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A DERIVED EQUIVALENCE BETWEEN CLUSTER EQUIVALENT ALGEBRAS

CLAIRE AMIOT

ABSTRACT. Let Q be an acyclic quiver. Associated with any element w of the Coxeter group of Q , triangulated categories $\underline{\text{Sub}}\Lambda_w$ were introduced in [BIRS09a]. There are shown to be triangle equivalent to generalized cluster categories \mathcal{C}_{Γ_w} associated to algebras Γ_w of global dimension ≤ 2 in [ART09]. For w satisfying a certain property, called co- c -sortable, other algebras A_w of global dimension ≤ 2 are constructed in [AIRT09] with a triangle equivalence $\mathcal{C}_{A_w} \simeq \underline{\text{Sub}}\Lambda_w$. The main result of this paper is to prove that the algebras Γ_w and A_w are derived equivalent when w is co- c -sortable. The proof uses the 2-APR-tilting theory introduced in [IO09].

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INTRODUCTION

Let k be an algebraically closed field. The cluster category \mathcal{C}_Q associated to an acyclic quiver Q has been introduced in [BMR⁺06]. It is defined as the orbit category $\mathcal{D}^b(kQ)/\mathbb{S}_2$, where \mathbb{S}_2 is the composition of the Serre functor \mathbb{S} of the bounded derived category $\mathcal{D}^b(kQ)$ of finitely presented kQ -modules with $[-2]$. This is a triangulated category [Kel05], with finite dimensional spaces of morphisms (Hom -finite for short), and with the 2-Calabi-Yau property: for any two objects X and Y in \mathcal{C}_Q there is a functorial isomorphism $\text{Hom}_{\mathcal{C}_Q}(X, Y) \simeq D\text{Hom}(Y, X[2])$ where

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D is $\mathbf{Hom}_k(-, k)$. This construction was motivated, via [MRZ03], by the theory of cluster algebras initiated by Fomin and Zelevinsky [FZ02]. Following another point of view, Geiss, Leclerc and Schröer have related in [GLS07a] and [GLS06] certain cluster algebras with the stable categories $\mathbf{mod}\Lambda$ where Λ is the preprojective algebra associated with a Dynkin quiver. These categories are also \mathbf{Hom} -finite, triangulated and 2-Calabi-Yau. Cluster categories \mathcal{C}_Q and stable categories $\mathbf{mod}\Lambda$ have both a special kind of objects called *cluster-tilting*. These are defined to be objects without selfextension and maximal for this property. They are very important since they are the analogs of clusters.

Therefore it is interesting to study \mathbf{Hom} -finite, 2-Calabi-Yau triangulated categories with cluster-tilting objects in general, and to find new such categories. To an acyclic quiver Q and to an element w of the Coxeter group of Q , Buan, Iyama, Reiten and Scott have associated in [BIRS09a] (see also [GLS08], [GLS07b]) a triangulated category $\mathbf{Sub}\Lambda_w$ where $\mathbf{Sub}\Lambda_w$ is a subcategory of the category of finite length modules over the preprojective algebra Λ associated with Q . These categories are \mathbf{Hom} -finite, 2-Calabi-Yau and have cluster-tilting objects. Moreover they generalize the previous categories: If Q is Dynkin and w is the element of maximal length of the Coxeter group, $\mathbf{Sub}\Lambda_w$ is equivalent to $\mathbf{mod}\Lambda$. For any acyclic Q which is not A_n with the linear orientation, if w is cc where c is the Coxeter element associated with the orientation of Q then $\mathbf{Sub}\Lambda_w$ is equivalent to the cluster category \mathcal{C}_Q .

More recently cluster categories have been generalized in [Ami08], replacing the finite dimensional hereditary algebras kQ by finite dimensional algebras A of global dimension ≤ 2 . The orbit category $\mathcal{D}^b(A)/\mathbb{S}_2$ is not triangulated in general. Therefore the *generalized cluster category* \mathcal{C}_A is defined to be the triangulated hull of the orbit category $\mathcal{D}^b(A)/\mathbb{S}_2$. This construction generalizes again the previous constructions. Indeed in [ART09] given an acyclic quiver Q and an element w in the Coxeter group, an algebra Γ_w of global dimension ≤ 2 is constructed with a triangle equivalence

$$\mathcal{C}_{\Gamma_w} \simeq \mathbf{Sub}\Lambda_w.$$

The algebra Γ_w is constructed by using a natural grading on the preprojective algebra Λ . With a very different point of view, it is shown in [AIRT09] that for a certain kind of words (containing the *adaptable* words in the sens of [GLS07b]), called *co- c -sortable*, where c is a Coxeter element, it is possible to construct a triangle equivalence

$$\mathcal{C}_{A_w} \simeq \mathbf{Sub}\Lambda_w$$

where A_w is the Auslander algebra of a finite torsion class in $\mathbf{mod}kQ$ naturally associated with the word w .

The aim of this paper is to link the algebras Γ_w and A_w when w is *co- c -sortable*. By definition of the generalized cluster category, if two algebras of global dimension ≤ 2 are derived equivalent their generalized cluster categories are triangle equivalent. However the converse is not true. Two algebras of global dimension ≤ 2 can have the same generalized cluster categories without being derived equivalent. We say in this case that they are *cluster equivalent* (see [AO]). The main result of this paper is that the algebras Γ_w and A_w are derived equivalent (Theorem 2.5). Moreover we explicitly describe this equivalence using 2-APR-tilting introduced in [IO09].

The paper is organized as follows. Section 1 is devoted to background definitions and results from [Ami08], [BIRS09a] and [BIRS09b]. We recall the definitions of the generalized cluster categories and of the categories $\mathbf{Sub}\Lambda_w$ and state some of their properties. In section 2 we recall results of [ART09] and [AIRT09]. We give the explicit construction of the algebra Γ_w of global dimension ≤ 2 and we described the finite torsion class associated to a *co- c -sortable* word w .

Sections 3 and 4 are devoted to the proof of the main theorem. We construct a tilting module over the algebra A_w in section 3, and we prove that its endomorphism algebra is isomorphic to Γ_w in section 4. In section 5 we give an example to illustrate the main theorem.

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Notations. Throughout k is a algebraically closed field and D is the usual duality $\text{Hom}_k(?, k)$.

By triangulated category, we mean k -linear triangulated category satisfying the Krull-Schmidt property. For all triangulated categories we will denote the shift functor by $[1]$. By Frobenius category we mean an exact k -category with enough projectives and injectives and where projectives and injectives coincide.

For a finite-dimensional k -algebra A , we denote by $\text{mod } A$ the category of finite-dimensional right A -modules. The category $\mathcal{D}^b(A)$ will be the bounded derived category of $\text{mod } A$. Its Serre functor $-\overset{L}{\otimes}_A DA$ will be denoted by \mathbb{S}_A (or \mathbb{S} if there is no danger of confusion). More generally, for an additive k -category \mathcal{M} we denote by $\text{mod } \mathcal{M}$ the category of finitely presented functors $\mathcal{M}^{op} \rightarrow \text{mod } k$.

1. BACKGROUND: PREPROJECTIVE ALGEBRAS AND GENERALIZED CLUSTER CATEGORIES

1.1. Generalized cluster categories. This section is devoted to recalling some results of [Ami08].

Let Γ be a finite dimensional k -algebra of global dimension ≤ 2 . We denote by $\mathcal{D}^b(\Gamma)$ the bounded derived category of finite dimensional Γ -modules. It has a Serre functor that we denote by \mathbb{S} . We denote by \mathbb{S}_2 the composition $\mathbb{S}[-2]$.

The *generalized cluster category* \mathcal{C}_Γ of Γ has been defined in [Ami08] as the triangulated hull of the orbit category $\mathcal{D}^b(\Gamma)/\mathbb{S}_2$. We will denote by π_Γ (or π if there is no danger of confusion) the triangle functor

$$\pi_\Gamma : \mathcal{D}^b(\Gamma) \twoheadrightarrow \mathcal{D}^b(\Gamma)/\mathbb{S}_2 \hookrightarrow \mathcal{C}_\Gamma$$

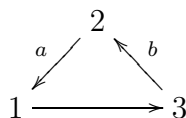
Theorem 1.1 (Theorem 4.10 of [Ami08]). *Let Γ be a finite dimensional algebra of global dimension ≤ 2 , and assume that the endomorphism algebra $\text{End}_{\mathcal{C}}(\pi_\Gamma)$ is finite dimensional. Then \mathcal{C}_Γ is a Hom-finite, triangulated 2-Calabi-Yau category and $\pi(\Gamma)$ is a cluster-tilting object.*

The construction of \mathcal{C}_Γ depends only on the derived category $\mathcal{D}^b(\Gamma)$. Thus we have the following consequence.

Proposition 1.2. *Let Γ and Γ' be two derived equivalent finite dimensional k -algebras of global dimension ≤ 2 . Then the categories \mathcal{C}_Γ and $\mathcal{C}_{\Gamma'}$ are triangle equivalent.*

However, the converse is not true in general.

Example 1.1. Let A be the algebra given by the quiver



with the relation $ab = 0$. Using Theorem 2.1 of [KR08], it is not hard to check that the generalized cluster category \mathcal{C}_A is triangle equivalent to the cluster category of \tilde{A}_2 . However,

the algebra A is not derived equivalent to the hereditary algebra $k\tilde{A}_2$. Indeed, since the quiver of A has an oriented cycle, A cannot be derived equivalent to $\mathcal{D}^b(k\tilde{A}_2)$.

This example leads us to introduce the following definition.

Definition 1.3. [AO] Two finite dimensional algebras A and B of global dimension ≤ 2 will be called *cluster equivalent* if there exists a triangle equivalence between their generalized cluster categories \mathcal{C}_A and \mathcal{C}_B .

1.2. Categories associated to a word. This section is devoted to recalling some results of [BIRS09a] and [BIRS09b].

Let Q be a finite quiver without oriented cycles. We denote by $Q_0 = \{1, \dots, n\}$ the set of vertices and by Q_1 the set of arrows. The preprojective algebra associated to Q is the algebra

$$k\overline{Q}/\langle \sum_{a \in Q_1} aa^* - a^*a \rangle$$

where \overline{Q} is the double quiver of Q , which is obtained from Q by adding to each arrow $a : i \rightarrow j \in Q_1$ an arrow $a^* : i \leftarrow j$ pointing in the opposite direction. We denote by Λ the completion of the preprojective algebra associated to Q and by $\text{f.l.}\Lambda$ the category of right Λ -modules of finite length.

For a vertex i in Q_0 we denote by \mathcal{I}_i the two-sided ideal $\Lambda(1 - e_i)\Lambda$, where e_i is the primitive idempotent of Λ associated to the vertex i . Let C_Q be the *Coxeter group* associated to Q . It is defined by the generators s_i , where $i \in Q_0$, and by the relations

- $s_i^2 = 1$,
- $s_i s_j = s_j s_i$ if there is no arrow between i and j ,
- $s_i s_j s_i = s_j s_i s_j$ if there is exactly one arrow between i and j .

A *reduced expression* $w = s_{u_1} \dots s_{u_l}$ of an element of C_Q is an expression of w with l as small as possible. When $s_{u_1} \dots s_{u_l}$ is a reduced expression of w , the integer l is said to be the *length* $l(w)$ of w .

Let $w = s_{u_1} \dots s_{u_l}$ be a reduced expression of an element in C_Q . For $p \leq l$ we denote by \mathcal{I}_{w_p} the two sided ideal $\mathcal{I}_{u_p} \mathcal{I}_{u_{p-1}} \dots \mathcal{I}_{u_1}$. We denote by Λ_w the algebra $\Lambda/\mathcal{I}_{w_l}$ and by $\text{Sub } \Lambda_w$ the subcategory of $\text{f.l.}\Lambda$ of finite direct sums of copies of submodules of Λ_w .

Let us recall Theorem III.2.8 of [BIRS09a].

Theorem 1.4 (Buan-Iyama-Reiten-Scott). *The category $\text{Sub } \Lambda_w$ is a Hom-finite Frobenius category and its stable category $\underline{\text{Sub}} \Lambda_w$ is 2-Calabi-Yau. The object $C_w = \bigoplus_{p=1}^l e_{u_p}(\Lambda/\mathcal{I}_{w_p})$ is a cluster-tilting object.*

The endomorphism algebra of C_w is described in terms of a quiver with relations in [BIRS09b]. Let us define the quiver Q_w as follows:

- vertices: $1, \dots, l(w)$.
- for each $i \in Q_0$, one arrow $t \leftarrow s$ if t and s are two consecutive vertices of type i (*i.e.* $u_s = u_t = i$) and $t < s$ (we call them *arrows going to the left*);
- for each $a : i \rightarrow j \in Q_1$, put $a : t \rightarrow s$ if t is a vertex of type i , s of type j , and if there is no vertex of type i between t and s and if s is the last vertex of type j before the next vertex of type i in the expression $w = s_{u_1} \dots s_{u_l}$ (we call them *Q -arrows*);
- for each $a : i \rightarrow j \in Q_1$, put $a^* : t \rightarrow s$ if t is of type j , s is of type i , if there is no vertex of type j between t and s and if s is the last vertex of type i before the next vertex of type j in the expression of w (we call them *Q^* -arrows*).

For i in Q_0 we define l_i to be the maximal integer such that $u_{l_i} = i$. We denote by \bar{Q}_w the full subquiver of Q_w whose vertices are not l_i .

For each Q -arrow $a : t \rightarrow s$ in \bar{Q}_w we denote by W_a the composition aa^*p if there is a (unique) Q^* -arrow $a^* : r \rightarrow t$ in \bar{Q}_w where $u_r = u_s$ and where p is the composition of arrows going to the left $r \leftarrow \cdots \leftarrow s$. Otherwise we put $W_a = 0$. For each Q^* -arrow $a^* : t \rightarrow r$ in \bar{Q}_w , we denote by W_{a^*} the composition a^*ap if there exists a (unique) Q -arrow $a : s \rightarrow t$ in \bar{Q}_w with $u_s = u_r$ and where p is the composition of arrows going to the left $s \leftarrow \cdots \leftarrow r$. Otherwise we put $W_{a^*} = 0$. Then let W_w be the sum

$$W_w = \sum_{a \text{ } Q\text{-arrow}} W_a - \sum_{a^* \text{ } Q^*\text{-arrow}} W_{a^*}.$$

It is a *potential* in the sense of [DWZ08], that is, a linear combination of cycles in \bar{Q}_w . For a cycle p in \bar{Q}_w and an arrow a in \bar{Q}_w , it is possible to define the partial derivative $\partial_a p$ as the sum $\partial_a p := \sum_{p=ua_v} vu$. The definition of the partial derivative can be extended by linearity to any potential.

The *Jacobian algebra* (see [DWZ08]) is defined as the algebra

$$Jac(\bar{Q}_w, W_w) := k\bar{Q}_w / \langle \partial_a W_w, a \in \bar{Q}_w \rangle.$$

Let us recall Theorem 6.8 of [BIRS09b].

Theorem 1.5 (Buan-Iyama-Reiten-Smith). *Let $w = s_{u_1} \dots s_{u_l}$ be a reduced expression of an element w of the Coxeter group C_Q . Let C_w be the cluster-tilting object of $\underline{\mathbf{Sub}}(\Lambda_w)$ (defined in Theorem 1.4) associated to this reduced expression. Then there is an algebra isomorphism*

$$Jac(\bar{Q}_w, W_w) \simeq \mathbf{End}_{\underline{\mathbf{Sub}}\Lambda_w}(C_w).$$

2. CATEGORIES ASSOCIATED TO A WORD AS GENERALIZED CLUSTER CATEGORIES

In this section we recall some results of [ART09] and [AIRT09] which describe some categories $\underline{\mathbf{Sub}}\Lambda_w$ as generalized cluster categories.

2.1. General words: Results of [ART09]. In [ART09] it is shown that for any element w of the Coxeter group, the category $\underline{\mathbf{Sub}}\Lambda_w$ is triangle equivalent to some generalized cluster category.

Let Q and Λ be as in the previous section. Let $w = s_{u_1} \dots s_{u_l}$ be a reduced expression of an element in the Coxeter group C_Q . Since the category $\underline{\mathbf{Sub}}\Lambda_w$ and the object C_w do not depend on the orientation of Q , we can assume that the orientation of Q satisfies the property

$$(*) \quad \text{if there exists } i \rightarrow j, \text{ then } l_i < l_j,$$

where l_i is the maximal integer such that $u_{l_i} = i$.

We define a grading on the quiver \bar{Q}_w . All arrows going to the left and all Q -arrows are defined to have degree 0. All Q^* -arrows are defined to have degree 1. It is then easy to see that the potential W_w is homogeneous of degree 1. Hence we get a grading on the Jacobian algebra $Jac(\bar{Q}_w, W_w)$, so on the algebra $B = \mathbf{End}_{\underline{\mathbf{Sub}}\Lambda_w}(C_w)$ by Theorem 1.5. Let $\Gamma := B_0$ be its part of degree zero.

Theorem 2.1 (Theorem 4.4 of [ART09]). *For any acyclic quiver Q and any element w in the Coxeter group of Q , the algebra Γ is of global dimension ≤ 2 and there exists a commutative*

diagram of triangle functors:

$$\begin{array}{ccc} \mathcal{D}^b(\Gamma) & \xrightarrow{F_\Gamma} & \underline{\text{Sub}}\Lambda_w, \\ \downarrow \pi_\Gamma & \nearrow f_\Gamma & \\ \mathcal{C}_\Gamma & & \end{array}$$

where f_Γ is an equivalence, and where $F_\Gamma(\Gamma) = C_w$.

2.2. Co- c -sortable words. In this section we recall some results of [AIRT09].

Let Q be a quiver without oriented cycles with n vertices. We assume that the orientation of Q satisfies

$$(**) \quad \text{if there exists } i \rightarrow j, \text{ then } i < j.$$

We denote by c the Coxeter element $s_1 \dots s_n$.

Definition 2.2. An element w of the Coxeter group of Q is called *co- c -sortable* if there exists a reduced expression of w of the form $w = c^{(m)} \dots c^{(1)}c^{(0)}$, where all $c^{(t)}$ are subwords of c whose supports satisfy

$$\text{supp}(c^{(m)}) \subseteq \text{supp}(c^{(m-1)}) \subseteq \dots \subseteq \text{supp}(c^{(1)}) \subseteq \text{supp}(c^{(0)}) \subseteq Q_0.$$

If $i \in Q_0$ is in the support of $c^{(t)}$, by abuse of notation, we will write $i \in c^{(t)}$.

Remark 2.1. The word w is co- c -sortable if and only if w^{-1} is c^{-1} -sortable in the sense of [Rea07].

Let $w = c^{(m)} \dots c^{(1)}c^{(0)}$ be a co- c -sortable word. Let $Q^{(1)}$ be the full subquiver of Q whose support is the same as $c^{(1)}$. Then the word $w' = c^{(m)} \dots c^{(1)}$ is co- $c^{(1)}$ -sortable as an element of the Coxeter group $C_{Q^{(1)}}$. For $t \geq 1$ and i in $c^{(t)}$, we define $kQ^{(1)}$ -modules $T_{(i,t)}$ by induction as follows:

- We put $T_{(i,1)} = e_i D(kQ^{(1)})$ for all $i \in c^{(1)}$.
- For $t \geq 2$, assume that we have defined $T_{(j,s)}$ for $1 \leq s \leq t-1$ and $j \in c^{(s)}$, and $T_{(n,t)}, \dots, T_{(i+1,t)}$. Then $T_{(i,t)}$ is defined to be the kernel of the map

$$f: E \rightarrow T_{(i,t-1)}$$

where f is a minimal right $\text{add}(\bigoplus_{j < i} T_{(j,t-1)} \oplus \bigoplus_{j > i} T_{(j,t)})$ -approximation.

For i in $Q_0^{(1)}$, we define m_i such that i is in $c^{(m_i)}$ but not in $c^{(m_i+1)}$. We define the $kQ^{(1)}$ -module

$$T := \bigoplus_{i \in Q_0^{(1)}} T_{(i,m_i)}.$$

Here is the dual version of results of [AIRT09].

Theorem 2.3. *Let w' , $Q^{(1)}$ and T be as above. Then*

- T is a tilting $kQ^{(1)}$ -module with finite torsion class;
- the torsion class $\mathcal{M} := \text{Fac} T = \{X \in \text{mod } kQ \text{ s.t. } \text{Ext}_{kQ^{(1)}}^1(T, X) = 0\}$ is the additive category $\text{add}\{T_{(i,t)}, t \geq 1, i \in c^{(t)}\}$;
- the sequences $0 \longrightarrow T_{(i,t)} \longrightarrow E \xrightarrow{f} T_{(i,t-1)} \longrightarrow 0$ are exact and are the almost split sequences of $\mathcal{M} = \text{Fac}(T)$.

We have also the following result which is a generalization of Theorem 5.21 of [Ami08].

Theorem 2.4 ([AIRT09]). *Let Q , $w = c^{(m)} \dots c^{(0)}$, $w' = c^{(m)} \dots c^{(1)}$, $Q^{(1)}$ and $T_{(i,t)}$ (for $t \geq 1$ and $i \in c^{(t)}$) be as above. Define the endomorphism algebra*

$$A := \text{End}_{kQ^{(1)}}\left(\bigoplus_{t \geq 1} \bigoplus_{i \in c^{(t)}} T_{(i,t)}\right).$$

Then the algebra A is of global dimension ≤ 2 and there exists a commutative diagram of triangle functors:

$$\begin{array}{ccc} \mathcal{D}^b(A) & \xrightarrow{F_A} & \underline{\text{Sub}}\Lambda_w, \\ \downarrow \pi_A & \nearrow f_A & \\ \mathcal{C}_A & & \end{array}$$

where f_A is an equivalence, and where $F_A(A) = C_w$.

2.3. Main result. From now on we assume that $w = c^{(m)} \dots c^{(1)}c^{(0)}$ is a co- c -sortable word. Combining Theorems 2.1 and 2.4, we get two algebras A and Γ of global dimension ≤ 2 , with the following diagram

$$\begin{array}{ccc} \mathcal{D}^b(A) & \overset{?}{\dashrightarrow} & \mathcal{D}^b(\Gamma) . \\ \pi_A \downarrow & \begin{array}{c} \nearrow F_A \\ \nwarrow F_\Gamma \end{array} & \downarrow \pi_\Gamma \\ \mathcal{C}_A & \xrightarrow{\sim} \underline{\text{Sub}}\Lambda_w \xleftarrow{\sim} & \mathcal{C}_\Gamma \end{array}$$

As we have seen in the first section, we do not automatically get a derived equivalence between A and Γ . The aim of this paper is to prove that there is an equivalence in this case and that this derived equivalence is given by a tilting module which is easy to describe. More precisely we will show the following.

Theorem 2.5. *Let $w = c^{(m)} \dots c^{(0)}$ be a co- c -sortable element in the Coxeter group of Q . Let Γ and A be the algebras of global dimension ≤ 2 associated to w defined in section 2.*

For $p \geq 1$ and $j \in c^{(p)}$ define the indecomposable projective A -module

$$P_{(j,p)} := \text{Hom}_{kQ^{(1)}}\left(\bigoplus_{t \geq 1} \bigoplus_{i \in c^{(t)}} T_{(i,t)}, T_{(j,p)}\right),$$

and the complex

$$M = \bigoplus_{p=1}^m \bigoplus_{j \in c^{(p)}} \mathbb{S}_2^{-p+1}(P_{(j,p)}),$$

where \mathbb{S}_2 is the autoequivalence $\mathbb{S}[-2]$ of $\mathcal{D}^b(A)$. Then we have

- (1) M is a tilting module;
- (2) $\text{End}_A(M) \simeq \Gamma$;
- (3) the functor $R\text{Hom}_A(M, -)$ makes the following diagram commute

$$\begin{array}{ccc} \mathcal{D}^b(A) & \xrightarrow{R\text{Hom}_A(M, -)} & \mathcal{D}^b(\Gamma) . \\ & \begin{array}{c} \searrow F_A \\ \swarrow F_\Gamma \end{array} & \\ & \underline{\text{Sub}}\Lambda_w & \end{array}$$

Remark 2.2. Note that we have equivalences

$$\begin{aligned} \text{gl. dim } A = 1 &\Leftrightarrow m = 1 \\ &\Leftrightarrow \text{gl. dim } \Gamma = 1 \\ &\Leftrightarrow A \simeq \Gamma \end{aligned}$$

Therefore we can assume from now that $m \geq 2$.

3. CONSTRUCTION OF A TILTING MODULE

This section consists of the proof of (1) of Theorem 2.5. We assume that $w = c^{(m)} \dots c^{(0)}$ is a co- c -sortable word with $m \geq 2$. For $t \geq 1$ and $i \in c^{(t)}$, we define $kQ^{(1)}$ -modules $T_{(i,t)}$ as in subsection 2.1. We define the tilting $kQ^{(1)}$ -module $T := \bigoplus_{i \in c^{(1)}} T_{(i,m_i)}$, where m_i is the maximal integer such that $i \in c^{(m_i)}$. We define

$$\mathcal{M} := \text{Fac}(T) = \text{add} \{T_{(i,t)}, 1 \leq t \leq m, i \in c^{(t)}\}, \text{ and } A := \text{End}_{kQ^{(1)}} \left(\bigoplus_{t \geq 1, i \in c^{(t)}} T_{(i,t)} \right).$$

The first subsection is devoted to basic lemmas on the category $\mathbf{mod} A$: Lemma 3.1 describes the spaces of morphisms between objects in $\mathbf{mod} A$, and Lemma 3.2 describes the action of the functor ${}_A \mathbb{S}_2$ on the projective indecomposables A -modules. It permits to prove (1) of Theorem 2.5 using 2-APR-tilting in subsection 3.2.

3.1. The category $\mathbf{mod} A$. The algebra A is defined as the endomorphism algebra of the sum of pairwise non-isomorphic indecomposable $kQ^{(1)}$ -modules in $\text{Fac}(T) = \mathcal{M}$. Thus the category $\mathbf{mod} A$ is equivalent to the category $\mathbf{mod} \mathcal{M}$ of finitely presented functors $\mathcal{M}^{op} \rightarrow \mathbf{mod} k$. Through this equivalence, indecomposable projective A -modules are the functors of the form $\mathcal{M}(-, X)$ where X is an indecomposable object in \mathcal{M} and indecomposable injective A -modules are the functors of the form $D\mathcal{M}(X, -)$ where X is an indecomposable object in \mathcal{M} .

The category \mathcal{M} , as a torsion class, has almost split sequences (cf Theorem 2.4). We will denote by τ its Auslander-Reiten translation. Moreover, since \mathcal{M} is a finite torsion class, any indecomposable object X of \mathcal{M} is isomorphic to some $\tau^p(e_i DkQ^{(1)})$ for some $p \geq 0$ and some $i \in Q_0^{(1)}$.

Let \mathcal{T} be a full subcategory of \mathcal{M} and X be an object of \mathcal{M} . We define the \mathcal{M} -module $\mathcal{M}(-, X)/[\mathcal{T}]$ as the cokernel of

$$\mathcal{M}(-, T) \rightarrow \mathcal{M}(-, X)$$

induced by a minimal right \mathcal{T} -approximation $T \rightarrow X$.

We first state a lemma which describes morphisms between objects in $\mathbf{mod} \mathcal{M}$ of the form $\mathcal{M}(-, X)/[\mathcal{T}]$ in terms of morphisms in \mathcal{M} .

Lemma 3.1. *Let \mathcal{M} be a k -linear category with finitely many indecomposables. Let A be the Auslander algebra of \mathcal{M} . Let \mathcal{T} and \mathcal{S} be full subcategories of \mathcal{M} , then we have an isomorphism for any X and Y in \mathcal{M} between*

$$\text{Hom}_A(\mathcal{M}(-, X)/[\mathcal{T}], \mathcal{M}(-, Y)/[\mathcal{S}])$$

and the space of commutative squares up to homotopy

$$\begin{array}{ccc} T & \xrightarrow{t} & X \\ \downarrow & & \downarrow \\ S & \xrightarrow{s} & Y \end{array}$$

where t (resp. s) is a minimal right \mathcal{T} (resp. \mathcal{S}) -approximation.

Proof. Let $t : T \rightarrow X$ be a minimal \mathcal{T} - approximation of X . Then the projective presentation of the module $\mathcal{M}(-, X)/[\mathcal{T}]$ is

$$\mathcal{M}(-, T) \xrightarrow{\mathcal{M}(-, t)} \mathcal{M}(-, X) \longrightarrow \mathcal{M}(-, X)/[\mathcal{T}] \longrightarrow 0$$

Thus the space of morphisms

$$\mathrm{Hom}_A(\mathcal{M}(-, X)/[\mathcal{T}], \mathcal{M}(-, Y)/[\mathcal{S}])$$

is isomorphic to the space of commutative squares

$$\begin{array}{ccc} \mathcal{M}(-, T) & \xrightarrow{\mathcal{M}(-, t)} & \mathcal{M}(-, X) \\ \downarrow & & \downarrow \\ \mathcal{M}(-, S) & \xrightarrow{\mathcal{M}(-, s)} & \mathcal{M}(-, Y) \end{array}$$

up to homotopy, where $s : S \rightarrow Y$ is a minimal right \mathcal{S} -approximation. By the Yoneda lemma, this is isomorphic to the space of commutative squares

$$\begin{array}{ccc} T & \xrightarrow{t} & X \\ \downarrow & & \downarrow \\ S & \xrightarrow{s} & Y \end{array}$$

up to homotopy. □

For $t \geq 1$ we define the subcategory \mathcal{T}_t of \mathcal{M} as

$$\mathcal{T}_t := \mathrm{add}(T \oplus \tau^{-1}T \oplus \cdots \oplus \tau^{-t+1}T).$$

Note that $\mathcal{T}_m = \mathcal{M}$ and by convention $\mathcal{T}_0 = \emptyset$.

We then have the following lemma which is a variant of Lemma 5.9 of [Ami08].

Lemma 3.2. *Let X be an object of \mathcal{M} and $n \geq 0$ such that $\tau^{-n}X \neq 0$, where τ is the AR-translation of the category \mathcal{M} . Then we have an isomorphism in $\mathrm{mod} \mathcal{M}$:*

$$\mathbb{S}_2^{-n}(\mathcal{M}(-, X)) \simeq \mathcal{M}(-, \tau^{-n}X)/[\mathcal{T}_n]$$

Proof. We prove this lemma by induction on n .

Let X be an indecomposable in \mathcal{M} such that $\tau^{-1}X$ is not zero. Since $\tau^{-1}X$ is in \mathcal{M} we have a short exact sequence

$$(1) \quad 0 \longrightarrow T_1 \longrightarrow T_0 \xrightarrow{f} \tau^{-1}X \longrightarrow 0$$

with T_0 and T_1 in $\mathcal{T}_1 = \mathbf{add}(T)$ and f a minimal right $\mathbf{add}(T)$ -approximation. The objects T_0 and T_1 are not zero since X is not zero and in \mathcal{M} . Thus we get an exact sequence in $\mathbf{mod}\mathcal{M}$

$$0 \longrightarrow \mathcal{M}(-, T_1) \longrightarrow \mathcal{M}(-, T_0) \longrightarrow \mathcal{M}(-, \tau^{-1}X) \longrightarrow \mathcal{M}(-, \tau^{-1}X)/[\mathcal{T}_1] \longrightarrow 0$$

which gives a projective resolution of the module $\mathcal{M}(-, \tau^{-1}X)/[\mathcal{T}_1]$. Therefore the object

$$\mathbb{S}_2(\mathcal{M}(-, \tau^{-1}X)/[\mathcal{T}_1])$$

is quasi-isomorphic to the complex

$$D\mathcal{M}(T_1, -) \longrightarrow D\mathcal{M}(T_0, -) \longrightarrow D\mathcal{M}(\tau^{-1}X, -)$$

By the short exact sequence (1) we get a long exact sequence in $\mathbf{mod}\mathcal{M}$:

$$D\mathrm{Ext}_{kQ(1)}^1(T_0, -)|_{\mathcal{M}} \rightarrow D\mathrm{Ext}_{kQ(1)}^1(\tau^{-1}X, -)|_{\mathcal{M}} \rightarrow D\mathcal{M}(T_1, -) \rightarrow D\mathcal{M}(T_0, -) \rightarrow D\mathcal{M}(\tau^{-1}X, -) \rightarrow 0.$$

By definition we have $\mathrm{Ext}_{kQ(1)}^1(T_0, \mathcal{M}) = 0$. Hence using the Auslander-Reiten formula

$$D\mathrm{Ext}_{\mathcal{M}}^1(\tau^{-1}X, -) \simeq \mathcal{M}(-, X),$$

we get the desired isomorphism

$$\mathbb{S}_2^{-1}(\mathcal{M}(-, X)) \simeq \mathcal{M}(-, \tau^{-1}X)/[\mathcal{T}_1].$$

Thus we have the assertion for $n = 1$.

Now let $n \geq 2$ and assume that for any Y such that $\tau^{-n+1}Y$ is not zero we have

$$\mathbb{S}_2^{-n+1}(\mathcal{M}(-, Y)) \simeq \mathcal{M}(-, \tau^{-n+1}Y)/[\mathcal{T}_{n-1}].$$

Let X be in \mathcal{M} such that $\tau^{-n}X$ is not zero. Then by the assertion for $n = 1$, we have an isomorphism

$$\mathbb{S}_2^{-n}(\mathcal{M}(-, X)) \simeq \mathbb{S}_2^{-n+1}(\mathcal{M}(-, \tau^{-1}X)/[\mathcal{T}_1]).$$

Since $\tau^{-n}X$ is not zero, the short exact sequence (1) yields a short exact sequence

$$0 \longrightarrow \tau^{-n+1}T_1 \longrightarrow \tau^{-n+1}T_0 \longrightarrow \tau^{-n}X \longrightarrow 0.$$

The objects $\tau^{-n+1}T_0$ and $\tau^{-n+1}T_1$ cannot be zero since $X, \tau^{-1}X, \dots, \tau^{-n}X$ are not zero.

By definition, the module $\mathcal{M}(-, \tau^{-1}X)/[\mathcal{T}_1]$ is quasi-isomorphic to the complex of projectives:

$$\mathcal{M}(-, T_1) \longrightarrow \mathcal{M}(-, T_0) \longrightarrow \mathcal{M}(-, \tau^{-1}X)$$

thus by the induction hypothesis $\mathbb{S}_2^{-n+1}(\mathcal{M}(-, \tau^{-1}X)/[\mathcal{T}_1])$ is quasi-isomorphic to the complex

$$\mathcal{M}(-, \tau^{-n+1}T_1)/[\mathcal{T}_{n-1}] \longrightarrow \mathcal{M}(-, \tau^{-n+1}T_0)/[\mathcal{T}_{n-1}] \longrightarrow \mathcal{M}(-, \tau^{-n}X)/[\mathcal{T}_{n-1}]$$

which is quasi-isomorphic to the module

$$\mathcal{M}(-, \tau^{-n}X)/[\mathcal{T}_n].$$

Hence we get the assertion for any $n \geq 1$.

□

3.2. 2-APR-tilting. The object M of Theorem 2.5 is constructed by applying powers of the functor \mathbb{S}_2 to summands of A . To prove that it is tilting, we will use the tool of 2-APR-tilting introduced by Iyama and Oppermann. The following result is Theorem 4.5 with Proposition 4.7 of [IO09].

Theorem 3.3 (Iyama-Oppermann). *Let Γ be a finite dimensional k -algebra of global dimension ≤ 2 . Suppose we can decompose $\Gamma = P \oplus Q$ as Γ -module in such a way that*

- (1) $\mathrm{Hom}_\Gamma(Q, P) = 0$;
- (2) $\mathrm{Ext}_\Gamma^{-1}(Q, \mathbb{S}_2^{-1}(P)) = 0$.

Then $T = \mathbb{S}_2^{-1}(P) \oplus Q$ is a tilting complex over Γ and $\mathrm{End}_\Gamma(T)$ is of global dimension ≤ 2 .

Applying recursively this theorem, we get the following.

Proposition 3.4. *Let Γ be a finite dimensional k -algebra of global dimension ≤ 2 . Suppose that we can decompose Γ as the sum $P_m \oplus \cdots \oplus P_1 \oplus P_0$ of Γ -modules in such a way that*

- (1) *for any s, t, j such that $s - j - t \geq 1$ and $j \geq 0$ the space $\mathrm{Hom}_\Gamma(P_t, \mathbb{S}_2^{-j} P_s)$ vanishes;*
- (2) *$\mathbb{S}_2^{-j} P_s$ is a module for $0 \leq j \leq s$.*

Then $T = \mathbb{S}_2^{-m} P_m \oplus \cdots \mathbb{S}_2^{-1} P_1 \oplus P_0$ is a tilting module and the algebra $\mathrm{End}_\Gamma(T)$ is of global dimension ≤ 2 .

Proof. We prove by induction on $i \geq 0$ that the object

$$T_i = \mathbb{S}_2^{-i}(P_m \oplus \cdots \oplus P_i) \oplus \mathbb{S}_2^{-i+1} P_{i-1} \oplus \cdots \oplus \mathbb{S}_2^{-1} P_1 \oplus P_0$$

is a tilting module over Γ and that the endomorphism algebra $\Gamma_i := \mathrm{End}_\Gamma(T_i)$ has global dimension ≤ 2 . This holds for $i = 0$ by hypothesis. Let us assume that this holds for an $i \geq 0$. The functor $F_i = R\mathrm{Hom}_\Gamma(T_i, -)$ yields a triangle equivalence

$$\begin{array}{ccc} F_i : \mathcal{D}^b(\Gamma) & \xrightarrow{\sim} & \mathcal{D}^b(\Gamma_i) \\ \text{\scriptsize \hookrightarrow} & & \text{\scriptsize \hookrightarrow} \\ \text{\scriptsize $\mathbb{S}_2 = -\otimes_\Gamma[-2]$} & & \text{\scriptsize } {}_i\mathbb{S}_2 = -\otimes_{\Gamma_i} D\Gamma_i[-2] \end{array}$$

which sends T_i to Γ_i . By the uniqueness of the Serre functor we have an isomorphism

$$F_i \circ \mathbb{S}_2 = R\mathrm{Hom}_\Gamma(T_i, - \otimes_\Gamma D\Gamma[-2]) \simeq R\mathrm{Hom}_\Gamma(T_i, -) \otimes_{\Gamma_i} D\Gamma_i[-2] = {}_i\mathbb{S}_2 \circ F_i.$$

We want to apply Theorem 3.3 to

$$P := F_i(\mathbb{S}_2^{-i}(P_m \oplus \cdots \oplus P_{i+1})) \text{ and } Q := F_i(\mathbb{S}_2^{-i}(P_i) \oplus \cdots \oplus \mathbb{S}_2^{-1}(P_1) \oplus P_0).$$

We have

$$\begin{aligned} \mathrm{Hom}_{\Gamma_i}(Q, P) &= \mathrm{Hom}_{\Gamma_i}(F_i(\mathbb{S}_2^{-i}(P_i) \oplus \cdots \oplus \mathbb{S}_2^{-1}(P_1) \oplus P_0), F_i(\mathbb{S}_2^{-i}(P_m \oplus \cdots \oplus P_{i+1}))) \\ &\simeq \mathrm{Hom}_\Gamma(\mathbb{S}_2^{-i}(P_i) \oplus \cdots \oplus \mathbb{S}_2^{-1}(P_1) \oplus P_0, \mathbb{S}_2^{-i}(P_m \oplus \cdots \oplus P_{i+1})) \\ &= 0 \text{ by (1)}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \mathrm{Ext}_{\Gamma_i}^{-1}(Q, {}_i\mathbb{S}_2^{-1} P) &= \mathrm{Ext}_{\Gamma_i}^{-1}(F_i(\mathbb{S}_2^{-i}(P_i) \oplus \cdots \oplus \mathbb{S}_2^{-1}(P_1) \oplus P_0), {}_i\mathbb{S}_2^{-1} F_i(\mathbb{S}_2^{-i}(P_m \oplus \cdots \oplus P_{i+1}))) \\ &\simeq \mathrm{Ext}_{\Gamma_i}^{-1}(F_i(\mathbb{S}_2^{-i}(P_i) \oplus \cdots \oplus \mathbb{S}_2^{-1}(P_1) \oplus P_0), F_i(\mathbb{S}_2^{-(i+1)}(P_m \oplus \cdots \oplus P_{i+1}))) \\ &\simeq \mathrm{Ext}_\Gamma^{-1}(\mathbb{S}_2^{-i}(P_i) \oplus \cdots \oplus \mathbb{S}_2^{-1}(P_1) \oplus P_0, \mathbb{S}_2^{-(i+1)}(P_m \oplus \cdots \oplus P_{i+1})) \end{aligned}$$

By (2), for $1 \leq j \leq s$ the object $\mathbb{S}_2^{-j} P_s$ is a module, hence the space $\mathrm{Ext}_\Gamma^{-1}(P_l, \mathbb{S}_2^{-j} P_s)$ vanishes for any l . Therefore the space $\mathrm{Ext}_{\Gamma_i}^{-1}(Q, {}_i\mathbb{S}_2^{-1} P)$ vanishes.

Thus by Theorem 3.3, ${}_i\mathbb{S}_2^{-1}(P) \oplus Q \simeq F_i(T_{i+1})$ is a tilting complex in $\mathcal{D}^b(\Gamma_i)$, hence T_{i+1} is a tilting complex in $\mathcal{D}^b(\Gamma)$, and a module by (2), and its endomorphism algebra $\Gamma_{i+1} = \text{End}_\Gamma(T_{i+1}) \simeq \text{End}_{\Gamma_i}(F_i(T_{i+1}))$ is of global dimension ≤ 2 . Thus we get the proposition. \square

We want to apply this result to the decomposition $A = \bigoplus_{t=1}^m (\bigoplus_{i \in c^{(t)}} P_{(i,t)})$ where $P_{(i,t)} := \mathcal{M}(-, T_{(i,t)})$ is the projective indecomposable A -module defined in Theorem 2.5.

Proposition 3.5. *The complex $M = \bigoplus_{t=1}^m \bigoplus_{i \in c^{(t)}} \mathbb{S}_2^{-t+1}(P_{(i,t)})$ is a tilting A -module.*

Proof. If i is in $c^{(t)}$, then $\tau^{-t+1}(T_{(i,t)})$ is isomorphic to $e_i D(kQ^{(1)})$. Now for $t = 1, \dots, m$, we denote by P_t the projective A -module $\bigoplus_{i \in c^{(t)}} P_{(i,t)}$. For i in $c^{(t)}$, the indecomposable projective $P_{(i,t)}$ is of the form $\mathcal{M}(-, T_{(i,t)})$. Thus we have $P_{(i,t)} = \mathcal{M}(-, \tau^{t-1}(e_i DkQ^{(1)}))$. Therefore we can write

$$P_t = \mathcal{M}(-, \tau^{t-1}(DkQ^{(1)})).$$

Note that if i is not in $c^{(t)}$ then $\tau^{t-1}(e_i DkQ^{(1)})$ is zero, thus the decomposition of P_t into indecomposables is given by

$$P_t = \bigoplus_{i \in c^{(t)}} \mathcal{M}(-, \tau^{t-1}(e_i DkQ^{(1)})) = \bigoplus_{i \in c^{(t)}} \mathcal{M}(-, T_{(i,t)}).$$

We want to apply Proposition 3.4 to the decomposition $A = P_m \oplus \dots \oplus P_1$. By Lemma 3.2, we know that for any $0 \leq j \leq s-1$, we have

$$\mathbb{S}_2^{-j} P_s \simeq \mathcal{M}(-, \tau^{s-1-j} DkQ^{(1)}) / [\mathcal{T}_j]$$

which is a module. Thus we have condition (2) of Proposition 3.4.

For $s-j-t \geq 1$ we have isomorphisms

$$\begin{aligned} \text{Hom}_A(P_t, \mathbb{S}_2^{-j} P_s) &= \text{Hom}_A(\mathcal{M}(-, \tau^{t-1}(DkQ^{(1)})), \mathbb{S}_2^{-j}(\mathcal{M}(-, \tau^{s-1}(DkQ^{(1)})))) \\ &= \text{Hom}_A(\mathcal{M}(-, \tau^{t-1}(DkQ^{(1)})), \mathcal{M}(-, \tau^{s-1-j}(DkQ^{(1)})) / [\mathcal{T}_j]) && \text{by Lemma 3.2} \\ &\simeq \mathcal{M}(\tau^{t-1}(DkQ^{(1)}), \tau^{s-j-1}(DkQ^{(1)})) / [\mathcal{T}_j] && \text{by Lemma 3.1} \\ &\simeq \mathcal{M}(DkQ^{(1)}, \tau^{s-j-t}(DkQ^{(1)})) / \text{add}(\tau^{1-t}T \oplus \dots \oplus \tau^{2-t-j}(T)) \end{aligned}$$

Since $s-j-t \geq 1$ the space $\mathcal{M}(DkQ^{(1)}, \tau^{s-j-t} DkQ^{(1)})$ vanishes. Hence we have condition (1) of Proposition 3.4. Therefore the complex

$$M = \bigoplus_{t=1}^m \mathbb{S}_2^{-t+1}(P_t) = \bigoplus_{t=1}^m \bigoplus_{i \in c^{(t)}} \mathbb{S}_2^{1-t}(P_{(i,t)})$$

is a tilting module. \square

4. COMPUTATION OF THE ENDOMORPHISM ALGEBRA

In this section, we prove (2) of Theorem 2.5, that is that the endomorphism algebra $\text{End}_A(M)$ is isomorphic to the algebra Γ defined in section 2.1. The strategy consists of describing these two algebras with a quiver and an ideal of relations.

Let $w = c^{(m)} \dots c^{(0)}$ be a co- c -sortable word, and define $w' = c^{(m)} \dots c^{(1)}$. Let $R_{w'}$ be the following quiver:

- its vertices are (i, t) where i is in $c^{(t)}$;
- for $i \in Q_0^{(1)}$, for $t \geq 1$ such that i is in $c^{(t+1)}$, we have an arrow $q_t^i : (i, t) \rightarrow (i, t+1)$;
- for any $a : i \rightarrow j$ in $Q_1^{(1)}$ such that $i, j \in c^{(t)}$ we have an arrow $a_t : (i, t) \rightarrow (j, t)$.

We define an ideal $\mathcal{J}_{w'}$ of relations on the path algebra $kR_{w'}$ generated by commutative squares

$$\begin{array}{ccc} (i, t) & \xrightarrow{q_t^i} & (i, t+1) \\ \downarrow a_t & & \downarrow a_{t+1} \\ (j, t) & \xrightarrow{q_t^j} & (j, t+1) \end{array}$$

when all these arrows are defined, and by zero relations

$$(i, t) \xrightarrow{a_t^i} (j, t) \xrightarrow{q_t^j} (j, t+1)$$

when i is not in $c^{(t+1)}$.

Lemma 4.1. *The algebra Γ is isomorphic to the algebra $kR_{w'}/\mathcal{J}_{w'}$.*

Proof. In the case where w is co- c -sortable, the quivers Q_w and \bar{Q}_w described in section 1 are much simpler. The orientation of Q satisfies condition $(*)$ of section 2.1 if and only if it satisfies $(**)$ of section 2.2. It is routine to check that if we remove the Q^* -arrows of \bar{Q}_w we get $R_{w'}$, and that the partial derivatives $\partial_{a^*} W_w$ where a^* is a Q^* -arrow are exactly the relations generating $\mathcal{J}_{w'}$. \square

Proposition 4.2. *There exists an algebra isomorphism*

$$G : kR_{w'}/\mathcal{J}_{w'} \rightarrow \text{End}_A\left(\bigoplus_{t=1}^m \bigoplus_{i \in c^{(t)}} \mathbb{S}_2^{-t+1}(P_{(i,t)})\right) = \text{End}_A(M).$$

Proof. We divide the proof in several steps.

Step 1: Definition of the map $G : kR_{w'}/\mathcal{J}_{w'} \rightarrow \text{End}_A(M)$.

We define G on the vertices of $R_{w'}$. For i in $c^{(t)}$ we put

$$G(i, t) = \mathbb{S}_2^{-t+1}(P_{(i,t)}).$$

Let t be integers $\leq m$, i be in $c^{(t)}$ and j be in $c^{(s)}$. By Lemma 3.2, we have an isomorphism $\text{Hom}_A(\mathbb{S}_2^{-t+1}(P_{(i,t)}), \mathbb{S}_2^{-s}(P_{(j,s)})) \simeq \text{Hom}_A(\mathcal{M}(-, e_i(DkQ^{(1)}))/[\mathcal{T}_{t-1}], \mathcal{M}(-, e_j(DkQ^{(1)}))/[\mathcal{T}_{s-1}])$.

It is isomorphic, by Lemma 3.1, to the space of commutative squares

$$\begin{array}{ccc} \mathcal{T}_{t-1} & \longrightarrow & e_i(DkQ^{(1)}) \\ \downarrow & & \downarrow \\ \mathcal{T}_{s-1} & \longrightarrow & e_j(DkQ^{(1)}) \end{array}$$

up to homotopy, where horizontal maps are minimal right \mathcal{T}_{t-1} (resp. \mathcal{T}_{s-1})-approximations.

Here is an easy result which will often be used in the proof.

Lemma 4.3. *For $t \geq 1$ we have equivalences*

- $i \notin c^{(t+1)}$,
- $\tau^t(e_i DkQ^{(1)}) = 0$,
- $e_i DkQ^{(1)} \in \mathcal{T}_t$.

Let i be in $Q_0^{(1)}$ and t be an integer ≥ 1 . Let $f : T_{t-1} \rightarrow e_i(DkQ^{(1)})$ be a minimal right \mathcal{T}_{t-1} -approximation and let $g : T_t \rightarrow e_i(DkQ^{(1)})$ be a minimal right \mathcal{T}_t -approximation. Since we have the inclusion $\mathcal{T}_{t-1} \subset \mathcal{T}_t$, then there exists a commutative square

$$U_t^i := \begin{array}{ccc} T_{t-1} & \xrightarrow{f} & e_i(DkQ^{(1)}) \\ \downarrow & & \parallel \\ T_t & \xrightarrow{g} & e_i(DkQ^{(1)}) \end{array} .$$

It is homotopic to zero if and only if $e_i(DkQ^{(1)})$ is in \mathcal{T}_t . This is equivalent to the fact that i is not in $c^{(t+1)}$ by Lemma 4.3. Thus for i in $c^{(t+1)}$ we define

$$G(q_t^i) = U_t^i \neq 0.$$

Let $a : i \rightarrow j$ be an arrow in $Q_1^{(1)}$ and t be an integer ≥ 1 . Let $f : T_{t-1} \rightarrow e_i(DkQ^{(1)})$ and $g : T'_{t-1} \rightarrow e_j(DkQ^{(1)})$ be minimal right \mathcal{T}_{t-1} -approximations. Then we have a commutative square

$$S(a)_t := \begin{array}{ccc} T_{t-1} & \xrightarrow{f} & e_i(DkQ^{(1)}) \\ \downarrow & & \downarrow a \\ T'_{t-1} & \xrightarrow{g} & e_j(DkQ^{(1)}) \end{array}$$

If this square is homotopic to zero then there exists a map $h : e_i(DkQ^{(1)}) \rightarrow T'_{t-1}$ such that $a = g \circ h$. Since a is an irreducible map, h is a section or g is a retraction. Thus $e_i(DkQ^{(1)})$ or $e_j(DkQ^{(1)})$ are in \mathcal{T}_{t-1} . By Lemma 4.3, this means that either $i \notin c^{(t)}$ or $j \notin c^{(t)}$.

Therefore for any $a : i \rightarrow j$ in Q_1 and for $t \geq 1$ such that $i, j \in c^{(t)}$, we put

$$G(a_t) = S(a)_t \neq 0.$$

Moreover for any i, j in $c^{(t)}$ we have a commutative diagram:

$$\begin{array}{ccccc} T_{t-1} & \xrightarrow{\quad} & e_i(DkQ^{(1)}) & \xrightarrow{\quad} & e_j(DkQ^{(1)}) \\ & \searrow & \parallel & & \downarrow \\ & & T'_{t-1} & \xrightarrow{\quad} & e_j(DkQ^{(1)}) \\ & \downarrow & \downarrow & & \parallel \\ T_t & \xrightarrow{\quad} & e_i(DkQ^{(1)}) & \xrightarrow{\quad} & e_j(DkQ^{(1)}) \\ & \searrow & \downarrow & & \downarrow \\ & & T'_t & \xrightarrow{\quad} & e_j(DkQ^{(1)}) \end{array}$$

This implies that in $\mathbf{End}_A(M)$ we have the relations:

$$U_t^j \circ S(a)_t = S(a)_{t+1} \circ U_t^i$$

if $S(a)_{t+1}$ is not zero, thus if i and j are in $c^{(t+1)}$. And we have the relation

$$U_t^j \circ S(a)_t = 0$$

if i is not in $c^{(t+1)}$.

Thus we get a morphism $G : kR_{w'}/\mathcal{J}_{w'} \rightarrow \mathbf{End}_A(\bigoplus_t \bigoplus_{i \in c^{(t)}} \mathbb{S}_2^{-t+1}(P_{(i,t)}) = \mathbf{End}_A(M)$.

Step 2: The map G is surjective.

We will show that the squares of the form $S(a)_t$ and U_t^i generate the algebra $\mathbf{End}_A(M)$.

Let α be a path in $Q^{(1)}$ from i to j . We denote by $length(\alpha)$ its length. For a commutative square

$$S := \begin{array}{ccc} T_{t-1} & \longrightarrow & e_i(DkQ^{(1)}) \\ \downarrow & & \downarrow \alpha \\ T_{s-1} & \longrightarrow & e_j(DkQ^{(1)}) \end{array}$$

let us define the size of S by

$$size(S) = s - t + length(\alpha).$$

We have for all integers $t \geq 1$, all i in $Q_0^{(1)}$ and all a in $Q_1^{(1)}$, $size(U_t^i) = size(S(a)_t) = 1$.

We will first show that the only non zero squares S with $size(S) \leq 0$ are the isomorphism and then $size(S) = 0$.

Let $s \leq t$ be two integers, i be in $c^{(t)}$ and j be in $c^{(s)}$. Suppose there is a commutative square

$$S := \begin{array}{ccc} T_{t-1} & \xrightarrow{t} & e_i(DkQ^{(1)}) \\ \downarrow u & & \downarrow \alpha \\ T_{s-1} & \xrightarrow{s} & e_j(DkQ^{(1)}) \end{array}$$

where α is non zero path, and where t (resp. s) is a minimal right \mathcal{T}_{t-1} (resp. \mathcal{T}_{s-1})-approximation. Since $s \leq t$, we have $\mathcal{T}_{s-1} \subset \mathcal{T}_{t-1}$. The approximation t is not zero, hence u is not zero and T_{t-1} must be in \mathcal{T}_{s-1} . Let $g : T'_{s-1} \rightarrow e_i(DkQ^{(1)})$ be a minimal right \mathcal{T}_{s-1} -approximation, then we have such a factorization:

$$\begin{array}{ccc} T_{t-1} & \xrightarrow{t} & e_i(DkQ^{(1)}) \\ \uparrow \downarrow & & \parallel \\ T'_{s-1} & \xrightarrow{g} & e_i(DkQ^{(1)}) \end{array}$$

This square gives an isomorphism between $\mathcal{M}(-, e_i(DkQ^{(1)}))/[\mathcal{T}_{s-1}]$ and $\mathcal{M}(-, e_i(DkQ^{(1)}))/[\mathcal{T}_{t-1}]$, thus we have $s = t$.

Finally we get that all squares of size < 0 are zero. Moreover, all squares of size 0 are isomorphisms and all squares of size 1 which are not homotopic to zero are the $S(a)_t$ and U_t^i .

Now we will show that all square S such that $size(S) \geq 2$ can be written as a composition of squares of size stricly smaller. Let $s \geq t$ be positive integers, $\alpha \neq 0$ be a path from i to j in $Q^{(1)}$. Let S be a non zero commutative square non homotopic to zero with $size(S) \geq 2$:

$$\begin{array}{ccc} T_{t-1} & \xrightarrow{t} & e_i(DkQ^{(1)}) \\ \downarrow u & & \downarrow \alpha \\ T_{s-1} & \xrightarrow{s} & e_j(DkQ^{(1)}) \end{array}$$

where t (resp. s) is a minimal right \mathcal{T}_{t-1} (resp. \mathcal{T}_{s-1})-approximation. Assume that s is $\geq t + 1$. Then we have a commutative diagram

$$\begin{array}{ccc}
& T_{t-1} & \xrightarrow{t} & e_i(DkQ^{(1)}) \\
& \swarrow & \downarrow & \parallel \\
T_t & \xrightarrow{t'} & e_i(DkQ^{(1)}) & \searrow \\
& \searrow & \downarrow & \downarrow \alpha \\
& T_{s-1} & \xrightarrow{s} & e_j(DkQ^{(1)})
\end{array}$$

where t' is a minimal right \mathcal{T}_t approximation. Thus the square S is the composition $B \circ U_t^i$ where

$$B := \begin{array}{ccc}
T_t & \xrightarrow{t'} & e_i(DkQ^{(1)}) \\
\downarrow & & \downarrow \alpha \\
T_{s-1} & \xrightarrow{s} & e_j(DkQ^{(1)})
\end{array} .$$

We have $size(B) = s - t - 1 + length(\alpha)$ and $size(U_t^i) = 1$.

If $s = t$ and if α is a composition of arrows $a_1 \circ \dots \circ a_n$ of $Q_1^{(1)}$ with $n \geq 2$, then we have $S = S(a_1)_t \circ B$ where $size(B) = length(\alpha) - 1$. Therefore $(U_t^i, S(a)_t)$ generate the algebra $End(\bigoplus_t \bigoplus_{i \in c^{(t)}} \mathbb{S}_2^{-t+1}(P_{(i,t)}))$ and the morphism G is surjective.

Step 3: The map G is injective

Let x be a linear combination of paths from (i, t) to (j, s) in $R_{w'}$ which is non zero in $kR_{w'}/\mathcal{J}_{w'}$. Then we have $s \geq t$, $i \in c^{(t)}$ and $j \in c^{(s)}$. The element x can be written as a sum $\sum_u x_u$ where for each u there is a path

$$\alpha_u := i = i_1 \xrightarrow{a^1} i_2 \xrightarrow{a^2} \dots \xrightarrow{a^{n-1}} i_n \xrightarrow{a^n} j$$

in $Q^{(1)}$ such that $i_l \in c^{(s)}$ for $l = 1, \dots, n$ and

$$x_u = \lambda_u q_t^i q_{t+1}^i \dots q_{s-1}^i a_s^n a_s^{n-1} \dots a_s^1$$

where λ_u is in the field k .

Now assume that $G(x)$ is a commutative square homotopic to zero

$$\begin{array}{ccc}
T_{t-1} & \longrightarrow & e_i(DkQ^{(1)}) \\
\downarrow & \swarrow h & \downarrow \alpha \\
T_{s-1} & \longrightarrow & e_j(DkQ^{(1)})
\end{array}$$

where $\alpha = \sum_u \lambda_u \alpha_u$ and where horizontal maps are minimal right \mathcal{T}_{t-1} and \mathcal{T}_{s-1} -approximations. Since $s \geq t$, we have a factorization:

$$\begin{array}{ccc}
& T_{t-1} & \xrightarrow{\quad} & e_i(DkQ^{(1)}) \\
& \swarrow & \downarrow & \parallel \\
T'_{s-1} & \xrightarrow{\quad} & e_i(DkQ^{(1)}) & \searrow \\
& \searrow & \downarrow & \downarrow \alpha \\
& T_{s-1} & \xrightarrow{\quad} & e_j(DkQ^{(1)})
\end{array}$$

Thus the square

$$\begin{array}{ccc} T'_{s-1} & \longrightarrow & e_i(DkQ^{(1)}) \\ \downarrow & \swarrow h & \downarrow \alpha \\ T_{s-1} & \longrightarrow & e_j(DkQ^{(1)}) \end{array}$$

is homotopic to zero.

Therefore for all u there exists a factorization

$$\alpha_u : e_i(DkQ^{(1)}) \xrightarrow{\beta_u} e_{i_{l(u)}}(DkQ^{(1)}) \xrightarrow{\gamma_u} e_j(DkQ^{(1)})$$

with $e_{i_{l(u)}}$ in \mathcal{T}_{s-1} . Thus by Lemma 4.3 $i_{l(u)}$ is not in $c^{(s)}$ for all u . This is a contradiction. Therefore the morphism G is injective. □

We are now able to prove Theorem 2.5. Combining Proposition 3.5 with Lemma 4.1 and Proposition 4.2, we get that M is a tilting module over A and that $\text{End}_A(M) \simeq \Gamma$. Therefore by Theorem 1.6 of [Hap87] we have a derived equivalence

$$R\text{Hom}_A(M, -) : \mathcal{D}^b(A) \xrightarrow{\sim} \mathcal{D}^b(\Gamma) .$$

We still have to prove that the diagram

$$\begin{array}{ccc} \mathcal{D}^b(A) & \xrightarrow{R\text{Hom}_A(M, -)} & \mathcal{D}^b(\Gamma) \\ \pi_A \downarrow & \searrow F_A & \swarrow F_\Gamma \\ \mathcal{C}_A & \longrightarrow & \underline{\text{Sub}}\Lambda_w \end{array}$$

commutes. The tilting A -module M is sent to Γ by the functor $R\text{Hom}_A(M, -)$, and thus to the cluster-tilting object C_w in $\underline{\text{Sub}}\Lambda_w$ by Theorem 2.1. By definition of the generalized cluster category, the objects $\pi_A(\mathbb{S}_2(X))$ and $\pi_A(X)$ are isomorphic in the category \mathcal{C}_A , therefore we have an isomorphism in \mathcal{C}_A

$$\pi_A(M) \simeq \pi_A(A).$$

Hence by Theorem 2.4 the object M is sent to the cluster-tilting object C_w in $\underline{\text{Sub}}\Lambda_w$. The triangle functors $\mathcal{D}^b(A) \rightarrow \underline{\text{Sub}}\Lambda_w$ and $\mathcal{D}^b(\Gamma) \rightarrow \underline{\text{Sub}}\Lambda_w$ are given by tensor products (see [AIRT09] and [ART09]). We can now conclude using the fact that two triangle functors which are tensor products and which coincide on a tilting object are isomorphic. Therefore the diagram above is commutative. And we finish the proof of Theorem 2.5.

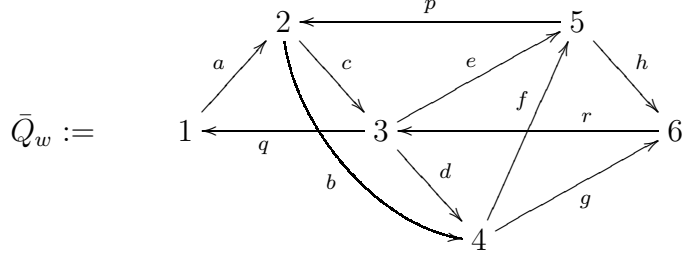
5. EXAMPLE

Let Q be the quiver

$$\begin{array}{ccc} & 2 & \\ \nearrow & & \searrow \\ 1 & \longrightarrow & 3 \end{array} .$$

Let w be the following word $w := s_3s_2s_3s_1s_2s_3s_1s_2s_3$. Then we have $w' = s_3s_2s_3s_1s_2s_3$ and the quiver $Q^{(1)}$ is Q . The endomorphism algebra in $\underline{\text{Sub}}\Lambda_w$ of the standard cluster-tilting object

C_w of Theorem 1.4 is the Jacobian algebra $Jac(\bar{Q}_w, W_w)$ (Theorem 1.5) where



and $W_w := gdr + her + fbp - ecp + caq$. The arrows $\{c, f, g, h\}$ are the Q -arrows, $\{a, b, d, e\}$ are the Q^* -arrows, and $\{p, q, r\}$ are the arrows going to the left.

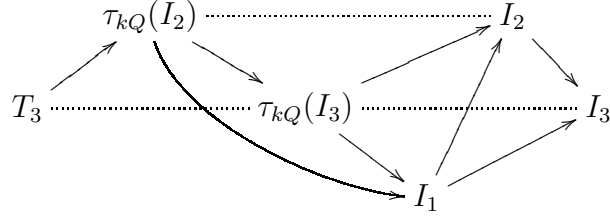
Then T is the module

$$T = \begin{matrix} 3 \\ 2 \\ 1 \end{matrix} \begin{matrix} 3 \\ 2 \\ 1 \end{matrix} \oplus \begin{matrix} 3 \\ 3 \\ 1 \end{matrix} \begin{matrix} 3 \\ 2 \\ 1 \end{matrix} \oplus \begin{matrix} 3 \\ 1 \end{matrix} = I_1 \oplus \tau_{kQ}(I_2) \oplus T_3$$

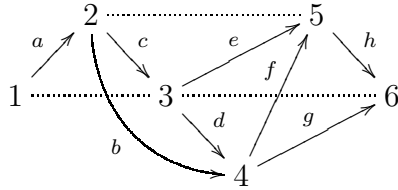
The torsion class $\mathcal{M} = \text{Fac}(T)$ has finitely many indecomposables, namely we have

$$\mathcal{M} = \left\{ \begin{matrix} 3 \\ 2 \\ 1 \end{matrix}, \begin{matrix} 3 \\ 3 \\ 1 \end{matrix}, \begin{matrix} 3 \\ 2 \\ 1 \end{matrix}, \begin{matrix} 3 \\ 3 \\ 1 \end{matrix}, \begin{matrix} 3 \\ 1 \end{matrix} \right\} = \{I_1, I_2, I_3, \tau_{kQ}(I_3), \tau_{kQ}(I_2), T_3\}$$

The Auslander-Reiten quiver of \mathcal{M} is



Therefore the algebra A is given by the quiver



with the relations $he - gd = 0$, $fb - ec = 0$ and $ca = 0$.

The projective indecomposable A -modules are

$$P_1 \oplus P_2 \oplus P_3 \oplus P_4 \oplus P_5 \oplus P_6 = \begin{matrix} 6 \\ 2 \\ 1 \end{matrix} \oplus \begin{matrix} 2 \\ 1 \end{matrix} \oplus \begin{matrix} 3 \\ 2 \end{matrix} \oplus \begin{matrix} 3 \\ 4 \\ 2 \\ 1 \end{matrix} \oplus \begin{matrix} 3 \\ 5 \\ 4 \\ 3 \\ 2 \end{matrix} \oplus \begin{matrix} 6 \\ 5 \\ 4 \\ 2 \\ 3 \\ 1 \end{matrix}.$$

The injective indecomposable A -modules are

$$I_1 \oplus I_2 \oplus I_3 \oplus I_4 \oplus I_5 \oplus I_6 = \begin{matrix} 6 \\ 4 \\ 1 \end{matrix} \oplus \begin{matrix} 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{matrix} \oplus \begin{matrix} 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{matrix} \oplus \begin{matrix} 6 \\ 5 \\ 4 \\ 3 \end{matrix} \oplus \begin{matrix} 6 \\ 5 \\ 4 \\ 6 \end{matrix} \oplus \begin{matrix} 6 \\ 5 \end{matrix} \oplus \begin{matrix} 6 \end{matrix}.$$

Let us compute $\mathbb{S}_2^{-1}(P_1) = \mathbb{S}_2^{-1}(1)$. The minimal injective resolution of P_1 is:

$$\begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix} \longrightarrow \begin{matrix} 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{matrix} \longrightarrow \begin{matrix} 6 \\ 5 \\ 4 \\ 3 \\ 5 \end{matrix} = (I_1 \longrightarrow I_2 \longrightarrow I_3)$$

Thus we get

$$\mathbb{S}_2^{-1}P_1 = (P_1 \longrightarrow P_2 \longrightarrow P_3) = (1 \longrightarrow \frac{2}{1} \longrightarrow \frac{3}{2})$$

which is quasi-isomorphic to the simple 3. Note that $P_1 \simeq \mathcal{M}(-, T_3)$, and that the support of the functor

$$\mathcal{M}(-, \tau_{\mathcal{M}}^{-1}(T_3))/[\text{add } T] = \mathcal{M}(-, \tau_{kQ}(I_3))/[\text{add}(I_1 \oplus \tau_{kQ}(I_2) \oplus T_3)]$$

is just $\tau_{kQ}(I_3)$ so viewed as a A -module, it is the simple 3. Therefore we have

$$\mathbb{S}_2^{-1}(\mathcal{M}(-, T_3)) \simeq \mathcal{M}(-, \tau_{\mathcal{M}}^{-1}T_3)/[\text{add } T] \text{ (cf. Lemma 3.2).}$$

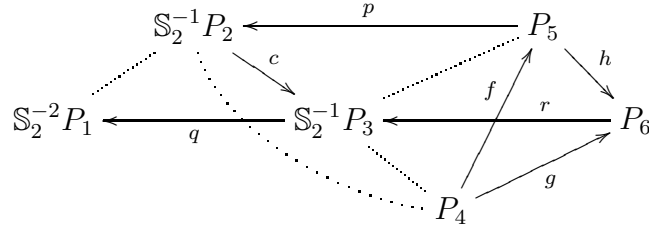
In the same way, we can check that

$$\mathbb{S}_2^{-2}(P_1) = 6, \quad \mathbb{S}_2^{-1}(P_2) = \frac{5}{3} \quad \text{and} \quad \mathbb{S}_2^{-1}(P_3) = \frac{6}{5}.$$

The object M of Theorem 2.5 is

$$M := \mathbb{S}_2^{-2}(P_1) \oplus \mathbb{S}_2^{-1}(P_2 \oplus P_3) \oplus (P_4 \oplus P_5 \oplus P_6).$$

The endomorphism algebra $\text{End}_A(M)$ is given by the quiver



with relations $rh - cp = 0$, $rg = 0$, $qc = 0$ and $pf = 0$. It is isomorphic to the algebra Γ defined in section 2.1.

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