partial_\theta(e^{\alpha(\rho)\cos\theta}\partial_\theta\psi) = \sin\theta\, e^{\alpha(\rho)\cos\theta}\,. \tag{3.48}$$

In fact in that case we have an explicit expression :

$$\psi_\alpha(\theta) = \frac{\pi}{\alpha}\frac{\int_0^\theta e^{-\alpha\cos\varphi}\mathrm{d}\varphi}{\int_0^\pi e^{-\alpha\cos\varphi}\mathrm{d}\varphi} - \frac{\theta}{\alpha}\,. \tag{3.49}$$

So if we define $h_\rho(\cos\theta) = \dfrac{\psi_\alpha(\theta)}{\sin\theta}$ (this is possible since this term is an even 2π-periodic function), so we also have $\psi_{\rho,\Omega}(\omega) = (\Omega\times\omega)\,h_\rho(\Omega\cdot\omega)$ if we denote by $\times$ the determinant of two vectors, and the notations are coherent in 2 and 3-dimensional framework, which will be convenient for the following.

Remark 3.3.2 *If we take in account the "ability to turn", we just replace $\alpha\mu$ in* (3.46) *by $\widehat{\alpha}(\rho,\mu)$ (for dimension 3). In dimension 2, we replace $\alpha(\rho)\cos\theta$ by $\widehat{\alpha}(\rho,\cos\theta)$ in* (3.48)*, and we still have an explicit expression :*

$$\psi_{\widehat{\alpha}}(\theta) = \psi^0_{\widehat{\alpha}}(\theta) - \frac{\psi^0_{\widehat{\alpha}}(\pi)}{\psi^\infty_{\widehat{\alpha}}(\pi)}\psi^\infty_{\widehat{\alpha}}(\theta), \tag{3.50}$$

where

$$\psi^0_{\widehat{\alpha}}(\theta) = -\int_0^\theta\left(\int_\varphi^\pi e^{\widehat{\alpha}(\rho,\cos\phi)}\sin\phi\,\mathrm{d}\phi\right)e^{-\widehat{\alpha}(\rho,\cos\varphi)}\,\mathrm{d}\varphi, \tag{3.51}$$

$$\psi^\infty_{\widehat{\alpha}}(\theta) = \int_0^\theta e^{-\widehat{\alpha}(\rho,\cos\varphi)}\,\mathrm{d}\varphi\,. \tag{3.52}$$

3.3.3 Computation of the limit as $\varepsilon \to 0$

The third and final important point is taking the limit $\varepsilon \to 0$ in the equation (3.23), after integrating against the collision invariants.

When $\varepsilon \to 0$, if f^ε does not present any pathologic behavior (for example all derivatives of f are bounded), we have that $Q(f^\varepsilon) \to 0$, so $f^\varepsilon \to M_{\rho,\Omega}$ (here we suppose that f^ε converges to some function f, and that this is also true for all the derivative we need). So we have $\rho^\varepsilon \to \rho$, and $\Omega^\varepsilon \to \Omega$.

For the mass equation, using the constant invariant and integrating, we get the conservation of mass (in fact we can even replace the $O(\varepsilon)$ by zero in this equation since in the original model (3.13) we have conservation of mass) :

$$\partial_t\rho^\varepsilon + \nabla_x\cdot j^\varepsilon = O(\varepsilon). \tag{3.53}$$

We get in the $\varepsilon \to 0$ limit :

$$\partial_t\rho + \nabla_x\cdot(c_1\rho\Omega) = 0, \tag{3.54}$$

where (see (3.40)) :

$$c_1 = \langle\cos\theta\rangle_{M_\alpha}. \tag{3.55}$$

The complications start for the equation on Ω. The method is the same, we multiply at the right by $\vec{\psi}_{\rho^\varepsilon,\Omega^\varepsilon}$, integrate on the sphere, and take the limit $\varepsilon \to 0$. Since $\vec{\psi}$ depends continuously on ρ and Ω, we obtain finally an expression with $\vec{\psi}_{\rho,\Omega}$.

We get an expression of the form $\Omega \times X = 0$ (equivalent to $(\mathrm{Id} - \Omega \otimes \Omega)\, X = 0$), where

$$X = \int_{\omega \in \mathbb{S}} (\partial_t(M_{\rho,\Omega}) + \omega \cdot \nabla_x(M_{\rho,\Omega}) + \kappa P(M_{\rho,\Omega}) + \widetilde{\kappa} R(M_{\rho,\Omega}))\, h_\rho(\omega \cdot \Omega)\, \omega \, \mathrm{d}\omega \,. \tag{3.56}$$

Here the computation is similar as [8] for some terms, but some additional work is required for the terms coming from the nonlinearity of $M_{\rho,\Omega}$ in ρ and the operators P and R.

We give the result of the computations under the form of a proposition :

Proposition 1 *$(Id - \Omega \otimes \Omega)\, X = 0$, where X is given in (3.56), is equivalent to*

$$\rho\, (\partial_t \Omega + c_2(\Omega \cdot \nabla)\Omega) + \lambda\, (Id - \Omega \otimes \Omega) \nabla_x \rho = 0, \tag{3.57}$$

where

$$\lambda(\rho) = \frac{1}{\alpha} + \rho \frac{\dot{\alpha}}{\alpha} \left((1 - n\, \widetilde{\kappa}\, d)\, \langle \cos\theta \rangle_{\widetilde{M}_\alpha} - \langle \cos\theta \rangle_{M_\alpha} + \widetilde{\kappa}\, \nu \langle \sin^2\theta \rangle_{\widetilde{M}_\alpha} \right), \tag{3.58}$$

$$c_2(\rho) = (1 - n\, \kappa\, d)\, \langle \cos\theta \rangle_{\widetilde{M}_\alpha} - \kappa\, \nu \langle \cos^2\theta \rangle_{\widetilde{M}_\alpha}, \tag{3.59}$$

with, if we are in dimension n (for $n = 2$ or $n = 3$) :

$$\langle \gamma(\cos\theta) \rangle_{\widetilde{M}_\alpha} = \frac{\int_0^\pi \gamma(\cos\theta) h_\rho(\cos\theta) e^{\alpha(\rho)\cos\theta}\, \sin^n\theta \, \mathrm{d}\theta}{\int_0^\pi h_\rho(\cos\theta) e^{\alpha(\rho)\cos\theta}\, \sin^n\theta \, \mathrm{d}\theta}. \tag{3.60}$$

This is actually the last part of theorem 1, with a precise definition for coefficients c_2 and λ, and this ends the derivation of the continuum model (2.11)-(2.12).

The computations to get this result are given in Annex A.1.2, we just detail briefly here the way we do it. The idea is first to get rid of the derivatives in ω coming from the operators P and R with an integration by parts on the sphere, and then to use chain rule to obtain an expression of the form

$$X = \sum A_i \left(\int_{\omega \in \mathbb{S}} \gamma_i(\omega \cdot \Omega)\, \omega^{\otimes j_i}\, M_{\rho,\Omega} \mathrm{d}\omega \right), \tag{3.61}$$

where A_i are simple linear operations only depending on the time and space derivatives of ρ and Ω (like taking the scalar product with $\nabla_x \rho$, for example), and γ_i are functions involving h_ρ for instance.

Finally we compute $(\mathrm{Id} - \Omega \otimes \Omega) \int_{\omega \in \mathbb{S}} \gamma(\omega \cdot \Omega)\, \omega^{\otimes j} M_{\rho,\Omega}\, \mathrm{d}\omega$ in spherical (or polar) coordinates in a basis where the last element is Ω, which simplifies a lot of terms.

Remark 3.3.3 *Here, we only give the results obtained in the case where ν depends on $\omega \cdot \Omega$ (and not on ρ) and where d is a constant. We get the same results, except that the*

constants are given (with analogous definitions) by :

$$c_1 = \langle \cos\theta \rangle_{M_{\widehat{\alpha}}} \tag{3.62}$$

$$c_2 = \left\langle \cos\theta(1 - \kappa(\nu\cos\theta - \nu'\sin^2\theta)) \right\rangle_{\widetilde{M}_{\widehat{\alpha}}} - \kappa d \left\langle n\cos\theta + \frac{\nu'}{\nu}((n+2)\cos^2\theta - 1) - \frac{\nu''}{\nu}\cos\theta\sin^2\theta \right\rangle_{\widetilde{M}_{\widehat{\alpha}}}, \tag{3.63}$$

$$\lambda = d\left\langle \frac{1}{\nu} \right\rangle_{\widetilde{M}_{\widehat{\alpha}}}. \tag{3.64}$$

Since ν is supposed to be positive, the constant λ is positive, and we will see in the next section that its possible change of sign with the dependence on ρ is important. This is why we focus on the dependance on ρ and not in $\omega \cdot \Omega$ in this article.

4 Properties of the macroscopic model

4.1 Hyperbolicity

The main result about model (2.11)-(2.12) is that if d or ν depends on ρ, the model may lose hyperbolicity in some regions of the state space.

It is not possible to do another scaling to get rid of c_1 (and have a more simple analysis of the system), like in [8], because c_1 depends on ρ.

We consider waves travelling along one direction, and denote by θ the angle between Ω and this direction. If we write $\widetilde{c_1(\rho)} = \dfrac{\mathrm{d}(\rho c_1(\rho))}{\mathrm{d}\rho}$, we have the following statement :

Theorem 2 *If $\lambda(\rho) \geqslant 0$, then the system (2.11)-(2.12) is hyperbolic. And if $\lambda < 0$, this system is hyperbolic if and only if*

$$|\tan\theta| < \frac{|c_2 - \widetilde{c_1}|}{2\sqrt{-\lambda c_1}}. \tag{4.65}$$

Here are the graphics of the boundary of the region of hyperbolicity (for $\theta \in [0, \frac{\pi}{2}]$, the system is hyperbolic in the lower part of the area delimited by the curve, and is not hyperbolic in the upper part) for ν of the form ρ^β, for different values of parameters d, β, κ and $\widetilde{\kappa}$.

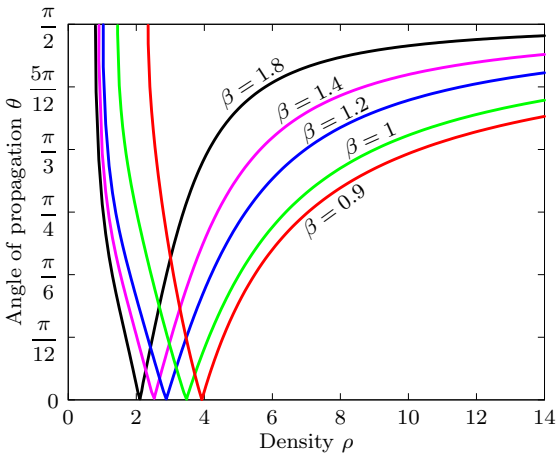

Figure 1: $d = 0.4$, $\kappa = \widetilde{\kappa} = 0.1$

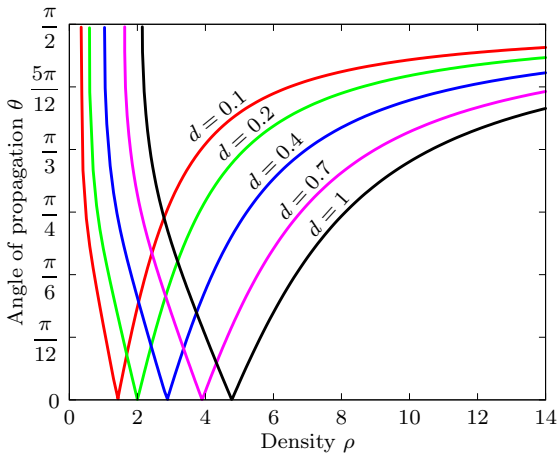

Figure 2: $\beta = 1.2$, $\kappa = \widetilde{\kappa} = 0.1$

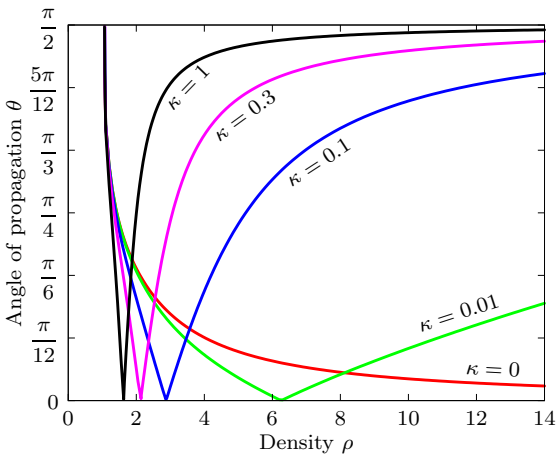

Figure 3: $\beta = 1.2$, $d = 0.4$, $\widetilde{\kappa} = 0.1$

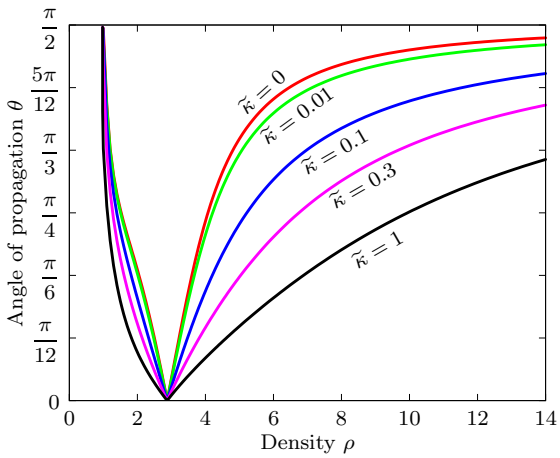

Figure 4: $\beta = 1.2$, $d = 0.4$, $\kappa = 0.1$

Before giving a proof of this theorem, we will discuss the interest and the problems due to the non-hyperbolicity.

The first thing to remark is that the model is not always well-posed. Indeed, in general, we cannot ensure that a solution will stay in the region of hyperbolicity for all time.

The property of hyperbolicity is linked with the fact that perturbation propagate with finite speed. Here the presence of a region of ellipticity means that we could have propagation with infinite speed across this region.

This leads to a second remark : it may be possible to construct non-classical shocks, using the crossing of a zone of non-hyperbolicity, see [14], and [13].

The interest is that me may construct some travelling waves, as observed in [5]. Actually we did not manage to construct such solutions yet, this is part of our future work.

We should also construct models with formation of coherent structures from such non-hyperbolic models, if we could use stabilisation with diffusion.

Here the expansion at higher order in ε in this model, in order to obtain diffusion terms in the macroscopic model becomes too much complicated to perform some study.

Proof of theorem 2

We compute, in dimension 3, in an fixed cartesian coordinate system $(\Omega_1, \Omega_2, \Omega_3)$ and use spherical coordinates (θ, ϕ) in this system. Then, $\Omega = (\sin\theta\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta)$. A simple algebra shows that (ρ, θ, ϕ) satisfy the system

$$\partial_t\rho + \partial_x(\rho c_1(\rho)\sin\theta\cos\phi) + \partial_y(\rho c_1(\rho)\sin\theta\sin\phi) + \partial_z(\rho c_1(\rho)\cos\theta) = 0. \tag{4.66}$$

$$\begin{aligned}\partial_t\theta + c_2(\rho)(\sin\theta\cos\phi\,\partial_x\theta + \sin\theta\sin\phi\,\partial_y\theta + \cos\theta\partial_z\theta) + \\ + \lambda\,(\cos\theta\cos\phi\,\partial_x\ln\rho + \cos\theta\sin\phi\,\partial_y\ln\rho - \sin\theta\,\partial_z\ln\rho) = 0.\end{aligned} \tag{4.67}$$

$$\begin{aligned}\partial_t\phi + c_2(\rho)(\sin\theta\cos\phi\,\partial_x\phi + \sin\theta\sin\phi\,\partial_y\phi + \cos\theta\partial_z\phi) + \\ + \lambda\,(-\sin\theta\sin\phi\,\partial_x\ln\rho + \sin\theta\cos\phi\,\partial_y\ln\rho) = 0.\end{aligned} \tag{4.68}$$

Supposing that ρ, θ, ϕ are independent of x and y amounts to looking at waves which propagate in the z direction at a solid angle (θ, ϕ) with the velocity director Ω. In this geometry, the system reads :

$$\partial_t\rho + \widetilde{c_1(\rho)}\cos\theta\,\partial_z\rho - c_1(\rho)\rho\sin\theta\,\partial_z\theta = 0. \tag{4.69}$$

$$\partial_t\theta + c_2(\rho)\cos\theta\,\partial_z\theta - \lambda\,\sin\theta\,\partial_z\ln\rho = 0. \tag{4.70}$$

$$\partial_t\phi + c_2(\rho)\cos\theta\,\partial_z\phi = 0, \tag{4.71}$$

where $\widetilde{c_1(\rho)} = \dfrac{\mathrm{d}(\rho c_1(\rho))}{\mathrm{d}\rho} = c_1(\rho) + \rho\dot{\alpha}\left(\langle\cos^2\theta\rangle_{M_\alpha} - \langle\cos\theta\rangle^2_{M_\alpha}\right)$.

This is a first order system of the form

$$\begin{pmatrix}\partial_t\rho\\ \partial_t\theta\\ \partial_t\phi\end{pmatrix} + A(\rho,\theta,\phi)\begin{pmatrix}\partial_z\rho\\ \partial_z\theta\\ \partial_z\phi\end{pmatrix} = 0, \tag{4.72}$$

with

$$A(\rho,\theta,\phi) = \begin{pmatrix}\widetilde{c_1(\rho)}\cos\theta & -c_1(\rho)\rho\sin\theta & 0\\ -\frac{\lambda\sin\theta}{\rho} & c_2(\rho)\cos\theta & 0\\ 0 & 0 & c_2(\rho)\cos\theta\end{pmatrix}, \tag{4.73}$$

and this system is hyperbolic in case $\lambda \geqslant 0$. The eigenvalues $\gamma_\pm$ et γ_0 are given by

$$\gamma_0 = c\cos\theta, \quad \gamma_\pm = \frac{1}{2}\left[(\widetilde{c_1} + c_2)\cos\theta \pm \left((c_2 - \widetilde{c_1})^2\cos^2\theta + 4\lambda c_1\sin^2\theta\right)^{1/2}\right], \tag{4.74}$$

In case of dimension 2, we get the same thing with only the eigenvalues $\gamma_\pm$.

Now if $\lambda < 0$, asking $\gamma_\pm$ to be real is exactly equivalent to the equation (4.65), and this ends the proof. $\diamond$

4.2 Influence of the anisotropy

The influence of the anisotropy on the final macroscopic model is only visible through the values of the speed c_2 and the parameter λ.

So this can have an impact on the regions of non-hyperbolicity, and the behaviour of the information on the orientation. In annex A.2, we perform an asymptotic study of this coefficients, when the noise is small.

If we suppose that $\rho\dfrac{\dot{\alpha}(\rho)}{\alpha(\rho)}$ is bounded with respect to $\mathrm{d}(\rho)$ (for instance a power law for α), and that $d(\rho)$ is a small parameter, we get the following expansions :

$$c_1(\rho) = 1 - \frac{1}{2\nu(\rho)} d(\rho) + O(d^2) \tag{4.75}$$

$$c_2(\rho) = 1 - \kappa\nu(\rho) - \left(\frac{3}{2\nu(\rho)} - \kappa \right) d(\rho) + O(d^2) \tag{4.76}$$

$$\lambda(\rho) = \frac{d(\rho)}{\nu(\rho)} \left(1 - \rho \frac{\dot{\alpha}(\rho)}{\alpha(\rho)} (1 - \widetilde{\kappa}\nu(\rho)) \right) + O(d^2). \tag{4.77}$$

First, this shows that in some cases we have actually $\lambda < 0$, for instance when $\nu = \nu_0 \rho^\beta$ with $\beta > 1$, $\widetilde{\kappa} = 0$ and d is a sufficiently small constant.

Then we see that when the noise is small, the difference between c_1 and c_2 is enhanced when κ is a large positive constant. This can be interpreted as follows : if the observation kernel is strongly directed forward, then the information on the orientation moves rapidly backward. This could be compared to results on modelling of traffic flows, where the speed of a congested phase depends on the distance of anticipation of the drivers (see [2, 4, 7]).

Remark 4.2.1 *We can also do an expansion in the more general case where ν depends on ρ and $\omega \cdot \Omega$. In annex A.2.1 we give tips to perform this expansion. Here we only give the final expansion in the case where ν and d do not depend on ρ. In this case we can suppose $\nu(1) = 1$ (up to a rescaling), and denote $\gamma = \left. \frac{\mathrm{d}\nu}{\mathrm{d}(\omega \cdot \Omega)} \right|_{\omega \cdot \Omega = 1}$. We finally get an expansion, as $d \to 0$, of the coefficients given by* (3.62)-(3.64)*, with $n = 2$:*

$$c_1(d) = 1 - \frac{d}{2} + O(d^2), \tag{4.78}$$

$$c_2 = 1 - \kappa + \left((1 + \frac{3}{2}\gamma)\kappa - \frac{3}{2} \right) d + O(d^2), \tag{4.79}$$

$$\lambda = d + \frac{3}{2}\gamma\, d^2 + O(d^3). \tag{4.80}$$

5 Conclusion

In this article, we have seen that the introduction of a dependance on some local density for some parameters at the microscopic level implies a significant change in the macroscopic limit : the possible loss of hyperbolicity in some regimes. The introduction of a non-isotropic kernel of observation is not sufficient to imply a strong difference of behavior for the continuum model. However, it enhances some properties, as the differences between the velocity of the fluid, and the velocity of the information on the orientation.

It is important to note that the method introduced in [8] works to derive the macroscopic model. In particular the concept of generalized collisional invariants is still valid, with some adaptations.

Some questions are left open. The limit here is formal, and we are still looking for an appropriate functional framework to obtain more precise results of convergence. The rigorous obtention of the kinetic version of the dynamical system of particles is also part of our future work.

Finally, the next step to this study consists in some numerical simulations, in order to see how the difference between c_2 and c_1 can be observed in simulations of the discrete dynamical system, or how the particles behave in the regions of non-hyperbolicity.

A Annex

A.1 Proof of some statements for section 3

A.1.1 Expansion of the local density and orientation

We recall the expressions of $\overline{\Omega}^\varepsilon$ and $\overline{\rho}^\varepsilon$:

$$\overline{\Omega}^\varepsilon(x,\omega,t) = \frac{\overline{J}^\varepsilon(x,\omega,t)}{|\overline{J}^\varepsilon(x,\omega,t)|}, \tag{A.81}$$

$$\overline{J}^\varepsilon(x,\omega,t) = \int_{y\in\mathbb{R}^n,\, \upsilon\in\mathbb{S}} K\left(\frac{|x-y|}{\varepsilon}, \frac{y-x}{|x-y|}\cdot\omega\right)\, \upsilon\, f^\varepsilon(y,\upsilon,t)\,\frac{\mathrm{d}y}{\varepsilon^n}\,\mathrm{d}\upsilon\,, \tag{A.82}$$

$$\overline{\rho}^\varepsilon(x,\omega,t) = \int_{y\in\mathbb{R}^n,\, \upsilon\in\mathbb{S}} \widetilde{K}\left(\frac{|x-y|}{\varepsilon}, \frac{y-x}{|x-y|}\cdot\omega\right)\, f^\varepsilon(y,\upsilon,t)\,\frac{\mathrm{d}y}{\varepsilon^n}\,\mathrm{d}\upsilon\,, \tag{A.83}$$

Lemma 1 *We have the following expansions :*

$$\overline{\Omega}^\varepsilon(x,\omega,t) = \Omega^\varepsilon(x,t) + \varepsilon\kappa\,(\omega\cdot\nabla_x)\,\Omega^\varepsilon(x,t) + O(\varepsilon^2)\,, \tag{A.84}$$

$$\overline{\rho}^\varepsilon(x,\omega,t) = \rho^\varepsilon(x,t) + \varepsilon\widetilde{\kappa}\,\omega\cdot\nabla_x\rho^\varepsilon(x,t) + O(\varepsilon^2)\,, \tag{A.85}$$

where κ is a constant, depending on the observation kernel K, and

$$\Omega^\varepsilon(x,t) = \frac{j^\varepsilon(x,t)}{|j^\varepsilon(x,t)|}, \qquad \textit{with} \qquad j^\varepsilon(x,t) = \int_{\upsilon\in\mathbb{S}_2} \upsilon\, f^\varepsilon(x,\upsilon,t)\,\mathrm{d}\upsilon\,, \tag{A.86}$$

$$\rho^\varepsilon(x,t) = \int_{\upsilon\in\mathbb{S}_2} f^\varepsilon(x,\upsilon,t)\,\mathrm{d}\upsilon\,, \tag{A.87}$$

Proof

After change of variable $y = x + \varepsilon\xi$, let us expand f at first order in ε in (A.82). We get

$$\overline{J}^\varepsilon(x,\omega,t) = \int_{\xi\in\mathbb{R}^n,\, \upsilon\in\mathbb{S}} K(|\xi|, |\xi|^{-1}\,\xi\cdot\omega)\,\upsilon\,(f^\varepsilon(x,\upsilon,t) + \varepsilon\,\xi\cdot\nabla_x f^\varepsilon(x,\upsilon,t) + O(\varepsilon^2))\,\mathrm{d}\xi\,\mathrm{d}\upsilon\,. \tag{A.88}$$

We have to compute

$$K_0(\omega) = \int_{\xi\in\mathbb{R}^n} K(|\xi|, |\xi|^{-1}\,\xi\cdot\omega)\mathrm{d}\xi \quad \text{and} \quad K_1(\omega) = \int_{\xi\in\mathbb{R}^n} K(|\xi|, |\xi|^{-1}\,\xi\cdot\omega)\,\xi\,\mathrm{d}\xi. \tag{A.89}$$

For any rotation R, change of variable $\tilde{\xi} = R(\xi)$ gives on one hand

$$K_0(\omega) = K_0(R(\omega))\,, \tag{A.90}$$

and so K_0 does not depend on ω.

On the other hand, we get

$$R(K_1(\omega)) = K_1(R(\omega))\,, \tag{A.91}$$

which shows that $K_1(\omega)$ is a vector invariant by any rotation of axe ω, so it is parallel to ω. Fix one vector e of $\mathbb{S}$, we have $K_1(e) = \alpha e$. Then taking one rotation mapping ω to e, we get $R(K_1(\omega)) = K_1(e) = \alpha e = R(\alpha\omega)$, so $K_1(\omega) = \alpha\omega$.

Let then $\kappa = \dfrac{\alpha}{K_0}$, and we have

$$\frac{\overline{J}^\varepsilon(x,\omega,t)}{K_0} = \int_{\upsilon\in\mathbb{S}} \upsilon\left(f^\varepsilon(x,\upsilon,t) + \varepsilon\,\kappa\,\omega\cdot\nabla_x f^\varepsilon(x,\upsilon,t)\right)\mathrm{d}\upsilon + O(\varepsilon^2) \tag{A.92}$$

$$= j^\varepsilon(x,t) + \varepsilon\,\kappa\,(\omega\cdot\nabla_x) j^\varepsilon(x,t) + O(\varepsilon^2)\,. \tag{A.93}$$

Putting (A.93) into (A.81), we get

$$\left|\frac{\overline{J}^\varepsilon(x,\omega,t)}{K_0}\right|^2 = |j^\varepsilon(x,t)|^2 + 2\,\varepsilon\,\kappa\, j^\varepsilon(x,t)\cdot(\omega\cdot\nabla_x) j^\varepsilon(x,t) + O(\varepsilon^2)\,, \tag{A.94}$$

so

$$\left|\frac{\overline{J}^\varepsilon(x,\omega,t)}{K_0}\right|^{-1} = \frac{1}{|j^\varepsilon(x,t)|}\left(1 - \frac{\varepsilon\,\kappa}{|j^\varepsilon(x,t)|^2}\, j^\varepsilon(x,t)\cdot(\omega\cdot\nabla_x) j^\varepsilon(x,t)\right) + O(\varepsilon^2)\,, \tag{A.95}$$

and finally

$$\begin{aligned}\overline{\Omega}^\varepsilon(x,\omega,t) = {}& \frac{j^\varepsilon(x,t)}{|j^\varepsilon(x,t)|} \\ & + \varepsilon\,\kappa\left(\frac{(\omega\cdot\nabla_x) j^\varepsilon(x,t)}{|j^\varepsilon(x,t)|} - \frac{j^\varepsilon(x,t)}{|j^\varepsilon(x,t)|}\cdot\frac{(\omega\cdot\nabla_x) j^\varepsilon(x,t)}{|j^\varepsilon(x,t)|}\frac{j^\varepsilon(x,t)}{|j^\varepsilon(x,t)|}\right) + O(\varepsilon^2).\end{aligned} \tag{A.96}$$

But we also have

$$(\omega\cdot\nabla_x)\Omega^\varepsilon(x,t) = \frac{(\omega\cdot\nabla_x) j^\varepsilon(x,t)}{|j^\varepsilon(x,t)|} + \left(\omega\cdot\nabla_x\left(\frac{1}{|j^\varepsilon(x,t)|}\right)\right) j^\varepsilon(x,t) \tag{A.97}$$

$$= \frac{(\omega\cdot\nabla_x) j^\varepsilon(x,t)}{|j^\varepsilon(x,t)|} - \frac{1}{|j^\varepsilon(x,t)|^3}\left(((\omega\cdot\nabla_x) j^\varepsilon(x,t))\cdot j^\varepsilon(x,t)\right) j^\varepsilon(x,t)\,. \tag{A.98}$$

Therefore

$$\overline{\Omega}^\varepsilon(x,\omega,t) = \Omega^\varepsilon(x,t) + \varepsilon\kappa\,(\omega\cdot\nabla_x)\,\Omega^\varepsilon(x,t) + O(\varepsilon^2)\,, \tag{A.99}$$

and this is the first part of the lemma.

After the same change of variable $y = x + \varepsilon\xi$ and expansion in (A.83), and using the same techniques, and the normalisation condition (2.8), we get

$$\overline{\rho}^\varepsilon(x,\omega,t) = \int_{\upsilon\in\mathbb{S}} \widetilde{\kappa} f^\varepsilon(x,\upsilon,t) + \varepsilon\,\widetilde{K}_1(\omega)\cdot\nabla_x f^\varepsilon(x,\upsilon,t)\,\mathrm{d}\upsilon + O(\varepsilon^2) \tag{A.100}$$

$$= \rho^\varepsilon(x,t) + \varepsilon\,\widetilde{\kappa}\,\omega\cdot\nabla_x\rho^\varepsilon(x,t) + O(\varepsilon^2)\,. \tag{A.101}$$

This is the second part of the lemma. $\diamond$

A.1.2 Proof of Proposition 1

We write α for $\alpha(\rho)$ in the following.

We have $(\mathrm{Id} - \Omega \otimes \Omega)\, X = 0$, where

$$X = \int_{\omega\in\mathbb{S}} (\partial_t(M_{\rho,\Omega}) + \omega\cdot\nabla_x(M_{\rho,\Omega}) + \kappa P(M_{\rho,\Omega}) + \widetilde{\kappa} R(M_{\rho,\Omega}))\, h_\rho(\omega\cdot\Omega)\,\omega\,\mathrm{d}\omega\,. \quad \text{(A.102)}$$

We first transform the terms with P and R, with an integration by parts on the sphere to get rid of the partial derivatives with respect to ω.

We recall that $P(M_{\rho,\Omega}) = \nu\,\nabla_\omega\cdot(M_{\rho,\Omega}G_\Omega)(\omega)$, with

$$G_\Omega(\omega) = (\nabla_x\Omega)^T\omega - (\nabla_x\Omega : \omega\otimes\omega)\,\omega. \quad \text{(A.103)}$$

Integrating by parts, we get, for a function a and a vector field A on the sphere $\mathbb{S}$,

$$\int_{\omega\in\mathbb{S}} \nabla_\omega\cdot A(\omega)\, a(\omega\cdot\Omega)\,\omega\mathrm{d}\omega = -\int_{\omega\in\mathbb{S}} (a(\omega\cdot\Omega)\,A(\omega) + A(\omega)\cdot\Omega\, a'(\omega\cdot\Omega)\,\omega)\,\mathrm{d}\omega \quad \text{(A.104)}$$

So we have

$$\int_{\omega\in\mathbb{S}} P(\rho M_{\rho,\Omega})h_\rho(\cos\theta)\omega\mathrm{d}\omega = -\nu\int_{\omega\in\mathbb{S}} B_{\rho,\Omega}(\omega)hM_{\rho,\Omega}\,\mathrm{d}\omega, \quad \text{(A.105)}$$

where the argument $\Omega\cdot\omega$ of the functions $M_{\rho,\Omega}$ and h_ρ is written $\cos\theta$, and omitted when no confusion is possible. Here the function $B_\Omega(\omega)$ is given by

$$B_\Omega(\omega) = h_\rho\, G_\Omega(\omega) + h'_\alpha\,\Omega\cdot G_\Omega(\omega)\omega \quad \text{(A.106)}$$

$$= h_\rho\,(\nabla_x\Omega)^T\omega - (h_\rho + \cos\theta h'_\alpha)(\nabla_x\Omega : \omega\otimes\omega)\,\omega, \quad \text{(A.107)}$$

because $((\nabla_x\Omega)^T\,\omega)\cdot\Omega = \omega\cdot(\nabla_x\Omega\,\Omega) = \omega\cdot(\frac{1}{2}\nabla_x|\Omega|^2) = 0$.

We have $R(M_{\rho,\Omega}) = \nabla_\omega\cdot\Big((\omega\cdot\nabla_x\rho)(\dot{\nu}F_\Omega M_{\rho,\Omega} - \dot{d}\nabla_\omega M_{\rho,\Omega})\Big)(\omega)$.

But since $\nabla_\omega M_{\rho,\Omega} = \alpha\, M_{\rho,\Omega}\,(\mathrm{Id} - \omega\otimes\omega)\Omega$, we get $R(M_{\rho,\Omega}) = d\dot{\alpha}\,\nabla_\omega\cdot(M_{\rho,\Omega}H_{\rho,\Omega})(\omega)$, with

$$H_{\rho,\Omega}(\omega) = (\omega\cdot\nabla_x)(\mathrm{Id} - \omega\otimes\omega)\Omega = \omega\cdot\nabla_x\rho\,\Omega - \cos\theta\,\omega\otimes\omega\nabla_x\rho\,. \quad \text{(A.108)}$$

We get with the same computations,

$$\int_{\omega\in\mathbb{S}} R(\rho M_{\rho,\Omega})h_\rho(\cos\theta)\omega\mathrm{d}\omega = -d\,\dot{\alpha}\int_{\omega\in\mathbb{S}} E_{\rho,\Omega}(\omega)M_{\rho,\Omega}\,\mathrm{d}\omega, \quad \text{(A.109)}$$

where the function $E_{\rho,\Omega}(\omega)$ is given by

$$E_{\rho,\Omega}(\omega) = h_\rho H_{\rho,\Omega}(\omega) + h'_\alpha\,\Omega\cdot H_{\rho,\Omega}(\omega)\omega \quad \text{(A.110)}$$

$$= h_\rho\,\omega\cdot\nabla_x\rho\,\Omega + (\sin^2\theta\, h'_\alpha - \cos\theta\, h_\rho)\,\omega\otimes\omega\nabla_x\rho\,. \quad \text{(A.111)}$$

Now we will use the chain rule to have all the partial derivatives with respect to t and x applied only on ρ and Ω.

We have

$$\partial_t(M_{\rho,\Omega}) + \omega \cdot \nabla_x(M_{\rho,\Omega}) = \\ M_{\rho,\Omega}\left[(\partial_t\rho + \omega\cdot\nabla_x\rho)\frac{\partial}{\partial\rho}\ln(M_{\rho,\Omega}) + \left(\frac{\partial}{\partial\Omega}\ln(M_{\rho,\Omega})\right)(\partial_t\Omega + (\omega\cdot\nabla_x)\Omega)\right]. \tag{A.112}$$

Elementary differential geometry gives the derivative of $\ln M_{\rho,\Omega}$ with respect to Ω acting on a tangent vector $\mathrm{d}\Omega$ to the sphere as follows:

$$\frac{\partial \ln M_{\rho,\Omega}}{\partial\Omega}(\mathrm{d}\Omega) = \alpha\,\omega\cdot\mathrm{d}\Omega. \tag{A.113}$$

Term in $\dfrac{\partial}{\partial\rho}$:

$$\frac{\partial \ln M_{\rho,\Omega}}{\partial\rho} = \frac{\mathrm{d}}{\mathrm{d}\rho}\ln C(\rho) + \frac{\mathrm{d}\alpha}{\mathrm{d}\rho}\,\omega\cdot\Omega \tag{A.114}$$

$$= \frac{1}{\rho} - \frac{\mathrm{d}}{\mathrm{d}\rho}\ln\left(\int_{\mathbb{S}} e^{\alpha\,\omega\cdot\Omega}\mathrm{d}\omega\right) + \dot\alpha\,\omega\cdot\Omega \tag{A.115}$$

$$= \frac{1}{\rho} - \frac{\dot\alpha\int_{\mathbb{S}}\omega\cdot\Omega e^{\alpha\,\omega\cdot\Omega}\mathrm{d}\omega}{\int_{\mathbb{S}} e^{\alpha\,\omega\cdot\Omega}\mathrm{d}\omega} + \dot\alpha\,\omega\cdot\Omega \tag{A.116}$$

$$= \frac{1}{\rho} + \dot\alpha(\cos\theta - \langle\cos\theta\rangle_{M_\alpha}), \tag{A.117}$$

Define

$$\gamma_1(\rho,\cos\theta) = \left[\frac{1}{\rho} + \dot\alpha\,(\cos\theta - \langle\cos\theta\rangle_{M_\alpha})\right]h_\rho, \tag{A.118}$$

$$\gamma_2(\rho,\cos\theta) = \alpha\,h_\rho, \tag{A.119}$$

$$\gamma_3(\rho,\cos\theta) = \kappa\,d\,\alpha\,(h_\rho + \cos\theta h'_\alpha), \tag{A.120}$$

$$\gamma_4(\rho,\cos\theta) = \widetilde\kappa\,d\,\dot\alpha\,(\sin^2\theta\,h'_\alpha - \cos\theta\,h_\rho), \tag{A.121}$$

$$\gamma_5(\rho,\cos\theta) = \widetilde\kappa\,d\,\dot\alpha\,h_\rho, \tag{A.122}$$

and we get

$$\begin{aligned} X = &\left(\int_{\omega\in\mathbb{S}}\gamma_1(\rho,\cos\theta)\,\omega\,M_{\rho,\Omega}\,\mathrm{d}\omega\right)\partial_t\rho \\ &- \left(\int_{\omega\in\mathbb{S}}\gamma_5(\rho,\cos\theta)\,\omega\,M_{\rho,\Omega}\,\mathrm{d}\omega\right)\cdot\nabla_x\rho\,\Omega \\ &- \kappa\,d\,(\nabla_x\Omega)^T\left(\int_{\omega\in\mathbb{S}}\gamma_2(\rho,\cos\theta)\,\omega\,M_{\rho,\Omega}\,\mathrm{d}\omega\right) \\ &+ \left(\int_{\omega\in\mathbb{S}}\gamma_2(\rho,\cos\theta)\,\omega\otimes\omega\,M_{\rho,\Omega}\,\mathrm{d}\omega\right)\partial_t\Omega \\ &+ \left(\int_{\omega\in\mathbb{S}}(\gamma_1-\gamma_4)(\rho,\cos\theta)\,\omega\otimes\omega\,M_{\rho,\Omega}\,\mathrm{d}\omega\right)\nabla_x\rho \\ &+ \left(\int_{\omega\in\mathbb{S}}(\gamma_2+\gamma_3)(\rho,\cos\theta)\,\omega\otimes\omega\otimes\omega\,M_{\rho,\Omega}\,\mathrm{d}\omega\right):\nabla_x\Omega. \end{aligned} \tag{A.123}$$

We denote by X_1, X_2, X_3, X_4, X_5 and X_6 these six terms. To do the computation we write ω in spherical coordinates (or polar in dimension 2).

First X_2, which is parallel to Ω, does not contribute to $(\mathrm{Id} - \Omega \otimes \Omega)\, X$.

Then, $\displaystyle\int_{\omega\in\mathbb{S}} \gamma\, M_{\rho_\Omega}\, \omega\, \mathrm{d}\omega = \langle \rho \cos\theta\, \gamma \rangle_{M_\alpha}\, \Omega$, and since $(\nabla_x \Omega)^T \Omega = (\Omega \cdot \nabla_x)\Omega$ is orthogonal to Ω, we have :

$$(\mathrm{Id} - \Omega \otimes \Omega)\, X_1 = 0, \tag{A.124}$$

$$(\mathrm{Id} - \Omega \otimes \Omega)\, X_3 = X_3 = \kappa\, d\, \rho\, \langle \cos\theta\, \gamma_2 \rangle_{M_\alpha} (\Omega \cdot \nabla_x)\Omega. \tag{A.125}$$

For X_4 and X_5, we have to compute $I_2(\gamma) = \int_{\omega\in\mathbb{S}} \omega \otimes \omega\, \gamma\, M_{\rho,\Omega}\, \mathrm{d}\omega$.

We get

$$\int_{\omega\in\mathbb{S}} \omega \otimes \omega\, \gamma\, M_{\rho,\Omega}\, \mathrm{d}\omega = \frac{\rho}{n-1} \langle \sin^2\theta\, \gamma \rangle_{M_\alpha} (\mathrm{Id} - \Omega \otimes \Omega) + \rho \langle \cos^2\theta\, \gamma \rangle_{M_\alpha} \Omega \otimes \Omega\,, \tag{A.126}$$

where n is the dimension (in dimension 3, we get a $\frac{1}{2}$ coefficient for the term in $\sin^2\theta$, coming from the mean of $\cos^2\phi$ or $\sin^2\phi$ in spherical coordinates).

Taking the orthogonal component to Ω, we only keep the first term. Knowing that $\partial_t \Omega$ is orthogonal to Ω, we finally get :

$$(\mathrm{Id} - \Omega \otimes \Omega)\, X_4 = \frac{\rho}{n-1} \langle \sin^2\theta\, \gamma_2 \rangle_{M_\alpha} \partial_t \Omega \tag{A.127}$$

$$(\mathrm{Id} - \Omega \otimes \Omega)\, X_5 = \frac{\rho}{n-1} \langle \sin^2\theta\, (\gamma_1 - \gamma_4) \rangle_{M_\alpha} (\mathrm{Id} - \Omega \otimes \Omega) \nabla_x \rho\,, \tag{A.128}$$

We have to compute the term with X_6. Same kind of computation, a little bit more complicated.

$$\begin{aligned}\int_{\omega\in\mathbb{S}} \omega \otimes \omega \otimes \omega\, \gamma\, M_{\rho,\Omega}\, \mathrm{d}\omega = \frac{\rho}{n-1} \langle \sin^2\theta \cos\theta\, \gamma \rangle_{M_\alpha} ((\mathrm{Id} - \Omega \otimes \Omega) \otimes \Omega + \\ + \Omega \otimes (\mathrm{Id} - \Omega \otimes \Omega) + [(\mathrm{Id} - \Omega \otimes \Omega) \otimes \Omega \otimes (\mathrm{Id} - \Omega \otimes \Omega)]_{:24}) \\ + \rho \langle \cos^3\theta\, \gamma \rangle_{M_\alpha}\, \Omega \otimes \Omega \otimes \Omega,\end{aligned} \tag{A.129}$$

where the index ': 24' indicates contraction of the indices 2 and 4. In other words, the tensor element $(\int_0^{2\pi} \omega \otimes \omega \otimes \omega\, \mathrm{d}\phi)_{ijk}$ equals $\pi \sin^2\theta \cos\theta$ when (i,j,k) equals any of the triples $(1,1,3)$, $(2,2,3)$, $(3,1,1)$, $(3,2,2)$, $(1,3,1)$, $(2,3,2)$, equals $2\pi \cos^3\theta$ when (i,j,k) is the triple $(3,3,3)$ and is equal to 0 otherwise (in dimension 3, and the analogous in dimension 2). In fact we want to compute $X_6 = \int_{\omega\in\mathbb{S}} (\omega \otimes \omega : \nabla_x \Omega)\, \gamma_2\, \omega M_{\rho,\Omega}\, \mathrm{d}\omega$, so, using Einstein's summation convention, the following formula follows:

$$\begin{aligned} X_6 &= \left(\int_{\omega\in\mathbb{S}} \omega \otimes \omega \otimes \omega\, \gamma\, M_{\rho,\Omega}\, \mathrm{d}\omega \right)_{ijk} \partial_{x_j} \Omega_k = \\ &= \frac{\rho}{n-1} \langle \sin^2\theta \cos\theta\, \gamma \rangle_{M_\alpha} ((\mathrm{Id} - \Omega \otimes \Omega)_{ij} \Omega_k \partial_{x_j} \Omega_k + \Omega_i (\mathrm{Id} - \Omega \otimes \Omega)_{jk} \partial_{x_j} \Omega_k + \\ &\qquad + (\mathrm{Id} - \Omega \otimes \Omega)_{ik} \Omega_j \partial_{x_j} \Omega_k) \\ &\qquad + \rho \langle \cos^3\theta\, \gamma \rangle_{M_\alpha}\, \Omega_i \Omega_j \Omega_k \partial_{x_j} \Omega_k. \end{aligned} \tag{A.130}$$

But since Ω is a unit vector, $\Omega_k \partial_{x_j} \Omega_k = \frac{1}{2}\partial_{x_j}(|\Omega|^2) = 0$ and the first and last terms in the sum vanish. The expression simplifies into:

$$\begin{aligned} X_6 = &\frac{\rho}{n-1}\langle \sin^2\theta \cos\theta\, \gamma\rangle_{M_\alpha}((\mathrm{Id} - \Omega\otimes\Omega) : (\nabla_x \Omega))\,\Omega + \\ &+ \frac{\rho}{n-1}\langle \sin^2\theta\cos\theta\,\gamma\rangle_{M_\alpha}\,(\mathrm{Id}-\Omega\otimes\Omega)((\Omega\cdot\nabla_x)\Omega). \end{aligned} \tag{A.131}$$

The first term is parallel to Ω. Besides, since Ω is a unit vector, $(\Omega\cdot\nabla)\Omega$ is normal to Ω. So, we finally get

$$(\mathrm{Id}-\Omega\otimes\Omega)X_6 = \frac{\rho}{n-1}\langle \sin^2\theta\cos\theta\,(\gamma_2+\gamma_3)\rangle_{M_\alpha}\,(\Omega\cdot\nabla_x)\Omega. \tag{A.132}$$

So we have that $(\mathrm{Id}-\Omega\otimes\Omega)X = 0$, is equivalent to

$$\begin{aligned} &\frac{\rho}{n-1}\langle \sin^2\theta\,(\gamma_1-\gamma_4)\rangle_{M_\alpha}(\mathrm{Id}-\Omega\otimes\Omega)\nabla_x\rho + \frac{\rho}{n-1}\langle\sin^2\theta\,\gamma_2\rangle_{M_\alpha}\partial_t\Omega \\ &+ \frac{\rho}{n-1}\langle \sin^2\theta\cos\theta\,(\gamma_2+\gamma_3) - (n-1)\kappa\, d\,\cos\theta\gamma_2\rangle_{M_\alpha}\,(\Omega\cdot\nabla_x)\Omega = 0. \end{aligned} \tag{A.133}$$

For any function $\gamma(\cos\theta)$ and $\sigma(\cos\theta)$, we denote by $\langle\gamma(\cos\theta)\rangle_{\sigma M_\alpha}$ the mean of γ following the "weight" $\sigma M_{\rho,\Omega}$, (which is consistant with our previous definition of $\langle\cdot\rangle_{M_\alpha}$) so we have $\langle\gamma(\cos\theta)\rangle_{\sigma M_\alpha} = \dfrac{\langle\sigma(\cos\theta)\gamma(\cos\theta)\rangle_{M_\alpha}}{\langle\sigma(\cos\theta)\rangle_{M_\alpha}}$.

We have the expression in dimension n (for $n=2$ or $n=3$) :

$$\langle\gamma(\cos\theta)\rangle_{\sigma M_\alpha} = \frac{\int_0^\pi \gamma(\cos\theta)\sigma(\cos\theta)e^{\alpha(\rho)\cos\theta}\,\sin^{n-2}\theta\,\mathrm{d}\theta}{\int_0^\pi \sigma(\cos\theta)e^{\alpha(\rho)\cos\theta}\,\sin^{n-2}\theta\,\mathrm{d}\theta}. \tag{A.134}$$

Dividing by $\frac{1}{n-1}\langle\gamma_2\sin^2\theta\rangle_{M_\alpha}$ in (A.133) we finally get :

$$\rho\,(\partial_t\Omega + c_2(\Omega\cdot\nabla)\Omega) + \lambda\,(\mathrm{Id}-\Omega\otimes\Omega)\nabla_x\rho = 0, \tag{A.135}$$

where the coefficients $\lambda(\rho)$ and $c_2(\rho)$ are given by $\lambda(\rho) = \rho\left\langle\dfrac{\gamma_1-\gamma_4}{\gamma_2}\right\rangle_{\sin^2\theta h_\rho M_\alpha}$ and by $c_2(\rho) = \left\langle\cos\theta\,\left(1+\dfrac{\gamma_3}{\gamma_2}\right) - (n-1)\kappa\, d\,\dfrac{\cos\theta}{\sin^2\theta}\right\rangle_{\sin^2\theta h_\rho M_\alpha}$.

So we have

$$\lambda = \frac{1}{\alpha} + \rho\frac{\dot\alpha}{\alpha}\left((1+\widetilde\kappa\, d)\,\langle\cos\theta\rangle_{\sin^2\theta h_\rho M_\alpha} - \widetilde\kappa\, d\left\langle\sin^2\theta\frac{h'_\alpha}{h_\rho}\right\rangle_{\sin^2\theta h_\rho M_\alpha} - \langle\cos\theta\rangle_{M_\alpha}\right), \tag{A.136}$$

$$c_2 = (1+\kappa\, d)\,\langle\cos\theta\rangle_{\sin^2\theta h_\rho M_\alpha} + \kappa\, d\left\langle\cos^2\theta\frac{h'_\alpha}{h_\rho} - (n-1)\frac{\cos\theta}{\sin^2\theta}\right\rangle_{\sin^2\theta h_\rho M_\alpha}. \tag{A.137}$$

We compute $\left\langle\gamma(\cos\theta)\dfrac{h'_\alpha}{h_\rho}\right\rangle_{\sin^2\theta h_\rho M_\alpha}$ using (A.134) and an integration by parts (we integrate the factor $\sin\theta\, h'_\alpha(\cos\theta)$, and since $\sin^{n-1}\theta\, h_\rho(\theta)$ cancels at $\theta=0$ and $\theta=\pi$, the boundary terms vanish) and we get

$$\left\langle\gamma(\cos\theta)\frac{h'_\alpha}{h_\rho}\right\rangle_{\sin^2\theta h_\rho M_\alpha} = \left\langle\frac{n-1}{\sin^2\theta}\cos\theta\gamma - \alpha\,\gamma - \gamma'\right\rangle_{\sin^2\theta h_\rho M_\alpha}. \tag{A.138}$$

Now we can simplify the expressions above (taking $\gamma(x) = x^2$ for c_2 and $\gamma(x) = 1 - x^2$ for λ).

$$\lambda = \frac{1}{\alpha} + \rho\frac{\dot\alpha}{\alpha}\left((1 - n\,\widetilde\kappa\,d)\,\langle\cos\theta\rangle_{\sin^2\theta h_\rho M_\alpha} - \langle\cos\theta\rangle_{M_\alpha} + \widetilde\kappa\,\nu\langle\sin^2\theta\rangle_{\sin^2\theta h_\rho M_\alpha}\right), \quad \text{(A.139)}$$

$$c_2 = (1 - n\,\kappa\,d)\,\langle\cos\theta\rangle_{\sin^2\theta h_\rho M_\alpha} - \kappa\,\nu\langle\cos^2\theta\rangle_{\sin^2\theta h_\rho M_\alpha}. \quad \text{(A.140)}$$

A.2 Asymptotics of the coefficients

We try to obtain an asymptotic expansion of c_1, c_2 and λ when the noise is small.

We recall the expressions (3.55), (3.59), (3.58) here :

$$c_1(\rho) = \langle\cos\theta\rangle_{M_{\alpha(\rho)}}, \quad c_2(\rho) = (1 - n\,\kappa\,d(\rho))\,\langle\cos\theta\rangle_{\widetilde M_{\alpha(\rho)}} - \kappa\,\nu(\rho)\langle\cos^2\theta\rangle_{\widetilde M_{\alpha(\rho)}}, \quad \text{(A.141)}$$

$$\lambda(\rho) = \frac{1}{\alpha(\rho)} + \rho\frac{\dot\alpha(\rho)}{\alpha(\rho)}\left((1 - n\,\widetilde\kappa\,d(\rho))\,\langle\cos\theta\rangle_{\widetilde M_{\alpha(\rho)}} - \langle\cos\theta\rangle_{M_{\alpha(\rho)}} + \widetilde\kappa\,\nu(\rho)\langle\sin^2\theta\rangle_{\widetilde M_{\alpha(\rho)}}\right). \quad \text{(A.142)}$$

with, if we are in dimension n (for $n = 2$ or $n = 3$) :

$$\langle\gamma(\cos\theta)\rangle_{M_{\alpha(\rho)}} = \frac{\int_0^\pi \gamma(\cos\theta)e^{\alpha(\rho)\cos\theta}\,\sin^{n-2}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{\alpha(\rho)\cos\theta}\,\sin^{n-2}\theta\,\mathrm{d}\theta}, \quad \text{(A.143)}$$

$$\langle\gamma(\cos\theta)\rangle_{\widetilde M_{\alpha(\rho)}} = \frac{\int_0^\pi \gamma(\cos\theta)h_\rho(\cos\theta)e^{\alpha(\rho)\cos\theta}\,\sin^{n}\theta\,\mathrm{d}\theta}{\int_0^\pi h_\rho(\cos\theta)e^{\alpha(\rho)\cos\theta}\,\sin^{n}\theta\,\mathrm{d}\theta}. \quad \text{(A.144)}$$

Since we do not have an explicit expression of h_ρ in dimension 3, the computations are not easy in that case, we do not have a lot of information on the solution of the elliptic problem (3.46).

So from now we will only study the case of dimension 2.

We only have to study the averages $\langle\cos\theta\rangle_{M_\alpha}$, $\langle\cos\theta\rangle_{\widetilde M_\alpha}$, and $\langle\cos^2\theta\rangle_{\widetilde M_\alpha}$ as $\alpha \to \infty$. This expressions only depend on α, and we can replace $h_\rho(\cos\theta)\sin\theta$ in (A.144) by ψ_α, which we recall the explicit expression (3.49) obtained in dimension 2 :

$$\psi_\alpha(\theta) = \frac{\pi}{\alpha}\frac{\int_0^\theta e^{-\alpha\cos\varphi}\mathrm{d}\varphi}{\int_0^\pi e^{-\alpha\cos\varphi}\mathrm{d}\varphi} - \frac{\theta}{\alpha}. \quad \text{(A.145)}$$

We will show the following result :

Proposition 2 *When $\alpha \to \infty$, we have the following expansions :*

$$\langle\cos\theta\rangle_{M_\alpha} = \frac{\int_0^\pi \cos\theta e^{\alpha\cos\theta}\,\mathrm{d}\theta}{\int_0^\pi e^{\alpha\cos\theta}\,\mathrm{d}\theta} = 1 - \frac{1}{2\alpha} + O\left(\frac{1}{\alpha^2}\right), \quad \text{(A.146)}$$

$$\langle\cos^k\theta\rangle_{\widetilde M_\alpha} = \frac{\int_0^\pi \cos^k\theta\,\psi_\alpha(\theta)e^{\alpha\cos\theta}\,\sin\theta\,\mathrm{d}\theta}{\int_0^\pi \psi_\alpha(\theta)e^{\alpha\cos\theta}\,\sin\theta\,\mathrm{d}\theta} = 1 - \frac{3k}{2\alpha} + O\left(\frac{1}{\alpha^2}\right). \quad \text{(A.147)}$$

Now if we suppose that $\nu(\rho)$ and $\rho\dfrac{\dot\alpha(\rho)}{\alpha(\rho)}$ are of order 1 (for instance a power law for α) , and that $d(\rho)$ is a small parameter, we can put these expansions in (A.141)-(3.58), and expand in term of the noise $d(\rho)$. We get

$$c_1(\rho) = 1 - \frac{1}{2\nu(\rho)} d(\rho) + O(d^2) \tag{A.148}$$

$$c_2(\rho) = 1 - \kappa\nu(\rho) - \left(\frac{3}{2\nu(\rho)} - \kappa\right) d(\rho) + O(d^2) \tag{A.149}$$

$$\lambda(\rho) = \frac{d(\rho)}{\nu(\rho)} \left(1 - \rho\frac{\dot\alpha(\rho)}{\alpha(\rho)}(1 - \widetilde\kappa\nu)\right) + O(d^2) \tag{A.150}$$

Proof of proposition 2.

We will use Watson's lemma, the proof of which is elementary, see [3] for examples and variants :

Lemma 2

Let p be a continuous fonction on $]0,T]$, with $T>0$. Let $I_\alpha(p) = \displaystyle\int_0^T p(t)e^{-\alpha t}\mathrm{d}t$.

Suppose that $p(t) = t^\beta \left(\displaystyle\sum_{k=0}^{N-1} a_k t^k + O(t^N)\right)$, with $\beta > -1$.

Alors $I_d(p) = \alpha^{-\beta-1}\left(\displaystyle\sum_{k=0}^{N-1} a_k \Gamma(\beta+k+1)\alpha^{-k} + O(\alpha^{-N})\right)$.

Now we do the change of variable $t = 1 - \cos\theta$ in the integrals.

We have

$$m_1(\alpha) = \int_0^\pi e^{\alpha\cos\theta}\,\mathrm{d}\theta = e^\alpha \int_0^2 \frac{1}{\sqrt{2t-t^2}} e^{-\alpha t}\,\mathrm{d}t \tag{A.151}$$

$$= \frac{e^\alpha}{\sqrt{2\alpha}}\left(\Gamma(\tfrac12) + \frac{1}{4\alpha}\Gamma(\tfrac32) + O\left(\frac{1}{\alpha^2}\right)\right), \tag{A.152}$$

and

$$m_2(\alpha) = \int_0^\pi \cos\theta e^{\alpha\cos\theta}\,\mathrm{d}\theta = e^\alpha \int_0^2 \frac{1-t}{\sqrt{2t-t^2}} e^{-\alpha t}\,\mathrm{d}t \tag{A.153}$$

$$= \frac{e^\alpha}{\sqrt{2\alpha}}\left(\Gamma(\tfrac12) - \frac{3}{4\alpha}\Gamma(\tfrac32) + O\left(\frac{1}{\alpha^2}\right)\right), \tag{A.154}$$

Since $\Gamma(\frac32) = \frac12\Gamma(\frac12)$, we get $\langle\cos\theta\rangle_{M_\alpha} = \dfrac{m_2(\alpha)}{m_1(\alpha)} = 1 - \dfrac{1}{2\alpha} + O\left(\dfrac{1}{\alpha^2}\right)$.

Let $\phi_\alpha(\theta) = \pi\dfrac{\int_0^\theta e^{\alpha\cos\varphi}\mathrm{d}\varphi}{\int_0^\pi e^{\alpha\cos\varphi}\mathrm{d}\varphi}$ for convenience.

We have $\langle\cos^k\theta\rangle_{\widetilde{M}_\alpha} = \dfrac{\int_0^\pi \cos^k\theta\,(\theta-\phi_\alpha(\theta))e^{\alpha\cos\theta}\,\sin\theta\,\mathrm{d}\theta}{\int_0^\pi(\theta-\phi_\alpha(\theta))e^{\alpha\cos\theta}\,\sin\theta\,\mathrm{d}\theta}$.

Again we compute

$$\widetilde{m_k}(\alpha) = \int_0^\pi \cos^k\theta\,(\theta - \phi_\alpha(\theta))e^{\alpha\cos\theta}\,\sin\theta\,\mathrm{d}\theta \tag{A.155}$$

$$= e^\alpha \int_0^2 (1-t)^k \cos^{-1}(1-t)e^{-\alpha t}\,\mathrm{d}t - e^\alpha \int_0^2 \phi_\alpha(\cos^{-1}(1-t))e^{-\alpha t}\,\mathrm{d}t \tag{A.156}$$

We have first that

$$|\phi_\alpha(\cos^{-1}(1-t))| = \left|\pi\frac{\int_0^\theta e^{\alpha\cos\varphi}\mathrm{d}\varphi}{\int_0^\pi e^{\alpha\cos\varphi}\mathrm{d}\varphi}\right| \tag{A.157}$$

$$\leqslant \frac{\pi\cos^{-1}(1-t)e^{-\alpha(1-t)}}{m_1(\alpha)} \tag{A.158}$$

$$\leqslant C\sqrt{\alpha}e^{\alpha t - 2\alpha} \tag{A.159}$$

So $\displaystyle\int_0^2 \phi_\alpha(\cos^{-1}(1-t))e^{-\alpha t}\,\mathrm{d}t = O\left(\frac{1}{\alpha^N}\right)$ for all $N > 0$.

Now since $\cos^{-1}(1-t) = 2\sin^{-1}\sqrt{\dfrac{t}{2}} = \sqrt{2t}\left(1 + \dfrac{t}{12} + O(t^2)\right)$, we can apply the lemma and we get $\widetilde{m_k}(\alpha) = \sqrt{2\alpha}e^\alpha\left(\Gamma(\frac{3}{2}) + (\frac{1}{12} - k)\dfrac{1}{\alpha}\Gamma(\frac{5}{2}) + O\left(\dfrac{1}{\alpha^2}\right)\right)$.

Then since $\Gamma(\frac{5}{2}) = \frac{3}{2}\Gamma(\frac{3}{2})$, and $\langle\cos^k\theta\rangle_{\widetilde{M}_\alpha} = \dfrac{\widetilde{m_k}}{\widetilde{m_0}}$, we get the result (A.147) and this ends the proof. $\diamond$

A.2.1 Tips for the general case

Here we give some tips to perform an asymptotic study of the coefficients when ν also depend on $\omega\cdot\Omega$. We are only able to do it in dimension 2.

We have to compute expressions of the form :

$$\langle\gamma(\cos\theta)\rangle_{M_{\widehat{\alpha}}} = \frac{\int_0^\pi \gamma(\cos\theta)e^{\widehat{\alpha}(\rho,\cos\theta)}\,\mathrm{d}\theta}{\int_0^\pi e^{\widehat{\alpha}(\rho,\cos\theta)}\,\mathrm{d}\theta}, \tag{A.160}$$

$$\langle\gamma(\cos\theta)\rangle_{\widetilde{M}_{\widehat{\alpha}}} = \frac{\int_0^\pi \gamma(\cos\theta)\psi_{\widehat{\alpha}}(\theta)e^{\widehat{\alpha}(\rho,\cos\theta)}\,\sin\theta\,\mathrm{d}\theta}{\int_0^\pi \psi_{\widehat{\alpha}}(\theta)e^{\widehat{\alpha}(\rho,\cos\theta)}\,\sin\theta\,\mathrm{d}\theta}, \tag{A.161}$$

where

$$\widehat{\alpha}(\rho,\mu) = \int_0^\mu \frac{\nu(\rho,\tau)}{d(\rho)}\mathrm{d}\tau \tag{A.162}$$

$$\psi_{\widehat{\alpha}}(\theta) = \psi^0_{\widehat{\alpha}}(\theta) - \frac{\psi^0_{\widehat{\alpha}}(\pi)}{\psi^\infty_{\widehat{\alpha}}(\pi)}\psi^\infty_{\widehat{\alpha}}(\theta), \tag{A.163}$$

with

$$\psi^0_{\widehat{\alpha}}(\theta) = -\int_0^\theta \left(\int_\varphi^\pi e^{\widehat{\alpha}(\rho,\cos\phi)}\sin\phi\,\mathrm{d}\phi\right)e^{-\widehat{\alpha}(\rho,\cos\varphi)}\,\mathrm{d}\varphi, \tag{A.164}$$

$$\psi^\infty_{\widehat{\alpha}}(\theta) = \int_0^\theta e^{-\widehat{\alpha}(\rho,\cos\varphi)}\,\mathrm{d}\varphi\,. \tag{A.165}$$

We want an expansion for a given ρ, when $d(\rho)$ is supposed to be a small parameter. So the dependance in ρ here is not important : we can write d for $d(\rho)$, $\nu(\cos\theta)$ for $\nu(\rho,\cos\theta)$, and $\dfrac{\sigma(\cos\theta)}{d}$ for $\widehat{\alpha}(\rho,\cos\theta)$, where $\sigma(\mu)=\displaystyle\int_0^\mu \nu(\tau)\mathrm{d}\tau$.

So we just have to expand expressions of the form $\displaystyle\int_0^\pi \gamma(\cos\theta)\exp\left(\frac{\sigma(\cos\theta)}{d}\right)\mathrm{d}\theta$ and $\displaystyle\int_0^\pi \gamma(\cos\theta)\psi(\theta)\exp\left(\frac{\sigma(\cos\theta)}{d}\right)\mathrm{d}\theta$, where $\psi(\theta)$ is given by

$$\psi(\theta)=\psi_0(\theta)-\frac{\psi_0(\pi)}{\psi_\infty(\pi)}\psi_\infty(\theta), \tag{A.166}$$

$$\psi_0(\theta)=-\int_0^\theta\left(\int_\varphi^\pi \sin\phi\exp\left(\frac{\sigma(\cos\phi)}{d}\right)\mathrm{d}\phi\right)\exp\left(-\frac{\sigma(\cos\varphi)}{d}\right)\mathrm{d}\varphi, \tag{A.167}$$

$$\psi_\infty(\theta)=\int_0^\theta \exp\left(-\frac{\sigma(\cos\varphi)}{d}\right)\mathrm{d}\varphi\,. \tag{A.168}$$

The first step consists in an expansion of the form $\psi(\theta)=\displaystyle\sum_{i=1}^{N-1} g_i(\cos\theta)d^i+R_{N,d}(\theta)$, where $R_{N,d}$ is sufficiently small, and the functions g_i do not depend on d. After that we will only have to expand terms of the form $\displaystyle\int_0^\pi \gamma(\cos\theta)\exp\left(\frac{\sigma(\cos\theta)}{d}\right)\mathrm{d}\theta$.

We integrate by parts N times the term under the integral in ϕ dans (A.167) :

$$\begin{aligned}\int_\varphi^\pi \sin\phi\exp\left(\frac{\sigma(\cos\phi)}{d}\right)\mathrm{d}\phi=\sum_{k=1}^{N}\left[\frac{-f_k(\cos\phi)\exp\left(\frac{\sigma(\cos\phi)}{d}\right)}{\nu(\cos\phi)}\right]_\varphi^\pi d^k+\\ +d^N\int_\varphi^\pi \sin\phi f_{N+1}(\cos\phi)\exp\left(\frac{\sigma(\cos\phi)}{d}\right)\mathrm{d}\phi,\end{aligned} \tag{A.169}$$

where the functions f_n are given by the following induction relation :

$$f_1(x)=1 \tag{A.170}$$

$$f_{n+1}(x)=-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f_n(x)}{\nu(x)}\right). \tag{A.171}$$

So if we define

$$\begin{aligned}\tau_N(\theta)=&-\int_0^\theta\sum_{k=1}^{N} d^k\,\frac{f_k(\cos\varphi)}{\nu(\cos\varphi)}d\varphi\\ &-d^N\int_0^\theta\int_\varphi^\pi \sin\phi\, f_{N+1}(\cos\phi)\exp\left(\frac{\sigma(\cos\phi)-\sigma(\cos\varphi)}{d}\right)\mathrm{d}\phi\,\mathrm{d}\varphi,\end{aligned} \tag{A.172}$$

we get :

$$\psi(\theta)=\tau_N(\theta)-\frac{\tau_N(\pi)}{\psi_\infty(\pi)}\psi_\infty(\theta). \tag{A.173}$$

And if we define

$$g_i(\cos\theta) = -\int_0^\theta \frac{f_i(\cos\varphi)}{\nu(\cos\varphi)}\mathrm{d}\varphi, \tag{A.174}$$

$$R_{N,d}(\theta) = \psi(\theta) - \sum_{i=1}^{N-1} g_i(\cos\theta)d^i \tag{A.175}$$

we obtain with some easy estimations

$$|R_{N,d}(\theta)| \leqslant C\left(|\theta|d^N + \exp\left(-\frac{\sigma(\cos\theta)}{d}\right)\right). \tag{A.176}$$

Now it remains to expand integrals of the form

$$S_d(r) = \int_0^\pi r(\cos\theta)\exp\left(\frac{\sigma(\cos\theta)}{d}\right)d\theta. \tag{A.177}$$

We do a change of variable, setting $t = \sigma(1) - \sigma(\cos\theta)$, and $a(t) = \sigma^{-1}(\sigma(1) - t)$, where σ^{-1} is the inverse function of σ (which is increasing since $\nu > 0$, actually we have $a(t) = \cos\theta$). We get :

$$S_d(r) = \exp\left(\frac{\sigma(1)}{d}\right)\int_0^{\sigma(1)-\sigma(-1)} \frac{r(a(t))e^{-\frac{t}{d}}}{\nu(a(t))\sqrt{1-a(t)^2}}dt. \tag{A.178}$$

To bypass problems with the upper bound of the integral, we set $T = \sigma(1)$ and restrict (since $\sigma(-1) < 0$) the integral to $]0, T]$. In any case, if $\theta \mapsto r(\cos\theta)$ is integrable, what remains is $O(1)$, and so for all N, it is $O(d^N e^{\frac{T}{d}})$. So we have

$$S_d(r) = e^{\frac{T}{d}}\int_0^T \frac{r(a(t))e^{-\frac{t}{d}}}{\nu(a(t))\sqrt{1-a(t)^2}}dt + O(d^N e^{\frac{T}{d}}),\ \forall N. \tag{A.179}$$

The function ν being positive, a has the same regularity as σ, and we can compute the derivatives by induction. We have $a'(t) = -\dfrac{1}{\nu(a(t))}$, and we realize then that $a^{(n)}$ nearly satisfies the same induction relation as f_n. We actually get

$$a^{(n)}(t) = -\frac{f_n(a(t))}{\nu(a(t))}, \tag{A.180}$$

which gives us the Taylor expansion of a at 0 up to order N (we have $a(0) = 1$).

$$a(t) = 1 - \sum_{k=1}^{N} \frac{f_n(1)}{\nu(1)n!}t^N + O(t^{N+1}). \tag{A.181}$$

So we can now have an expansion of $\dfrac{r(a(t))}{\nu(a(t))\sqrt{1-a(t)^2}}$ at 0 and we can use lemma 2 (Watson's lemma) in (A.179) to perform the last computations.

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