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ON THE SPECTRAL ANALYSIS OF MANY-BODY SYSTEMS

MONDHER DAMAK AND VLADIMIR GEORGESCU

ABSTRACT. We describe the essential spectrum and prove the Mourre estimate for quantum particle systems interacting through k -body forces and creation-annihilation processes which do not preserve the number of particles. For this we compute the “Hamiltonian algebra” of the system, i.e. the C^* -algebra \mathcal{C} generated by the Hamiltonians we want to study, and show that, as in the N -body case, it is graded by a semilattice. Hilbert C^* -modules graded by semilattices are involved in the construction of \mathcal{C} . For example, if we start with an N -body system whose Hamiltonian algebra is \mathcal{C}_N and then we add field type couplings between subsystems, then the many-body Hamiltonian algebra \mathcal{C} is the imprimitivity algebra of a graded Hilbert \mathcal{C}_N -module.

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1. INTRODUCTION

1.1. The quantum systems studied in this paper are obtained by coupling a certain number (finite or infinite) of N -body systems. A (standard) N -body system consists of a fixed number N of particles which interact through k -body forces which preserve N (arbitrary $1 \leq k \leq N$). The many-body type interactions include forces which allow the system to make transitions between states with different numbers of particles. These transitions are realized by creation-annihilation processes as in quantum field theory.

The Hamiltonians we want to analyze are rather complex objects and standard Hilbert space techniques seem to us inefficient in this situation. Our approach is based on the observation that the C^* -algebra \mathcal{C} generated by a class of physically interesting Hamiltonians often has a quite simple structure which allows one to describe its quotient with respect to the ideal of compact operators in rather explicit terms [GI1, GI2]. From this one can deduce certain important spectral properties of the Hamiltonians. We refer to \mathcal{C} as the *Hamiltonian algebra* (or C^* -algebra of Hamiltonians) of the system.

The main difficulty in this algebraic approach is to isolate the correct C^* -algebra. This is especially problematic in the present situations since it is not a priori clear how to define the couplings between the various N -body systems but in very special situations. It is rather remarkable that the C^* -algebra generated by a small class of elementary and natural Hamiltonians will finally prove to be a fruitful choice. These elementary Hamiltonians are analogs of the Pauli-Fierz Hamiltonians.

The purpose of the preliminary Section 2 is to present this approach in the simplest but physically important case when the configuration spaces of the N -body systems are Euclidean spaces. We start with a fundamental example, the standard N -body case. Then we describe the many-body formalism in the Euclidean case and we state our main results on the spectral analysis of the corresponding Hamiltonians.

There is one substantial simplification in the Euclidean case: each subspace has a canonical supplement, the subspace orthogonal to it. This plays a role in the way we present the framework in Section 2. However, the main constructions and results do not depend on the existence of a supplement but to see this requires more sophisticated tools from the theory of crossed product C^* -algebras and Hilbert C^* -modules which are not apparent in this introductory part. In the rest of the paper we consider many-body type couplings of systems whose configuration space is an arbitrary abelian locally compact group. One of the simplest nontrivial physically interesting cases covered by this framework is that when the configuration spaces of the N -body systems are discrete groups, e.g. discretizations \mathbb{Z}^D of \mathbb{R}^D .

1.2. We summarize now the content of the paper. Section 2 starts with a short presentation of the standard N -body formalism, the rest of the section being devoted to a rather detailed description of our framework and main results in the case when the configuration spaces of the N -body subsystems are Euclidean spaces. These results are proven in a more general and natural setting in the rest of the paper. In Section 3 we recall some facts concerning C^* -algebras graded by a semilattice \mathcal{S} (we take here into account the results of Athina Mageira's thesis [Ma1]) and then we present some results on \mathcal{S} -graded Hilbert C^* -modules. This notion, due to Georges Skandalis [Ska], proved to be very natural and useful in our context: thanks to it many results can be expressed in a simple and systematic way thus giving a new and interesting perspective to the subject (this is discussed in more detail in [DaG4]). The heart of the paper is Section 4, where we define the many-body Hamiltonian algebra \mathcal{C} in a general setting and prove that it is naturally graded by a certain semilattice \mathcal{S} . In Section 5 we give alternative descriptions of the components of \mathcal{C} which are important for the affiliation criteria presented in Section 6, where we point out a large class of self-adjoint operators affiliated to the many-body algebra. The \mathcal{S} -graded structure of \mathcal{C} gives then an HVZ type description of the essential spectrum for all these operators. The main result of Section 7 is the proof of the Mourre estimate for nonrelativistic many-body Hamiltonians. Finally, an Appendix is devoted to the question of generation of some classes of C^* -algebras by "elementary" Hamiltonians.

1.3. Notations. We recall some notations and terminology. If \mathcal{E}, \mathcal{F} are normed spaces then $L(\mathcal{E}, \mathcal{F})$ is the space of bounded operators $\mathcal{E} \rightarrow \mathcal{F}$ and $K(\mathcal{E}, \mathcal{F})$ the subspace consisting of compact operators. If \mathcal{G} is a third normed space and $(e, f) \mapsto ef$ is a bilinear map $\mathcal{E} \times \mathcal{F} \rightarrow \mathcal{G}$ then $\mathcal{E}\mathcal{F}$ is the linear subspace of \mathcal{G} generated by the elements ef with $e \in \mathcal{E}, f \in \mathcal{F}$ and $\mathcal{E} \cdot \mathcal{F}$ is its closure. If $\mathcal{E} = \mathcal{F}$ then we set $\mathcal{E}^2 = \mathcal{E} \cdot \mathcal{E}$. Two unusual abbreviations are convenient: by *lspan* and *clspan* we mean "linear span" and "closed linear span" respectively. If \mathcal{A}_i are subspaces of a normed space then $\sum_i^c \mathcal{A}_i$ is the clspan of $\cup_i \mathcal{A}_i$. If X is a locally compact topological space then $C_o(X)$ is the space of continuous complex functions which tend to zero at infinity and $C_c(X)$ the subspace of functions with compact support.

By *ideal* in a C^* -algebra we mean a closed self-adjoint ideal. A $*$ -homomorphism between two C^* -algebras will be called *morphism*. We write $\mathcal{A} \simeq \mathcal{B}$ if the C^* -algebras \mathcal{A}, \mathcal{B} are isomorphic and $\mathcal{A} \cong \mathcal{B}$ if they are canonically isomorphic (the isomorphism should be clear from the context).

A self-adjoint operator H on a Hilbert space \mathcal{H} is *affiliated* to a C^* -algebra \mathcal{A} of operators on \mathcal{H} if $(H + i)^{-1} \in \mathcal{A}$; then $\varphi(H) \in \mathcal{A}$ for all $\varphi \in C_o(\mathbb{R})$. If \mathcal{A} is the closed linear span of the elements $\varphi(H)A$ with $\varphi \in C_o(\mathbb{R})$ and $A \in \mathcal{A}$, we say that H is *strictly affiliated* to \mathcal{A} . The C^* -algebra generated by a set \mathcal{E} of self-adjoint operators is the smallest C^* -algebra such that each $H \in \mathcal{E}$ is affiliated to it.

We now recall the definition of \mathcal{S} -graded C^* -algebras following [Ma2]. Here \mathcal{S} is a *semilattice*, i.e. a set equipped with an order relation \leq such that the lower bound $\sigma \wedge \tau$ of each couple of elements σ, τ exists. We say that a subset \mathcal{T} of \mathcal{S} is a *sub-semilattice* of \mathcal{S} if $\sigma, \tau \in \mathcal{T} \Rightarrow \sigma \wedge \tau \in \mathcal{T}$. The set \mathcal{S} of all closed

subgroups of a locally compact abelian group is a semilattice for the order relation given by set inclusion. The semilattices which are of main interest for us are (inductive limits of) sub-semilattices of \mathcal{S} .

A C^* -algebra \mathcal{A} is called \mathcal{S} -graded if a linearly independent family $\{\mathcal{A}(\sigma)\}_{\sigma \in \mathcal{S}}$ of C^* -subalgebras of \mathcal{A} has been given such that $\sum_{\sigma \in \mathcal{S}}^c \mathcal{A}(\sigma) = \mathcal{A}$ and $\mathcal{A}(\sigma)\mathcal{A}(\tau) \subset \mathcal{A}(\sigma \wedge \tau)$ for all σ, τ . The algebras $\mathcal{A}(\sigma)$ are the *components* of \mathcal{A} . It is useful to note that some of the algebras $\mathcal{A}(\sigma)$ could be zero. If \mathcal{T} is a sub-semilattice of \mathcal{S} and $\mathcal{A}(\sigma) = \{0\}$ for $\sigma \notin \mathcal{T}$ we say that \mathcal{A} is *supported* by \mathcal{T} ; then \mathcal{A} is in fact \mathcal{T} -graded. Reciprocally, any \mathcal{T} -graded C^* -algebra becomes \mathcal{S} -graded if we set $\mathcal{A}(\sigma) = \{0\}$ for $\sigma \notin \mathcal{T}$.

1.4. Note. The preprint [DaG4] is a preliminary version of this paper. We decided to change the title because the differences between the two versions are rather important: the preliminaries concerning the theory of Hilbert C^* -modules and the role of the imprimitivity algebra of a Hilbert C^* -module in the spectral analysis of many-body systems are now reduced to a minimum; on the other hand, the Euclidean case and the spectral theory of the corresponding Hamiltonians are treated in more detail.

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2. EUCLIDEAN FRAMEWORK: MAIN RESULTS

2.1. The Hamiltonian algebra of a standard N -body system. Consider a system of N particles moving in the physical space \mathbb{R}^d . In the nonrelativistic case the Hamiltonian is of the form

$$H = \sum_{j=1}^N P_j^2/2m_j + \sum_{j=1}^N V_j(x_j) + \sum_{j < k} V_{jk}(x_j - x_k) \quad (2.1)$$

where m_1, \dots, m_N are the masses of the particles, $x_1, \dots, x_N \in \mathbb{R}^d$ their positions, and $P_j = -i\nabla_{x_j}$ their momenta. In the simplest situation the potentials V_j, V_{jk} are real continuous functions with compact support on \mathbb{R}^d . The state space of the system is the Hilbert space $L^2(X)$ with $X = (\mathbb{R}^d)^N$.

Let $P = (P_1, \dots, P_N)$, this is a set of commuting self-adjoint operators on $L^2(X)$ and so $h(P)$ is a well defined self-adjoint operator for any real Borel function h on X . In what follows we replace the kinetic energy part $\sum_j P_j^2/2m_j$ in (2.1) by an operator $h(P)$ with h continuous and divergent at infinity. Denote $x = (x_1, \dots, x_N)$ the points of X and let us consider the linear subspaces of X defined as follows: $X_j = \{x \in X \mid x_j = 0\}$ if $1 \leq j \leq N$ and $X_{jk} = \{x \in X \mid x_j = x_k\}$ for $j < k$. Let π_j and π_{jk} be the natural maps of X onto the quotient spaces X/X_j and X/X_{jk} respectively (the Euclidean structure of X allows us to identify these abstract spaces with the subspaces of X orthogonal to X_j and X_{jk} , but this is irrelevant here). Then H may be written in the form

$$H = h(P) + \sum_j v_j \circ \pi_j(x) + \sum_{j < k} v_{jk} \circ \pi_{jk}(x) \quad (2.2)$$

for some real functions $v_j \in \mathcal{C}_c(X/X_j)$ and $v_{jk} \in \mathcal{C}_c(X/X_{jk})$. Thus Hamiltonians of the form (2.2) are natural objects in the N -body problem. Note that there should be no privileged origin in the momentum space, so if we accept $h(P)$ as an admissible kinetic energy operator then $h(P + p)$ should also be admissible for any $p = (p_1, \dots, p_N) \in (\mathbb{R}^d)^N$.

Let $\mathcal{S} \equiv \mathcal{S}(X)$ be the set of linear subspaces of X equipped with the order relation $Y \leq Z \Leftrightarrow Y \subset Z$. Then \mathcal{S} is a semilattice with $Y \wedge Z = Y \cap Z$. If $Y \in \mathcal{S}$ then we realize $\mathcal{C}_o(X/Y)$ as a C^* -algebra of operators on $L^2(X)$ by associating to v the operator of multiplication by $v \circ \pi_Y$, where $\pi_Y : X \rightarrow X/Y$ is the canonical surjection. The following fact is easy to prove: *if $\mathcal{S} \subset \mathcal{S}$ is finite then $\mathcal{C}(\mathcal{S}) := \sum_{Y \in \mathcal{S}} \mathcal{C}_o(X/Y)$ is a direct topological sum and is an algebra if and only if \mathcal{S} is a sub-semilattice of \mathcal{S} ; in this case, $\mathcal{C}(\mathcal{S})$ is an \mathcal{S} -graded C^* -algebra.*

In the next proposition \mathcal{S} is the semilattice of subspaces of X generated by the X_j and X_{jk} , i.e. the set of subspaces of X obtained by taking arbitrary intersections of subspaces of the form X_j and X_{jk} . We denote \mathcal{T}_X the C^* -algebra of operators on $L^2(X)$ of the form $\varphi(P)$ with $\varphi \in \mathcal{C}_o(X)$.

Proposition 2.1. *Let $h : X \rightarrow \mathbb{R}$ be continuous with $h(x) \rightarrow \infty$ if $x \rightarrow \infty$ and let H_p be the self-adjoint operator (2.2) with $h(P)$ replaced by $h(P + p)$. Then the C^* -algebra generated by the operators H_p when p runs over $(\mathbb{R}^d)^N$ and v_j and v_{jk} run over the set of real functions in $C_c(X/X_j)$ and $C_c(X/X_{jk})$ respectively is $\mathcal{C} = \mathcal{C}(\mathcal{S}) \cdot \mathcal{T}_X = \sum_{Y \in \mathcal{S}} \mathcal{C}_o(X/Y) \cdot \mathcal{T}_X$. Moreover, \mathcal{C} is \mathcal{S} -graded by this decomposition.*

This has been proved in [DaG1]. More general results of this nature are presented in Appendix 8. Observe that we decided to fix the function h which represents the kinetic energy but not the potentials v_j, v_{jk} . However, as a consequence of Proposition 2.1, if we allow h to vary we get the same algebra.

Proposition 2.1 provides a basic example of ‘‘Hamiltonian algebra’’. We mention that \mathcal{C} is the crossed product of the C^* -algebra $\mathcal{C}(\mathcal{S})$ by the natural action of the additive group X , so it is a natural mathematical object. We shall see in a more general context that the set of self-adjoint operators affiliated to it is much larger than expected (cf. Theorem 2.13 for example).

If we are in the nonrelativistic case and $V_j = 0$ for all j then the center of mass of the system moves freely and it is more convenient to eliminate it and to take as origin of the reference system the center of mass of the N particles. Then the configuration space X is the set of points $x = (x_1, \dots, x_N) \in (\mathbb{R}^d)^N$ such that $\sum_k m_k x_k = 0$. Proposition 2.1 remains valid if \mathcal{S} is conveniently defined, see §2.3.

The following ‘‘generalized’’ class of N -body systems is suggested by results from [Ma1, Ma3].

Definition 2.2. *An N -body structure on a locally compact abelian group X is a set \mathcal{S}_X of closed subgroups such that $X \in \mathcal{S}_X$ and such that for all $Y, Z \in \mathcal{S}_X$ the following three conditions are satisfied: (i) $Y \cap Z \in \mathcal{S}$; (ii) the subgroups Y, Z of X are compatible; (iii) if $Y \supsetneq Z$ then Y/Z is not compact.*

X must be thought as configuration space of the system. The notion of compatible subgroups is defined in Subsection 4.3 (if X is a σ -compact topological space this means that $Y + Z$ is a closed subgroup). We shall see that the Hamiltonian algebra associated to such an N -body system is an interesting object:

$$\mathcal{C}_X(\mathcal{S}_X) := \mathcal{C}(\mathcal{S}_X) \cdot \mathcal{T}_X \cong \mathcal{C}(\mathcal{S}_X) \rtimes X \quad \text{where} \quad \mathcal{C}(\mathcal{S}_X) = \sum_{Y \in \mathcal{S}_X}^c \mathcal{C}_o(X/Y). \quad (2.3)$$

Here $\mathcal{T}_X \cong \mathcal{C}_o(X^*)$ is the group C^* -algebra of X and \rtimes means crossed product.

Example 2.3. This framework covers an interesting extension of the standard N -body setting. Assume that X is a finite dimensional real vector space. In the standard framework the semilattice \mathcal{S} consists of linear subspaces of X but here we allow them to be closed additive subgroups. The closed additive subgroups of X are of the form $Y = E + L$ where E is a vector subspace of X and L is a lattice in a vector subspace F of X such that $E \cap F = \{0\}$. More precisely, $L = \sum_k \mathbb{Z} f_k$ where $\{f_k\}$ is a basis in F . Thus F/L is a torus and if G is a third vector subspace such that $X = E \oplus F \oplus G$ then the space $X/Y \simeq (F/L) \oplus G$ is a cylinder with F/L as basis.

2.2. The Euclidean many-body algebra. We introduce here an abstract framework which allows us to study couplings between several N -body systems of the type considered above. A concrete and physically interesting example may be found in §2.3.

Let \mathcal{X} be a real prehilbert space. Let $\mathcal{S}(\mathcal{X})$ be the set of finite dimensional subspaces of \mathcal{X} equipped with the order relation given by set inclusion. This is a lattice with $X \wedge Y = X \cap Y$ and $X \vee Y = X + Y$, but only the semilattice structure is relevant for what follows.

Each finite dimensional subspace $X \subset \mathcal{X}$ is equipped with the Euclidean structure induced by \mathcal{X} hence the Hilbert space $\mathcal{H}_X = L^2(X)$ and the C^* -algebras $\mathcal{L}_X = L(\mathcal{H}_X)$ and $\mathcal{K}_X = K(\mathcal{H}_X)$ are well defined. The group algebra \mathcal{T}_X is defined as the closure in \mathcal{L}_X of the set of operators of convolution with functions of class $\mathcal{C}_c(X)$. If $O = \{0\}$ then $\mathcal{H}_O = \mathbb{C}$ and $\mathcal{L}_O = \mathcal{K}_O = \mathcal{T}_O = \mathbb{C}$ by convention.

We denote $h(P)$ the operator on \mathcal{H}_X given by $\mathcal{F}_X^{-1} M_h \mathcal{F}_X$, where \mathcal{F}_X is the Fourier transformation and M_h is the operator of multiplication by the function $h : X \rightarrow \mathbb{C}$. Then $\mathcal{T}_X = \{\psi(P) \mid \psi \in \mathcal{C}_o(X)\}$. We use the notation $P = P_X$ if the space X has to be specified.

If X, Y are finite dimensional subspaces of \mathcal{X} we set $\mathcal{L}_{XY} = L(\mathcal{H}_Y, \mathcal{H}_X)$ and $\mathcal{K}_{XY} = K(\mathcal{H}_Y, \mathcal{H}_X)$. We define a closed subspace $\mathcal{T}_{XY} \subset \mathcal{L}_{XY}$ as follows. If $\varphi \in \mathcal{C}_c(X+Y)$ then one may easily check that $(T_{XY}(\varphi)f)(x) = \int_Y \varphi(x-y)f(y)dy$ defines a continuous operator $\mathcal{H}_Y \rightarrow \mathcal{H}_X$. Let

$$\mathcal{T}_{XY} = \text{norm closure of the set of operators } T_{XY}(\varphi) \text{ with } \varphi \in \mathcal{C}_c(X+Y). \quad (2.4)$$

Clearly $\mathcal{T}_{XX} = \mathcal{T}_X$. The space \mathcal{T}_{XY} is a ‘‘concrete’’ realization of the Hilbert C^* -module introduced by Philip Green to show the Morita equivalence of the crossed products $\mathcal{C}_o(Z/Y) \rtimes X$ and $\mathcal{C}_o(Z/X) \rtimes Y$ where $Z = X + Y$ (this has been noticed by Georges Skandalis, see Remark 4.8 for more details).

Now we fix a sub-semilattice $\mathcal{S} \subset \mathcal{S}$, i.e. we assume that $X \cap Y \in \mathcal{S}$ if $X, Y \in \mathcal{S}$. This set completely determines the many-body system and the class of Hamiltonians that we intend to study. For each $X \in \mathcal{S}$ the Hilbert space \mathcal{H}_X is thought as the state space of an N -body system with X as configuration space. We define the state space of the many-body system as the Hilbertian direct sum

$$\mathcal{H} \equiv \mathcal{H}_{\mathcal{S}} = \oplus_{X \in \mathcal{S}} \mathcal{H}_X. \quad (2.5)$$

We have a natural embedding $\mathcal{L}_{XY} \subset L(\mathcal{H})$ for all $X, Y \in \mathcal{S}$. Let $\mathcal{L} \equiv \mathcal{L}_{\mathcal{S}}$ be the closed linear span of the subspaces \mathcal{L}_{XY} . Clearly \mathcal{L} is a C^* -subalgebra of $L(\mathcal{H})$ which is equal to $L(\mathcal{H})$ if and only if \mathcal{S} is finite. We will be interested in subspaces \mathcal{R} of \mathcal{L} constructed as follows: for each couple X, Y we are given a closed subspace $\mathcal{R}_{XY} \subset \mathcal{L}_{XY}$ and $\mathcal{R} \equiv (\mathcal{R}_{XY})_{X, Y \in \mathcal{S}} = \sum_{X, Y \in \mathcal{S}}^c \mathcal{R}_{XY}$ where \sum^c means closure of the sum. Note that $\mathcal{K} \equiv \mathcal{K}_{\mathcal{S}} = (\mathcal{K}_{XY})_{X, Y \in \mathcal{S}} = K(\mathcal{H})$.

Theorem 2.4. *Let $\mathcal{T} \equiv \mathcal{T}_{\mathcal{S}} = (\mathcal{T}_{XY})_{X, Y \in \mathcal{S}}$. Then \mathcal{T} is a closed self-adjoint subspace of \mathcal{L} and $\mathcal{C} \equiv \mathcal{C}_{\mathcal{S}} = \mathcal{T}^2$ is a non-degenerate C^* -algebra of operators on \mathcal{H} .*

We say that \mathcal{C} is the *Hamiltonian algebra of the many-body system \mathcal{S}* . This terminology will be justified later on: we shall see that physically interesting many-body Hamiltonians are self-adjoint operators affiliated to \mathcal{C} . Moreover, in a quite precise way, \mathcal{C} is the smallest C^* -algebra with this property. For the purposes of this paper we define a *many-body Hamiltonian as a self-adjoint operator affiliated to \mathcal{C}* .

We now equip \mathcal{C} with an \mathcal{S} -graded C^* -algebras structure. This structure will play a central role in the spectral analysis of self-adjoint operators affiliated to \mathcal{C} . We often say ‘‘graded’’ instead of \mathcal{S} -graded.

To define the grading we need new objects. If $Y \not\subset X$ we set $\mathcal{C}_X(Y) = \{0\}$. If $Y \subset X$ then we define $\mathcal{C}_X(Y)$ as the set of continuous functions on X which are invariant under translations in the Y directions and tend to zero in the Y^\perp directions. This is a C^* -algebra of bounded uniformly continuous functions on X canonically isomorphic with $\mathcal{C}_o(X/Y)$ where X/Y is the orthogonal of Y in X . Thus

$$\mathcal{C}_X(Y) \cong \mathcal{C}_o(X/Y) \text{ if } Y \subset X \quad \text{and} \quad \mathcal{C}_X(Y) = \{0\} \text{ if } Y \not\subset X. \quad (2.6)$$

Let $\mathcal{C}_X \equiv \mathcal{C}_X(\mathcal{S}) := \sum_{Y \in \mathcal{S}}^c \mathcal{C}_X(Y)$, this is a C^* -algebra of bounded uniformly continuous functions on X . We embed it in \mathcal{L}_X by identifying a function φ with the operator on \mathcal{H}_X of multiplication by φ . Then

$$\mathcal{C} \equiv \mathcal{C}_{\mathcal{S}} = \oplus_{X \in \mathcal{S}} \mathcal{C}_X \quad (2.7)$$

is a C^* -algebra of operators on \mathcal{H} included in \mathcal{L} . For each $Z \in \mathcal{S}$ we define a C^* -subalgebra of \mathcal{C} by

$$\mathcal{C}(Z) \equiv \mathcal{C}_{\mathcal{S}}(Z) = \oplus_X \mathcal{C}_X(Z) = \oplus_{X \supset Z} \mathcal{C}_X(Z). \quad (2.8)$$

It is easy to see that the family $\{\mathcal{C}(Z)\}_{Z \in \mathcal{S}}$ defines a graded C^* -algebra structure on \mathcal{C} .

Theorem 2.5. *We have $\mathcal{C} = \mathcal{T} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{T}$. For each $Z \in \mathcal{S}$ the space $\mathcal{C}(Z) = \mathcal{T} \cdot \mathcal{C}(Z) = \mathcal{C}(Z) \cdot \mathcal{T}$ is a C^* -subalgebra of \mathcal{C} and the family $\{\mathcal{C}(Z)\}_{Z \in \mathcal{S}}$ defines a graded C^* -algebra structure on \mathcal{C} .*

In particular we get $\mathcal{C} = (\mathcal{C}_{XY})_{X, Y \in \mathcal{S}}$ with (the second and fourth equalities are not obvious):

$$\mathcal{C}_{XY} = \mathcal{T}_{XY} \cdot \mathcal{C}_Y = \sum_{Z \subset X \cap Y}^c \mathcal{T}_{XY} \cdot \mathcal{C}_Y(Z) \quad (2.9)$$

$$= \mathcal{C}_X \cdot \mathcal{T}_{XY} = \sum_{Z \subset X \cap Y}^c \mathcal{C}_X(Z) \cdot \mathcal{T}_{XY}. \quad (2.10)$$

The C^* -algebras $\mathcal{C}(Z)$ and $\mathcal{C}(Z)$ ‘‘live’’ in the closed subspace $\mathcal{H}_{\geq Z} = \oplus_{X \supset Z} \mathcal{H}_X$ of \mathcal{H} . More precisely, they leave invariant $\mathcal{H}_{\geq Z}$ and their restriction to its orthogonal subspace is zero. Moreover, if we denote $\mathcal{L}_{\geq Z} = (\mathcal{L}_{X, Y})_{X, Y \supset Z} \subset L(\mathcal{H}_{\geq Z})$ then clearly $\mathcal{C}(Z)$ and $\mathcal{C}(Z)$ are subalgebras of $\mathcal{L}_{\geq Z}$.

Remark 2.6. The diagonal element $\mathcal{C}_{XX} \equiv \mathcal{C}_X$ of the “matrix” \mathcal{C} is given by

$$\mathcal{C}_X = \mathcal{C}_X \cdot \mathcal{T}_X = \sum_{Z \subset X}^c \mathcal{C}_o(X/Z) \cdot \mathcal{T}_X \subset \mathcal{L}_X. \quad (2.11)$$

This C^* -algebra is the *Hamiltonian algebra of the (generalized) N -body system* associated to the semilattice $\mathcal{S}_X = \{Z \in \mathcal{S} \mid Z \subset X\}$ of subspaces of X , cf. (2.3). The non-diagonal elements \mathcal{C}_{XY} are Hilbert C^* -bimodules which define the coupling between the N -body type systems X and Y .

Remark 2.7. Note that if we take \mathcal{S} equal to the set of all finite dimensional subspaces of \mathcal{X} then we get a graded C^* -algebra \mathcal{C} canonically associated to the prehilbert space \mathcal{X} . According to a remark in §1.3, any other choice of \mathcal{S} would give us a graded C^* -subalgebra of this one.

Remark 2.8. The algebra \mathcal{C} is not adapted to symmetry considerations, in particular in applications to physical systems consisting of particles one has to assume them distinguishable. The Hamiltonian algebra for systems of identical particles interacting through field type forces (both bosonic and fermionic case) is constructed in [Geo]. We mention that a quantum field model without symmetry considerations corresponds to the case when \mathcal{S} is a distributive relatively ortho-complemented lattice.

2.3. Particle systems with conserved total mass. We give now a physically interesting example of the preceding abstract construction. We shall describe the many-body system associated to N “elementary particles” of masses m_1, \dots, m_N moving in the physical space \mathbb{R}^d without external fields. We shall get *a system in which the the total mass is conserved but not the number of particles.*

We go back the framework of §2.1 but assume that the particles interact only through 2-body forces. Then

$$H = \sum_{j=1}^N P_j^2/2m_j + \sum_{j < k} V_{jk}(x_j - x_k). \quad (2.12)$$

and the center of mass of the system moves freely so it is convenient to eliminate it. This is a standard procedure that we sketch now, cf. [ABG, DeG1] for a detailed discussion of the formalism. We take as origin of the reference system the center of mass of the N particles so the configuration space X is the set of points $x = (x_1, \dots, x_N) \in (\mathbb{R}^d)^N$ such that $\sum_k m_k x_k = 0$. We equip X with the scalar product $\langle x|y \rangle = \sum_{k=1}^N 2m_k x_k y_k$. The advantage is that the reduced Hamiltonian, the operator acting in $L^2(X)$ naturally associated to the expression (2.12), is $\Delta_X + \sum_{j < k} V_{jk}(x_j - x_k)$ where Δ_X is the Laplacian associated to this scalar product. We denote by the same symbol H this reduced operator.

The first step is to describe the C^* -algebra generated by these Hamiltonians, i.e. to get the analog of Proposition 2.1 in the present context. Thus we have to describe the semilattice of subspaces of X generated by the $X_{(jk)} := X_{jk} \cap X$. We give the result below and refer to the Appendix §8.2 for proofs.

A partition σ of the set $\{1, \dots, N\}$ is also called cluster decomposition. Then the sets of the partition are called clusters. A cluster $a \in \sigma$ is thought as a “composite particle” of mass $m_a = \sum_{k \in a} m_k$. Let $|\sigma|$ be the number of clusters of σ . We interpret σ as a system of $|\sigma|$ particles with masses m_a hence its configuration space should be the set of $x = (x_a)_{a \in \sigma} \in (\mathbb{R}^d)^{|\sigma|}$ such that $\sum_a m_a x_a = 0$ equipped with a scalar product similar to that defined above.

Let X_σ be the set of $x \in X$ such that $x_i = x_j$ if i, j belong to the same cluster and let us equip X_σ with the scalar product induced by X . Then there is an obvious isometric identification of X_σ with the configuration space of the system σ as defined above. The advantage now is that all the spaces X_σ are isometrically embedded in the same X . We equip the set \mathfrak{S} of partitions with the order relation: $\sigma \leq \tau$ if and only if “ τ is finer than σ ” (this is opposite to the usual convention). Then $\sigma \leq \tau$ is equivalent to $X_\sigma \subset X_\tau$ and $X_\sigma \cap X_\tau = X_{\sigma \wedge \tau}$. Thus we see that \mathfrak{S} is isomorphic as semilattice with the set $\mathcal{S} = \{X_\sigma \mid \sigma \in \mathfrak{S}\}$ of subspaces of X with inclusion as order relation. Now it is easy to check that \mathcal{S} coincides with the semilattice of subspaces of X generated by the $X_{(jk)}$.

We abbreviate $\mathcal{H}_\sigma = \mathcal{H}_{X_\sigma} = L^2(X_\sigma)$. According to the identifications made above, this is the state space of a system of $|\sigma|$ particles with masses $(m_a)_{a \in \sigma}$.

Now we may apply our construction to \mathcal{S} . We get a system whose state space is $\mathcal{H} = \bigoplus_\sigma \mathcal{H}_\sigma$. If the system is in a state $u \in \mathcal{H}_\sigma$ then it consists of $|\sigma|$ particles of masses m_a . Note that $\min \mathfrak{S}$ is the partition

consisting of only one cluster $\{1, \dots, N\}$ with mass $M = m_1 + \dots + m_N$. Since there are no external fields and we decided to eliminate the motion of the center of mass, this system must be the vacuum. And its state space is indeed $\mathcal{H}_{\min \mathcal{S}} = \mathbb{C}$. The algebra \mathcal{C} in this case predicts usual inter-cluster interactions associated, for examples, to potentials defined on $X^\sigma = X/X_\sigma$, but also interactions which force the system to make a transition from a “phase” σ to a “phase” τ . In other terms, the system of $|\sigma|$ particles with masses $(m_a)_{a \in \sigma}$ is transformed into a system of $|\tau|$ particles with masses $(m_b)_{b \in \tau}$. Thus the number of particles varies from 1 to N but the total mass is constant and equal to M .

2.4. Natural morphisms and essential spectrum. We return to the general case. Sub-semilattices \mathcal{T} of \mathcal{S} define many-body type subsystems (this is discussed in more detail in §2.5). The spectral properties of the total many-body Hamiltonian are described in terms of a special class of such subsystems.

Each $X \in \mathcal{S}$ determines a new many-body system $\mathcal{S}_{\geq X} = \{Y \in \mathcal{S} \mid Y \supset X\}$ whose state space is $\mathcal{H}_{\geq X}$. Let $\mathcal{C}_{\geq X}$ be the corresponding Hamiltonian algebra $\mathcal{C}_{\mathcal{S}_{\geq X}}$. It is easy to see that $\mathcal{C}_{\geq X} = \sum_{Y \supset X}^c \mathcal{C}(Y)$. Thus $\mathcal{C}_{\geq X}$ is a C^* -subalgebra of \mathcal{C} which lives and is non-degenerate on the subspace $\mathcal{H}_{\geq X}$ of \mathcal{H} . We mention one fact: if $\Pi_{\geq X}$ is the orthogonal projection $\mathcal{H} \rightarrow \mathcal{H}_{\geq X}$, then $\Pi_{\geq X} \mathcal{C} \Pi_{\geq X}$ is an \mathcal{S} -graded C^* -subalgebra of $\mathcal{C}_{\mathcal{S}}$ and we have $\mathcal{C}_{\geq X} \subset \Pi_{\geq X} \mathcal{C} \Pi_{\geq X}$ strictly in general.

Then the general theory of graded C^* -algebras implies that *there is a unique linear continuous projection $\mathcal{P}_{\geq X} : \mathcal{C} \rightarrow \mathcal{C}_{\geq X}$ such that $\mathcal{P}_{\geq X}(T) = 0$ if $T \in \mathcal{C}(Y)$ with $Y \not\supset X$ and this projection is a morphism.* These are the *natural morphisms* of the graded algebra \mathcal{C} .

This extends to unbounded operators as follows: if H is a self-adjoint operator on \mathcal{H} strictly affiliated to \mathcal{C} then there is a unique self-adjoint operator $H_{\geq X} \equiv \mathcal{P}_{\geq X}(H)$ on $\mathcal{H}_{\geq X}$ such that $\mathcal{P}_{\geq X}(\varphi(H)) = \varphi(H_{\geq X})$ for all $\varphi \in \mathcal{C}_o(\mathbb{R})$. If H is only affiliated to \mathcal{C} then $H_{\geq X}$ could be not densely defined.

Assume that the semilattice \mathcal{S} has a smallest element $\min \mathcal{S}$. Then $X \in \mathcal{S}$ is called *atom* if the only element of \mathcal{S} strictly included in X is $\min \mathcal{S}$. Let $\mathcal{P}(\mathcal{S})$ be the set of atoms of \mathcal{S} . We say that \mathcal{S} is *atomic* if each of its elements distinct from $\min \mathcal{S}$ contains an atom. The following HVZ type theorem is an immediate consequence Theorem 3.2. The symbol $\bar{\cup}$ means “closure of union”.

Theorem 2.9. *If H is a self-adjoint operator on \mathcal{H} strictly affiliated to \mathcal{C} then for each $X \in \mathcal{S}$ there is a unique self-adjoint operator $H_{\geq X} \equiv \mathcal{P}_{\geq X}(H)$ on $\mathcal{H}_{\geq X}$ such that $\mathcal{P}_{\geq X}(\varphi(H)) = \varphi(H_{\geq X})$ for all $\varphi \in \mathcal{C}_o(\mathbb{R})$. The operator $H_{\geq X}$ is strictly affiliated to $\mathcal{C}_{\geq X}$. If $O \in \mathcal{S}$ and \mathcal{S} is atomic then*

$$\text{Sp}_{\text{ess}}(H) = \bar{\cup}_{X \in \mathcal{P}(\mathcal{S})} \text{Sp}(H_{\geq X}). \quad (2.13)$$

The theorem remains valid for operators which are only affiliated to \mathcal{C} but then we must allow them to be non-densely defined.

2.5. Subsystems and subhamiltonians. If \mathcal{T} is an arbitrary subset of \mathcal{S} then the Hilbert space $\mathcal{H}_{\mathcal{T}} = \bigoplus_{X \in \mathcal{T}} \mathcal{H}_X$ is well defined and naturally embedded as a closed subspace of $\mathcal{H}_{\mathcal{S}}$. Let $\Pi_{\mathcal{T}}$ be the orthogonal projection of \mathcal{H} onto $\mathcal{H}_{\mathcal{T}}$. Note that the definition of $\mathcal{T}_{\mathcal{S}}$ makes sense for any set \mathcal{S} of finite dimensional subspaces, in particular we may replace \mathcal{S} by \mathcal{T} . Then we have:

$$\mathcal{T}_{\mathcal{T}} = \Pi_{\mathcal{T}} \mathcal{T}_{\mathcal{S}} \Pi_{\mathcal{T}}, \quad \mathcal{C}_{\mathcal{T}}^{\mathcal{T}} := \Pi_{\mathcal{T}} \mathcal{C}_{\mathcal{S}} \Pi_{\mathcal{T}} = \bigoplus_{X \in \mathcal{T}} \mathcal{C}_X(\mathcal{S}), \quad (2.14)$$

$$\mathcal{C}_{\mathcal{S}}^{\mathcal{T}} := \Pi_{\mathcal{T}} \mathcal{C}_{\mathcal{S}} \Pi_{\mathcal{T}} = \Pi_{\mathcal{T}} \mathcal{T}_{\mathcal{S}} \cdot \mathcal{C}_{\mathcal{S}} \Pi_{\mathcal{T}} = \mathcal{T}_{\mathcal{T}} \cdot \mathcal{C}_{\mathcal{S}}^{\mathcal{T}} \quad (2.15)$$

From this we easily get that $\mathcal{C}_{\mathcal{S}}^{\mathcal{T}} = \sum_{X, Y \in \mathcal{T}}^c \mathcal{C}_{XY}$ is an \mathcal{S} -graded C^* -subalgebra of \mathcal{C} supported by the ideal generated by \mathcal{T} in \mathcal{S} (an ideal is a subset \mathcal{J} of \mathcal{S} such that $X \subset Y \in \mathcal{J} \Rightarrow X \in \mathcal{J}$). The operators affiliated to $\mathcal{C}_{\mathcal{S}}^{\mathcal{T}}$ are affiliated to $\mathcal{C}_{\mathcal{S}}$, so are many-body Hamiltonians in our sense.

The case when \mathcal{T} is a sub-semilattice of \mathcal{S} is interesting. Indeed, then \mathcal{T} defines a many-body system whose Hamiltonian algebra $\mathcal{C}_{\mathcal{T}}$ is a \mathcal{T} -graded C^* -algebra of operators on the Hilbert space $\mathcal{H}_{\mathcal{T}}$. We emphasize that *this algebra does not coincide with $\mathcal{C}_{\mathcal{S}}^{\mathcal{T}}$* . We always have $\mathcal{C}_{\mathcal{T}} \subset \mathcal{C}_{\mathcal{S}}^{\mathcal{T}}$ but *the inclusion is strict unless \mathcal{T} is an ideal of \mathcal{S}* . This is clear because $\mathcal{C}_{\mathcal{T}} \subset \mathcal{C}_{\mathcal{S}}^{\mathcal{T}}$ strictly in general.

The simplest sub-semilattices are the chains (totally ordered subsets). Then $\mathcal{H}_{\mathcal{T}}$ has a structure analogous to a Fock space. Nonrelativistic Hamiltonians affiliated to such C^* -algebras $\mathcal{C}_{\mathcal{S}}^{\mathcal{T}}$ have been studied before in [SSZ]. If we take $\mathcal{T} = \{X\}$ for an arbitrary $X \in \mathcal{S}$ then the associated subsystem has \mathcal{H}_X as state space and its Hamiltonian algebra is just $\mathcal{C}_X = \mathcal{C}_X(\mathcal{S}) \cdot \mathcal{T}_X$, the Hamiltonian algebra of the (generalized) N -body system determined by the semi-lattice \mathcal{S}_X . We refer to §4.6 and to the Example 2.12 for other simple but instructive examples of subsystems.

2.6. Intrinsic descriptions. We give two explicit descriptions of $\mathcal{C}(Z)$ as an algebra of operators on $\mathcal{H}_{\geq Z}$. Since \mathcal{C} is the closure of the sum of the algebras $\mathcal{C}(Z)$, these descriptions allow one to check rather easily whether a self-adjoint operator is affiliated to \mathcal{C} or not. Both theorems are consequences of more general results in Section 5.

For any vector $a \in \mathcal{X}$ and any finite dimensional subspace X of \mathcal{X} we define two unitary operators in \mathcal{H}_X by $(U_a f)(x) = f(x + a_X)$ and $(V_a f)(x) = e^{i\langle a|x \rangle} f(x)$ where a_X is the orthogonal projection of x on X . Then $\{U_a\}_{a \in \mathcal{X}}$ and $\{V_a\}_{a \in \mathcal{X}}$ are strongly continuous representations of (the additive group) \mathcal{X} on \mathcal{H}_X such that $U_a = 1 \Leftrightarrow V_a = 1 \Leftrightarrow a \perp X$. The direct sum over $X \in \mathcal{S}$ of these representations give representations of \mathcal{X} on \mathcal{H} for which we use the same notations.

Theorem 2.10. $\mathcal{C}(Z)$ is the set of $T \in \mathcal{L}_{\geq Z}$ such that

- (i) $U_a^* T U_a = T$ for all $a \in Z$ and $\|T(V_a - 1)\| \rightarrow 0$ if $a \rightarrow 0$ in Z^\perp ,
- (ii) $\|T(U_a - 1)\| \rightarrow 0$ and $\|V_a^* T V_a - T\| \rightarrow 0$ if $a \rightarrow 0$ in \mathcal{X} .

Let \mathcal{S}/Z be the set of subspaces of $\mathcal{X}/Z = Z^\perp$ of the form X/Z with $X \in \mathcal{S}, X \supset Z$. Clearly \mathcal{S}/Z is a semilattice of finite dimensional subspaces of \mathcal{X}/Z so the Hilbert space $\mathcal{H}_{\mathcal{S}/Z}$ and the corresponding algebra of compact operators $\mathcal{K}_{\mathcal{S}/Z}$ are well defined. If $X \supset Z$ then $X = Z \oplus X/Z$ so we have a canonical factorization $\mathcal{H}_X = \mathcal{H}_Z \otimes \mathcal{H}_{X/Z}$. Thus $\mathcal{H}_{\geq Z} = \mathcal{H}_Z \otimes \mathcal{H}_{\mathcal{S}/Z}$.

Theorem 2.11. $\mathcal{C}(Z) = \mathcal{T}_Z \otimes \mathcal{K}_{\mathcal{S}/Z}$ relatively to the factorization $\mathcal{H}_{\geq Z} = \mathcal{H}_Z \otimes \mathcal{H}_{\mathcal{S}/Z}$.

2.7. Factorization properties. For $Z \subset X \cap Y$ we have $X = Z \oplus (X/Z)$ and $Y = Z \oplus (Y/Z)$ hence we have canonical factorizations

$$\mathcal{H}_X = \mathcal{H}_Z \otimes \mathcal{H}_{X/Z} \quad \text{and} \quad \mathcal{H}_Y = \mathcal{H}_Z \otimes \mathcal{H}_{Y/Z}. \quad (2.16)$$

Relatively to these factorizations, we get from Theorem 2.11:

$$\mathcal{C}_{XY}(Z) = \mathcal{T}_Z \otimes \mathcal{K}_{X/Z, Y/Z} \cong \mathcal{C}_o(Z^*; \mathcal{K}_{X/Z, Y/Z}). \quad (2.17)$$

The tensor product (and those below) is in the category of Hilbert modules, cf. §3.4. We have written Z^* above in spite of the canonical Euclidean isomorphism $Z^* \cong Z$ in order to stress that we consider functions of momentum not of position. For any X, Y we set

$$X/Y = X/(X \cap Y) = X \ominus (X \cap Y) \quad (2.18)$$

and so we have

$$X/Z = X/(X \cap Y) \oplus (X \cap Y)/Z = X/Y \oplus (X \cap Y)/Z \quad (2.19)$$

and similarly for Y/Z . Then from (2.17) and (7.16) we get the finer factorization:

$$\mathcal{C}_{XY}(Z) = \mathcal{T}_Z \otimes \mathcal{K}_{(X \cap Y)/Z} \otimes \mathcal{K}_{X/Y, Y/X}. \quad (2.20)$$

In particular, we get

$$\mathcal{C}_{XY} = \mathcal{C}_{X \cap Y} \otimes \mathcal{K}_{X/Y, Y/X} \quad (2.21)$$

relatively to the tensor factorizations

$$\mathcal{H}_X = \mathcal{H}_{X \cap Y} \otimes \mathcal{H}_{X/Y} \quad \text{and} \quad \mathcal{H}_Y = \mathcal{H}_{X \cap Y} \otimes \mathcal{H}_{Y/X}. \quad (2.22)$$

Since $\mathcal{K}_{X/Y, O} \cong \mathcal{H}_{X/Y}$ in the special case $Z \subset Y \subset X$ we have

$$\mathcal{C}_{XY} = \mathcal{C}_Y \otimes \mathcal{H}_{X/Y} \quad \text{and} \quad \mathcal{C}_{XY}(Z) = \mathcal{T}_Z \otimes \mathcal{K}_{Y/Z} \otimes \mathcal{H}_{X/Y}. \quad (2.23)$$

Example 2.12. These factorizations give us the possibility of expressing quite explicitly the Hamiltonian algebra of some subsystems. We refer to §4.6 for more general situations and consider here sub-semilattices of the form $\mathcal{T} = \{X, Y\}$ with $X \supset Y$. This is a toy model, an N -body system coupled to one of its subsystems, and can be nicely formulated in a purely abstract setting, cf. Proposition 3.10. We have $\mathcal{H}_{\mathcal{T}} = \mathcal{H}_X \oplus \mathcal{H}_Y$ with $\mathcal{H}_X = \mathcal{H}_Y \otimes \mathcal{H}_{X/Y}$. From (2.23) we have

$$\mathcal{C}_{\mathcal{S}}^{\mathcal{T}} = \begin{pmatrix} \mathcal{C}_X & \mathcal{C}_Y \otimes \mathcal{H}_{X/Y} \\ \mathcal{C}_Y \otimes \mathcal{H}_{X/Y}^* & \mathcal{C}_Y \end{pmatrix}.$$

where $\mathcal{H}_{X/Y}^*$ has a natural meaning (see §2.9). The grading is defined for $Z \in \mathcal{S}_X$ by

(1) If $Z \subset Y$ then

$$\mathcal{C}_{\mathcal{S}}^{\mathcal{T}}(Z) = \begin{pmatrix} \mathcal{C}_X(Z) & \mathcal{C}_Y(Z) \otimes \mathcal{H}_{X/Y} \\ \mathcal{C}_Y(Z) \otimes \mathcal{H}_{X/Y}^* & \mathcal{C}_Y(Z) \end{pmatrix}.$$

(2) If $Z \not\subset Y$ then

$$\mathcal{C}_{\mathcal{S}}^{\mathcal{T}}(Z) = \begin{pmatrix} \mathcal{C}_X(Z) & 0 \\ 0 & 0 \end{pmatrix}.$$

If $\mathcal{T} = \{X, O\}$ we get a version of the Friedrichs model: an N -body system coupled to the vacuum. The case when \mathcal{T} is an arbitrary chain (a totally ordered subset of \mathcal{S}) is very similar. The case $\mathcal{T} = \{X, Y\}$ with not comparable X, Y is more complicated and is treated in §4.6 in a more general setting.

2.8. Examples of many-body Hamiltonians. Here we use Theorems 2.11 and 2.10 to construct self-adjoint operators strictly affiliated to \mathcal{C} . For simplicity, *in this and the next subsections* \mathcal{S} is assumed finite. If \mathcal{S} is infinite then an assumption of the same nature as the non-zero mass condition in quantum field theory models is needed to ensure that the kinetic energy operator K is affiliated to \mathcal{C} .

The Hamiltonians will be of the form $H = K + I$ where the self-adjoint operator K is the kinetic energy and I is an interaction term bounded in form sense by K . More precisely, I is a symmetric sesquilinear form on the domain of $|K|^{1/2}$ which is continuous, i.e. satisfies

$$\pm I \leq \mu |K + ia| \quad \text{for some real numbers } \mu, a. \quad (2.24)$$

H and K are matrices of operators, e.g. $H = (H_{XY})_{X, Y \in \mathcal{S}}$ where H_{XY} is defined on a subspace of \mathcal{H}_Y and has values in \mathcal{H}_X and the relation $H_{XY}^* = H_{YX}$ holds at least formally. By construction K is given by a diagonal matrix, so $K_{XY} = 0$ if $X \neq Y$, and we set $K_X = K_{XX}$. The interaction will be a matrix of sesquilinear forms. Then $H_{XX} = K_X + I_{XX}$ will be an N -body type Hamiltonian, i.e. a self-adjoint operator affiliated to \mathcal{C}_X , cf. Remark 2.6. The non-diagonal elements $H_{XY} = I_{XY}$ define the interaction between the systems X and Y . We give now a rigorous construction of such Hamiltonians.

(a) For each X we choose a kinetic energy operator $K_X = h_X(P)$ for the system having X as configuration space. The function $h_X : X \rightarrow \mathbb{R}$ is continuous and such that $h_X(x) \rightarrow \infty$ if $x \rightarrow \infty$. We stress that there are no relations between the kinetic energies of the systems corresponding to different X . Denote \mathcal{G}_X^2 the domain of K_X equipped with the graph norm and let \mathcal{G}_X^s ($s \in \mathbb{R}$) be the scale of Hilbert spaces associated to it, e.g. $\mathcal{G}_X^0 = \mathcal{H}_X$, $\mathcal{G}_X^1 = D(K_X^{1/2})$ is the form domain of K_X , and \mathcal{G}_X^{-1} its adjoint space.

(b) The total kinetic energy of the system is by definition $K = \oplus_X K_X$. We call this a *standard kinetic energy operator*. Then the spaces \mathcal{G}^s of the scale determined by the domain \mathcal{G}^2 of K can be identified with direct sums $\mathcal{G}^s \simeq \oplus_X \mathcal{G}_X^s$. In particular this holds for the form domain $\mathcal{G}^1 = D(K^{1/2})$ and for its adjoint space \mathcal{G}^{-1} . Note that we may also introduce the operators $K_{\geq X} = \oplus_{Y \supset X} K_Y$ and the associated spaces $\mathcal{G}_{\geq X}^s$. If $s > 0$ we have $\mathcal{G}_{\geq X}^s = \mathcal{G}^s \cap \mathcal{H}_{\geq X}$.

(c) The simplest type of interactions that we may consider are given by symmetric elements I of the multiplier algebra of \mathcal{C} . Then $H = K + I$ is strictly affiliated to \mathcal{C} and $\mathcal{P}_{\geq X}(H) = K_{\geq X} + \mathcal{P}_{\geq X}(I)$ where $\mathcal{P}_{\geq X}$ is extended to the multiplier algebras as explained in [Lac, p. 18].

(d) In order to cover singular interactions (form bounded but not necessarily operator bounded by K) we assume that the functions h_X are equivalent to regular weights. This is a quite weak assumption, cf. page 40. For example, it suffices that $c'|x|^\alpha \leq h_X(x) \leq c''|x|^\alpha$ for large x where $c', c'', \alpha > 0$ are numbers depending on X . Then U_a, V_a induce continuous operators in each of the spaces $\mathcal{G}_X^s, \mathcal{G}^s, \mathcal{G}_{\geq X}^s$.

(e) The interaction will be of the form $I = \sum_{Z \in \mathcal{S}} I(Z)$ where the $I(Z)$ are continuous symmetric sesquilinear forms on \mathcal{G}^1 such that $I(Z) \geq -\mu_Z K - \nu$ for some positive numbers μ_Z and ν with $\sum_Z \mu_Z < 1$. Then the form sum $K + I$ defines a self-adjoint operator H on \mathcal{H} .

(f) We identify $I(Z)$ with a symmetric operator $\mathcal{G}^1 \rightarrow \mathcal{G}^{-1}$ and we assume that $I(Z)$ is supported by the subspace $\mathcal{H}_{\geq Z}$. In other terms, $I(Z)$ is the sesquilinear form on \mathcal{G}^1 associated to an operator $I(Z) : \mathcal{G}_{\geq Z}^1 \rightarrow \mathcal{G}_{\geq Z}^{-1}$. Moreover, we assume that this last operator satisfies

$$U_a I(Z) = I(Z) U_a \text{ if } a \in Z, \quad I(Z)(V_a - 1) \rightarrow 0 \text{ if } a \rightarrow 0 \text{ in } Z^\perp, \quad V_a^* I(Z) V_a \rightarrow I(Z) \text{ if } a \rightarrow 0 \quad (2.25)$$

where the limits hold in norm in $L(\mathcal{G}_{\geq Z}^2, \mathcal{G}_{\geq Z}^{-1})$.

Note that the first part of condition (f), saying that $I(Z)$ is supported by $\mathcal{H}_{\geq Z}$, is equivalent to an estimate of the form $\pm I(Z) \leq \mu K_{\geq Z} + \nu \Pi_{\geq Z}$ for some positive numbers μ, ν . See also Remark 2.15.

Theorem 2.13. *The Hamiltonian H is a self-adjoint operator strictly affiliated to \mathcal{C} , we have $H_{\geq X} = K_{\geq X} + \sum_{Z \geq X} I(Z)$, and $\text{Sp}_{\text{ess}}(H) = \bigcup_{X \in \mathcal{P}(\mathcal{S})} \text{Sp}(H_{\geq X})$.*

Remark 2.14. We required the h_X to be bounded from below only for the simplicity of the statements. Moreover, a simple extension of the formalism allows one to treat particles with arbitrary spin. Indeed, if E is a complex Hilbert then Theorem 2.5 remains true if \mathcal{C} is replaced by $\mathcal{C}^E = \mathcal{C} \otimes K(E)$ and the $\mathcal{C}(Z)$ by $\mathcal{C}(Z) \otimes K(E)$. If E is the spin space then it is finite dimensional and one obtains \mathcal{C}^E exactly as above by replacing the $\mathcal{H}(X)$ by $\mathcal{H}(X) \otimes E = L^2(X; E)$. Then one may consider instead of scalar kinetic energy functions h self-adjoint operator valued functions $h : X^* \rightarrow L(E)$. For example, we may take as one particle kinetic energy operators the Pauli or Dirac Hamiltonians.

Remark 2.15. We give here a second, more explicit version of condition (f). Since $I(Z)$ is a continuous symmetric operator $\mathcal{G}^1 \rightarrow \mathcal{G}^{-1}$ we may represent it as a matrix $I(Z) = (I_{XY}(Z))_{X, Y \in \mathcal{S}}$ of continuous operators $I_{XY}(Z) : \mathcal{G}_Y^1 \rightarrow \mathcal{G}_X^{-1}$ with $I_{XY}(Z)^* = I_{YX}(Z)$. We take $I_{XY}(Z) = 0$ if $Z \not\subset X \cap Y$ and if $Z \subset X \cap Y$ we assume $V_a^* I_{XY}(Z) V_a \rightarrow I_{XY}(Z)$ if $a \rightarrow 0$ in $X + Y$ and

$$U_a I_{XY}(Z) = I_{XY}(Z) U_a \text{ if } a \in Z, \quad I_{XY}(Z)(V_a - 1) \rightarrow 0 \text{ if } a \rightarrow 0 \text{ in } Y/Z. \quad (2.26)$$

The limits should hold in norm in $L(\mathcal{G}_Y^2, \mathcal{G}_X^{-1})$.

The operators $I_{XY}(Z)$ satisfying (2.26) are described in more detail in Proposition 7.7. In the next example we consider the simplest situation which is useful in the nonrelativistic case.

If E is an Euclidean space and s is a real number let \mathcal{H}_E^s be the Sobolev space defined by the norm

$$\|u\|_{\mathcal{H}^s} = \|(1 + \Delta_E)^{s/2} u\|$$

where Δ_E is the (positive) Laplacian associated to the Euclidean space E . The space \mathcal{H}_E^s is equipped with two continuous representations of E , a unitary one induced by $\{U_x\}_{x \in E}$ and a non-unitary one induced by $\{V_x\}_{x \in E}$. If $E = O := \{0\}$ we define $\mathcal{H}_E^s = \mathbb{C}$.

Definition 2.16. If E, F are Euclidean spaces and $T : \mathcal{H}_E^s \rightarrow \mathcal{H}_F^t$ is a linear map, we say that T is *small at infinity* if there is $\varepsilon > 0$ such that when viewed as a map $\mathcal{H}_E^{s+\varepsilon} \rightarrow \mathcal{H}_F^t$ the operator T is compact.

By the closed graph theorem T is continuous and the compactness property holds for all $\varepsilon > 0$. If $E = O$ or $F = O$ then we consider that all the operators $T : \mathcal{H}_E^s \rightarrow \mathcal{H}_F^t$ are small at infinity.

Example 2.17. Due to assumption (d) the form domains of K_X and K_Y are Sobolev spaces, for example $\mathcal{G}_X^1 = \mathcal{H}_X^1$ and $\mathcal{G}_Y^1 = \mathcal{H}_Y^1$. Let $I_{XY}^Z : \mathcal{H}_{Y/Z}^t \rightarrow \mathcal{H}_{X/Z}^{-s}$ be a linear small at infinity map. Then we may take $I_{XY}(Z) = 1_Z \otimes I_{XY}^Z$ relatively to the tensor factorizations (2.16).

We make now some comments to clarify the conditions **(a)** - **(f)**. Assume, more generally, that \mathcal{C} is a C^* -algebra of operators on a Hilbert space \mathcal{H} and that K is a self-adjoint operator on \mathcal{H} affiliated to \mathcal{C} . Let I be a continuous symmetric sesquilinear form on the domain of $|K|^{1/2}$. Then for small real ν the form sum $K + \nu I$ is a self-adjoint operator H_ν . If H_ν is affiliated to \mathcal{C} for small ν , and since the derivative with respect to ν at zero of $(H_\nu + i)^{-1}$ exists in norm, we get $(K + i)^{-1}I(K + i)^{-1} \in \mathcal{C}$. This clearly implies $\langle K \rangle^{-2}I\langle K \rangle^{-2} \in \mathcal{C}$. Since $\langle K \rangle^{-1/2}I\langle K \rangle^{-1/2}$ is a bounded operator, the map $z \mapsto \langle K \rangle^{-z}I\langle K \rangle^{-z}$ is holomorphic on $\Re z > 1/2$ hence we get

$$\langle K \rangle^{-\alpha}I\langle K \rangle^{-\alpha} \in \mathcal{C} \text{ if } \alpha > 1/2. \quad (2.27)$$

Reciprocally, if K is strictly affiliated to \mathcal{C} (and K as defined at **(b)** has this property) then Theorem 2.8 from [DaG3] says that $\langle K \rangle^{-1/2}I\langle K \rangle^{-\alpha} \in \mathcal{C}$ suffices to ensure that $H = K + I$ is strictly affiliated to \mathcal{C} under a quite general condition needed to make this operator well defined (this is the role of assumption **(e)** above). Condition **(f)** is formulated such as to imply $\langle K \rangle^{-1/2}I\langle K \rangle^{-1} \in \mathcal{C}$. To simplify the statement we added condition **(d)** which implies that the spaces \mathcal{G}^s are stable under the group V_a . Formally

$$((K)^{-1/2}I\langle K \rangle^{-1})_{XY} = \langle K_X \rangle^{-1/2}I_{XY}\langle K_Y \rangle^{-1}.$$

So this should belong to $\mathcal{C}_{XY} = \sum_{Z \subset X \cap Y} \mathcal{C}_{XY}(Z)$. Thus I_{XY} must be a sum of terms $I_{XY}(Z)$ with

$$\langle K_X \rangle^{-1/2}I_{XY}(Z)\langle K_Y \rangle^{-1} \in \mathcal{C}_{XY}(Z).$$

Conditions **(d)** and **(f)** are formulated such as this to hold, cf. Remark 2.15 and Theorem 2.10.

2.9. Pauli-Fierz Hamiltonians. The next result is an a priori argument which supports our interpretation of \mathcal{C} as Hamiltonian algebra of a many-body system: we show that \mathcal{C} is the C^* -algebra generated by a simple class of Hamiltonians which have a natural quantum field theoretic interpretation. For simplicity we state this only for finite \mathcal{S} .

For each couple $X, Y \in \mathcal{S}$ such that $X \supset Y$ we have $\mathcal{H}_X = \mathcal{H}_Y \otimes \mathcal{H}_{X/Y}$. Then we define $\Phi_{XY} \subset \mathcal{L}_{XY}$ as the closed linear subspace consisting of ‘‘creation operators’’ associated to states from $\mathcal{H}_{X/Y}$, i.e. operators $a^*(\theta) : \mathcal{H}_Y \rightarrow \mathcal{H}_X$ with $\theta \in \mathcal{H}_{X/Y}$ which act as $u \mapsto u \otimes \theta$. We set $\Phi_{YX} = \Phi_{XY}^* \subset \mathcal{L}_{YX}$, this is the space of ‘‘annihilation operators’’ $a(\theta) = a^*(\theta)^*$ defined by $\mathcal{H}_{X/Y}$. This defines Φ_{XY} when X, Y are comparable, i.e. $X \supset Y$ or $X \subset Y$, which we abbreviate by $X \sim Y$. If $X \not\sim Y$ then we take $\Phi_{XY} = 0$. Note that $\Phi_{XX} = \mathbb{C}1_X$, where 1_X is the identity operator on \mathcal{H}_X . We have

$$\mathcal{I}_X \cdot \Phi_{XY} = \Phi_{XY} \cdot \mathcal{I}_Y = \mathcal{I}_{XY} \quad \text{if } X \sim Y. \quad (2.28)$$

Now let $\Phi = (\Phi_{XY})_{X, Y \in \mathcal{S}} \subset \mathcal{L}$. This is a closed self-adjoint linear space of bounded operators on \mathcal{H} . A symmetric element $\phi \in \Phi$ will be called *field operator*. Giving such a ϕ is equivalent to giving a family $\theta = (\theta_{XY})_{X \supset Y}$ of elements $\theta_{XY} \in \mathcal{H}_{X/Y}$, the components of the operator $\phi \equiv \phi(\theta)$ being given by: $\phi_{XY} = a^*(\theta_{XY})$ if $X \supset Y$, $\phi_{XY} = a(\theta_{YX})$ if $X \subset Y$, and $\phi_{XY} = 0$ if $X \not\sim Y$.

The operators of the form $K + \phi$, where K is a standard kinetic energy operator and $\phi \in \Phi$ is a field operator, will be called *Pauli-Fierz Hamiltonians*.

Theorem 2.18. *If \mathcal{S} is finite then \mathcal{C} is the C^* -algebra generated by the Pauli-Fierz Hamiltonians.*

Thus \mathcal{C} is generated by a class of Hamiltonians involving only elementary field type interactions. On the other hand, we have seen before that the class of Hamiltonians affiliated to \mathcal{C} is very large and covers N -body systems interacting between themselves with field type interactions. We emphasize that the k -body type interactions *inside* each of the N -body subsystems are generated by pure field interactions.

2.10. Nonrelativistic Hamiltonians and Mourre estimate. We prove the Mourre estimate only for non-relativistic many-body systems. There are serious difficulties when the kinetic energy is not a quadratic form even in the much simpler case of N -body Hamiltonians, but see [Der1, Ger1, DaG2] for some partial results which could be extended to our setting. Note that the quantum field case is much easier from this point of view because of the special nature of the interactions [DeG2, Ger2, Geo].

Let \mathcal{S} be a finite semilattice of subspaces of \mathcal{X} . Recall that for $X \in \mathcal{S}$ we denote \mathcal{S}/X the set of subspaces $Y/X = Y \cap X^\perp$ with $Y \in \mathcal{S}_{\geq X}$. This is a finite semilattice of subspaces of \mathcal{X} which contains O . Hence the Hilbert space $\mathcal{H}_{\mathcal{S}/X}$ and the C^* -algebra $\mathcal{C}_{\mathcal{S}/X}$ are well defined by our general rules and (cf. §2.7):

$$\mathcal{H}_{\geq X} = \mathcal{H}_X \otimes \mathcal{H}_{\mathcal{S}/X} \quad \text{and} \quad \mathcal{C}_{\geq X} = \mathcal{T}_X \otimes \mathcal{C}_{\mathcal{S}/X}. \quad (2.29)$$

Denote Δ_X the (positive) Laplacian associated to the Euclidean space X with the convention $\Delta_O = 0$. We have $\Delta_X = h_X(P)$ with $h_X(x) = \|x\|^2$. We set $\Delta \equiv \Delta_{\mathcal{S}} = \oplus_X \Delta_X$ and define $\Delta_{\geq X}$ similarly. If $Y \supset X$ then $\Delta_Y = \Delta_X \otimes 1 + 1 \otimes \Delta_{Y/X}$ hence $\Delta_{\geq X} = \Delta_X \otimes 1 + 1 \otimes \Delta_{\mathcal{S}/X}$. The domain and form domain of the operator $\Delta_{\mathcal{S}}$ are given by $\mathcal{H}_{\mathcal{S}}^2$ and $\mathcal{H}_{\mathcal{S}}^1$ where $\mathcal{H}_{\mathcal{S}}^s \equiv \mathcal{H}^s = \oplus_X \mathcal{H}^s(X)$ for any real s .

We define nonrelativistic many-body Hamiltonian by extending to the present setting [ABG, Def. 9.1]. We consider only strictly affiliated operators to avoid working with not densely defined operators. Note that the general case of affiliated operators covers interesting physical situations (hard-core interactions).

Definition 2.19. A nonrelativistic many-body Hamiltonian of type \mathcal{S} is a bounded from below self-adjoint operator $H = H_{\mathcal{S}}$ on $\mathcal{H} = \mathcal{H}_{\mathcal{S}}$ which is strictly affiliated to $\mathcal{C} = \mathcal{C}_{\mathcal{S}}$ and has the following property: for each $X \in \mathcal{S}$ there is a bounded from below self-adjoint operator $H_{\mathcal{S}/X}$ on $\mathcal{H}_{\geq X}$ such that

$$\mathcal{P}_{\geq X}(H) \equiv H_{\geq X} = \Delta_X \otimes 1 + 1 \otimes H_{\mathcal{S}/X} \quad (2.30)$$

relatively to the tensor factorization $\mathcal{H}_{\geq X} = \mathcal{H}_X \otimes \mathcal{H}_{\mathcal{S}/X}$.

Then each $H_{\mathcal{S}/X}$ is a nonrelativistic many-body Hamiltonian of type \mathcal{S}/X . Indeed, the argument from [ABG, p. 415] extends in a straightforward way to the present situation.

Remark 2.20. If X is a maximal element in \mathcal{S} then $\mathcal{S}/X = \{O\}$ hence $\mathcal{H}_{\mathcal{S}/X} = \mathcal{H}_O = \mathbb{C}$ and H_O will necessarily be a real number. Then we get $\mathcal{H}_{\geq X} = \mathcal{H}_X$, $\mathcal{C}_{\geq X} = \mathcal{T}_X$, and $H_{\geq X} = \Delta_X + H_O$ on \mathcal{H}_X .

Remark 2.21. Since \mathcal{S} is a finite semilattice, it has a least element $\min \mathcal{S}$. If $\mathcal{S}_o = \mathcal{S}/\min \mathcal{S}$, we get

$$\mathcal{H}_{\mathcal{S}} = \mathcal{H}_{\min \mathcal{S}} \otimes \mathcal{H}_{\mathcal{S}_o}, \quad \mathcal{C}_{\mathcal{S}} = \mathcal{T}_{\min \mathcal{S}} \otimes \mathcal{C}_{\mathcal{S}_o}, \quad H_{\mathcal{S}} = \Delta_{\min \mathcal{S}} \otimes 1 + 1 \otimes H_{\mathcal{S}_o}. \quad (2.31)$$

Now we give an HVZ type description of the essential spectrum of a nonrelativistic many-body Hamiltonian. For a more detailed statement, see the proof.

Theorem 2.22. Denote $\tau_X = \inf H_{\mathcal{S}/X}$ the bottom of the spectrum of $H_{\mathcal{S}/X}$. Then

$$\text{Sp}_{\text{ess}}(H) = [\tau, \infty[\quad \text{with} \quad \tau = \min\{\tau_X \mid X \text{ is minimal in } \mathcal{S} \setminus \{O\}\}. \quad (2.32)$$

Proof: From (2.30) we get

$$\text{Sp}(H_{\geq X}) = [0, \infty[+ \text{Sp}(H_{\mathcal{S}/X}) = [\tau_X, \infty[\quad \text{if } X \neq O. \quad (2.33)$$

In particular, if $O \notin \mathcal{S}$ then by taking $X = \min \mathcal{S}$ in (2.31) we get

$$\text{Sp}(H) = \text{Sp}_{\text{ess}}(H) = [\inf H_{\mathcal{S}_o}, \infty[. \quad (2.34)$$

If $O \in \mathcal{S}$ then Theorem 2.9 implies

$$\text{Sp}_{\text{ess}}(H) = [\tau, \infty[\quad \text{with} \quad \tau = \min_{X \in \mathcal{P}(\mathcal{S})} \tau_X. \quad (2.35)$$

The relation (2.32) expresses (2.34) and (2.35) in a unified way. \square

For $X \in \mathcal{S}$ we consider the dilation group $W_\tau = e^{i\tau D}$ defined on \mathcal{H}_X by (set $n = \dim X$):

$$(W_\tau u)(x) = e^{n\tau/4} u(e^{\tau/2} x), \quad 2iD = x \cdot \nabla_x + n/2 = \nabla_x \cdot x - n/2. \quad (2.36)$$

Let $D_O = 0$. We keep the same notation for the unitary operator $\oplus_X W_\tau$ on the direct sum $\mathcal{H} = \oplus_X \mathcal{H}_X$ and we do not indicate explicitly the dependence on X or \mathcal{S} of W_τ and D unless this is really needed. Note that D has factorization properties similar to that of the Laplacian, e.g. $D_{\geq X} = D_X \otimes 1 + 1 \otimes D_{\mathcal{S}/X}$.

We refer to Subsection 7.1 for terminology related to the Mourre estimate. We take D as conjugate operator and we denote by $\widehat{\rho}_H(\lambda)$ the best constant (which could be infinite) in the Mourre estimate at point λ . The *threshold set* $\tau(H)$ of H with respect to D is the set where $\widehat{\rho}_H(\lambda) \leq 0$. If A is a real set then we define $N_A : \mathbb{R} \rightarrow [-\infty, \infty[$ by $N_A(\lambda) = \sup\{x \in A \mid x \leq \lambda\}$ with the convention $\sup \emptyset = -\infty$. Denote $\text{ev}(T)$ the set of eigenvalues of an operator T .

Theorem 2.23. *Let $H = H_{\mathcal{S}}$ be a nonrelativistic many-body Hamiltonian of type \mathcal{S} and of class $C_u^1(D)$. Then $\tau(H)$ is a closed countable real set given by*

$$\tau(H) = \bigcup_{X \neq O} \text{ev}(H_{\mathcal{S}/X}). \quad (2.37)$$

The eigenvalues of H which do not belong to $\tau(H)$ are of finite multiplicity and may accumulate only to points from $\tau(H)$. We have $\widehat{\rho}_H(\lambda) = \lambda - N_{\tau(H)}(\lambda)$ for all real λ .

We emphasize that if $O \notin \mathcal{S}$ the threshold set

$$\tau(H) = \bigcup_{X \in \mathcal{S}} \text{ev}(H_{\mathcal{S}/X}) \quad (2.38)$$

is very rich although the spectrum of $H = \Delta_{\min \mathcal{S}} \otimes 1 + 1 \otimes H_{\mathcal{S}_o}$ is purely absolutely continuous.

Remark 2.24. We thus see that there is no difference between nonrelativistic N -body and many-body Hamiltonians from the point of view of their channel structure. The formulas which give the essential spectrum and the threshold set relevant in the Mourre estimate are identical, cf. (2.35) and (2.37). This is due to the fact that both Hamiltonian algebras are graded by the same semilattice \mathcal{S} .

2.11. Examples of nonrelativistic many-body Hamiltonians. Let $H = K + I$ with kinetic energy $K = \Delta$. Hence $\mathcal{G}^1 = \mathcal{H}^1 = \oplus_X \mathcal{H}_X^1$ and $\mathcal{G}^{-1} = \mathcal{H}^{-1} = \oplus_X \mathcal{H}_X^{-1}$ with the notations of §2.8. The interaction term is an operator $I : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ given by a sum $I = \sum_{Z \in \mathcal{S}} I(Z)$ where each $I(Z)$ is defined with the help of the tensor factorization $\mathcal{H}_{\geq Z} = \mathcal{H}_Z \otimes \mathcal{H}_{\mathcal{S}/Z}$.

Proposition 2.25. *Let $I^Z : \mathcal{H}_{\mathcal{S}/Z}^1 \rightarrow \mathcal{H}_{\mathcal{S}/Z}^{-1}$ be symmetric and small at infinity and let $I(Z) := 1_Z \otimes I^Z$ which is naturally defined as a symmetric operator $\mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$. Assume that $I(Z) \geq -\mu_Z \Delta - \nu$ for some numbers $\mu_Z, \nu \geq 0$ with $\sum \mu_Z < 1$. Then $H = \Delta + I$ defined in the quadratic form sense is a nonrelativistic many-body Hamiltonian of type \mathcal{S} and we have $H_{\geq X} = \Delta_{\geq X} + \sum_{Z \supset X} I(Z)$.*

The first condition on I^Z can be stated in terms of its coefficients as follows: if $Z \subset X \cap Y$ then the operator $I_{XY}^Z : \mathcal{H}_{Y/Z}^1 \rightarrow \mathcal{H}_{X/Z}^{-1}$ is small at infinity and such that $(I_{XY}^Z)^* = I_{YX}^Z$. On the other hand, note that if the operators $I^Z : \mathcal{H}_{\mathcal{S}/Z}^1 \rightarrow \mathcal{H}_{\mathcal{S}/Z}^{-1}$ are compact then they are small at infinity and for any $\mu > 0$ there is a number ν such that $\pm I(Z) \leq \mu \Delta_{\mathcal{S}} + \nu$ for all Z . The more general smallness at infinity condition covers second order perturbations of $\Delta_{\mathcal{S}}$.

In the next proposition we give examples of nonrelativistic operators of class $C_u^1(D)$. The operator H is constructed as in Proposition 2.25 but we consider only interactions which are relatively bounded in operator sense with respect to the kinetic energy such as to force the domain of H to be equal to the domain of Δ , hence to $\mathcal{H}^2 = \oplus_X \mathcal{H}_X^2$. Since this space is stable under the action of the operators W_τ , we shall get a simple condition for H to be of class $C_u^1(D)$.

Proposition 2.26. *For each $Z \in \mathcal{S}$ assume that $I^Z : \mathcal{H}_{\mathcal{S}/Z}^2 \rightarrow \mathcal{H}_{\mathcal{S}/Z}$ is compact and symmetric as operator on $\mathcal{H}_{\mathcal{S}/Z}$ and that $[D, I^Z] : \mathcal{H}_{\mathcal{S}/Z}^2 \rightarrow \mathcal{H}_{\mathcal{S}/Z}^{-2}$ is compact. Then the conditions of Proposition 2.25 are fulfilled and each operator $I(Z) : \mathcal{H}^2 \rightarrow \mathcal{H}$ is Δ -bounded with relative bound zero. The operator H is self-adjoint on \mathcal{H}^2 and of class $C_u^1(D)$.*

So for the coefficients I_{XY}^Z we ask $I_{XY}^Z = 0$ if $Z \not\subset X \cap Y$ and if $Z \subset X \cap Y$ then $(I_{XY}^Z)^* \supset I_{YX}^Z$ and

$$I_{XY}^Z : \mathcal{H}_{Y/Z}^2 \rightarrow \mathcal{H}_{X/Z} \text{ and } [D, I_{XY}^Z] : \mathcal{H}_{Y/Z}^2 \rightarrow \mathcal{H}_{X/Z}^{-2} \text{ are compact operators.} \quad (2.39)$$

The expression $[D, I_{XY}^Z] = D_{X/Z}I_{XY}^Z - I_{XY}^Z D_{Y/Z}$ is not really a commutator. Indeed, if we denote $E = (X \cap Y)/Z$, so $Y/Z = E \oplus (Y/X)$ and $X/Z = E \oplus (X/Y)$, then $\mathcal{H}_{X/Z} = \mathcal{H}_E \otimes \mathcal{H}_{X/Y}$ and $\mathcal{H}_{Y/Z} = \mathcal{H}_E \otimes \mathcal{H}_{Y/X}$. Hence the relation $D_{X/Z} = D_E \otimes 1 + 1 \otimes D_{X/Y}$ and a similar one for Y/Z give

$$[D, I_{XY}^Z] = [D_E, I_{XY}^Z] + D_{X/Y}I_{XY}^Z - I_{XY}^Z D_{Y/X}.$$

The first term above is a commutator and so is of a different nature than the next two. Since $I_{XY}^Z D_{Y/X}$ is a restriction of $(D_{Y/X}I_{YX}^Z)^*$ it is clear that the second part of condition (2.39) follows from:

$$[D_E, I_{XY}^Z] \text{ and } D_{X/Y}I_{XY}^Z \text{ are compact operators } \mathcal{H}_{Y/Z}^2 \rightarrow \mathcal{H}_{X/Z}^{-2} \text{ for all } X, Y, Z. \quad (2.40)$$

We consider some simple examples of operators I_{XY}^Z to clarify the difference with respect to the N -body situation (see §7.4 for details and generalizations). If E, F are Euclidean spaces we denote

$$\mathcal{K}_{FE}^2 = K(\mathcal{H}_E^2, \mathcal{H}_F) \quad \text{and} \quad \mathcal{K}_E^2 = \mathcal{K}_{E,E}^2 = K(\mathcal{H}_E^2, \mathcal{H}_E). \quad (2.41)$$

Denote $X \boxplus Y = X/Y \oplus Y/X$ and embed $L^2(X \boxplus Y) \subset \mathcal{K}_{X/Y, Y/X}$ by identifying a Hilbert-Schmidt operator with its kernel. Then

$$L^2(X \boxplus Y; \mathcal{K}_E^2) \subset \mathcal{K}_E^2 \otimes \mathcal{K}_{X/Y, Y/X} \subset \mathcal{K}_{X/Z, Y/Z}^2.$$

Thus $I_{XY}^Z \in L^2(X \boxplus Y; \mathcal{K}_E^2)$ is a simple example of operator satisfying the first part of condition (2.39). Such an I_{XY}^Z acts as follows: if $u \in \mathcal{H}_{Y/Z}^2 \subset L^2(Y/X; \mathcal{H}_E^2)$ then

$$I_{XY}^Z u \in \mathcal{H}_{X/Z} = L^2(X/Y; \mathcal{H}_E) \quad \text{is given by} \quad (I_{XY}^Z u)(x') = \int_{Y/X} I_{XY}^Z(x', y') u(y') dy'.$$

Now we consider (2.40). Since $(x', y') \mapsto [D_E, I_{XY}^Z(x', y')]$ is the kernel of the operator $[D_E, I_{XY}^Z]$, if

$$[D_E, I_{XY}^Z] \in L^2(X \boxplus Y; K(\mathcal{H}_E^2, \mathcal{H}_E^{-2}))$$

then $[D_E, I_{XY}^Z]$ is a compact operator $\mathcal{H}_{Y/Z}^2 \rightarrow \mathcal{H}_{X/Z}^{-2}$. For the term $D_{X/Y}I_{XY}^Z$ it suffices to require the compactness of the operator

$$D_{X/Y}I_{XY}^Z = 1_E \otimes D_{X/Y} \cdot I_{XY}^Z : \mathcal{H}_{Y/Z}^2 \rightarrow \mathcal{H}_E \otimes \mathcal{H}_{X/Y}^{-2}.$$

From (2.36) we see that this is a condition on the kernel $x' \cdot \nabla_{x'} I_{XY}^Z(x', y')$. For example, it suffices that the operator $\langle Q_{X/Y} \rangle I_{XY}^Z : \mathcal{H}_{Y/Z}^2 \rightarrow \mathcal{H}_{X/Z}$ be compact, which is a short range assumption. In summary:

Example 2.27. For each $Z \subset X \cap Y$ let $I_{XY}^Z \in L^2(X \boxplus Y; \mathcal{K}_E^2)$ such that the adjoint of $I_{XY}^Z(x', y')$ is an extension of $I_{YX}^Z(y', x')$. Assume that kernel $[D_E, I_{XY}^Z(x', y')]$ belongs to $L^2(X \boxplus Y; K(\mathcal{H}_E^2, \mathcal{H}_E^{-2}))$ and that the kernel $x' \cdot \nabla_{x'} I_{XY}^Z(x', y')$ defines a compact operator $\mathcal{H}_{Y/Z}^2 \rightarrow \mathcal{H}_{X/Z}^{-2}$. Then (2.39) is fulfilled.

Example 2.28. Here we consider the particular case $Y \subset X$ to see the structure of a generalized creation operator which appears in this context. For each $Z \subset Y$ let $I_{XY}^Z \in \mathcal{K}_{Y/Z}^2 \otimes \mathcal{H}_{X/Y}$, where the tensor product is a kind of weak version of $L^2(X/Y; \mathcal{K}_{Y/Z}^2)$ discussed in §3.4. Furthermore, assume that $[D_{Y/Z}, I_{XY}^Z] \in K(\mathcal{H}^2(Y/Z), \mathcal{H}_{Y/Z}^{-2}) \otimes \mathcal{H}_{X/Y}$ and $D_{X/Y}I_{XY}^Z \in \mathcal{K}_{Y/Z}^2 \otimes \mathcal{H}_{X/Y}^{-2}$. Then (2.39) holds.

2.12. Boundary values of the resolvent. Theorem 2.23 has important consequences in the spectral theory of the operator H : we shall use it together with [ABG, Theorem 7.4.1] to show that H has no singular continuous spectrum and to prove the existence of the boundary values of its resolvent in the class of weighted L^2 spaces that we define now. Let $\mathcal{H}_{s,p} = \oplus_X L_{s,p}^2(X)$ where the $L_{s,p}^2(X)$ are the Besov spaces associated to the position observable on X (these are obtained from the usual Besov spaces associated to $L^2(X)$ by a Fourier transformation). Note that $\mathcal{H}_s = \mathcal{H}_{s,2}$ is the Fourier transform of the Sobolev space \mathcal{H}^s . Let \mathbb{C}_+ be the open upper half plane and $\mathbb{C}_+^H = \mathbb{C}_+ \cup (\mathbb{R} \setminus \tau(H))$. If we replace

the upper half plane by the lower one we similarly get the sets \mathbb{C}_- and \mathbb{C}_-^H . We define two holomorphic maps $R_\pm : \mathbb{C}_\pm \rightarrow L(\mathcal{H})$ by $R_\pm(z) = (H - z)^{-1}$ and note that we have continuous embeddings

$$L(\mathcal{H}) \subset L(\mathcal{H}_{1/2,1}, \mathcal{H}_{-1/2,\infty}) \subset L(\mathcal{H}_s, \mathcal{H}_{-s}) \quad \text{if } s > 1/2$$

so we may consider R_\pm as maps with values in $L(\mathcal{H}_{1/2,1}, \mathcal{H}_{-1/2,\infty})$.

Theorem 2.29. *If H is of class $C^{1,1}(D)$ then its singular continuous spectrum is empty and the holomorphic maps $R_\pm : \mathbb{C}_\pm \rightarrow L(\mathcal{H}_{1/2,1}, \mathcal{H}_{-1/2,\infty})$ extend to weak* continuous functions \bar{R}_\pm on \mathbb{C}_\pm^H . The maps $\bar{R}_\pm : \mathbb{C}_\pm^H \rightarrow L(\mathcal{H}_s, \mathcal{H}_{-s})$ are norm continuous if $s > 1/2$.*

This result is optimal both with regard to the regularity of the Hamiltonian relatively to the conjugate operator D and to the Besov spaces in which we establish the existence of the boundary values of the resolvent. The class $C^{1,1}(D)$ will be discussed and its optimality will be made precise in §7.6 but we give some examples below.

We state first the simplest sufficient condition: *assume that H is as in Proposition 2.25 and that its domain is equal to \mathcal{H}^2 ; if $[D, [D, I^Z]] \in L(\mathcal{H}_{S/Z}^2, \mathcal{H}_{S/Z}^{-2})$ for all Z then H is of class $C^{1,1}(D)$.* This follows from Theorem 6.3.4 in [ABG]. The condition on $[D, [D, I^Z]]$ can easily be written in terms of the coefficients I_{XY}^Z by arguments similar to those of §2.11. Refinements allow the addition of long range and short range interactions as in [ABG, §9.4.2].

Let $\xi : \mathbb{R} \rightarrow \mathbb{R}$ be of class C^∞ and such that $\xi(\lambda) = 0$ if $\lambda \leq 1$ and $\xi(\lambda) = 1$ if $\lambda \geq 2$. For each Euclidean space X and real $r \geq 1$ we denote ξ_X^r the operator of multiplication by the function $x \mapsto \xi(|x|/r)$ on any Sobolev space over X . Then we define $\xi_S^r = \oplus_{X \in \mathcal{S}} \xi_X^r$ considered as operator on \mathcal{H}_S^s for any real s .

Definition 2.30. Let $T : \mathcal{H}_S^2 \rightarrow \mathcal{H}_S$ be a symmetric operator. We say that T is a *long range interaction* if $[D, T] \in L(\mathcal{H}_S^2, \mathcal{H}_S^{-1})$ and $\int_1^\infty \|\xi_S^r[D, T]\|_{\mathcal{H}_S^2 \rightarrow \mathcal{H}_S^{-1}} dr / r < \infty$. We say that T is a *short range interaction* if $\int_1^\infty \|\xi_S^r[D, T]\|_{\mathcal{H}_S^2 \rightarrow \mathcal{H}_S} dr < \infty$.

Theorem 2.31. *Assume that $H = \Delta_S + \sum_{Z \in \mathcal{S}} 1_Z \otimes I^Z$ where each $I^z : \mathcal{H}_{S/Z}^2 \rightarrow \mathcal{H}_{S/Z}$ is symmetric, compact, and is the sum of a long range and a short range interaction. Then H is a nonrelativistic many-body Hamiltonian of class $C^{1,1}(D)$, hence the conclusions of Theorem 2.29 are true.*

Scattering channels may be defined in a natural way in the context of the theorem. If the long range interactions are absent we expect that asymptotic completeness holds.

3. GRADED HILBERT C^* -MODULES

3.1. Graded C^* -algebras. The natural framework for the systems considered in this paper is that of C^* -algebras graded by semilattices. We refer to [Ma2, Ma3] for a detailed study of this class of algebras.

Let \mathcal{S} be a semilattice and \mathcal{A} a graded C^* -algebra. Following [Ma2] we say that $\mathcal{B} \subset \mathcal{A}$ is a *graded C^* -subalgebra* if \mathcal{B} is a C^* -subalgebra of \mathcal{A} equal to $\sum_\sigma^c \mathcal{B} \cap \mathcal{A}(\sigma)$. Then \mathcal{B} has a natural graded C^* -algebra structure: $\mathcal{B}(\sigma) = \mathcal{B} \cap \mathcal{A}(\sigma)$. If \mathcal{B} is also an ideal of \mathcal{A} then \mathcal{B} is a *graded ideal*.

A subset \mathcal{T} of a semilattice \mathcal{S} is a *sub-semilattice* of \mathcal{S} if $\sigma, \tau \in \mathcal{T} \Rightarrow \sigma \wedge \tau \in \mathcal{T}$. We say that \mathcal{T} is an *ideal* of \mathcal{S} if $\sigma \leq \tau \in \mathcal{T} \Rightarrow \sigma \in \mathcal{T}$. If \mathcal{A} is an \mathcal{S} -graded C^* -algebra and $\mathcal{T} \subset \mathcal{S}$ let $\mathcal{A}(\mathcal{T}) = \sum_{\sigma \in \mathcal{T}}^c \mathcal{A}(\sigma)$ (if \mathcal{T} is finite the sum is already closed). If \mathcal{T} is a sub-semilattice or an ideal then clearly $\mathcal{A}(\mathcal{T})$ is a C^* -subalgebra or an ideal of \mathcal{A} respectively.

We say that \mathcal{A} is *supported by a sub-semilattice \mathcal{T}* if $\mathcal{A} = \mathcal{A}(\mathcal{T})$, i.e. $\mathcal{A}(\sigma) = \{0\}$ for $\sigma \notin \mathcal{T}$. Then \mathcal{A} is also \mathcal{T} -graded. The smallest sub-semilattice with this property will be called *support* of \mathcal{A} . If \mathcal{T} is a sub-semilattice of \mathcal{S} and \mathcal{A} is a \mathcal{T} -graded algebra then \mathcal{A} is \mathcal{S} -graded: set $\mathcal{A}(\sigma) = \{0\}$ for $\sigma \in \mathcal{S} \setminus \mathcal{T}$.

The next result is obvious if \mathcal{S} is finite. For the general case, see the proof of Proposition 3.3 in [DaG3].

Proposition 3.1. *Let \mathcal{T} be a sub-semilattice of \mathcal{S} such that $\mathcal{T}' = \mathcal{S} \setminus \mathcal{T}$ is an ideal. Then $\mathcal{A}(\mathcal{T})$ is a C^* -subalgebra of \mathcal{A} , $\mathcal{A}(\mathcal{T}')$ is an ideal of \mathcal{A} , and $\mathcal{A} = \mathcal{A}(\mathcal{T}) + \mathcal{A}(\mathcal{T}')$ with $\mathcal{A}(\mathcal{T}) \cap \mathcal{A}(\mathcal{T}') = \{0\}$. In particular, the natural linear projection $\mathcal{P}(\mathcal{T}) : \mathcal{A} \rightarrow \mathcal{A}(\mathcal{T})$ is a morphism.*

If \mathcal{T} is a sub-semilattice then \mathcal{T}' is an ideal if and only if \mathcal{T} is a filter (i.e. $\sigma \geq \tau \in \mathcal{T} \Rightarrow \sigma \in \mathcal{T}$). Thus if \mathcal{S} is finite then the only sub-semilattices which have this property are the $\mathcal{S}_{\geq \sigma}$ introduced below.

The simplest sub-semilattices are the chains (totally ordered subsets). If $\sigma \in \mathcal{S}$ and

$$\mathcal{S}_{\geq \sigma} = \{\tau \in \mathcal{S} \mid \tau \geq \sigma\}, \quad \mathcal{S}_{\not\geq \sigma} = \mathcal{S}'_{\geq \sigma} = \{\tau \in \mathcal{S} \mid \tau \not\geq \sigma\}, \quad \mathcal{S}_{\leq \sigma} = \{\tau \in \mathcal{S} \mid \tau \leq \sigma\} \quad (3.1)$$

then $\mathcal{S}_{\geq \sigma}$ is a sub-semilattice and $\mathcal{S}_{\not\geq \sigma}$ and $\mathcal{S}_{\leq \sigma}$ are ideals. So $\mathcal{A}_{\geq \sigma} \equiv \mathcal{A}(\mathcal{S}_{\geq \sigma})$ is a graded C^* -subalgebra of \mathcal{A} supported by $\mathcal{S}_{\geq \sigma}$ and $\mathcal{A}(\mathcal{S}_{\not\geq \sigma})$ is a graded ideal supported by $\mathcal{S}_{\not\geq \sigma}$ such that

$$\mathcal{A} = \mathcal{A}_{\geq \sigma} + \mathcal{A}(\mathcal{S}_{\not\geq \sigma}) \quad \text{with} \quad \mathcal{A}_{\geq \sigma} \cap \mathcal{A}(\mathcal{S}_{\not\geq \sigma}) = \{0\}. \quad (3.2)$$

The projection morphism $\mathcal{P}_{\geq \sigma} : \mathcal{A} \rightarrow \mathcal{A}_{\geq \sigma}$ defined by (3.2) is the unique linear continuous map $\mathcal{P}_{\geq \sigma} : \mathcal{A} \rightarrow \mathcal{A}$ such that $\mathcal{P}_{\geq \sigma} A = A$ if $A \in \mathcal{A}(\tau)$ for some $\tau \geq \sigma$ and $\mathcal{P}_{\geq \sigma} A = 0$ otherwise.

\mathcal{S} is called *atomic* if it has a smallest element $o \equiv \min \mathcal{S}$ and if each $\sigma \neq o$ is minorated by an atom. We denote by $\mathcal{P}(\mathcal{S})$ the set of atoms of \mathcal{S} . If \mathcal{T} is an ideal of \mathcal{S} and \mathcal{S} is atomic then \mathcal{T} is atomic, we have $\min \mathcal{T} = \min \mathcal{S}$, and $\mathcal{P}(\mathcal{T}) = \mathcal{P}(\mathcal{S}) \cap \mathcal{T}$. This next result is also easy to prove [DaG3].

Theorem 3.2. *If \mathcal{S} is atomic then $\mathcal{P}A = (\mathcal{P}_{\geq \alpha} A)_{\alpha \in \mathcal{P}(\mathcal{S})}$ defines a morphism $\mathcal{P} : \mathcal{A} \rightarrow \prod_{\alpha \in \mathcal{P}(\mathcal{S})} \mathcal{A}_{\geq \alpha}$ with $\mathcal{A}(o)$ as kernel. This gives us a canonical embedding*

$$\mathcal{A} / \mathcal{A}(o) \subset \prod_{\alpha \in \mathcal{P}(\mathcal{S})} \mathcal{A}_{\geq \alpha}. \quad (3.3)$$

We call this ‘‘theorem’’ because it has important consequences in the spectral theory of many-body Hamiltonians: it allows us to compute their essential spectrum and to prove the Mourre estimate.

We assume that \mathcal{S} is atomic so that \mathcal{A} comes equipped with a remarkable ideal $\mathcal{A}(o)$. Then for $A \in \mathcal{A}$ we define its *essential spectrum* (relatively to $\mathcal{A}(o)$) by the formula

$$\text{Sp}_{\text{ess}}(A) \equiv \text{Sp}(\mathcal{P}A). \quad (3.4)$$

In our concrete examples \mathcal{A} is represented on a Hilbert space \mathcal{H} and $\mathcal{A}(o) = K(\mathcal{H})$, so we get the usual Hilbertian notion of essential spectrum.

In order to extend this to unbounded operators it is convenient to define an *observable affiliated to \mathcal{A}* as a morphism $H : \mathcal{C}_o(\mathbb{R}) \rightarrow \mathcal{A}$. We set $\varphi(H) \equiv H(\varphi)$. If \mathcal{A} is realized on \mathcal{H} then a self-adjoint operator on \mathcal{H} such that $(H + i)^{-1} \in \mathcal{A}$ is said to be affiliated to \mathcal{A} ; then $H(\varphi) = \varphi(H)$ defines an observable affiliated to \mathcal{A} (see Appendix A in [DaG3] for a precise description of the relation between observables and self-adjoint operators affiliated to \mathcal{A}). The spectrum of an observable is by definition the support of the morphism H :

$$\text{Sp}(H) = \{\lambda \in \mathbb{R} \mid \varphi \in \mathcal{C}_o(\mathbb{R}), \varphi(\lambda) \neq 0 \Rightarrow \varphi(H) \neq 0\}. \quad (3.5)$$

Now note that $\mathcal{P}H \equiv \mathcal{P} \circ H$ is an observable affiliated to the quotient algebra $\mathcal{A} / \mathcal{A}(o)$ so we may define the essential spectrum of H as the spectrum of $\mathcal{P}H$. Explicitly, we get:

$$\text{Sp}_{\text{ess}}(H) = \{\lambda \in \mathbb{R} \mid \varphi \in \mathcal{C}_o(\mathbb{R}), \varphi(\lambda) \neq 0 \Rightarrow \varphi(H) \notin \mathcal{A}(o)\}. \quad (3.6)$$

Now the first assertion of the next theorem follows immediately from Theorem 3.2. For the second assertion, see the proof of Theorem 2.10 in [DaG2]. By $\bar{\cup}$ we denote the closure of the union.

Theorem 3.3. *Let \mathcal{S} be atomic. If H is an observable affiliated to \mathcal{A} then $H_{\geq \alpha} = \mathcal{P}_{\geq \alpha} H$ is an observable affiliated to $\mathcal{A}_{\geq \alpha}$ and we have:*

$$\text{Sp}_{\text{ess}}(H) = \bar{\cup}_{\alpha \in \mathcal{P}(\mathcal{S})} \text{Sp}(H_{\geq \alpha}). \quad (3.7)$$

If for each $A \in \mathcal{A}$ the set of $\mathcal{P}_{\geq \alpha} A$ with $\alpha \in \mathcal{P}(\mathcal{S})$ is compact in \mathcal{A} then the union in (3.7) is closed.

3.2. Hilbert C^* -modules. Some basic knowledge of the theory of Hilbert C^* -modules is useful but not indispensable for understanding our constructions. We translate here the necessary facts in a purely Hilbert space language. Our main reference for the general theory of Hilbert C^* -modules is [Lac] but see also [Bla, RW]. The examples of interest in this paper are the “concrete” Hilbert C^* -modules described below as Hilbert C^* -submodules of $L(\mathcal{E}, \mathcal{F})$. We recall, however, the general definition.

If \mathcal{A} is a C^* -algebra then a *Banach \mathcal{A} -module* is a Banach space \mathcal{M} equipped with a continuous bilinear map $\mathcal{A} \times \mathcal{M} \ni (A, M) \mapsto MA \in \mathcal{M}$ such that $(MA)B = M(AB)$. We denote $\mathcal{M} \cdot \mathcal{A}$ the clspan of the elements MA with $A \in \mathcal{A}$ and $M \in \mathcal{M}$. By the Cohen-Hewitt theorem [FeD] for each $N \in \mathcal{M} \cdot \mathcal{A}$ there are $A \in \mathcal{A}$ and $M \in \mathcal{M}$ such that $N = MA$, in particular $\mathcal{M} \cdot \mathcal{A} = \mathcal{M}\mathcal{A}$. Note that by module we mean “right module” but the Cohen-Hewitt theorem is also valid for left Banach modules.

Let \mathcal{A} be a C^* -algebra. A (right) *Hilbert \mathcal{A} -module* is a Banach \mathcal{A} -module \mathcal{M} equipped with an \mathcal{A} -valued sesquilinear map $\langle \cdot | \cdot \rangle \equiv \langle \cdot | \cdot \rangle_{\mathcal{A}}$ which is positive (i.e. $\langle M | M \rangle \geq 0$) \mathcal{A} -sesquilinear (i.e. $\langle M | NA \rangle = \langle M | N \rangle A$) and such that $\|M\| \equiv \|\langle M | M \rangle\|^{1/2}$. Then $\mathcal{M} = \mathcal{M}\mathcal{A}$. The clspan of the elements $\langle M | M \rangle$ is an ideal of \mathcal{A} denoted $\langle \mathcal{M} | \mathcal{M} \rangle$. One says that \mathcal{M} is *full* if $\langle \mathcal{M} | \mathcal{M} \rangle = \mathcal{A}$. If \mathcal{A} is an ideal of a C^* -algebra \mathcal{C} then \mathcal{M} is equipped with an obvious structure of Hilbert \mathcal{C} -module. Left Hilbert \mathcal{A} -modules are defined similarly.

If \mathcal{M}, \mathcal{N} are Hilbert \mathcal{A} -modules and $(M, N) \in \mathcal{M} \times \mathcal{N}$ then $M' \mapsto N \langle M | M' \rangle$ is a linear continuous map $\mathcal{M} \rightarrow \mathcal{N}$ denoted $|N \rangle \langle M |$ or NM^* . The closed linear subspace of $L(\mathcal{M}, \mathcal{N})$ generated by these elements is denoted $\mathcal{K}(\mathcal{M}, \mathcal{N})$. There is a unique antilinear isometric map $T \mapsto T^*$ of $\mathcal{K}(\mathcal{M}, \mathcal{N})$ onto $\mathcal{K}(\mathcal{N}, \mathcal{M})$ which sends $|N \rangle \langle M |$ into $|M \rangle \langle N |$. The space $\mathcal{K}(\mathcal{M}) \equiv \mathcal{K}(\mathcal{M}, \mathcal{M})$ is a C^* -algebra called *imprimitivity algebra* of the Hilbert \mathcal{A} -module \mathcal{M} .

Assume that \mathcal{N} is a closed subspace of a Hilbert \mathcal{A} -module \mathcal{M} and let $\langle \mathcal{N} | \mathcal{N} \rangle$ be the clspan of the elements $\langle N | N \rangle$ in \mathcal{A} . If \mathcal{N} is an \mathcal{A} -submodule of \mathcal{M} then it inherits an obvious Hilbert \mathcal{A} -module structure from \mathcal{M} . If \mathcal{N} is not an \mathcal{A} -submodule of \mathcal{M} it may happen that there is a C^* -subalgebra $\mathcal{B} \subset \mathcal{A}$ such that $\mathcal{N}\mathcal{B} \subset \mathcal{N}$ and $\langle \mathcal{N} | \mathcal{N} \rangle \subset \mathcal{B}$. Then clearly we get a Hilbert \mathcal{B} -module structure on \mathcal{N} . On the other hand, it is clear that such a \mathcal{B} exists if and only if $\mathcal{N} \langle \mathcal{N} | \mathcal{N} \rangle \subset \mathcal{N}$ and then $\langle \mathcal{N} | \mathcal{N} \rangle$ is a C^* -subalgebra of \mathcal{A} . Under these conditions we say that \mathcal{N} is a *Hilbert C^* -submodule* of the Hilbert \mathcal{A} -module \mathcal{M} . Then \mathcal{N} inherits a Hilbert $\langle \mathcal{N} | \mathcal{N} \rangle$ -module structure and this defines the C^* -algebra $\mathcal{K}(\mathcal{N})$. Moreover, if \mathcal{B} is as above then $\mathcal{K}(\mathcal{N}) = \mathcal{K}_{\mathcal{B}}(\mathcal{N})$.

If \mathcal{N} is a closed subspace of a Hilbert \mathcal{A} -module \mathcal{M} then let $\mathcal{K}(\mathcal{N} | \mathcal{M})$ be the closed subspace of $\mathcal{K}(\mathcal{M})$ generated by the elements NN^* with $N \in \mathcal{N}$. It is easy to prove that if \mathcal{N} is a Hilbert C^* -submodule of \mathcal{M} then $\mathcal{K}(\mathcal{N} | \mathcal{M})$ is a C^* -subalgebra of $\mathcal{K}(\mathcal{M})$ and the map $T \mapsto T|_{\mathcal{N}}$ sends $\mathcal{K}(\mathcal{N} | \mathcal{M})$ onto $\mathcal{K}(\mathcal{N})$ and is an isomorphism of C^* -algebras. Then we identify $\mathcal{K}(\mathcal{N} | \mathcal{M})$ with $\mathcal{K}(\mathcal{N})$.

If \mathcal{E}, \mathcal{F} are Hilbert spaces then we equip $L(\mathcal{E}, \mathcal{F})$ with the Hilbert $L(\mathcal{E})$ -module structure defined as follows: the C^* -algebra $L(\mathcal{E})$ acts to the right by composition and we take $\langle M | N \rangle = M^*N$ as inner product, where M^* is the usual adjoint of the operator M . Note that $L(\mathcal{E}, \mathcal{F})$ is also equipped with a natural left Hilbert $L(\mathcal{F})$ -module structure: this time the inner product is MN^* .

If $\mathcal{M} \subset L(\mathcal{E}, \mathcal{F})$ is a linear subspace then $\mathcal{M}^* \subset L(\mathcal{F}, \mathcal{E})$ is the set of adjoint operators M^* with $M \in \mathcal{M}$. Clearly $\mathcal{M}_1 \subset \mathcal{M}_2 \Rightarrow \mathcal{M}_1^* \subset \mathcal{M}_2^*$. If \mathcal{G} is a third Hilbert spaces and $\mathcal{N} \subset L(\mathcal{F}, \mathcal{G})$ is a linear subspace then $(\mathcal{N} \cdot \mathcal{M})^* = \mathcal{M}^* \cdot \mathcal{N}^*$. In particular, if $\mathcal{E} = \mathcal{F} = \mathcal{G}$, $\mathcal{M} = \mathcal{M}^*$, and $\mathcal{N} = \mathcal{N}^*$ then $\mathcal{M} \cdot \mathcal{N} \subset \mathcal{N} \cdot \mathcal{M}$ is equivalent to $\mathcal{M} \cdot \mathcal{N} = \mathcal{N} \cdot \mathcal{M}$.

Now let $\mathcal{M} \subset L(\mathcal{E}, \mathcal{F})$ be a closed linear subspace. Then \mathcal{M} is a Hilbert C^* -submodule of $L(\mathcal{E}, \mathcal{F})$ if and only if $\mathcal{M} \cdot \mathcal{M}^* \cdot \mathcal{M} \subset \mathcal{M}$.

These are the “concrete” Hilbert C^* -modules we are interested in. It is clear that \mathcal{M}^* will be a Hilbert C^* -submodule of $L(\mathcal{F}, \mathcal{E})$. We mention that \mathcal{M}^* is canonically identified with the left Hilbert \mathcal{A} -module $\mathcal{K}(\mathcal{M}, \mathcal{A})$ dual to \mathcal{M} .

Proposition 3.4. *Let \mathcal{E}, \mathcal{F} be Hilbert spaces and let \mathcal{M} be a Hilbert C^* -submodule of $L(\mathcal{E}, \mathcal{F})$. Then $\mathcal{A} \equiv \mathcal{M}^* \cdot \mathcal{M}$ and $\mathcal{B} \equiv \mathcal{M} \cdot \mathcal{M}^*$ are C^* -algebras of operators on \mathcal{E} and \mathcal{F} respectively and \mathcal{M} is equipped with a canonical structure of $(\mathcal{B}, \mathcal{A})$ imprimitivity bimodule.*

For the needs of this paper the last assertion of the proposition could be interpreted as a definition.

Proposition 3.5. *Let \mathcal{N} be a C^* -submodule of $L(\mathcal{E}, \mathcal{F})$ such that $\mathcal{N} \subset \mathcal{M}$ and $\mathcal{N}^* \cdot \mathcal{N} = \mathcal{M}^* \cdot \mathcal{M}$, $\mathcal{N} \cdot \mathcal{N}^* = \mathcal{M} \cdot \mathcal{M}^*$. Then $\mathcal{N} = \mathcal{M}$.*

Proof: If $M \in \mathcal{M}$ and $N \in \mathcal{N}$ then $MN^* \in \mathcal{B} = \mathcal{N} \cdot \mathcal{N}^*$ and $\mathcal{N}\mathcal{N}^*\mathcal{N} \subset \mathcal{N}$ hence $MN^*N \in \mathcal{N}$. Since $\mathcal{N}^* \cdot \mathcal{N} = \mathcal{A}$ we get $MA \in \mathcal{N}$ for all $A \in \mathcal{A}$. Let A_i be an approximate identity for the C^* -algebra \mathcal{A} . Since one can factorize $M = M'A'$ with $M' \in \mathcal{M}$ and $A' \in \mathcal{A}$ the sequence $MA_i = M'A'A_i$ converges to $M'A' = M$ in norm. Thus $M \in \mathcal{N}$. \square

Proposition 3.6. *Let $\mathcal{E}, \mathcal{F}, \mathcal{H}$ be Hilbert spaces and let $\mathcal{M} \subset L(\mathcal{H}, \mathcal{E})$ and $\mathcal{N} \subset L(\mathcal{H}, \mathcal{F})$ be Hilbert C^* -submodules. Let \mathcal{A} be a C^* -algebra of operators on \mathcal{H} such that $\mathcal{M}^* \cdot \mathcal{M}$ and $\mathcal{N}^* \cdot \mathcal{N}$ are ideals of \mathcal{A} and let us view \mathcal{M} and \mathcal{N} as Hilbert \mathcal{A} -modules. Then $\mathcal{K}(\mathcal{M}, \mathcal{N}) \cong \mathcal{N} \cdot \mathcal{M}^*$ the isometric isomorphism being determined by the condition $|N\rangle\langle M| = NM^*$.*

3.3. Graded Hilbert C^* -modules. This is due to Georges Skandalis [Ska] (see also Remark 4.28).

Definition 3.7. Let \mathcal{S} be a semilattice and \mathcal{A} an \mathcal{S} -graded C^* -algebra. A Hilbert \mathcal{A} -module \mathcal{M} is an \mathcal{S} -graded Hilbert \mathcal{A} -module if a linearly independent family $\{\mathcal{M}(\sigma)\}_{\sigma \in \mathcal{S}}$ of closed subspaces of \mathcal{M} is given such that $\sum_{\sigma} \mathcal{M}(\sigma)$ is dense in \mathcal{M} and:

$$\mathcal{M}(\sigma)\mathcal{A}(\tau) \subset \mathcal{M}(\sigma \wedge \tau) \quad \text{and} \quad \langle \mathcal{M}(\sigma) | \mathcal{M}(\tau) \rangle \subset \mathcal{A}(\sigma \wedge \tau) \quad \text{for all } \sigma, \tau \in \mathcal{S}. \quad (3.8)$$

Note that \mathcal{A} equipped with its canonical Hilbert \mathcal{A} -module structure is an \mathcal{S} -graded Hilbert \mathcal{A} -module. (3.8) implies that each $\mathcal{M}(\sigma)$ is a Hilbert $\mathcal{A}(\sigma)$ -module and if $\sigma \leq \tau$ then $\mathcal{M}(\sigma)$ is an $\mathcal{A}(\tau)$ -module.

From (3.8) we also see that the imprimitivity algebra $\mathcal{K}(\mathcal{M}(\sigma))$ of the Hilbert $\mathcal{A}(\sigma)$ -module $\mathcal{M}(\sigma)$ is naturally identified with the clspan in $\mathcal{K}(\mathcal{M})$ of the elements MM^* with $M \in \mathcal{M}(\sigma)$. Thus $\mathcal{K}(\mathcal{M}(\sigma))$ is identified with a C^* -subalgebra of $\mathcal{K}(\mathcal{M})$. We use this identification below.

Theorem 3.8. *If \mathcal{M} is a graded Hilbert \mathcal{A} -module then $\mathcal{K}(\mathcal{M})$ becomes a graded C^* -algebra if we define $\mathcal{K}(\mathcal{M})(\sigma) = \mathcal{K}(\mathcal{M}(\sigma))$. If $M \in \mathcal{M}(\sigma)$ and $N \in \mathcal{M}(\tau)$ then there are elements M' and N' in $\mathcal{M}(\sigma \wedge \tau)$ such that $MN^* = M'N'^*$; in particular $MN^* \in \mathcal{K}(\mathcal{M})(\sigma \wedge \tau)$.*

Proof: As explained before, $\mathcal{K}(\mathcal{M})(\sigma)$ are C^* -subalgebras of $\mathcal{K}(\mathcal{M})$. To show that they are linearly independent, let $T(\sigma) \in \mathcal{K}(\mathcal{M})(\sigma)$ such that $T(\sigma) = 0$ but for a finite number of σ and assume $\sum_{\sigma} T(\sigma) = 0$. Then for each $M \in \mathcal{M}$ we have $\sum_{\sigma} T(\sigma)M = 0$. Note that the range of $T(\sigma)$ is included in $\mathcal{M}(\sigma)$. Since the linear spaces $\mathcal{M}(\sigma)$ are linearly independent we get $T(\sigma)M = 0$ for all σ and M hence $T(\sigma) = 0$ for all σ .

We now prove the second assertion of the proposition. Since $\mathcal{M}(\sigma)$ is a Hilbert $\mathcal{A}(\sigma)$ -module there are $M_1 \in \mathcal{M}(\sigma)$ and $S \in \mathcal{A}(\sigma)$ such that $M = M_1S$, cf. the Cohen-Hewitt theorem or Lemma 4.4 in [Lac]. Similarly, $N = N_1T$ with $N_1 \in \mathcal{M}(\tau)$ and $T \in \mathcal{A}(\tau)$. Then $MN^* = M_1(ST^*)N_1^*$ and $ST^* \in \mathcal{A}(\sigma \wedge \tau)$ so we may factorize it as $ST^* = UV^*$ with $U, V \in \mathcal{A}(\sigma \wedge \tau)$, hence $MN^* = (M_1U)(N_1V)^*$. By using (3.8) we see that $M' = M_1U$ and $N' = N_1V$ belong to $\mathcal{M}(\sigma \wedge \tau)$. In particular, we have $MN^* \in \mathcal{K}(\mathcal{M})(\sigma \wedge \tau)$ if $M \in \mathcal{M}(\sigma)$ and $N \in \mathcal{M}(\tau)$.

Observe that the assertion we just proved implies that $\sum_{\sigma} \mathcal{K}(\mathcal{M})(\sigma)$ is dense in $\mathcal{K}(\mathcal{M})$. It remains to see that $\mathcal{K}(\mathcal{M})(\sigma)\mathcal{K}(\mathcal{M})(\tau) \subset \mathcal{K}(\mathcal{M})(\sigma \wedge \tau)$. For this it suffices that $M\langle M|N\rangle N^*$ be in $\mathcal{K}(\mathcal{M})(\sigma \wedge \tau)$ if $M \in \mathcal{M}(\sigma)$ and $N \in \mathcal{M}(\tau)$. Since $\langle M|N\rangle \in \mathcal{A}(\sigma \wedge \tau)$ we may write $\langle M|N\rangle = ST^*$ with $S, T \in \mathcal{A}(\sigma \wedge \tau)$ so $M\langle M|N\rangle N^* = (MS)(NT)^* \in \mathcal{K}(\mathcal{M})(\sigma \wedge \tau)$ by (3.8). \square

We recall that the direct sum of a family $\{\mathcal{M}_i\}$ of Hilbert \mathcal{A} -modules is defined as follows: $\oplus_i \mathcal{M}_i$ is the space of elements $(M_i)_i \in \prod_i \mathcal{M}_i$ such that the series $\sum_i \langle M_i | M_i \rangle$ converges in \mathcal{A} equipped with the natural \mathcal{A} -module structure and with the \mathcal{A} -valued inner product defined by

$$\langle (M_i)_i | (N_i)_i \rangle = \sum_i \langle M_i | N_i \rangle. \quad (3.9)$$

The algebraic direct sum of the \mathcal{A} -modules \mathcal{M}_i is dense in $\oplus_i \mathcal{M}_i$.

It is easy to check that if each \mathcal{M}_i is graded and if we set $\mathcal{M}(\sigma) = \oplus_i \mathcal{M}_i(\sigma)$ then \mathcal{M} becomes a graded Hilbert \mathcal{A} -module. For example, if \mathcal{N} is a graded Hilbert \mathcal{A} -module then $\mathcal{N} \oplus \mathcal{A}$ is a graded Hilbert \mathcal{A} -module and so the linking algebra $\mathcal{K}(\mathcal{N} \oplus \mathcal{A})$ is equipped with a graded algebra structure. We recall [RW, p. 50-52] that we have a natural identification

$$\mathcal{K}(\mathcal{N} \oplus \mathcal{A}) = \begin{pmatrix} \mathcal{K}(\mathcal{N}) & \mathcal{N} \\ \mathcal{N}^* & \mathcal{A} \end{pmatrix} \quad (3.10)$$

and by Theorem 3.8 this is a graded algebra whose σ -component is equal to

$$\mathcal{K}(\mathcal{N}(\sigma) \oplus \mathcal{A}(\sigma)) = \begin{pmatrix} \mathcal{K}(\mathcal{N}(\sigma)) & \mathcal{N}(\sigma) \\ \mathcal{N}(\sigma)^* & \mathcal{A}(\sigma) \end{pmatrix}. \quad (3.11)$$

If \mathcal{N} is a C^* -submodule of $L(\mathcal{E}, \mathcal{F})$ and if we set $\mathcal{N}^* \cdot \mathcal{N} = \mathcal{A}$, $\mathcal{N} \cdot \mathcal{N}^* = \mathcal{B}$ then the linking algebra $\begin{pmatrix} \mathcal{B} & \mathcal{M} \\ \mathcal{M}^* & \mathcal{A} \end{pmatrix}$ of \mathcal{M} is a C^* -algebra of operators on $\mathcal{F} \oplus \mathcal{E}$.

Some of the graded Hilbert C^* -modules which we shall use later on will be constructed as follows.

Proposition 3.9. *Let \mathcal{E}, \mathcal{F} be Hilbert spaces and let $\mathcal{M} \subset L(\mathcal{E}, \mathcal{F})$ be a Hilbert C^* -submodule, so that $\mathcal{A} \equiv \mathcal{M}^* \cdot \mathcal{M} \subset L(\mathcal{E})$ is a C^* -algebra and \mathcal{M} is a full Hilbert \mathcal{A} -module. Let \mathcal{C} be a C^* -algebra of operators on \mathcal{E} graded by the family of C^* -subalgebras $\{\mathcal{C}(\sigma)\}_{\sigma \in \mathcal{S}}$. Assume that we have*

$$\mathcal{A} \cdot \mathcal{C}(\sigma) = \mathcal{C}(\sigma) \cdot \mathcal{A} \equiv \mathcal{C}(\sigma) \text{ for all } \sigma \in \mathcal{S} \quad (3.12)$$

and that the family $\{\mathcal{C}(\sigma)\}$ of subspaces of $L(\mathcal{F})$ is linearly independent. Then the $\mathcal{C}(\sigma)$ are C^* -algebras of operators on \mathcal{E} and $\mathcal{C} = \sum_{\sigma}^c \mathcal{C}(\sigma)$ is a C^* -algebra graded by the family $\{\mathcal{C}(\sigma)\}$. If $\mathcal{N}(\sigma) \equiv \mathcal{M} \cdot \mathcal{C}(\sigma)$ then $\mathcal{N} = \sum_{\sigma}^c \mathcal{N}(\sigma)$ is a full Hilbert \mathcal{C} -module graded by $\{\mathcal{N}(\sigma)\}$.

Proof: We have

$$\mathcal{C}(\sigma) \cdot \mathcal{C}(\tau) = \mathcal{A} \cdot \mathcal{C}(\sigma) \cdot \mathcal{A} \cdot \mathcal{C}(\tau) = \mathcal{A} \cdot \mathcal{A} \cdot \mathcal{C}(\sigma) \cdot \mathcal{C}(\tau) \subset \mathcal{A} \cdot \mathcal{C}(\sigma \wedge \tau) = \mathcal{C}(\sigma \wedge \tau).$$

This proves that the $\mathcal{C}(\sigma)$ are C^* -algebras and that \mathcal{C} is \mathcal{S} -graded. Then:

$$\mathcal{N}(\sigma) \cdot \mathcal{C}(\tau) = \mathcal{M} \cdot \mathcal{C}(\sigma) \cdot \mathcal{C}(\tau) \cdot \mathcal{A} \subset \mathcal{M} \cdot \mathcal{C}(\sigma \wedge \tau) \cdot \mathcal{A} = \mathcal{M} \cdot \mathcal{A} \cdot \mathcal{C}(\sigma \wedge \tau) = \mathcal{M} \cdot \mathcal{C}(\sigma \wedge \tau) = \mathcal{N}(\sigma \wedge \tau)$$

and

$$\mathcal{N}(\sigma)^* \cdot \mathcal{N}(\tau) = \mathcal{C}(\sigma) \cdot \mathcal{M}^* \cdot \mathcal{M} \cdot \mathcal{C}(\tau) = \mathcal{C}(\sigma) \cdot \mathcal{A} \cdot \mathcal{C}(\tau) = \mathcal{A} \cdot \mathcal{C}(\sigma) \cdot \mathcal{C}(\tau) \subset \mathcal{A} \cdot \mathcal{C}(\sigma \wedge \tau) = \mathcal{C}(\sigma \wedge \tau).$$

Observe that this computation also gives $\mathcal{N}(\sigma)^* \cdot \mathcal{N}(\sigma) = \mathcal{C}(\sigma)$. Then

$$(\sum_{\sigma} \mathcal{N}(\sigma)^*) (\sum_{\sigma} \mathcal{N}(\sigma)) = \sum_{\sigma, \tau} \mathcal{N}(\sigma)^* \cdot \mathcal{N}(\tau) \subset \sum_{\sigma, \tau} \mathcal{C}(\sigma \wedge \tau) \subset \sum_{\sigma} \mathcal{C}(\sigma)$$

and by the preceding remark we get $\mathcal{N}^* \cdot \mathcal{N} = \mathcal{C}$ so \mathcal{N} is a full Hilbert \mathcal{C} -module. To show the grading property it suffices to prove that the family of subspaces $\mathcal{N}(\sigma)$ is linearly independent. Assume that $\sum N(\sigma) = 0$ with $N(\sigma) \in \mathcal{N}(\sigma)$ and $N(\sigma) = 0$ for all but a finite number of σ . Assuming that there are non-zero elements in this sum, let τ be a maximal element of the set of σ such that $N(\sigma) \neq 0$. From $\sum_{\sigma_1, \sigma_2} N(\sigma_1)^* N(\sigma_2) = 0$ and since $N(\sigma_1)^* N(\sigma_2) \in \mathcal{C}(\sigma_1 \wedge \sigma_2)$ we get $\sum_{\sigma_1 \wedge \sigma_2 = \sigma} N(\sigma_1)^* N(\sigma_2) = 0$ for each σ . Take here $\sigma = \tau$ and observe that if $\sigma_1 \wedge \sigma_2 = \tau$ and $\sigma_1 > \tau$ or $\sigma_2 > \tau$ then $N(\sigma_1)^* N(\sigma_2) = 0$. Thus $N(\tau)^* N(\tau) = 0$ so $N(\tau) = 0$. But this contradicts the choice of τ , so $N(\sigma) = 0$ for all σ . \square

3.4. Tensor products. In this subsection we collect some facts concerning tensor products which are useful in what follows. We recall the definition of the tensor product of a Hilbert space \mathcal{E} and a C^* -algebra \mathcal{A} in the category of Hilbert C^* -modules, cf. [Lac]. We equip the algebraic tensor product $\mathcal{E} \odot \mathcal{A}$ with the obvious right \mathcal{A} -module structure and with the \mathcal{A} -valued sesquilinear map given by

$$\langle \sum_{u \in \mathcal{E}} u \otimes A_u | \sum_{v \in \mathcal{E}} v \otimes B_v \rangle = \sum_{u,v} \langle u | v \rangle A_u^* B_v \quad (3.13)$$

where $A_u = B_u = 0$ outside a finite set. Then the completion of $\mathcal{E} \odot \mathcal{A}$ for the norm $\|M\| := \|(M|M)\|^{1/2}$ is a full Hilbert \mathcal{A} -module denoted $\mathcal{E} \otimes \mathcal{A}$. Clearly its imprimitivity algebra is

$$\mathcal{K}(\mathcal{E} \otimes \mathcal{A}) = \mathcal{K}(\mathcal{E}) \otimes \mathcal{A}. \quad (3.14)$$

If \mathcal{A} is \mathcal{S} -graded then $\mathcal{E} \otimes \mathcal{A}$ is equipped with an obvious structure of \mathcal{S} -graded Hilbert \mathcal{A} -module.

If \mathcal{A} is realized on a Hilbert space \mathcal{F} then one has a natural isometric embedding $\mathcal{E} \otimes \mathcal{A} \subset L(\mathcal{F}, \mathcal{E} \otimes \mathcal{F})$. Indeed, there is a unique linear map $\mathcal{E} \otimes \mathcal{A} \rightarrow L(\mathcal{F}, \mathcal{E} \otimes \mathcal{F})$ which associates to $u \otimes A$ the function $f \mapsto u \otimes (Af)$ and due to (3.13) this map is an isometry. Thus the Hilbert \mathcal{A} -module $\mathcal{E} \otimes \mathcal{A}$ is realized as a Hilbert C^* -submodule of $L(\mathcal{F}, \mathcal{E} \otimes \mathcal{F})$, the dual module is realized as the set of adjoint operators $(\mathcal{E} \otimes \mathcal{A})^* \subset L(\mathcal{E} \otimes \mathcal{F}, \mathcal{E})$, and one clearly has

$$(\mathcal{E} \otimes \mathcal{A})^* \cdot (\mathcal{E} \otimes \mathcal{A}) = \mathcal{A}, \quad (\mathcal{E} \otimes \mathcal{A}) \cdot (\mathcal{E} \otimes \mathcal{A})^* = \mathcal{K}(\mathcal{E}) \otimes \mathcal{A}. \quad (3.15)$$

If X is a locally compact space equipped with a Radon measure then $L^2(X) \otimes \mathcal{A}$ is the completion of $\mathcal{C}_c(X; \mathcal{A})$ for the norm $\|\int_X F(x)^* F(x) dx\|^{1/2}$. Note that $L^2(X; \mathcal{A}) \subset L^2(X) \otimes \mathcal{A}$ strictly in general, cf. the example below. If $\mathcal{A} \subset L(\mathcal{F})$ then the norm on $L^2(X) \otimes \mathcal{A}$ is

$$\|\int_X F(x)^* F(x) dx\|^2 = \sup_{f \in \mathcal{F}, \|f\|=1} \int_X \|F(x)f\|^2 dx. \quad (3.16)$$

If Y is a locally compact space then $\mathcal{E} \otimes \mathcal{C}_o(Y) \cong \mathcal{C}_o(Y; \mathcal{E})$. Hence $L^2(X) \otimes \mathcal{C}_o(Y)$ is the completion of $\mathcal{C}_c(X \times Y)$ for the norm $\sup_{y \in Y} (\int_X |F(x,y)|^2 dx)^{1/2}$. Assume that $X = Y$ is a locally compact abelian group and let $f \in L^\infty(X)$ with compact support and $g \in L^2(X)$. It is easy to check that $F(x,y) = f(x)g(x+y)$ is an element of $\mathcal{C}_o(X; L^2(X)) = L^2(X) \otimes \mathcal{C}_o(X)$ but if $F(x, \cdot) = f(x)U_x g$ is not zero then it does not belong to $\mathcal{C}_o(X)$ and is not even a bounded function if g is not. Thus the elements of $L^2(X) \otimes \mathcal{A}$ can not be realized as bounded operator valued (equivalence classes of) functions on X .

More generally, if $\mathcal{F}', \mathcal{F}''$ are Hilbert spaces and $\mathcal{M} \subset L(\mathcal{F}', \mathcal{F}'')$ is a closed subspace then we define $L^2(X) \otimes \mathcal{M}$ as the completion of the space $\mathcal{C}_c(X; \mathcal{M})$ for a norm similar to (3.16). We clearly have $L^2(X) \otimes \mathcal{M} \subset L(\mathcal{F}', L^2(X) \otimes \mathcal{F}'')$ isometrically and $L^2(X; \mathcal{M}) \subset L^2(X) \otimes \mathcal{M}$ continuously.

If $\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}$ are Hilbert spaces and $\mathcal{M} \subset L(\mathcal{E}, \mathcal{F})$ and $\mathcal{N} \subset L(\mathcal{G}, \mathcal{H})$ are closed linear subspaces then we denote $\mathcal{M} \otimes \mathcal{N}$ the closure in $L(\mathcal{E} \otimes \mathcal{G}, \mathcal{F} \otimes \mathcal{H})$ of the algebraic tensor product of \mathcal{M} and \mathcal{N} . Now suppose that \mathcal{M} is a C^* -submodule of $L(\mathcal{E}, \mathcal{F})$ and that \mathcal{N} is a C^* -submodule of $L(\mathcal{G}, \mathcal{H})$ and let $\mathcal{A} = \mathcal{M}^* \cdot \mathcal{M}$ and $\mathcal{B} = \mathcal{N}^* \cdot \mathcal{N}$. Then \mathcal{M} is a Hilbert \mathcal{A} -module and \mathcal{N} is a Hilbert \mathcal{B} -module hence the exterior tensor product, denoted temporarily $\mathcal{M} \otimes_{\text{ext}} \mathcal{N}$, is well defined in the category of Hilbert C^* -modules [Lac] and is a Hilbert $\mathcal{A} \otimes \mathcal{B}$ -module. On the other hand, it is easy to check that $(\mathcal{M} \otimes \mathcal{N})^* = \mathcal{M}^* \otimes \mathcal{N}^*$ and then that $\mathcal{M} \otimes \mathcal{N}$ is a Hilbert C^* -submodule of $L(\mathcal{E} \otimes \mathcal{G}, \mathcal{F} \otimes \mathcal{H})$ such that $(\mathcal{M} \otimes \mathcal{N})^* \cdot (\mathcal{M} \otimes \mathcal{N}) = \mathcal{A} \otimes \mathcal{B}$. Finally, it is clear that $L(\mathcal{E} \otimes \mathcal{G}, \mathcal{F} \otimes \mathcal{H})$ and $\mathcal{M} \otimes_{\text{ext}} \mathcal{N}$ induce the same $\mathcal{A} \otimes \mathcal{B}$ -valued inner product on the algebraic tensor product of \mathcal{M} and \mathcal{N} . Thus we get a canonical isometric isomorphism $\mathcal{M} \otimes_{\text{ext}} \mathcal{N} = \mathcal{M} \otimes \mathcal{N}$.

As an application we give now an abstract version of the "toy models" described in Example 2.12. Let \mathcal{E}, \mathcal{F} be Hilbert spaces and let us define $\mathcal{H} = (\mathcal{E} \otimes \mathcal{F}) \oplus \mathcal{F}$. Let \mathcal{A} and \mathcal{B} be C^* -algebras of operators on \mathcal{F} and $\mathcal{E} \otimes \mathcal{F}$ respectively. We embed $\mathcal{E} \otimes \mathcal{A} \subset L(\mathcal{F}, \mathcal{E} \otimes \mathcal{F})$ as above. We simplify notation and denote $\mathcal{E}^* \otimes \mathcal{A} := (\mathcal{E} \otimes \mathcal{A})^* \subset L(\mathcal{E} \otimes \mathcal{F}, \mathcal{F})$ the dual module.

Proposition 3.10. *Let \mathcal{S} be a semilattice and \mathcal{T} an ideal of \mathcal{S} . Assume that the C^* -algebras \mathcal{A} and \mathcal{B} are \mathcal{S} -graded and that we have $\mathcal{A}(\sigma) = \{0\}$ if $\sigma \notin \mathcal{T}$ and $\mathcal{B}(\tau) = \mathcal{K}(\mathcal{E}) \otimes \mathcal{A}(\tau)$ for $\tau \in \mathcal{T}$. Then*

$$\mathcal{C} = \begin{pmatrix} \mathcal{B} & \mathcal{E} \otimes \mathcal{A} \\ \mathcal{E}^* \otimes \mathcal{A} & \mathcal{A} \end{pmatrix}. \quad (3.17)$$

is an \mathcal{S} -graded C^* -algebra if we define its components as follows:

$$\mathcal{C}(\sigma) = \begin{pmatrix} \mathcal{B}(\sigma) & \mathcal{E} \otimes \mathcal{A}(\sigma) \\ \mathcal{E}^* \otimes \mathcal{A}(\sigma) & \mathcal{A}(\sigma) \end{pmatrix} \quad \text{for all } \sigma \in \mathcal{S}. \quad (3.18)$$

Proof: Observe that if we set $\mathcal{T}' = \mathcal{S} \setminus \mathcal{T}$ then

$$\mathcal{C} = \begin{pmatrix} K(\mathcal{E}) \otimes \mathcal{A} & \mathcal{E} \otimes \mathcal{A} \\ \mathcal{E}^* \otimes \mathcal{A} & \mathcal{A} \end{pmatrix} + \begin{pmatrix} \mathcal{B}(\mathcal{T}') & 0 \\ 0 & 0 \end{pmatrix} = \mathcal{K}(\mathcal{N} \oplus \mathcal{A}) + \begin{pmatrix} \mathcal{B}(\mathcal{T}') & 0 \\ 0 & 0 \end{pmatrix} \quad (3.19)$$

where $\mathcal{N} = \mathcal{E} \otimes \mathcal{A}$ is an \mathcal{S} -graded Hilbert \mathcal{A} -module, cf. (3.10) and (3.14). It is easy to see that the family $\{\mathcal{C}(\sigma)\}$ is linearly independent and that \mathcal{C} is the closure of its sum. By taking into account (3.11) we see that it suffices to show that $\mathcal{C}(\sigma)\mathcal{C}(\tau) \subset \mathcal{C}(\sigma \wedge \tau)$ if $\sigma \in \mathcal{T}'$ and $\tau \in \mathcal{T}$. After computing the coefficients of the matrices we see that it suffices to check that $\mathcal{B}(\sigma) \cdot \mathcal{E} \otimes \mathcal{A}(\tau) \subset \mathcal{E} \otimes \mathcal{A}(\sigma \wedge \tau)$. But:

$$\begin{aligned} \mathcal{B}(\sigma) \cdot \mathcal{E} \otimes \mathcal{A}(\tau) &= \mathcal{B}(\sigma) \cdot K(\mathcal{E}) \otimes \mathcal{A}(\tau) \cdot \mathcal{E} \otimes \mathcal{A}(\tau) = \mathcal{B}(\sigma) \cdot \mathcal{B}(\tau) \cdot \mathcal{E} \otimes \mathcal{A}(\tau) \\ &\subset \mathcal{B}(\sigma \wedge \tau) \cdot \mathcal{E} \otimes \mathcal{A}(\tau) = K(\mathcal{E}) \otimes \mathcal{A}(\sigma \wedge \tau) \cdot \mathcal{E} \otimes \mathcal{A}(\tau) \subset \mathcal{E} \otimes \mathcal{A}(\sigma \wedge \tau) \end{aligned}$$

which finishes the proof. \square

The extension to an increasing family of ideals $\mathcal{T}_1 \subset \mathcal{T}_2 \cdots \subset \mathcal{S}$ is straightforward.

4. THE MANY-BODY C^* -ALGEBRA

In this section we introduce the many-body C^* -algebra and describe its main properties (in particular, we prove the theorems 2.4 and 2.5). Subsection 4.4 contains some preparatory material on concrete realizations of Hilbert C^* -modules which implement the Morita equivalence between some crossed products.

4.1. Notations. Let X be a locally compact abelian group with operation denoted additively equipped with a Haar measures dx . We abbreviate this by saying that X is an lca group. We set $\mathcal{L}_X \equiv L(L^2(X))$ and $\mathcal{K}_X \equiv K(L^2(X))$ and note that these are C^* -algebras independent of the choice of the measure on X . If Y is a second lca group we shall use the abbreviations

$$\mathcal{L}_{XY} = L(L^2(Y), L^2(X)) \quad \text{and} \quad \mathcal{K}_{XY} = K(L^2(Y), L^2(X)). \quad (4.1)$$

We denote by $\varphi(Q)$ the operator in $L^2(X)$ of multiplication by a function φ and if X has to be explicitly specified we set $Q = Q_X$. The bounded uniformly continuous functions on X form a C^* -algebra $\mathcal{C}_b^u(X)$ which contains the algebras $\mathcal{C}_c(X)$ and $\mathcal{C}_o(X)$. The map $\varphi \mapsto \varphi(Q)$ is an embedding $\mathcal{C}_b^u(X) \subset \mathcal{L}_X$.

The group C^* -algebra \mathcal{T}_X of X is the closed linear subspace of \mathcal{L}_X generated by the convolution operators of the form $(\varphi * f)(x) = \int_X \varphi(x-y)f(y)dy$ with $\varphi \in \mathcal{C}_c(X)$. Observe that $f \mapsto \varphi * f$ is equal to $\int_X \varphi(-a)U_a da$ where U_a is the unitary translation operator on $L^2(X)$ defined by $(U_a f)(x) = f(x+a)$.

Let X^* be the group dual to X with operation denoted additively[†]. If $k \in X^*$ we define a unitary operator V_k on $L^2(X)$ by $(V_k u)(x) = k(x)u(x)$. The Fourier transform of an integrable measure μ on X is defined by $(F\mu)(k) = \int \bar{k}(x)\mu(dx)$. Then F induces a bijective map $L^2(X) \rightarrow L^2(X^*)$ hence a canonical isomorphism $S \mapsto F^{-1}SF$ of \mathcal{L}_{X^*} onto \mathcal{L}_X . If ψ is a function on X^* we set $\psi(P) \equiv \psi(P_X) = F^{-1}M_\psi F$, where $M_\psi = \psi(Q_{X^*})$ is the operator of multiplication by ψ on $L^2(X^*)$. The map $\psi \mapsto \psi(P)$ gives an isomorphism $\mathcal{C}_o(X^*) \cong \mathcal{T}_X$.

If $Y \subset X$ is a closed subgroup then $\pi_Y : X \rightarrow X/Y$ is the canonical surjection. We embed $\mathcal{C}_b^u(X/Y) \subset \mathcal{C}_b^u(X)$ with the help of the injective morphism $\varphi \mapsto \varphi \circ \pi_Y$. So $\mathcal{C}_b^u(X/Y)$ is identified with the set of functions $\varphi \in \mathcal{C}_b^u(X)$ such that $\varphi(x+y) = \varphi(x)$ for all $x \in X$ and $y \in Y$.

In particular, $\mathcal{C}_o(X/Y)$ is identified with the set of continuous functions φ on X such that $\varphi(x+y) = \varphi(x)$ for all $x \in X$ and $y \in Y$ and such that for each $\varepsilon > 0$ there is a compact $K \subset X$ such that

[†] Then $(k+p)(x) = k(x)p(x)$, $0(x) = 1$, and the element $-k$ of X^* represents the function \bar{k} . In order to avoid such strange looking expressions one might use the notation $k(x) = [x, k]$.

$|\varphi(x)| < \varepsilon$ if $x \notin K + Y$. By $x/Y \rightarrow \infty$ we mean $\pi_Y(x) \rightarrow \infty$, so the last condition is equivalent to $\varphi(x) \rightarrow 0$ if $x/Y \rightarrow \infty$. For coherence with later notations we set

$$\mathcal{C}_X(Y) = \mathcal{C}_o(X/Y) \quad (4.2)$$

Observe that to an element $y \in Y$ we may associate a translation operator U_y in $L^2(X)$ and another translation operator in $L^2(Y)$. However, in order not to overcharge the writing we shall denote the second operator also by U_y . The restriction map $k \mapsto k|_Y$ is a continuous surjective group morphism $X^* \rightarrow Y^*$ with kernel equal to $Y^\perp = \{k \in X^* \mid k(y) = 1 \forall y \in Y\}$ which defines the canonical identification $Y^* \cong X^*/Y^\perp$. We denote by the same symbol V_k the operator of multiplication by the character $k \in X^*$ in $L^2(X)$ and by the character $k|_Y \in Y^*$ in $L^2(Y)$.

We shall write $X = Y \oplus Z$ if X is the direct sum of the two closed subgroups Y, Z equipped with compatible Haar measures, in the sense that $dx = dy \otimes dz$. Then $L^2(X) = L^2(Y) \otimes L^2(Z)$ as Hilbert spaces and $\mathcal{K}_X = \mathcal{K}_Y \otimes \mathcal{K}_Z$ and $\mathcal{C}_X(Y) = 1 \otimes \mathcal{C}_o(Z)$ as C^* -algebras.

Let $O = \{0\}$ be the trivial group equipped with the Haar measure of total mass 1. Then $L^2(O) = \mathbb{C}$.

4.2. Crossed products. Let X be a locally compact abelian group. A C^* -subalgebra $\mathcal{A} \subset \mathcal{C}_b^u(X)$ stable under translations will be called X -algebra. The *crossed product of \mathcal{A} by the action of X* is an abstractly defined C^* -algebra $\mathcal{A} \rtimes X$ canonically identified with the C^* -algebra of operators on $L^2(X)$ given by

$$\mathcal{A} \rtimes X \equiv \mathcal{A} \cdot \mathcal{T}_X = \mathcal{T}_X \cdot \mathcal{A}. \quad (4.3)$$

Crossed products of the form $\mathcal{C}_X(Y) \rtimes X$ where Y is a closed subgroup of X play an important role in the many-body problem. To simplify notations we set

$$\mathcal{C}_X(Y) = \mathcal{C}_X(Y) \rtimes X = \mathcal{C}_X(Y) \cdot \mathcal{T}_X = \mathcal{T}_X \cdot \mathcal{C}_X(Y). \quad (4.4)$$

If $X = Y \oplus Z$ and if we identify $L^2(X) = L^2(Y) \otimes L^2(Z)$ then $\mathcal{T}_X = \mathcal{T}_Y \otimes \mathcal{T}_Z$ hence

$$\mathcal{C}_X(Y) = \mathcal{T}_Y \otimes \mathcal{K}_Z. \quad (4.5)$$

A useful ‘‘symmetric’’ description of $\mathcal{C}_X(Y)$ is contained in the next lemma. Let $Y^{(2)}$ be the closed subgroup of $X^2 \equiv X \oplus X$ consisting of elements of the form (y, y) with $y \in Y$.

Lemma 4.1. $\mathcal{C}_X(Y)$ is the closure of the set of integral operators with kernels $\theta \in \mathcal{C}_c(X^2/Y^{(2)})$.

Proof: Let \mathcal{C} be the norm closure of the set of integral operators with kernels $\theta \in \mathcal{C}_b^u(X^2)$ having the properties: (1) $\theta(x + y, x' + y) = \theta(x, x')$ for all $x, x' \in X$ and $y \in Y$; (2) $\text{supp } \theta \subset K_\theta + Y$ for some compact $K_\theta \subset X^2$. We show $\mathcal{C} = \mathcal{C}_X(Y)$. Observe that the map in X^2 defined by $(x, x') \mapsto (x - x', x')$ is a topological group isomorphism with inverse $(x_1, x_2) \mapsto (x_1 + x_2, x_2)$ and sends the subgroup $Y^{(2)}$ onto the subgroup $\{0\} \oplus Y$. This map induces an isomorphism $X^2/Y^{(2)} \simeq X \oplus (X/Y)$. Thus any $\theta \in \mathcal{C}_c(X^2/Y^{(2)})$ is of the form $\theta(x, x') = \tilde{\theta}(x - x', x')$ for some $\tilde{\theta} \in \mathcal{C}_c(X \oplus (X/Y))$. Thus \mathcal{C} is the closure in \mathcal{L}_X of the set of operators of the form $(Tu)(x) = \int_X \tilde{\theta}(x - x', x')u(x')dx'$. Since we may approximate $\tilde{\theta}$ with linear combinations of functions of the form $a \otimes b$ with $a \in \mathcal{C}_c(X)$, $b \in \mathcal{C}_c(X/Y)$ we see that \mathcal{C} is the clspan of the set of operators of the form $(Tu)(x) = \int_X a(x - x')b(x')u(x')dx'$. But this clspan is $\mathcal{T}_X \cdot \mathcal{C}_X(Y) = \mathcal{C}_X(Y)$. \square

4.3. Compatible subgroups. If X, Y is an arbitrary pair of lca groups then $X \oplus Y$ is the set $X \times Y$ equipped with the product topology and group structure. If X, Y are closed subgroups of an lca group G and if the map $Y \oplus Z \rightarrow Y + Z$ defined by $(y, z) \mapsto y + z$ is open, we say that they are *compatible subgroups of G* . In this case $Y + Z$ is a closed subgroup of X .

Remark 4.2. If G is σ -compact then X, Y are compatible if and only if $X + Y$ is closed. Indeed, a continuous surjective morphism between two locally compact σ -compact groups is open and a subgroup H of a locally compact group G is closed if and only if H is locally compact for the induced topology, see Theorems 5.11 and 5.29 in [HRe]. We thank Loïc Dubois and Benoit Pausader for enlightening discussions on this matter.

The importance of the compatibility condition in the context of graded C^* -algebras has been pointed out in [Ma1, Lemma 6.1.1] and one may find there several descriptions of this condition (see also Lemma 3.1 from [Ma3]). We quote two of them. Let X/Y be the image of X in G/Y considered as a subgroup of G/Y equipped with the induced topology. The group $X/(X \cap Y)$ is equipped with the locally compact quotient topology and we have a natural map $X/(X \cap Y) \rightarrow X/Y$ which is a bijective continuous group morphism. Then X, Y are compatible if and only if the following equivalent conditions are satisfied:

$$\text{the natural map } X/(X \cap Y) \rightarrow X/Y \text{ is a homeomorphism,} \quad (4.6)$$

$$\text{the natural map } G/(X \cap Y) \rightarrow G/X \times G/Y \text{ is closed.} \quad (4.7)$$

If \mathcal{A} is a G -algebra let $\mathcal{A}|_X$ be the set of restrictions to X of the functions from \mathcal{A} . This is an X -algebra.

Lemma 4.3. *If X, Y are compatible subgroups of G then*

$$\mathcal{C}_G(X) \cdot \mathcal{C}_G(Y) = \mathcal{C}_G(X \cap Y) \quad (4.8)$$

$$\mathcal{C}_G(Y)|_X = \mathcal{C}_X(X \cap Y). \quad (4.9)$$

The second relation remains valid for the subalgebras \mathcal{C}_c .

Proof: The fact that the inclusion \subset in (4.8) is equivalent to the compatibility of X and Y is shown in Lemma 6.1.1 from [Ma1], so we only have to prove that the equality holds. Let $E = (G/X) \times (G/Y)$. If $\varphi \in \mathcal{C}_o(G/X)$ and $\psi \in \mathcal{C}_o(G/Y)$ then $\varphi \otimes \psi$ denotes the function $(s, t) \mapsto \varphi(s)\psi(t)$, which belongs to $\mathcal{C}_o(E)$. The subspace generated by the functions of the form $\varphi \otimes \psi$ is dense in $\mathcal{C}_o(E)$ by the Stone-Weierstrass theorem. If F is a closed subset of E then, by the Tietze extension theorem, each function in $\mathcal{C}_c(F)$ extends to a function in $\mathcal{C}_c(E)$, so the restrictions $(\varphi \otimes \psi)|_F$ generate a dense linear subspace of $\mathcal{C}_o(F)$. Let us denote by π the map $x \mapsto (\pi_X(x), \pi_Y(x))$, so π is a group morphism from G to E with kernel $V = X \cap Y$. Then by (4.7) the range F of π is closed and the quotient map $\tilde{\pi} : G/V \rightarrow F$ is a continuous and closed bijection, hence is a homeomorphism. So $\theta \mapsto \theta \circ \tilde{\pi}$ is an isometric isomorphism of $\mathcal{C}_o(F)$ onto $\mathcal{C}_o(G/V)$. Hence for $\varphi \in \mathcal{C}_o(G/X)$ and $\psi \in \mathcal{C}_o(G/Y)$ the function $\theta = (\varphi \otimes \psi) \circ \tilde{\pi}$ belongs to $\mathcal{C}_o(G/V)$, it has the property $\theta \circ \pi_V = \varphi \circ \pi_X \cdot \psi \circ \pi_Y$, and the functions of this form generate a dense linear subspace of $\mathcal{C}_o(G/V)$.

Now we prove (4.9). Recall that we identify $\mathcal{C}_G(Y)$ with a subset of $\mathcal{C}_b^u(G)$ by using $\varphi \mapsto \varphi \circ \pi_Y$ so in terms of φ the restriction map which defines $\mathcal{C}_G(Y)|_X$ is just $\varphi \mapsto \varphi|_{X/Y}$. Thus we have a canonical embedding $\mathcal{C}_G(Y)|_X \subset \mathcal{C}_b^u(X/Y)$ for an arbitrary pair X, Y . Then the continuous bijective group morphism $\theta : X/(X \cap Y) \rightarrow X/Y$ allows us to embed $\mathcal{C}_G(Y)|_X \subset \mathcal{C}_b^u(X/(X \cap Y))$. That the range of this map is not $\mathcal{C}_X(X \cap Y)$ in general is clear from the example $G = \mathbb{R}, X = \pi\mathbb{Z}, Y = \mathbb{Z}$. But if X, Y are compatible then X/Y is closed in G/Y , so $\mathcal{C}_G(Y)|_X = \mathcal{C}_o(X/Y)$ by the Tietze extension theorem, and θ is a homeomorphism, hence we get (4.9). \square

Lemma 4.4. *If X, Y are compatible subgroups of G then $X^2 = X \oplus X$ and $Y^{(2)} = \{(y, y) \mid y \in Y\}$ is a compatible pair of closed subgroups of $G^2 = G \oplus G$.*

Proof: Let $D = X^2 \cap Y^{(2)} = \{(x, x) \mid x \in X \cap Y\}$. Due to (4.6) it suffices to show that the natural map $Y^{(2)}/D \rightarrow Y^{(2)}/X^2$ is a homeomorphism. Here $Y^{(2)}/X^2$ is the image of $Y^{(2)}$ in $G^2/X^2 \cong (G/X) \oplus (G/X)$, more precisely it is the subset of pairs (a, a) with $a = \pi_X(z)$ and $z \in Y$, equipped with the topology induced by $(G/X) \oplus (G/X)$. Thus the natural map $Y/X \rightarrow Y^{(2)}/X^2$ is a homeomorphism. On the other hand, the natural map $Y/(X \cap Y) \rightarrow Y^{(2)}/D$ is clearly a homeomorphism. To finish the proof note that $Y/(X \cap Y) \rightarrow Y/X$ is a homeomorphism because X, Y is a regular pair. \square

Lemma 4.5. *Let X, Y be compatible subgroups of an lca group G and let X^\perp, Y^\perp be their orthogonals in G^* . Then $(X \cap Y)^\perp = X^\perp + Y^\perp$ and the closed subgroups X^\perp, Y^\perp of G^* are compatible.*

Proof: $X + Y$ is closed and, since $(x, y) \mapsto (x, -y)$ is a homeomorphism, the map $S : X \oplus Y \rightarrow X + Y$ defined by $S(x, y) = x + y$ is an open surjective morphism. Then from the Theorem 9.5, Chapter 2 of [Gur] it follows that the adjoint map S^* is a homeomorphism between $(X + Y)^*$ and its range. In

particular its range is a locally compact subgroup for the topology induced by $X^* \oplus Y^*$ hence is a closed subgroup of $X^* \oplus Y^*$, see Remark 4.2. We have $(X + Y)^\perp = X^\perp \cap Y^\perp$, cf. 23.29 in [HRe]. Thus from $X^* \cong G^*/X^\perp$ and similar representations for Y^* and $(X + Y)^*$ we see that

$$S^* : G^*/(X^\perp \cap Y^\perp) \rightarrow G^*/X^\perp \oplus G^*/Y^\perp$$

is a closed map. But S^* is clearly the natural map involved in (4.7), hence the pair X^\perp, Y^\perp is regular. Finally, note that $(X \cap Y)^\perp$ is always equal to the closure of the subgroup $X^\perp + Y^\perp$, cf. 23.29 and 24.10 in [HRe], and in our case $X^\perp + Y^\perp$ is closed. \square

4.4. Green Hilbert C^* -modules. Let X, Y be a compatible pair of closed subgroups of a locally compact abelian group G . Then the subgroup $X + Y$ of G generated by $X \cup Y$ is also closed. If we identify $X \cap Y$ with the closed subgroup D of $X \oplus Y$ consisting of the elements of the form (z, z) with $z \in X \cap Y$ then the quotient group $X \uplus Y \equiv (X \oplus Y)/(X \cap Y)$ is locally compact and the map

$$\phi : X \oplus Y \rightarrow X + Y \quad \text{defined by} \quad \phi(x, y) = x - y \quad (4.10)$$

is an open continuous surjective group morphism $X \oplus Y \rightarrow X + Y$ with $X \cap Y$ as kernel. Hence the group morphism $\phi^\circ : X \uplus Y \rightarrow X + Y$ induced by ϕ is a homeomorphism.

Since $\mathcal{C}_c(X \uplus Y) \subset \mathcal{C}_b^u(X \oplus Y)$ the elements $\theta \in \mathcal{C}_c(X \uplus Y)$ are functions $\theta : X \times Y \rightarrow \mathbb{C}$ and we may think of them as kernels of integral operators.

Lemma 4.6. *If $\theta \in \mathcal{C}_c(X \uplus Y)$ then $(T_\theta u)(x) = \int_Y \theta(x, y)u(y)dy$ defines an operator in \mathcal{L}_{XY} with norm $\|T_\theta\| \leq C \sup |\theta|$ where C depends only on a compact which contains the support of θ .*

Proof: By the Schur test

$$\|T_\theta\|^2 \leq \sup_{x \in X} \int_Y |\theta(x, y)|dy \cdot \sup_{y \in Y} \int_X |\theta(x, y)|dx.$$

Let $K \subset X$ and $L \subset Y$ be compact sets such that $(K \times L) + D$ contains the support of θ . Thus if $\theta(x, y) \neq 0$ then $x \in z + K$ and $y \in z + L$ for some $z \in X \cap Y$ hence $\int_Y |\theta(x, y)|dy \leq \sup |\theta| \lambda_Y(L)$. Similarly $\int_X |\theta(x, y)|dx \leq \sup |\theta| \lambda_X(K)$. \square

Definition 4.7. \mathcal{T}_{XY} is the norm closure in \mathcal{L}_{XY} of the set of operators T_θ as in Lemma 4.6.

Remark 4.8. If $X \supset Y$ then \mathcal{T}_{XY} is a ‘‘concrete’’ realization of the Hilbert C^* -module introduced by Rieffel in [Rie] which implements the Morita equivalence between the group C^* -algebra $C^*(Y)$ and the crossed product $\mathcal{C}_o(X/Y) \rtimes X$. More precisely, \mathcal{T}_{XY} is a Hilbert $C^*(Y)$ -module and its imprimitivity algebra is canonically isomorphic with $\mathcal{C}_o(X/Y) \rtimes X$. If X, Y is an arbitrary couple of compatible subgroups of G then we defined \mathcal{T}_{XY} such that $\mathcal{T}_{XY} = \mathcal{T}_{XG} \cdot \mathcal{T}_{GY}$. On the other hand, from (4.24) we get $\mathcal{T}_{XY} = \mathcal{T}_{XE} \cdot \mathcal{T}_{EY}$ with $E = X \cap Y$, hence \mathcal{T}_{XY} is naturally a Hilbert $(\mathcal{C}_o(X/E) \rtimes X, \mathcal{C}_o(Y/E) \rtimes Y)$ imprimitivity bimodule. It has been noticed by Georges Skandalis that \mathcal{T}_{XY} is in fact a ‘‘concrete’’ realization of a Hilbert C^* -module introduced by Green to show the Morita equivalence of the C^* -algebras $\mathcal{C}_o(Z/Y) \rtimes X$ and $\mathcal{C}_o(Z/X) \rtimes Y$ where we take $Z = X + Y$, cf. [Wil, Example 4.13].

We give now an alternative definition of \mathcal{T}_{XY} . If $\varphi \in \mathcal{C}_c(G)$ we define $T_{XY}(\varphi) : \mathcal{C}_c(Y) \rightarrow \mathcal{C}_c(X)$ by

$$(T_{XY}(\varphi)u)(x) = \int_Y \varphi(x - y)u(y)dy. \quad (4.11)$$

This operator depends only the restriction $\varphi|_{X+Y}$ hence, by the Tietze extension theorem, we could take $\varphi \in \mathcal{C}_c(Z)$ instead of $\varphi \in \mathcal{C}_c(G)$, where Z is any closed subgroup of G containing $X \cup Y$.

Proposition 4.9. *$T_{XY}(\varphi)$ extends to a bounded operator $L^2(Y) \rightarrow L^2(X)$, also denoted $T_{XY}(\varphi)$, and for each compact $K \subset G$ there is a constant C such that if $\text{supp } \varphi \subset K$*

$$\|T_{XY}(\varphi)\| \leq C \sup_{x \in G} |\varphi(x)|. \quad (4.12)$$

The adjoint operator is given by $T_{XY}(\varphi)^ = T_{YX}(\varphi^*)$ where $\varphi^*(x) = \bar{\varphi}(-x)$. The space \mathcal{T}_{XY} coincides with the closure in \mathcal{L}_{XY} of the set of operators of the form $T_{XY}(\varphi)$.*

Proof: The set $X + Y$ is closed in G hence the restriction map $\mathcal{C}_c(G) \rightarrow \mathcal{C}_c(X + Y)$ is surjective. On the other hand, the map $\phi^\circ : X \uplus Y \rightarrow X + Y$, defined after (4.10), is a homeomorphism so it induces an isomorphism $\varphi \rightarrow \varphi \circ \phi^\circ$ of $\mathcal{C}_c(X + Y)$ onto $\mathcal{C}_c(X \uplus Y)$. Clearly $T_{XY}(\varphi) = T_\theta$ if $\theta = \varphi \circ \phi$, so the proposition follows from Lemma 4.6. \square

We discuss now some properties of the spaces \mathcal{T}_{XY} . We set $\mathcal{T}_{XY}^* \equiv (\mathcal{T}_{XY})^* \subset \mathcal{L}_{YX}$.

Proposition 4.10. *We have $\mathcal{T}_{XX} = \mathcal{T}_X$ and:*

$$\mathcal{T}_{XY}^* = \mathcal{T}_{YX} \quad (4.13)$$

$$\mathcal{T}_{XY} = \mathcal{T}_{XY} \cdot \mathcal{T}_Y = \mathcal{T}_X \cdot \mathcal{T}_{XY} \quad (4.14)$$

$$\mathcal{A}|_X \cdot \mathcal{T}_{XY} = \mathcal{T}_{XY} \cdot \mathcal{A}|_Y \quad (4.15)$$

where \mathcal{A} is an arbitrary G -algebra.

Proof: The relations $\mathcal{T}_{XX} = \mathcal{T}_X$ and (4.13) are obvious. Now we prove the first equality in (4.14) (then the second one follows by taking adjoints). If $C(\eta)$ is the operator of convolution in $L^2(Y)$ with $\eta \in \mathcal{C}_c(Y)$ then a short computation gives

$$T_{XY}(\varphi)C(\eta) = T_{XY}(T_{GY}(\varphi)\eta) \quad (4.16)$$

for $\varphi \in \mathcal{C}_c(G)$. Since $T_{GY}(\varphi)\eta \in \mathcal{C}_c(G)$ we get $T_{XY}(\varphi)C(\eta) \in \mathcal{T}_{GX}$, so $\mathcal{T}_{XY} \cdot \mathcal{T}_Y \subset \mathcal{T}_{XY}$. The converse follows by a standard approximation argument.

Let $\varphi \in \mathcal{C}_c(G)$ and $\theta \in \mathcal{A}$. We shall denote by $\theta(Q_X)$ the operator of multiplication by $\theta|_X$ in $L^2(X)$ and by $\theta(Q_Y)$ that of multiplication by $\theta|_Y$ in $L^2(Y)$. Choose some $\varepsilon > 0$ and let V be a compact neighborhood of the origin in G such that $|\theta(z) - \theta(z')| < \varepsilon$ if $z - z' \in V$. There are functions $\alpha_k \in \mathcal{C}_c(G)$ with $0 \leq \alpha_k \leq 1$ such that $\sum_k \alpha_k = 1$ on the support of φ and $\text{supp } \alpha_k \subset z_k + V$ for some points z_k . Below we shall prove:

$$\|T_{XY}(\varphi)\theta(Q_Y) - \sum_k \theta(Q_X - z_k)T_{XY}(\varphi\alpha_k)\| \leq \varepsilon \|T_{XY}(|\varphi|)\|. \quad (4.17)$$

This implies $\mathcal{T}_{XY} \cdot \mathcal{A}|_Y \subset \mathcal{A}|_X \cdot \mathcal{T}_{XY}$. If we take adjoints, use (4.13) and interchange X and Y in the final relation, we obtain $\mathcal{A}|_X \cdot \mathcal{T}_{XY} = \mathcal{T}_{XY} \cdot \mathcal{A}|_Y$ hence the proposition is proved. For $u \in \mathcal{C}_c(X)$ we have:

$$\begin{aligned} (T_{XY}(\varphi)\theta(Q_Y)u)(x) &= \int_Y \varphi(x-y)\theta(y)u(y)dy = \sum_k \int_Y \varphi(x-y)\alpha_k(x-y)\theta(y)u(y)dy \\ &= \sum_k \int_Y \varphi(x-y)\alpha_k(x-y)\theta(x-z_k)u(y)dy + (Ru)(x) \\ &= \sum_k (\theta(Q_X - z_k)T_{XY}(\varphi\alpha_k)u)(x) + (Ru)(x). \end{aligned}$$

We can estimate the remainder as follows

$$|(Ru)(x)| = \left| \sum_k \int_Y \varphi(x-y)\alpha_k(x-y)[\theta(y) - \theta(x-z_k)]u(y)dy \right| \leq \varepsilon \int_Y |\varphi(x-y)u(y)|dy.$$

because $x - z_k - y \in V$. This proves (4.17). \square

Proposition 4.11. *\mathcal{T}_{XY} is a Hilbert C^* -submodule of \mathcal{L}_{XY} and*

$$\mathcal{T}_{XY}^* \cdot \mathcal{T}_{XY} = \mathcal{C}_Y(X \cap Y), \quad \mathcal{T}_{XY} \cdot \mathcal{T}_{XY}^* = \mathcal{C}_X(X \cap Y). \quad (4.18)$$

Thus \mathcal{T}_{XY} is a $(\mathcal{C}_X(X \cap Y), \mathcal{C}_Y(X \cap Y))$ imprimitivity bimodule.

Proof: Due to (4.13), to prove the first relation in (4.18) we have to compute the clspan \mathcal{C} of the operators $T_{XY}(\varphi)T_{YX}(\psi)$ with φ, ψ in $\mathcal{C}_c(G)$. We recall the notation $G^2 = G \oplus G$, this is a locally compact abelian group and $X^2 = X \oplus X$ is a closed subgroup. Let us choose functions $\varphi_k, \psi_k \in \mathcal{C}_c(G)$ and let

$\Phi = \sum_k \varphi_k \otimes \psi_k \in \mathcal{C}_c(G^2)$. If $\psi_k^\dagger(x) = \psi_k(-x)$, then $\sum_k T_{XY}(\varphi_k)T_{YX}(\psi_k^\dagger)$ is an integral operator on $L^2(X)$ with kernel $\theta_X = \theta|_{X^2}$ where $\theta : G^2 \rightarrow \mathbb{C}$ is given by

$$\theta(x, x') = \int_Y \Phi(x + y, x' + y)dy.$$

Since the set of decomposable functions is dense in $\mathcal{C}_c(G^2)$ in the inductive limit topology, an easy approximation argument shows that \mathcal{C} contains all integral operators with kernels of the same form as θ_X but with arbitrary $\Phi \in \mathcal{C}_c(G^2)$. Let $Y^{(2)}$ be the closed subgroup of G^2 consisting of the elements (y, y) with $y \in Y$. Then $K = \text{supp } \Phi \subset G^2$ is a compact, θ is zero outside $K + Y^{(2)}$, and $\theta(a+b) = \theta(a)$ for all $a \in G^2, b \in Y^{(2)}$. Thus $\theta \in \mathcal{C}_c(G^2/Y^{(2)})$, with the usual identification $\mathcal{C}_c(G^2/Y^{(2)}) \subset \mathcal{C}_b^u(G^2)$. From Proposition 2.48 in [Foll] it follows that reciprocally, any function θ in $\mathcal{C}_c(G^2/Y^{(2)})$ can be represented in terms of some Φ in $\mathcal{C}_c(G^2)$ as above. Thus \mathcal{C} is the closure of the set of integral operators on $L^2(X)$ with kernels of the form θ_X with $\theta \in \mathcal{C}_c(G^2/Y^{(2)})$. According to Lemma 4.4, the pair of subgroups $X^2, Y^{(2)}$ is regular, so we may apply Lemma 4.3 to get $\mathcal{C}_c(G^2/Y^{(2)})|_{X^2} = \mathcal{C}_c(X^2/D)$ where $D = X^2 \cap Y^{(2)} = \{(x, x) \mid x \in X \cap Y\}$. But by Lemma 4.1 the norm closure in \mathcal{L}_X of the set of integral operators with kernel in $\mathcal{C}_c(X^2/D)$ is $\mathcal{C}_X/(X \cap Y)$. This proves (4.18).

It remains to prove that \mathcal{I}_{XY} is a Hilbert C^* -submodule of \mathcal{L}_{XY} , i.e. that we have

$$\mathcal{I}_{XY} \cdot \mathcal{I}_{XY}^* \cdot \mathcal{I}_{XY} = \mathcal{I}_{XY}. \quad (4.19)$$

The first identity in (4.18) and (4.14) imply

$$\mathcal{I}_{XY} \cdot \mathcal{I}_{XY}^* \cdot \mathcal{I}_{XY} = \mathcal{I}_{XY} \cdot \mathcal{I}_Y \cdot \mathcal{C}_Y(X \cap Y) = \mathcal{I}_{XY} \cdot \mathcal{C}_Y(X \cap Y).$$

From Lemma 4.3 we get

$$\mathcal{C}_Y(X \cap Y) = \mathcal{C}_G(X \cap Y)|_Y = \mathcal{C}_G(X)|_Y \cdot \mathcal{C}_G(Y)|_Y = \mathcal{C}_G(X)|_Y$$

because $\mathcal{C}_G(Y)|_Y = \mathbb{C}$. Then by using Proposition 4.10 we obtain

$$\mathcal{I}_{XY} \cdot \mathcal{C}_Y(X \cap Y) = \mathcal{I}_{XY} \cdot \mathcal{C}_G(X)|_Y = \mathcal{C}_G(X)|_X \cdot \mathcal{I}_{XY} = \mathcal{I}_{XY}$$

because $\mathcal{C}_G(X)|_X = \mathbb{C}$. □

Corollary 4.12. *We have*

$$\mathcal{I}_{XY} = \mathcal{I}_{XY} \mathcal{I}_Y = \mathcal{I}_{XY} \mathcal{C}_Y(X \cap Y) \quad (4.20)$$

$$= \mathcal{I}_X \mathcal{I}_{XY} = \mathcal{C}_X(X \cap Y) \mathcal{I}_{XY}. \quad (4.21)$$

Proof: If \mathcal{M} is a Hilbert \mathcal{A} -module then $\mathcal{M} = \mathcal{M}\mathcal{A}$ hence $\mathcal{I}_{XY} = \mathcal{I}_{XY} \mathcal{C}_Y(X \cap Y)$ by Proposition 4.11. The space $\mathcal{C}_Y(X \cap Y)$ is a \mathcal{I}_Y -bimodule and $\mathcal{C}_Y(X \cap Y) = \mathcal{C}_Y(X \cap Y) \cdot \mathcal{I}_Y$ by (4.4) hence we get $\mathcal{C}_Y(X \cap Y) = \mathcal{C}_Y(X \cap Y) \mathcal{I}_Y$ by the Cohen-Hewitt theorem. This proves the first equality in (4.20) and the other ones are proved similarly. □

If \mathcal{G} is a set of closed subgroups of G then the *semilattice generated by \mathcal{G}* is the set of finite intersections of elements of \mathcal{G} .

Proposition 4.13. *Let X, Y, Z be closed subgroups of G such that any two subgroups from the semilattice generated by the family $\{X, Y, Z\}$ are compatible. Then:*

$$\mathcal{I}_{XZ} \cdot \mathcal{I}_{ZY} = \mathcal{I}_{XY} \cdot \mathcal{C}_Y(Y \cap Z) = \mathcal{C}_X(X \cap Z) \cdot \mathcal{I}_{XY} \quad (4.22)$$

$$= \mathcal{I}_{XY} \cdot \mathcal{C}_Y(X \cap Y \cap Z) = \mathcal{C}_X(X \cap Y \cap Z) \cdot \mathcal{I}_{XY}. \quad (4.23)$$

In particular, if $Z \supset X \cap Y$ then

$$\mathcal{I}_{XZ} \cdot \mathcal{I}_{ZY} = \mathcal{I}_{XY}. \quad (4.24)$$

Proof: We first prove (4.24) in the particular case $Z = G$. As in the proof of Proposition 4.11 we see that $\mathcal{T}_{XG} \cdot \mathcal{T}_{GY}$ is the the closure in \mathcal{L}_{XY} of the set of integral operators with kernels $\theta_{XY} = \theta|_{X \times Y}$ where $\theta : G^2 \rightarrow \mathbb{C}$ is given by

$$\theta(x, y) = \int_G \sum_k \varphi_k(x - z) \psi_k(z - y) dz = \int_G \sum_k \varphi_k(x - y - z) \psi_k(z) dz \equiv \xi(x - y)$$

where $\varphi_k, \psi_k \in \mathcal{C}_c(G)$ and $\xi = \sum_k \varphi_k * \psi_k$ convolution product on G . Since $\mathcal{C}_c(G) * \mathcal{C}_c(G)$ is dense in $\mathcal{C}_c(G)$ in the inductive limit topology, the space $\mathcal{T}_{XG} \cdot \mathcal{T}_{GY}$ is the the closure of the set of integral operators with kernels $\theta(x, y) = \xi(x - y)$ with $\xi \in \mathcal{C}_c(G)$. By Proposition 4.9 this is \mathcal{T}_{XY} .

Now we prove (4.22). From (4.24) with $Z = G$ and (4.18) we get:

$$\mathcal{T}_{XZ} \cdot \mathcal{T}_{ZY} = \mathcal{T}_{XG} \cdot \mathcal{T}_{GZ} \cdot \mathcal{T}_{ZG} \cdot \mathcal{T}_{GY} = \mathcal{T}_{XG} \cdot \mathcal{C}_G(Z) \cdot \mathcal{T}_G \cdot \mathcal{T}_{GY}.$$

Then from Proposition (4.10) and Lemma 4.3 we get:

$$\mathcal{C}_G(Z) \cdot \mathcal{T}_G \cdot \mathcal{T}_{GY} = \mathcal{C}_G(Z) \cdot \mathcal{T}_{GY} = \mathcal{T}_{GY} \cdot \mathcal{C}_G(Z)|_Y = \mathcal{T}_{GY} \cdot \mathcal{C}_Y(Y \cap Z).$$

We obtain (4.22) by using once again (4.24) with $Z = G$ and taking adjoints. On the other hand, the relation $\mathcal{T}_{XY} = \mathcal{T}_{XY} \cdot \mathcal{C}_Y(X \cap Y)$ holds because of (4.20), so we have

$$\mathcal{T}_{XY} \cdot \mathcal{C}_Y(Y \cap Z) = \mathcal{T}_{XY} \cdot \mathcal{C}_Y(X \cap Y) \cdot \mathcal{C}_Y(Y \cap Z) = \mathcal{T}_{XY} \cdot \mathcal{C}_Y(X \cap Y \cap Z)$$

where we also used (4.8) and the fact that $X \cap Y, Z \cap Y$ are compatible. Finally, to get (4.24) for $Z \supset X \cap Y$ we use once again (4.18). \square

The object of main interest for us is introduced in the next definition.

Definition 4.14. If X, Y are compatible subgroups and Z is a closed subgroup of $X \cap Y$ then we set

$$\mathcal{C}_{XY}(Z) := \mathcal{T}_{XY} \cdot \mathcal{C}_Y(Z) = \mathcal{C}_X(Z) \cdot \mathcal{T}_{XY}. \quad (4.25)$$

The equality above follows from (4.15) with $\mathcal{A} = \mathcal{C}_G(Z)$. We clearly have $\mathcal{C}_{XY}(X \cap Y) = \mathcal{T}_{XY}$ and $\mathcal{C}_{XX}(Y) = \mathcal{C}_X(Y)$ if $X \supset Y$. Moreover

$$\mathcal{C}_{XY}^*(Z) := \mathcal{C}_{XY}(Z)^* = \mathcal{C}_{YX}(Z) \quad (4.26)$$

because of (4.13).

Theorem 4.15. $\mathcal{C}_{XY}(Z)$ is a Hilbert C^* -submodule of \mathcal{L}_{XY} such that

$$\mathcal{C}_{XY}^*(Z) \cdot \mathcal{C}_{XY}(Z) = \mathcal{C}_Y(Z) \quad \text{and} \quad \mathcal{C}_{XY}(Z) \cdot \mathcal{C}_{XY}^*(Z) = \mathcal{C}_X(Z). \quad (4.27)$$

In particular, $\mathcal{C}_{XY}(Z)$ is a $(\mathcal{C}_X(Z), \mathcal{C}_Y(Z))$ imprimitivity bimodule.

Proof: By using (4.26), the definition (4.25), and (4.8) we get

$$\begin{aligned} \mathcal{C}_{XY}(Z) \cdot \mathcal{C}_{YX}(Z) &= \mathcal{C}_X(Z) \cdot \mathcal{T}_{XY} \cdot \mathcal{T}_{YX} \cdot \mathcal{C}_X(Z) \\ &= \mathcal{C}_X(Z) \cdot \mathcal{C}_X(X \cap Y) \cdot \mathcal{T}_X \cdot \mathcal{C}_X(Z) \\ &= \mathcal{C}_X(Z) \cdot \mathcal{T}_X \cdot \mathcal{C}_X(Z) = \mathcal{C}_X(Z) \cdot \mathcal{T}_X \end{aligned}$$

which proves the second equality in (4.27). The first one follows by interchanging X and Y . \square

4.5. Many-body systems. Here we give a formal definition of the notion of ‘‘many-body system’’ then define and discuss the Hamiltonian algebra associated to it.

Let \mathcal{S} be a set of locally compact abelian groups with the following property: for any $X, Y \in \mathcal{S}$ there is $Z \in \mathcal{S}$ such that X and Y are compatible subgroups of Z . Note that this implies the following: if $Y \subset X$ then the topology and the group structure of Y coincide with those induced by X .

If \mathcal{S} is a set of σ -compact locally compact abelian groups then the compatibility assumption is equivalent to the following more explicit condition: for any $X, Y \in \mathcal{S}$ there is $Z \in \mathcal{S}$ such that X and Y are closed subgroups of Z and $X + Y$ is closed in Z .

Definition 4.16. A *many-body system* is a couple (\mathcal{S}, λ) where:

- (i) $\mathcal{S} \subset \mathcal{S}$ is a subset such that $X, Y \in \mathcal{S} \Rightarrow X \cap Y \in \mathcal{S}$ and if $X \supsetneq Y$ then X/Y is not compact,
- (ii) λ is a map $X \mapsto \lambda_X$ which associates a Haar measures λ_X on X to each $X \in \mathcal{S}$.

We identify $\mathcal{S} = (\mathcal{S}, \lambda)$ so the choice of Haar measures is implicit. Note that the Hilbert space $\mathcal{H}_{\mathcal{S}}$ and the C^* -algebra $\mathcal{C}_{\mathcal{S}}$ that we introduce below depend on λ but different choices give isomorphic objects. Each $X \in \mathcal{S}$ is equipped with a Haar measure so the Hilbert spaces $\mathcal{H}_X = L^2(X)$ are well defined. If $Y \subset X$ are in \mathcal{S} then X/Y is equipped with the quotient measure so $\mathcal{H}_{X/Y} = L^2(X/Y)$ is well defined.

Example: Let \mathcal{S} the set of all finite dimensional vector subspaces of a vector space over an infinite locally compact field and let \mathcal{S} be any subset of \mathcal{S} such that $X, Y \in \mathcal{S} \Rightarrow X \cap Y \in \mathcal{S}$.

For each $X \in \mathcal{S}$ let \mathcal{S}_X be the set of $Y \in \mathcal{S}$ such that $Y \subset X$. This is an N -body system with X as configuration space in the sense of Definition 2.2. Then by Lemma 4.3 the space

$$\mathcal{C}_X := \sum_{Y \in \mathcal{S}_X}^c \mathcal{C}_X(Y) \quad (4.28)$$

is an X -algebra so the crossed product $\mathcal{C}_X \rtimes X$ is well defined and we clearly have

$$\mathcal{C}_X := \mathcal{C}_X \rtimes X \equiv \mathcal{C}_X \cdot \mathcal{T}_X = \sum_{Y \in \mathcal{S}_X}^c \mathcal{C}_X(Y). \quad (4.29)$$

The C^* -algebra \mathcal{C}_X is realized on the Hilbert space \mathcal{H}_X and we think of it as the Hamiltonian algebra of the N -body system determined by \mathcal{S}_X .

Theorem 4.17. The C^* -algebras \mathcal{C}_X and \mathcal{C}_X are \mathcal{S}_X -graded by the decompositions (4.28) and (4.29).

The theorem is a particular case of results due to A. Mageira, cf. Propositions 6.1.2, 6.1.3 and 4.2.1 in [Ma1] (or see [Ma3]). We mention that the results in [Ma1, Ma3] are much deeper since the groups are allowed to be noncommutative and the treatment is so that the second part of condition (i) is not needed. The case when \mathcal{S} consists of linear subspaces of a finite dimensional real vector space has been considered in [BG1, DaG1] and the corresponding version of Theorem 4.17 is proved there by elementary means.

Definition 4.18. If $X, Y \in \mathcal{S}$ then $\mathcal{C}_{XY} := \mathcal{T}_{XY} \cdot \mathcal{C}_Y = \mathcal{C}_X \cdot \mathcal{T}_{XY}$.

In particular $\mathcal{C}_{XX} = \mathcal{C}_X$ is a C^* -algebra of operators on \mathcal{H}_X . For $X \neq Y$ the space \mathcal{C}_{XY} is a closed linear space of operators $\mathcal{H}_Y \rightarrow \mathcal{H}_X$ canonically associated to the semilattice of groups $\mathcal{S}_{X \cap Y}$, cf. (4.34). We call these spaces *coupling modules* because they are Hilbert C^* -modules and determine the way the systems corresponding to X and Y are allowed to interact.

For each pair $X, Y \in \mathcal{S}$ with $X \supset Y$ we set

$$\mathcal{C}_X^Y := \sum_{Z \in \mathcal{S}_Y}^c \mathcal{C}_X(Z). \quad (4.30)$$

This is also an X -algebra so we may define $\mathcal{C}_X^Y = \mathcal{C}_X^Y \rtimes X$ and we have

$$\mathcal{C}_X^Y := \mathcal{C}_X^Y \rtimes X = \sum_{Z \in \mathcal{S}_Y}^c \mathcal{C}_X(Z). \quad (4.31)$$

If $X = Y \oplus Z$ then $\mathcal{C}_X^Y \simeq \mathcal{C}_Y \otimes 1$ and $\mathcal{C}_X^Z \simeq \mathcal{C}_Y \otimes \mathcal{T}_Z$.

Lemma 4.19. Let $X \in \mathcal{S}$ and $Y \in \mathcal{S}_X$. Then

$$\mathcal{C}_X^Y = \mathcal{C}_X(Y) \cdot \mathcal{C}_X \quad \text{and} \quad \mathcal{C}_X^Y = \mathcal{C}_X(Y) \cdot \mathcal{C}_X = \mathcal{C}_X \cdot \mathcal{C}_X(Y). \quad (4.32)$$

Moreover, for all $Y, Z \in \mathcal{S}_X$ we have

$$\mathcal{C}_X^Y \cdot \mathcal{C}_X^Z = \mathcal{C}_X^{Y \cap Z} \quad \text{and} \quad \mathcal{C}_X^Y \cdot \mathcal{C}_X^Z = \mathcal{C}_X^{Y \cap Z}. \quad (4.33)$$

Proof: The abelian case follows from (4.8) and a straightforward computation. For the crossed product algebras we use $\mathcal{C}_X(Y) \cdot \mathcal{C}_X = \mathcal{C}_X(Y) \cdot \mathcal{C}_X \cdot \mathcal{T}_X$ and the first relation in (4.32) for example. \square

Lemma 4.20. For arbitrary $X, Y \in \mathcal{S}$ we have

$$\mathcal{C}_X \cdot \mathcal{T}_{XY} = \mathcal{T}_{XY} \cdot \mathcal{C}_Y = \mathcal{T}_{XY} \cdot \mathcal{C}_Y^{X \cap Y} = \mathcal{C}_X^{X \cap Y} \cdot \mathcal{T}_{XY}. \quad (4.34)$$

Proof: If $G \in \mathcal{S}$ contains $X \cup Y$ then clearly

$$\mathcal{C}_X \cdot \mathcal{T}_{XY} = \sum_{Z \in \mathcal{S}_X}^c \mathcal{C}_X(Z) \cdot \mathcal{T}_{XY} = \sum_{Z \in \mathcal{S}_X}^c \mathcal{C}_G(Z)|_X \cdot \mathcal{T}_{XY}.$$

From (4.15) and (4.9) we get

$$\mathcal{C}_G(Z)|_X \cdot \mathcal{T}_{XY} = \mathcal{T}_{XY} \cdot \mathcal{C}_Y(Y \cap Z).$$

Since $Y \cap Z$ runs over $\mathcal{S}_{X \cap Y}$ when Z runs over \mathcal{S}_X we obtain $\mathcal{C}_X \cdot \mathcal{T}_{XY} = \mathcal{T}_{XY} \cdot \mathcal{C}_Y^{X \cap Y}$. Similarly $\mathcal{T}_{XY} \cdot \mathcal{C}_Y = \mathcal{C}_X^{X \cap Y} \cdot \mathcal{T}_{XY}$. On the other hand $\mathcal{C}_X^{X \cap Y} = \mathcal{C}_G^{X \cap Y}|_X$ and similarly with X, Y interchanged, hence $\mathcal{C}_X^{X \cap Y} \cdot \mathcal{T}_{XY} = \mathcal{T}_{XY} \cdot \mathcal{C}_Y^{X \cap Y}$ because of (4.15). \square

Proposition 4.21. *Let $X, Y, Z \in \mathcal{S}$. Then $\mathcal{C}_{XY}^* = \mathcal{C}_{YX}$ and*

$$\mathcal{C}_{XZ} \cdot \mathcal{C}_{ZY} = \mathcal{C}_{XY} \cdot \mathcal{C}_Y^{X \cap Y \cap Z} = \mathcal{C}_X^{X \cap Y \cap Z} \cdot \mathcal{C}_{XY} \subset \mathcal{C}_{XY}. \quad (4.35)$$

In particular $\mathcal{C}_{XZ} \cdot \mathcal{C}_{ZY} = \mathcal{C}_{XY}$ if $Z \supset X \cap Y$.

Proof: The first assertion follows from (4.13). From the Definition 4.18 and Proposition 4.13 we then get

$$\begin{aligned} \mathcal{C}_{XZ} \cdot \mathcal{C}_{ZY} &= \mathcal{C}_X \cdot \mathcal{T}_{XZ} \cdot \mathcal{T}_{ZY} \cdot \mathcal{C}_Y = \mathcal{C}_X \cdot \mathcal{T}_{XY} \cdot \mathcal{C}_Y(X \cap Y \cap Z) \cdot \mathcal{C}_Y \\ &= \mathcal{T}_{XY} \cdot \mathcal{C}_Y \cdot \mathcal{C}_Y(X \cap Y \cap Z) \cdot \mathcal{C}_Y = \mathcal{T}_{XY} \cdot \mathcal{C}_Y(X \cap Y \cap Z) \cdot \mathcal{C}_Y. \end{aligned}$$

But $\mathcal{C}_Y(X \cap Y \cap Z) \cdot \mathcal{C}_Y = \mathcal{C}_Y^{X \cap Y \cap Z}$ by Lemma 4.19. For the last inclusion in (4.35) we use the obvious relation $\mathcal{C}_Y^{X \cap Y \cap Z} \cdot \mathcal{C}_Y \subset \mathcal{C}_Y$. The last assertion of the proposition follows from (4.34). \square

The following theorem is a consequence of the results obtained so far.

Theorem 4.22. *\mathcal{C}_{XY} is a Hilbert C^* -submodule of \mathcal{L}_{XY} such that*

$$\mathcal{C}_{XY}^* \cdot \mathcal{C}_{XY} = \mathcal{C}_Y^{X \cap Y} \text{ and } \mathcal{C}_{XY} \cdot \mathcal{C}_{XY}^* = \mathcal{C}_X^{X \cap Y}. \quad (4.36)$$

In particular, \mathcal{C}_{XY} is a $(\mathcal{C}_X^{X \cap Y}, \mathcal{C}_Y^{X \cap Y})$ imprimitivity bimodule.

We recall the conventions

$$X, Y \in \mathcal{S} \text{ and } Y \not\subset X \Rightarrow \mathcal{C}_X(Y) = \mathcal{C}_X(Y) = \{0\}, \quad (4.37)$$

$$X, Y, Z \in \mathcal{S} \text{ and } Z \not\subset X \cap Y \Rightarrow \mathcal{C}_{XY}(Z) = \{0\}. \quad (4.38)$$

From now on by ‘‘graded’’ we mean \mathcal{S} -graded. Then $\mathcal{C}_X = \sum_{Y \in \mathcal{S}}^c \mathcal{C}_X(Y)$ is a graded C^* -algebra supported by the ideal \mathcal{S}_X of \mathcal{S} , in particular it is a graded ideal in \mathcal{C}_X . With the notations of Subsection 3.1 the algebra $\mathcal{C}_X^Y = \mathcal{C}_X(\mathcal{S}_Y)$ is a graded ideal of \mathcal{C}_X supported by \mathcal{S}_Y . Similarly for \mathcal{C}_X and \mathcal{C}_X^Y .

Since $\mathcal{C}_X^{X \cap Y}$ and $\mathcal{C}_Y^{X \cap Y}$ are ideals in \mathcal{C}_X and \mathcal{C}_Y respectively, Theorem 4.22 allows us to equip \mathcal{C}_{XY} with (right) Hilbert \mathcal{C}_Y -module and left Hilbert \mathcal{C}_X -module structures (which are not full in general).

Theorem 4.23. *The Hilbert \mathcal{C}_Y -module \mathcal{C}_{XY} is graded by the family of C^* -submodules $\{\mathcal{C}_{XY}(Z)\}_{Z \in \mathcal{S}}$.*

Proof: We use Proposition 3.9 with $\mathcal{M} = \mathcal{T}_{XY}$ and $\mathcal{C}_Y(Z)$ as algebras $\mathcal{C}(\sigma)$. Then $\mathcal{A} = \mathcal{C}_Y(X \cap Y)$ by (4.18) hence $\mathcal{A} \cdot \mathcal{C}_Y(Z) = \mathcal{C}_Y(Z)$ and the conditions of the proposition are satisfied. \square

Remark 4.24. The following more precise statement is a consequence of the Theorem 4.23: the Hilbert $\mathcal{C}_Y^{X \cap Y}$ -module \mathcal{C}_{XY} is $\mathcal{S}_{X \cap Y}$ -graded by the family of C^* -submodules $\{\mathcal{C}_{XY}(Z)\}_{Z \in \mathcal{S}_{X \cap Y}}$.

Finally, we may construct the C^* -algebra \mathcal{C} which is of main interest for us, the many-body Hamiltonian algebra. We shall describe it as an algebra of operators on the Hilbert space

$$\mathcal{H} \equiv \mathcal{H}_{\mathcal{S}} = \bigoplus_{X \in \mathcal{S}} \mathcal{H}_X \quad (4.39)$$

which is a kind of Boltzmann-Fock space (without symmetrization or anti-symmetrization) determined by the semilattice \mathcal{S} . Note that if the zero group $O = \{0\}$ belongs to \mathcal{S} then \mathcal{H} contains $\mathcal{H}_O = \mathbb{C}$ as a

subspace, this is the vacuum sector. Let Π_X be the orthogonal projection of \mathcal{H} onto \mathcal{H}_X and let us think of its adjoint Π_X^* as the natural embedding $\mathcal{H}_X \subset \mathcal{H}$. Then for any pair $X, Y \in \mathcal{S}$ we identify

$$\mathcal{C}_{XY} \equiv \Pi_X^* \mathcal{C}_{XY} \Pi_Y \subset L(\mathcal{H}). \quad (4.40)$$

Thus we realize $\{\mathcal{C}_{XY}\}_{X,Y \in \mathcal{S}}$ as a linearly independent family of closed subspaces of $L(\mathcal{H})$ such that $\mathcal{C}_{XY}^* = \mathcal{C}_{YX}$ and $\mathcal{C}_{XZ} \mathcal{C}_{ZY} \subset \mathcal{C}_{XY}$ for all $X, Y, Z, Z' \in \mathcal{S}$. Then by what we proved before, especially Proposition 4.21, the space $\sum_{X,Y \in \mathcal{S}} \mathcal{C}_{XY}$ is a $*$ -subalgebra of $L(\mathcal{H})$ hence its closure

$$\mathcal{C} \equiv \mathcal{C}_{\mathcal{S}} = \sum_{X,Y \in \mathcal{S}}^c \mathcal{C}_{XY}. \quad (4.41)$$

is a C^* -algebra of operators on \mathcal{H} . Note that one may view \mathcal{C} as a matrix $(\mathcal{C}_{XY})_{X,Y \in \mathcal{S}}$.

In a similar way one may associate to the spaces \mathcal{T}_{XY} a closed self-adjoint subspace $\mathcal{T} \subset L(\mathcal{H})$. It is also useful to define a new subspace $\mathcal{T}^\circ \subset L(\mathcal{H})$ by $\mathcal{T}_{XY}^\circ = \mathcal{T}_{XY}$ if $X \sim Y$ and $\mathcal{T}^\circ = \{0\}$ if $X \not\sim Y$. Here $X \sim Y$ means $X \subset Y$ or $Y \subset X$. Clearly \mathcal{T}° is a closed self-adjoint linear subspace of \mathcal{T} . Finally, let \mathcal{C} be the diagonal C^* -algebra $\mathcal{C} \equiv \bigoplus_X \mathcal{C}_X$ of operators on \mathcal{H} .

Theorem 4.25. *We have $\mathcal{C} = \mathcal{T} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{T} = \mathcal{T} \cdot \mathcal{T} = \mathcal{T}^\circ \cdot \mathcal{T}^\circ$.*

Proof: The first two equalities are an immediate consequence of the Definition 4.18. To prove the third equality we use Proposition 4.13, more precisely the relation

$$\mathcal{T}_{XZ} \cdot \mathcal{T}_{ZY} = \mathcal{T}_{XY} \cdot \mathcal{C}_Y(X \cap Y \cap Z) = \mathcal{C}_{XY}(X \cap Y \cap Z)$$

which holds for any X, Y, Z . Then

$$\sum_Z^c \mathcal{T}_{XZ} \cdot \mathcal{T}_{ZY} = \sum_Z^c \mathcal{C}_{XY}(X \cap Y \cap Z) = \sum_Z^c \mathcal{C}_{XY}(Z) = \mathcal{C}_{XY}$$

which is equivalent to $\mathcal{T} \cdot \mathcal{T} = \mathcal{C}$. Now we prove the last equality in the proposition. We have

$$\sum_Z^c \mathcal{T}_{XZ}^\circ \cdot \mathcal{T}_{ZY}^\circ = \text{closure of the sum } \sum_{\substack{Z \sim X \\ Z \sim Y}} \mathcal{T}_{XZ} \cdot \mathcal{T}_{ZY}.$$

In the last sum we have four possibilities: $Z \supset X \cup Y$, $X \supset Z \supset Y$, $Y \supset Z \supset X$, and $Z \subset X \cap Y$. In the first three cases we have $Z \supset X \cap Y$ hence $\mathcal{T}_{XZ} \cdot \mathcal{T}_{ZY} = \mathcal{T}_{XY}$ by (4.24). In the last case we have $\mathcal{T}_{XZ} \cdot \mathcal{T}_{ZY} = \mathcal{T}_{XY} \cdot \mathcal{C}_Y(Z)$ by (4.22). This proves $\mathcal{T}^\circ \cdot \mathcal{T}^\circ = \mathcal{C}$. \square

Finally, we are able to equip \mathcal{C} with an \mathcal{S} -graded C^* -algebra structure.

Theorem 4.26. *For each $Z \in \mathcal{S}$ the space $\mathcal{C}(Z) := \sum_{X,Y \in \mathcal{S}}^c \mathcal{C}_{XY}(Z)$ is a C^* -subalgebra of \mathcal{C} . The family $\{\mathcal{C}(Z)\}_{Z \in \mathcal{S}}$ defines a graded C^* -algebra structure on \mathcal{C} .*

Proof: We first prove the following relation:

$$\mathcal{C}_{XZ}(E) \cdot \mathcal{C}_{ZY}(F) = \mathcal{C}_{XY}(E \cap F) \quad \text{if } X, Y, Z \in \mathcal{S} \text{ and } E \subset X \cap Z, F \subset Y \cap Z. \quad (4.42)$$

From Definition 4.14, Proposition 4.13, relations (4.8) and (4.15), and $F \subset Y \cap Z$, we get

$$\begin{aligned} \mathcal{C}_{XZ}(E) \cdot \mathcal{C}_{ZY}(F) &= \mathcal{C}_X(E) \cdot \mathcal{T}_{XZ} \cdot \mathcal{T}_{ZY} \cdot \mathcal{C}_Y(F) \\ &= \mathcal{C}_X(E) \cdot \mathcal{T}_{XY} \cdot \mathcal{C}_Y(Y \cap Z) \cdot \mathcal{C}_Y(F) \\ &= \mathcal{C}_X(E) \cdot \mathcal{T}_{XY} \cdot \mathcal{C}_Y(F) \\ &= \mathcal{T}_{XY} \cdot \mathcal{C}_Y(Y \cap E) \cdot \mathcal{C}_Y(F) \\ &= \mathcal{T}_{XY} \cdot \mathcal{C}_Y(Y \cap E \cap F). \end{aligned}$$

At the next to last step we used $\mathcal{C}_X(E) = \mathcal{C}_G(E)|_X$ for some $G \in \mathcal{S}$ containing both X and Y and then (4.15), (4.9). Finally, we use $\mathcal{C}_Y(Y \cap E \cap F) = \mathcal{C}_Y(E \cap F)$ and the Definition 4.14. This proves (4.42). Due to the conventions (4.37), (4.38) we now get from (4.42) for $E, F \in \mathcal{S}$

$$\sum_{Z \in \mathcal{S}} \mathcal{C}_{XZ}(E) \cdot \mathcal{C}_{ZY}(F) = \mathcal{C}_{XY}(E \cap F).$$

Thus $\mathcal{C}(E)\mathcal{C}(F) \subset \mathcal{C}(E \cap F)$, in particular $\mathcal{C}(E)$ is a C^* -algebra. It remains to be shown that the family of C^* -algebras $\{\mathcal{C}(E)\}_{E \in \mathcal{S}}$ is linearly independent. Let $A(E) \in \mathcal{C}(E)$ such that $A(E) = 0$ but for a finite number of E and assume that $\sum_E A(E) = 0$. Then for all $X, Y \in \mathcal{S}$ we have $\sum_E \Pi_X A(E) \Pi_Y^* =$

0. Clearly $\Pi_X A(E) \Pi_Y^* \in \mathcal{C}_{XY}(E)$ hence from Theorem 4.23 we get $\Pi_X A(E) \Pi_Y^* = 0$ for all X, Y so $A(E) = 0$ for all E . \square

4.6. Subsystems. We now point out some interesting subalgebras of \mathcal{C} . If $\mathcal{T} \subset \mathcal{S}$ is any subset let

$$\mathcal{C}_{\mathcal{S}}^{\mathcal{T}} \equiv \sum_{X, Y \in \mathcal{T}}^c \mathcal{C}_{XY} \quad \text{and} \quad \mathcal{H}_{\mathcal{T}} \equiv \oplus_{X \in \mathcal{T}} \mathcal{H}_X. \quad (4.43)$$

Note that the sum defining $\mathcal{C}_{\mathcal{S}}^{\mathcal{T}}$ is already closed if \mathcal{T} is finite and that $\mathcal{C}_{\mathcal{S}}^{\mathcal{T}}$ is a C^* -algebra which lives on the subspace $\mathcal{H}_{\mathcal{T}}$ of \mathcal{H} . In fact, if $\Pi_{\mathcal{T}}$ is the orthogonal projection of \mathcal{H} onto $\mathcal{H}_{\mathcal{T}}$ then

$$\mathcal{C}_{\mathcal{S}}^{\mathcal{T}} = \Pi_{\mathcal{T}} \mathcal{C}_{\mathcal{S}} \Pi_{\mathcal{T}} \quad (4.44)$$

and this is a C^* -algebra because $\mathcal{C} \Pi_{\mathcal{T}} \mathcal{C} \subset \mathcal{C}$ by Proposition 4.21. It is easy to check that $\mathcal{C}_{\mathcal{S}}^{\mathcal{T}}$ is a graded C^* -subalgebra of \mathcal{C} supported by the ideal $\bigcup_{X \in \mathcal{T}} \mathcal{S}_X$ generated by \mathcal{T} in \mathcal{S} . Indeed, we have

$$\mathcal{C}_{\mathcal{S}}^{\mathcal{T}} \cap \mathcal{C}(E) = \left(\sum_{X, Y \in \mathcal{T}}^c \mathcal{C}_{XY} \right) \cap \left(\sum_{X, Y \in \mathcal{S}}^c \mathcal{C}_{XY}(E) \right) = \sum_{X, Y \in \mathcal{T}}^c \mathcal{C}_{XY}(E).$$

It is clear that \mathcal{C} is the inductive limit of the increasing family of C^* -algebras $\mathcal{C}_{\mathcal{S}}^{\mathcal{T}}$ with finite \mathcal{T} .

If $\mathcal{T} = \{X\}$ then $\mathcal{C}_{\mathcal{S}}^{\mathcal{T}}$ is just \mathcal{C}_X . If $\mathcal{T} = \{X, Y\}$ with distinct X, Y we get a simple but nontrivial situation. Indeed, we shall have $\mathcal{H}_{\mathcal{T}} = \mathcal{H}_X \oplus \mathcal{H}_Y$ and $\mathcal{C}_{\mathcal{S}}^{\mathcal{T}}$ may be thought as a matrix

$$\mathcal{C}_{\mathcal{S}}^{\mathcal{T}} = \begin{pmatrix} \mathcal{C}_X & \mathcal{C}_{XY} \\ \mathcal{C}_{YX} & \mathcal{C}_Y \end{pmatrix}.$$

The grading is now explicitly defined as follows:

(1) If $E \subset X \cap Y$ then

$$\mathcal{C}_{\mathcal{S}}^{\mathcal{T}}(E) = \begin{pmatrix} \mathcal{C}_X(E) & \mathcal{C}_{XY}(E) \\ \mathcal{C}_{YX}(E) & \mathcal{C}_Y(E) \end{pmatrix}.$$

(2) If $E \subset X$ and $E \not\subset Y$ then

$$\mathcal{C}_{\mathcal{S}}^{\mathcal{T}}(E) = \begin{pmatrix} \mathcal{C}_X(E) & 0 \\ 0 & 0 \end{pmatrix}.$$

(3) If $E \not\subset X$ and $E \subset Y$ then

$$\mathcal{C}_{\mathcal{S}}^{\mathcal{T}}(E) = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{C}_Y(E) \end{pmatrix}.$$

The case when \mathcal{T} is of the form \mathcal{S}_X for some $X \in \mathcal{S}$ is especially interesting. We denote $\mathcal{C}_X^{\#} \equiv \mathcal{C}_{\mathcal{S}_X}$ and we say that the \mathcal{S}_X -graded C^* -algebra is the *unfolding* of the algebra \mathcal{C}_X . More explicitly

$$\mathcal{C}_X^{\#} \equiv \sum_{Y, Z \in \mathcal{S}_X}^c \mathcal{C}_{YZ}. \quad (4.45)$$

The self-adjoint operators affiliated to \mathcal{C}_X live on the Hilbert space \mathcal{H}_X and are (an abstract version of) Hamiltonians of an N -particle system \mathcal{S} with a fixed N (the configuration space is X and N is the number of levels of the semilattice \mathcal{S}_X). The unfolding $\mathcal{C}_X^{\#}$ lives on the ‘‘Boltzmann-Fock space’’ $\mathcal{H}_{\mathcal{S}_X}$ and is obtained by adding interactions which couple the subsystems of \mathcal{S} which have the groups $Y \in \mathcal{S}_X$ as configuration spaces and \mathcal{C}_Y as Hamiltonian algebras.

Clearly $\mathcal{C}_X^{\#} \subset \mathcal{C}_Y^{\#}$ if $X \subset Y$ and \mathcal{C} is the inductive limit of the algebras $\mathcal{C}_X^{\#}$. Below we give an interesting alternative description of $\mathcal{C}_X^{\#}$.

Theorem 4.27. *Let $\mathcal{N}_X = \oplus_{Y \in \mathcal{S}_X} \mathcal{C}_{YX}$ be the direct sum of the Hilbert \mathcal{C}_X -modules \mathcal{C}_{YX} equipped with the direct sum graded structure. Then $\mathcal{K}(\mathcal{N}_X) \cong \mathcal{C}_X^{\#}$ the isomorphism being such that the graded structure on $\mathcal{K}(\mathcal{N}_X)$ defined in Theorem 3.8 is transported into that of $\mathcal{C}_X^{\#}$. In other terms, $\mathcal{C}_X^{\#}$ is the imprimitivity algebra of the full Hilbert \mathcal{C}_X -module \mathcal{N}_X and \mathcal{C}_X and $\mathcal{C}_X^{\#}$ are Morita equivalent.*

Proof: If $Y \subset X$ then $\mathcal{C}_{YX}^* \cdot \mathcal{C}_{YX} = \mathcal{C}_X^Y$ and \mathcal{C}_{YX} is a full Hilbert \mathcal{C}_X^Y -module. Since the \mathcal{C}_X^Y are ideals in \mathcal{C}_X and their sum over $Y \in \mathcal{S}_X$ is equal to \mathcal{C}_X we see that \mathcal{N}_X becomes a full Hilbert graded \mathcal{C}_X -module supported by \mathcal{S}_X , cf. Section 3. By Theorem 3.8 the imprimitivity C^* -algebra $\mathcal{K}(\mathcal{N}_X)$ is equipped with a canonical \mathcal{S}_X -graded structure.

We shall make a comment on $\mathcal{K}(\mathcal{M})$ in the more general the case when $\mathcal{M} = \bigoplus_i \mathcal{M}_i$ is a direct sum of Hilbert \mathcal{A} -modules \mathcal{M}_i , cf. §3.3. First, it is clear that we have

$$\mathcal{K}(\mathcal{M}) = \sum_{ij}^c \mathcal{K}(\mathcal{M}_j, \mathcal{M}_i) \cong (\mathcal{K}(\mathcal{M}_j, \mathcal{M}_i))_{ij}.$$

Now assume that $\mathcal{E}, \mathcal{E}_i$ are Hilbert spaces such that \mathcal{A} is a C^* -algebra of operators on \mathcal{E} and \mathcal{M}_i is a Hilbert C^* -submodule of $L(\mathcal{E}, \mathcal{E}_i)$ such that $\mathcal{A}_i \equiv \mathcal{M}_i^* \cdot \mathcal{M}_i$ is an ideal of \mathcal{A} . Then by Proposition 3.6 we have $\mathcal{K}(\mathcal{M}_j, \mathcal{M}_i) \cong \mathcal{M}_i \cdot \mathcal{M}_j^* \subset L(\mathcal{E}_j, \mathcal{E}_i)$.

In our case we take

$$i = Y \in \mathcal{S}_X, \quad \mathcal{M}_i = \mathcal{C}_{YX}, \quad \mathcal{A} = \mathcal{C}_X, \quad \mathcal{E} = \mathcal{H}_X, \quad \mathcal{E}_i = \mathcal{H}_Y, \quad \mathcal{A}_i = \mathcal{C}_X^Y.$$

Then we get

$$\mathcal{K}(\mathcal{M}_j, \mathcal{M}_i) \equiv \mathcal{K}(\mathcal{C}_{ZX}, \mathcal{C}_{YX}) \cong \mathcal{C}_{YX} \cdot \mathcal{C}_{ZX}^* = \mathcal{C}_{YX} \cdot \mathcal{C}_{XZ} = \mathcal{C}_{YZ}$$

by Proposition 4.21. □

Remark 4.28. We understood the role in our work of the imprimitivity algebra of a Hilbert C^* -module thanks to a discussion with Georges Skandalis: he recognized (a particular case of) the main C^* -algebra \mathcal{C} we have constructed as the imprimitivity algebra of a certain Hilbert C^* -module. Theorem 4.27 is a reformulation of his observation and of his abstract construction of graded Hilbert C^* -modules in the present framework (at the time of the discussion our definition of \mathcal{C} was rather different because we were working in a tensor product formalism). More generally, if \mathcal{M} is a full Hilbert \mathcal{A} -module then the imprimitivity C^* -algebra $\mathcal{K}(\mathcal{M})$ could also be interpreted as Hamiltonian algebra of a system related in some natural way to the initial one. For example, this is a natural method of “second quantizing” N -body systems, i.e. introducing interactions which couple subsystems corresponding to different cluster decompositions of the N -body systems. This is clear in the physical N -body situation discussed in §2.3

5. AN INTRINSIC DESCRIPTION

We begin with some preliminary facts on crossed products. Let X be a locally compact abelian group. The next result, due to Landstad [Lad], gives an “intrinsic” characterization of crossed products of X -algebras by the action of X . We follow the presentation from [GI3, Theorem 3.7] which takes advantage of the fact that X is abelian.

Theorem 5.1. *A C^* -algebra $\mathcal{A} \subset \mathcal{L}_X$ is a crossed product if and only for each $A \in \mathcal{A}$ we have:*

- if $k \in X^*$ then $V_k^* A V_k \in \mathcal{A}$ and $\lim_{k \rightarrow 0} \|V_k^* A V_k - A\| = 0$,
- if $x \in X$ then $U_x A \in \mathcal{A}$ and $\lim_{x \rightarrow 0} \|(U_x - 1)A\| = 0$.

In this case one has $\mathcal{A} = \mathcal{A} \rtimes X$ for a unique X -algebra $\mathcal{A} \subset C_b^u(X)$ and this algebra is given by

$$\mathcal{A} = \{\varphi \in C_b^u(X) \mid \varphi(Q)S \in \mathcal{A} \text{ and } \bar{\varphi}(Q)S \in \mathcal{A} \text{ for all } S \in \mathcal{T}_X\}. \quad (5.1)$$

Note that the second condition above is equivalent to $\mathcal{T}_X \cdot \mathcal{A} = \mathcal{A}$, cf. Lemma 8.1.

The following consequence of Landstad’s theorem is an intrinsic description of $\mathcal{C}_X(Y)$.

Theorem 5.2. *$\mathcal{C}_X(Y)$ is the set of $A \in \mathcal{L}_X$ such that $U_y^* A U_y = A$ for all $y \in Y$ and:*

- (1) $\|U_x^* A U_x - A\| \rightarrow 0$ if $x \rightarrow 0$ in X and $\|V_k^* A V_k - A\| \rightarrow 0$ if $k \rightarrow 0$ in X^* ,
- (2) $\|(U_x - 1)A\| \rightarrow 0$ if $x \rightarrow 0$ in X and $\|(V_k - 1)A\| \rightarrow 0$ if $k \rightarrow 0$ in Y^\perp .

By “ $k \rightarrow 0$ in Y^\perp ” we mean: $k \in Y^\perp$ and $k \rightarrow 0$. Note that the second condition above is equivalent to:

$$\text{there are } \theta \in \mathcal{T}_X, \psi \in \mathcal{C}_X(Y) \text{ and } B, C \in \mathcal{L}_X \text{ such that } A = \theta(P)B = \psi(Q)C. \quad (5.2)$$

For the proof, use $Y^\perp \cong (X/Y)^*$ and apply Lemma 8.1. In particular, the last factorization shows that for each $\varepsilon > 0$ there is a compact set $M \subset X$ such that $\|\chi_V(Q)A\| < \varepsilon$, where $V = X \setminus (M + Y)$.

Proof of Theorem 5.2: Let $\mathcal{A} \subset \mathcal{L}_X$ be the set of operators A satisfying the conditions from the statement of the theorem. We first prove that \mathcal{A} satisfies the two conditions of Theorem 5.1. Let $A \in \mathcal{A}$. We have to show that $A_p \equiv V_p^*AV_p \in \mathcal{A}$ and $\|V_p^*AV_p - A\| \rightarrow 0$ as $p \rightarrow 0$. From the commutation relations $U_xV_p = p(x)V_pU_x$ we get $\|(U_x - 1)A_p\| = \|(U_x - p(x))A\| \rightarrow 0$ if $x \rightarrow 0$ and the second part of condition 1 of the theorem is obviously satisfied by A_p . Then for $y \in Y$

$$U_y^*A_pU_y = U_y^*V_p^*AV_pU_y = V_p^*U_y^*AU_yV_p = V_p^*AV_p = A_p.$$

Condition 2 is clear so we have $A_p \in \mathcal{A}$ and the fact that $\|V_p^*AV_p - A\| \rightarrow 0$ as $p \rightarrow 0$ is obvious. That A satisfies the second Landstad condition, namely that for each $a \in X$ we have $U_aA \in \mathcal{A}$ and $\|(U_a - 1)A\| \rightarrow 0$ as $a \rightarrow 0$, is also clear because $\|[U_a, V_k]\| \rightarrow 0$ as $k \rightarrow 0$.

Now we have to find the algebra \mathcal{A} defined by (5.1). Assume that $\varphi \in \mathcal{C}_b^u(X)$ satisfies $\varphi(Q)S \in \mathcal{A}$ for all $S \in \mathcal{T}_X$. Since $U_y^*\varphi(Q)U_y = \varphi(Q - y)$ we get $(\varphi(Q) - \varphi(Q - y))S = 0$ for all such S and all $y \in Y$, hence $\varphi(Q) - \varphi(Q - y) = 0$ which means $\varphi \in \mathcal{C}_b^u(X/Y)$. We shall prove that $\varphi \in \mathcal{C}_X(Y)$ by reductio ad absurdum.

If $\varphi \notin \mathcal{C}_X(Y)$ then there is $\mu > 0$ and there is a sequence of points $x_n \in X$ such that $x_n/Y \rightarrow \infty$ and $|\varphi(x_n)| > 2\mu$. From the uniform continuity of φ we see that there is a compact neighborhood K of zero in X such that $|\varphi| > \mu$ on $\bigcup_n(x_n + K)$. Let K' be a compact neighborhood of zero such that $K' + K' \subset K$ and let us choose two positive not zero functions $\psi, f \in \mathcal{C}_c(K')$. We define $S \in \mathcal{T}_X$ by $Su = \psi * u$ and recall that $\text{supp } Su \subset \text{supp } \psi + \text{supp } u$. Thus $\text{supp } SU_{x_n}^*f \subset K' + x_n + K' \subset x_n + K$. Now let V be as in the remarks after (5.2). Since $\pi_Y(x_n) \rightarrow \infty$ we have $x_n + K \subset V$ for n large enough, hence

$$\|\chi_V(Q)\varphi(Q)SU_{x_n}^*f\| \geq \mu\|SU_{x_n}^*f\| = \mu\|Sf\| > 0.$$

On the other hand, for each $\varepsilon > 0$ one can choose V such that $\|\chi_V(Q)\varphi(Q)S\| < \varepsilon$. Then we shall have $\|\chi_V(Q)\varphi(Q)SU_{x_n}^*f\| \leq \varepsilon\|f\|$ so $\mu\|Sf\| \leq \varepsilon\|f\|$ for all $\varepsilon > 0$ which is absurd. \square

We now give a similar characterization of $\mathcal{C}_{XY}(Z)$ where X, Y is a compatible pair of closed subgroups of an lca group G .

Theorem 5.3. $\mathcal{C}_{XY}(Z)$ is the set of $T \in \mathcal{L}_{XY}$ satisfying the following conditions:

- (1) $U_z^*TU_z = T$ if $z \in Z$ and $\|V_k^*TV_k - T\| \rightarrow 0$ if $k \rightarrow 0$ in $(X + Y)^*$
- (2) $\|(U_x - 1)T\| \rightarrow 0$ if $x \rightarrow 0$ in X and $\|T(U_y - 1)\| \rightarrow 0$ if $y \rightarrow 0$ in Y ,
- (3) $\|(V_k - 1)T\| \rightarrow 0$ if $k \rightarrow 0$ in $(X/Z)^*$ and $\|T(V_k - 1)\| \rightarrow 0$ if $k \rightarrow 0$ in $(Y/Z)^*$.

Before the proof we make some preliminary comments. We think of $X + Y$ as a closed subgroup of $G \in \mathcal{S}$ which contains X and Y as closed subgroups. Each character $k \in (X + Y)^*$ defines by restriction a character $k|_X \in X^*$ and the map $k \mapsto k|_X$ is a continuous open surjection. And similarly if X is replaced by Y . In (1) the operator V_k acts in $L^2(X)$ as multiplication by $k|_X$ and in $L^2(Y)$ as multiplication by $k|_Y$. In the first part of (3) we take $k \in X^*$ and identify $(X/Z)^*$ with the orthogonal of Z in X^* and similarly for the second part.

Assumptions (2) and (3) of Theorem 5.3 are decay conditions in certain directions in P and Q space. Indeed, by Lemma 8.1 condition (2) is equivalent to:

$$\text{there are } S_1 \in \mathcal{T}_X, S_2 \in \mathcal{T}_Y \text{ and } R_1, R_2 \in \mathcal{L}_{XY} \text{ such that } T = S_1R_1 = R_2S_2. \quad (5.3)$$

Recall that $\mathcal{T}_X \cong \mathcal{C}_0(X^*)$ for example. Then condition (3) is equivalent to:

$$\text{there are } S_1 \in \mathcal{C}_X(Z), S_2 \in \mathcal{C}_Y(Z) \text{ and } R_1, R_2 \in \mathcal{L}_{XY} \text{ such that } T = S_1R_1 = R_2S_2. \quad (5.4)$$

Proof of Theorem 5.3: The set \mathcal{C} of all the operators satisfying the conditions of the theorem is clearly a closed subspace of \mathcal{L}_{XY} . We have $\mathcal{C}_{X,Y}(Z) \subset \mathcal{C}$ because (5.3), (5.4) are satisfied by any $T \in \mathcal{C}_{XY}(Z)$ as a consequence of Theorem 4.15. Then we get:

$$\mathcal{C}_Y(Z) = \mathcal{C}_{XY}^*(Z) \cdot \mathcal{C}_{XY}(Z) \subset \mathcal{C}^* \cdot \mathcal{C}, \quad \mathcal{C}_X(Z) = \mathcal{C}_{XY}(Z) \cdot \mathcal{C}_{XY}^*(Z) \subset \mathcal{C} \cdot \mathcal{C}^*.$$

We prove that equality holds in both these relations. We show, for example, that $A \equiv TT^*$ belongs to $\mathcal{C}_X(Z)$ if $T \in \mathcal{C}$ and for this we shall use Theorem 5.2 with Y replaced by Z . That $U_z^*AU_z = A$ for $z \in Z$ is clear. From (5.3) we get $A = S_1R_1R_1^*S_1^*$ with $S_1 \in \mathcal{T}_X$ hence $\|(U_x - 1)A\| \rightarrow 0$ and $\|A(U_x - 1)\| \rightarrow 0$ as $x \rightarrow 0$ in X are obvious and imply $\|U_x^*AU_x - A\| \rightarrow 0$. Then (5.4) implies $A = \psi(Q)C$ with $\psi \in \mathcal{C}_X(Z)$ and bounded C hence (5.2) is satisfied.

That $\mathcal{C}\mathcal{C}_Y(Z) \subset \mathcal{C}$ is easily proven because $T = SA$ has the properties (5.3) and (5.4) if S belongs to \mathcal{C} and A to $\mathcal{C}_Y(Z)$, cf. Theorem 5.2. From what we have shown above we get $\mathcal{C}\mathcal{C}^*\mathcal{C} \subset \mathcal{C}\mathcal{C}_Y(Z) \subset \mathcal{C}$ so \mathcal{C} is a Hilbert C^* -submodule of \mathcal{L}_{XY} . On the other hand, $\mathcal{C}_{XY}(Z)$ is a Hilbert C^* -submodule of \mathcal{L}_{XY} such that $\mathcal{C}_{XY}^*(Z) \cdot \mathcal{C}_{XY}(Z) = \mathcal{C}^* \cdot \mathcal{C}$ and $\mathcal{C}_{XY}(Z) \cdot \mathcal{C}_{XY}^*(Z) = \mathcal{C} \cdot \mathcal{C}^*$. Since $\mathcal{C}_{XY}(Z) \subset \mathcal{C}$ we get $\mathcal{C} = \mathcal{C}_{XY}(Z)$ from Proposition 3.5. \square

If $Z = X \cap Y$ then Theorem 5.3 gives an intrinsic description of the space \mathcal{T}_{XY} . For example:

Corollary 5.4. *If $X \supset Y$ then \mathcal{T}_{XY} is the set of $T \in \mathcal{L}_{XY}$ satisfying $U_y^*TU_y = T$ if $y \in Y$ and such that: $U_xT \rightarrow T$ if $x \rightarrow 0$ in X , $V_k^*TV_k \rightarrow T$ if $k \rightarrow 0$ in X^* and $V_kT \rightarrow T$ if $k \rightarrow 0$ in Y^\perp .*

In the rest of this section we describe the structure of the objects introduced in Section 4 when the subgroups are complemented, e.g. if S consists of finite dimensional vector spaces.

We say that Z is *complemented in X* if $X = Z \oplus E$ for some closed subgroup E of X . If X, Z are equipped with Haar measures then X/Z is equipped with the quotient Haar measure and we have $E \simeq X/Z$. If Z is complemented in X and Y then $\mathcal{C}_{XY}(Z)$ can be expressed as a tensor product.

Proposition 5.5. *If Z is complemented in X and Y then*

$$\mathcal{C}_{XY}(Z) \simeq \mathcal{T}_Z \otimes \mathcal{K}_{X/Z, Y/Z}. \quad (5.5)$$

If $Y \subset X$ then $\mathcal{T}_{XY} \simeq \mathcal{T}_Y \otimes L^2(X/Y)$ tensor product of Hilbert C^ -modules.*

Proof: Note first that the tensor product in (5.5) is interpreted as the exterior tensor product of the Hilbert C^* -modules \mathcal{T}_Z and $\mathcal{K}_{X/Z, Y/Z}$. Let $X = Z \oplus E$ and $Y = Z \oplus F$ for some closed subgroups E, F . Then, as explained in §3.4, we may also view the tensor product as the norm closure in the space of continuous operators from $L^2(Y) \simeq L^2(Z) \otimes L^2(F)$ to $L^2(X) \simeq L^2(Z) \otimes L^2(E)$ of the linear space generated by the operators of the form $T \otimes K$ with $T \in \mathcal{T}_Z$ and $K \in \mathcal{K}_{EF}$.

We now show that under the conditions of the proposition $X + Y \simeq Z \oplus E \oplus F$ algebraically and topologically. The natural map $\theta : Z \oplus E \oplus F \rightarrow Z + E + F = X + Y$ is a continuous bijective morphism, we have to prove that it is open. Since X, Y are compatible, the map (4.10) is a continuous open surjection. If we represent $X \oplus Y \simeq Z \oplus Z \oplus E \oplus F$ then this map becomes $\phi(a, b, c, d) = (a - b) + c + d$. Let $\psi = \xi \oplus \text{id}_E \oplus \text{id}_F$ where $\xi : Z \oplus Z \rightarrow Z$ is given by $\xi(a, b) = a - b$. Then ξ is continuous surjective and open because if U is an open neighborhood of zero in Z then $U - U$ is also an open neighborhood of zero. Thus $\psi : (Z \oplus Z) \oplus E \oplus F \rightarrow Z \oplus E \oplus F$ is a continuous open surjection and $\phi = \theta \circ \psi$. So if V is open in $Z \oplus E \oplus F$ then there is an open $U \subset Z \oplus Z \oplus E \oplus F$ such that $V = \psi(U)$ and then $\theta(V) = \theta \circ \psi(U) = \phi(U)$ is open in $Z + E + F$.

Thus we may identify $L^2(Y) \simeq L^2(Z) \otimes L^2(F)$ and $L^2(X) \simeq L^2(Z) \otimes L^2(E)$ and we must describe the norm closure of the set of operators $T_{XY}(\varphi)\psi(Q)$ with $\varphi \in C_c(X + Y)$ (cf. the remark after (4.11) and the fact that $X + Y$ is closed) and $\psi \in C_o(Y/Z)$. Since $X + Y \simeq Z \oplus E \oplus F$ and $Y = Z \oplus F$ it suffices to describe the clspan of the operators $T_{XY}(\varphi)\psi(Q)$ with $\varphi = \varphi_Z \otimes \varphi_E \otimes \varphi_F$ and $\varphi_Z, \varphi_E, \varphi_F$

continuous functions with compact support on Z, E, F respectively and $\psi = 1 \otimes \eta$ where 1 is the function identically equal to 1 on Z and $\eta \in \mathcal{C}_o(F)$. Then, if $x = (a, c) \in Z \times E$ and $y = (b, d) \in Z \times F$, we get:

$$(T_{XY}(\varphi)\psi(Q)u)(a, c) = \int_{Z \times F} \varphi_Z(a-b)\varphi_E(c)\varphi_F(d)\eta(d)u(b, d)dbdd.$$

But this is just $C(\varphi_Z) \otimes |\varphi_E\rangle\langle\bar{\eta}\bar{\varphi}_F|$ where $|\varphi_E\rangle\langle\bar{\eta}\bar{\varphi}_F|$ is a rank one operator $L^2(F) \rightarrow L^2(E)$ and $C(\varphi_Z)$ is the operator of convolution by φ_Z on $L^2(Z)$. \square

If $X \cap Y$ is complemented in X and Y then \mathcal{C}_{XY} can be expressed (non canonically) as a tensor product.

Proposition 5.6. *If $X \cap Y$ is complemented in X and Y then*

$$\mathcal{C}_{XY} \simeq \mathcal{C}_{X \cap Y} \otimes \mathcal{H}_{X/Y, Y/X}.$$

In particular, if $X \supset Y$ then $\mathcal{C}_{XY} \simeq \mathcal{C}_Y \otimes \mathcal{H}_{X/Y}$.

Proof: If $X = (X \cap Y) \oplus E$ and $Y = (X \cap Y) \oplus F$ then we have to show that $\mathcal{C}_{XY} \simeq \mathcal{C}_{X \cap Y} \otimes \mathcal{H}_{EF}$ where the tensor product may be interpreted either as the exterior tensor product of the Hilbert C^* -modules $\mathcal{C}_{X \cap Y}$ and \mathcal{H}_{EF} or as the norm closure in the space of continuous operators from $L^2(Y) \simeq L^2(X \cap Y) \otimes L^2(F)$ to $L^2(X) \simeq L^2(X \cap Y) \otimes L^2(E)$ of the algebraic tensor product of $\mathcal{C}_{X \cap Y}$ and \mathcal{H}_{EF} . From Proposition 5.5 with $Z = X \cap Y$ we get $\mathcal{T}_{XY} \simeq \mathcal{T}_{X \cap Y} \otimes \mathcal{H}_{EF}$. The relations (4.34) and the Definition 4.18 imply $\mathcal{C}_{XY} = \mathcal{T}_{XY} \cdot \mathcal{C}_Y^{X \cap Y}$ and we clearly have

$$\mathcal{C}_Y^{X \cap Y} = \sum_{Z \subset X \cap Y}^c \mathcal{C}_Y(Z) \simeq \sum_{Z \subset X \cap Y}^c \mathcal{C}_{X \cap Y}(Z) \otimes \mathcal{C}_o(F) \simeq \mathcal{C}_{X \cap Y} \otimes \mathcal{C}_o(F).$$

Then we get

$$\mathcal{C}_{XY} \simeq \mathcal{T}_{X \cap Y} \otimes \mathcal{H}_{EF} \cdot \mathcal{C}_{X \cap Y} \otimes \mathcal{C}_o(F) = (\mathcal{T}_{X \cap Y} \cdot \mathcal{C}_{X \cap Y}) \otimes (\mathcal{H}_{EF} \cdot \mathcal{C}_o(F))$$

and this is $\mathcal{C}_{X \cap Y} \otimes \mathcal{H}_{EF}$. \square

If Z is complemented in X and Y then Theorem 5.3 can be improved. We shall describe this improvement only in the Euclidean case which will be useful in our treatment of nonrelativistic Hamiltonians. Thus below we assume that X, Y are subspaces of an Euclidean space (see §2.10 for notations). Note that V_k is the operator of multiplication by the function $x \mapsto e^{i(x|k)}$ where the scalar product $\langle x|k \rangle$ is well defined for any x, k in the ambient space \mathcal{X} .

Theorem 5.7. *$\mathcal{C}_{XY}(Z)$ is the set of $T \in \mathcal{L}_{XY}$ satisfying:*

- (1) $U_z^* T U_z = T$ for $z \in Z$ and $\|V_z^* T V_z - T\| \rightarrow 0$ if $z \rightarrow 0$ in Z ,
- (2) $\|(U_x - 1)T\| \rightarrow 0$ if $x \rightarrow 0$ in X and $\|(V_k - 1)T\| \rightarrow 0$ if $k \rightarrow 0$ in X/Z .

Remark 5.8. Condition 2 may be replaced by

- (2') $\|T(U_y - 1)\| \rightarrow 0$ if $y \rightarrow 0$ in Y and $\|T(V_k - 1)\| \rightarrow 0$ if $k \rightarrow 0$ in Y/Z .

This will be clear from the next proof.

Proof: Let $\mathcal{F} \equiv \mathcal{F}_Z$ be the Fourier transformation in the space Z , this is a unitary operator in the space $L^2(Z)$ which interchanges the position and momentum observables Q_Z, P_Z . We denote also by \mathcal{F} the operators $\mathcal{F} \otimes 1_{\mathcal{H}_{X/Z}}$ and $\mathcal{F} \otimes 1_{\mathcal{H}_{Y/Z}}$ which are unitary operators in the spaces \mathcal{H}_X and \mathcal{H}_Y due to (2.16). If $S = \mathcal{F} T \mathcal{F}^{-1}$ then S satisfies the following conditions:

- (i) $V_z^* S V_z = S$ for $z \in Z$, $\|(V_z - 1)S\| \rightarrow 0$ if $z \rightarrow 0$ in Z , and $\|U_z S U_z^* - S\| \rightarrow 0$ if $z \rightarrow 0$ in Z ;
- (ii) $\|(U_x - 1)S\| \rightarrow 0$ and $\|(V_x - 1)S\| \rightarrow 0$ if $x \rightarrow 0$ in X/Z .

For the proof, observe that the first part of condition (2) may be written as the conjunction of the two relations $\|(U_z - 1)T\| \rightarrow 0$ if $z \rightarrow 0$ in Z and $\|(U_x - 1)T\| \rightarrow 0$ if $x \rightarrow 0$ in X/Z . We shall work in the representations

$$\mathcal{H}_X = L^2(Z; \mathcal{H}_{X/Z}) \quad \text{and} \quad \mathcal{H}_Y = L^2(Z; \mathcal{H}_{Y/Z}). \quad (5.6)$$

From the relation $V_z^* S V_z = S$ for all $z \in Z$ it follows that there is a bounded weakly measurable function $S(\cdot) : Z \rightarrow \mathcal{L}_{X/Z, Y/Z}$ such that in the representations (5.6) S is the operator of multiplication by $S(\cdot)$. Then $\|U_z S U_z^* - S\| \rightarrow 0$ if $z \rightarrow 0$ in Z means that the function $S(\cdot)$ is uniformly continuous. And clearly $\|(V_z - 1)S\| \rightarrow 0$ if $z \rightarrow 0$ in Z is equivalent to the fact that $S(\cdot)$ tends to zero at infinity. Thus we see that $S(\cdot) \in \mathcal{C}_0(Z; \mathcal{L}_{X/Z, Y/Z})$. The condition (ii) can now be written

$$\sup_{z \in Z} (\|(U_x - 1)S(z)\| + \|(V_x - 1)S(z)\|) \rightarrow 0 \quad \text{if } x \rightarrow 0 \text{ in } X/Z.$$

From the Riesz-Kolmogorov theorem it follows that each $S(z)$ is a compact operator. Thus we have $S(\cdot) \in \mathcal{C}_0(Z; \mathcal{H}_{X/Z, Y/Z})$ which implies $T \in \mathcal{C}_{XY}(Z)$ by Proposition 5.5. \square

Remark 5.9. Since $S(\cdot)$ is continuous and tends to zero at infinity, for each $\varepsilon > 0$ there are points $z_1, \dots, z_n \in Z$ and complex functions $\varphi_1, \dots, \varphi_n \in \mathcal{C}_c(Z)$ such that

$$\|S(z) - \sum_k \varphi_k(z) S(z_k)\| \leq \varepsilon \quad \forall z \in Z.$$

The operators $S(z_k)$ being compact, applying once again the Riesz-Kolmogorov theorem we get

$$\sup_{z \in Z} (\|S(z)(U_y - 1)\| + \|S(z)(V_y - 1)\|) \rightarrow 0 \quad \text{if } y \rightarrow 0 \text{ in } Y/Z.$$

This explains why the second parts of conditions (2) and (3) of Theorem 5.3 is not needed.

6. AFFILIATED OPERATORS

In this section we give examples of self-adjoint operators affiliated to the algebra \mathcal{C} constructed in Section 4 and then we give a formula for their essential spectrum. We refer to §3.1 for terminology and basic results related to the notion of affiliation that we use and to [ABG, GI1, DaG3] for details.

We recall that a self-adjoint operator H on a Hilbert space \mathcal{H} is *strictly affiliated* to a C^* -algebra of operators \mathcal{A} on \mathcal{H} if $(H + i)^{-1} \in \mathcal{A}$ (then $\varphi(H) \in \mathcal{A}$ for all $\varphi \in \mathcal{C}_0(\mathbb{R})$) and if \mathcal{A} is the clspan of the elements $\varphi(H)A$ with $\varphi \in \mathcal{C}_0(\mathbb{R})$ and $A \in \mathcal{A}$. This class of operators has the advantage that each time \mathcal{A} is non-degenerately represented on a Hilbert space \mathcal{H}' with the help of a morphism $\mathcal{P} : \mathcal{A} \rightarrow L(\mathcal{H}')$, the observable $\mathcal{P}H$ is represented by a usual (densely defined) self-adjoint operator on \mathcal{H}' .

The diagonal algebra

$$\mathcal{T}_d \equiv (\mathcal{T}_S)_d = \bigoplus_{X \in \mathcal{S}} \mathcal{T}_X \tag{6.1}$$

has a simple physical interpretation: this is the C^* -algebra generated by the kinetic energy operators. Since $\mathcal{C}_{XX} = \mathcal{C}_X \supset \mathcal{C}_X(X) = \mathcal{T}_X$ we see that \mathcal{T}_d is a C^* -subalgebra of \mathcal{C} . From (4.25), (4.20), (4.21) and the Cohen-Hewitt theorem we get

$$\mathcal{C}(Z) \mathcal{T}_d = \mathcal{T}_d \mathcal{C}(Z) = \mathcal{C}(Z) \quad \forall Z \in \mathcal{S} \quad \text{and} \quad \mathcal{C} \mathcal{T}_d = \mathcal{T}_d \mathcal{C} = \mathcal{C}. \tag{6.2}$$

In other terms, \mathcal{T}_d acts non-degenerately[†] on each $\mathcal{C}(Z)$ and on \mathcal{C} . It follows that a self-adjoint operator strictly affiliated to \mathcal{T}_d is also strictly affiliated to \mathcal{C} .

For each $X \in \mathcal{S}$ let $h_X : X^* \rightarrow \mathbb{R}$ be a continuous function such that $|h_X(k)| \rightarrow \infty$ if $k \rightarrow \infty$ in X^* . Then the self-adjoint operator $K_X \equiv h_X(P)$ on \mathcal{H}_X is strictly affiliated to \mathcal{T}_X and the norm of $(K_X + i)^{-1}$ is equal to $\sup_k (h_X^2(k) + 1)^{-1/2}$. Let $K \equiv \bigoplus_{X \in \mathcal{S}} K_X$, this is a self-adjoint operator \mathcal{H} . Clearly K is affiliated to \mathcal{T}_d if and only if

$$\lim_{X \rightarrow \infty} \sup_k (h_X^2(k) + 1)^{-1/2} = 0 \tag{6.3}$$

and then K is strictly affiliated to \mathcal{T}_d (the set \mathcal{S} is equipped with the discrete topology). If the functions h_X are positive this means that $\min h_X$ tends to infinity when $X \rightarrow \infty$. One could avoid such a condition by considering an algebra larger than \mathcal{C} such as to contain $\prod_{X \in \mathcal{S}} \mathcal{T}_X$, but we shall not develop this here.

[†] Note that if \mathcal{S} has a largest element \mathcal{X} then the algebra $\mathcal{C}(\mathcal{X})$ acts on each $\mathcal{C}(Z)$ but this action is degenerate.

Now let $H = K + I$ with $I \in \mathcal{C}$ (or in the multiplier algebra) a symmetric element. Then

$$(\lambda - H)^{-1} = (\lambda - K)^{-1} (1 - I(\lambda - K)^{-1})^{-1} \quad (6.4)$$

if $\lambda \notin \text{Sp}(H) \cup \text{Sp}(K)$. Thus H is strictly affiliated to \mathcal{C} . We interpret H as the Hamiltonian of our system of particles when the kinetic energy is K and the interactions between particles are described by I . Even in the simple case $I \in \mathcal{C}$ these interactions are of a very general nature being a mixture of N -body and quantum field type interactions (which involve creation and annihilation operators so the number of particles is not preserved).

We shall now use Theorem 3.3 in order to compute the essential spectrum of an operator like H . The case of unbounded interactions will be treated later on. Let $\mathcal{C}_{\geq E}$ be the C^* -subalgebra of \mathcal{C} determined by $E \in \mathcal{S}$ according to the rules of §3.1. More explicitly, we set

$$\mathcal{C}_{\geq E} = \sum_{F \supset E}^c \mathcal{C}(F) \cong (\sum_{F \supset E}^c \mathcal{C}_{XY}(F))_{X \cap Y \supset E} \quad (6.5)$$

and note that $\mathcal{C}_{\geq E}$ lives on the subspace $\mathcal{H}_{\geq E} = \bigoplus_{X \supset E} \mathcal{H}_X$ of \mathcal{H} . Since in the second sum from (6.5) the group F is such that $E \subset F \subset X \cap Y$ the algebra $\mathcal{C}_{\geq E}$ is strictly included in the algebra $\mathcal{C}_{\mathcal{T}}$ obtained by taking $\mathcal{T} = \{F \in \mathcal{S} \mid F \supset E\}$ in (4.43).

Let $\mathcal{P}_{\geq E}$ be the canonical idempotent morphism of \mathcal{C} onto $\mathcal{C}_{\geq E}$ introduced in §3.1. We consider the self-adjoint operator on the Hilbert space $\mathcal{H}_{\geq E}$ defined as follows:

$$H_{\geq E} = K_{\geq E} + I_{\geq E} \quad \text{where} \quad K_{\geq E} = \bigoplus_{X \supset E} K_X \quad \text{and} \quad I_{\geq E} = \mathcal{P}_{\geq E} I. \quad (6.6)$$

Then $H_{\geq E}$ is strictly affiliated to $\mathcal{C}_{\geq E}$ and it follows easily from (6.4) that

$$\mathcal{P}_{\geq E} \varphi(H) = \varphi(H_{\geq E}) \quad \forall \varphi \in \mathcal{C}_o(\mathbb{R}). \quad (6.7)$$

Now let us assume that the group $O = \{0\}$ belongs to \mathcal{S} . Then we have

$$\mathcal{C}(O) = K(\mathcal{H}). \quad (6.8)$$

Indeed, from (4.25) we get $\mathcal{C}_{XY}(O) = \mathcal{T}_{XY} \cdot \mathcal{C}_o(Y) = \mathcal{K}_{XY}$ which implies the preceding relation. If we also assume that \mathcal{S} is atomic and we denote $\mathcal{P}(\mathcal{S})$ its set of atoms, then from Theorem 3.2 we get a canonical embedding

$$\mathcal{C}/K(\mathcal{H}) \subset \prod_{E \in \mathcal{P}(\mathcal{S})} \mathcal{C}_{\geq E} \quad (6.9)$$

defined by the morphism $\mathcal{P} \equiv (\mathcal{P}_{\geq E})_{E \in \mathcal{P}(\mathcal{S})}$. Then from (3.7) we obtain:

$$\text{Sp}_{\text{ess}}(H) = \overline{\bigcup_{E \in \mathcal{P}(\mathcal{S})} \text{Sp}(H_{\geq E})}. \quad (6.10)$$

Our next purpose is to prove a similar formula for a certain class of unbounded interactions I .

Let $\mathcal{G} \equiv \mathcal{G}_{\mathcal{S}} = D(|K|^{1/2})$ be the form domain of K equipped with the graph topology. Then $\mathcal{G} \subset \mathcal{H}$ continuously and densely so after the Riesz identification of \mathcal{H} with its adjoint space \mathcal{H}^* we get the usual scale $\mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^*$ with continuous and dense embeddings. Let us denote

$$\langle K \rangle = |K + i| = \sqrt{K^2 + 1}. \quad (6.11)$$

Then $\langle K \rangle^{1/2}$ is a self-adjoint operator on \mathcal{H} with domain \mathcal{G} and $\langle K \rangle$ induces an isomorphism $\mathcal{G} \rightarrow \mathcal{G}^*$. The following result is a straightforward consequence of Theorem 2.8 and Lemma 2.9 from [DaG3].

Theorem 6.1. *Let $I : \mathcal{G} \rightarrow \mathcal{G}^*$ be a continuous symmetric operator and let us assume that there are real numbers μ, a with $0 < \mu < 1$ such that one of the following conditions is satisfied:*

- (i) $\pm I \leq \mu |K + ia|$,
- (ii) K is bounded from below and $I \geq -\mu |K + ia|$.

Let $H = K + I$ be the form sum of K and I , so H has as domain the set of $u \in \mathcal{G}$ such that $Ku + Iu \in \mathcal{H}$ and acts as $Hu = Ku + Iu$. Then H is a self-adjoint operator on \mathcal{H} . If there is $\alpha > 1/2$ such that $\langle K \rangle^{-1/2} I \langle K \rangle^{-\alpha} \in \mathcal{C}$ then H is strictly affiliated to \mathcal{C} . If $O \in \mathcal{S}$ and the semilattice \mathcal{S} is atomic then

$$\text{Sp}_{\text{ess}}(H) = \overline{\bigcup_{E \in \mathcal{P}(\mathcal{S})} \text{Sp}(H_{\geq E})}. \quad (6.12)$$

The last assertion of the theorem follows immediately from Theorem 3.3 and is a general version of the HVZ theorem. In order to have a more explicit description of the observables $H_{\geq E} \equiv \mathcal{P}_{\geq E}H$ we now prove an analog of Theorem 3.5 from [DaG3]. We cannot use that theorem in our context for three reasons: first we did not suppose that \mathcal{S} has a maximal element, then even if \mathcal{S} has a maximal element \mathcal{X} the action of the corresponding algebra $\mathcal{C}(\mathcal{X})$ on the algebras $\mathcal{C}(E)$ is degenerate, and finally our “free” operator K is not affiliated to $\mathcal{C}(\mathcal{X})$.

Theorem 6.2. *For each $E \in \mathcal{S}$ let $I(E) \in L(\mathcal{G}, \mathcal{G}^*)$ be a symmetric operator such that:*

- (i) $\langle K \rangle^{-1/2} I(E) \langle K \rangle^{-\alpha} \in \mathcal{C}(E)$ for some $\alpha > 1/2$ independent of E ,
- (ii) there are real positive numbers μ_E, a such that either $\pm I(E) \leq \mu_E |K + ia|$ for all E or K is bounded from below and $I(E) \geq -\mu_E |K + ia|$ for all E ,
- (iii) we have $\sum_E \mu_E \equiv \mu < 1$ and the series $\sum_E I(E) \equiv I$ is norm summable in $L(\mathcal{G}, \mathcal{G}^*)$.

Let us set $I_{\geq E} = \sum_{F \geq E} I(F)$. Define the self-adjoint operator $H = K + I$ on \mathcal{H} as in Theorem 6.1 and define similarly the self-adjoint operator $H_{\geq E} = K_{\geq E} + I_{\geq E}$ on $\mathcal{H}_{\geq E}$. Then the operator H is strictly affiliated to \mathcal{C} , the operator $H_{\geq E}$ is strictly affiliated to $\mathcal{C}_{\geq E}$, and we have $\mathcal{P}_{\geq E}H = H_{\geq E}$.

Proof: We shall consider only the case when $\pm I(E) \leq \mu_E |K + ia|$ for all E . The more singular situation when K is bounded from below but there is no restriction on the positive part of the operators $I(E)$ (besides summability) is more difficult but the main idea has been explained in [DaG3].

We first make some comments to clarify the definition of the operators H and $H_{\geq E}$. Observe that our assumptions imply $\pm I \leq \mu |K + ia|$ hence if we set

$$\Lambda \equiv |K + ia|^{-1/2} = (K^2 + a^2)^{-1/4} \in \mathcal{T}_d$$

then we obtain

$$\pm \langle u | Iu \rangle \leq \mu \langle u | |K + ia| u \rangle = \mu \| |K + ia|^{1/2} u \|^2 = \mu \|\Lambda^{-1} u\|^2$$

which is equivalent to $\pm \Lambda I \Lambda \leq \mu$ or $\|\Lambda I \Lambda\| \leq \mu$. In particular we may use Theorem 6.1 in order to define the self-adjoint operator H . Moreover, we have

$$\langle K \rangle^{-1/2} I \langle K \rangle^{-\alpha} = \sum_E \langle K \rangle^{-1/2} I(E) \langle K \rangle^{-\alpha} \in \mathcal{C}$$

because the series is norm summable in $L(\mathcal{H})$. Thus H is strictly affiliated to \mathcal{C} .

In order to define $H_{\geq E}$ we first make a remark on $I_{\geq E}$. If we set $\mathcal{G}_X = D(|K_X|^{-1/2})$ and if we equip \mathcal{G} and \mathcal{G}_X with the norms

$$\|u\|_{\mathcal{G}} = \|\langle K \rangle^{1/2} u\|_{\mathcal{H}} \quad \text{and} \quad \|u\|_{\mathcal{G}_X} = \|\langle K_X \rangle^{1/2} u\|_{\mathcal{H}_X}$$

respectively then clearly $\mathcal{G} = \oplus_X \mathcal{G}_X$ and $\mathcal{G}^* = \oplus_X \mathcal{G}_X^*$ where the sums are Hilbertian direct sums and \mathcal{G}^* and \mathcal{G}_X^* are equipped with the dual norms. Then each $I(F)$ may be represented as a matrix $I(F) = (I_{XY}(F))_{X,Y \in \mathcal{S}}$ of continuous operators $I_{XY}(F) : \mathcal{G}_Y \rightarrow \mathcal{G}_X^*$. Clearly

$$\langle K \rangle^{-1/2} I(F) \langle K \rangle^{-\alpha} = \left(\langle K_X \rangle^{-1/2} I_{XY}(F) \langle K_Y \rangle^{-\alpha} \right)_{X,Y \in \mathcal{S}}$$

and since by assumption (i) this belongs to $\mathcal{C}(F)$ we see that $I_{XY}(F) = 0$ if $X \not\geq F$ or $Y \not\geq F$. Now fix E and let $F \supset E$. Then, when viewed as a sesquilinear form, $I(F)$ is supported by the subspace $\mathcal{H}_{\geq E}$ and has domain $\mathcal{G}_{\geq E} = D(|K_{\geq E}|^{1/2})$. It follows that $I_{\geq E}$ is a sesquilinear form with domain $\mathcal{G}_{\geq E}$ supported by the subspace $\mathcal{H}_{\geq E}$ and may be thought as an element of $L(\mathcal{G}_{\geq E}, \mathcal{G}_{\geq E}^*)$ such that $\pm I_{\geq E} \leq \mu |K_{\geq E} + ia|$ because $\sum_{F \supset E} \mu_F \leq \mu$. To conclude, we may now define $H_{\geq E} = K_{\geq E} + I_{\geq E}$ exactly as in the case of H and get a self-adjoint operator on $\mathcal{H}_{\geq E}$ strictly affiliated to $\mathcal{C}_{\geq E}$. Note that this argument also gives

$$\langle K \rangle^{-1/2} I(F) \langle K \rangle^{-1/2} = \langle K_{\geq E} \rangle^{-1/2} I(F) \langle K_{\geq E} \rangle^{-1/2}. \quad (6.13)$$

It remains to be shown that $\mathcal{P}_{\geq E}H = H_{\geq E}$. If we set $R \equiv (ia - H)^{-1}$ and $R_{\geq E} \equiv (ia - H_{\geq E})^{-1}$ then this is equivalent to $\mathcal{P}_{\geq E}R = R_{\geq E}$. Let us set

$$U = |ia - K|(ia - K)^{-1} = \Lambda^{-2}(ia - K)^{-1}, \quad J = \Lambda I \Lambda U.$$

Then U is a unitary operator and $\|J\| < 1$, so we get a norm convergent series expansion

$$R = (ia - K - I)^{-1} = \Lambda U (1 - \Lambda I \Lambda U)^{-1} \Lambda = \sum_{n \geq 0} \Lambda U J^n \Lambda$$

which implies

$$\mathcal{P}_{\geq E}(R) = \sum_{n \geq 0} \mathcal{P}_{\geq E}(\Lambda U J^n \Lambda)$$

the series being norm convergent. Thus it suffices to prove that for each $n \geq 0$

$$\mathcal{P}_{\geq E}(\Lambda U J^n \Lambda) = \Lambda_{\geq E} (J_{\geq E})^n \Lambda_{\geq E} \quad (6.14)$$

where $J_{\geq E} = \Lambda_{\geq E} I_{\geq E} \Lambda_{\geq E} U_{\geq E}$. Here $\Lambda_{\geq E}$ and $U_{\geq E}$ are associated to $K_{\geq E}$ in the same way Λ and K are associated to K . For $n = 0$ this is obvious because $\mathcal{P}_{\geq E} K = K_{\geq E}$. If $n = 1$ this is easy because

$$\begin{aligned} \Lambda U J \Lambda &= \Lambda U \Lambda I \Lambda U \Lambda = (ia - K)^{-1} I (ia - K)^{-1} \\ &= [(ia - K)^{-1} \langle K \rangle^{1/2}] \cdot [\langle K \rangle^{-1/2} I \langle K \rangle^{-\alpha}] \cdot [\langle K \rangle^{\alpha} (ia - K)^{-1}] \end{aligned} \quad (6.15)$$

and it suffices to note that $\mathcal{P}_{\geq E}(\langle K \rangle^{-1/2} I(F) \langle K \rangle^{-\alpha}) = 0$ if $F \not\supset E$ and to use (6.13) for $F \supset E$.

To treat the general case we make some preliminary remarks. If $J(F) = \Lambda I(F) \Lambda U$ then $J = \sum_F J(F)$ where the convergence holds in norm on \mathcal{H} because of the condition (iii). Then we have a norm convergent expansion

$$\Lambda U J^n \Lambda = \sum_{F_1, \dots, F_n \in \mathcal{S}} \Lambda U J(F_1) \dots J(F_n) \Lambda.$$

Assume that we have shown $\Lambda U J(F_1) \dots J(F_n) \Lambda \in \mathcal{C}(F_1 \cap \dots \cap F_n)$. Then we get

$$\mathcal{P}_{\geq E}(\Lambda U J^n \Lambda) = \sum_{F_1 \geq E, \dots, F_n \geq E} \Lambda U J(F_1) \dots J(F_n) \Lambda \quad (6.16)$$

because if one F_k does not contain E then the intersection $F_1 \cap \dots \cap F_n$ does not contain E hence $\mathcal{P}_{\geq E}$ applied to the corresponding term gives 0. Because of (6.13) we have $J(F) = \Lambda_{\geq E} I(F) \Lambda_{\geq E} U_{\geq E}$ if $F \supset E$ and we may replace everywhere in the right hand side of (6.16) Λ and U by $\Lambda_{\geq E}$ and $U_{\geq E}$. This clearly proves (6.14).

Now we prove the stronger fact $\Lambda U J(F_1) \dots J(F_n) \in \mathcal{C}(F_1 \cap \dots \cap F_n)$. If $n = 1$ this follows from a slight modification of (6.15): the last factor on the right hand side of (6.15) is missing but is not needed. Assume that the assertion holds for some n . Since K is strictly affiliated to \mathcal{T}_d and \mathcal{T}_d acts non-degenerately on each $\mathcal{C}(F)$ we may use the Cohen-Hewitt theorem to deduce that there is $\varphi \in \mathcal{C}_o(\mathbb{R})$ such that $\Lambda U J(F_1) \dots J(F_n) = T \varphi(K)$ for some $T \in \mathcal{C}(F_1 \cap \dots \cap F_n)$. Then

$$\Lambda U J(F_1) \dots J(F_n) J(F_{n+1}) = T \varphi(K) J(F_{n+1})$$

hence it suffices to prove that $\varphi(K) J(F) \in \mathcal{C}(F)$ for any $F \in \mathcal{S}$ and any $\varphi \in \mathcal{C}_o(\mathbb{R})$. But the set of φ which have this property is a closed subspace of $\mathcal{C}_o(\mathbb{R})$ which clearly contains the functions $\varphi(\lambda) = (\lambda - z)^{-1}$ if z is not real hence is equal to $\mathcal{C}_o(\mathbb{R})$. \square

Remark 6.3. Choosing $\alpha > 1/2$ allows one to consider perturbations of K which are of the same order as K , e.g. in the N -body situations one may add to the Laplacian Δ on operator like $\nabla^* M \nabla$ where the function M is bounded measurable and has the structure of an N -body type potential, cf. [DaG3, DeIf].

The only assumption of Theorem 6.2 which is really relevant is $\langle K \rangle^{-1/2} I(E) \langle K \rangle^{-\alpha} \in \mathcal{C}(E)$. We shall give below more explicit conditions which imply it. If we change notation $E \rightarrow Z$ and use the formalism introduced in the proof of Theorem 6.2 we have

$$I(Z) = (I_{XY}(Z))_{X, Y \in \mathcal{S}} \quad \text{with} \quad I_{XY}(Z) : \mathcal{G}_Y \rightarrow \mathcal{G}_X^* \text{ continuous.} \quad (6.17)$$

We are interested in conditions on $I_{XY}(Z)$ which imply

$$\langle K_X \rangle^{-1/2} I_{XY}(Z) \langle K_X \rangle^{-\alpha} \in \mathcal{C}_{XY}(Z). \quad (6.18)$$

For this we shall use Theorem 5.3 which gives a simple intrinsic characterization of $\mathcal{C}_{XY}(Z)$.

The construction which follows is interesting only if X is not a discrete group, otherwise X^* is compact and many conditions are trivially satisfied. We shall use weights only in order to avoid imposing on the functions h_X regularity conditions stronger than continuity.

A positive function w on X^* is a *weight* if $\lim_{k \rightarrow \infty} w(k) = \infty$ and $w(k+p) \leq \omega(k)w(p)$ for some function ω on X^* and all k, p . We say that w is *regular* if one may choose ω such that $\lim_{k \rightarrow 0} \omega(k) = 1$. The example one should have in mind when X is an Euclidean space is $w(k) = \langle k \rangle^s$ for some $s > 0$. Note that we have $\omega(-k)^{-1} \leq w(k+p)w(p)^{-1} \leq \omega(k)$ hence if w is a regular weight then

$$\theta(k) \equiv \sup_{p \in X^*} \frac{|w(k+p) - w(p)|}{w(p)} \implies \lim_{k \rightarrow 0} \theta(k) = 0. \quad (6.19)$$

It is clear that if w is a regular weight and $\sigma \geq 0$ is a real number then w^σ is also a regular weight.

We say that two functions f, g defined on a neighborhood of infinity of X^* are *equivalent* and we write $f \sim g$ if there are numbers a, b such that $a|f(k)| \leq |g(k)| \leq b|f(k)|$. Then $|f|^\sigma \sim |g|^\sigma$ for all $\sigma > 0$.

We denote $\mathcal{G}_X^\sigma = D(|K_X|^{\sigma/2})$ and $\mathcal{G}_X^{-\sigma} \equiv (\mathcal{G}_X^\sigma)^*$ with $\sigma \geq 1$. In particular $\mathcal{G}_X^1 = \mathcal{G}_X$ and $\mathcal{G}_X^{-1} = \mathcal{G}_X^*$.

Proposition 6.4. *Assume that h_X, h_Y are equivalent to regular weights. Let $Z \subset X \cap Y$ and let $I_{XY}(Z)$ be a continuous map $\mathcal{G}_Y \rightarrow \mathcal{G}_X^*$ such that*

- (1) $U_z I_{XY}(Z) = I_{XY}(Z) U_z$ if $z \in Z$ and $V_k^* I_{XY}(Z) V_k \rightarrow I_{XY}(Z)$ if $k \rightarrow 0$ in $(X+Y)^*$,
- (2) $(U_x - 1) I_{XY}(Z) \rightarrow 0$ if $x \rightarrow 0$ in X and $(V_k - 1) I_{XY}(Z) \rightarrow 0$ if $k \rightarrow 0$ in $(X/Z)^*$,

where the limits hold in norm in $L(\mathcal{G}_Y^\sigma, \mathcal{G}_X^{-1})$ for some $\sigma \geq 1$. Then (6.18) holds with $\alpha = \sigma/2$.

Proof: We begin with some general comments on weights. Let w be a regular weight and let \mathcal{G}_X be the domain of the operator $w(P)$ in \mathcal{H}_X equipped with the norm $\|w(P)u\|$. Then \mathcal{G}_X is a Hilbert space and if \mathcal{G}_X^* is its adjoint space then we get a scale of Hilbert spaces $\mathcal{G}_X \subset \mathcal{H}_X \subset \mathcal{G}_X^*$ with continuous and dense embeddings. Since U_x commutes with $w(P)$ it is clear that $\{U_x\}_{x \in X}$ induces strongly continuous unitary representation of X on \mathcal{G}_X and \mathcal{G}_X^* . Then

$$\|V_k u\|_{\mathcal{G}_X} = \|w(k+P)u\| \leq \omega(k)\|u\|_{\mathcal{G}_X}$$

from which it follows that $\{V_k\}_{k \in X^*}$ induces by restriction and extension strongly continuous representations of X^* in \mathcal{G}_X and \mathcal{G}_X^* . Moreover, as operators on \mathcal{H}_X we have

$$\begin{aligned} |V_k^* w(P)^{-1} V_k - w(P)^{-1}| &= |w(k+P)^{-1} - w(P)^{-1}| = |w(k+P)^{-1}(w(P) - w(k+P))w(P)^{-1}| \\ &\leq \omega(-k)|(w(P) - w(k+P))w(P)^{-2}| \leq \omega(-k)\theta(k)w(P)^{-1}. \end{aligned} \quad (6.20)$$

Now let w_X, w_Y be regular weights equivalent to $|h_X|^{1/2}, |h_Y|^{1/2}$ and let us set $S = I_{XY}(Z)$. Then

$$\langle K_X \rangle^{-1/2} S \langle K_Y \rangle^{-\alpha} = \langle K_X \rangle^{-1/2} w_X(P) \cdot w_X(P) S w_Y(P)^{-2\alpha} \cdot w_Y(P)^{2\alpha} \langle K_Y \rangle^{-\alpha}$$

and $\langle h_X \rangle^{-1/2} w_X, \langle h_Y \rangle^{-\alpha} w_Y^{2\alpha}$ and their inverses are bounded continuous functions on X, Y . Since $\mathcal{C}_{XY}(Z)$ is a non-degenerate left \mathcal{T}_X -module and right \mathcal{T}_Y -module we may use the Cohen-Hewitt theorem to deduce that (6.18) is equivalent to

$$w_X(P)^{-1} I_{XY}(Z) w_Y(P)^{-\sigma} \in \mathcal{C}_{XY}(Z) \quad (6.21)$$

where $\sigma = 2\alpha$. To simplify notations we set $W_X = w_X(P), W_Y = w_Y^\sigma(P)$. We also omit the index X or Y for the operators W_X, W_Y since their value is obvious from the context. In order to show $W^{-1} S W^{-1} \in \mathcal{C}_{XY}(Z)$ we check the conditions of Theorem 5.3 with $T = W^{-1} S W^{-1}$. The first part of condition (2) of the theorem is verified by the hypothesis (2) of the present proposition. We may assume $\sigma > 1$ and then hence the second part of condition (2) of the theorem follows from

$$\|T(U_y - 1)\| \leq \|W^{-1} I_{XY}(Z) w_Y^{-1}(P)\| \rightarrow 0 \| (U_y - 1) w_Y^{1-\sigma}(P) \| \quad \text{if } y \rightarrow 0.$$

To check the second part of condition (1) of the theorem set $W_k = V_k^* W V_k$ and $S_k = V_k^* S V_k$ and write

$$\begin{aligned} V_k^* T V_k - T &= W_k^{-1} S_k W_k^{-1} - W^{-1} S W^{-1} \\ &= (W_k^{-1} - W^{-1}) S_k W_k^{-1} + W^{-1} (S_k - S) W_k^{-1} + W^{-1} S (W_k^{-1} - W^{-1}). \end{aligned}$$

Now if we use (6.20) and set $\xi(k) = \omega(-k)\theta(k)$ we get

$$\|V_k^*TV_k - T\| \leq \xi(k)\|W^{-1}S_kW_k^{-1}\| + \|W^{-1}(S_k - S)W^{-1}\|\|WW_k^{-1}\| + \xi(k)\|W^{-1}SW^{-1}\|$$

which clearly tends to zero if $k \rightarrow 0$. Condition (3) of Theorem 5.3 follows by a similar argument. \square

Now let H be defined according to the algorithm of §2.8. Then condition (i) of Theorem 6.2 will be satisfied for all $\alpha > 1/2$. Indeed, from Proposition 6.4 we get $\langle K \rangle^{-1/2}\Pi_{\mathcal{T}}I(Z)\Pi_{\mathcal{T}}\langle K \rangle^{-\alpha} \in \mathcal{C}(Z)$ for any finite \mathcal{T} and this operator converges in norm to $\langle K \rangle^{-1/2}I(Z)\langle K \rangle^{-\alpha}$. Thus all conditions of Theorem 6.2 are fulfilled by the Hamiltonian $H = K + I$ and so H is strictly affiliated to \mathcal{C} .

7. THE MOURRE ESTIMATE

7.1. Proof of the Mourre estimate. From now on we work in the framework of the second part of Section 2, so we assume that \mathcal{S} is a *finite* semilattice of finite dimensional subspaces of an Euclidean space. In this subsection we prove the Mourre estimate for nonrelativistic Hamiltonians. The strategy of the proof is that introduced in [BG2] and further developed in [ABG, DaG2] (graded C^* -algebras over infinite semilattices and dispersive Hamiltonians are considered in Section 5 from [DaG2]). We choose the generator D of the dilation group W_τ in \mathcal{H} as conjugate operator for reasons explained below. For special types of interactions, similar to those occurring in quantum field models, which are allowed by our formalism, better choices can be made, but at a technical level there is nothing new in that with respect to [Geo] (these special interactions correspond to distributive semilattices \mathcal{S}).

The dilations implement a group of automorphisms of the C^* -algebra \mathcal{C} which is compatible with the grading, i.e. it leaves invariant each component $\mathcal{C}(Z)$ of \mathcal{C} . In fact, it is clear that $W_\tau^*\mathcal{C}_{XY}(Z)W_\tau = \mathcal{C}_{XY}(Z)$ for all X, Y, Z hence $W_\tau^*\mathcal{C}(Z)W_\tau = \mathcal{C}(Z)$. This fact plays a fundamental role in the proof of the Mourre estimate for operators affiliated to \mathcal{C} and explains the choice of D as conjugate operator. Moreover, for each $T \in \mathcal{C}$ the map $\tau \mapsto W_\tau^*TW_\tau$ is norm continuous. We can compute explicitly the function $\hat{\rho}_H$ thanks to the relation

$$W_\tau^*\Delta_XW_\tau = e^\tau\Delta_X \quad \text{or} \quad [\Delta_X, iD] = \Delta_X \quad (7.1)$$

We say that a self-adjoint operator H is of class $C^1(D)$ or of class $C_u^1(D)$ if $W_\tau^*RW_\tau$ as a function of τ is of class C^1 strongly or in norm respectively. Here $R = (H - z)^{-1}$ for some z outside the spectrum of H . The formal relation

$$[D, R] = R[H, D]R \quad (7.2)$$

can be given a rigorous meaning as follows. If H is of class $C^1(D)$ then the intersection \mathcal{D} of the domains of the operators H and D is dense in $D(H)$ and the sesquilinear form with domain \mathcal{D} associated to the formal expression $HD - DH$ is continuous for the topology of $D(H)$ so extends uniquely to a continuous sesquilinear form on the domain of H which is denoted $[H, D]$. This defines the right hand side of (7.2). The left hand side can be defined for example as $i\frac{d}{d\tau}W_\tau^*RW_\tau|_{\tau=0}$.

For Hamiltonians as those considered here it is easy to decide that H is of class $C^1(D)$ in terms of properties of the commutator $[H, D]$. Moreover, the following is easy to prove: *if H is affiliated to \mathcal{C} then H is of class $C_u^1(D)$ if and only if H is of class $C^1(D)$ and $[R, D] \in \mathcal{C}$.*

Let H be of class $C^1(D)$ and $\lambda \in \mathbb{R}$. Then for each $\theta \in \mathcal{C}_c(\mathbb{R})$ with $\theta(\lambda) \neq 0$ one may find a real number a and a compact operator K such that

$$\theta(H)^*[H, iD]\theta(H) \geq a|\theta(H)|^2 + K. \quad (7.3)$$

Definition 7.1. The upper bound $\hat{\rho}_H(\lambda)$ of the numbers a for which such an estimate holds is *the best constant in the Mourre estimate for H at λ* . The *threshold set* of H (relative to D) is the closed real set

$$\tau(H) = \{\lambda \mid \hat{\rho}_H(\lambda) \leq 0\} \quad (7.4)$$

One says that D is *conjugate to H* at λ if $\hat{\rho}_H(\lambda) > 0$.

The set $\tau(H)$ is closed because the function $\widehat{\rho}_H : \mathbb{R} \rightarrow]-\infty, \infty]$ is lower semicontinuous.

To each closed real set A we associate the function $N_A : \mathbb{R} \rightarrow [-\infty, \infty[$ defined by

$$N_A(\lambda) = \sup\{x \in A \mid x \leq \lambda\}. \quad (7.5)$$

We make the convention $\sup \emptyset = -\infty$. Thus N_A may take the value $-\infty$ if and only if A is bounded from below and then $N_A(\lambda) = -\infty$ if and only if $\lambda < \min A$. The function N_A is further discussed during the proof of Lemma 7.3.

Nonrelativistic many-body Hamiltonians have been introduced in Definition 2.19. Let $\text{ev}(T)$ be the set of eigenvalues of an operator T .

Theorem 7.2. *Let \mathcal{S} be finite and let $H = H_{\mathcal{S}}$ be a nonrelativistic many-body Hamiltonian of class $C_u^1(D)$. Then $\widehat{\rho}_H(\lambda) = \lambda - N_{\tau(H)}(\lambda)$ for all real λ and $\tau(H)$ is a closed countable real set given by*

$$\tau(H) = \bigcup_{X \neq O} \text{ev}(H_{\mathcal{S}/X}). \quad (7.6)$$

Proof: We first treat the case $O \in \mathcal{S}$. We need some facts which are discussed in detail in Sections 7.2, 8.3 and 8.4 from [ABG] (see pages 51–61 in [BG2] for a shorter presentation).

(i) For each real λ let $\rho_H(\lambda)$ be the upper bound of the numbers a for which an estimate like (7.3) but with $K = 0$ holds. This defines a lower semicontinuous function $\rho_H : \mathbb{R} \rightarrow]-\infty, \infty]$ hence the set $\varkappa(H) = \{\lambda \mid \rho_H(\lambda) \leq 0\}$ is a closed real set called *critical set* of H (relative to D). We clearly have $\rho_H \leq \widehat{\rho}_H$ and so $\tau(H) \subset \varkappa(H)$.

(ii) Let $\mu(H)$ be the set of eigenvalues of H such that $\widehat{\rho}_H(\lambda) > 0$. Then $\mu(H)$ is a discrete subset of $\text{ev}(H)$ consisting of eigenvalues of finite multiplicity. This is essentially the virial theorem.

(iii) There is a simple and rather unexpected relation between the functions ρ_H and $\widehat{\rho}_H$: they are “almost” equal. In fact, $\rho_H(\lambda) = 0$ if $\lambda \in \mu(H)$ and $\rho_H(\lambda) = \widehat{\rho}_H(\lambda)$ otherwise. In particular

$$\varkappa(H) = \tau(H) \cup \text{ev}(H) = \tau(H) \sqcup \mu(H) \quad (7.7)$$

where \sqcup denotes disjoint union.

(iv) This step is easy but rather abstract and the C^* -algebra setting really comes into play. We assume that H is affiliated to our algebra \mathcal{C} . The preceding arguments did not require more than the $C^1(D)$ class. Now we require H to be of class $C_u^1(D)$. Then the operators $H_{\geq X}$ are also of class $C_u^1(D)$ and we have the important relation (Theorem 8.4.3 in [ABG] or Theorem 4.4 in [BG2])

$$\widehat{\rho}_H = \min_{X \in \mathcal{P}(\mathcal{S})} \rho_{H_{\geq X}}.$$

To simplify notations we adopt the abbreviations $\rho_{H_{\geq X}} = \rho_{\geq X}$ and instead of $X \in \mathcal{P}(\mathcal{S})$ we write $X \succ O$, which should be read “ X covers O ”. For coherence with later notations we also set $\widehat{\rho}_H = \widehat{\rho}_{\mathcal{S}}$. So (7.8) may be written

$$\widehat{\rho}_{\mathcal{S}} = \min_{X \succ O} \rho_{\geq X}. \quad (7.8)$$

(v) From (7.1) and (2.30) we get

$$H_{\geq X} = \Delta_X \otimes 1 + 1 \otimes H_{\mathcal{S}/X}, \quad [H_{\geq X}, iD] = \Delta_X \otimes 1 + 1 \otimes [D, iH_{\mathcal{S}/X}].$$

Recall that we denote D the generator of the dilation group independently of the space in which it acts. We note that the formal argument which gives the second relation above can easily be made rigorous but this does not matter here. Indeed, since $H_{\geq X}$ is of class $C_u^1(D)$ and by using the first relation above, one can easily show that $H_{\mathcal{S}/X}$ is also of class $C_u^1(D)$ (see the proof of Lemma 9.4.3 in [ABG]). Now we may use Theorem 8.3.6 from [ABG] to get

$$\rho_{\geq X}(\lambda) = \inf_{\lambda_1 + \lambda_2 = \lambda} (\rho_{\Delta_X}(\lambda_1) + \rho_{\mathcal{S}/X}(\lambda_2))$$

where $\rho_{S/X} = \rho_{H_{S/X}}$. But clearly if $X \neq O$ we have $\rho_{\Delta_X}(\lambda) = \infty$ if $\lambda < 0$ and $\rho_{\Delta_X}(\lambda) = \lambda$ if $\lambda \geq 0$. Thus we get

$$\rho_{\geq X}(\lambda) = \inf_{\mu \leq \lambda} (\lambda - \mu + \rho_{S/X}(\mu)) = \lambda - \sup_{\mu \leq \lambda} (\mu - \rho_{S/X}(\mu)). \quad (7.9)$$

(vi) Now from (7.8) and (7.9) we get

$$\lambda - \widehat{\rho}_S(\lambda) = \max_{X \succ O} \sup_{\mu \leq \lambda} (\mu - \rho_{S/X}(\mu)). \quad (7.10)$$

Finally, we are able to prove the formula $\widehat{\rho}_H(\lambda) = \lambda - N_{\tau(H)}(\lambda)$ by induction over the semilattice \mathcal{S} . In other terms, we assume that the formula is correct if H is replaced by $H_{S/X}$ for all $X \neq O$ and we prove it for $H = H_{S/O}$. So we have to show that the right hand side of (7.10) is equal to $N_{\tau(H)}(\lambda)$.

According to step (iii) above we have $\rho_{S/X}(\mu) = 0$ if $\mu \in \mu(H_{S/X})$ and $\rho_{S/X}(\mu) = \widehat{\rho}_{S/X}(\mu)$ otherwise. Since by the explicit expression of $\widehat{\rho}_{S/X}$ this is a positive function and since $\rho_H(\lambda) \leq 0$ is always true if λ is an eigenvalue, we get $\mu - \rho_{S/X}(\mu) = \mu$ if $\mu \in \text{ev}(H_{S/X})$ and

$$\mu - \rho_{S/X}(\mu) = \mu - \widehat{\rho}_{S/X}(\mu) = N_{\tau(H_{S/X})}(\mu)$$

otherwise. From the first part of Lemma 7.3 below we get

$$\sup_{\mu \leq \lambda} (\mu - \rho_{S/X}(\mu)) = N_{\text{ev}(H_{S/X}) \cup \tau(H_{S/X})}.$$

If we use the second part of Lemma 7.3 then we see that

$$\max_{X \succ O} \sup_{\mu \leq \lambda} (\mu - \rho_{S/X}(\mu)) = \max_{X \succ O} N_{\text{ev}(H_{S/X}) \cup \tau(H_{S/X})}$$

is the N function of the set

$$\bigcup_{X \succ O} (\text{ev}(H_{S/X}) \cup \tau(H_{S/X})) = \bigcup_{X \succ O} \left(\text{ev}(H_{S/X}) \bigcup \bigcup_{Y \succ X} \text{ev}(H_{S/Y}) \right) = \bigcup_{X \succ O} \text{ev}(H_{S/X})$$

which finishes the proof of $\widehat{\rho}_H(\lambda) = \lambda - N_{\tau(H)}(\lambda)$ hence the proof of Theorem 7.2 in the case $O \in \mathcal{S}$.

No assume $O \notin \mathcal{S}$ and let $E = \min \mathcal{S}$. Then $O \in \mathcal{S}/E$ so we may use the preceding result for $H_{S/E}$. Moreover, we have $H = \Delta_E \otimes 1 + 1 \otimes H_{S/E}$. Thus $\text{ev}(H) = \emptyset$, $\widehat{\rho}_H = \rho_H$, and we may use a relation similar to (7.9) to get

$$\lambda - \widehat{\rho}_H(\lambda) = \sup_{\mu \leq \lambda} (\mu - \rho_{S/E}(\mu)).$$

By what we have shown before we have $\mu - \rho_{S/E}(\mu) = N_{\tau(H_{S/E})}(\mu)$ if $\mu \notin \mu(H_{S/E})$ and otherwise $\mu - \rho_{S/E}(\mu) = \mu$. From Lemma 7.3 we get $\lambda - \widehat{\rho}_H(\lambda) = N_{\tau(H_{S/E}) \cup \mu(H_{S/E})}$. But from (7.7) we get $\tau(H_{S/E}) \cup \mu(H_{S/E}) = \tau(H_{S/E}) \cup \text{ev}(H_{S/E})$. From (7.6) we get

$$\tau(H_{S/E}) = \bigcup_{Y \in \mathcal{S}/E, Y \neq O} \text{ev}(H_{(S/E)/Y}) = \bigcup_{X \in \mathcal{S}, X \neq E} \text{ev}(H_{S/X})$$

because if we write $Y = X/E$ with $X \in \mathcal{S}$, $X \neq E$ then $(\mathcal{S}/E)/(X/E) = \mathcal{S}/X$. Finally,

$$\tau(H_{S/E}) \cup \text{ev}(H_{S/E}) = \bigcup_{X \in \mathcal{S}} \text{ev}(H_{S/X})$$

which proves the Theorem in the case $O \notin \mathcal{S}$. \square

It remains to show the following fact which was used above.

Lemma 7.3. *If A and $A \cup B$ are closed and if M is the function given by $M(\mu) = N_A(\mu)$ for $\mu \notin B$ and $M(\mu) = \mu$ for $\mu \in B$ then $\sup_{\mu \leq \lambda} M(\mu) = N_{A \cup B}(\lambda)$. If A, B are closed then $\sup(N_A, N_B) = N_{A \cup B}$.*

Proof: The last assertion of the lemma is easy to check, we prove the first one. Observe first that the function N_A has the following properties:

- (i) N_A is increasing and right-continuous,
- (ii) $N_A(\lambda) = \lambda$ if $\lambda \in A$,
- (iii) N_A is locally constant and $N_A(\lambda) < \lambda$ on $A^c \equiv \mathbb{R} \setminus A$.

Indeed, the first assertion in (i) and assertion (ii) are obvious. The second part of (i) follows from the more precise and easy to prove fact

$$N_A(\lambda + \varepsilon) \leq N_A(\lambda) + \varepsilon \quad \text{for all real } \lambda \text{ and } \varepsilon > 0. \quad (7.11)$$

A connected component of the open set A^c is necessarily an open interval of one of the forms $] -\infty, y[$ or $]x, y[$ or $]x, \infty[$ with $x, y \in A$. On the first interval (if such an interval appears) N_A is equal to $-\infty$ and on the second or the third one it is clearly constant and equal to $N_A(x)$. We also note that the function N_A is characterized by the properties (i)–(iii).

Thus, if we denote $N(\lambda) = \sup_{\mu \leq \lambda} M(\mu)$, then it will suffice to show that the function N satisfies the conditions (i)–(iii) with A replaced by $A \cup B$. Observe that $M(\mu) \leq \mu$ and the equality holds if and only if $\mu \in A \cup B$. Thus N is increasing, $N(\lambda) \leq \lambda$, and $N(\lambda) = \lambda$ if $\lambda \in A \cup B$.

Now assume that λ belongs to a bounded connected component $]x, y[$ of $A \cup B$ (the unbounded case is easier to treat). If $x < \mu < y$ then $\mu \notin B$ so $M(\mu) = N_A(\mu)$ and $]x, y[$ is included in a connected component of A^c hence $M(\mu) = N_A(x)$. Then $N(\lambda) = \max(\sup_{\nu \leq x} M(\nu), N_A(x))$ hence N is constant on $]x, y[$. Here we have $M(\nu) \leq \nu \leq x$ so if $x \in A$ then $N_A(x) = x$ and we get $N(\lambda) = x$. If $x \in B \setminus A$ then $M(x) = x$ so $\sup_{\nu \leq x} M(\nu) = x$ and $N_A(x) < x$ hence $N(\lambda) = x$. Since $x \in A \cup B$ one of these two cases is certainly realized and the same argument gives $N(x) = x$. Thus the value of N on $]x, y[$ is $N(x)$ so N is right continuous on $]x, y[$. Thus we proved that N is locally constant and right continuous on the complement of $A \cup B$ and also that $N(\lambda) < \lambda$ there.

It remains to be shown that N is right continuous at each point of $\lambda \in A \cup B$. We show that (7.11) hold with N_A replaced by N . If $\mu \leq \lambda$ then $M(\mu) \leq \mu \leq \lambda = M(\lambda)$ hence we have

$$N(\lambda + \varepsilon) = \sup_{\lambda \leq \mu \leq \lambda + \varepsilon} M(\mu).$$

But $M(\mu)$ above is either $N_A(\mu)$ either μ . In the second case $\mu \leq \lambda + \varepsilon$ and in the first case

$$N_A(\mu) \leq N_A(\lambda + \varepsilon) \leq N_A(\lambda) + \varepsilon \leq \lambda + \varepsilon.$$

Thus we certainly have $N(\lambda + \varepsilon) \leq \lambda + \varepsilon$ and $\lambda = N(\lambda)$ because $\lambda \in A \cup B$. \square

7.2. A general class of interactions. The rest of this section is devoted to some technical questions. Our main purpose is to clarify the structure of the interactions in the Euclidean case.

The following compactness criterion will be useful. This is a consequence of the Riesz-Kolmogorov theorem and of the argument page 40 involving the regularity of the weight. Let E, F be arbitrary Euclidean space and $s, t \in \mathbb{R}$.

Proposition 7.4. *An operator $T \in L(\mathcal{H}_E^s, \mathcal{H}_F^t)$ is compact if and only if one of the next two equivalent conditions is satisfied, where $\|\cdot\|$ is the norm in $L(\mathcal{H}_E^s, \mathcal{H}_F^t)$:*

- (i) $\|(U_x - 1)T\| + \|(V_x - 1)T\| \rightarrow 0$ if $x \rightarrow 0$ in F ,
- (ii) $\|T(U_x - 1)\| + \|T(V_x - 1)\| \rightarrow 0$ if $x \rightarrow 0$ in E .

We denote $L^\circ(\mathcal{H}_E^s, \mathcal{H}_F^t)$ the set of small at infinity operators, cf. Definition 2.16. Clearly $L^\circ(\mathcal{H}_E^s, \mathcal{H}_F^t)$ is a closed subspace of $L(\mathcal{H}_E^s, \mathcal{H}_F^t)$.

Corollary 7.5. *An operator $T \in L(\mathcal{H}_E^s, \mathcal{H}_F^t)$ is small at infinity if and only if $\lim_{k \rightarrow 0} T(V_k - 1) = 0$ in norm in $L(\mathcal{H}_E^{s+\varepsilon}, \mathcal{H}_F^t)$ for some $\varepsilon > 0$. Then this holds for all $\varepsilon > 0$.*

Indeed, the first part of condition (ii) of Proposition 7.4 (s replaced by $s + \varepsilon$) is automatically satisfied.

We now give a Sobolev space version of Proposition 6.4 which uses the weights $\langle \cdot \rangle^s$ and is convenient in applications. By using Theorem 5.7 instead of Theorem 5.3 in the proof of Proposition 6.4 we get:

Proposition 7.6. *Let $s, t > 0$ and $Z \subset X \cap Y$. Let $I_{XY}(Z) \in L(\mathcal{H}_Y^t, \mathcal{H}_X^{-s})$ such that the following relations hold in norm in $L(\mathcal{H}_Y^{t+\varepsilon}, \mathcal{H}_X^{-s})$ for some $\varepsilon > 0$:*

- (1) $U_z I_{XY}(Z) = I_{XY}(Z) U_z$ if $z \in Z$ and $V_z^* I_{XY}(Z) V_z \rightarrow I_{XY}(Z)$ if $z \rightarrow 0$ in Z ,
(2) $I_{XY}(Z)(V_y - 1) \rightarrow 0$ if $y \rightarrow 0$ in Y/Z .

If h_X, h_Y are continuous real functions on X, Y such that $h_X(x) \sim \langle x \rangle^{2s}$ and $h_Y(y) \sim \langle y \rangle^{2t}$ and if we set $K_X = h_X(P), K_Y = h_Y(P)$ then $\langle K_X \rangle^{-1/2} I_{XY}(Z) \langle K_Y \rangle^{-\alpha} \in \mathcal{C}_{XY}(Z)$ if $\alpha > 1/2$.

Our next purpose is to discuss in more detail the structure of the operators $I_{XY}(Z)$ from Proposition 7.6. For this we make a Fourier transformation \mathcal{F}_Z in the Z variable as in the proof of Theorem 5.7.

We fix X, Y, Z with $Z \subset X \cap Y$, use the tensor factorizations (2.16) and make identifications like $\mathcal{H}_Z \otimes \mathcal{H}_{X/Z} = L^2(Z; \mathcal{H}_{X/Z})$. Thus $\mathcal{H}_X = \mathcal{H}_Z \otimes \mathcal{H}_{X/Z}$ and $\Delta_X = \Delta_Z \otimes 1 + 1 \otimes \Delta_{X/Z}$ hence if $s \geq 0$

$$\mathcal{H}^s(X) = \mathcal{H}(Z; \mathcal{H}^s(X/Z)) \cap \mathcal{H}^s(Z; \mathcal{H}_{X/Z}) = (\mathcal{H}_Z \otimes \mathcal{H}^s(X/Z)) \cap (\mathcal{H}^s(Z) \otimes \mathcal{H}_{X/Z}) \quad (7.12)$$

where our notations are extended to vector-valued Sobolev spaces. Clearly

$$\mathcal{F}_Z \langle P_X \rangle^s \mathcal{F}_Z^{-1} = \int_Z^\oplus (1 + |k|^2 + |P_{X/Z}|^2)^{s/2} dk. \quad (7.13)$$

Then from (2.17) and $\mathcal{I}_Z = \mathcal{F}_Z^{-1} \mathcal{C}_o(Z) \mathcal{F}_Z$ we get

$$\mathcal{C}_{XY}(Z) = \mathcal{I}_Z \otimes \mathcal{K}_{X/Z, Y/Z} = \mathcal{F}_Z^{-1} \mathcal{C}_o(Z; \mathcal{K}_{X/Z, Y/Z}) \mathcal{F}_Z.$$

To each weakly measurable map $I_{XY}^Z : Z \rightarrow L(\mathcal{H}_{Y/Z}^t, \mathcal{H}_{X/Z}^{-s})$ such that

$$\sup_k \|(1 + |k| + |P_{X/Z}|)^{-s} I_{XY}^Z(k) (1 + |k| + |P_{Y/Z}|)^{-t}\| < \infty. \quad (7.14)$$

we associate a continuous operator $I_{XY}(Z) : \mathcal{H}_Y^t \rightarrow \mathcal{H}_X^{-s}$ by the relation

$$\mathcal{F}_Z I_{XY}(Z) \mathcal{F}_Z^{-1} \equiv \int_Z^\oplus I_{XY}^Z(k) dk. \quad (7.15)$$

The following fact is known: a continuous operator $T : \mathcal{H}_Y^t \rightarrow \mathcal{H}_X^{-s}$ is of the preceding form if and only if $U_a T = T U_a$ for all $a \in Z$. From the preceding results we get (notations are as in Remark 2.15):

Proposition 7.7. *Let $X, Y, Z \in \mathcal{S}$ with $Z \subset X \cap Y$ and assume that $\mathcal{G}_X^1 = \mathcal{H}_X^s$ and $\mathcal{G}_Y^1 = \mathcal{H}_Y^t$. An operator $I_{XY}(Z) : \mathcal{H}_Y^t \rightarrow \mathcal{H}_X^{-s}$ satisfies the conditions of Remark 2.15 if and only if it is of the form (7.15) with a norm continuous function $I_{XY}^Z : Z \rightarrow L^\circ(\mathcal{H}_{Y/Z}^t, \mathcal{H}_{X/Z}^{-s})$ satisfying (7.14).*

7.3. Auxiliary results. In this subsection we collect some useful technical results. Let $\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}$ be Hilbert spaces. Note that we have a canonical identification (tensor products are discussed in §3.4)

$$K(\mathcal{E}, \mathcal{F}) \otimes K(\mathcal{G}, \mathcal{H}) \cong K(\mathcal{E} \otimes \mathcal{G}, \mathcal{F} \otimes \mathcal{H}), \quad \text{in particular } K(\mathcal{E}, \mathcal{F} \otimes \mathcal{H}) \cong K(\mathcal{E}, \mathcal{F}) \otimes K(\mathcal{H}). \quad (7.16)$$

Assume that we have continuous injective embeddings $\mathcal{E} \subset \mathcal{G}$ and $\mathcal{F} \subset \mathcal{G}$. We equip $\mathcal{E} \cap \mathcal{F}$ with the intersection topology defined by the norm $(\|g\|_{\mathcal{E}}^2 + \|g\|_{\mathcal{F}}^2)^{1/2}$, hence $\mathcal{E} \cap \mathcal{F}$ becomes a Hilbert space continuously embedded in \mathcal{G} .

Lemma 7.8. *The map $K(\mathcal{E}, \mathcal{H}) \times K(\mathcal{F}, \mathcal{H}) \rightarrow K(\mathcal{E} \cap \mathcal{F}, \mathcal{H})$ which associates to $S \in K(\mathcal{E}, \mathcal{H})$ and $T \in K(\mathcal{F}, \mathcal{H})$ the operator $S|_{\mathcal{E} \cap \mathcal{F}} + T|_{\mathcal{E} \cap \mathcal{F}} \in K(\mathcal{E} \cap \mathcal{F}, \mathcal{H})$ is surjective. Thus, slightly formally,*

$$K(\mathcal{E} \cap \mathcal{F}, \mathcal{H}) = K(\mathcal{E}, \mathcal{H}) + K(\mathcal{F}, \mathcal{H}). \quad (7.17)$$

Proof: It is clear that the map is well defined. Let $R \in K(\mathcal{E} \cap \mathcal{F}, \mathcal{H})$, we have to show that there are S, T as in the statement of the proposition such that $R = S|_{\mathcal{E} \cap \mathcal{F}} + T|_{\mathcal{E} \cap \mathcal{F}}$. Observe that the norm on $\mathcal{E} \cap \mathcal{F}$ has been chosen such that the linear map $g \mapsto (g, g) \in \mathcal{E} \oplus \mathcal{F}$ be an isometry with range a closed linear subspace \mathcal{I} . Consider R as a linear map $\mathcal{I} \rightarrow \mathcal{H}$ and extend it to the orthogonal of \mathcal{I} by zero. The so defined map $\tilde{R} : \mathcal{I} \rightarrow \mathcal{H}$ is clearly compact. Let S, T be defined by $S e = \tilde{R}(e, 0)$ and $T f = \tilde{R}(0, f)$. Clearly $S \in K(\mathcal{E}, \mathcal{H})$ and $T \in K(\mathcal{F}, \mathcal{H})$ and if $g \in \mathcal{E} \cap \mathcal{F}$ then

$$Sg + Tg = \tilde{R}(g, 0) + \tilde{R}(0, g) = \tilde{R}(g, g) = Rg$$

which proves the lemma. \square

We give some applications. If E, F are Euclidean spaces and $s > 0$ is real then

$$\mathcal{H}_{E \oplus F}^s = (\mathcal{H}_E^s \otimes \mathcal{H}_F) \cap (\mathcal{H}_E \otimes \mathcal{H}_F^s) \quad (7.18)$$

hence Lemma 7.8 gives for an arbitrary Hilbert space \mathcal{H}

$$K(\mathcal{H}_{E \oplus F}^s, \mathcal{H}) = K(\mathcal{H}_E^s \otimes \mathcal{H}_F, \mathcal{H}) + K(\mathcal{H}_E \otimes \mathcal{H}_F^s, \mathcal{H}). \quad (7.19)$$

If \mathcal{H} itself is a tensor product $\mathcal{H} = \mathcal{H}' \otimes \mathcal{H}''$ then we can combine this with (7.16) and get

$$K(\mathcal{H}_{E \oplus F}^s, \mathcal{H}' \otimes \mathcal{H}'') = K(\mathcal{H}_E^s, \mathcal{H}') \otimes K(\mathcal{H}_F, \mathcal{H}'') + K(\mathcal{H}_E, \mathcal{H}') \otimes K(\mathcal{H}_F^s, \mathcal{H}''). \quad (7.20)$$

Consider now a triplet X, Y, Z such that $Z \subset X \cap Y$ and denote

$$E = (X \cap Y)/Z \quad \text{and} \quad X \boxplus Y = X/Y \times Y/X. \quad (7.21)$$

Then $Y/Z = E \oplus (Y/X)$ and $X/Z = E \oplus (X/Y)$ hence by using (7.20) we get for example

$$\mathcal{H}_{Y/Z} = \mathcal{H}_E \otimes \mathcal{H}_{Y/X} \quad \text{and} \quad \mathcal{H}_{X/Z} = \mathcal{H}_E \otimes \mathcal{H}_{X/Y} \quad (7.22)$$

$$\mathcal{H}_{Y/Z}^2 = (\mathcal{H}_E^2 \otimes \mathcal{H}_{Y/X}) \cap (\mathcal{H}_E \otimes \mathcal{H}_{Y/X}^2) \quad (7.23)$$

$$\mathcal{H}_{X/Z}^{-2} = \mathcal{H}_E^{-2} \otimes \mathcal{H}_{X/Y} + \mathcal{H}_E \otimes \mathcal{H}_{X/Y}^{-2}. \quad (7.24)$$

By using once again (7.20) and the notations introduced in (2.41), we get

$$\mathcal{K}_{X/Z, Y/Z}^2 = \mathcal{K}_E^2 \otimes \mathcal{K}_{X/Y, Y/X} + \mathcal{K}_E \otimes \mathcal{K}_{X/Y, Y/X}^2. \quad (7.25)$$

We identify a Hilbert-Schmidt operator with its kernel, so $L^2(X \boxplus Y) \subset \mathcal{K}_{X/Y, Y/X}$ is the subspace of Hilbert-Schmidt operators. The we have a strict inclusion

$$L^2(X \boxplus Y; \mathcal{K}_E^2) \subset \mathcal{K}_E^2 \otimes \mathcal{K}_{X/Y, Y/X} \quad (7.26)$$

7.4. First order regularity conditions. In the next two subsections we consider interactions as in Proposition 2.26 and give explicit conditions on the I_{XY}^Z such that H be of class $C_u^1(D)$. We recall that the assumptions of Proposition 2.26 can be stated as follows: for all $Z \subset X \cap Y$

$$I_{XY}^Z : \mathcal{H}_{Y/Z}^2 \rightarrow \mathcal{H}_{X/Z} \quad \text{is compact and satisfies} \quad (I_{XY}^Z)^* \supset I_{YX}^Z, \quad (7.27)$$

$$[D, I_{XY}^Z] : \mathcal{H}_{Y/Z}^2 \rightarrow \mathcal{H}_{X/Z}^{-2} \quad \text{is compact.} \quad (7.28)$$

If (7.27) is satisfied then by duality and interpolation we get

$$I_{XY}^Z : \mathcal{H}_{Y/Z}^\theta \rightarrow \mathcal{H}_{X/Z}^{\theta-2} \quad \text{is a compact operator for all } 0 \leq \theta \leq 2, \quad (7.29)$$

in particular the operator $[D, I_{XY}^Z] \equiv D_{X/Z} I_{XY}^Z - I_{XY}^Z D_{Y/Z}$ restricted to the space of functions in $\mathcal{H}_{Y/Z}^2$ with compact support has values in the space of functions locally in $\mathcal{H}_{X/Z}^{-1}$. We use, for example, the relation $D_{X/Z} = D_E \otimes 1 + 1 \otimes D_{X/Y}$ relatively to (7.22) to decompose this operator as follows:

$$\begin{aligned} [D, I_{XY}^Z] &= (D_E + D_{X/Y}) I_{XY}^Z - I_{XY}^Z (D_E + D_{Y/X}) \\ &= [D_E, I_{XY}^Z] + D_{X/Y} I_{XY}^Z - I_{XY}^Z D_{Y/X}. \end{aligned} \quad (7.30)$$

Since $I_{XY}^Z D_{Y/X} \subset (D_{Y/X} I_{YX}^Z)^*$ if (7.27) is satisfied then condition (7.28) follows from:

$$[D_E, I_{XY}^Z] \text{ and } D_{X/Y} I_{XY}^Z \text{ are compact operators } \mathcal{H}_{Y/Z}^2 \rightarrow \mathcal{H}_{X/Z}^{-2} \text{ for all } X, Y, Z. \quad (7.31)$$

According to (7.25) the first part of condition (7.27) is equivalent to

$$I_{XY}^Z = J + J' \text{ for some } J \in \mathcal{K}_E^2 \otimes \mathcal{K}_{X/Y, Y/X} \text{ and } J' \in \mathcal{K}_E \otimes \mathcal{K}_{X/Y, Y/X}^2. \quad (7.32)$$

As a particular case, from (7.26) we obtain the example discussed in §2.11. The compactness conditions (7.31) are conditions on the kernels $[D_E, I_{XY}^Z(x', y')]$ and $x' \cdot \nabla_{x'} I_{XY}^Z(x', y')$ of the operators $[D_E, I_{XY}^Z]$ and $D_{X/Y} I_{XY}^Z$. Note that a condition on $I_{XY}^Z D_{Y/X}$ is a requirement on the kernel $y' \cdot \nabla_{y'} I_{XY}^Z(x', y')$.

7.5. Creation-annihilation type interactions. To see the relation with the creation-annihilation type interactions characteristic to quantum field models we consider now the case when $Y \subset X$ strictly. Then

$$\mathcal{C}_{XY} = \mathcal{C}_Y \otimes \mathcal{H}_{X/Y}, \quad \mathcal{C}_{XY}(Z) = \mathcal{C}_Y(Z) \otimes \mathcal{H}_{X/Y}, \quad \mathcal{H}_X = \mathcal{H}_Y \otimes \mathcal{H}_{X/Y} \quad (7.33)$$

where the first two tensor product have to be interpreted as explained in §3.4. In particular we have

$$L^2(X/Y; \mathcal{C}_Y) \subset \mathcal{C}_{XY} \quad \text{and} \quad L^2(X/Y; \mathcal{C}_Y(Z)) \subset \mathcal{C}_{XY}(Z) \quad \text{strictly.} \quad (7.34)$$

If $Z \subset Y$ then $X = Z \oplus (Y/Z) \oplus (X/Y)$ and $X/Z = (Y/Z) \oplus (X/Y)$ hence $\mathcal{H}_{X/Z} = \mathcal{H}_{Y/Z} \otimes \mathcal{H}_{X/Y}$ and thus the operator I_{XY}^Z is just a compact operator

$$I_{XY}^Z : \mathcal{H}_{Y/Z}^2 \rightarrow \mathcal{H}_{Y/Z} \otimes \mathcal{H}_{X/Y}. \quad (7.35)$$

If $\mathcal{E}, \mathcal{F}, \mathcal{G}$ are Hilbert spaces then $K(\mathcal{E}, \mathcal{F} \otimes \mathcal{G}) \cong K(\mathcal{E}, \mathcal{F}) \otimes \mathcal{G}$. Hence (7.35) means

$$I_{XY}^Z \in \mathcal{K}_{Y/Z}^2 \otimes \mathcal{H}_{X/Y}. \quad (7.36)$$

Let $\mathcal{I}_{XY} = \sum_{Z \subset X \cap Y} 1_Z \otimes \mathcal{K}_{X/Z, Y/Z}^2$, where the sum is direct and closed in \mathcal{K}_{XY}^2 . A usual nonrelativistic N -body Hamiltonian associated to the semilattice \mathcal{S}_X of subspaces of X is of the form $\Delta_X + I_X$ with $I_X \in \mathcal{I}_X \equiv \mathcal{I}_{XX}$. Thus the interaction which couples the X and Y systems is of the form

$$I_{XY} = \sum_{Z \in \mathcal{S}_Y} 1_Z \otimes I_{XY}^Z \in \mathcal{I}_Y \otimes \mathcal{H}_{X/Y}. \quad (7.37)$$

In particular we may take $I_{XY} \in L^2(X/Y; \mathcal{I}_Y)$, but we stress that the space $\mathcal{I}_Y \otimes \mathcal{H}_{X/Y}$ is much larger (see §3.4). More explicitly, a square integrable function $I_{XY} : X/Y \rightarrow \mathcal{I}_Y$ determines an operator $I_{XY} : \mathcal{H}_Y^2 \rightarrow \mathcal{H}_X$ by the following rule: it associates to $u \in \mathcal{H}^2(Y)$ the function $y' \mapsto I_{XY}(y')u$ which belongs to $L^2(X/Y; \mathcal{H}_{X/Y}) = \mathcal{H}_X$. We may also write

$$(I_{XY}u)(x) = (I_{XY}(y')u)(y) \quad \text{where } x \in X = Y \oplus X/Y \text{ is written as } x = (y, y'). \quad (7.38)$$

We say that the operator valued function I_{XY} is the kernel of the operator I_{XY} . The adjoint $I_{YX} = I_{XY}^*$ is an integral operator in the y' variable (like an annihilation operator). Indeed, if $v \in \mathcal{H}_X$ is thought as a map $y' \mapsto v(y') \in \mathcal{H}_Y$ then we have $I_{YX}v = \int_{X/Y} I_{XY}^*(y')v(y')dy'$ at least formally.

The particular case when the function I_{XY} is factorizable clarifies the connection with the quantum field type interactions: let I_{XY} be a finite sum $I_{XY} = \sum_i V_Y^i \otimes \phi_i$ where $V_Y^i \in \mathcal{I}_Y$ and $\phi_i \in \mathcal{H}_{X/Y}$, then

$$I_{XY}u = \sum_i (V_Y^i u) \otimes \phi_i \quad \text{as an operator } I_{XY} : \mathcal{H}_Y^2 \rightarrow \mathcal{H}_X = \mathcal{H}_Y \otimes \mathcal{H}_{X/Y}. \quad (7.39)$$

This is a sum of N -body type interactions V_Y^i tensorized with operators which create particles in states ϕ_i .

The conditions on the ‘‘commutator’’ $[D, I_{XY}]$ may be written in terms of the kernel of I_{XY} . The relation (7.30) becomes $[D, I_{XY}] = [D_Y, I_{XY}] + D_{X/Y}I_{XY}$. The operator D_Y acts only on the variable y and $D_{X/Y}$ acts only on the variable y' . Thus $[D_Y, I_{XY}]$ and $D_{X/Y}I_{XY}$ are operators of the same nature as I_{XY} but more singular: the kernel of $[D_Y, I_{XY}]$ is the function $y' \mapsto [D_Y, I_{XY}(y')]$ and that of $2iD_{X/Y}I_{XY}$ is the function $y' \mapsto (y' \cdot \nabla_{y'} + n/2)I_{XY}(y')$. Thus, to get (7.28) it suffices to require two conditions on the kernel I_{XY} , one on $[D_Y, I_{XY}(y')]$ and a second one on $y' \cdot \nabla_{y'}I_{XY}(y')$.

If we decompose I_{XY} as in (7.37) with $I_{XY}^Z : \mathcal{H}_{Y/Z}^2 \rightarrow \mathcal{H}_{Y/Z} \otimes \mathcal{H}_{X/Y}$ compact then the (formal) kernel of I_{XY}^Z is a $\mathcal{K}_{Y/Z}^2$ valued map on X/Y . We require that $[D_{Y/Z}, I_{XY}^Z]$ and $D_{X/Y}I_{XY}^Z$ be compact operators $\mathcal{H}_{Y/Z}^2 \rightarrow \mathcal{H}_{X/Z}^2$. From (7.12) and $X/Z = (Y/Z) \oplus (X/Y)$ we get

$$\mathcal{H}_{X/Z}^2 = (\mathcal{H}_{Y/Z} \otimes \mathcal{H}_{X/Y}^2) \cap (\mathcal{H}_{Y/Z}^2 \otimes \mathcal{H}_{X/Y}), \quad \mathcal{H}_{X/Z}^{-2} = \mathcal{H}_{Y/Z} \otimes \mathcal{H}_{X/Y}^{-2} + \mathcal{H}_{Y/Z}^{-2} \otimes \mathcal{H}_{X/Y}$$

which are helpful in checking these compactness requirements.

7.6. Besov regularity classes. We recall some facts concerning the Besov type regularity class $C^{1,1}(D)$; we refer to [ABG] for details on these matters. Since the conjugate operator D is fixed we shall not indicate it in the notation from now on. An operator $T \in L(\mathcal{H})$ is of class $C^{1,1}$ if

$$\int_0^1 \|W_{2\varepsilon}^* T W_{2\varepsilon} - 2W_\varepsilon^* T W_\varepsilon + T\| \frac{d\varepsilon}{\varepsilon^2} \equiv \int_0^1 \|(\mathcal{W}_\varepsilon - 1)^2 T\| \frac{d\varepsilon}{\varepsilon^2} < \infty \quad (7.40)$$

where \mathcal{W}_ε is the automorphism of $L(\mathcal{H})$ defined by $\mathcal{W}_\varepsilon T = W_\varepsilon^* T W_\varepsilon$. The condition (7.40) implies that T is of class C_u^1 and is just slightly more than this. Indeed, T is of class C^1 or C_u^1 if and only if the limit

$$\lim_{\tau \rightarrow 0} \int_\tau^1 (\mathcal{W}_\varepsilon - 1)^2 T \frac{d\varepsilon}{\varepsilon^2}$$

exists strongly or in norm respectively. The following subclass of $C^{1,1}$ is useful in applications: T is called of class C^{1+} if T is of class C^1 , so the commutator $[D, T]$ is a bounded operator, and

$$\int_0^1 \|W_\varepsilon^* [D, T] W_\varepsilon - [D, T]\| \frac{d\varepsilon}{\varepsilon} < \infty. \quad (7.41)$$

Then $C^{1+} \subset C^{1,1}$. The class most frequently used in the context of the Mourre theorem is C^2 : this is the set of $T \in C^1$ such that $[D, T] \in C^1$. But $[D, T] \in C^1$ if and only if

$$\|W_\varepsilon^* [D, T] W_\varepsilon - [D, T]\| \leq C|\varepsilon| \quad \text{for some constant } C \text{ and all real } \varepsilon$$

hence this condition is much stronger than the Dini type condition (7.41). A self-adjoint operator H is of class $C^{1,1}$, C^{1+} or C^2 if its resolvent is of class $C^{1,1}$, C^{1+} or C^2 respectively.

We now consider a Hamiltonian as in Proposition 2.26 and discuss conditions which ensure that H is of class $C^{1,1}$. An important point is that the domain \mathcal{H}^2 of H is stable under the dilation group W_τ . Then Theorem 6.3.4 from [ABG] implies that H is of class $C^{1,1}$ if and only if

$$\int_0^1 \|(\mathcal{W}_\varepsilon - 1)^2 H\|_{\mathcal{H}^2 \rightarrow \mathcal{H}^{-2}} \frac{d\varepsilon}{\varepsilon^2} < \infty. \quad (7.42)$$

As above $\mathcal{W}_\varepsilon H = W_\varepsilon^* H W_\varepsilon$ hence $(\mathcal{W}_\varepsilon - 1)^2 H = W_{2\varepsilon}^* H W_{2\varepsilon} - 2W_\varepsilon^* H W_\varepsilon + H$. We have $H = \Delta + I$ and due to (7.1) the relation (7.42) is trivially verified by the kinetic part Δ of H hence we are only interested in conditions on I which ensure that (7.42) is satisfied with H replaced by I . If this is the case, by a slight abuse of language we say that I is of class $C^{1,1}$. In terms of the coefficients I_{XY} , this means

$$\int_0^1 \|(\mathcal{W}_\varepsilon - 1)^2 I_{XY}^Z\|_{\mathcal{H}_{Y/Z}^2 \rightarrow \mathcal{H}_{X/Z}^{-2}} \frac{d\varepsilon}{\varepsilon^2} < \infty \quad \text{for all } X, Y, Z. \quad (7.43)$$

We recall one fact (see [ABG, Ch. 5]). Let $I : \mathcal{H}^2 \rightarrow \mathcal{H}^{-2}$ be an arbitrary linear continuous operator. Then $[D, I] : \mathcal{H}_c^2 \rightarrow \mathcal{H}_{\text{loc}}^{-3}$ is well defined and I is of class C^1 (in an obvious sense) if and only if this operator is the restriction of a continuous map $\mathcal{H}^2 \rightarrow \mathcal{H}^{-2}$, which will be denoted also $[D, I]$. We say that I is of class C^{1+} if this condition is satisfied and

$$\int_0^1 \|W_\varepsilon^* [D, I] W_\varepsilon - [D, I]\|_{\mathcal{H}^2 \rightarrow \mathcal{H}^{-2}} \frac{d\varepsilon}{\varepsilon} < \infty. \quad (7.44)$$

As before, if I is of class C^{1+} then it is of class $C^{1,1}$. In terms of the coefficients I_{XY}^Z this means

$$\int_0^1 \|W_\varepsilon^* [D, I_{XY}^Z] W_\varepsilon - [D, I_{XY}^Z]\|_{\mathcal{H}_{Y/Z}^2 \rightarrow \mathcal{H}_{X/Z}^{-2}} \frac{d\varepsilon}{\varepsilon} < \infty. \quad (7.45)$$

Such a condition should be imposed on each of the three terms in the decomposition (7.30) separately.

The techniques developed in §7.5.3 and on pages 425–429 from [ABG] can be used to get more concrete conditions. The only new fact with respect to the N -body situation as treated there is that \mathcal{W}_τ when considered as an operator on $\mathcal{L}_{X/Z, Y, Z}$ factorizes in a product of three commuting operators. Indeed, if we write $\mathcal{H}_{Y/Z} = \mathcal{H}_E \otimes \mathcal{H}_{Y/X}$ and $\mathcal{H}_{X/Z} = \mathcal{H}_E \otimes \mathcal{H}_{X/Y}$ then we get $\mathcal{W}_\tau(T) = W_{-\tau}^{X/Y} \mathcal{W}_\tau^E(T) W_\tau^{Y/X}$ where this time we indicated by an upper index the space to which the operator is related and, for example, we identified $W_\tau^{Y/X} = 1_E \otimes W_\tau^{Y/X}$. To check the $C^{1,1}$ property in this context one may use:

Proposition 7.9. *If $T \in \mathcal{L} := L(\mathcal{H}_{Y/Z}^2, \mathcal{H}_{X/Z}^2)$ then $\int_0^1 \|(\mathcal{W}_\varepsilon - 1)^2 T\|_{\mathcal{L}} d\varepsilon / \varepsilon^2 < \infty$ follows from*

$$\int_0^1 \left(\|(\mathcal{W}_\varepsilon^{X/Y} - 1)^2 T\|_{\mathcal{L}} + \|(\mathcal{W}_\varepsilon^E - 1)^2 T\|_{\mathcal{L}} + \|T(\mathcal{W}_\varepsilon^{Y/X} - 1)^2\|_{\mathcal{L}} \right) \frac{d\varepsilon}{\varepsilon^2} < \infty. \quad (7.46)$$

Proof: We shall interpret $\int_0^1 \|(\mathcal{W}_\varepsilon - 1)^2 T\|_{\mathcal{L}} d\varepsilon / \varepsilon^2 < \infty$ in terms of real interpolation theory. Let L_τ be the operator of left multiplication by $W_{-\tau}^{X/Y}$ and N_τ the operator of right multiplication by $W_\tau^{Y/X}$ on $\mathcal{L}_{X/Z, Y/Z}$. If we also set $M_\tau = \mathcal{W}_\tau^E$ then we get three commuting operators L_τ, M_τ, N_τ on $\mathcal{L}_{X/Z, Y/Z}$ such that $\mathcal{W}_\tau = L_\tau M_\tau N_\tau$. Then it is easy to check a Dini type condition like (7.45) by using

$$\mathcal{W}_\tau - 1 = (L_\tau - 1)M_\tau N_\tau + (M_\tau - 1)N_\tau + (N_\tau - 1). \quad (7.47)$$

On the other hand, observe that $\mathcal{W}_\tau, L_\tau, M_\tau, N_\tau$ are one parameter groups of operators on the Banach space \mathcal{L} . These groups are not continuous in the ordinary sense but this does not really matter, in fact we are in the setting of [ABG, Ch. 5]. The main point is that the integral $\int_0^1 \|(\mathcal{W}_\varepsilon - 1)^2 T\|_{\mathcal{L}} d\varepsilon / \varepsilon^2$ is finite if and only if $\int_0^1 \|(\mathcal{W}_\varepsilon - 1)^6 T\|_{\mathcal{L}} d\varepsilon / \varepsilon^2$ is finite (see Theorem 3.4.6 in [ABG]; this is where real interpolation comes into play). Now by taking the sixth power of (7.47) and developing the right hand side we easily get the result, cf. the formula on top of page 132 of [ABG]. \square

The proof of Theorem 2.31 is based on an extension of Propositions 9.4.11 and 9.4.12 from [ABG] to the present context. Since the argument is very similar, we do not enter into details. We mention only that the operator D can be written as $4D = P \cdot Q + Q \cdot P$ where $P = \oplus_X P_X$ and $Q = \oplus_X Q_X$ are suitably interpreted. The proofs in [ABG] depend only on this structure.

8. APPENDIX: HAMILTONIAN ALGEBRAS

We prove here some results on C^* -algebras generated by certain classes of “elementary” Hamiltonians.

8.1. Let X be a locally compact abelian group and let $\{U_x\}_{x \in X}$ be a strongly continuous unitary representation of X on a Hilbert space \mathcal{H} . Then one can associate to it a Borel regular spectral measure E on X^* with values projectors on \mathcal{H} such that $U_x = \int_{X^*} k(x) E(dk)$ and this allows us to define for each Borel function $\psi : X^* \rightarrow \mathbb{C}$ a normal operator on \mathcal{H} by the formula $\psi(P) = \int_{X^*} \psi(k) E(dk)$. The set $\mathcal{T}_X(\mathcal{H})$ of all the operators $\psi(P)$ with $\psi \in C_0(X^*)$ is clearly a non-degenerate C^* -algebra of operators on \mathcal{H} . The following result, which will be useful in several contexts, is an easy consequence of the Cohen-Hewitt factorization theorem, see Lemma 3.8 from [GI3]. Consider an operator $A \in L(\mathcal{H})$.

Lemma 8.1. $\lim_{x \rightarrow 0} \|(U_x - 1)A\| = 0$ if and only if $A = \psi(P)B$ for some $\psi \in C_0(X^*)$ and $B \in L(\mathcal{H})$.

We say that an operator $S \in L(\mathcal{H})$ is of class $C^0(P)$ if the map $x \mapsto U_x S U_x^*$ is norm continuous.

Lemma 8.2. *Let $S \in L(\mathcal{H})$ be of class $C^0(P)$ and let $T \in \mathcal{T}_X(\mathcal{H})$. Then for each $\varepsilon > 0$ there is $Y \subset X$ finite and there are operators $T_y \in \mathcal{T}_X(\mathcal{H})$ such that $\|ST - \sum_{y \in Y} T_y U_y S U_y^*\| < \varepsilon$.*

Proof: It suffices to assume that $T = \psi(P)$ where ψ has a Fourier transform integrable on X , so that $T = \int_X U_x \widehat{\psi}(x) dx$, and then to use a partition of unity on X and the uniform continuity of the map $x \mapsto U_x S U_x^*$ (see the proof of Lemma 2.1 in [DaG1]). \square

We say that a subset \mathcal{B} of $L(\mathcal{H})$ is X -stable if $U_x S U_x^* \in \mathcal{B}$ whenever $S \in \mathcal{B}$ and $x \in X$. From Lemma 8.2 we see that if \mathcal{B} is an X -stable real linear space of operators of class $C^0(P)$ then

$$\mathcal{B} \cdot \mathcal{T}_X(\mathcal{H}) = \mathcal{T}_X(\mathcal{H}) \cdot \mathcal{B}.$$

Since the C^* -algebra \mathcal{A} generated by \mathcal{B} is also X -stable and consists of operators of class $C^0(P)$

$$\mathcal{A} \equiv \mathcal{A} \cdot \mathcal{T}_X(\mathcal{H}) = \mathcal{T}_X(\mathcal{H}) \cdot \mathcal{A} \quad (8.48)$$

is a C^* -algebra. The operators U_x implement a norm continuous action of X by automorphisms of the algebra \mathcal{A} so the C^* -algebra crossed product $\mathcal{A} \rtimes X$ is well defined and the algebra \mathcal{A} is a quotient of this crossed product.

A function h on X^* is called p -periodic for some non-zero $p \in X^*$ if $h(k+p) = h(k)$ for all $k \in X^*$.

Proposition 8.3. *Let \mathcal{V} be an X -stable set of symmetric bounded operators of class $C^0(P)$ and such that $\lambda\mathcal{V} \subset \mathcal{V}$ if $\lambda \in \mathbb{R}$. Denote \mathcal{A} the C^* -algebra generated by \mathcal{V} and define \mathcal{A} by (8.48). Let $h : X^* \rightarrow \mathbb{R}$ be continuous, not p -periodic if $p \neq 0$, and such that $|h(k)| \rightarrow \infty$ as $k \rightarrow \infty$. Then \mathcal{A} is the C^* -algebra generated by the self-adjoint operators of the form $h(P+k) + V$ with $k \in X^*$ and $V \in \mathcal{V}$.*

Proof: Denote $K = h(P+k)$ and let $R_\lambda = (z - K - \lambda V)^{-1}$ with z not real and λ real. Let \mathcal{C} be the C^* -algebra generated by such operators (with varying k and V). By taking $V = 0$ we see that \mathcal{C} will contain the C^* -algebra generated by the operators R_0 . By the Stone-Weierstrass theorem this algebra is $\mathcal{T}_X(\mathcal{H})$ because the set of functions $p \rightarrow (z - h(p+k))^{-1}$ where k runs over X^* separates the points of X^* . The derivative with respect to λ at $\lambda = 0$ of R_λ exists in norm and is equal to $R_0 V R_0$, so $R_0 V R_0 \in \mathcal{C}$. Since $\mathcal{T}_X \subset \mathcal{C}$ we get $\phi(P)V\psi(P) \in \mathcal{C}$ for all $\phi, \psi \in \mathcal{C}_0(X^*)$ and all $V \in \mathcal{V}$. Since V is of class $C^0(P)$ we have $(U_x - 1)V\psi(P) \sim V(U_x - 1)\psi(P) \rightarrow 0$ in norm as $x \rightarrow 0$ from which we get $\phi(P)V\psi(P) \rightarrow S\psi(P)$ in norm as $\phi \rightarrow 1$ conveniently. Thus $V\psi(P) \in \mathcal{C}$ for V, ψ as above. This implies $V_1 \cdots V_n \psi(P) \in \mathcal{C}$ for all $V_1, \dots, V_n \in \mathcal{V}$. Indeed, assuming $n = 2$ for simplicity, we write $\psi = \psi_1 \psi_2$ with $\psi_i \in \mathcal{C}_0(X^*)$ and then Lemma 8.2 allows us to approximate $V_2 \psi_1(P)$ in norm with linear combinations of operators of the form $\phi(P)V_2^x$ where the V_2^x are translates of V_2 . Since \mathcal{C} is an algebra we get $V_1 \phi(P)V_2^x \psi_2(P) \in \mathcal{C}$ hence passing to the limit we get $V_1 V_2 \psi(P) \in \mathcal{C}$. Thus we proved $\mathcal{A} \subset \mathcal{C}$. The converse inclusion follows from a series expansion of R_λ in powers of V . \square

The next two corollaries follow easily from Proposition 8.3. We take $\mathcal{H} = L^2(X)$ which is equipped with the usual representations U_x, V_k of X and X^* respectively. Let $W_\xi = U_x V_k$ with $\xi = (x, k)$ be the phase space translation operator, so that $\{W_\xi\}$ is a projective representation of the phase space $\Xi = X \oplus X^*$. Fix some classical kinetic energy function h as in the statement of Proposition 8.3 and let the classical potential $v : X \rightarrow \mathbb{R}$ be a bounded uniformly continuous function. Then the quantum Hamiltonian will be $H = h(P) + v(Q) \equiv K + V$. Since the origins in the configuration and momentum spaces X and X^* have no special physical meaning one may argue [Be1, Be2] that $W_\xi H W_\xi^* = h(P-k) + v(Q+x)$ is a Hamiltonian as good as H for the description of the evolution of the system. It is not clear to us whether the algebra generated by such Hamiltonians (with h and v fixed) is in a natural way a crossed product. On the other hand, it is natural to say that the coupling constant in front of the potential is also a variable of the system and so the Hamiltonians $H_\lambda = K + \lambda V$ with any real λ are as relevant as H . Then we may apply Proposition 8.3 with \mathcal{V} equal to the set of operators of the form $\lambda v(Q+x)$. Thus:

Corollary 8.4. *Let $v \in C_b^u(X)$ real and let \mathcal{A} be the C^* -subalgebra of $C_b^u(X)$ generated by the translates of v . Let $h : X^* \rightarrow \mathbb{R}$ be continuous, not p -periodic if $p \neq 0$, and such that $|h(k)| \rightarrow \infty$ as $k \rightarrow \infty$. Then the C^* -algebra generated by the self-adjoint operators of the form $W_\xi H_\lambda W_\xi^*$ with $\xi \in \Xi$ and real λ is the crossed product $\mathcal{A} \rtimes X$.*

Now let \mathcal{T} be a set of closed subgroups of X such that the semilattice \mathcal{S} generated by it (i.e. the set of finite intersections of elements of \mathcal{T}) consists of pairwise compatible subgroups. Set $\mathcal{C}_X(\mathcal{S}) = \sum_{Y \in \mathcal{S}}^c \mathcal{C}_X(Y)$. From (4.8) it follows that this is the C^* -algebra generated by $\sum_{Y \in \mathcal{T}} \mathcal{C}_X(Y)$.

Corollary 8.5. *Let h be as in Corollary 8.4. Then the C^* -algebra generated by the self-adjoint operators of the form $h(P+k) + v(Q)$ with $k \in X^*$ and $v \in \sum_{Y \in \mathcal{T}} \mathcal{C}_X(Y)$ is the crossed product $\mathcal{C}_X(\mathcal{S}) \rtimes X$.*

Remark 8.6. Proposition 8.3 and Corollaries 8.4 and 8.5 remain true and are easier to prove if we consider the C^* -algebra generated by the operators $h(P) + V$ with all $h : X^* \rightarrow \mathbb{R}$ continuous and such that $|h(k)| \rightarrow \infty$ as $k \rightarrow \infty$. If in Proposition 8.3 we take $\mathcal{H} = L^2(X; E)$ with E a finite dimensional Hilbert space (describing the spin degrees of freedom) then the operators $H_0 = h(P)$ with $h : X \rightarrow L(E)$ a continuous symmetric operator valued function such that $\|(h(k) + i)^{-1}\| \rightarrow 0$ as

$k \rightarrow \infty$ are affiliated to \mathcal{A} hence also their perturbations $H_0 + V$ where V satisfies the criteria from [DaG3], for example.

8.2. We consider the framework of §2.3 and use Corollary 8.5 to prove that the Hamiltonian algebra of a nonrelativistic N -body system is generated in a natural way by the operators of the form (2.12). To state a precise result it suffices to consider the reduced Hamiltonians (for which we keep the notation H).

Let \mathfrak{S}_2 be the set of cluster decompositions which contain only one nontrivial cluster which consists of exactly two elements. This cluster is of the form $\{j, k\}$ for a unique pair of numbers $1 \leq j < k \leq N$ and we denote by (jk) the corresponding cluster decomposition. The map $x \mapsto x_j - x_k$ sends X onto \mathbb{R}^d and has $X_{(jk)}$ as kernel hence $V_{jk}(x_j - x_k) = V_{(jk)} \circ \pi_{(jk)}(x)$ where $V_{(jk)} : X/X_{(jk)} \rightarrow \mathbb{R}$ is continuous with compact support and $\pi_{(jk)} : X \rightarrow X/X_{(jk)}$ is the canonical surjection.

Thus the reduced Hamiltonians corresponding to (2.12) are the operators on \mathcal{H}_X of the form

$$\Delta_X + \sum_{\sigma \in \mathfrak{S}_2} V_\sigma \circ \pi_\sigma \quad \text{with } V_\sigma : X/X_\sigma \rightarrow \mathbb{R} \text{ continuous with compact support.} \quad (8.49)$$

These operators must be affiliated to the Hamiltonian algebra of the N -body system. On the other hand, if a Hamiltonian $h(P) + V$ is considered as physically admissible then $h(P + k) + V$ should be admissible too because the zero momentum $k = 0$ should not play a special role. In other terms, translations in momentum space should leave invariant the set of admissible Hamiltonians. Hence it is natural to consider the smallest C^* -algebra $\mathcal{C}_X(\mathcal{S})$ such that the operators (8.49) are affiliated to it and which is stable under translations in momentum space. But this algebra is exactly the crossed product

$$\mathcal{C}_X = \mathcal{C}_X \rtimes X = \mathcal{C}_X \cdot \mathcal{T}_X \quad \text{with } \mathcal{C}_X = \sum_{\sigma} \mathcal{C}_X(X_\sigma).$$

Indeed, it is clear that the semilattice generated by \mathfrak{S}_2 is \mathfrak{S} so we may apply Corollary 8.5.

8.3. Here we prove Theorem 2.18.

Let \mathcal{C}' be the C^* -algebra generated by the operators of the form $(z - K - \phi)^{-1}$ where z is a not real number, K is a standard kinetic energy operator, and ϕ is a symmetric field operator. With the notation (6.1) we easily get $\mathcal{T}_d \subset \mathcal{C}'$. If $\lambda \in \mathbb{R}$ then $\lambda\phi$ is also a field operator so $(z - K - \lambda\phi)^{-1} \in \mathcal{C}'$. By taking the derivative with respect to λ at $\lambda = 0$ of this operator we get $(z - K)^{-1}\phi(z - K)^{-1} \in \mathcal{C}$. Since $(z - K)^{-1} = \oplus_X (z - h_X(P))^{-1}$ (recall that P is the momentum observable independently of the group X) and since $\mathcal{T}_d \subset \mathcal{C}'$ we get $S\phi(\theta)T \in \mathcal{C}'$ for all $S, T \in \mathcal{T}_d$ and $\theta = (\theta_{XY})_{X \supset Y}$.

Let $\mathcal{C}'_{XY} = \Pi_X \mathcal{C}' \Pi_Y \subset \mathcal{L}_{XY}$ be the components of the algebra \mathcal{C}' and let us fix $X \supset Y$. Then we get $\varphi(P)a^*(u)\psi(P) \in \mathcal{C}'_{XY}$ for all $\varphi \in \mathcal{C}_o(X^*)$, $\psi \in \mathcal{C}_o(Y^*)$, and $u \in \mathcal{H}_{X/Y}$. The clspan of the operators $a^*(u)\psi(P)$ is \mathcal{T}_{XY} , see Proposition 5.5 and the comments after (3.16), and from (4.14) we have $\mathcal{T}_X \cdot \mathcal{T}_{XY} = \mathcal{T}_{XY}$. Thus the clspan of the operators $\varphi(P)a^*(u)\psi(P)$ is \mathcal{T}_{XY} for each $X \supset Y$ and then we get $\mathcal{T}_{XY} \subset \mathcal{C}'_{XY}$. By taking adjoints we get $\mathcal{T}_{XY} \subset \mathcal{C}'_{XY}$ if $X \sim Y$.

Now recall that the subspace $\mathcal{T}^\circ \subset L(\mathcal{H})$ defined by $\mathcal{T}_{XY}^\circ = \mathcal{T}_{XY}$ if $X \sim Y$ and $\mathcal{T}^\circ = \{0\}$ if $X \not\sim Y$ is a closed self-adjoint linear subspace of \mathcal{T} and that $\mathcal{T}^\circ \cdot \mathcal{T}^\circ = \mathcal{C}$, cf. Theorem 4.25. By what we proved before we have $\mathcal{T}^\circ \subset \mathcal{C}'$ hence $\mathcal{C} \subset \mathcal{C}'$. The converse inclusions is easy to prove. This finishes the proof of Theorem 2.18.

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