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Binomial extensions of Simplicial ideals and reduction number

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Abtracts

In this article, we define a class of binomial ideals associated to a simplicial complex. This class of ideals appears in the presentation of fiber cones of codimension 2 lattice ideals [HM], and in the work of Barile and Morales [BM2], [BM3], [BM4]. We compute the reduction number of Binomial extensions of Simplicial ideals. This extends all the previous results in this area.

Introduction

According to the classification resulting from the successive contributions by Del Pezzo, Bertini, and Xambó (see [EG] for literature), the equidimensional algebraic subsets $X \subset \mathbb{P}^n$ of minimal degree which are connected in codimension one are of three types: quadric hypersurfaces, the cone over the Veronese surface in $\mathbb{P}^5$, and unions $X = \cup_{i=0}^{n} X_i$ of scrolls embedded in linear subspaces such that for all $i = 1, \ldots, n-1$, we have:

$$(X_1 \cup \ldots \cup X_i) \cap X_{i+1} = \mathrm{Span}(X_1 \cup \ldots \cup X_i) \cap \mathrm{Span}(X_{i+1}). \qquad (*)$$

Homologically, varieties of minimal degree were characterized by Eisenbud – Goto [EG] (see Theorem 2.1).

Under the algebraic point of view, the condition $(*)$ was considered at first in [BM2], and later in [BM3], the authors give a complete constructive characterization of the ideals defining varieties of unions of scrolls satisfying the above condition $(*)$.

Later Eisenbud–Green–Hulek–Popescu [EGHP], define a *linearly joined* sequence of varieties, as an union of varieties satisfying the condition $(*)$. They prove that an algebraic set $X \subset \mathbb{P}^r$ is 2–regular if, and only if, $X = X_1 \cup \ldots X_n$, with $X_1, \ldots, X_n$ is a sequence of varieties of minimal degree.

Recall that the homogeneous coordinate ring of a scroll is of the type $A = S/I_2(M)$, where S is the polynomial ring $S = k[T_{i,j} \mid 1 \leq i \leq l \text{ and } 1 \leq \mathrm{j} \leq \mathrm{s_i} + 1]$, and $I_2(M)$ is the ideal generated by 2×2 minors of the matrix $M = (M_1 \mid M_2 \mid \ldots \mid M_l)$, with each M_u is the generic catalecticant matrix

$$M_u = \begin{pmatrix} T_{u,1} & T_{u,2} & \ldots & T_{u,s_u-1} & T_{u,s_u} \\ T_{u,2} & T_{u,3} & \ldots & T_{u,s_u} & T_{u,s_u+1} \end{pmatrix}.$$

We call an ideal of type $I_2(M)$ a scroll ideal.

In this article, we define a class of binomial ideals associated to a simplicial complex. This class of ideals appears in the presentation of fiber cones of codimension 2 lattice ideals [HM], and in the work of Barile and Morales [BM2], [BM3], [BM4]. Let $\triangle$ be a simplicial complex over a set of vertices $V_\triangle = \{x_1, x_2, \ldots, x_n\}$. We will call proper facet a facet F_l with a star of some edges belonging only to F_l (called also *proper edges*). To each *proper facet* of $\triangle$, we associate a set of points $Y^{(l)}$ (which can be empty), and a scroll ideal I_l of variables in $Y^{(l)}$ and in vertex set of these proper edges. The new simplicial complex obtained from $\triangle$ and the sets $Y^{(l)}$ is called an *extension complex*, and denoted by $\overline{\triangle}$.

The *binomial extension of a simplicial ideal* $\mathcal{B}_{\overline{\triangle}}$ associated to $\overline{\triangle}$ is defined to be the one generated by all I_l and the Stanley–Reisner ideal of $\overline{\triangle}$.

The aim of this article is to prove that binomial extension of simplicial ideal is a good generalization of Stanley–Reisner theory to the case of binomial ideals.

In the first section, we will define the class of binomial extension of simplicial ideals, we will give the prime decomposition. From this, we deduce that our class of ideals, in fact, defines an union of scrolls along linear spaces.

In Section 2, we study the reduction number of binomial extension of simplicial ideals. Our aim is to extend the results of Barile and Morales : to describe explicitly the reduction ideals through the complexes. In [BM1], they described a class of square–free monomial ideals whose reduction number is 1 by coloring the graph of a simplicial complex. In [BM2], Barile and Morales considered a class of binomial ideals, which indeed are particular cases of binomial extension of simplicial ideals $\mathcal{B}_{\overline{\triangle}}$ where $\mathcal{G}_\triangle$ is a generalized d–tree and each vertex belongs to at most two extension facets. They proved that this class of ideals is of reduction number 1, and an explicit expression of the reduction is given.

In the case of binomial extension of simplicial ideals, we have the following theorem:

Theorem 0.1. *If the graph associated to $\triangle$ admits a good $(d+1)$–coloration, and in addition, for each proper facet F the origin of the star of proper edges belongs only to F, then the ring $K[\mathbf{x},\mathbf{y}]/\mathcal{B}_{\overline{\triangle}}$ has reduction number 1.*

In this case, the reduced graph associated to $\overline{\triangle}$ admits also a good $(d+1)$–coloration, and

$$(g_1,\ g_2, \ldots,\ g_{d+1})\mathfrak{m}_{\overline{\triangle}} + \mathcal{B}_{\overline{\triangle}} = \mathfrak{m}_{\overline{\triangle}}^2,$$

where $\mathfrak{m}_{\overline{\triangle}} = (\mathbf{x},\mathbf{y})$ is the irrelevant ideal of $K[\mathbf{x},\mathbf{y}]$, and g_i is the sum of all variables with color i.

Theorem 0.2. *Let $\mathcal{G}_\triangle$ be a generalized d–tree. Then we can find a good $(d+1)$–coloration for the reduced graph associated to $\overline{\triangle}$, such that*

$$(g_1,\ g_2, \ldots,\ g_{d+1})\mathfrak{m}_{\overline{\triangle}} + \mathcal{B}_{\overline{\triangle}} = \mathfrak{m}_{\overline{\triangle}}^2,$$

where $\mathfrak{m}_{\overline{\triangle}} = (\mathbf{x},\mathbf{y})$ is the irrelevant maximal ideal of the polynomial ring $K[\mathbf{x},\mathbf{y}]$, and g_i is the sum of all variables with color i.

1 Simplicial ideals and binomial extension of simplicial ideals

binomial extension of simplicial ideals is an extension of Stanley-Reisner monomial ideals. It associates an ideal to a simplicial complex and a family of ideals indexed by a set of its facets. In this article, we will consider some particular cases, which define in fact an union of scrolls, and study some properties of these binomial extension of simplicial ideals.

First of all, let us recall some definitions.

A *simplicial complex* $\triangle$ over a vertex set $V_\triangle = \{x_1, x_2, \ldots, x_n\}$ is a collection of subsets of $V_\triangle$ with the property that:

- For all i, the set $\{x_i\}$ is in $\triangle$,
- If $F \in \triangle$ and $G \subset F$, then $G \in \triangle$.

An element of a simplicial complex $\triangle$ is called a *face* of $\triangle$. The dimension of a face F of $\triangle$, denoted by $\dim F$, is defined to be $|F| -1$, where $|F|$ denotes the number of vertices in F. The dimension of $\triangle$, denoted by $\dim \triangle$, is defined to be the maximal dimension of a face in $\triangle$. The maximal faces of $\triangle$ under inclusion are called *facets* of $\triangle$.

Let us remark that by taking all faces of dimension 0 and 1 of $\triangle$, i.e. all vertices and edges, we associate to $\triangle$ a simple graph $\mathcal{G}_\triangle$. An arbitrary facet F of $\triangle$ becomes a completed subgraph of $\mathcal{G}_\triangle$, so called a $|F|$-clique of $\mathcal{G}_\triangle$.

Notation 1.1. We denote by (E) (resp. $\mathcal{K}[E]$) the ideal generated by E (resp. the polynomial ring with the variables in E). And for a facet F, let F° denote the set of the points of F which belong to no other facets of $\triangle$. To a set of vertices $V_\triangle = \{x_1, x_2, \ldots, x_n\}$, one associate a ring of polynomials $R = \mathcal{K}[x_1, x_2, \ldots, x_n]$ (here, by abuse of notation, we use the x_i's to denote both the vertices in $V_\triangle$ and the variables in the polynomial ring). It is known that to a simplicial complex $\triangle$ on this vertex set, there is an associated ideal, called the Stanley – Reisner ideal, defined as follows

$$\mathcal{I}_\triangle = (x_{i_1} x_{i_2} \cdots x_{i_r} \mid i_1 < i_2 < \cdots < i_r, \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} \notin \triangle)\,.$$

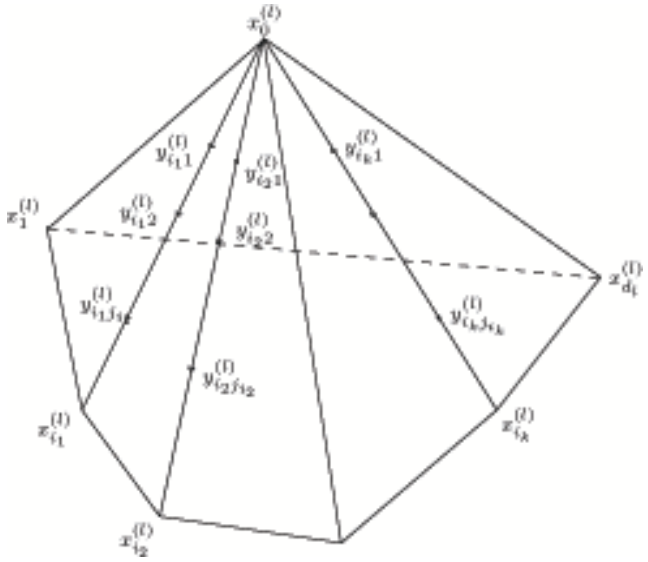
This ideal is generated by monomials.

Now, we introduce the definition of a binomial extension of a simplicial complex.

Definition 1.2. A facet F_l of dimension d_l of $\triangle$ is said to be *proper* if F contains k_l edges $\langle x_0^{(l)},\ x_{i_1}^{(l)}\rangle, \langle x_0^{(l)},\ x_{i_2}^{(l)}\rangle, \ldots, \langle x_0^{(l)},\ x_{i_{k_l}}^{(l)}\rangle$ which belong uniquely to F. If it is the case, these edges are called the *proper edges* of F.

To each proper edge $\langle x_0^{(l)},\ x_{i_j}^{(l)}\rangle$ of a facet F_l of $\triangle$, we associate a set of points $Y_j^{(l)}$ (which can be empty). The simplex which is the product of $Y^{(l)} := \cup Y_j^{(l)}$ and F_l is called *the extension of* F_l by $Y^{(l)}$. By abuse of notation, we use F_l to denote this new facet (without any confusion). Let us remark that this extension can be trivial for some facets of $\triangle$. Let $\overline{\triangle}$ be the complex which facets are all the extension facets of $\triangle$. We call this complex an *extension complex* of $\triangle$.

Let $\overline{\triangle}$ be an extension complex constructed by $\triangle$ and a set of points. We associate to $\overline{\triangle}$ a polynomial ring $\mathcal{R} := \mathcal{K}[V_{\overline{\triangle}}] = \mathcal{K}[\mathbf{x}, \mathbf{y}]$. We will denote by $x_{i_j}^{(l)}$ the vertices in $F_l \cap \triangle$ and by $y_{ij_m}^{(l)}$ the vertices in $F_l \setminus \triangle$.



Definition 1.3. To each non trivial extension facet F_l of $\overline{\triangle}$ we associate the prime ideals $\mathcal{I}_l, \mathcal{J}_l$, where

- $\mathcal{I}_l = 0$ if $Y^{(l)} = \emptyset$,
- if $Y^{(l)} := \{y_{ijm}^{(l)}\} \neq \emptyset$, the $\mathcal{I}_l$ is the ideal generated by the 2×2 minors of the matrix:

$$M_l := \left(\begin{array}{cccc|ccc|c|ccc} x_0^{(l)} & y_{i_1 1}^{(l)} & \dots & y_{i_1 j_{i_1}}^{(l)} & y_{i_2 1}^{(l)} & \dots & y_{i_2 j_{i_2}}^{(l)} & \dots & y_{i_{k_l} 1}^{(l)} & \dots & y_{i_{k_l} j_{i_{k_l}}}^{(l)} \\ y_{i_1 1}^{(l)} & y_{i_1 2}^{(l)} & \dots & x_{i_1}^{(l)} & y_{i_2 1}^{(l)} & \dots & x_{i_2}^{(l)} & \dots & y_{i_{k_l} 2}^{(l)} & \dots & x_{i_{k_l}}^{(l)} \end{array} \right).$$

$\mathcal{J}_l = \Big(\mathcal{I}_l, (V_{\overline{\triangle}} \setminus F_l)\Big) \subset \mathcal{K}[V_{\overline{\triangle}}]$, where $V_{\overline{\triangle}} \setminus F_l$ denote the set of vertices of $\overline{\triangle}$ which are not in F_l.

The *binomial extension of simplicial ideal* $\mathcal{B}_{\overline{\triangle}} \subset \mathcal{K}[V_{\overline{\triangle}}]$ is defined by

$$\mathcal{B}_{\overline{\triangle}} = \left(\sum_{F_l \text{ facet of } \overline{\triangle}} \mathcal{I}_l, \mathcal{I}_{\overline{\triangle}} \right),$$

where $\mathcal{I}_{\overline{\triangle}}$ is the Stanley–Reisner ideal associated to the simplicial complex $\overline{\triangle}$.

The couple $\triangle(\mathcal{B}_{\overline{\triangle}}) := (\overline{\triangle}, \mathcal{B}_{\overline{\triangle}})$ is called a *binomial extension* of $\triangle$.

It is well–known that the Stanley–Reisner ideal of a simplicial complex admits a decomposition into prime ideals corresponding to the facets of the complex (each ideal is generated by the variables which are not in the correspondent facet). We will prove the same property for the ideal $\mathcal{B}_{\overline{\triangle}}$.

We have the Primary decomposition of $\mathcal{B}_{\overline{\triangle}}$ in the ring $\mathcal{K}[V_{\overline{\triangle}}]$:

Proposition 1.4.

$$\mathcal{B}_{\overline{\triangle}} = \bigcap_{F_l \text{ facet of } \overline{\triangle}} \mathcal{J}_l.$$

Before proving the proposition, let us remark that

- $x_0^{(l)} x_{i_s}^{(l)} \in (V_{\overline{\triangle}} \setminus F_i)$ for all proper edge $\langle x_0^{(l)}, x_{i_s}^{(l)} \rangle$ of F_l ($s = \overline{1, k_l}$), and all facet F_i of $\overline{\triangle}$, $F_i \neq F_l$, since either $x_0^{(l)} \in V_{\overline{\triangle}} \setminus F_i$ or $x_{i_s}^{(l)} \in V_{\overline{\triangle}} \setminus F_i$.
- $(F_l^\circ) \subset (V_{\overline{\triangle}} \setminus F_i)$, for all facet F_i of $\overline{\triangle}$, $F_i \neq F_l$.

It implies that for all facet F_i of $\overline{\triangle}$, $F_i \neq F_l$ we have

$$\mathcal{I}_l \subset (V_{\overline{\triangle}} \setminus F_i) \tag{$*$}$$

Proof: First of all, recall a well-known fact that if I, J are disjoint sets of variables and $\mathcal{I} = (I)\,, \mathcal{J} = (J)$ then $\mathcal{I} \cap \mathcal{J} = (pq \mid p \in I, q \in J\,)\,.$

We will prove the proposition by induction on the number m of facets of $\overline{\triangle}$. The case $m = 1$ is trivial. If $m > 1$, then we denote by F_m the mth facet. Denote by $\overline{\triangle'}$ the complex constructed by $m-1$ facets of $\overline{\triangle}$, and denote by $\mathcal{B}_{\overline{\triangle'}}$ the binomial ideal associated to $\overline{\triangle'}$, and $\mathcal{J}_i'$ the prime ideal associated to the ith facet of $\overline{\triangle'}$. Remark that $\mathcal{J}_i = (\mathcal{J}_i',\ x \mid x \in F_m^\circ)$ for all $(i = 1, 2, \ldots, m-1)$. By induction, we have $\mathcal{B}_{\overline{\triangle'}} = \bigcap \mathcal{J}_i'$. We have that $\mathcal{J}_m = \left(\mathcal{I}_m, V_{\overline{\triangle}} \setminus F_m\right)$, and

$$\mathcal{B}_{\overline{\triangle'}} = \left(\sum_{1 \le k \le m-1} \mathcal{I}_k, \mathcal{I}_{\overline{\triangle'}} \right), \text{ and } \mathcal{B}_{\overline{\triangle}} = \left(\sum_{1 \le k \le m} \mathcal{I}_k + \mathcal{I}_m, \mathcal{I}_{\overline{\triangle}} \right).$$

For the Stanley–Reisner ideal $\mathcal{I}_{\overline{\triangle}}$, it is known that:

$$\left(\mathcal{I}_{\overline{\triangle'}}\ ,\ F_m^\circ\right) \bigcap \left(V_{\overline{\triangle}} \setminus F_m\right) = \mathcal{I}_{\overline{\triangle}}. \tag{α}$$

In addition, since $\mathcal{I}_k \subset \left(V_{\overline{\triangle}} \setminus F_m\right)$ for all $k \neq m$, we have:

$$\sum_{1 \le k \le m-1} \mathcal{I}_k \subset \left(V_{\overline{\triangle}} \setminus F_m\right). \tag{β}$$

Moreover, since $\mathcal{I}_m \subset \left(V_{\overline{\triangle}} \setminus F_{m'}\right)$ for all facet $F_{m'} \neq F_m$ in $\overline{\triangle}$, we have:

$$\mathcal{I}_m \subset \bigcap_{m' \neq m} \left(V_{\overline{\triangle}} \setminus F_{m'}\right) = \left(\mathcal{I}_{\overline{\triangle'}}, F_m^\circ\right). \tag{γ}$$

It implies that:

$$\mathcal{B}_{\overline{\triangle}} \subset \left(\mathcal{B}_{\overline{\triangle'}},\ F_m^\circ\right) \bigcap \left(\mathcal{I}_m, V_{\overline{\triangle}} \setminus F_m\right).$$

Now, we will prove the other inclusion. If $r \in \left(\mathcal{B}_{\overline{\triangle'}},\ F_m^\circ\right) \bigcap \left(\mathcal{I}_m, V_{\overline{\triangle}} \setminus F_m\right)$, then $r = u + v = p + q$, where $u \in \sum_{1 \le k \le m-1} \mathcal{I}_k$, $v \in \left(\mathcal{I}_{\overline{\triangle'}},\ F_m^\circ\right)$, $p \in \mathcal{I}_m$, and $q \in \left(V_{\overline{\triangle}} \setminus F_m\right)$. Due to (β) and (γ), we have

$$v - p = q - u \in \left(V_{\overline{\triangle}} \setminus F_m\right) \cap \left(\mathcal{I}_{\overline{\triangle'}}, F_m^\circ\right) \overset{(\alpha)}{=} \mathcal{I}_{\overline{\triangle}}.$$

Hence,

$$r = u + (v - p) + p \in \sum_{1 \le k \le m-1} \mathcal{I}_k + \mathcal{I}_{\overline{\triangle}} + \mathcal{I}_m = \mathcal{B}_{\overline{\triangle}}.$$

From this it follows that:

$$\mathcal{B}_{\overline{\triangle}} = \left(\mathcal{B}_{\overline{\triangle}r},\ F_m^\circ\right) \bigcap \left(\mathcal{I}_m, V_{\overline{\triangle}} \setminus F_m\right).$$

From the induction hypotheses, we deduce that

$$\mathcal{B}_{\overline{\triangle}} = \left(\bigcap_{i=1}^{m-1} \mathcal{J}_i',\ F_m^\circ\right) \bigcap \mathcal{J}_m = \bigcap_{i=1}^{m-1} (\mathcal{J}_i',\ F_m^\circ) \bigcap \mathcal{J}_m = \bigcap_{i=1}^{m-1} \mathcal{J}_i \bigcap \mathcal{J}_m.$$

The proposition is proved. □

Remark 1.5. For all facet F_l of $\overline{\triangle}$, the ideal $\mathcal{J}_l$ is prime and the ring $\mathcal{K}[V_{\overline{\triangle}}]/\mathcal{J}_l$ is of dimension $1 + d_l$, where d_l is the dimension of F_l.

We deduce from that a corollary on the dimension of $\mathcal{B}_{\overline{\triangle}}$ as follows:

Corollary 1.6. $\dim(\mathcal{K}[V_{\overline{\triangle}}]/\mathcal{B}_{\overline{\triangle}}) = 1 + \dim(\overline{\triangle})$.

2 Reduction number one

First, we recall a theorem of Eisenbud–Goto [EG]:

Theorem 2.1. *Let R be a reduced graded ring, defining an algebraic projective variety. Then we have:*

(1) *R is Cohen-Macaulay and $e(R) = 1 + \operatorname{codim} R$, where $e(R)$ is the multiplicity of R;*

$\Rightarrow$ *(2) R admits a $2-$linear resolution;*

$\Rightarrow$ *(3) $r(R) = 1$;*

$\Rightarrow$ *(4) $e(R) \le 1 + \operatorname{codim} R$*

Moreover, if R is Cohen–Macaulay, then the above implications are equivalences.

Definition 2.2. A *generalized $d-$tree* on a set of vertices V is a graph defined recursively by the following properties:

(a) A complete graph on $d + 1$ elements of V is a generalized $d-$tree.

(b) Let G be a graph on the set V. Assume that there exists a vertex $v \in V$ such that:

1. The restriction G' of G on $V' = V \setminus \{v\}$ is a generalized $d-$tree,
2. There is a subset $V'' \subset V'$ with $1 \le j \le d$ vertexes such that the restriction of G on V'' is a complete graph, and
3. G is the graph generated by G' and the complete graph on $V'' \cup \{v\}$.

The vertex v as above is called a *extremal*.

If $j = d$, then we say that G is a $d-$*tree*.

Remark 2.3. Let $\Delta(G)$ the "clique complex" of G, i.e. the simplicial complex whose vertices are the ones of G and the facets are the simplexes with support on the complete subgraphs of G. M. Morales associate to $\Delta(G)$ a graph $H(G)$ whose vertices are the facets of $\Delta(G)$, and an edge of $H(G)$ links two vertices such that the intersection of their associated facets is non–empty. He proved that G is a generalized $d-$tree if and only if $H(G)$ is a tree.

The following theorems are proved by Fröberg [Fr]:

Theorem 2.4. *The Stanley–Reisner ring of a simplicial complex Δ is a Cohen–Macaulay ring of minimal degree if and only if*

1. *The graph $\mathcal{G}_\triangle$ is a $d-$tree, and*
2. *Δ is a clique complex of $\mathcal{G}_\triangle$, i.e. $\Delta = \Delta(\mathcal{G}_\triangle)$.*

Theorem 2.5. *The Stanley–Reisner ring of a simplicial complex Δ admits a $2-$linear resolution if and only if*

1. *The graph $\mathcal{G}_\triangle$ is a generalized $d-$tree, and*
2. *$\Delta = \Delta(\mathcal{G}_\triangle)$.*

Definition 2.6. Let $\mathcal{R}$ be the polynomial ring of n variables on the field $\mathcal{K}$. Let $\mathcal{I} \subset \mathcal{R}$ be a homogeneous graded ideal under the standard graduation and $d = \dim \mathcal{R}/\mathcal{I}$. A set of linear forms $\{g_1,\ g_2, \ldots,\ g_d\}$ is a *reduction* of $\mathcal{R}/\mathcal{I}$, if

$$(g_1,\ g_2, \ldots,\ g_d)\mathfrak{m}^\rho = \mathfrak{m}^{\rho+1}(\mod \mathcal{I})$$

where $\mathfrak{m}$ is the irrelevant maximal ideal of $\mathcal{R}$.

The smallest number ρ for all the possible reductions is called *the reduction number* of $\mathcal{R}/\mathcal{I}$.

In [BM1], Barile and Morales described a class of square–free monomial ideals whose reduction number is 1. First of all, we recall some definitions.

Definition 2.7. A $(d+1)$-*coloration* of a graph $\mathcal{G}$ is a partition of the vertex set $V_\mathcal{G}$ into $d+1$ subsets, which are called "class of colors", such that two neighbors in $\mathcal{G}$ belong to different classes of colors. For each vertex $x \in \mathcal{G}$, we denote by $\mathcal{C}(x)$ the class containing x.

A $(d+1)$-coloration of $\mathcal{G}$ is *good* if every cycle of $\mathcal{G}$ is colored by at least three colors. Remark that this definition is considered only in the case where $d \geq 2$.

Let us recall that by taking all faces of dimension 0 and 1 of a simplicial complex $\triangle$, i.e. all vertices and edges, we associate to $\triangle$ a simple graph $\mathcal{G}_\triangle$.

Proposition 2.8. *[BM1, Theorem 1.1] Let $\triangle$ be a simplicial complex of dimension d. Denote by $R_\triangle$ the associated Stanley–Reisner ring. Assume that $\mathcal{G}_\triangle$ admits a good*

$(d+1)$–coloration. Let $\mathcal{C}_1,\ \mathcal{C}_2,\dots,\mathcal{C}_{d+1}$ sign the classes of colors and for each $i = 1,2,\dots d+1$, put

$$g_i = \sum_{x_j \in \mathcal{C}_i} x_j.$$

Then $g_1,\ g_2,\dots,\ g_{d+1}$ is a system of parameters of $R_\triangle$. In particular, the reduction number of $R_\triangle$ is 1.

In [BM2], Barile and Morales considered a class of binomial ideals defining an union of scrolls, which indeed are binomial extension of simplicial ideals $\mathcal{B}_{\overline{\triangle}}$ where $\mathcal{G}_\triangle$ is a generalized d–tree and each vertex belongs to at most two extension facets. They proved that this class of ideals is of reduction number 1, and an explicit expression of the reduction is given.

In this section, we will extend these results to the binomial extension of simplicial ideals.

Notation 2.9. Let $F_l = \left\{x_0^{(l)}, x_1^{(l)}, \dots, x_{i_1}^{(l)}, \dots,\ , x_{i_{k_l}}^{(l)}, ..., x_{d_l}^{(l)}\right\}$ be an extension facet of $\triangle$ by the points $\{y_{i_j m}^{(l)}\}$in the proper edges $\left\{\langle x_0^{(l)},\ x_{i_1}^{(l)}\rangle, \dots,\ \langle x_0^{(l)},\ x_{i_{k_l}}^{(l)}\rangle\right\}$, and we denote $I_l = \left\{x_0^{(l)}, x_{i_1}^{(l)}, \dots,\ , x_{i_{k_l}}^{(l)}\right\}$.

For the binomial extension of simplicial ideals, it is not necessary to color all $\mathcal{G}_{\overline{\triangle}}$. In fact, for each extension facet F_l, it is sufficient to color the extremal points in each bloc of the associated matrix M_l. The graph obtained from these points is defined as follows:

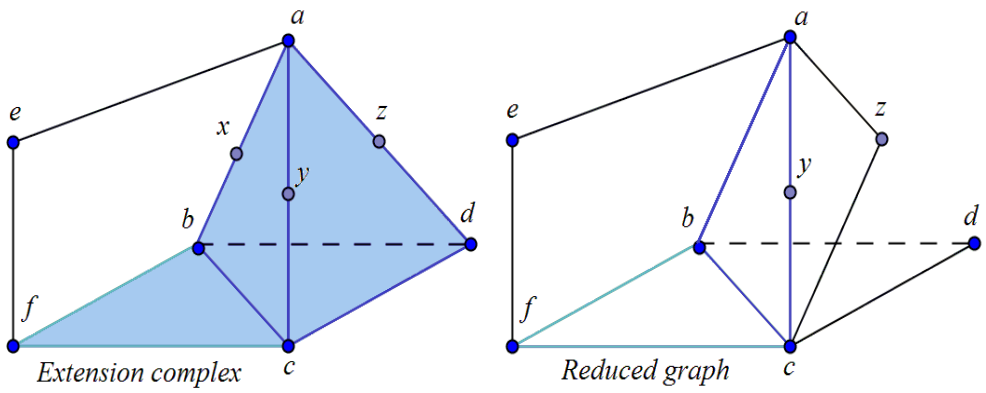
Definition 2.10. The *reduced graph*, denoted by $\widetilde{\mathcal{G}}_{(\overline{\triangle},\mathcal{B})}$, is given by:

- The vertex set $V_{\widetilde{\mathcal{G}}_{(\overline{\triangle},\mathcal{B})}}$ consists of the points of $\triangle$ and the points $y_{i_j 1}^{(l)}$ (with $j \geq 2$) for all extension facet F_l of $\overline{\triangle}$.
- The set of edges is

$$E_{\widetilde{\mathcal{G}}_{(\overline{\triangle},\mathcal{B})}} := \bigcup_{F_l \in \overline{\triangle}\mathrm{s.t}\Upsilon^{(l)} \neq \emptyset} \left\{\left(E_\triangle \setminus \{\langle x_0^{(l)},\ x_{i_j}^{(l)}\rangle\}_{j=2}^{k_l}\right) \cup \{\langle x_0^{(l)},\ y_{i_j 1}^{(l)}\rangle, \langle x_{i_2}^{(l)},\ y_{i_j 1}^{(l)}\rangle\}_{j=2}^{k_l}\right\}.$$

Example 2.11. Consider the following binomial extension with the binomial ideal associated to the facet $F = [a,b,c,d]$ extended by x,y,z is generated by 2×2 minors of the matrix

$$M = \left\{\begin{array}{cc|c|c} a & x & y & z \\ x & b & c & d \end{array}\right\}.$$



Then, the reduced graph associated to this extension complex is as in the figure.

Lemma 2.12. *Let $\mathcal{B}_M$ be the ideal generated by all the 2×2 minors of the matrix M*

$$M := \left(\begin{array}{cccc|ccc|c|ccc} x_0 & y_{1,1} & \dots & y_{1,j_1} & y_{2,1} & \dots & y_{2,j_2} & \dots & y_{k,1} & \dots & y_{k,j_k} \\ y_{1,1} & y_{1,2} & \dots & x_1 & y_{2,2} & \dots & x_2 & \dots & y_{k,2} & \dots & x_k \end{array} \right).$$

and $\mathfrak{m}' = \{x_0, ..., x_k\} \cup Y$ be the set of variables in the matrix M, then for any two distinct variables $x, y \in \mathfrak{m}'$ with $y \in Y$ the product xy is equivalent modulo $\mathcal{B}_M$ to one of the following monomials:

1. $x_1 x_0$,
2. $x_m p, p \in Y_n$
3. $q y_{n,1}, q \in Y_m$, $n \geq 2$,
4. $x_0 y_{1,j}$,
5. $x_0 y_{n,1}$,

Where $1 \leq m \leq n \leq k$

Proof. We can assume that x is a variable in the $m-$block of M, and y is a variable in the $n-$block of M, with $m \leq n$. We have the following cases:

1. $1 \leq m < n \leq k$
   (a) $x = x_m$,
   In this case we have the monomials $x_m p$, with $m \geq 1, p \in Y_n$,
   (b) x any and $y = y_{n,1}$,
   In this case we have the monomials $x_0 y_{n,1}, q y_{n,1}, x_m y_{n,1}$ with $q \in Y_m, m \geq 1$.
   (c) $x = y_{m,u}, y = y_{n,v}$, or the case $x = y_{m,u}, y = x_n$. We only consider the first case, the proof of the second case is similar.

$$xy = y_{m,u} y_{n,v} = (y_{m,u} y_{n,v} - y_{m,u+1} y_{n,v-1}) + y_{m,u+1} y_{n,v-1}$$

$$= y_{m,u+1} y_{n,v-1} \mod \mathcal{B}_M = ... = \begin{cases} x_m p & \mod \mathcal{B}_M, p \in Y_n \\ q y_{n,1} & \mod \mathcal{B}_M, q \in Y_n \end{cases}.$$

2. $m = n > 1$

   (a) $x = y_{m,1}$. In this case we have the monomials $y_{m,1}p, y_{m,1}x_m$ with $p \in Y_m$.

   (b) $y = x_m$. In this case we have the monomials px_m with $p \in Y_m$.

   (c) $x = y_{m,u}, y = y_{m,v}$, with $u < v$. In this case we have

$$xy = y_{m,u}y_{m,v} = (y_{m,u}y_{m,v} - y_{m,u-1}y_{m,v+1}) + y_{m,u-1}y_{m,v+1}$$

$$= y_{m,u-1}y_{m,v+1} \mod \mathcal{B}_M = ... = \begin{cases} x_m p & \mod \mathcal{B}_M, p \in Y_m \\ q y_{m,1} & \mod \mathcal{B}_M, q \in Y_m \end{cases}.$$

3. $m = n = 1$

   (a) $x = x_0$. In this case we have the monomials $x_0 p$ with $p \in Y_1$.

   (b) $x = x_1$. In this case we have the monomials px_1 with $p \in Y_1$.

   (c) $x = y_{1,u}, y = y_{1,v}$.

$$xy = y_{1,u}y_{1,v} = (y_{1,u}y_{1,v} - y_{1,u-1}y_{1,v+1}) + y_{1,u-1}y_{1,v+1}$$

$$= y_{1,u-1}y_{1,v+1} \mod \mathcal{B}_M = ... = \begin{cases} x_1 p & \mod \mathcal{B}_M, p \in Y_1 \\ x_1 p & \mod \mathcal{B}_M, p \in Y_1 \\ x_0 x_1 & \mod \mathcal{B}_M, p \in Y_1 \end{cases}.$$

Definition 2.13. We will say that $\widetilde{\mathcal{G}}_{(\overline{\triangle},\mathcal{B})}$ admits a binomial-coloration if $\widetilde{\mathcal{G}}_{(\overline{\triangle},\mathcal{B})}$ admits a (d + 1)-coloration $\widetilde{\mathcal{C}}$ such that for every facet F_l:

1. $\widetilde{\mathcal{C}}(x_0^{(l)}) \cap F_l = \{x_0^{(l)}, x_2^{(l)}\}$,
2. $\widetilde{\mathcal{C}}(y_{j1}^{(l)}) \cap F_l = \{y_{j1}^{(l)}, x_{j+1}^{(l)}\}$ pour tout $(j = \overline{2, k_l - 1})$,
3. $\widetilde{\mathcal{C}}(y_{k,1}^{(l)}) \cap F_l = \{y_{k,1}^{(l)}\}$,
4. For all the other vertices $x \in \widetilde{\mathcal{G}}_{(\overline{\triangle},\mathcal{B})} \cap F_l$, $\widetilde{\mathcal{C}}(x) \cap F_l = \{x\}$.

Proposition 2.14. *Suppose that $\widetilde{\mathcal{G}}_{(\overline{\triangle},\mathcal{B})}$ admits a binomial-coloration $\widetilde{\mathcal{C}}$, we set*

$$g_i = \sum_{x \in \widetilde{\mathcal{C}}_i} x, \quad \mathbb{G} := (g_1,\ g_2, \ldots,\ g_{d+1}).$$

Consider a facet F_l with a nonzero associated scroll matrix.

1. *If $x_0^{(l)} y_{n,1}^{(l)} \in \mathbb{G}\mathfrak{m} + \mathcal{B}_{\overline{\triangle}}$ for any $2 \leq n \leq k_l$, then*

$$xy \in \mathbb{G}\mathfrak{m}_{\overline{\triangle}} + \mathcal{B}_{\overline{\triangle}}$$

   for any variables $x \neq y \in F_l$, excepts for the products $x_0^{(l)} x_1^{(l)}, y_{1,u}^{(l)} y_{1,v}^{(l)}$, $x_0^{(l)} y^{(l)}$, for $y^{(l)} \in F_l$ but not appearing in M_l, and $x^{(l)} y^{(l)}$, for $x^{(l)}, y^{(l)} \in F_l \cap \triangle \setminus \{x_0^{(l)}\}$.

2. *If* $x_0^{(l)} \in F_l^\circ$, *then*

$$xy \in \mathbb{G}\mathfrak{m}_{\overline{\triangle}} + \mathcal{B}_{\overline{\triangle}}$$

for any variables $x \neq y \in F_l$, *excepts for the products* $x^{(l)}y^{(l)}$, *for* $x^{(l)}, y^{(l)} \in F_l \cap \triangle \setminus \{x_0^{(l)}\}$.

Proof.- We call this facet F, we also delete all scripts l from the variables defining vertices in F and the associated matrix M. from now on, we will denote by $\equiv$ the equivalence relation introduced by $\mathbb{G}\mathfrak{m} + \mathcal{B}_{\overline{\triangle}}$. We have the following:

Remark 2.15. 1. If x, y are two distinct element in F such that $\mathcal{C}(x) \cap F = \{x\}$, and y belongs only to the facet F, (i.e. $y \in F^\circ$), then $xy \equiv 0$, since we can write

$$xy = g_x y - \sum_{z \in C(x), z \neq x} zy,$$

but y belongs only to the facet F and $z \notin F$, so $zy \equiv 0$, hence $xy \equiv 0$.

2. If x, y are two distinct element in F such that $\mathcal{C}(x) \cap F = \{x, x'\}$, and $y \in F^\circ$ then $xy \equiv x'y$.

We have the following cases:

1. x doesn't appears in the matrix M and $y \in Y$,
2. x, y appear in the matrix M, but one of them belongs to Y,
3. $x = x_0$, and $y = x_m$, for some $1 \leq m \leq k$

Now we consider each case:

1. If x doesn't appears in the matrix M and $y \in Y$, then $\mathcal{C}(x) \cap F = \{x\}$, the Remark 2.15 applies so $xy \equiv 0$.

2. If x, y appear in the matrix M, but one of them belongs to Y.

   By applying the Lemma 2.12 the monomial xy is equivalent modulo $\mathbb{G}m + \mathcal{B}_{\overline{\triangle}}$ to one of the following monomials:

   (a) For $1 \leq m \leq n \leq k$, $x_m p, p \in Y_n$, or $qy_{n,1}, q \in Y_m$, $n \geq 2$,

   a-1) The remark 2.15 applies to the monomial $x_1 p, p \in Y_n, n \geq 1$, so it belongs to $\mathbb{G}m + \mathcal{B}_{\overline{\triangle}}$.

   a-2) If we are in the first case then by hypothesis $x_0 y_{n,1} \equiv 0$. If we are in the second case the following argument is also true when $q = x_0$. Consider the case $qy_{n,1}, q \in Y_1, n \geq 2$.
   If $n = k$, and $q \in Y_1$ the remark 2.15 applies, so $qy_{k,1} \equiv 0$ for any $q \in Y \cup \{x_0\}, q \neq y_{k,1}$. So we may assume that $2 \leq n < k$. In this case $C(y_{n,1}) \cap F = \{y_{n,1}, x_{n+1}\}$, by applying the remark 2.15 we have:

   $$qy_{n,1} \equiv -qx_{n+1},$$

   so by using the binomial relations in the matrix M we have

   $$qy_{n,1} \equiv -qx_{n+1} \equiv \begin{cases} x_1 p', p' \in Y_{n+1} \\ q' y_{n+1,1}, q' \in Y_1 \end{cases}.$$

Since the case $x_1p, p \in Y_n, n \geq 1$ was considered in the item b-1), after a finite number of steps we have either $qy_{n,1} \equiv 0$ or $qy_{n,1} \equiv q'y_{k,1}$, but it was proved before that $q'y_{k,1} \equiv 0$, so $qy_{n,1} \equiv 0$.

a-3) We now consider the monomial x_2p, with $p \in Y_n$, $n \geq 2$. Since $C(x_2) \cap F = \{x_0, x_2\}$, by applying the remark 2.15 we have

$$x_2p \equiv -x_0p.$$

By using the binomial relations in the matrix M we have

$$x_0p \equiv \begin{cases} x_0y_{n,1} \\ x_1p', p' \in Y_n \\ qy_{n,1}, q \in Y_1 \end{cases},$$

So this case is done by taking care of the previous cases.

a-4) We consider the monomials x_mp, for $2 < m \leq n \leq k$, and the monomials $qy_{n,1}$ with $q \in Y_m$, $2 \leq m < n \leq k$. Since $\widetilde{\mathcal{C}}(x_m) \cap F) = \{x_m, y_{m-1,1}\}$ by applying the remark 2.15 we have

$$x_mp \equiv -y_{m-1,1}p.$$

By the proof of the Lemma 2.12, case 1.c, we have

$$y_{m-1,1}p \equiv \begin{cases} x_{m-1}p & p \in Y_n \\ qy_{n,1} & q \in Y_{m-1} \end{cases}.$$

We have either

$$x_mp \equiv x_{m-1}p \equiv ... \equiv x_2p,$$

or

$$x_mp \equiv qy_{n,1},$$

for some $q \in Y_m$, $2 \leq m < n \leq k$.
So it should be enough to consider the monomial $qy_{n,1}$ with $q \in Y_m$, $2 \leq m < n \leq k$. The case $n = k$ was considered in b-3). So we may assume $2 \leq m < n < k$. By applying the remark 2.15 we have

$$qy_{n,1} \equiv -qx_{n+1},$$

and by using the binomial relations in M

$$qx_{n+1} \equiv \begin{cases} x_mp & p \in Y_{n+1} \\ q'y_{n+1,1} & q' \in Y_m \end{cases}.$$

So after a finite number of steps we will have:

- Either $qy_{n,1} \equiv x_2p$, $p \in Y_n, n \geq 3$, yet considered in b-1) or
- $qy_{n,1} \equiv q'y_{k,1}$, $q' \in Y_m$, yet considered in b-2).

(b) If x, y belongs to the first block of M then we have to consider the following monomials $x_1y_{1,j}, x_0y_{1,j}, x_0x_1$ or $y_{1,u}y_{1,v}$. For the monomial $x_0y_{1,j}$, by applying the remark 2.15 we have

$$x_0y_{1,j} \equiv -x_2y_{1,j},$$

which is equivalent modulo the binomial relations in the matrix M either to the monomial x_1p, with $p \in Y_2$, or to the monomial $qy_{2,1}$, with $q \in Y_1$. The monomial x_1p, with $p \in Y_2$, was yet considered in the first item, and the the monomial $qy_{2,1}$, with $q \in Y_1$ was considered in the second item.

The monomial $x_1y_{1,j}$, was consider before. By the Lemma 2.12 the monomial $y_{1,u}y_{1,v}$ is equivalent modulo $\mathcal{B}_{\overline{\triangle}}$ to one of the monomials $x_1y_{1,j}, x_0y_{1,j}, x_0x_1$. If $x_0 \in F^\circ$ then the remark 2.15 applies to x_1x_0 so we have $x_1x_0 \equiv 0$. So this subcase is done.

3. We consider the monomial x_0x_m, where x_m appears in the matrix M, $m > 1$. By the proof of the Lemma 2.12, 1.c, for any $1 < m$, we have either $x_0x_m \equiv x_1p$, with $p \in Y_m$, or $x_0x_m \equiv qy_{m,1}$, with $q \in Y_1$. Both monomials were considered in the previous items and we have seen that they belong to $\mathbb{G}m + \mathcal{B}_{\overline{\triangle}}$. The proposition is proved. □

Proposition 2.16. *Suppose that $\widetilde{\mathcal{G}}_{(\overline{\triangle},\mathcal{B})}$ admits a binomial-coloration $\widetilde{\mathcal{C}}$. Let $\widetilde{\mathcal{C}}_1, \widetilde{\mathcal{C}}_2, \ldots, \widetilde{\mathcal{C}}_{d+1}$ the classes of colors of the vertices of $\widetilde{\mathcal{G}}_{(\overline{\triangle},\mathcal{B})}$, we set*

$$g_i = \sum_{x \in \widetilde{\mathcal{C}}_i} x, \quad \mathbb{G} := (g_1,\ g_2, \ldots,\ g_{d+1}).$$

1. *Suppose that this coloration on the graph $\widetilde{\mathcal{G}}'$, with set of edges $E_{\widetilde{\mathcal{G}}'} := E_{\widetilde{\mathcal{G}}_{(\overline{\triangle},\mathcal{B})}} \cap E_{\mathcal{G}_\Delta}$, is a good $d+1-$coloration $\widetilde{\mathcal{C}}$, that is every cycle has more than three colors. Let $\mathfrak{m}_{\overline{\triangle}} = (\mathbf{x}, \mathbf{y})$ be the maximal ideal of the polynomial ring $\mathcal{R} := k[\mathbf{x}, \mathbf{y}]$.*
2. *Suppose that for every facet F_l we have either:*
   (a) $x_0^{(l)} y_{n,1}^{(l)} \in \mathbb{G}\mathfrak{m} + \mathcal{B}_{\overline{\triangle}}$ *for any* $2 \leq n \leq k_l$.
   (b) $x_0^{(l)} \in F^\circ$.

Then we have :

$$\mathfrak{m}_{\overline{\triangle}}^2 = \mathbb{G}\mathfrak{m}_{\overline{\triangle}} + \mathcal{B}_{\overline{\triangle}}.$$

In particular, the reduction number of $\mathcal{R}/\mathcal{B}_{\overline{\triangle}}$ is 1.

Proof.- We consider a monomial $xy \in \mathfrak{m}^2$. We want to prove that $xy \in \mathbb{G}\mathfrak{m} + \mathcal{B}_{\overline{\triangle}}$.

- We have two cases:

A. The variables x, y are distinct. We remark that if $x, y \in \mathfrak{m}$ are not in the same facet then $xy \in \mathcal{B}_{\overline{\triangle}}$, hence $xy \equiv 0$.

• We can assume that the variables x, y belong to the same facet F_l. We fix this facet and we call it F, we also delete all scripts l from the variables defining vertices in F and the associated matrix M. By applying the Proposition 2.14, we have to study the cases :$x = x_0$, and $y = x_m$, for some $k < m \leq d$ or $x = x_l$, and $y = x_m$, for some $1 \leq l < m \leq d$.

B. The variables x, y coincide.

We consider now in detail both cases.

A. If x_m doesn't appears in the matrix M or $m = 1$, then the edge $\langle x_0, x_m\rangle$ belongs to $\mathcal{G}' := E_{\widetilde{\mathcal{G}}_{(\overline{\triangle},\mathcal{B})}} \cap E_{\triangle}$. So it is enough to consider a monomial $xy = x_l x_m$, such that $\langle x_l, x_m\rangle$ belongs to $\mathcal{G}'$. In this case $C(x) \cap C(y) = \emptyset$ and the edge $\langle x, y\rangle$ belongs to $\mathcal{G}'$. We have

$$xy = g_x y - \sum_{z\in C(x), z\neq x} zy,$$

that is

$$xy \equiv - \sum_{z\in C(x), zy\notin \mathcal{B}_{\overline{\triangle}}} zy.$$

Let remark that the condition $zy \notin \mathcal{B}_{\overline{\triangle}}$ implies that y, z belongs to the same facet and are distinct since they have distinct colors. Taking care of all solved cases we can assume that $z = x_s^{l'}, y = x_t^{l'}$ belongs to some facet $F_{l'}$ and the edge $\langle z, y\rangle$ belongs to $\mathcal{G}'$. By applying again the same argument we have:

$$xy \equiv \sum_{z\in C(x), zy\notin \mathcal{B}_{\overline{\triangle}}} \sum_{w\in C(y), zw\notin \mathcal{B}_{\overline{\triangle}}} zw.$$

If it rests some monomials in the sum, we redo the algorithm. We will prove that the algorithm stops after a finite number of steps. Assume the opposite that the algorithm will never stop. Then, there exists an infinite chain of edges $\langle x, y\rangle$, $\langle y, z\rangle$, $\langle z, y_1\rangle$, $\langle y_1, z_1\rangle$, $\ldots$. Since the number of the variables is finite, we must have a cycle in this chain, i.e. we have a cycle in $\mathcal{G}'$. Moreover, each point of this cycle is colored by either the color of x or the one of y (here, we have $\mathcal{C}(x) \neq \mathcal{C}(y)$ because x and y form an edge of $\mathcal{G}'$. This is a contradiction to the fact that $\mathcal{G}'$ admits a good coloration.

Hence, the algorithm will stop, i.e.:

$$xy \in (g_1,\ g_2, \ldots,\ g_{d+1})\mathfrak{m}_{\overline{\triangle}} + \mathcal{B}_{\overline{\triangle}}.$$

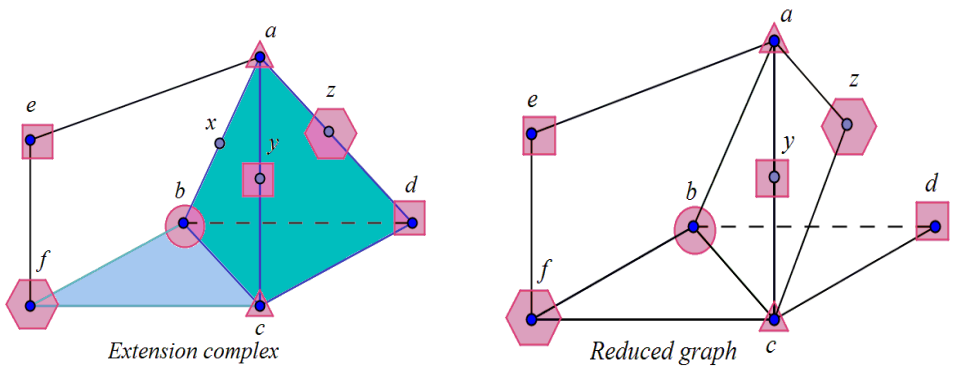
B. To finish the proof, we consider the case $x = y$.

If x is a vertex colored, then by replacing x by g_x, one has

$$x^2 = xg_x - \sum_{z\in \mathcal{C}(x), z\neq x} xz.$$

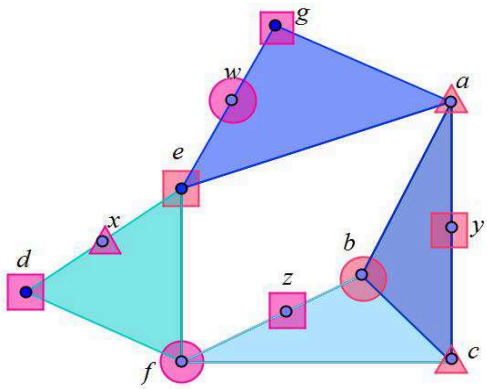
We have that z and x have the same color, but they are distinct, so $xz \equiv 0$ by the case A). Hence, $x^2 \equiv 0$.
If x is not colored, then x appears in some scroll matrix and we have a binomial $x^2 - yz \in \mathcal{B}_{\overline{\triangle}}$, with $y \neq z$, so $x^2 \equiv yz \equiv 0$, by the above cases. The proposition is proved. □

Example 2.17. Our Proposition applies to the example 2.11 colored as follows.



Example 2.18. Our Proposition applies to the following complex 2.18 colored as showed, and extended by the scroll matrices:

$$M_1 := \left(\begin{array}{c|c} a & y \\ b & c \end{array} \right), M_2 := \left(\begin{array}{c|c} f & z \\ c & b \end{array} \right), M_3 := \left(\begin{array}{c|c} d & x \\ f & e \end{array} \right), M_4 := \left(\begin{array}{c|c} g & w \\ a & e \end{array} \right)$$



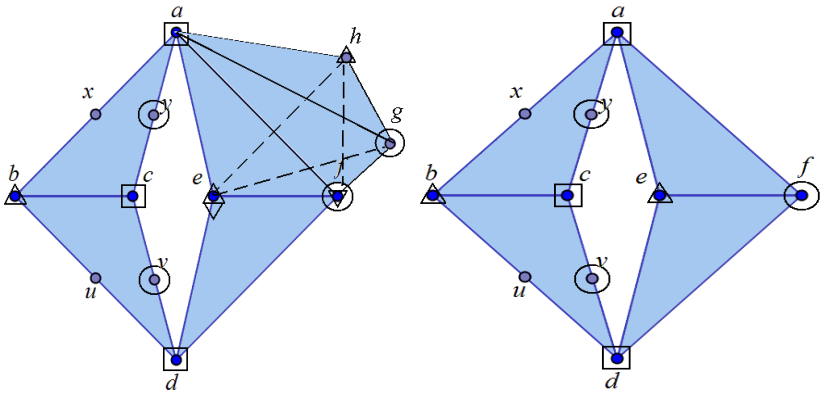
Example 2.19. Our Proposition applies to the complex represented by the picture in the left, but our Proposition cannot be applied to the complex represented by the picture in the right, in this case we have that the degree of the projective variety defined by this extended complex is 8, the codimension is 7. On the other hand by a computation we can check that the ideal $(a + c + d, b + e, f + v + y)$ is a reduction with reduction number two. The complex is extended by the following matrices:

$$M_1 := \left(\begin{array}{cc|c} a & x & y \\ x & b & c \end{array} \right)$$

associated to the facet $[\ a,\ b,\ c]$ extended by $x,\ y$;

$$M_2 := \left(\begin{array}{cc|c} d & u & v \\ u & b & c \end{array} \right)$$

associated to the facet $[\ b,\ c,\ d]$ extended by $u,\ v$.



If $\mathcal{G}_\triangle$ is a generalized d–tree, then in [BM1] it was constructed an explicit reduction for the quotient by the Stanley-Reisner ideal associated to $\triangle$. The aim of the following proposition is to prove that the reduction number of $\mathcal{R}/\mathcal{B}_{\overline{\triangle}}$ is 1, and to give an explicit expression of the reduction.

Proposition 2.20. *Let $\mathcal{G}_\triangle$ be a generalized d–tree. Then $\widetilde{\mathcal{G}}_{(\overline{\triangle},\mathcal{B})}$ admits a binomial–coloration. Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{d+1}$ denote the classes of colors. Put*

$$g_i = \sum_{x \in \mathcal{C}_i} x.$$

Then, we have:

$$(g_1,\ g_2, \ldots,\ g_{d+1})\mathfrak{m}_{\overline{\triangle}} + \mathcal{B}_{\overline{\triangle}} = \mathfrak{m}^2_{\overline{\triangle}},$$

where $\mathfrak{m}_{\overline{\triangle}} = (\mathbf{x}, \mathbf{y})$ is the irrelevant maximal ideal of the polynomial ring $\mathcal{R} := \mathcal{K}[\mathbf{x}, \mathbf{y}]$.

Proof: The proof will be by induction on the number λ of facets of $\overline{\triangle}$.

The case $\lambda = 1$ is a particular case of the Proposition 2.16. See also [BM2].

Assume that the proposition is true for $\lambda \geq 1$, we will prove it for $\lambda + 1$. Since $\mathcal{G}_\triangle$ is a generalized d–tree, one can find a facet F such that its associated vertex in $H(\mathcal{G}_\triangle)$ is a leaf. Consider $(\overline{\triangle'}, \mathcal{B}')$ the extension complex constructed by the λ facets different from F in $\overline{\triangle}$. We put $U = F \cap V_{\overline{\triangle'}}$. The graph $\mathcal{G}_{\triangle'}$ is also a generalized d–tree. By induction, the graph $\widetilde{\mathcal{G}}_{(\overline{\triangle'},\mathcal{B}')}$ admits a good $(d+1)$–coloration as in the proposition, and for all $xy \in \mathfrak{m}^2_{\overline{\triangle'}}$, we have:

$$xy \in (g'_1,\ g'_2, \ldots,\ g'_{d+1})\mathfrak{m}_{\overline{\triangle'}} + \mathcal{B}_{\overline{\triangle'}},$$

where $\mathcal{C}'_i$, g'_i are the ith class of color and the correspondent sum. We have two cases:

I) $F \cap V_\triangle = F$.

II) $F \cap V_\triangle \neq F$.

For each case, we will color the points in $F^\circ \cap V_{\widetilde{\mathcal{G}}_{(\overline{\triangle},\mathcal{B})}}$, and we will define the sums g_i.

I): $F \cap V_\triangle = F$. We can suppose that $F = \{x_0, x_1,\ x_2, \ldots,\ x_n\}$, and $U = \{x_1,\ x_2, \ldots,\ x_m\}$ $(1 \leq m \leq n \leq d)$, and $x_i \in \mathcal{C}'_i$ for all $i = \overline{1, m}$. To obtain $(d+1)$–coloration which verifies the proposition, it is sufficient to color the points $x_i \notin U$ by arbitrary colors $\mathcal{C}'_j$ with

$j > m$. For example, we can color x_i by the ith color ($\forall i = \overline{m+1,n}$), and x_0 by the $(d+1)^{th}$ color. Hence, we have:

$$\mathcal{C}_i = \mathcal{C}'_i \text{ and } g_i = g'_i \text{ for all } i = \overline{1,m} \text{ or } i = \overline{n+1,d},$$
$$\mathcal{C}_j = \mathcal{C}'_j \cup \{x_j\} \text{ and } g_j = g'_j + x_j, j = \overline{m+1,n},$$
$$\mathcal{C}_{d+1} = \mathcal{C}'_{d+1} \cup \{x_0\} \text{ and } g_{d+1} = g'_{d+1} + x_0.$$

II): $F \cap V_\triangle \neq F$. Let M be the matrix associated to F :

$$M := \left(\begin{array}{cccc|ccc|c|ccc} x_0 & y_{1,1} & \dots & y_{1,j_1} & y_{2,1} & \dots & y_{2,j_2} & \dots & y_{l,1} & \dots & y_{k,j_k} \\ y_{11} & y_{1,2} & \dots & x_1 & y_{2,2} & \dots & x_2 & \dots & y_{k,2} & \dots & x_k \end{array} \right).$$

We can assume that $F \cap V_\triangle = \{x_0, x_1, x_2, \dots, x_l\}$ $(1 \leq k \leq l \leq d)$. Let remark that since by construction the propers edges of F are not in $\overline{\triangle'}$, we have either $x_0 \in F^\circ$, or $x_i \in F^\circ$ $\forall i = \overline{1,k}$. So we have to consider two sub-cases:

II-1) $x_0 \in F^\circ$, i.e. $x_0 \notin U$: In order to color $\widetilde{\mathcal{G}}_{(\overline{\triangle},\mathcal{B})}$, we color each point $x \in V_\triangle \setminus \{U, x_0\}$ by a color not used in U, and we define :

- the color of x_0 is the same of x_2 ;
- the color of $y_{(j-1)1}$ is the color of x_j for all $j = \overline{3,k}$.
- the color of $y_{k,1}$ is the $(l+1)^{\text{i-th}}$ color not used in F.

We can renumbering the classes of colors in such a way that $x_i \in \mathcal{C}'_i$ for $i = \overline{1,l}$. Then, we have

$$\begin{array}{rcl} g_2 & = & \begin{cases} g'_2 + x_0, \text{ if } x_2 \in U, \\ g'_2 + x_2 + x_0, \text{ if } x_2 \notin U; \end{cases} \\ g_j & = & \begin{cases} g'_j + y_{(j-1)1} \text{ for } j = \overline{3,l} \text{ such that } x_j \in U, \\ g'_j + x_j + y_{(j-1)1} \text{ for } j = \overline{3,l} \text{ such that } x_j \notin U, \\ g'_j + x_j \text{ for } j \in \{1, l+1, ..., k\}, \text{ such that } x_n \notin U; \end{cases} \\ g_{l+1} & = & g'_{l+1} + y_{k1}; \\ g_j & = & g'_j \text{ for all other indices } j \text{ ;} \end{array}$$

II-2) If $x_0 \notin F^\circ$: In this case $x_i \in F^\circ$ for all $i = \overline{1,k}$. We can suppose that $U = \{x_0, x_s, \dots, x_l\}$ with $k < s \leq l$, and that $x_j \in \mathcal{C}_j$ for all $j = \overline{s,l}$, and $x_0 \in \mathcal{C}_2$. We put:

$$\begin{array}{rcl} \mathcal{C}_1 & = & \mathcal{C}'_1 \cup \{x_1\}; \\ \mathcal{C}_2 & = & \mathcal{C}'_2 \cup \{x_2\}; \\ \mathcal{C}_i & = & \mathcal{C}'_i \cup \{x_i, y_{(i-1)1}\} \text{ for all } i = \overline{3,k}; \\ \mathcal{C}_t & = & \mathcal{C}'_t \cup \{x_t\} \text{ for all } t = \overline{k+1, s-1}; \\ \mathcal{C}_j & = & \mathcal{C}'_j \text{ for all } j \geq s; \\ \mathcal{C}_{l+1} & = & \mathcal{C}'_{l+1} \cup \{y_{k,1}\}. \end{array}$$

Then we have:

$$\begin{array}{rcl} g_1 & = & g'_1 + x_1; \\ g_2 & = & g'_2 + x_2; \\ g_i & = & g'_i + x_i + y_{(i-1)1} \text{ for all } i = \overline{3,k}; \\ g_t & = & g'_t + x_t \text{ for all } t = \overline{l+1, s-1}; \\ g_j & = & g'_j \text{ for all } j \geq s \text{ and } j \neq k+1; \\ g_{l+1} & = & g'_{l+1} + y_{k1}. \end{array}$$

Let us remark that in all the cases, for all j the support of $g_j - g'_j$ is contained in F°.

A) First we will prove that $xy \equiv 0$ for any $x \in F^\circ, x \neq y$ and $y \in F$.

• Case I) We have that $\mathcal{C}(y) \cap F$ contains only y, so by applying the Remark 2.15, we have $xy \equiv 0$.

• Case II-1) since $x_0 \in F^\circ$, then by the Proposition 2.14 $xy \equiv 0$ for any variables $x \neq y \in F$, excepts for the products xy, for $x, y \in F \cap \triangle \setminus \{x_0\}$. Since we are interested in the monomials $x \in F^\circ$, and $y \in F \cap \triangle$, we have to consider the following cases:

II-1-a) $x \in F^\circ$, and $x = x_u, y = x_v$ appear in M, $u, v \neq 0$. By applying the Remark 2.15, since $x_u \in F^\circ$, we have either

$$x_u x_v \equiv \begin{cases} 0 & \text{if } v = 1, \\ x_u x_0 \equiv 0 & \text{if } v = 2, \\ x_u y_{(v-1)1} \equiv 0 & \text{if } v > 2. \end{cases}$$

II-1-b) $x \in F^\circ$. If x appears in M, and y doesn't appears in M, since $\mathcal{C}(y) \cap F = \{y\}$, we get by the Remark 2.15,that $xy \equiv 0$.

II-1-c) $x \in F^\circ$. If x doesn't appears in M and y appears in M, since $\mathcal{C}(x) \cap F = \{x\}$, then due to the Remark 2.15, it is sufficient to check the case where $y = x_v$ with $2 \leq v \leq k$. But in this case, one has also that either $xx_v \equiv xy_{(v-1)1} \equiv 0$ or $xx_v \equiv xx_0 \equiv 0$.

II-1-d) $x \in F^\circ$, both x, y don't appear in M: One has $\mathcal{C}(y) \cap F = \{y\}$, so $xy \equiv 0$ by to the Remark 2.15.

• Case II-2) First we prove that $x_0 y_{n,1} \equiv 0$, for any $n \geq 2$. By the Remark 2.15, we have $x_0 y_{n,1} \equiv x_2 y_{n,1}$. If $n = k$ since $x_2 \in F^\circ$, and $\widetilde{\mathcal{C}}(y_{k,1}) \cap F = \{y_{k,1}\}$, Remark 2.15 we have $x_2 y_{k,1} \equiv 0$. If $n < k$ then $x_2 y_{n,1} \equiv x_2 x_{n+1}$, but $x_{n+1} \in F^\circ$ so $x_2 x_{n+1} \equiv x_0 x_{n+1}$. By using the binomial relations in the matrix M, we will have that

$$x_0 x_{n+1} \equiv \begin{cases} x_1 q & q \in Y_{n+1} \\ p y_{n+1,1} & p \in Y_1. \end{cases}$$

Now by the Remark 2.15 $x_1 q \equiv 0$, $p y_{n+1,1} \equiv 0$ if $n+1 = 0$, and $p y_{n+1,1} \equiv p x_{n+1,1}$ if $n+1 < k$. by applying the binomial relations in the matrix and the Remark 2.15 after a finite number of steps we will have $x_0 y_{n,1} \equiv 0$.

By using the Proposition 2.14, we have $xy \equiv 0$ for any variables $x \neq y \in F_l$, excepts for the products $x_0 x_1, y_{1,u} y_{1,v}$, $x_0 y$, for $y \in F$ but not appearing in M, and xy for $x, y \in F_l \cap \triangle \setminus \{x_0\}$. So we need to consider the following cases:

1. $x_0 x_1 \equiv -x_2 x_1 \equiv -x_2 g x_1 \equiv 0$. This case also will imply that
2. $x_0 x_m$, $x_m \in F^\circ$ doesn't appears in M. $x_0 x_m \equiv -x_2 x_m \equiv -x_2 g x_m \equiv 0$.
3. $x_m x_n, m, n > 0$, $x_m \in F^\circ$, both x_m, x_n appear in M, this implies $x_n \in F^\circ$. We can assume that $n \geq 2$, so $x_m x_n \equiv x_m y_{n-1,1}$ if $n > 2$, or $x_m x_n \equiv x_m x_0$, both monomial are equivalent to 0.
4. $x_m x_n, m, n > 0$, $x_m \in F^\circ$, x_m appears in M but x_n doesn't appears in M. Then $x_m x_n \equiv -x_m g x_n \equiv 0$.

5. $x_m x_n, m, n > 0$, $x_m \in F^\circ$, x_m doesn't appears in M but x_n appears in M. Then $x_m x_n \equiv -g x_m x_n \equiv 0$.
6. $x_m x_n, m, n > 0$, $x_m \in F^\circ$, both x_m, x_n don't appear in M. Then $x_m x_n \equiv -x_m g x_n \equiv 0$.

B) • $x, y \in V_{\overline{\triangle'}}$, By induction, one has $xy = \sum_{i=1}^{d+1} m_i g_i' \mod \mathcal{B}_{\overline{\triangle'}}$ with $m_i \in \mathfrak{m}_{\overline{\triangle'}}$. But

$$\sum_{i=1}^{d+1} m_i g_i' = \sum_{i=1}^{d+1} m_i g_i - \sum_{i=1}^{d+1} m_i (g_i - g_i').$$

Since the support of $g_i - g_i'$ is in F° and $\mathrm{Supp}(g_i - g_i') \cap \mathrm{Supp}(m_i) = \emptyset$, due to the precedent cases

$$m_i(g_i - g_i') \in (g_1,\ g_2, \ldots,\ g_{d+1})\mathfrak{m}_{\overline{\triangle}} + \mathcal{B}_{\overline{\triangle}}.$$

It implies that xy verifies $(*)$.

C) • $x = y \in F^\circ$, In this case, if in addition x is not colored, modulo $\mathcal{B}_{\overline{\triangle}}$ (see Lemma 2.12), we re–obtain one of cases above. If x is colored, we replace x by the sum g_x of all variables in the class of color of x, we will be in the case $x \neq y$. Hence the proposition is proved. □

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