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► **To cite this version:**

Jean-Yves Brou. Asymptotically efficient estimators of the drift function in ergodic diffusion processes. 2008. <hal-00437558>

**HAL Id: hal-00437558**

**<https://hal.science/hal-00437558v1>**

Preprint submitted on 3 Dec 2009

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# Asymptotically efficient estimators of the drift function in ergodic diffusion processes

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## Abstract

This paper deals with the pointwise estimation of the drift function of an ergodic diffusion using the absolute error loss. The optimal convergence rate and the sharp asymptotic lower bound are found for the minimax risk. An asymptotically efficient kernel estimator is constructed.

*Key words:* Asymptotic efficiency; Drift; Ergodic diffusion; Minimax; Nonparametric estimation

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## 1 Introduction

This paper is devoted to the problem of estimating the drift coefficient of an ergodic diffusion, solution of the stochastic differential equation

$$dX_t = S(X_t)dt + dB_t, \quad 0 \leq t \leq T \quad (1)$$

where  $(B_t)_{t \geq 0}$  is a scalar standard Brownian motion. Assuming that we observe the continuous data  $(X_t, 0 \leq t \leq T)$  and knowing the smoothness of  $S$ , we want to estimate the function  $S$  at a fixed point  $x_0 \in \mathbb{R}$ .

Model (1) finds applications in numerous fields, namely financial mathematics, econometrics, stochastic control, filtering and others (see for example Aït-Sahalia (2002), Jiang and Knight (1997) and Liptser and Shiryaev (1978)). There is a lot of papers studying the problem of nonparametric estimation of the drift of a diffusion. For our model (1) Banon (1978) proved the consistency whereas Pham (1981) considered the convergence rate of kernel estimators. The reader is referred for instance to Pinsker (1980), Efröimovich and Pinsker (1984), Tsybakov (1997) and Spokoiny (2000) in which time-continuous signal estimation problems are handled.

Our current purpose is to estimate the drift function at a single point using the absolute error loss to quantify the performance of an estimator through its corresponding risk. We are first interested in obtaining the exact asymptotic behavior of the minimax risk and particularly in finding an asymptotic lower bound of it. Secondly we aim at constructing an estimator of the drift coefficient for which the risk is asymptotically bounded from above by the same constant as for the minimax risk. Such an estimator will be called asymptotically efficient (see Ibragimov and Has'minskii (1981)).

The problem of sharp estimation of the drift function in diffusion models has already been treated for some Sobolev classes: Dalalyan and Kutoyants (2002) with known smoothness then Dalalyan (2005) with an unknown one proposed asymptotically efficient estimators of the drift coefficient in model (1) for an  $L^2$ -type risk.

The drift estimation in some Hölder classes was handled as well. Galtchouk and Pergamenshchikov (2004) achieved the optimal convergence rate of the minimax risk for the  $L^2([a, b], dx)$ -loss function as the regularity of the drift function is known but as it remains unknown too. The optimal convergence rate of the minimax risk is also obtained for the pointwise estimation of the drift function with unknown smoothness and for the absolute error loss in Galtchouk and Pergamenshchikov (2001). For any positive power of the absolute error loss and the same Hölder class, the sharp constant for the local minimax risk is given in Galtchouk and Pergamenshchikov (2005) as well as an asymptotically efficient estimator of the drift coefficient with known regularity.

We consider here the pointwise estimation of the drift function belonging to

a Hölder class with known smoothness using the absolute error loss. For this problem Galtchouk and Pergamenshchikov (2006) gave the sharp asymptotic lower bound for the local minimax risk and an asymptotically efficient kernel estimator. More precisely they assumed that the drift function belonged to a neighborhood centred on a Lipschitz function. The neighborhood consists in the centre plus a function satisfying a weak Hölder condition (involving a weak Hölder constant) and having a small norm. The asymptotic results were given with the time of observations tending to infinity, the weak Hölder constant and the diameter of the neighborhood to zero. We propose to find the sharp asymptotic lower bound for the minimax risk and an asymptotically efficient estimator without making the neighborhood tend to its centre but by keeping its diameter constant.

This paper takes the following organization. In the next section we describe the problem and define explicitly the function class, the neighborhoods and the risk of an estimator. The sequential procedure and the main results are given in section 3, their proofs in section 4.

## 2 Statement of the problem

In model (1) we are interested in the estimation of the unknown function  $S$  at a fixed point  $x_0 \in \mathbb{R}$  assuming that  $S$  lies in a neighborhood of a function  $S_0$  belonging to

$$\Sigma_{L,M} := \left\{ f : |f(0)| \leq L, -L \leq \frac{f(x) - f(y)}{x - y} \leq -M, \forall x, y \in \mathbb{R} \right\},$$

with  $0 < M < L$ .

As mentioned in the introduction we need to construct neighborhoods of a function  $S_0$  belonging to  $\Sigma_{L,M}$  in order to define the risk of an estimator of  $S(x_0)$  with  $S$  lying in this vicinity. Let  $S_0 \in \Sigma_{L,M}$  and set

$$\mathcal{U}_{\delta,\beta}(S_0) = \left\{ S : S = S_0 + D, D \in \mathcal{H}_{x_0}^w(\delta, \beta) \right\}$$

where

$$\begin{aligned} \mathcal{H}_{x_0}^w(\delta, \beta) = \{ D \text{ differentiable} : \sup_{x \in \mathbb{R}} (|D(x)| + |\dot{D}(x)|) \leq B; \\ \forall h > 0, \left| \int_{-1}^1 (D(x_0 + zh) - D(x_0)) dz \right| \leq \delta h^\beta \}, \end{aligned}$$

with  $\beta \in ]1; 2[$ ,  $0 < \delta < 1$  and  $0 < B < M$ .

**Remark 2.1** *If  $S_0 \in \Sigma_{L,M}$  one has  $\mathcal{U}_{\delta,\beta}(S_0) \subset \Sigma_{L+B, M-B}$ , so that for all  $S \in \mathcal{U}_{\delta,\beta}(S_0)$  the process  $(X_t)_{t \geq 0}$  is ergodic and there exists an ergodic density*

(see Gihman and Skorohod (1968)) given by

$$q_S(x) = \frac{\exp\left(2 \int_0^x S(z) dz\right)}{\int_{-\infty}^{+\infty} \exp\left(2 \int_0^y S(z) dz\right) dy}.$$

Recall that the ergodicity of a process guarantees that it returns to any neighborhood of  $x_0$  infinitely many times.

The risk of an estimator  $\tilde{S}_T(x_0)$  of  $S(x_0)$  is defined by

$$\mathcal{R}_{\delta, \beta}(\tilde{S}_T, S_0) = \sup_{S \in \mathcal{U}_{\delta, \beta}(S_0)} \sqrt{2q_S(x_0)} \varphi_T \mathbb{E}_S |\tilde{S}_T(x_0) - S(x_0)|, \quad \varphi_T = T^{\beta/(2\beta+1)}.$$

### 3 Main results

The asymptotic lower bound of the minimax risk is given in the following theorem.

**Theorem 3.1** *If  $S_0 \in \Sigma_{L, M}$  we have for any  $\delta \in ]0; 1[$*

$$\liminf_{T \rightarrow \infty} \inf_{\tilde{S}} \mathcal{R}_{\delta, \beta}(\tilde{S}_T, S_0) \geq \mathbb{E}|\xi|, \quad \xi \sim \mathcal{N}(0, 1),$$

where the infimum is taken over all estimators of  $S(x_0)$ .

In order to exhibit an asymptotically efficient estimator of  $S(x_0)$  we begin with estimating the ergodic density at the point  $x_0$  through the observations  $\{X_t, t \leq t_0\}$ , where  $t_0 = T^{2\gamma}$ ,  $\gamma_* < \gamma < 1/2$  and  $\gamma_* = \frac{\alpha}{2\beta+1}$ . Let

$$\hat{q}_T(x_0) = \frac{1}{2t_0 l} \int_0^{t_0} Q\left(\frac{X_t - x_0}{l}\right) dt,$$

with  $Q = \mathbb{I}_{[-1; 1]}$  and  $l = l_T = o(1/\sqrt{T})$  as  $T \rightarrow \infty$ .

Then for  $H > 0$  we define the sequential procedure  $(\tau_H, S_T^*(x_0))$  as

$$\begin{aligned} \tau_H &= \inf\{t \geq t_0 : \int_{t_0}^t Q\left(\frac{X_t - x_0}{h}\right) dt \geq H\}, \\ S_T^*(x_0) &= \frac{1}{H} \int_{t_0}^{\tau_H} Q\left(\frac{X_t - x_0}{h}\right) dX_t \mathbb{I}_{\{\tau_H \leq T\}}. \end{aligned} \quad (2)$$

We choose the bandwidth  $h = h_T = T^{-1/(2\beta+1)}$  and the level  $H = H_T = (T - t_0)(2\tilde{q}_T(x_0) - \varepsilon_T)h$ , where  $\tilde{q}_T(x_0) = \max(\hat{q}_T(x_0), \nu_T^{-1/2})$ ,  $\varepsilon_T = 1/(\nu_T T^{\gamma_*})$  and  $\nu_T = \ln T$ .

The following lemmas proved in Galtchouk and Pergamenshchikov (2006) give some properties of  $\hat{q}_T(x_0)$ ,  $\tilde{q}_T(x_0)$  and  $\tau_H$ .

**Lemma 3.2** *There exist two constants  $\kappa > 0$  and  $T_* > 0$  such that for all  $T \geq T_*$  and all  $\lambda > 1/T$ , we have*

$$\sup_{S \in \Sigma_{L+B, M-B}} \mathbb{P}_S (|\hat{q}_T(x_0) - q_S(x_0)| > \lambda) \leq 2e^{-\kappa\lambda^2 t_0}.$$

**Lemma 3.3** *There exists a constant  $T_* > 0$  such that*

$$\mu^* = \sup_{T \geq T_*} \sup_{S \in \Sigma_{L+B, M-B}} \sqrt{t_0} \mathbb{E}_S |\tilde{q}_T(x_0) - q_S(x_0)| < \infty.$$

**Lemma 3.4** *One has*

$$\limsup_{T \rightarrow \infty} \sup_{S \in \Sigma_{L+B, M-B}} \mathbb{E}_S \left| \frac{1}{\tilde{q}_T(x_0)} - \frac{1}{q_S(x_0)} \right| = 0.$$

**Lemma 3.5** *For any  $p > 0$ ,*

$$\lim_{T \rightarrow \infty} T^p \sup_{S \in \mathcal{U}_{\delta, \beta}(S_0)} \mathbb{P}_S(\tau_H > T) = 0.$$

The centre of the considered neighborhood needs to verify an additional condition described by the following definition.

**Definition 3.6** *A function  $f$  satisfies the "zero-constant" Hölder condition with an exponent  $\iota > 0$  at the point  $x_0$  if*

$$\lim_{y \rightarrow x_0} \frac{f(y) - f(x_0)}{|y - x_0|^\iota} = 0.$$

An example of such a function can be found in Galtchouk and Pergamenshchikov (2006).

We are now able to give the upper bound of the risk for the estimator (2).

**Theorem 3.7** *Let  $S_0 \in \Sigma_{L, M}$ ,  $\beta = 1 + \alpha$ ,  $\alpha \in ]0; 1[$  and assume that  $\dot{S}_0$  satisfies the "zero-constant" Hölder condition with exponent  $\alpha > 0$  at the point  $x_0$ . Then one has*

$$\limsup_{\delta \rightarrow 0} \limsup_{T \rightarrow \infty} \mathcal{R}_{\delta, \beta}(S_T^*(x_0), S_0) \leq \mathbb{E}|\xi|, \quad \xi \sim \mathcal{N}(0, 1).$$

## 4 Proofs

### 4.1 Proof of theorem 3.1

For  $u > 0$ , denote  $S_u(x) = S_0(x) + uD_\nu(x)$ , where

$$\begin{aligned} D_\nu(x) &= \varphi_T^{-1} V_\nu \left( \frac{x - x_0}{h} \right), \\ V_\nu(x) &= \nu^{-1} \int_{-\infty}^{\infty} \tilde{Q}_\nu(u) g \left( \frac{u - x}{\nu} \right) du, \\ \tilde{Q}_\nu(u) &= \mathbb{I}_{\{|u| \leq 1 - 2\nu\}} + 2\mathbb{I}_{\{1 - 2\nu \leq |u| \leq 1 - \nu\}}, \\ g(z) &= \begin{cases} c \exp(-(1 - z^2)^{-1}), & |z| \leq 1; \\ 0, & |z| > 1, \end{cases} \end{aligned}$$

with  $0 < \nu < 1/4$ ; the normalizing constant  $c > 0$  is such that  $\int_{-1}^1 g(z) dz = 1$ . Now fix  $b > 0$  and  $\delta > 0$ . We can easily see that there exists  $T_{b,\nu} > 0$  such that for any  $|u| \leq b$  and  $T \geq T_{b,\nu}$  one has  $S_u \in \mathcal{U}_{\delta,\beta}(S_0)$ .

Furthermore we may write for all  $T \geq T_{b,\nu}$

$$\begin{aligned} \mathcal{R}_{\delta,\beta}(\tilde{S}_T, S_0) &\geq \sup_{|u| \leq b} \varphi_T \sqrt{2q_{S_u}(x_0)} \mathbb{E}_{S_u} |\tilde{S}_T(x_0) - S_u(x_0)| \\ &\geq \frac{1}{2b} \int_{-b}^b \sqrt{2q_{S_u}(x_0)} \mathbb{E}_{S_u} \Psi_{a,T}(\tilde{S}_T, S_u) du \\ &=: I_T(a, b), \end{aligned}$$

with  $\Psi_{a,T}(\tilde{S}_T, S) = v_a \left( \varphi_T(\tilde{S}_T(x_0) - S(x_0)) \right)$  and  $v_a(x) = a \wedge |x|$ ,  $a > 0$ .

Denoting  $\mathbb{P}_S$  the distribution of the process  $(X_t)$  in  $C([0; T])$  as the drift function is  $S$ , we have thanks to Galtchouk and Pergamenshchikov (2006, lemma 4.2)

$$\rho_T(u) := \frac{d\mathbb{P}_{S_u}}{d\mathbb{P}_{S_0}} = \exp\left\{u\Delta_T - \frac{1}{2}u^2\sigma_\nu^2 + r_T(u)\right\}, \quad \forall u > 0,$$

where  $\Delta_T = \int_0^T D_\nu(X_t) dB_t$  and  $\sigma_\nu^2 = q_{S_0}(x_0) \int_{-1}^1 V_\nu^2(z) dz$ .

Moreover

$$\Delta_T \xrightarrow[T \rightarrow \infty]{\mathcal{L}_{\mathbb{P}_{S_0}}} \xi_\nu, \quad \xi_\nu \sim \mathcal{N}(0, \sigma_\nu^2) \quad \text{et} \quad r_T(u) \xrightarrow[T \rightarrow \infty]{\mathbb{P}_{S_0}} 0, \quad (3)$$

and we can assume that  $\xi_\nu$  and  $\Delta_T$  are independent.

Then write

$$\begin{aligned} I_T(a, b) &= \frac{1}{2b} \int_{-b}^b \sqrt{2q_{S_u}(x_0)} \mathbb{E}_{S_0} \Psi_{a,T}(\tilde{S}_T, S_u) \rho_T(u) du \\ &\geq \frac{1}{2b} \int_{-b}^b \sqrt{2q_{S_u}(x_0)} \mathbb{E}_{S_0} \mathbb{I}_{B_d} \Psi_{a,T}(\tilde{S}_T, S_u) \rho_T^0(u) du + \delta_T^0(a, b) \\ &=: J_T(a, b) + \delta_T^0(a, b), \end{aligned}$$

where  $B_d = \{|\Delta_T| \leq d\}$ ,  $d = \sigma_\nu^2(b - \sqrt{b})$ ,  $\rho_T^0(u) = \exp(u\Delta_T - u^2\sigma_\nu^2/2)$  and

$$\delta_T^0(a, b) = \frac{1}{2b} \int_{-b}^b \sqrt{2q_{S_u}(x_0)} \mathbb{E}_{S_0} \mathbb{I}_{B_d} \Psi_{a,T}(\tilde{S}_T, S_u) (\rho_T(u) - \rho_T^0(u)) du.$$

The family  $\{\rho_T(u), T > 0\}$  is uniformly integrable. Actually for any  $T > 0$ , we have immediately  $\mathbb{E}_{S_0} \rho_T(u) = 1$ ,  $\mathbb{E} \eta(u) = 1$  and using (3) we can show that

$$\rho_T(u) \xrightarrow[T \rightarrow \infty]{\mathcal{L}} \eta(u) = \exp(u\xi_\nu - u^2\sigma_\nu^2/2), \quad \xi_\nu \sim \mathcal{N}(0, \sigma_\nu^2).$$

The uniform integrability criteria from Ibragimov and Has'minskii (1981, p. 364) allows us to conclude.

Since  $\rho_T^0(u)$  is bounded on  $B_d$  and  $\Psi_{a,T}(\tilde{S}_T, S_u)$  is bounded from above by  $a$ , the family  $\{\mathbb{I}_{B_d} \Psi_{a,T}(\tilde{S}_T, S_u) (\rho_T(u) - \rho_T^0(u)), T > 0\}$  is uniformly integrable too.

Rewrite

$$\rho_T(u) - \rho_T^0(u) = \exp\left(u\Delta_T - \frac{1}{2}u^2\sigma_\nu^2 + r_T(u)\right) - \exp\left(u\xi_\nu - u^2\sigma_\nu^2/2\right).$$

On the one hand we can prove via (3) that  $\exp(r_T(u)) \xrightarrow[T \rightarrow \infty]{\mathbb{P}_{S_0}} 1$ . On the other hand using in particular the independence between  $\Delta_T$  and  $\xi_\nu$ , we can show that  $\exp(u(\xi_\nu - \Delta_T)) \xrightarrow[T \rightarrow \infty]{\mathbb{P}_{S_0}} 1$ .

Then we get from the preceding results

$$\left(\rho_T(u) - \rho_T^0(u)\right) \Psi_{a,T}(\tilde{S}_T, S_u) \mathbb{I}_{B_d} \xrightarrow[T \rightarrow \infty]{\mathbb{P}_{S_0}} 0.$$

Taking into account the uniform integrability of the corresponding random variables family, we obtain for all  $|u| \leq b$ ,

$$\mathbb{E}_{S_0} \left| \left(\rho_T(u) - \rho_T^0(u)\right) \Psi_{a,T}(\tilde{S}_T, S_u) \mathbb{I}_{B_d} \right| \xrightarrow[T \rightarrow \infty]{} 0.$$



On the event  $B_d$ , one has  $\rho_T(u) \leq \exp(|u|d)$  and  $\rho_T^0(u) \leq \exp(|u|d - u^2\sigma_\nu^2/2)$ . Hence  $\mathbb{E}_{S_0}(\rho_T(u) - \rho_T^0(u))\Psi_{a,T}(\tilde{S}_T, S_u)\mathbb{I}_{B_d} \leq 2a \exp(|u|d)$ , which is an integrable function on  $[-b; b]$ . Then bounded convergence yields

$$\delta_T^0(a, b) \xrightarrow{T \rightarrow \infty} 0. \quad (4)$$

Now we consider the quantity

$$\frac{1}{2b} \int_{-b}^b \mathbb{E}_{S_0} \mathbb{I}_{B_d} \Psi_{a,T}(\tilde{S}_T, S_u) \left( \sqrt{2q_{S_u}(x_0)} - \sqrt{2q_{S_0}(x_0)} \right) \rho_T^0(u) du.$$

Recall that  $S_u(z) = S_0(z) + uD_\nu(z)$ ,  $S_u(x_0) = S_0(x_0) + u\varphi_T^{-1}$  and specify that  $\|D_\nu\|_\infty \leq 3\varphi_T^{-1}$ . We set

$$c_T(y) := \exp\left(2 \int_0^y u D_\nu(z) dz\right),$$

and rewrite

$$q_{S_u}(x_0) = \frac{\exp\left(2 \int_0^{x_0} S_0(z) dz\right) c_T(x_0)}{\int_{-\infty}^{\infty} \exp\left(2 \int_0^y S_0(z) dz\right) c_T(y) dy}.$$

By bounded convergence one gets

$$\exp\left(2 \int_0^{x_0} u D_\nu(z) dz\right) \xrightarrow{T \rightarrow \infty} 1 \quad \text{and} \quad c_T(y) \xrightarrow{T \rightarrow \infty} 1, \quad \forall y \in \mathbb{R}.$$

Note that if  $y > 0$ ,

$$\exp(-Ly^2 + 2Ly) \leq \exp\left(2 \int_0^y S_0(z) dz\right) \leq \exp(-My^2 - 2Ly);$$

and if  $y < 0$ ,

$$\exp(-Ly^2 - 2Ly) \leq \exp\left(2 \int_0^y S_0(z) dz\right) \leq \exp(-My^2 + 2Ly).$$

Moreover for all  $y \in \mathbb{R}$ , we have  $|c_T(y)| \leq \exp(6|u||y|)$ , so that the function  $y \mapsto \exp\left(2 \int_0^y S_0(z) dz\right) c_T(y)$  is bounded from above by an integrable function on  $\mathbb{R}$  which does not depend on  $T$ . We conclude by bounded convergence that

$$q_{S_u}(x_0) \xrightarrow{T \rightarrow \infty} q_{S_0}(x_0), \quad (5)$$

and also that

$$\left| \frac{1}{2b} \int_{-b}^b \mathbb{E}_{S_0} \mathbb{I}_{B_d} \Psi_{a,T}(\tilde{S}_T, S_u) \left( \sqrt{2q_{S_u}(x_0)} - \sqrt{2q_{S_0}(x_0)} \right) \rho_T^0(u) du \right| \xrightarrow{T \rightarrow \infty} 0.$$

Finally we have proved the following equality

$$\liminf_{T \rightarrow \infty} \frac{1}{2b} \int_{-b}^b \mathbb{E}_{S_0} \mathbb{I}_{B_d} \Psi_{a,T}(\tilde{S}_T, S_u) \sqrt{2q_{S_0}(x_0)} \rho_T^0(u) du = \liminf_{T \rightarrow \infty} J_T(a, b).$$

At this point rewrite

$$\rho_T^0(u) = \zeta_T \exp(-\sigma_\nu^2(u - \tilde{\Delta}_T)/2), \quad \zeta_T = \exp(\tilde{\Delta}_T^2/2\sigma_\nu^2), \quad \tilde{\Delta}_T = \Delta_T/\sigma_\nu^2$$

and put  $g_T = \varphi_T(\tilde{S}_T(x_0) - S_0(x_0))$ ,  $\tilde{g}_T = g_T - \tilde{\Delta}_T$ . Then one successively has

$$\begin{aligned} & \frac{1}{2b} \int_{-b}^b \mathbb{E}_{S_0} \mathbb{I}_{B_d} \Psi_{a,T}(\tilde{S}_T, S_u) \sqrt{2q_{S_0}(x_0)} \rho_T^0(u) du \\ &= \mathbb{E}_{S_0} \mathbb{I}_{B_d} \zeta_T \frac{1}{2b} \int_{-b}^b v_a(u - g_T) \exp\left(-\sigma_\nu^2(u - \tilde{\Delta}_T)^2/2\right) \sqrt{2q_{S_0}(x_0)} du \\ &= \mathbb{E}_{S_0} \mathbb{I}_{B_d} \zeta_T \frac{1}{2b} \int_{-b-\tilde{\Delta}_T}^{b-\tilde{\Delta}_T} v_a(u - \tilde{g}_T) \exp\left(-\sigma_\nu^2 u^2/2\right) \sqrt{2q_{S_0}(x_0)} du \\ &\geq \mathbb{E}_{S_0} \mathbb{I}_{B_d} \zeta_T \frac{1}{2b} \int_{-\sqrt{b}}^{\sqrt{b}} v_a(u - \tilde{g}_T) \exp\left(-\sigma_\nu^2 u^2/2\right) \sqrt{2q_{S_0}(x_0)} du \\ &\geq \mathbb{E}_{S_0} \mathbb{I}_{B_d} \zeta_T \frac{1}{2b} \int_{-\sqrt{b}}^{\sqrt{b}} v_a(u) \exp\left(-\sigma_\nu^2 u^2/2\right) \sqrt{2q_{S_0}(x_0)} du, \end{aligned}$$

the second inequality arising from Anderson's lemma (see Ibragimov and Has'minskii, 1981, Lemma 10.2, p.157).

Thus

$$\begin{aligned} & \frac{1}{2b} \int_{-b}^b \mathbb{E}_{S_0} \mathbb{I}_{B_d} \Psi_{a,T}(\tilde{S}_T, S_u) \sqrt{2q_{S_0}(x_0)} \rho_T^0(u) du \\ &\geq \mathbb{E}_{S_0} \mathbb{I}_{B_d} \zeta_T \frac{1}{2b} \int_{-\sqrt{b}}^{\sqrt{b}} |u| \exp\left(-\sigma_\nu^2 u^2/2\right) \sqrt{2q_{S_0}(x_0)} du + \delta_T^1(a, b), \end{aligned}$$

where

$$\delta_T^1(a, b) = \mathbb{E}_{S_0} \mathbb{I}_{B_d} \zeta_T \frac{1}{2b} \int_{-\sqrt{b}}^{\sqrt{b}} (v_a(u) - |u|) \exp\left(-\sigma_\nu^2 u^2/2\right) \sqrt{2q_{S_0}(x_0)} du.$$

Noticing that

$$\lim_{T \rightarrow \infty} \mathbb{E}_{S_0} \mathbb{I}_{B_d} \zeta_T = 2\sigma_\nu(b - \sqrt{b})/\sqrt{2\pi},$$

one obtains for all  $b > 0$ ,

$$\lim_{a \rightarrow \infty} \liminf_{T \rightarrow \infty} \delta_T^1(a, b) = 0,$$

and

$$\liminf_{a \rightarrow \infty} \liminf_{T \rightarrow \infty} I_T(a, b) \geq \frac{b - \sqrt{b}}{b} \frac{\sqrt{2q_{S_0}(x_0)}\sigma_\nu}{\sqrt{2\pi}} \int_{-\sqrt{b}}^{\sqrt{b}} |u| \exp(-u^2\sigma_\nu^2/2) du.$$

Remarking here that  $\sigma_\nu \rightarrow \sqrt{2q_{S_0}(x_0)}$  as  $\nu \rightarrow 0$  and limiting  $b \rightarrow \infty$  before  $\nu \rightarrow 0$  in the last inequality yield

$$\liminf_{T \rightarrow \infty} \mathcal{R}_{\delta, \beta}(\tilde{S}_T, S_0) \geq \mathbb{E}|\xi|, \quad \xi \sim \mathcal{N}(0, 1).$$

Eventually we have shown that

$$\liminf_{T \rightarrow \infty} \inf_{\tilde{S}} \mathcal{R}_{\delta, \beta}(\tilde{S}_T, S_0) \geq \mathbb{E}|\xi|, \quad \xi \sim \mathcal{N}(0, 1).$$

□

#### 4.2 Proof of theorem 3.7

Let  $S \in \mathcal{U}_{\delta, \beta}(S_0)$  and parse the error of estimation as

$$S_T^*(x_0) - S(x_0) = \left( B_T - G_T + \frac{\xi_T}{\sqrt{H_T}} \right) \mathbb{I}_{\{\tau_H \leq T\}} - S(x_0) \mathbb{I}_{\{\tau_H > T\}},$$

where

$$\begin{aligned} B_T &= \frac{1}{H_T} \int_{t_0}^T Q\left(\frac{X_t - x_0}{h}\right) (S(X_t) - S(x_0)) dt, \\ G_T &= \frac{1}{H_T} \int_{\tau_H}^T Q\left(\frac{X_t - x_0}{h}\right) (S(X_t) - S(x_0)) dt, \\ \xi_T &= \frac{1}{\sqrt{H_T}} \int_{t_0}^T Q\left(\frac{X_t - x_0}{h}\right) dB_t. \end{aligned}$$

First we want to show that

$$\lim_{\delta \rightarrow 0} \limsup_{T \rightarrow \infty} \sup_{S \in \mathcal{U}_{\delta, \beta}(S_0)} \mathbb{E}_S \sqrt{2q_S(x_0) \varphi_T} |B_T| \mathbb{I}_{\{\tau_H \leq T\}} = 0. \quad (6)$$

We begin with writing

$$B_T = \frac{T}{(2\tilde{q}_T(x_0) - \varepsilon_T)(T - t_0)} \left( \frac{T - t_0}{T} m(f_h) + \frac{1}{\sqrt{T}} \Delta_{t_0, T}(f_h) \right), \quad (7)$$

where  $f_h(y) = \phi_h(y)(S(y) - S(x_0))$ ,  $\phi_h(y) = \frac{1}{h} Q\left(\frac{y - x_0}{h}\right)$ ,  $m(f) = \int f(y) q_S(y) dy$  and  $\Delta_{t_0, T}(f) = \frac{1}{\sqrt{T}} \int_{t_0}^T (f(X_t) - m(f)) dt$ .

We can rewrite the term  $m(f_h)$  as

$$\begin{aligned}
m(f_h) &= \frac{1}{h} \int_{-\infty}^{\infty} Q\left(\frac{y-x_0}{h}\right) (S(y) - S(x_0)) q_S(y) dy \\
&= \int_{-1}^1 (S(x_0 + hz) - S(x_0)) q_S(x_0 + hz) dz \\
&= \int_{-1}^1 (S(x_0 + hz) - S(x_0)) (q_S(x_0 + hz) - q_S(x_0)) dz \\
&\quad + q_S(x_0) \int_{-1}^1 (S(x_0 + hz) - S(x_0)) dz \\
&=: m_1(h) + q_S(x_0) m_0(h).
\end{aligned}$$

Put  $r(y) := S_0(y) - S_0(x_0) - \dot{S}_0(x_0)(y - x_0)$ . Since  $\int_{-1}^1 \dot{S}_0(x_0) z h dz = 0$  and  $S \in \mathcal{U}_{\delta, \beta}(S_0)$ , we get

$$\begin{aligned}
|m_0(h)| &= \left| \int_{-1}^1 (S(x_0 + zh) - S(x_0)) dz \right| \\
&= \left| \int_{-1}^1 r(x_0 + zh) dz + \int_{-1}^1 (D(x_0 + zh) - D(x_0)) dz \right| \\
&\leq 2 \sup_{|u| \leq h} \frac{|r(x_0 + u)|}{|u|^\beta} h^\beta + \delta h^\beta =: (2r^*(h) + \delta) h^\beta.
\end{aligned}$$

Using the "zero-constant" Hölder condition one can show that  $\lim_{h \rightarrow 0} r^*(h) = 0$ . Consequently we obtain  $\varphi_T |m_0(h)| \leq (2r^*(h) + \delta) \varphi_T h^\beta = 2r^*(h) + \delta$  and

$$\lim_{\delta \rightarrow 0} \lim_{T \rightarrow \infty} \varphi_T |m_0(h)| = 0. \quad (8)$$

Moreover denoting  $C = C(L, M, B)$  a constant which depends on these parameters but which is not necessarily the same each time it will appear, we have

$$\begin{aligned}
\varphi_T |m_1(h)| &\leq \varphi_T \int_{-1}^1 |S(x_0 + zh) - S(x_0)| |q_S(x_0 + zh) - q_S(x_0)| dz \\
&\leq C \varphi_T h^2 = C h^{2-\beta} \xrightarrow{T \rightarrow \infty} 0,
\end{aligned} \quad (9)$$

because  $|q_S(x_0 + zh) - q_S(x_0)| = |zh \dot{q}_S(c_z)| \leq C |z| h$  and  $|S(x_0 + zh) - S(x_0)| \leq (L + B) |z| h$ .

Thanks to inequality (A.1) in Galtchouk and Pergamenshchikov (2006), there exists a constant  $\kappa > 0$  such that for all  $\lambda > 0$ :

$$\sup_{T \geq 1} \sup_{0 \leq t_0 \leq T} \sup_{S \in \Sigma_{L+B, M-B}} \mathbb{P}_S (|\Delta_{t_0, T}(f_h)| > \lambda) \leq 2e^{-\kappa \lambda^2}. \quad (10)$$

Then it is easy to see that  $\varphi_T T^{-1/2} \mathbb{E}_S |\Delta_{t_0, T}(f_h)|$  tends to zero as  $T \rightarrow \infty$  uniformly in  $S \in \mathcal{U}_{\delta, \beta}(S_0)$ .

Set  $q_*(x_0) := \inf_{S \in \Sigma_{L+B, M-B}} q_S(x_0)$ . Then  $q_*(x_0) > 0$  and we can write for sufficiently large  $T$

$$\frac{1}{2\tilde{q}_T(x_0) - \varepsilon_T} \leq \frac{1}{\tilde{q}_T(x_0)} \leq \left| \frac{1}{\tilde{q}_T(x_0)} - \frac{1}{q_S(x_0)} \right| + \frac{1}{q_*(x_0)}.$$

Using lemma 3.4, that is the reason why we get

$$\lim_{T \rightarrow \infty} \sup_{S \in \mathcal{U}_{\delta, \beta}(S_0)} \mathbb{E}_S \frac{\varphi_T}{\sqrt{T}(2\tilde{q}_T(x_0) - \varepsilon_T)} |\Delta_{t_0, T}(f_h)| = 0. \quad (11)$$

Thanks to lemma 3.4 again, (9) shows that

$$\lim_{T \rightarrow \infty} \sup_{S \in \mathcal{U}_{\delta, \beta}(S_0)} \mathbb{E}_S \frac{\varphi_T}{2\tilde{q}_T(x_0) - \varepsilon_T} |m_1(h)| = 0. \quad (12)$$

Finally since  $\sup_{S \in \mathcal{U}_{\delta, \beta}(S_0)} q_S(x_0) < \infty$ , (8) and lemma 3.4 prove that

$$\lim_{\delta \rightarrow 0} \lim_{T \rightarrow \infty} \sup_{S \in \mathcal{U}_{\delta, \beta}(S_0)} \mathbb{E}_S \frac{\varphi_T}{2\tilde{q}_T(x_0) - \varepsilon_T} q_S(x_0) |m_0(h)| = 0. \quad (13)$$

Hence assertion (6) proceeds from (11), (12) and (13).

Now let us show that for all  $\delta \in (0, 1)$

$$\lim_{T \rightarrow \infty} \sup_{S \in \mathcal{U}_{\delta, \beta}(S_0)} \mathbb{E}_S \sqrt{2q_S(x_0)} \varphi_T |G_T| \mathbb{I}_{(\tau_H \leq T)} = 0. \quad (14)$$

First remark that

$$m(\phi_h) - 2q_S(x_0) = \int_{-1}^1 (q_S(x_0 + uh) - q_S(x_0)) du.$$

By the Taylor formula, for all  $u \in [-1, 1]$  there exists  $\theta \in (0, 1)$  such that

$$q_S(x_0 + uh) - q_S(x_0) = \dot{q}_S(x_0)uh + \frac{1}{2}u^2h^2\ddot{q}_S(x_0 + \theta uh),$$

therefore we obtain

$$\sup_{S \in \mathcal{U}_{\delta, \beta}(S_0)} |m(\phi_h) - 2q_S(x_0)| = \sup_{S \in \mathcal{U}_{\delta, \beta}(S_0)} \frac{1}{2} \left| \int_{-1}^1 u^2 h^2 \ddot{q}_S(x_0 + \theta uh) du \right| \leq Ch^2, \quad (15)$$

for a certain constant  $C = C(L, M, B)$ .

Then we have successively

$$\begin{aligned}
|G_T| &\leq \frac{1}{H_T} \int_{\tau_H}^T h \phi_h(X_t) |S(X_t) - S(x_0)| dt \\
&\leq \frac{1}{H_T} \int_{\tau_H}^T h^2 (L + B) \phi_h(X_t) dt \leq \frac{(L + B)h^2}{H_T} \left( \int_{t_0}^T \phi_h(X_t) dt - \frac{H_T}{h} \right) \\
&\leq \frac{(L + B)h^2}{H_T} \left( \sqrt{T} \Delta_{t_0, T}(\phi_h) + (T - t_0)m(\phi_h) - (T - t_0)(2\tilde{q}_T(x_0) - \varepsilon_T) \right) \\
&\leq \frac{(L + B)h^2}{H_T} \left( \sqrt{T} \Delta_{t_0, T}(\phi_h) + (T - t_0)|2\tilde{q}_T(x_0) - m(\phi_h)| + \varepsilon_T(T - t_0) \right).
\end{aligned}$$

So (15) and the definition of  $H_T$  yield

$$|G_T| \leq \frac{(L + B)h}{\tilde{q}_T(x_0)} \left( \frac{\sqrt{T} \Delta_{t_0, T}(\phi_h)}{T - t_0} + 2|\tilde{q}_T(x_0) - q_S(x_0)| + Ch^2 + \varepsilon_T \right).$$

As a consequence of lemma 3.3 and the definition of  $\tilde{q}_T(x_0)$ , we get

$$\mathbb{E}_S |G_T| \mathbb{I}_{(\tau_H \leq T)} \leq (L + B) \sqrt{\nu_T} \left( \frac{h\sqrt{T}}{T - t_0} \mathbb{E}_S |\Delta_{t_0, T}(\phi_h)| + Ch^3 + h\varepsilon_T + 2\mu^* \frac{h}{\sqrt{t_0}} \right).$$

Since (10) implies that  $\mathbb{E}_S |\Delta_{t_0, T}(\phi_h)|$  is uniformly bounded in  $S \in \mathcal{U}_{\delta, \beta}(S_0)$  and  $T > 0$ , we can easily prove (14).

Eventually since  $\xi_T$  is a Gaussian standard random variable, one has

$$\begin{aligned}
&\left| \frac{\sqrt{2q_S(x_0)} \varphi_T \mathbb{E}_S \frac{|\xi_T|}{\sqrt{H_T}}}{\sqrt{H_T}} - \mathbb{E}_S |\xi| \right| = \left| \frac{\sqrt{2q_S(x_0)} T^{\beta/(2\beta+1)}}{\sqrt{(T - t_0)(2\tilde{q}_T(x_0) - \varepsilon_T)h}} - 1 \right| \mathbb{E} |\xi| \\
&= \left| \frac{\sqrt{2q_S(x_0)} (1 - \frac{t_0}{T})^{-1/2}}{\sqrt{2\tilde{q}_T(x_0) - \varepsilon_T}} - \frac{\sqrt{2q_S(x_0)}}{\sqrt{2\tilde{q}_T(x_0)}} + \frac{\sqrt{2q_S(x_0)}}{\sqrt{2\tilde{q}_T(x_0)}} - 1 \right| \mathbb{E} |\xi| \\
&\leq \left| \frac{\sqrt{2q_S(x_0)} (1 - \frac{t_0}{T})^{-1/2}}{\sqrt{2\tilde{q}_T(x_0) - \varepsilon_T}} - \frac{\sqrt{2q_S(x_0)}}{\sqrt{2\tilde{q}_T(x_0)}} \right| \mathbb{E} |\xi| + \left| \frac{\sqrt{2q_S(x_0)}}{\sqrt{2\tilde{q}_T(x_0)}} - 1 \right| \mathbb{E} |\xi|. \quad (16)
\end{aligned}$$

It is not difficult to show that  $2\tilde{q}_T(x_0) - \varepsilon_T \sim 2\tilde{q}_T(x_0)$  and  $(1 - t_0/T)^{1/2} \sim 1$

as  $T \rightarrow \infty$ . Consequently we have

$$\frac{(1 - \frac{t_0}{T})^{-1/2}}{\sqrt{2\tilde{q}_T(x_0) - \varepsilon_T}} - \frac{1}{\sqrt{2\tilde{q}_T(x_0)}} = o\left(\frac{1}{\sqrt{\tilde{q}_T(x_0)}}\right) = o((\ln T)^{-1/2}) \xrightarrow{T \rightarrow \infty} 0$$

uniformly on  $\mathcal{U}_{z_0, \delta}(S_0)$ .

Thanks to lemma 3.4, the second part of (16) tends to 0 uniformly on  $\mathcal{U}_{\delta, \beta}(S_0)$  as  $T \rightarrow \infty$ . Combining this with (6), (14) and lemma 3.5 finishes the proof of theorem 3.7.  $\square$

## References

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