High-order sufficient conditions for configuration tracking of affine connection control systems

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High order sufficient conditions for tracking

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Abstract

In this paper, we study under which conditions the trajectories of a mechanical control system can track any curve on the configuration manifold. We focus on systems that can be represented as forced affine connection control systems and we generalize the sufficient conditions for tracking known in the literature. The sufficient conditions are expressed in terms of convex cones of vector fields defined through particular brackets of the control vector fields of the system. The tracking control laws obtained by our constructions depend on several parameters. By imposing suitable asymptotic conditions on such parameters, we construct algorithmically one-parameter tracking control laws. The theory is supported by examples of control systems associated with elliptic hovercrafts and ellipsoidal submarines.

1   Introduction

The tracking problem has gained an increasing interest, mainly because of its applications to robot manipulators as for instance to control the position of underwater vehicles [6] and hovercrafts [15]. The tracking problem appears when a particular trajectory has to be followed by a control system, but there is no control law for the control system that makes this trajectory admissible. Then, the best that can be expected is to find a control law, typically oscillatory, that defines a good enough approximation of the target trajectory.

The mathematical background in the tracking problem includes the averaging theory [16] as explained, for instance, in [5]. The averaging techniques transform differential equations difficult to solve into other differential equations whose solutions approximate fairly well the solutions to the first set of equations. This is useful to approximate solutions to differential equations that depend on time or on parameters.

Differential geometry has provided a suitable framework to study in an intrinsic way typical mechanical control systems in engineering as, for instance, underwater submarines, aircraft models, hovercrafts and so on [5]. Here, we focus on forced affine connection control systems and generalize the sufficient conditions for tracking a trajectory that exist in the literature from a geometric viewpoint [5]. The existent results can be interpreted as first–order sufficient conditions because only the control vector fields and particular brackets, called symmetric products, between them get involved in the statement of the sufficient conditions. However, our conditions need longer symmetric products and so they are said to be of order higher than two.

In [6] it was observed that the tracking is possible for specific underwater vehicles, even though they do not satisfy the geometric sufficient conditions known in the literature. That motivates our research in order to obtain more general geometric sufficient conditions that ensure the tracking property for a wider range of control systems.

The chances to be able to track a target trajectory are related to some controllability requirement and to the avoidance of "bad" directions. In an informal way, these "bad" directions have to be interpreted as directions that will not make possible to have the starting point in the interior of the reachable set. (We refer to [5] for an accurate description of the obstructions to controllability in terms of the symmetric products and of particular vector–valued quadratic forms.) The sufficient conditions for being able to track unfeasible trajectories are also related to how nonholonomicity allows to enlarge the set of admissible velocities for the control system. In this regard, constructions of convex cones [4], the

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above–mentioned vector–valued quadratic forms [5] and some techniques similar to the ones in [12] have been useful for obtaining the constructions considered here.

The paper is organized as follows. Section 2 contains the necessary background in forced affine connection control systems and in chronological calculus [2]. Section 3 defines properly the notion of trackability and reviews the geometric sufficient conditions in the literature [5]. Sections 4 and 5 contain the main contributions of the paper and some examples to justify the utility of these results.

2 Notation and preliminaries

Denote by $\mathbb{N}$ the set of *positive* natural numbers and write $\mathbb{N}_0$ for $\mathbb{N} \cup \{0\}$. Fix $n \in \mathbb{N}$. From now on, Q is a n–dimensional smooth manifold and $\mathfrak{X}(Q)$ denotes the set of smooth vector fields on Q. All the vector fields are considered smooth, unless otherwise stated. Let $\tau_Q \colon TQ \to Q$ be the canonical tangent projection, a *vector field X on Q defined along τ_Q* is a mapping $X \colon TQ \to TQ$ such that $\tau_Q \circ X = \tau_Q$.

2.1 Affine connection control systems

The trajectories $\gamma \colon I \to Q$ of a Lagrangian mechanical systems on a manifold Q are minimizers of the action functional

$$A_L(\gamma) = \int_I L(t, \dot{\gamma}(t)) \mathrm{d}t$$

associated with a Lagrangian function $L \colon \mathbb{R} \times TQ \to \mathbb{R}$.

The solutions to this variational problem must satisfy the well–known Euler–Lagrange equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \ldots, n, \qquad (1)$$

where (q^i, v^i) are local coordinates for TQ. Here we consider controlled Euler–Lagrange equations obtained by modifying the right–hand side on the above equation, as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i} = \sum_{a=1}^k u_a Y_a^i, \quad i = 1, \ldots, n,$$

with $u_a \colon I \to \mathbb{R}$, $Y_a^i \colon Q \to \mathbb{R}$.

When the manifold Q is endowed with the Riemannian structure given by a Riemannian metric g and the Lagrangian function $L_g(v_q) = \frac{1}{2}g(v_q, v_q)$ is

considered, the solutions to (1) turn out to be the geodesics of the Levi–Civita affine connection ∇^g associated with the Riemannian metric. (See [5] for more details and for many examples of mechanical control systems that fit in this description.)

When control forces are added to the geodesic equations we obtain an affine connection control system

$$\nabla_{\dot{\gamma}(t)}^g \dot{\gamma}(t) = \sum_{a=1}^k u_a(t) Y_a(\gamma(t)),$$

with Y_a being vector fields on Q.

The notion of affine connection control system can be extended without the need of the Levi–Civita connection.

Definition 2.1. *An **affine connection** is a mapping*

$$\nabla \colon \mathfrak{X}(Q) \times \mathfrak{X}(Q) \longrightarrow \mathfrak{X}(Q)$$
$$(X, Y) \longmapsto \nabla(X, Y) = \nabla_X Y,$$

satisfying the following properties:

1. *∇ is $\mathbb{R}$–linear on X and on Y;*

2. *$\nabla_{fX} Y = f \nabla_X Y$ for every $f \in \mathcal{C}^\infty(Q)$;*

3. *$\nabla_X f Y = f \nabla_X Y + (Xf) Y$, for every $f \in \mathcal{C}^\infty(Q)$. (Here Xf denotes the derivative of f in the direction X.)*

The mapping $\nabla_X Y$ is called the *covariant derivative of Y with respect to X*. Given local coordinates (q^i) on Q, the *Christoffel symbols for the affine connection* in these coordinates are given by

$$\nabla_{\frac{\partial}{\partial q^j}} \frac{\partial}{\partial q^r} = \sum_{i=1}^n \Gamma_{jr}^i \frac{\partial}{\partial q^i}.$$

From the properties of the affine connection, we have

$$\nabla_X Y = \sum_{i,j,r=1}^n \left(X^j \frac{\partial Y^i}{\partial q^j} + \Gamma_{jr}^i X^j Y^r\right) \frac{\partial}{\partial q^i},$$

where $X = \sum_{i=1}^n X^i \partial/\partial q^i$ and $Y = \sum_{i=1}^n Y^i \partial/\partial q^i$.

Definition 2.2. *A **forced affine connection control system (FACCS)** is a control mechanical system given by $\Sigma = (Q, \nabla, Y, \mathscr{Y}, U)$ where*

- *Q is a smooth n–dimensional manifold called the configuration manifold,*

- *Y is a time-dependent vector field along the projection $\tau_Q \colon TQ \to Q$, measurable and bounded with respect to the time and affine with respect to the velocities,*

- $\mathscr{Y}$ is a set of k control vector fields on Q, and

- $U \subseteq \mathbb{R}^k$.

A trajectory $\gamma \colon I \subset \mathbb{R} \to Q$ is **admissible for Σ** if $\dot{\gamma} \colon I \to TQ$ is absolutely continuous and there exists a measurable and bounded control $u \colon I \to U$ such that the dynamical equations of the control system Σ

$$\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = Y(t, \dot{\gamma}(t)) + \sum_{a=1}^{k} u_a(t) Y_a(\gamma(t)), \quad (2)$$

are fulfilled (almost everywhere).

The vector field Y includes all the non-controlled external forces; e.g., the potential and the non–potential forces. The assumption that Y is affine with respect to the velocities means that, in every local system of coordinates (q^i, v^i) on TQ, Y can be written as

$$Y(t, v_q) = Y_0(t, q) + \sum_{i=1}^{n} v^i Y^i(t, q).$$

Equation (2) can be rewritten as a first–order control–affine system on TQ,

$$\dot{\Upsilon}(t) = Z(\Upsilon(t)) + Y^V(t, \Upsilon(t)) + \sum_{a=1}^{k} u_a(t) Y_a^V(\Upsilon(t)), \quad (3)$$

where $\Upsilon \colon I \to TQ$ is such that $\tau_Q \circ \Upsilon = \gamma$, Z is the geodesic spray associated to the affine connection on Q and Y_a^V denotes the vertical lift of the vector field Y_a (see [1] for more details).

Apart from the usual Lie bracket that provides $\mathfrak{X}(Q)$ with a Lie algebra structure, the following product of elements in $\mathfrak{X}(Q)$ associated to ∇ can be introduced.

Definition 2.3. *The **symmetric product** is the map*

$$\begin{aligned} \langle \cdot : \cdot \rangle \colon \mathfrak{X}(Q) \times \mathfrak{X}(Q) &\longrightarrow \mathfrak{X}(Q) \\ (X, Y) &\longmapsto \nabla_X Y + \nabla_Y X. \end{aligned}$$

It can be proved that

$$[Y_a^V, [Z, Y_b^V]] = \langle Y_a : Y_b \rangle^V \quad (4)$$

(see [5]).

2.2 Chronological calculus

We recall in this section some notion of chronological calculus, which is used later as a tool in the study of the asymptotic behavior of endpoint mappings depending on parameters. For a comprehensive discussion and for the proofs of all results stated in this section see [2]. In the sequel all vector fields, autonomous and non–autonomous, are assumed to be complete. The behavior of non–complete vector fields on compact sets can be studied by considering suitable cut–off procedures.

Given a non–autonomous vector field X_τ on some manifold M, where τ denotes the time variable and the map $(\tau, q) \mapsto X_\tau(q)$ is assumed to be smooth with respect to q and measurable bounded with respect to τ, we denote by $\overrightarrow{\exp} \int_0^t X_\tau \mathrm{d}\tau$ the diffeomorphism of M onto itself corresponding to the flow from time 0 to time t of X_τ. Hence, $\overrightarrow{\exp} \int_0^t X_\tau \mathrm{d}\tau(\bar{q})$ is the evaluation at time t of the solution to the non–autonomous Cauchy problem

$$\dot{q}(\tau) = X_\tau(q(\tau)), \quad q(0) = \bar{q}.$$

Any diffeomorphism $P \colon M \to M$ defines an isomorphism $\mathrm{Ad}P$ of $\mathfrak{X}(M)$ through the rule

$$\mathrm{Ad}P(Y)(q) = P_*(Y(P^{-1}(q))),$$

where P_* denotes the pushforward by P.

If $P = \overrightarrow{\exp} \int_0^t X_\tau \mathrm{d}\tau$, then we write

$$\mathrm{Ad}P = \overrightarrow{\exp} \int_0^t \mathrm{ad}X_\tau \mathrm{d}\tau.$$

If $\mathrm{ad}^{(m)} X_\tau(Y) = 0$ for some $m \in \mathbb{N}$ and every τ, then

$$\begin{aligned} &\overrightarrow{\exp} \int_0^t \mathrm{ad}X_\tau \mathrm{d}\tau(Y) = Y + \int_0^t [X_\tau, Y]\mathrm{d}\tau \\ &+ \int_0^t \int_0^{\tau_1} [X_{\tau_2}, [X_{\tau_1}, Y]]\mathrm{d}\tau_2 \mathrm{d}\tau_1 + \cdots + \\ &\int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{m-2}} \mathrm{ad}X_{\tau_{m-1}} \circ \cdots \circ \mathrm{ad}X_{\tau_1} Y \mathrm{d}\tau_{m-1} \cdots \mathrm{d}\tau_1. \end{aligned} \quad (5)$$

In the framework of chronological calculus the flow of the sum of two non–autonomous vector fields can be conveniently represented by the following variation formula

$$\begin{aligned} &\overrightarrow{\exp} \int_0^t (X_\tau + Y_\tau)\mathrm{d}\tau = \\ &\overrightarrow{\exp} \int_0^t X_\tau \mathrm{d}\tau \circ \overrightarrow{\exp} \int_0^t \left(\overrightarrow{\exp} \int_0^\tau \mathrm{ad}X_s \mathrm{d}s \right) Y_\tau \mathrm{d}\tau. \end{aligned} \quad (6)$$

Let us recall a useful result for the convergence of flows of non–autonomous vector fields. It states, roughly speaking, that the flows converge if the vector fields converge in integral sense. For further results and a discussion on this kind of convergence from the point of view of ordinary differential equations and control theory, see [3, 11].

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Lemma 2.4. *[2, Lemma 8.10] Let Z_τ^j, $j \in \mathbb{N}$, and Z_τ, $\tau \in [0, t_1]$, be non–autonomous vector fields on M, bounded with respect to τ, and let these vector fields have a compact support. If*

$$\int_0^t Z_\tau^j \mathrm{d}\tau \to \int_0^t Z_\tau \mathrm{d}\tau, \quad j \to \infty,$$

then

$$\overrightarrow{\exp} \int_0^t Z_\tau^j \mathrm{d}\tau \to \overrightarrow{\exp} \int_0^t Z_\tau \mathrm{d}\tau, \quad j \to \infty,$$

both convergences being uniform with respect to $(t, q) \in [0, t_1] \times M$ and uniform with all derivatives with respect to $q \in M$.

Another, even more standard, result on the convergence of flows is the following, that we find useful to state as a separate lemma. Its difference from the previous one can be described as follows: if the dependence on time of the Z_τ^j is prescribed independently on j (it would be enough that they converge uniformly), then the uniform convergence of flows is guaranteed by the uniform convergence of the vector fields, even without any knowledge about the convergence of their derivatives with respect to the state variables. For completeness, we provide a brief proof for it.

Lemma 2.5. *Let $\lambda_1, \ldots, \lambda_m \in L^\infty([0, t_1], \mathbb{R})$. Let Z_τ^j, $n \in \mathbb{N}$, and Z_τ, $\tau \in [0, t_1]$, be non–autonomous vector fields on M of the form $Z_\tau^j = \lambda_1(\tau) Z_1^j + \cdots + \lambda_m(\tau) Z_m^j$ and $Z_\tau = \lambda_1(\tau) Z_1 + \cdots + \lambda_m(\tau) Z_m$, with $Z_i^j, Z_i \in \mathfrak{X}(M)$ with compact support. If $Z_i^j \to Z_i$, $j \to \infty$, then*

$$\overrightarrow{\exp} \int_0^t Z_\tau^j \mathrm{d}\tau \to \overrightarrow{\exp} \int_0^t Z_\tau \mathrm{d}\tau, \quad j \to \infty,$$

both convergences being uniform (the first with respect to the state q and the second with respect to (t, q)).

Proof. Using the variation formula (6) with $X_\tau = Z_\tau$ and $Y_\tau = Z_\tau^j - Z_\tau$, we get

$$\overrightarrow{\exp} \int_0^t Z_\tau^j \mathrm{d}\tau = \overrightarrow{\exp} \int_0^t Z_\tau \mathrm{d}\tau \circ \\ \overrightarrow{\exp} \int_0^t \left(\overrightarrow{\exp} \int_0^\tau \mathrm{ad} Z_s \mathrm{d}s\right) (Z_\tau^j - Z_\tau) \mathrm{d}\tau. \quad (7)$$

Since $\left(\overrightarrow{\exp} \int_0^\tau Z_s \mathrm{d}s\right)_*$, $\tau \in [0, t_1]$, is a compact family of operators, then the last diffeomorphism in (7) converges uniformly to the identity for $t \in [0, t_1]$ as j goes to infinity. ∎

3 Tracking problem

The need of trackability appears when one tries to follow a particular trajectory on the configuration manifold, called *reference* or *target* trajectory, which is not a solution of the FACCS considered. A trajectory is successfully tracked if there exist solutions to the FACCS that approximate it arbitrarily well.

Consider any distance $\mathrm{d} \colon Q \times Q \to \mathbb{R}$ on Q whose corresponding metric topology coincides with the topology on Q. From now on I will denote the interval $[0, t_1]$, with $t_1 > 0$.

Definition 3.1. *A control system Σ is **trackable** if for every continuous curve $\gamma \colon I \to Q$, for every $v \in T_{\gamma(0)} Q$ and for every strictly positive tolerance ϵ, there exist a control $u^\epsilon \in L^\infty(I, U)$ such that the solution $\xi^\epsilon \colon I \to Q$ to Σ corresponding to u^ϵ and with initial condition $\dot\xi^\epsilon(0) = v$ verifies*

$$\mathrm{d}(\gamma(t), \xi^\epsilon(t)) < \epsilon$$

for every time $t \in I$.

Remark 3.2. Since any continuous curve can be uniformly approximated, with arbitrary precision, by a smooth curve having a prescribed tangent vector at its initial point, then Σ is trackable if and only if every curve $\gamma \colon I \to Q$ on Q of class $\mathcal{C}^\infty$ is **trackable for** Σ, i.e., for every $\epsilon > 0$, there exist $u^\epsilon \in L^\infty(I, U)$ such that the solution $\xi^\epsilon \colon I \to Q$ to Σ corresponding to u^ϵ and with initial condition $\dot\xi^\epsilon(0) = \dot\gamma(0)$ verifies $\mathrm{d}(\gamma(t), \xi^\epsilon(t)) < \epsilon$ for every $t \in I$.

In order to give some insights into particular sufficient conditions for tracking, we are going to review a result in the literature.

Theorem 3.3. *[5, Theorem 12.26] Let $\Sigma = (Q, \nabla, Y, \mathscr{Y}, U)$ be a FACCS such that*

- *the distribution generated by $\mathscr{Y} = \{Y_1, \ldots, Y_k\}$ is regular, that is, it is a subbundle of TQ;*

- *$\langle Y_a : Y_a \rangle \in \mathrm{span}_{\mathcal{C}^\infty(Q)} \mathscr{Y}$ for every $a \in \{1, \ldots, k\}$, i.e., $\langle Y_a : Y_a \rangle = \sum_{b=1}^k \sigma_{ab} Y_b$, where $\sigma_{ab} \in \mathcal{C}^\infty(Q)$;*

- *the distribution $\mathrm{Sym}^{(1)}(\mathscr{Y})$ defined by $\mathrm{Sym}^{(1)}(\mathscr{Y})_q = \mathrm{span}_{\mathbb{R}} \mathscr{Y}_q + \mathrm{span}_{\mathbb{R}} \{\langle W : Z \rangle(q) | W, Z \in \mathscr{Y}\}$ is the entire tangent bundle TQ.*

Let $\gamma_{\mathrm{ref}} \colon I \to Q$ be a reference trajectory of class $\mathcal{C}^3$. Define the functions $u_{\mathrm{ref},a}, u_{\mathrm{ref},bc} \colon I \to \mathbb{R}$,

$a, b, c \in \{1, \dots, k\}$, $b < c$, as solutions of class $\mathcal{C}^1$ to

$$\nabla_{\dot{\gamma}_{\mathrm{ref}}(t)} \dot{\gamma}_{\mathrm{ref}}(t) - Y(t, \dot{\gamma}_{\mathrm{ref}}(t))$$
$$= \sum_{a=1}^{k} u_{\mathrm{ref},a}(t) Y_a(\gamma_{\mathrm{ref}}(t))$$
$$+ \sum_{b,c=1, b<c}^{k} u_{\mathrm{ref},bc}(t) \langle Y_b : Y_c \rangle (\gamma_{\mathrm{ref}}(t)).$$

Define $u_{\mathrm{slow}} \colon I \times TQ \to \mathbb{R}^k$, $u_{\mathrm{osc}} \colon \mathbb{R} \times I \times TQ \to \mathbb{R}^k$ by

$$u_{\mathrm{slow},a}(t, v_q) = u_{\mathrm{ref},a}(t) +$$
$$\sum_{b=1}^{k} \left(b - 1 + \sum_{c=b+1}^{k} \frac{(u_{\mathrm{ref},bc}(t))^2}{4} \right) \sigma_{ab}(q),$$
$$u_{\mathrm{osc},a}(\tau, t, v_q) = \sum_{c=1}^{a-1} \varphi_{\mathrm{lo}(c,a)}(\tau)$$
$$- \frac{1}{2} \sum_{c=a+1}^{k} u_{\mathrm{ref},ac}(t) \varphi_{\mathrm{lo}(a,c)}(\tau),$$

where $\mathrm{lo}(a, b) = \sum_{j=1}^{a-1} (k - j) + (b - a)$ for $(a, b) \in \{1, \dots, k\}^2$, $a < b$, and for $i \in \mathbb{N}$

$$\varphi_i \colon \mathbb{R} \to \mathbb{R}, \quad t \mapsto \frac{4\pi i}{T} \cos\left(\frac{2\pi i}{T} t \right). \tag{8}$$

Then γ_{ref} is trackable for Σ and moreover the solutions ξ^ϵ of Σ, $\epsilon > 0$, corresponding to the controls

$$u^\epsilon \colon \mathbb{R} \times TQ \longrightarrow U \subset \mathbb{R}^k$$
$$(t, v_q) \longmapsto u_{\mathrm{slow}}(t, v_q) + \frac{1}{\epsilon} u_{\mathrm{osc}}\left(\frac{t}{\epsilon}, t, v_q \right)$$

with initial condition $\dot{\xi}^\epsilon(0) = \dot{\gamma}_{\mathrm{ref}}(0)$ are such that $\mathrm{d}(\gamma_{\mathrm{ref}}(t), \xi^\epsilon(t))$ tends to zero as ϵ goes to zero uniformly with respect to $t \in I$.

Remark 3.4. Observe that under the hypotheses of Theorem 3.3 not only the tracking is guaranteed, but also the tracking control law is given explicitly.

4 A more general tracking result

The idea of the following construction is to identify, given the set $\mathscr{Y}$, a larger set of control vector fields $\mathscr{K}_1$ such that every trajectory solution of the FACCS obtained by replacing $\mathrm{span}_{\mathcal{C}^\infty(Q)} \mathscr{Y}$ by $\mathscr{K}_1$ can be tracked by solutions of the original FACCS Σ. Repeating the construction on $\mathscr{K}_1$ we obtain an even larger family $\mathscr{K}_2$ and so on. If eventually $\mathscr{K}_l(q) = T_q Q$ for every $q \in Q$ for some $l \in \mathbb{N}$, then we can show that the system is trackable.

In order to generalize the sufficient conditions for tracking given in Theorem 3.3, we construct the following set of vector fields on Q:

$$\mathscr{K}_0 = \overline{\mathrm{span}_{\mathcal{C}^\infty(Q)} \mathscr{Y}},$$
$$\mathscr{K}_l = \overline{\mathscr{K}_{l-1} - \mathrm{co}\{ \langle Z : Z \rangle \mid Z \in \mathrm{L}(\mathscr{K}_{l-1}) \}}, \tag{9}$$

for $l \in \mathbb{N}$, where, for $A \subset \mathfrak{X}(Q)$, $\mathrm{L}(A) = A \cap (-A)$, $\mathrm{co}(A)$ denotes the convex hull of A, and $\overline{A}$ is the closure of A in $\mathfrak{X}(Q)$ with respect to the topology of the uniform convergence on compact sets. For $A \subset \mathfrak{X}(Q)$ we also write

$$A(q) = \{ Y(q) \mid Y \in A \} \subseteq T_q Q.$$

Proposition 4.1. For every $l \in \mathbb{N}_0$, $\mathscr{K}_l$ is a convex cone of $\mathscr{X}(Q)$. In particular, for every $q \in Q$, $\mathscr{K}_l(q)$ is a convex cone with vertex at $0 \in T_q Q$.

Proof. The proposition is proved by induction having in mind that the set $\mathscr{K}_l$ is a convex cone if it contains all conic combinations of elements of $\mathscr{K}_l$. Remember that a conic combination of elements of $\mathscr{K}_l$ is of the form $\lambda_1 W_1 + \cdots + \lambda_r W_r$ with $\lambda_i \in \mathbb{R}_{\geq 0}$, $W_i \in \mathscr{K}_l$ for every $i \in \{1, \dots, r\}$.

First, $\mathscr{K}_0$ is subspace of $\mathscr{X}(Q)$ and thus it is a convex cone.

The induction step consists of proving that if $\mathscr{K}_i$ is a convex cone, so is $\mathscr{K}_{i+1}$. First notice that, if $W \in L(\mathscr{K}_i)$ and $\lambda \in \mathbb{R}$, then, by the induction hypothesis, $\lambda W \in L(\mathscr{K}_i)$ and thus $\lambda^2 \langle W : W \rangle$ belongs to $\{ \langle Z : Z \rangle \mid Z \in L(\mathscr{K}_i) \}$. Hence, $\mathscr{K}_{i+1}$ is the closure of the sum of two convex sets invariant by multiplication by any non-negative scalar. Therefore, $\mathscr{K}_{i+1}$ is itself a convex cone. ∎

Given two functions $\alpha, \beta : [0, T] \to \mathbb{R}$, let

$$\Lambda_T(\alpha, \beta) = \frac{1}{2T} \int_0^T \left(\int_0^\tau \alpha(s) \mathrm{d}s \right) \left(\int_0^\tau \beta(s) \mathrm{d}s \right) \mathrm{d}\tau.$$

We say that a *sequence* of smooth T-periodic functions $\psi_j : \mathbb{R} \to \mathbb{R}$, $j \in \mathbb{N}$, is Λ_T–*orthonormal and zero-mean* if $\int_0^T \psi_j(\tau) = 0$ for every $j \in \mathbb{N}$ and $\Lambda_T(\psi_j, \psi_m) = \delta_{jm}$ for $j, m \in \mathbb{N}$, where δ_{jm} denotes the Kronecker delta.

For instance, the sequence defined in (8), is Λ_T-orthonormal and zero-mean.

Proposition 4.2. Let $\Sigma = (Q, \nabla, Y, \mathscr{Y}, \mathbb{R}^k)$ be a FACCS and fix $v_q \in TQ$, $T > 0$, $n_1, \dots, n_k \in \mathbb{N}$, and $w = (w_1, \dots, w_k) \in L^\infty(I, \mathbb{R}^k)$. For every $\epsilon > 0$ denote by $\xi^\epsilon : I \to Q$ the solution of

$$\nabla_{\dot{\xi}^\epsilon(t)} \dot{\xi}^\epsilon(t) = Y(t, \dot{\xi}^\epsilon(t))$$
$$+ \sum_{a=1}^{k} \frac{1}{\epsilon} \psi_{n_a} \left(\frac{t}{\epsilon} \right) w_a(t) Y_a(\xi^\epsilon(t)),$$

with initial condition $\dot{\xi}^\epsilon(0) = v_q$, where $(\psi_j)_{j\in\mathbb{N}}$ is a Λ_T-orthonormal and zero-mean sequence. Let also $\gamma : I \to Q$ be the solution of

$$\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = Y(t, \dot{\gamma}(t))$$
$$- \sum_{a,b=1}^{k} \delta_{n_a n_b} w_a(t) w_b(t) \langle Y_a : Y_b \rangle(\gamma(t)), \qquad (10)$$

with initial condition $\dot{\gamma}(0) = v_q$. Then there exist $C, \epsilon_0 > 0$ such that $\mathrm{d}(\gamma(t), \xi^\epsilon(t)) < C\epsilon$ for every $t \in I$ and every $\epsilon \in (0, \epsilon_0)$.

Proof. The proof of this proposition follows from Theorem 9.32 in [5] by considering the function $u^a(t/\epsilon, t)$ that appears there as $\psi_{n_a}(t/\epsilon) w_a(t)$. ∎

Remark 4.3. Theorem 3.3 follows directly from Proposition 4.2 (and is obtained as such in [5]). Therefore, some of the hypotheses of Theorem 3.3 can easily be relaxed: first of all the distribution generated by $\mathscr{Y}$ need not be a subbundle of TQ. Moreover, the hypothesis that $\mathrm{Sym}^{(1)}(\mathscr{Y}) = TQ$ can be replaced by the requirement that

$$\nabla_{\dot{\gamma}_{\mathrm{ref}}(t)}\dot{\gamma}_{\mathrm{ref}}(t) - Y(t, \dot{\gamma}_{\mathrm{ref}}(t)) \in \mathrm{Sym}^{(1)}(\mathscr{Y})_{\gamma_{\mathrm{ref}}(t)}$$

for every $t \in I$.

Theorem 4.4. *Let* $\Sigma = (Q, \nabla, Y, \mathscr{Y}, \mathbb{R}^k)$ *be a FACCS. Fix a reference trajectory* $\gamma_{\mathrm{ref}} : I \to Q$ *of class* $\mathcal{C}^\infty$. *If for every* $t \in I$ *there exists* $l \in \mathbb{N}$ *such that* $\mathscr{K}_l(\gamma_{\mathrm{ref}}(t)) = T_{\gamma_{\mathrm{ref}}(t)}Q$, *then* γ_{ref} *is track-able. Therefore, if for every* $q \in Q$ *there exists* $l \in \mathbb{N}$ *such that* $\mathscr{K}_l(q) = T_qQ$, *then the control system* Σ *is trackable.*

Proof. First of all, notice that the last part of the statement (the trackability of Σ) directly follows from the first one (the trackability of γ_{ref}), which is proved below.

As the reference trajectory γ_{ref} is defined on a compact set and γ_{ref} is continuous, then $\mathrm{Im}\gamma_{\mathrm{ref}}$ is a closed compact set. By hypothesis, for every $t \in I$ there exists $l \in \mathbb{N}$ such that $\mathscr{K}_l(\gamma_{\mathrm{ref}}(t)) = T_{\gamma_{\mathrm{ref}}(t)}Q$. So there exist $n + 1$ vector fields in $\mathscr{K}_l$ whose conic combinations at $\gamma_{\mathrm{ref}}(t)$ give the whole tangent space $T_{\gamma_{\mathrm{ref}}(t)}Q$. The smoothness of these vector fields guarantees that this is still true in an open neighborhood U_t of $\gamma_{\mathrm{ref}}(t)$. In this way we construct an open cover of $\mathrm{Im}\gamma_{\mathrm{ref}}$. As $\mathrm{Im}\gamma_{\mathrm{ref}}$ is compact, there exists a finite open subcover of $\{U_t\}_{t\in I}$ given by $U_{t_1}, \ldots, U_{t_r}$. For each t_i there exists a different $l_i \in \mathbb{N}$ such that $\mathscr{K}_{l_i}(\gamma_{\mathrm{ref}}(t)) = T_{\gamma_{\mathrm{ref}}(t)}Q$ for every $t \in \gamma_{\mathrm{ref}}^{-1}(U_{t_i})$. Then $l = \max\{l_1, \ldots, l_r\}$ satisfies $\mathscr{K}_l(\gamma_{\mathrm{ref}}(t)) = T_{\gamma_{\mathrm{ref}}(t)}Q$ for every $t \in I$.

Moreover, there exists a partition of unity subordinated to the finite open subcover $\{U_{t_i}\}_{i=1,\ldots,r}$ (see [10]) that allows us to define a finite set of smooth global vector fields Z_a in $\mathscr{K}_l$ such that

$$\nabla_{\dot{\gamma}_{\mathrm{ref}}(t)}\dot{\gamma}_{\mathrm{ref}}(t) - Y(t, \dot{\gamma}_{\mathrm{ref}}(t)) = \sum_{a=1}^{N_l} \lambda_a(t) Z_a(\gamma_{\mathrm{ref}}(t))$$
$$(11)$$

for $t \in I$, $N_l \in \mathbb{N}$, where $\lambda_a : I \to [0, +\infty)$ is of class $\mathcal{C}^\infty$.

We introduce for a curve on Q the notion of having a *regular parameterization on* $\mathscr{K}_i$. A curve $\gamma : I \to Q$ admits a *regular parameterization on* $\mathscr{K}_i$ if there exist $N_i \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_{N_i} \in \mathcal{C}(I)$ and $Z_1, \ldots, Z_{N_i} \in \mathscr{K}_i$ such that $\lambda_a(t) \geq 0$ for every $t \in I$ and $a \in \{1, \ldots, N_i\}$ and

$$\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) - Y(t, \dot{\gamma}(t)) = \sum_{a=1}^{N_i} \lambda_a(t) Z_a(\gamma(t)).$$

Fix $\epsilon > 0$ and let $\xi_l = \gamma_{\mathrm{ref}}$. Then ξ_l admits a regular parameterization on $\mathscr{K}_l$. We are going to prove by induction that there exists a finite sequence $\{\xi_0, \xi_1, \ldots, \xi_l\}$ of curves on Q such that each ξ_i satisfies $\dot{\xi}_i(0) = \dot{\gamma}_{\mathrm{ref}}(0)$, admits a regular parameterization on $\mathscr{K}_i$ and $\mathrm{d}(\xi_i(t), \xi_{i+1}(t)) < \epsilon/l$ for every $i = 0, \ldots, l - 1$ and every $t \in I$. The induction step claims that: for $i = 0, \ldots, l - 1$ if there exists $\xi_{i+1} : I \to Q$ admitting a regular parameterization on $\mathscr{K}_{i+1}$, then there exists $\xi_i : I \to Q$ with $\dot{\xi}_{i+1}(0) = \dot{\xi}_i(0)$ admitting a regular parameterization on $\mathscr{K}_i$ and satisfying $\mathrm{d}(\xi_i(t), \xi_{i+1}(t)) < \epsilon/l$ for every $t \in I$.

Let us prove the induction step for i. Let $Z_1, \ldots, Z_{N_{i+1}} \in \mathscr{K}_{i+1}$ be the vector fields that determine the regular parameterization of ξ_{i+1}. Here we split the proof in two steps, considering first the special case:

1. $Z_a = F_a - G_a$ with $F_a \in \mathscr{K}_i$ and $G_a \in \mathrm{co}\{\langle Z : Z \rangle \mid Z \in \mathrm{L}(\mathscr{K}_i)\}$ for every $a = 1, \ldots, N_{i+1}$,

and then the general case

2. $Z_a \in \overline{\mathscr{K}_i - \mathrm{co}\{\langle Z : Z \rangle \mid Z \in \mathrm{L}(\mathscr{K}_i)\}}$.

Let us study case 1: ξ_{i+1} admits the parameterization

$$\nabla_{\dot{\xi}_{i+1}(t)}\dot{\xi}_{i+1}(t) - Y(t, \dot{\xi}_{i+1}(t)) =$$
$$\sum_{a=1}^{N_{i+1}} \lambda_a(t)(F_a - G_a)(\xi_{i+1}(t)). \qquad (12)$$

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Each G_a is given by $\sum_{b=1}^{N_a} \alpha_{a,b}\langle G_{a,b} \colon G_{a,b}\rangle$ where $G_{a,b} \in \mathrm{L}(\mathscr{K}_i)$ and $\alpha_{a,b} \geq 0$. Then (12) becomes

$$\nabla_{\dot{\xi}_{i+1}(t)}\dot{\xi}_{i+1}(t) - Y(t, \dot{\xi}_{i+1}(t)) = \sum_{a=1}^{N_{i+1}} \Big(\lambda_a(t)$$
$$F_a(\xi_{i+1}(t)) - \sum_{b=1}^{N_a} \lambda_a(t)\alpha_{a,b}\langle G_{a,b}\colon G_{a,b}\rangle(\xi_{i+1}(t))\Big).$$
$$(13)$$

The dynamics described in (13) are of the same form as in (10) as long as we take $\mathscr{Y} = \{G_{a,b}\}_{a=1,\dots,N_{i+1};b=1,\dots,N_a}$, and $\delta_{n_{ab}n_{a'b'}} = \delta_{aa'}\delta_{bb'}$, $w_{a,b} = \sqrt{\lambda_a \alpha_{a,b}}$.

Then by Proposition 4.2 the solutions to (13) can be approximated by solutions to

$$\nabla_{\dot{\xi}_i^{\epsilon_i}(t)}\dot{\xi}_i^{\epsilon_i}(t) - Y(t, \dot{\xi}_i^{\epsilon_i}(t)) = \sum_{a=1}^{N_{i+1}} \Big(\lambda_a(t)F_a(\xi_i^{\epsilon_i}(t))$$
$$+ \sum_{b=1}^{N_a} \frac{1}{\epsilon_i}\psi_{n_{ab}}\Big(\frac{t}{\epsilon_i}\Big)w_{a,b}(t)G_{a,b}(\xi_i^{\epsilon_i}(t))\Big)$$
$$(14)$$

being $(a,b) \mapsto n_{ab}$ any injective map from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ and $(\psi_j)_{j\in\mathbb{N}}$ a Λ_T-orthonormal and zero-mean sequence. More precisely, there exist $C_i, \epsilon_{i,0} > 0$ such that $\mathrm{d}(\xi_i^{\epsilon_i}(t),\xi_{i+1}(t)) < C_i\epsilon_i$ for every $t \in I$, $\epsilon_i \in (0,\epsilon_{i,0})$. In particular, we can choose ϵ_i such that $C_i\epsilon_i < \epsilon/l$.

The finite linear combination of elements in $\mathscr{K}_i(\xi_i^{\epsilon_i}(t))$ for every $t \in I$ on the right–hand side of (14) does not necessarily satisfy the non-negativeness of the coefficients. However, $G_{a,b} \in \mathrm{L}(\mathscr{K}_i)$, so $-G_{a,b} \in \mathrm{L}(\mathscr{K}_i)$. Then we can rewrite the coefficients of $G_{a,b}$ as follows:

$$\psi_{n_{ab}}\Big(\frac{t}{\epsilon_i}\Big)w_{a,b}(t)G_{a,b} = \max\Big\{0, \psi_{n_{ab}}\Big(\frac{t}{\epsilon_i}\Big)\Big\}$$
$$w_{a,b}(t)G_{a,b} + \max\Big\{0, -\psi_{n_{ab}}\Big(\frac{t}{\epsilon_i}\Big)\Big\}w_{a,b}(t)(-G_{a,b}).$$

Thus all the coefficients are continuous and non-negative. We can conclude that $\xi_i^{\epsilon_i}$ admits a regular parameterization on $\mathscr{K}_i$. We define then $\xi_i = \xi_i^{\epsilon_i}$ and the induction step has been proved for i in the case 1.

Let us turn to case 2. We recall that the closure appearing in (9) is considered with respect to the topology of the uniform convergence on compact sets. Then there exist two sequences $F_a^j \in \mathscr{K}_i$ and $G_a^j \in \mathrm{co}\,\{\langle Z\colon Z\rangle \mid Z \in \mathrm{L}(\mathscr{K}_i)\}$ such that $F_a^j - G_a^j$ converges uniformly to Z_a on a neighborhood of the curve ξ_{i+1}. For every $j \in \mathbb{N}$ let $\xi_{i+1}^j \colon I \to Q$ be the

solution to

$$\nabla_{\dot{\xi}_{i+1}^j(t)}\dot{\xi}_{i+1}^j(t) - Y(t, \dot{\xi}_{i+1}^j(t))$$
$$= \sum_{a=1}^{N_{i+1}} \lambda_a(t)(F_a^j - G_a^j)(\xi_{i+1}^j(t))$$
$$(15)$$

satisfying $\dot{\xi}_{i+1}^j(0) = \dot{\xi}_{i+1}(0)$. Then, thanks to Lemma 2.5, ξ_{i+1}^j converges to ξ_{i+1} uniformly on I as j tends to infinity. Take $\bar{j}$ large enough such that

$$\mathrm{d}(\xi_{i+1}^{\bar{j}}(t),\xi_{i+1}(t)) < \frac{\epsilon}{2l}$$

for every $t \in I$. We can apply to $\xi_{i+1}^{\bar{j}}$ the same reasoning as in the first case. Then solutions to (15) are approximated by solutions to (14) replacing $\xi_i^{\epsilon_i}$ by $\xi_i^{\bar{j},\epsilon_i}$, F_a by $F_a^{\bar{j}}$ and $G_{a,b}$ by $G_{a,b}^{\bar{j}}$. In other words, there exist $C_i, \epsilon_{i,0} > 0$ such that $\mathrm{d}(\xi_i^{\bar{j},\epsilon_i}(t),\xi_{i+1}^{\bar{j}}(t)) < C_i\epsilon_i$ for every $t \in I$, $\epsilon_i \in (0,\epsilon_{i,0})$. Again, ϵ_i can be chosen in such a way that $C_i\epsilon_i < \frac{\epsilon}{2l}$ and we define $\xi_i = \xi_i^{\bar{j},\epsilon_i}$. Thus,

$$\begin{aligned}\mathrm{d}(\xi_i(t),\xi_{i+1}(t)) &\leq \mathrm{d}(\xi_i^{\bar{j},\epsilon_i}(t),\xi_{i+1}^{\bar{j}}(t))\\ &+ \mathrm{d}(\xi_{i+1}^{\bar{j}}(t),\xi_{i+1}(t)) < \tfrac{\epsilon}{l}\end{aligned}$$

and ξ_i admits a regular parameterization on $\mathscr{K}_i$. Hence, the induction step has been proved for i.

After the induction, we end up with a curve ξ_0 on Q admitting a regular parameterization on $\mathscr{K}_0$ and such that

$$\mathrm{d}(\xi_0(t),\gamma_{\mathrm{ref}}(t)) \leq \sum_{i=0}^{l-1} \mathrm{d}(\xi_i(t),\xi_{i+1}(t)) < l\frac{\epsilon}{l} = \epsilon$$

for every $t \in I$. (Recall that $\xi_l = \gamma_{\mathrm{ref}}$.) Moreover, by compactness of I,

$$\bar{\epsilon} = \epsilon - \max_{t\in I}\mathrm{d}(\xi_0(t),\gamma_{\mathrm{ref}}(t)) > 0.$$

As $\mathscr{K}_0 = \overline{\mathrm{span}_{\mathcal{C}^\infty(Q)}\mathscr{Y}}$, we conclude from Lemma 2.5 that there exist $\theta_{a,j} \in \mathcal{C}^\infty(Q)$, $a \in \{1,\dots,N_0\}$, $j \in \{1,\dots,k\}$, such that the solution to

$$\nabla_{\dot{\xi}(t)}\dot{\xi}(t) - Y(t, \dot{\xi}(t))$$
$$= \sum_{a=1}^{N_0}\sum_{j=1}^{k} \lambda_a(t)\theta_{a,j}(\xi(t))Y_j(\xi(t))$$

with initial condition $\dot{\xi}(0) = \dot{\xi}_0(0) = \dot{\gamma}_{\mathrm{ref}}(0)$ satisfies $\mathrm{d}(\xi(t),\xi_0(t)) < \bar{\epsilon}$ for every $t \in I$. Thus, $\mathrm{d}(\xi(t),\gamma_{\mathrm{ref}}(t)) < \epsilon$ for every $t \in I$.

Since ξ is an admissible trajectory for Σ, we conclude that γ_{ref} is trackable for Σ with the tracking control law given by $u_j(t) = \sum_{a=1}^{N_0}\lambda_a(t)\theta_{a,j}(\xi(t))$, $j = 1,\dots,k$. ∎

Corollary 4.5. *Let* $\Sigma = (Q, \nabla, Y, \mathscr{Y}, \mathbb{R}^k)$ *be a FACCS. Define the following set of vector fields on Q for $l \in \mathbb{N}$:*

$$\begin{aligned}
\mathscr{H}_0 &= \operatorname{span}_{\mathcal{C}^\infty(Q)} \mathscr{Y}, \\
\mathscr{H}_l &= \mathscr{H}_{l-1} - \operatorname{co}\{\langle Z : Z \rangle \mid Z \in \mathrm{L}(\mathscr{H}_{l-1})\}.
\end{aligned} \tag{16}$$

Fix a smooth reference trajectory $\gamma_{\mathrm{ref}} : I \to Q$. If for every $t \in I$ there exists $l \in \mathbb{N}$ such that $\mathscr{H}_l(\gamma_{\mathrm{ref}}(t)) = T_{\gamma_{\mathrm{ref}}(t)}Q$, then γ_{ref} is trackable. Therefore, if for every $q \in Q$ there exists $l \in \mathbb{N}$ such that $\mathscr{H}_l(q) = T_q Q$, then the control system Σ is trackable.

Proof. Let us prove by induction that

$$\mathscr{H}_i \subseteq \mathscr{K}_i \quad i \in \mathbb{N}_0.$$

It is trivial that $\mathscr{H}_0 \subseteq \mathscr{K}_0$ (see (9) and (16)).

The claim now is that if $\mathscr{H}_i \subseteq \mathscr{K}_i$, then $\mathscr{H}_{i+1} \subseteq \mathscr{K}_{i+1}$. By definition, an element in $\mathscr{H}_{i+1}$ is of the form $F - G$ with $F \in \mathscr{H}_i$ and $G \in \operatorname{co}\{\langle Z : Z \rangle \mid Z \in \mathrm{L}(\mathscr{H}_i)\}$. Since $\mathscr{H}_i \subseteq \mathscr{K}_i$, then $F \in \mathscr{K}_i$ and $\mathrm{L}(\mathscr{H}_i) \subseteq \mathrm{L}(\mathscr{K}_i)$. So $G \in \operatorname{co}\{\langle Z : Z \rangle \mid Z \in \mathrm{L}(\mathscr{K}_i)\}$. We can conclude that $\mathscr{H}_{i+1} \subseteq \mathscr{K}_{i+1}$.

By hypotheses, for every $t \in I$ there exists $l \in \mathbb{N}$ such that $\mathscr{H}_l(\gamma_{\mathrm{ref}}(t)) = T_{\gamma_{\mathrm{ref}}(t)}Q$. As $\mathscr{H}_l \subseteq \mathscr{K}_l$, we have $\mathscr{K}_l(\gamma_{\mathrm{ref}}(t)) = T_{\gamma_{\mathrm{ref}}(t)}Q$. The hypotheses of Theorem 4.4 are satisfied, so the result holds. ∎

Remark 4.6. If one follows the proof of Theorem 4.4 under the stronger hypotheses of Corollary 4.5, then each step of the induction procedure is of the type considered in case 1. This will be important in the next section, where we will turn such procedure in an algorithmic construction.

Corollary 4.7. *Let* $\Sigma = (Q, \nabla, Y, \mathscr{Y}, \mathbb{R}^k)$ *be a FACCS. Define the following sets of vector fields for $l \in \mathbb{N}$,*

$$\begin{aligned}
\mathscr{Z}_0 &= \mathscr{Y}, \\
\mathscr{Z}_l &= \mathscr{Z}_{l-1} \cup \{\langle Z_a : Z_b \rangle \mid Z_a, Z_b \in \mathscr{Z}_{l-1}\}.
\end{aligned} \tag{17}$$

If there exists $l \in \mathbb{N}$ such that $\operatorname{span}_{\mathbb{R}} \mathscr{Z}_l(q) = T_q Q$ for all $q \in Q$ and for each $i \in \{0, \dots, l-1\}$, for each $Z \in \mathscr{Z}_i$, $\langle Z : Z \rangle \in \operatorname{span}_{\mathcal{C}^\infty(Q)} \mathscr{Z}_i$, then the system Σ is trackable.

Proof. Let us prove by induction that

$$\operatorname{span}_{\mathcal{C}^\infty(Q)} \mathscr{Z}_i \subseteq \mathscr{K}_i, \quad i \in \mathbb{N}_0.$$

Once this inclusion is proved Theorem 4.4 guarantees the trackability of the system.

It is trivial by definition that $\operatorname{span}_{\mathcal{C}^\infty(Q)} \mathscr{Z}_0 \subseteq \mathscr{K}_0$.

Assume that $\operatorname{span}_{\mathcal{C}^\infty(Q)} \mathscr{Z}_i \subseteq \mathscr{K}_i$ and let us prove the inclusion for i. Since $\mathscr{K}_{i+1}$ is a convex cone by Proposition 4.1 and $\operatorname{span}_{\mathcal{C}^\infty(Q)} \mathscr{Z}_i \subset \mathscr{K}_{i+1}$, it is enough to prove that $\alpha \langle Z_a : Z_b \rangle$ belongs to $\mathscr{K}_{i+1}$ for $Z_a, Z_b \in \mathscr{Z}_i$ and $\alpha \in \mathcal{C}^\infty(Q)$. Thanks to (4) and to the hypotheses on the symmetric products of elements of $\mathscr{Z}_i$,

$$\begin{aligned}
\langle \alpha Z_a : Z_b \rangle &\in \alpha \langle Z_a : Z_b \rangle + \operatorname{span}_{\mathcal{C}^\infty(Q)} \mathscr{Z}_i, \\
\langle \alpha Z_a : \alpha Z_a \rangle, \langle Z_b : Z_b \rangle &\in \operatorname{span}_{\mathcal{C}^\infty(Q)} \mathscr{Z}_i.
\end{aligned}$$

Hence the symmetric product

$$\langle \alpha Z_a - Z_b : \alpha Z_a - Z_b \rangle$$

belongs to $-2\alpha \langle Z_a : Z_b \rangle + \operatorname{span}_{\mathcal{C}^\infty(Q)} \mathscr{Z}_i$. Thus, $\alpha \langle Z_a : Z_b \rangle$ belongs to $\operatorname{span}_{\mathcal{C}^\infty(Q)} \mathscr{Z}_i - \frac{1}{2} \langle \alpha Z_a - Z_b : \alpha Z_a - Z_b \rangle$, which is contained in $\mathscr{K}_{i+1}$. ∎

Remark 4.8. The proof above actually shows that, under the assumptions of Corollary 4.7, $\operatorname{span}_{\mathcal{C}^\infty(Q)} \mathscr{Z}_i \subseteq \mathscr{H}_i$. It is easy to check that, in addition, $\mathscr{H}_i = \operatorname{span}_{\mathcal{C}^\infty(Q)} \mathscr{Z}_i$.

Remark 4.9. The main interest of Corollary 4.7 is that its hypotheses are formulated in terms of a finite set of vector fields, in contrast with the infinite family of vector fields considered in Theorem 4.4 and Corollary 4.5.

4.1 Examples

Let us consider some examples of mechanical systems for which the above results guarantee the trackability, but Theorem 3.3 could not guarantee it.

4.1.1 Hovercraft

Consider an elliptic hovercraft moving on the surface of a fluid, identified with $\mathbb{R}^2$. The configuration manifold is $Q = S^1 \times \mathbb{R}^2$ with local coordinates (θ, x_1, x_2) where θ is the attitude and (x_1, x_2) is the position of the center of symmetry of the hovercraft. Let ω and (v_1, v_2) be the standard angular and linear velocity, respectively, of the hovercraft with respect to a body–fixed coordinate frame attached at the center of symmetry of the body and whose axes coincide with those of the ellipse. Assume that the center of mass according to that body–fixed coordinate frame is on the horizontal axis and is different

from the center of symmetry. Then the added inertia matrix is the following 3×3 symmetric matrix:

$$\mathcal{M} = \left(\begin{array}{c|c|c} a & 0 & c \\ \hline 0 & e & 0 \\ \hline c & 0 & e \end{array} \right),$$

with $a, c, e > 0$ (see [9, 13] for more details). Denote the corresponding impulse vector by (Π, P_1, P_2) that is related to the velocities through the inertia matrix $\mathcal{M}$ as follows

$$\begin{pmatrix} \Pi \\ P_1 \\ P_2 \end{pmatrix} = \mathcal{M} \begin{pmatrix} \omega \\ v_1 \\ v_2 \end{pmatrix}.$$

The dynamics of the systems governed by the Kirchhoff equations in dimension 2 with two controls are

$$\begin{pmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \omega \\ x \\ y \end{pmatrix},$$

$$\begin{pmatrix} \dot{\omega} \\ \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} = \mathcal{M}^{-1} \begin{pmatrix} P \cdot v^\perp \\ \omega P^\perp \end{pmatrix} + \begin{pmatrix} u_1 \\ 0 \\ u_2 \end{pmatrix},$$

where $w^\perp = (-w_2, w_1)$ denotes the rotation by $\pi/2$ of a vector $w = (w_1, w_2)$ in $\mathbb{R}^2$. The control vector fields are $Y_1 = (1, 0, 0)$ and $Y_2 = (0, 0, 1)$. They correspond to the external torque, usually called yaw, and the external force, usually called surge, applied to the body. Let $\mathscr{Y} = \{Y_1, Y_2\}$.

The drift term Z appearing in (3) is, for the mechaincal system considered here, the vector field corresponding to the uncontrolled Kirchhoff's equations (notice that $Y = 0$). Using (4) we can compute $\langle Y_1 : Y_1 \rangle = (0, 2c/e, 0)$. The sufficient conditions for tracking given by Theorem 3.3 are not satisfied because $\langle Y_1 : Y_1 \rangle \notin \mathrm{span}_{\mathcal{C}^\infty(Q)} \{Y_1, Y_2\}$.

However, due to Corollary 4.5 tracking is possible because

$$\mathscr{H}_1(q) = T_q Q \;\forall\, q \in Q.$$

Indeed, for $w_1, w_2 \in \mathcal{C}^\infty(Q)$,

$$\langle w_1 Y_1 + w_2 Y_2 : w_1 Y_1 + w_2 Y_2 \rangle =$$

$$\frac{1}{e}(0, 2cw_1^2 + 2ew_1w_2, 0) + 2 \overbrace{\sum_{i,j=1}^{2} w_i Y_i(w_j) Y_j}^{\in \mathrm{span}_{\mathcal{C}^\infty(Q)} \mathscr{Y}},$$

where $Y_i(w_j)$ denotes the Lie derivative of w_j with respect to Y_i. In particular, taking as w_1 any nonzero constant function, we get

$$\mathrm{span}_{\mathcal{C}^\infty(Q)}(0, 1, 0) \subset \mathscr{H}_0 - \mathrm{co}\left\{ \langle Z : Z \rangle \mid Z \in \mathrm{L}(\mathscr{H}_0) \right\}.$$

So we conclude that $\mathscr{H}_1(q) = T_q Q$ for all $q \in Q$.

4.1.2 Submarine

Let us apply Corollary 4.7 to determine the trackability of a particular control system describing the motion of a submarine. The system corresponds to the case $\gamma = 0$ considered in [6]. It models a neutrally buoyant ellipsoid vehicle immersed in a infinite volume fluid that is inviscid, incompressible and whose motion is irrotational. The dynamics are obtained through Kirchhoff equations [9] and have a particularly simple form due to some symmetry assumption on the distribution of mass (see [6] for details and also [14] for general overview of control motion in a potential fluid).

Consider the coordinates (ω, v) for the angular and linear velocity of the ellipsoid with respect to a body–fixed coordinate frame. Then the impulse (Π, P) of the system is given by

$$\begin{pmatrix} \Pi \\ P \end{pmatrix} = \mathcal{M} \begin{pmatrix} \omega \\ v \end{pmatrix}$$

where, under the symmetry assumptions mentioned above,

$$\mathcal{M} = \mathrm{diag}(J_1, J_1, J_3, M_1, M_2, M_3)$$

with $M_1 \neq M_2$, where $\mathrm{diag}(J_1, J_1, J_3)$ is the usual inertia matrix and M_1, M_2, M_3 take into account the mass of the submarine and the added masses due to the action of the fluid.

The configuration manifold Q for this problem is the Special Euclidean group or the group of rigid motions $SE(3)$, which is homeomorphic to $\mathbb{R}^3 \times SO(3)$. Let $(r, A) \in SE(3)$ be the position and the attitude of the ellipsoid. Denote by $S: \mathbb{R}^3 \to \mathfrak{so}(3)$ the linear bijection between $\mathbb{R}^3$ and the linear algebra $\mathfrak{so}(3)$ of $SO(3)$ such that

$$S(x_1, x_2, x_3) = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$$

The dynamics of the controlled system are given by

$$\frac{\mathrm{d}r}{\mathrm{d}t} = Av, \quad \frac{\mathrm{d}A}{\mathrm{d}t} = AS(\omega), \qquad (18)$$

and

$$\frac{\mathrm{d}\Pi}{\mathrm{d}t} = \Pi \times \omega + P \times v + \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}, \quad \frac{\mathrm{d}P}{\mathrm{d}t} = P \times \omega + \begin{pmatrix} 0 \\ 0 \\ u_3 \end{pmatrix}. \qquad (19)$$

The control vector fields are $Y_1 = \partial/\partial\Pi_1$, $Y_2 = \partial/\partial\Pi_2$ and $Y_3 = \partial/\partial P_3$. They correspond to a linear acceleration along one of the three axes of the

submarine and to two angular accelerations around the other two axes.

Due to Theorem 1.2 in [6], we know that system (18)–(19) is trackable.

This case cannot be recovered from Theorem 3.3 because

$$\mathrm{Sym}^{(1)}(\mathscr{Y}) \neq TQ.$$

Indeed, it can be computed that

$$\langle Y_1 \colon Y_2 \rangle = 0, \langle Y_2 \colon Y_3 \rangle = \frac{M_3}{M_1} \frac{\partial}{\partial P_1},$$

$$\langle Y_1 \colon Y_3 \rangle = -\frac{M_3}{M_2} \frac{\partial}{\partial P_2}.$$

However, this case is covered by Corollary 4.7 with $l = 2$, because

$$0 = \langle Y_1 \colon Y_1 \rangle = \langle Y_2 \colon Y_2 \rangle = \langle Y_3 \colon Y_3 \rangle =$$
$$= \langle\langle Y_2 \colon Y_3 \rangle \colon \langle Y_2 \colon Y_3 \rangle\rangle = \langle\langle Y_1 \colon Y_3 \rangle \colon \langle Y_1 \colon Y_3 \rangle\rangle,$$

$$\langle\langle Y_2 \colon Y_3 \rangle \colon \langle Y_1 \colon Y_3 \rangle\rangle = -\frac{M_3^2}{J_3}\left(\frac{1}{M_1} - \frac{1}{M_2}\right)\frac{\partial}{\partial \Pi_3}.$$

Thus, $\langle Z \colon Z \rangle = 0$ for every $Z \in \mathscr{Z}_2$ and $T_q Q = \mathrm{span}_{\mathbb{R}} \mathscr{Z}_2(q) = \mathrm{span}_{\mathbb{R}}\{Y_1(q), Y_2(q), Y_3(q), \langle Y_2 \colon Y_3 \rangle(q), \langle Y_1 \colon Y_3 \rangle(q), \langle\langle Y_2 \colon Y_3 \rangle \colon \langle Y_1 \colon Y_3 \rangle\rangle(q)\}$ for all $q \in Q$.

The model studied in [6] can therefore be handled with the techniques proposed here. In particular, we can obtain for it one–parameter tracking control laws, as explained in the next section.

5 One–parameter tracking control laws

The aim of this section is to provide an algorithmic implementation of the results obtained in the previous one about the existence of controls yielding tracking. This will be done separately under the hypotheses of Corollaries 4.5 and 4.7, using two different algorithms. The first one is based on the procedure proposed in the proof of Theorem 4.4, while the second one exploits the construction proposed in [5] and recalled in Theorem 3.3.

In both cases we will consider a reference trajectory $\gamma_{\mathrm{ref}} : I \to Q$, which is assumed to be of class $\mathcal{C}^\infty$.

A simple, albeit crucial, fact that will be used several times in the following sections is stated in the lemma below.

Lemma 5.1. *If $f \colon \mathbb{R} \times I \to \mathbb{R}$, $(\tau, s) \mapsto f(\tau, s)$, is smooth on $\mathbb{R} \times I$ and T-periodic with respect to τ,*

then

$$\int_0^t f(s/\hat{\epsilon}, s)\mathrm{d}s = \int_0^t \bar{f}(s)ds + O(\hat{\epsilon}\|f\|_\infty)$$
$$+ O(\hat{\epsilon}\|\partial_2 f\|_\infty)$$

for $\hat{\epsilon}$ close to zero, where $\bar{f}(s) = (1/T)\int_0^T f(\tau, s)\mathrm{d}\tau$ and ∂_2 denotes the partial derivative with respect to the second variable.

5.1 Case $\mathscr{H}$

As noticed in Remark 4.6, the hypotheses of Corollary 4.5 guarantee that every step of the induction argument proposed in the proof of Theorem 4.4 falls in the framework of case 1 (see page 6). Hence, starting from a parameterization

$$\nabla_{\dot{\gamma}_{\mathrm{ref}}(t)}\dot{\gamma}_{\mathrm{ref}}(t) - Y(t, \dot{\gamma}_{\mathrm{ref}}(t)) = \sum_{a=1}^{N_l} \lambda_a(t)Z_a^l(\gamma_{\mathrm{ref}}(t))$$

of $\gamma_{\mathrm{ref}} \colon I \to Q$, with $Z_a^l \in \mathscr{H}_l$ and λ_a smooth and non-negative on I for every $a = 1, \dots, N_l$, we can construct algorithmically a l-parameter family of admissible trajectories $\xi^{\epsilon_1, \dots, \epsilon_l}$ of Σ with $\epsilon_1, \dots, \epsilon_l > 0$ such that $\xi_l = \gamma_{\mathrm{ref}}$, the uniform limit

$$\xi_i^{\epsilon_1, \dots, \epsilon_{l-i}} = \lim_{\epsilon_l \to 0} \lim_{\epsilon_{l-1} \to 0} \cdots \lim_{\epsilon_{l-i+1} \to 0} \xi^{\epsilon_1, \dots, \epsilon_l} \quad (20)$$

exists for every $i = 1, \dots, l$ and every $\epsilon_1, \dots, \epsilon_{l-i} > 0$ and satisfies

$$\nabla_{\dot{\xi}_i^{\epsilon_1, \dots, \epsilon_{l-i}}(t)}\dot{\xi}_i^{\epsilon_1, \dots, \epsilon_{l-i}}(t) - Y(t, \dot{\xi}_i^{\epsilon_1, \dots, \epsilon_{l-i}}(t))$$
$$\in \mathscr{H}_i(\xi_i^{\epsilon_1, \dots, \epsilon_{l-i}}(t)). \quad (21)$$

We also write $\xi_0^{\epsilon_1, \dots, \epsilon_l}$ for $\xi^{\epsilon_1, \dots, \epsilon_l}$. It is important to notice that the order of the limits in (20) cannot in general be reversed.

Let us recall to which extent the construction is algorithmic. Fix any injective map $j \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. By backward recursion on i, if $\xi_i^{\epsilon_1, \dots, \epsilon_{l-i}}$ satisfies

$$\nabla_{\dot{\xi}_i^{\epsilon_1, \dots, \epsilon_{l-i}}(t)}\dot{\xi}_i^{\epsilon_1, \dots, \epsilon_{l-i}}(t) - Y(t, \dot{\xi}_i^{\epsilon_1, \dots, \epsilon_{l-i}}(t))$$
$$= \sum_{a=1}^{N_i} \lambda_a^{\epsilon_1, \dots, \epsilon_{l-i}}(t)Z_a^i(\xi_i^{\epsilon_1, \dots, \epsilon_{l-i}}(t)) \quad (22)$$

with $\lambda_a^{\epsilon_1, \dots, \epsilon_{l-i}} \in \mathcal{C}(I, [0, +\infty))$ and $Z_a^i \in \mathscr{H}_i$, then

$\xi_{i-1}^{\epsilon_1,\ldots,\epsilon_{l-i+1}}$ is defined as the solution to

$$\nabla_{\dot{\xi}_{i-1}^{\epsilon_1,\ldots,\epsilon_{l-i+1}}(t)}\dot{\xi}_{i-1}^{\epsilon_1,\ldots,\epsilon_{l-i+1}}(t)$$
$$- Y(t,\dot{\xi}_{i-1}^{\epsilon_1,\ldots,\epsilon_{l-i+1}}(t))$$
$$= \sum_{a=1}^{N_i} \left(\lambda_a^{\epsilon_1,\ldots,\epsilon_{l-i}}(t) F_a^{i-1}(\xi_{i-1}^{\epsilon_1,\ldots,\epsilon_{l-i+1}}(t)) + \right.$$
$$\frac{1}{\epsilon_{l-i+1}} \sum_{b=1}^{\hat{N}_a^i} \psi_{j(a,b)}\left(\frac{t}{\epsilon_{l-i+1}}\right) \sqrt{\lambda_a^{\epsilon_1,\ldots,\epsilon_{l-i}}(t)\alpha_{a,b}^{i-1}}$$
$$\left. G_{a,b}^{i-1}(\xi_{i-1}^{\epsilon_1,\ldots,\epsilon_{l-i+1}}(t)) \right) \tag{23}$$

with $\dot{\xi}_{i-1}^{\epsilon_1,\ldots,\epsilon_{l-i+1}}(0) = \dot{\gamma}_{\mathrm{ref}}(0)$, where

$$Z_a^i = F_a^{i-1} - \sum_{b=1}^{\hat{N}_a^i} \alpha_{a,b}^{i-1}\langle G_{a,b}^{i-1} : G_{a,b}^{i-1}\rangle$$

and $F_a^{i-1} \in \mathscr{H}_{i-1}$, $G_{a,b}^{i-1} \in \mathrm{L}(\mathscr{H}_{i-1})$, $\alpha_{a,b}^{i-1} \geq 0$. Recall that $(\psi_j)_{j\in\mathbb{N}}$ is a Λ_T-orthonormal and zero-mean sequence, for some $T > 0$. Each $\lambda_a^{\epsilon_1,\ldots,\epsilon_{l-i+1}}(\cdot)$ is either equal to some $\lambda_{\tilde{a}}^{\epsilon_1,\ldots,\epsilon_{l-i}}(\cdot)$ or is of the form

$$\frac{\sqrt{\lambda_{\tilde{a}}^{\epsilon_1,\ldots,\epsilon_{l-i}}(\cdot)\alpha_{\tilde{a},b}^{i-1}}}{\epsilon_{l-i+1}} \max\left\{\upsilon\psi_{j(\tilde{a},b)}\left(\frac{\cdot}{\epsilon_{l-i+1}}\right), 0\right\}$$

with υ equal to 1 or -1.

We choose $(\psi_j)_{j\in\mathbb{N}}$ as follows: we require ψ_1 to be positive in $(0, T/2)$ and to annihilate, together with all its derivatives, at 0 and $T/2$. We also require it to satisfy

$$\psi_1\left(\frac{T}{2} - t\right) = \psi_1(t), \quad t \in \left[0, \frac{T}{2}\right],$$

and we extend it by

$$\psi_1(t) = -\psi_1\left(t - \frac{T}{2}\right), \quad t \in \left[\frac{T}{2}, T\right],$$

and by T-periodicity over $\mathbb{R}$. Finally we normalize ψ_1 in such a way that $\Lambda_T(\psi_1, \psi_1) = 1$. Then we define ψ_j by

$$\psi_j(t) = 2^j \psi_1(2^j t).$$

Such choice of $(\psi_j)_{j\in\mathbb{N}}$ is motivated by the property that, for every choice of $j \in \mathbb{N}$, $l \in \mathbb{N}_0$ and $\upsilon \in \{-1, 1\}$, the function $t \mapsto \sqrt[2^l]{\max\{\upsilon\psi_j(t), 0\}}$ is smooth. In particular, by backward recursion, each $\lambda_a^{\epsilon_1,\ldots,\epsilon_i}$ is smooth and is the product of functions of the type $\sqrt[2^l]{\frac{1}{\epsilon_h}\max\left\{\upsilon\psi_j\left(\frac{\cdot}{\epsilon_h}\right), 0\right\}}$ and of $\sqrt[2^m]{\lambda_b}$ for some $m \in \mathbb{N}_0$ and some $b \in \{1,\ldots,N_l\}$.

An important consequence of this factorization, which will be exploited in the proof of Theorem 5.2, is that the derivatives of $\sqrt{\lambda_a^{\epsilon_1,\ldots,\epsilon_i}}$ with respect to time can be bounded by a finite constant depending explicitly on $\epsilon_1,\ldots,\epsilon_{l-1}$.

The smoothness of $\lambda_a^{\epsilon_1,\ldots,\epsilon_i}$, moreover, allows us to consider $\lambda_a^{\epsilon_1,\ldots,\epsilon_i}Z_a^{l-i}$ as a smooth vector field on the extended manifold $\mathbb{R} \times Q$ and, similarly, $\lambda_a^{\epsilon_1,\ldots,\epsilon_{l-i}}(Z_a^i)^V$ as a smooth vector field on the extended manifold $\mathbb{R} \times TQ$.

Summing up, the trajectories of the l-parameter family $\xi_0^{\epsilon_1,\ldots,\epsilon_l} = \xi^{\epsilon_1,\ldots,\epsilon_l}$ are driven by a l-parameter family of control laws $u^{\epsilon_1,\ldots,\epsilon_l} \in \mathcal{C}^\infty(I, \mathbb{R}^k)$ depending smoothly on $\epsilon_1,\ldots,\epsilon_l$. The construction of Theorem 4.4 can be summarized as follows: given $\epsilon > 0$, if

$$0 < \epsilon_l \ll \epsilon_{l-1} \ll \cdots \ll \epsilon_1 \ll 1 \tag{24}$$

then $\mathrm{d}(\xi^{\epsilon_1,\ldots,\epsilon_l}(t), \gamma_{\mathrm{ref}}(t)) < \epsilon$ for every $t \in I$. Our aim is here to quantify the relations in (24). More precisely, we introduce $l - 1$ functions $\eta_2,\ldots,\eta_l : (0, +\infty) \to (0, +\infty)$ and we look for asymptotic conditions on their convergence to zero at zero such that

$$\lim_{\epsilon\to 0}\mathrm{d}(\xi^{\epsilon,\eta_2(\epsilon),\eta_3\circ\eta_2(\epsilon),\ldots,\eta_l\circ\cdots\circ\eta_2(\epsilon)}(t), \gamma_{\mathrm{ref}}(t)) = 0 \tag{25}$$

uniformly with respect to $t \in I$. Let $\hat{\eta}_i = \eta_i \circ \cdots \circ \eta_2$ for $i = 2,\ldots,l$ and define $\hat{\eta}_1$ as the identity on $(0, +\infty)$. We say that $\epsilon \mapsto u^{\hat{\eta}_1(\epsilon),\ldots,\hat{\eta}_l(\epsilon)}$ is a *one–parameter tracking control law for* γ_{ref} if (25) holds true.

Theorem 5.2. *Let* $\Sigma = (Q, \nabla, Y, \mathscr{Y}, \mathbb{R}^k)$ *be a FACCS. Let* $\mathscr{H}_i$, $i \in \mathbb{N}_0$, *be defined as in (16). Fix a reference trajectory* $\gamma_{\mathrm{ref}} \in \mathcal{C}^\infty(I, Q)$ *and assume that there exists* $l \in \mathbb{N}$ *such that*

$$\nabla_{\dot{\gamma}_{\mathrm{ref}}(t)}\dot{\gamma}_{\mathrm{ref}}(t) - Y(t, \dot{\gamma}_{\mathrm{ref}}(t)) \in \mathscr{H}_l(\gamma_{\mathrm{ref}}(t))$$

for every $t \in I$. *Construct* $\xi^{\epsilon_1,\ldots,\epsilon_l}$, $u^{\epsilon_1,\ldots,\epsilon_l}$ *and* $\hat{\eta}_i$ *as above. If* $\eta_i : (0, +\infty) \to (0, +\infty)$ *satisfies* $\limsup_{\epsilon\to 0}\eta_i(\epsilon)/\epsilon^3 < \infty$ *for every* $i = 2,\ldots,l$, *then* $\epsilon \mapsto u^{\hat{\eta}_1(\epsilon),\ldots,\hat{\eta}_l(\epsilon)}$ *is a one–parameter tracking control law for* γ_{ref}.

Proof. The first step of the proof consists in estimating the order with respect to ϵ of the L^∞-norm of the time-dependent parameters appearing in the parameterization (22). We write ϵ_i for $\hat{\eta}_i(\epsilon)$. Denoting by C any constant not depending on the ϵ_j, it is easy to check by backward induction on $i = 0,\ldots,l$ that

$$\|\lambda_a^{\epsilon_1,\ldots,\epsilon_i}\|_\infty \leq \frac{C}{\sqrt[2^{i-1}]{\epsilon_1}\sqrt[2^{i-2}]{\epsilon_2}\cdots\sqrt{\epsilon_{i-1}}\epsilon_i}.$$

Exploiting the factorization of $\lambda_a^{\epsilon_1,\ldots,\epsilon_i}$ described above we get, in addition,

$$\left\|\frac{\mathrm{d}^m}{\mathrm{d}t^m}\sqrt{\lambda_a^{\epsilon_1,\ldots,\epsilon_i}}\right\|_\infty \leq \frac{C}{\sqrt[2]{\epsilon_1}\,\sqrt[2^{i-1}]{\epsilon_2}\cdots\sqrt[4]{\epsilon_{i-1}}\,\epsilon_i^{m+\frac{1}{2}}}$$

for $m \in \mathbb{N}_0$.

Consider the extended system on $\mathbb{R} \times TQ$ associated with (3) where the time is the new variable. In the following computations we write, using the notation introduced in (23),

$$\begin{aligned}
X^e &= (1, Z + Y^V), \\
\Phi_a^i &= (0, (F_a^i)^V), \\
\Gamma_{a,b}^i &= (0, (G_{a,b}^i)^V),
\end{aligned}$$

all seen as vector fields on $\mathbb{R} \times TQ$. We also define $\gamma_{\mathrm{ref}}^e(0) = (0, \dot\gamma_{\mathrm{ref}}(0))$ and, given a smooth function $\lambda : I \to \mathbb{R}$, we write λ^V to denote a smooth function on $\mathbb{R} \times TQ$ such that $\lambda^V(t,v) = \lambda(t)$ for every $t \in I$ and every $v \in TQ$. In particular, we define

$$\begin{aligned}
\theta_{a,b}^i &= \left(\sqrt{\lambda_a^{\epsilon_1,\ldots,\epsilon_i}\alpha_{a,b}^{l-i-1}}\right)^V, \\
\dot\theta_{a,b}^i &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\lambda_a^{\epsilon_1,\ldots,\epsilon_i}\alpha_{a,b}^{l-i-1}}\right)^V.
\end{aligned}$$

Then, applying (5) and (6),

$$\begin{aligned}
&(t, \dot\xi_0^{\epsilon_1,\ldots,\epsilon_l}(t)) = \\
&\overrightarrow{\exp}\int_0^t (X^e + \sum_{a=1}^{N_0}\lambda_a^{\epsilon_1,\ldots,\epsilon_l}(s)(0,(Z_a^0)^V))\mathrm{d}s\,\gamma_{\mathrm{ref}}^e(0) \\
&= \overrightarrow{\exp}\int_0^t \Big(X^e + \sum_{a=1}^{N_1}\lambda_a^{\epsilon_1,\ldots,\epsilon_{l-1}}(s)\Phi_a^0 \\
&+ \frac{1}{\epsilon_l}\sum_{a=1}^{N_1}\sum_{b=1}^{\hat N_a^1}\psi_{j(a,b)}(\tfrac{s}{\epsilon_l})\sqrt{\lambda_a^{\epsilon_1,\ldots,\epsilon_{l-1}}(s)\alpha_{a,b}^0} \\
&\Gamma_{a,b}^0\Big)\mathrm{d}s\,\gamma_{\mathrm{ref}}^e(0) \\[4pt]
&= \overrightarrow{\exp}\int_0^t \frac{1}{\epsilon_l}\sum_{a=1}^{N_1}\sum_{b=1}^{\hat N_a^1}\psi_{j(a,b)}(\tfrac{s}{\epsilon_l})\theta_{a,b}^{l-1}\Gamma_{a,b}^0\mathrm{d}s\circ \\
&\overrightarrow{\exp}\int_0^t \Big(X^e + \sum_{a=1}^{N_1}\lambda_a^{\epsilon_1,\ldots,\epsilon_{l-1}}(s)\Phi_a^0 \\
&+ \sum_{a=1}^{N_1}\sum_{b=1}^{\hat N_a^1}\int_0^s \frac{1}{\epsilon_l}\psi_{j(a,b)}(\tfrac{s_0}{\epsilon_l})\mathrm{d}s_0(\theta_{a,b}^{l-1}[\Gamma_{a,b}^0, X^e] \\
&- \dot\theta_{a,b}^{l-1}\Gamma_{a,b}^0) - \sum_{a,a'=1}^{N_1}\sum_{b=1}^{\hat N_a^1}\sum_{b'=1}^{\hat N_{a'}^1}\int_0^s \frac{1}{\epsilon_l}\psi_{j(a,b)}(\tfrac{s_1}{\epsilon_l}) \\
&\int_0^{s_1}\frac{1}{\epsilon_l}\psi_{j(a',b')}(\tfrac{s_0}{\epsilon_l})\mathrm{d}s_0\mathrm{d}s_1\theta_{a,b}^{l-1}\theta_{a',b'}^{l-1} \\
&(0, \langle G_{a,b}^0 \colon G_{a',b'}^0\rangle^V)\Big)\mathrm{d}s\gamma_{\mathrm{ref}}^e(0).
\end{aligned}$$

From now on, let us denote by $\mathscr{V}$ any *vertical flow*, i.e., any flow on $\mathbb{R} \times TQ$ preserving the base point on Q.

Notice that, by construction,

$$\begin{aligned}
&\sum_{a=1}^{N_1}\lambda_a^{\epsilon_1,\ldots,\epsilon_{l-1}}(s)\Phi_a^0 + \\
&- \sum_{a,a'=1}^{N_1}\sum_{b=1}^{\hat N_a^1}\sum_{b'=1}^{\hat N_{a'}^1}\Lambda_T(\psi_{j(a,b)}, \psi_{j(a',b')}) \\
&\sqrt{\lambda_a^{\epsilon_1,\ldots,\epsilon_{l-1}}(s)\lambda_{a'}^{\epsilon_1,\ldots,\epsilon_{l-1}}(s)\alpha_{a,b}^0\alpha_{a',b'}^0} \\
&(0, \langle G_{a,b}^0 \colon G_{a',b'}^0\rangle^V) = \sum_{a=1}^{N_1}\lambda_a^{\epsilon_1,\ldots,\epsilon_l}(s)(0, (Z_a^1)^V).
\end{aligned}$$

Then,

$$\begin{aligned}
&(t, \dot\xi_0^{\epsilon_1,\ldots,\epsilon_l}(t)) = \\
&\mathscr{V}\circ\overrightarrow{\exp}\int_0^t \Big(X^e + \sum_{a=1}^{N_1}\lambda_a^{\epsilon_1,\ldots,\epsilon_{l-1}}(s)(0, (Z_a^1)^V) \\
&+ \sum_{a=1}^{N_1}\sum_{b=1}^{\hat N_a^1}\int_0^s \frac{1}{\epsilon_l}\psi_{j(a,b)}(\tfrac{s_0}{\epsilon_l})\mathrm{d}s_0(\theta_{a,b}^{l-1}[\Gamma_{a,b}^0, X^e] \\
&- \dot\theta_{a,b}^{l-1}\Gamma_{a,b}^0) \\
&+ \sum_{a,a'=1}^{N_1}\sum_{b=1}^{\hat N_a^1}\sum_{b'=1}^{\hat N_{a'}^1}(\Lambda_T(\psi_{j(a,b)}, \psi_{j(a',b')}) \\
&- \int_0^s \frac{1}{\epsilon_l}\psi_{j(a,b)}(\tfrac{s_1}{\epsilon_l})\int_0^{s_1}\frac{1}{\epsilon_l}\psi_{j(a',b')}(\tfrac{s_0}{\epsilon_l})\mathrm{d}s_0\mathrm{d}s_1) \\
&(0, \langle G_{a,b}^0 \colon G_{a',b'}^0\rangle^V)\theta_{a,b}^{l-1}\theta_{a',b'}^{l-1}\Big)\mathrm{d}s\gamma_{\mathrm{ref}}^e(0).
\end{aligned}$$

Applying iteratively the same computation as above, one ends up with

$$\begin{aligned}
&(t, \dot\xi_0^{\epsilon_1,\ldots,\epsilon_l}(t)) = \\
&\mathscr{V}\circ\overrightarrow{\exp}\int_0^t (X^e + \lambda_a(s)(0, (Z_a^l)^V) + \mathscr{T}(s))\,\mathrm{d}s\,\gamma_{\mathrm{ref}}^e(0),
\end{aligned}$$

where $\mathscr{T}(s)$ is a sum of terms of the form $\zeta(s)V$ where V is a vector field on $\mathbb{R} \times TQ$ independent of the ϵ_j, while $\zeta : I \to \mathbb{R}$ is smooth, depends on the ϵ_j and is of one of the following four types:

$$\begin{aligned}
\zeta_1(s) &= \int_0^s \frac{1}{\epsilon_i}\psi_{j(a,b)}(\tfrac{s_0}{\epsilon_i})\mathrm{d}s_0\sqrt{\lambda_a^{\epsilon_1,\ldots,\epsilon_{i-1}}(s)\alpha_{a,b}^{l-i}}, \\
\zeta_2(s) &= -\int_0^s \frac{1}{\epsilon_i}\psi_{j(a,b)}(\tfrac{s_0}{\epsilon_i})\mathrm{d}s_0\frac{\mathrm{d}}{\mathrm{d}s}\sqrt{\lambda_a^{\epsilon_1,\ldots,\epsilon_{i-1}}(s)\alpha_{a,b}^{l-i}}, \\
\zeta_3(s) &= \frac{1}{1+\delta_{bc}}\Big[2(\Lambda_T(\psi_{j(a,b)}, \psi_{j(a',b')})) \\
&- \left(\int_0^s \frac{1}{\epsilon_i}\psi_{j(a,b)}(\tfrac{s_0}{\epsilon_i})\mathrm{d}s_0\right)\left(\int_0^s \frac{1}{\epsilon_i}\psi_{j(a',b')}(\tfrac{s_0}{\epsilon_i})\mathrm{d}s_0\right)\Big] \\
&\sqrt{\lambda_a^{\epsilon_1,\ldots,\epsilon_{i-1}}(s)\lambda_{a'}^{\epsilon_1,\ldots,\epsilon_{i-1}}(s)\alpha_{a,b}^{l-i}\alpha_{a',b'}^{l-i}}
\end{aligned}$$

for $i = 1, \ldots, l$, or

$$\begin{aligned}
\zeta_4(s) &= -\left(\int_0^s \frac{1}{\epsilon_j}\psi_{j(a,b)}(\tfrac{s_0}{\epsilon_j})\mathrm{d}s_0\right) \\
&\left(\int_0^s \frac{1}{\epsilon_i}\psi_{j(a',b')}(\tfrac{s_0}{\epsilon_i})\mathrm{d}s_0\right) \\
&\sqrt{\lambda_a^{\epsilon_1,\ldots,\epsilon_{j-1}}(s)\lambda_{a'}^{\epsilon_1,\ldots,\epsilon_{i-1}}(s)\alpha_{a,b}^{l-j}\alpha_{a',b'}^{l-i}}
\end{aligned}$$

for $i > j$, $i, j \in \{1, \ldots, l\}$.

According to Lemma 2.4, the theorem is proved if we show that, for every ζ of one of the four types introduced above, $\int_0^t \zeta(s)\mathrm{d}s$ converges to zero uniformly with respect to $t \in I$ as ϵ goes to zero.

This can be done by applying Lemma 5.1. Taking for instance

$$f(\tau, s) = \int_0^\tau \psi_{j(a,b)}(s_0)\mathrm{d}s_0\sqrt{\lambda_a^{\epsilon_1,\ldots,\epsilon_{i-1}}(s)\alpha_{a,b}^{l-i}},$$

and $\hat\epsilon = \epsilon_i$ leads to

$$\int_0^t \zeta_1(s)\mathrm{d}s \leq C\frac{\epsilon_i}{\sqrt[2^{i-1}]{\epsilon_1}\,\sqrt[2^{i-2}]{\epsilon_2}\cdots\sqrt[4]{\epsilon_{i-2}}\,\epsilon_{i-1}^{\frac{3}{2}}}.$$

Similarly,

$$\int_0^t \zeta_2(s)\mathrm{d}s \leq C\frac{\epsilon_i}{\sqrt[2^{i-1}]{\epsilon_1}\,\sqrt[2^{i-2}]{\epsilon_2}\cdots\sqrt[4]{\epsilon_{i-2}}\,\epsilon_{i-1}^{\frac{5}{2}}}.$$

Taking

$$f(\tau, s) = \frac{\left(\int_0^\tau \psi_{j(a,b)}(s_0)\mathrm{d}s_0\right)\left(\int_0^\tau \psi_{j(a',b')}(s_0)\mathrm{d}s_0\right)}{\sqrt{\lambda_a^{\epsilon_1,\dots,\epsilon_{i-1}}(s)\lambda_{a'}^{\epsilon_1,\dots,\epsilon_{i-1}}(s)\alpha_{a,b}^{l-i}\alpha_{a',b'}^{l-i}}}$$

we obtain

$$\int_0^t \zeta_3(s)\mathrm{d}s \le C\frac{\epsilon_i}{2^{i-2}\sqrt{\epsilon_1}\,2^{i-3}\sqrt{\epsilon_2}\cdots\sqrt{\epsilon_{i-2}}\epsilon_{i-1}^2}.$$

Finally, with

$$f(\tau, s) = \frac{\left(\int_0^{s/\epsilon_j} \psi_{j(a,b)}(s_0)\mathrm{d}s_0\right)\left(\int_0^\tau \psi_{j(a',b')}(s_0)\mathrm{d}s_0\right)}{\sqrt{\lambda_a^{\epsilon_1,\dots,\epsilon_{j-1}}(s)\lambda_{a'}^{\epsilon_1,\dots,\epsilon_{i-1}}(s)\alpha_{a,b}^{l-j}\alpha_{a',b'}^{l-i}}}$$

we get

$$\int_0^t \zeta_4(s)\mathrm{d}s \le C\frac{\frac{\epsilon_i}{2^{j-1}\sqrt{\epsilon_1}\,2^{j-2}\sqrt{\epsilon_2}\cdots\sqrt[4]{\epsilon_{j-1}}\sqrt{\epsilon_{j-1}}}}{\frac{1}{2^{i-1}\sqrt{\epsilon_1}\,2^{i-2}\sqrt{\epsilon_2}\cdots\sqrt[4]{\epsilon_{i-2}}\epsilon_{i-1}^{\frac{3}{2}}}}.$$

Notice that the upper bound for $\int_0^t \zeta_2(s)\mathrm{d}s$ is the one growing faster as ϵ goes to zero. Indeed, in order to compare it with the one for $\int_0^t \zeta_3(s)\mathrm{d}s$ it suffices to notice that

$$\frac{2^{i-1}\sqrt{\epsilon_1}\,2^{i-2}\sqrt{\epsilon_2}\cdots\sqrt[4]{\epsilon_{i-2}}\epsilon_{i-1}^{\frac{5}{2}}}{2^{i-2}\sqrt{\epsilon_1}\,2^{i-3}\sqrt{\epsilon_2}\cdots\sqrt{\epsilon_{i-2}}\epsilon_{i-1}^2} = $$

$$\sqrt{\frac{\epsilon_{i-1}}{\epsilon_{i-2}}}\sqrt[4]{\frac{\epsilon_{i-2}}{\epsilon_{i-3}}}\cdots\,{}^{2^{i-2}}\!\!\sqrt{\frac{\epsilon_2}{\epsilon_1}}\,{}^{2^{i-1}}\!\!\sqrt{\epsilon_1}$$

converges to zero as ϵ goes to zero.

Hence, for each ζ as above, there exists $i = 1,\dots,l$ such that

$$\int_0^t \zeta(s)\mathrm{d}s \le C\frac{\epsilon_i}{2^{i-1}\sqrt{\epsilon_1}\,2^{i-2}\sqrt{\epsilon_2}\cdots\sqrt[4]{\epsilon_{i-2}}\epsilon_{i-1}^{\frac{5}{2}}}$$

and we are left to notice that

$$\frac{\epsilon_i}{2^{i-1}\sqrt{\epsilon_1}\,2^{i-2}\sqrt{\epsilon_2}\cdots\sqrt[4]{\epsilon_{i-2}}\epsilon_{i-1}^{\frac{5}{2}}} = $$

$$\frac{\epsilon_i}{\epsilon_{i-1}^3}\sqrt{\frac{\epsilon_{i-1}}{\epsilon_{i-2}}}\sqrt[4]{\frac{\epsilon_{i-2}}{\epsilon_{i-3}}}\cdots\,{}^{2^{i-2}}\!\!\sqrt{\frac{\epsilon_2}{\epsilon_1}}\,{}^{2^{i-1}}\!\!\sqrt{\epsilon_1}$$

tends to zero as ϵ goes to zero. ∎

Remark 5.3. The hypothesis $\limsup_{\epsilon\to 0}\eta_i(\epsilon)/\epsilon^3 < \infty$ has been chosen, in the statement of Theorem 5.2, because of its simplicity. However, we can weaken it by requiring that $\limsup_{\epsilon\to 0}\eta_i(\epsilon)/\epsilon^{\frac{5}{2}+a} < \infty$ with $a = \frac{\sqrt{5}}{2} - 1 \simeq 0.12$. Indeed, with this choice of a,

$$\frac{\epsilon_j^a}{2^j\sqrt{\epsilon_1}\,2^{j-1}\sqrt{\epsilon_2}\cdots\sqrt[4]{\epsilon_{j-1}}} = \left(\frac{\epsilon_j}{\epsilon_{j-1}^{\frac{5}{2}+a}}\right)^a$$

$$\left(\frac{\epsilon_{j-1}^a}{2^{j-1}\sqrt{\epsilon_1}\,2^{j-2}\sqrt{\epsilon_2}\cdots\sqrt[4]{\epsilon_{j-2}}}\right)^{\frac{1}{2}},$$

so that

$$\frac{\epsilon_i}{2^{i-1}\sqrt{\epsilon_1}\,2^{i-2}\sqrt{\epsilon_2}\cdots\sqrt[4]{\epsilon_{i-2}}\epsilon_{i-1}^{\frac{5}{2}}} = $$

$$\frac{\epsilon_i}{\epsilon_{i-1}^{\frac{5}{2}+a}}\left(\frac{\epsilon_{i-1}}{\epsilon_{i-2}^{\frac{5}{2}+a}}\right)^a\left(\frac{\epsilon_{i-2}}{\epsilon_{i-3}^{\frac{5}{2}+a}}\right)^{\frac{a}{2}}\cdots\left(\frac{\epsilon_2}{\epsilon_1^{\frac{5}{2}+a}}\right)^{\frac{a}{2^{i-3}}}\epsilon_1^{\frac{a}{2^{i-2}}}.$$

Hence, each $\int_0^t \zeta(s)\mathrm{d}s$ goes to zero uniformly with respect to $t \in I$ as ϵ tends to zero.

5.2 Case $\mathscr{Z}$

Analogously to Section 5.1, the aim is to provide an algorithmic implementation of the results obtained in Section 4 about the existence of controls yielding tracking but this time under the hypotheses of Corollary 4.7. Instead of adopting the algorithmic scheme on which the proof of Theorem 4.4 is based, as done in Section 5.1, we rely here on the iteration of the scheme proposed in [5] and recalled in Theorem 3.3 (see also Remark 4.3). The advantage is that, under the more restrictive hypotheses of Corollary 4.7, we can base the iteration scheme on the Λ_T-orthonormal and zero-mean sequence defined in (8) using trigonometric functions, which is more convenient for numerical implementation than the sequence $(\psi_j)_{j\in\mathbb{N}}$ constructed in the previous section.

We start from a parameterization

$$\nabla_{\dot\gamma_{\mathrm{ref}}(t)}\dot\gamma_{\mathrm{ref}}(t) - Y(t,\dot\gamma_{\mathrm{ref}}(t)) = \sum_{a=1}^{N_l}\lambda_a(t)Z_a^l(\gamma_{\mathrm{ref}}(t)) \tag{26}$$

of $\gamma_{\mathrm{ref}}\colon I \to Q$, with $Z_a^l \in \mathscr{Z}_l$ and λ_a smooth on I for every $a = 1,\dots,N_l$. (For the definition of $\mathscr{Z}_l$, see (17).) As in Section 5.1, we can construct algorithmically a l-parameter family of admissible trajectories $\xi^{\epsilon_1,\dots,\epsilon_l}$ of Σ with $\epsilon_1,\dots,\epsilon_l > 0$ such that $\xi_l = \gamma_{\mathrm{ref}}$, the uniform limit in (20) exists for every $i = 1,\dots,l$ and every $\epsilon_1,\dots,\epsilon_{l-i} > 0$, and, instead of (21), it satisfies

$$\nabla_{\dot\xi_i^{\epsilon_1,\dots,\epsilon_{l-i}}(t)}\dot\xi_i^{\epsilon_1,\dots,\epsilon_{l-i}}(t) - Y(t,\dot\xi_i^{\epsilon_1,\dots,\epsilon_{l-i}}(t))$$
$$\in \mathrm{span}_{\mathbb{R}}(\mathscr{Z}_i(\xi_i^{\epsilon_1,\dots,\epsilon_{l-i}}(t))).$$

The algorithm works by applying, at each step, the construction of Theorem 3.3 with $\mathscr{Z}_i$ as $\mathscr{Y}$ (see Remark 4.3). Then by backward recursion on i, the

solutions to

$$\nabla_{\dot{\xi}_i^{\epsilon_1,\ldots,\epsilon_{l-i}}(t)}\dot{\xi}_i^{\epsilon_1,\ldots,\epsilon_{l-i}}(t) - Y(t,\dot{\xi}_i^{\epsilon_1,\ldots,\epsilon_{l-i}}(t))$$

$$= \sum_{a=1}^{N_i} \lambda_a^{\epsilon_1,\ldots,\epsilon_{l-i}}(t) Z_a^{i-1}(\xi_i^{\epsilon_1,\ldots,\epsilon_{l-i}}(t))$$

$$+ \sum_{b,c=1,b<c}^{N_i} \lambda_{b,c}^{\epsilon_1,\ldots,\epsilon_{l-i}}(t)\langle Z_b^{i-1}: Z_c^{i-1}\rangle(\xi_i^{\epsilon_1,\ldots,\epsilon_{l-i}}(t))$$

for $Z_a^{i-1}, Z_b^{i-1}, Z_c^{i-1} \in \mathscr{Z}_{i-1}$ are trackable by solutions to

$$\nabla_{\dot{\xi}_{i-1}^{\epsilon_1,\ldots,\epsilon_{l-i+1}}(t)}\dot{\xi}_{i-1}^{\epsilon_1,\ldots,\epsilon_{l-i+1}}(t) - Y(t,\dot{\xi}_i^{\epsilon_1,\ldots,\epsilon_{l-i+1}}(t))$$

$$= \sum_{a=1}^{N_{i-1}} \left(u_{\mathrm{slow},a}^{\epsilon_1,\ldots,\epsilon_{l-i}}(t) \right.$$

$$\left. + \frac{1}{\epsilon_{l-i+1}} u_{\mathrm{osc},a}^{\epsilon_1,\ldots,\epsilon_{l-i}}(t/\epsilon_{l-i+1},t)\right) Z_a^{i-1}(\xi_{i-1}^{\epsilon_1,\ldots,\epsilon_{l-i+1}}(t))$$
(27)

where $u_{\mathrm{slow},a}^{\epsilon_1,\ldots,\epsilon_{l-i}}$ and $u_{\mathrm{osc},a}^{\epsilon_1,\ldots,\epsilon_{l-i}}$ are constructed as in Theorem 3.3. Notice that

$$u_{\mathrm{slow},a}^{\epsilon_1,\ldots,\epsilon_{l-i}}(t) + \frac{1}{\epsilon_{l-i+1}} u_{\mathrm{osc},a}^{\epsilon_1,\ldots,\epsilon_{l-i}}(t/\epsilon_{l-i+1},t)$$

plays the role of a $\lambda_{\hat{a}}^{\epsilon_1,\ldots,\epsilon_{l-i+1}}(t)$ or a $\lambda_{\hat{b},\hat{c}}^{\epsilon_1,\ldots,\epsilon_{l-i+1}}(t)$ at the next step.

We have

$$\|u_{\mathrm{slow},a}^{\epsilon_1,\ldots,\epsilon_{l-i}}\|_\infty \le C \max_{b,c}(\|\lambda_b^{\epsilon_1,\ldots,\epsilon_{l-i}}\|_\infty, \|\lambda_{b,c}^{\epsilon_1,\ldots,\epsilon_{l-i}}\|_\infty^2),$$

$$\|u_{\mathrm{osc},a}^{\epsilon_1,\ldots,\epsilon_{l-i}}\|_\infty \le C \max_{b,c}(\|\lambda_{b,c}^{\epsilon_1,\ldots,\epsilon_{l-i}}\|_\infty).$$

Given $\alpha, \beta: [0,T] \times I \to \mathbb{R}$, define Λ_T as

$$\Lambda_T(t,\alpha,\beta) =$$
$$\frac{1}{2T}\int_0^T \left(\int_0^\tau \alpha(s,t)\mathrm{d}s\right)\left(\int_0^\tau \beta(s,t)\mathrm{d}s\right)\mathrm{d}\tau.$$

The construction of u_{slow} and u_{osc} is such that

$$\sum_{a=1}^{N_{i-1}} u_{\mathrm{slow},a}^{\epsilon_1,\ldots,\epsilon_{l-i}}(t) Z_a^{i-1}$$
$$- \sum_{b,c=1}^{N_{i-1}} \Lambda_T(t, u_{\mathrm{osc},b}^{\epsilon_1,\ldots,\epsilon_{l-i}}, u_{\mathrm{osc},c}^{\epsilon_1,\ldots,\epsilon_{l-i}})\langle Z_b^{i-1}: Z_c^{i-1}\rangle$$
$$= \sum_{a=1}^{N_{i-1}} \lambda_a^{\epsilon_1,\ldots,\epsilon_{l-i}}(t) Z_a^{i-1}$$
$$+ \sum_{b,c=1,b<c}^{N_{i-1}} \lambda_{b,c}^{\epsilon_1,\ldots,\epsilon_{l-i}}(t)\langle Z_b^{i-1}: Z_c^{i-1}\rangle$$

(see [5]). As in Section 5.1, our aim is here to quantify the relations in (24). We will use the same notations for η_i and $\hat{\eta}_i$ as in Section 5.1. The notion of being a one–parameter tracking control law for γ_{ref} is again defined through (25).

Theorem 5.4. *Let* $\Sigma = (Q, \nabla, Y, \mathscr{Y}, \mathbb{R}^k)$ *be a FACCS. Let* $\mathscr{Z}_i$, $i \in \mathbb{N}_0$, *be defined as in (17) satisfying the same hypotheses as in Corollary 4.7. Fix a reference trajectory* $\gamma_{\mathrm{ref}} \in \mathcal{C}^\infty(I,Q)$ *and assume that there exists* $l \in \mathbb{N}$ *such that*

$$\nabla_{\dot{\gamma}_{\mathrm{ref}}(t)}\dot{\gamma}_{\mathrm{ref}}(t) - Y(t,\dot{\gamma}_{\mathrm{ref}}(t)) \in \mathscr{Z}_l(\gamma_{\mathrm{ref}}(t))$$

for every $t \in I$. *Construct* $\xi^{\epsilon_1,\ldots,\epsilon_l}$, $u^{\epsilon_1,\ldots,\epsilon_l}$ *and* $\hat{\eta}_i$ *as above. If* $\eta_i: (0,+\infty) \to (0,+\infty)$ *satisfies* $\limsup_{\epsilon\to 0}\eta_i(\epsilon)/\epsilon^4 < \infty$ *for every* $i = 2,\ldots,l$, *then* $\epsilon \mapsto u^{\hat{\eta}_1(\epsilon),\ldots,\hat{\eta}_l(\epsilon)}$ *is a one–parameter tracking control law for* γ_{ref}.

Proof. The first step of the proof consists in estimating the order with respect to ϵ of the L^∞–norm of the time–dependent parameters appearing in the parameterization (27). Once more we write ϵ_i for $\hat{\eta}_i(\epsilon)$. By induction it can be proved that

$$\|\partial_2^m u_{\mathrm{osc},a}^{\epsilon_1,\ldots,\epsilon_i}\|_\infty \le C(\epsilon_1\cdots\epsilon_{i-1}\epsilon_i^{m+1})^{-1}$$

for every $m \in \mathbb{N}_0$.

Consider, as in the proof of Theorem 5.2, the extended system on $\mathbb{R} \times TQ$ associated with (3). In the following computations we write

$$V_a^i = (0, (Z_a^i)^V),$$
$$W_{b,c}^i = (0, \langle Z_b^i: Z_c^i\rangle^V).$$

We also write $u_{\mathrm{osc},a}^{\epsilon_1,\ldots,\epsilon_i}(\sigma)$ to denote $(u_{\mathrm{osc},a}^{\epsilon_1,\ldots,\epsilon_i}(\sigma,\cdot))^V$, i.e., $u_{\mathrm{osc},a}^{\epsilon_1,\ldots,\epsilon_i}(\sigma)$ is the smooth function on $\mathbb{R} \times TQ$ such that

$$u_{\mathrm{osc},a}^{\epsilon_1,\ldots,\epsilon_i}(\sigma)(t,v) = u_{\mathrm{osc},a}^{\epsilon_1,\ldots,\epsilon_i}(\sigma,t).$$

Then, applying (5) and (6),

$$(t,\dot{\xi}_0^{\epsilon_1,\ldots,\epsilon_l}(t)) = \overrightarrow{\exp}\int_0^t \Big(X^e$$
$$+ \sum_{a=1}^{N_0} u_{\mathrm{slow},a}^{\epsilon_1,\ldots,\epsilon_{l-1}}(s)V_a^0$$
$$+ \frac{1}{\epsilon_l}\sum_{a=1}^{N_0} u_{\mathrm{osc},a}^{\epsilon_1,\ldots,\epsilon_{l-1}}(s/\epsilon_l,s)V_a^0\Big)\mathrm{d}s\,\gamma_{\mathrm{ref}}^e(0)$$
$$= \overrightarrow{\exp}\int_0^t \frac{1}{\epsilon_l}\sum_{a=1}^{N_0} u_{\mathrm{osc},a}^{\epsilon_1,\ldots,\epsilon_{l-1}}(s/\epsilon_l)V_a^0\mathrm{d}s \circ$$
$$\overrightarrow{\exp}\int_0^t \Big(X^e + \sum_{a=1}^{N_0} u_{\mathrm{slow},a}^{\epsilon_1,\ldots,\epsilon_{l-1}}(s)V_a^0$$
$$+ \sum_{a=1}^{N_0}\int_0^s \frac{1}{\epsilon_l}u_{\mathrm{osc},a}^{\epsilon_1,\ldots,\epsilon_{l-1}}(s_0/\epsilon_l)\mathrm{d}s_0[V_a^0, X^e]$$
$$- \sum_{a=1}^{N_0}\int_0^s \frac{1}{\epsilon_l}\partial_2 u_{\mathrm{osc},a}^{\epsilon_1,\ldots,\epsilon_{l-1}}(s_0/\epsilon_l)\mathrm{d}s_0 V_a^0$$
$$- \sum_{b,c=1}^{N_0}\int_0^s \frac{1}{\epsilon_l}u_{\mathrm{osc},b}^{\epsilon_1,\ldots,\epsilon_{l-1}}(s_1/\epsilon_l)$$
$$\int_0^{s_1}\frac{1}{\epsilon_l}u_{\mathrm{osc},c}^{\epsilon_1,\ldots,\epsilon_{l-1}}(s_0/\epsilon_l)\mathrm{d}s_0\mathrm{d}s_1 W_{b,c}^0\Big)\mathrm{d}s\,\gamma_{\mathrm{ref}}^e(0)$$

Denoting by $\mathscr{V}$ any vertical flow, we have

$$(t, \dot{\xi}_0^{\epsilon_1,\ldots,\epsilon_l}(t)) = \mathscr{V} \circ \overrightarrow{\exp} \int_0^t \Big(X^e$$
$$+ \sum_{a=1}^{N_0} \lambda_a^{\epsilon_1,\ldots,\epsilon_{l-1}}(s) V_a^0$$
$$+ \sum_{b,c=1,\, b<c}^{N_0} \lambda_{b,c}^{\epsilon_1,\ldots,\epsilon_{l-1}}(s) W_{b,c}^0$$
$$+ \int_0^s \frac{1}{\epsilon_l} \sum_{a=1}^{N_0} u_{\mathrm{osc},a}^{\epsilon_1,\ldots,\epsilon_{l-1}}(s_0/\epsilon_l, s) \mathrm{d}s_0 [V_a^0, X^e]$$
$$- \sum_{a=1}^{N_0} \int_0^s \frac{1}{\epsilon_l} \partial_2 u_{\mathrm{osc},a}^{\epsilon_1,\ldots,\epsilon_{l-1}}(s_0/\epsilon_l, s) \mathrm{d}s_0 V_a^0$$
$$+ \sum_{b,c=1}^{N_0} (\Lambda_T(s, u_{\mathrm{osc},b}^{\epsilon_1,\ldots,\epsilon_{l-1}}, u_{\mathrm{osc},c}^{\epsilon_1,\ldots,\epsilon_{l-1}}) -$$
$$\int_0^s \frac{1}{\epsilon_l} u_{\mathrm{osc},b}^{\epsilon_1,\ldots,\epsilon_{l-1}}(s_1/\epsilon_l, s) \int_0^{s_1} \frac{1}{\epsilon_l} u_{\mathrm{osc},c}^{\epsilon_1,\ldots,\epsilon_{l-1}}(s_0/\epsilon_l, s)$$
$$\mathrm{d}s_0 \mathrm{d}s_1) W_{b,c}^0 \Big) \mathrm{d}s\, \gamma_{\mathrm{ref}}^e(0).$$

Noticing that

$$\sum_{a=1}^{N_0} \lambda_a^{\epsilon_1,\ldots,\epsilon_{l-1}}(s) V_a^0 + \sum_{b,c=1}^{N_0} (\lambda_{b,c}^{\epsilon_1,\ldots,\epsilon_{l-1}}(s) W_{b,c}^0$$
$$= \sum_{\hat{a}=1}^{N_1} \Big(u_{\mathrm{slow},\hat{a}}^{\epsilon_1,\ldots,\epsilon_{l-2}}(s) + \frac{1}{\epsilon_{l-1}} u_{\mathrm{osc},\hat{a}}^{\epsilon_1,\ldots,\epsilon_{l-2}}(s/\epsilon_{l-1}, s) \Big) V_{\hat{a}}^1$$

and applying iteratively the same computation as above, one ends up with

$$(t, \dot{\xi}_0^{\epsilon_1,\ldots,\epsilon_l}(t)) =$$
$$\mathscr{V} \circ \overrightarrow{\exp} \int_0^t \left(X^e + \lambda_a(s) V_a^l + \mathscr{T}(s) \right) \mathrm{d}s\, \gamma_{\mathrm{ref}}^e(0),$$

where $\mathscr{T}(s)$ is a sum of terms of the form $\zeta(s) V$ where $V \in \mathscr{X}(\mathbb{R} \times TQ)$ is independent of the ϵ_j, while $\zeta \in \mathcal{C}^\infty(I, \mathbb{R})$ is of one of the following four types:

$$\zeta_1 = \int_0^s \frac{1}{\epsilon_i} u_{\mathrm{osc},a}^{\epsilon_1,\ldots,\epsilon_{i-1}}(s_0/\epsilon_i, s) \mathrm{d}s_0,$$
$$\zeta_2 = - \int_0^s \frac{1}{\epsilon_i} \partial_2 u_{\mathrm{osc},a}^{\epsilon_1,\ldots,\epsilon_{i-1}}(s_0/\epsilon_i, s) \mathrm{d}s_0,$$
$$\zeta_3 = \frac{1}{1+\delta_{bc}} \Big[2\Lambda_T(s, u_{\mathrm{osc},b}^{\epsilon_1,\ldots,\epsilon_{i-1}}, u_{\mathrm{osc},c}^{\epsilon_1,\ldots,\epsilon_{i-1}})$$
$$- \Big(\int_0^s \frac{1}{\epsilon_i} u_{\mathrm{osc},b}^{\epsilon_1,\ldots,\epsilon_{i-1}}(s_0/\epsilon_i, s) \mathrm{d}s_0 \Big)$$
$$\Big(\int_0^s \frac{1}{\epsilon_i} u_{\mathrm{osc},c}^{\epsilon_1,\ldots,\epsilon_{i-1}}(s_0/\epsilon_i, s) \mathrm{d}s_0 \Big) \Big],$$

for $i = 1, \ldots, l$, and

$$\zeta_4 = - \Big(\int_0^s \frac{1}{\epsilon_j} u_{\mathrm{osc},b}^{\epsilon_1,\ldots,\epsilon_{j-1}}(s_0/\epsilon_j, s) \mathrm{d}s_0 \Big)$$
$$\Big(\int_0^s \frac{1}{\epsilon_i} u_{\mathrm{osc},c}^{\epsilon_1,\ldots,\epsilon_{i-1}}(s_0/\epsilon_i, s) \mathrm{d}s_0 \Big),$$

for $i > j$, $i, j \in \{1, \ldots, l\}$.

We are left to prove that every $\int_0^t \zeta(s) \mathrm{d}s$ converges to zero uniformly with respect to t as ϵ goes to zero (Lemma 2.4).

Applying Lemma 5.1 with

$$f(\tau, s) = \int_0^\tau u_{\mathrm{osc},a}^{\epsilon_1,\ldots,\epsilon_{i-1}}(s_0, s) \mathrm{d}s_0$$

and $\hat{\epsilon} = \epsilon_i$ leads to

$$\int_0^t \zeta_1(s) \mathrm{d}s \leq C \epsilon_i (\epsilon_1 \cdots \epsilon_{i-2} \epsilon_{i-1}^2)^{-1}.$$

Similarly,

$$\int_0^t \zeta_2(s) \mathrm{d}s \leq C (\epsilon_1 \cdots \epsilon_{i-2} \epsilon_{i-1}^3)^{-1} \epsilon_i.$$

Taking

$$f(\tau, s) = \Big(\int_0^\tau u_{\mathrm{osc},b}^{\epsilon_1,\ldots,\epsilon_{i-1}}(s_0, s) \mathrm{d}s_0 \Big)$$
$$\Big(\int_0^\tau u_{\mathrm{osc},c}^{\epsilon_1,\ldots,\epsilon_{i-1}}(s_0, s) \mathrm{d}s_0 \Big),$$

we obtain

$$\int_0^t \zeta_3(s) \mathrm{d}s \leq C (\epsilon_1 \cdots \epsilon_{i-2})^{-2} \epsilon_{i-1}^{-3} \epsilon_i.$$

Finally, with

$$f(\tau, s) = \Big(\int_0^{s/\epsilon_j} u_{\mathrm{osc},b}^{\epsilon_1,\ldots,\epsilon_{j-1}}(s_0, s) \mathrm{d}s_0 \Big)$$
$$\Big(\int_0^\tau u_{\mathrm{osc},c}^{\epsilon_1,\ldots,\epsilon_{i-1}}(s_0, s) \mathrm{d}s_0 \Big),$$

we have

$$\int_0^t \zeta_4(s) \mathrm{d}s \leq C (\epsilon_1 \cdots \epsilon_{j-2} \epsilon_{j-1})^{-1} (\epsilon_1 \cdots \epsilon_{i-2} \epsilon_{i-1}^2)^{-1} \epsilon_i.$$

Hence, each ζ satisfies

$$\int_0^t \zeta(s) \mathrm{d}s \leq \frac{C\epsilon_i}{(\epsilon_1 \cdots \epsilon_{i-2})^2 \epsilon_{i-1}^3} = \frac{C\epsilon_i}{\epsilon_{i-1}^4} \frac{\epsilon_{i-1}}{(\epsilon_1 \cdots \epsilon_{i-2})^2}$$

and it is easy to prove by recurrence that $\epsilon_{i-1}(\epsilon_1 \cdots \epsilon_{i-2})^{-2}$ tends to zero as ϵ goes to zero. ∎

Remark 5.5. As in Remark 5.3, the hypothesis $\limsup_{\epsilon\to 0} \eta_i(\epsilon)/\epsilon^4 < \infty$ in the statement of Theorem 5.4 can be weakened by requiring that $\limsup_{\epsilon\to 0} \eta_i(\epsilon)/\epsilon^{3+a} < \infty$ with $a = \sqrt{3} - 1 \simeq 0.73$. Indeed, with this choice of a,

$$\frac{\epsilon_i^a}{\epsilon_1^2 \epsilon_2^2 \cdots \epsilon_{i-1}^2} = \left(\frac{\epsilon_i}{\epsilon_{i-1}^{3+a}} \right)^a \left(\frac{\epsilon_{i-1}^a}{\epsilon_1^2 \cdots \epsilon_{i-2}^2} \right),$$

so that

$$\frac{\epsilon_i}{\epsilon_1^2 \cdots \epsilon_{i-2}^2 \epsilon_{i-1}^3} =$$
$$\frac{\epsilon_i}{\epsilon_{i-1}^{3+a}} \left(\frac{\epsilon_{i-1}}{\epsilon_{i-2}^{3+a}} \right)^a \left(\frac{\epsilon_{i-2}}{\epsilon_{i-3}^{3+a}} \right)^a \cdots \left(\frac{\epsilon_2}{\epsilon_1^{3+a}} \right)^a \epsilon_1^a.$$

Hence, each $\int_0^t \zeta(s) \mathrm{d}s$ goes to zero uniformly with respect to $t \in I$ as ϵ tends to zero.

Under special assumptions the relations in (24) that have been quantified in Theorem 5.4 can be reduced up to $\lim_{\epsilon\to 0} \eta_i(\epsilon)/\epsilon^2 = 0$ as stated in the following corollary and illustrated in the numerical simulation included in Section 5.2.1. The required assumptions are stated in terms of the number of steps in the algorithm and of the coefficients providing the parametrization of the reference trajectory.

Corollary 5.6. Let $\Sigma = (Q, \nabla, Y, \mathscr{Y}, \mathbb{R}^k)$ be a FACCS. Let $\mathscr{Z}_i$, $i \in \mathbb{N}_0$, be defined as in (17) and assume that $\mathrm{span}_{\mathbb{R}} \mathscr{Z}_2(q) = T_q Q$ for all $q \in Q$

and that for each $i \in \{0, 1\}$ and each $Z \in \mathscr{Z}_i$, $\langle Z : Z \rangle(q) \in \operatorname{span}_{\mathbb{R}}(\mathscr{Z}_i(q))$. Fix a reference trajectory $\gamma_{\mathrm{ref}} \in \mathcal{C}^{\infty}(I, Q)$ such that the coefficients λ_a associated with γ_{ref} as in (26) are constant functions. Construct $\xi^{\epsilon_1, \epsilon_2}$, $u^{\epsilon_1, \epsilon_2}$ and $\hat{\eta}_i$ as above. If $\eta_2 : (0, +\infty) \to (0, +\infty)$ satisfies $\lim_{\epsilon \to 0} \eta_2(\epsilon)/\epsilon^2 = 0$, then $\epsilon \mapsto u^{\epsilon, \eta_2(\epsilon)}$ is a one–parameter tracking control law for γ_{ref}.

Proof. The hypotheses of the corollary and the expressions of the controls appearing in Theorem 3.3 guarantee that the oscillatory controls are as follows:

$$u_{\mathrm{osc},a}(\tau, t) \in \operatorname{span}_{\mathbb{R}}\{\varphi_{j_0}(\tau)\},$$

$$u_{\mathrm{osc},a}^{\epsilon_1}(\tau, t) \in \operatorname{span}_{\mathbb{R}}\{\varphi_{j_0}(\tau), \frac{1}{\epsilon_1}\varphi_{j_1}(t/\epsilon_1)\varphi_{j_2}(\tau)\},$$

where j_0, j_1, j_2 vary in $\mathbb{N}$.

Thus the integrals of all the terms $\mathscr{T}$ that appear in the proof of Theorem 5.4 only contain product of trigonometric functions, more specifically cosines and sines and converge to zero uniformly with respect to t as ϵ goes to zero if η_2 is such that $\lim_{\epsilon \to 0} \eta_2(\epsilon)/\epsilon^2 = 0$. ∎

5.2.1 Numerical simulation: submarine

In this section, we illustrate the method to obtain a one–parameter control law in a concrete situation that fulfills the assumptions in Corollary 5.6. The algorithm described in Section 5.2 has been implemented with Scilab.

For our example, we have the submarine presented in Section 4.1.2 whose dynamics are given by (18) and (19). We consider the same kind of inertia matrix as in Section 4.1.2, taking

$$J_1 = 1, \quad J_3 = 3, \quad M_1 = 1, \quad M_2 = 2, \quad M_3 = 3.$$

We recall that in the case under consideration Theorem 3.3 cannot be applied. However, our method provides a one–parameter control law that solves the tracking problem. The trajectory to be tracked is given by

$$r(t) = (-t, -t, -t), \quad A_{33}(t) = 1, \quad t \in [0, 1]$$

with initial condition $r(0) = (0, 0, 0)$, $A(0) = \mathrm{Id}$, being Id the identity 3×3 matrix, $\Pi(0) = (0, 0, 0)$, $P(0) = (-1, -2, -3)$. Thus there are degrees of freedom in the attitude of the submarine, but the target position of the center of the submarine is fully determined.
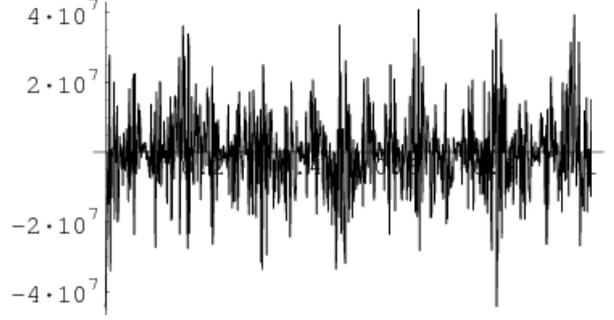

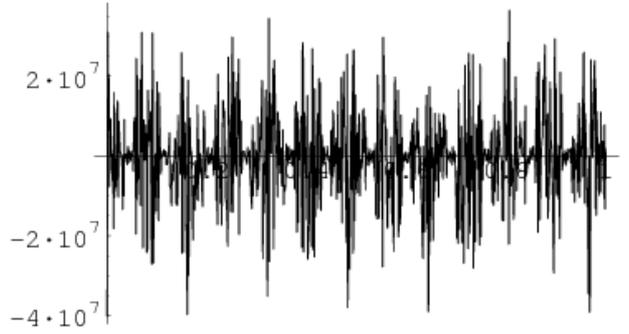Figure 1: The control law $u_1^{\epsilon, \eta_2(\epsilon)}$.

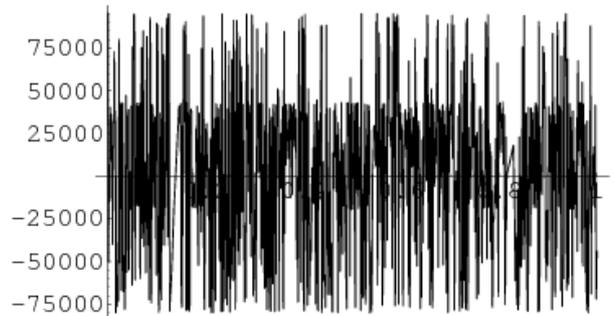Figure 2: The control law $u_2^{\epsilon, \eta_2(\epsilon)}$.

Figure 3: The control law $u_3^{\epsilon, \eta_2(\epsilon)}$.

In this implementation we take $\eta_2(\epsilon) = \epsilon^{2.5}$ because the considered reference trajectory satisfies the hypotheses of Corollary 5.6.

First, we compute the one–parameter control laws $u_1^{\epsilon_1,\epsilon_2}$, $u_2^{\epsilon_1,\epsilon_2}$ and $u_3^{\epsilon_1,\epsilon_2}$ as described in the proof of Theorem 5.4. Then we fix $\epsilon = 1/39 \approx 0.0256$, so that $\eta_2(\epsilon) \approx 0.0001$. The corresponding control laws are represented in Figures 1, 2 and 3. By construction, the controls are highly oscillatory.
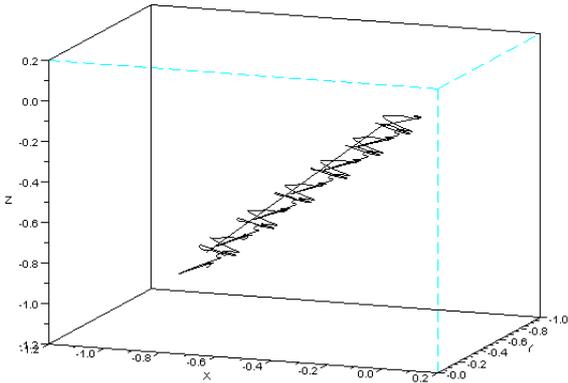

Figure 4: Evolution of the position of the center of the submarine with respect to time. The target trajectory is the non–oscillating curve.
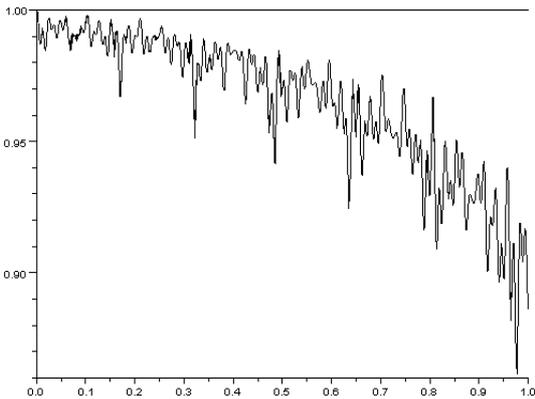

Figure 5: Evolution of A_{33} entry of the attitude matrix with respect to time. The target trajectory is the non–oscillating curve.

Then we integrate the dynamics of the system using the numerical integrator *stiff* included in Scilab. As a result, Figures 4 and 5 show that the target trajectory, corresponding with the non–oscillating line, is tracked by the oscillating curve. The error of the approximation, computed by the supremum distance, is $\mathrm{d}((r_{\mathrm{ref}}(t), A_{33,\mathrm{ref}}(t)), (r(t), A_{33}(t))) \approx$

0.1903.

6   Conclusions

The previously known sufficient conditions for tracking were given in terms of finite sets of vector fields, as reviewed in Section 3. Here we have constructed a sequence of infinite family of vector fields that defines a sequence of convex cones suitable for characterizing trackability (Theorem 4.4 and Corollary 4.5). Different convex cones, (9) and (16), have been considered. Under additional assumptions, using the cones in (16) and a particular sequence of finite families of vector fields, it is possible to recover the sufficient conditions for tracking already known in the literature [5], see Corollary 4.7. However, our constructions not only recover the previously known results, but they also extend them, as shown in Section 4.1.

The sequence of families of vector fields in Corollaries 4.5 and 4.7 are also suitable for constructing a one–parameter tracking control law (Theorems 5.2 and 5.4). It remains as future work to generalize the construction of one–parameter tracking control laws when the sets (9), that include the closure, are considered.

Another future research line is the study of the complexity for the control–affine systems considered in this work. Apart from tracking non–admissible trajectories, one could impose more requirements on the solution to the tracking problem, as for instance, to save energy. The complexity provides a good tool to formulate this kind of problems and, so far, has been only studied for control–linear systems [7, 8].

Acknowledgements

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