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Tree expansions in time-dependent perturbation theory

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The computational complexity of time-dependent perturbation theory is well-known to be largely combinatorial whatever the chosen expansion method and family of parameters (combinatorial sequences, Goldstone and other Feynman-type diagrams...). We show that a very efficient perturbative expansion, both for theoretical and numerical purposes, can be obtained through an original parametrization by trees and generalized iterated integrals. We emphasize above all the simplicity and naturality of the new approach that links perturbation theory with classical and recent results in enumerative and algebraic combinatorics. These tools are applied to the adiabatic approximation and the effective Hamiltonian. We prove perturbatively and non-perturbatively the convergence of Morita's generalization of the Gell-Mann and Low wavefunction. We show that summing all the terms associated to the same tree leads to an utter simplification where the sum is simpler than any of its terms. Finally, we recover the time-independent equation for the wave operator and we give an explicit non-recursive expression for the term corresponding to an arbitrary tree.

I. INTRODUCTION

Effective Hamiltonians provide a way to determine the low-energy eigenvalues of a (possibly infinite dimensional) Hamiltonian by diagonalizing a matrix defined in a subspace of small dimension, called the *model space* and hereafter denoted by M . Because of this appealing feature, effective Hamiltonians are used in nuclear, atomic, molecular, chemical and solid-state physics¹.

These theories are plagued with a tremendous combinatorial complexity because of the presence of folded diagrams (to avoid singularities of the adiabatic limit), partial resummations, subtle “linkedness” properties and the exponential growth of the number of graphs with the order of perturbation. This complexity has two consequences: on the one hand, few results are proved in the mathematical sense of the word, on the other hand, it is difficult to see what is the underlying structure of the perturbative expansion that could lead to useful resummations and non-perturbative approximations.

To avoid these pitfalls, we take a bird's-eye view of the problem and consider a general time-dependent Hamiltonian $H(t)$. This way, we disentangle the problem from the various particular forms that can be given to the Hamiltonian and which have lead in the past to various perturbative expansions. To be precise, take the example of fermions in molecular systems. The Coulomb interaction between the electrons (say V) can be viewed as a perturbation of a “free Hamiltonian” modeling the interaction with the nuclei (in the Born-Oppenheimer approximation). One can take advantage of the particular form of V (which is a linear combination of products of two creation and two annihilation operators in the second quantization picture) to represent the perturbative expansions using a given family of Feynman diagrams (see e.g. ref. 2 for such a family). However, the general results on perturbative expansions (such as the convergence of the time-dependent wave operator) do not depend on such a particular choice.

Thus, we consider an Hamiltonian $H(t)$ and we build its evolution operator $U(t, t_0)$, which is the solution of the Schrödinger equation

$$i\frac{\partial U(t, t_0)}{\partial t} = H(t)U(t, t_0), \quad (1)$$

with the boundary condition $U(t_0, t_0) = 1$. In perturbation theory, $H(t) = e^{-\epsilon|t|}e^{iH_0t}Ve^{-iH_0t}$ is the interaction Hamiltonian in the interaction picture (here H_0 and V stand respectively for the “free” and interaction terms of the initial Hamiltonian) and singularities show up in the adiabatic limit ($t_0 \rightarrow -\infty$ and $\epsilon \rightarrow 0$). Morita discovered² that, in this setting, the *time-dependent wave operator*

$$\Omega(t, t_0) = U(t, t_0)P(PU(t, t_0)P)^{-1},$$

where P is the projection onto the model space M , has no singularity in the adiabatic limit. Moreover, the wave

operator determines the effective Hamiltonian because

$$H_{\text{eff}} = \lim_{\epsilon \rightarrow 0} PH\Omega(0, -\infty). \quad (2)$$

However, as we have already alluded to, the effective computation of these operators raises several combinatorial and analytical problems that have been addressed in a long series of articles (several of which will be referred to in the present article).

In the first sections of the paper, we consider a general time-dependent Hamiltonian $H(t)$ (not necessarily in the interaction picture). In this broader setting, Jolicard³ found that the time-dependent wave operator provides also a powerful description of the evolution of quantum systems (see ref. 4 for applications). Then, we derive three (rigorously proven) series expansions of the wave operator. The first one is classical and can be physically interpreted as the replacement of causality (i.e. the Heaviside step function $\theta(t-t')$) by a “propagator” $\theta_P(t-t') = \theta(t-t')Q - \theta(t'-t)P$, where $Q = 1 - P$. This “propagator” is causal out of the model space ($\theta_P(t-t') = 0$ for $t < t'$ on the image of Q) and anticausal on it, like the Feynman propagator of quantum field theory⁵. However, this sum of causal and anticausal orderings is cumbersome to use in practice. A second series expansion is obtained by writing the wave operator as a sum of integrals over all possible time orderings of the Hamiltonians $H(t_i)$ (see Sect. IIB). This expansion, parametrized by all the permutations (or equivalent families), is used in many-body theory and gives rise to a large number of complicated terms. The third expansion is obtained by noticing that some time orderings can be added to give simpler expressions. This series is naturally indexed by trees and is the main new tool developed in the present paper. Among others, we derive a very simple recurrence relation for the terms of the series. We also show that the very structure of the corresponding generalized iterated integrals showing up in the expansion is interesting on its own. These integrals carry naturally a rich algebraic structure that is connected to several recent results in the field of combinatorial Hopf algebras and noncommutative symmetric functions. The corresponding algebraic results that point out in the direction of the existence of a specific Lie theory for effective Hamiltonians (generalizing the usual Lie theory) are gathered in an Appendix.

In the last sections of the paper, we restrict $H(t)$ to the interaction picture and we consider the adiabatic limit. We first prove that the adiabatic limit exists non perturbatively. We show that the effective Hamiltonian defined by eq. (2) has the expected properties. Then, we expand the series and we give a rigorous (but lengthy) proof that the term corresponding to each time ordering has an adiabatic limit. Then, we consider the series indexed by trees and we give a short and easy proof of the existence of that limit. Finally, we provide a direct rule to calculate the term corresponding to a given tree and establish the connection between the time-dependent approach and the time-independent equations discovered by Lindgren⁶ and Kvasnička⁷.

The existence of this series indexed by trees can be useful in many ways: (i) It describes a sort of superstructure that is common to all many-body theories without knowing the exact form of the interaction Hamiltonian; (ii) It considerably simplifies the manipulation of the general term of the series by providing a powerful recurrence relation; (iii) It provides simple algorithms to calculate the terms of the series; (iv) The number of trees of order n , $\frac{1}{n+1}\binom{2n}{n} \approx \frac{4^n}{n^{\frac{3}{2}}\sqrt{\pi}}$ being subexponential, it improves the convergence of the series^{8,9}; (v) It can deal with problems where the Hamiltonian H_0 is not quadratic. Indeed, many-body theories most often require the Hamiltonian H_0 to be free, i.e. to be a quadratic function of the fields¹⁰. As noticed by Bulaevskii¹¹, this is not the good point of view for some applications. For example, in the microscopic theory of the interaction of radiation with matter, it is natural to take for H_0 the Hamiltonian describing electrons and nuclei in Coulomb interaction¹², the perturbation being the interaction with the transverse electric field. In that case, quadratic free Hamiltonians many-body theories break down whereas our approach is still valid. Actually, it is precisely for that reason that we originally developed the tree series approach; (vi) Last, but not least, the tree-theoretical approach connects many-body theories with a large field of knowledge that originates in the “birth of modern bijective combinatorics” in the seventies with in particular the seminal works of Foata, Schützenberger and Viennot^{13,14}. See e.g. ref. 15 for a survey of the modern combinatorial theory of tree-like structures.

From the physical point of view, the tree expansion is particularly interesting in the adiabatic limit. Indeed, the denominator of each of its terms is a product of $E_i^Q - E_j^P$ factors, where E_j^P is the energy of a state in the model space M and E_j^Q the energy of a state not belonging to M . In the usual many-body expansions, the denominators are products of $\sum_i E_i^Q - \sum_j E_j^P$ factors, where the sums contain various numbers of elements (corresponding therefore to multiple transitions between low-energy and excited levels). In that sense, the tree expansion is the simplest possible because each term is a product of single transitions between two states.

II. TIME-DEPENDENT HAMILTONIAN AND COMBINATORICS

We consider a time-dependent Hamiltonian $H(t)$ (i.e. a self-adjoint operator on a Hilbert space \mathcal{H}) and the evolution operator $U(t, t_0)$ defined in eq. (1). Since we are interested in the combinatorial aspects of the problem, we consider the simple case where $H(t)$ is a strongly continuous map from \mathbb{R} into the bounded self-adjoint operators on \mathcal{H} . In that case, the Picard-Dyson series

$$1 + \sum_{n=1}^{\infty} (-i)^n \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n H(t_1) \dots H(t_n),$$

converges in the uniform operator topology to $U(t, t_0)$ and $U(t, t_0)$ is a jointly strongly continuous two-parameter family of unitaries on \mathcal{H} (see section X.12 of ref. 16).

Following Morita², Jolicard³ established a connection between the evolution operator and the effective Hamiltonian approach by defining

$$\Omega(t, t_0) = U(t, t_0) P (P U(t, t_0) P)^{-1},$$

where P is a projection operator onto M and $(P U(t, t_0) P)^{-1}$ is the inverse of $P U(t, t_0) P$ as a map from $M = P\mathcal{H}$ to itself. The condition of invertibility of $P U(t, t_0) P$ is discussed in ref. 17. We assume from now on that the condition is satisfied and we define three expansions for $\Omega(t, t_0)$.

A. First expression for Ω

We start by proving an elegant expression for Ω , that was stated by Michels and Suttorp¹⁸ and Dmitriev and Solnyshkina¹⁹.

Theorem 1

$$\Omega(t, t_0) = P + Q \sum_{n=1}^{\infty} (-i)^n \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n H(t_1) \theta_P(t_1 - t_2) H(t_2) \dots \theta_P(t_{n-1} - t_n) H(t_n) P, \quad (3)$$

where $Q = 1 - P$ and $\theta_P(t) = \theta(t)Q - \theta(-t)P$, with θ the Heaviside step function.

Proof. We first rewrite the Picard-Dyson series as $U(t, t_0) = 1 + \sum_n U_n(t, t_0)$ with $U_1(t, t_0) = -i \int_{t_0}^t dt_1 H(t_1)$ and, for $n > 1$,

$$U_n(t, t_0) = (-i)^n \int_{t_0}^t dt_1 \dots \int_{t_0}^{t_{n-1}} dt_n H(t_1) \theta(t_1 - t_2) \dots \theta(t_{n-1} - t_n) H(t_n).$$

Then, by using $\theta(t) + \theta(-t) = 1$, we notice that

$$\theta(t) = P + \theta(t) - \theta(t)P - \theta(-t)P = P + \theta_P(t).$$

Now, we replace $\theta(t)$ by the sum of operators $P + \theta_P(t)$ in the expression for $U_n(t, t_0)$. This gives us 2^{n-1} terms with various numbers of P and θ_P . Denote by $C_n(t, t_0)$ the term with no P (with the particular case $C_1(t, t_0) = U_1(t, t_0)$). Take then any other term. There is an index i such that the first P from the left occurs after $H(t_i)$. Therefore, the integrand of this term is

$$H(t_1) \theta_P(t_1 - t_2) \dots \theta_P(t_{i-1} - t_i) H(t_i) P H(t_{i+1}) \dots$$

Observe that the integral over t_1, \dots, t_i is independent from the integral over t_{i+1}, \dots, t_n . The first integral gives $C_i(t, t_0)$, the second integral is a term of the Picard-Dyson series for $U_{n-i}(t, t_0)$. Thus, the sum of the 2^{n-1} terms yields

$$U_n(t, t_0) = C_n(t, t_0) + \sum_{i=1}^{n-1} C_i(t, t_0) P U_{n-i}(t, t_0).$$

If we denote by $K(t, t_0)$ the sum of all the $C_n(t, t_0)$ with $n > 0$, we obtain $U = 1 + K + KP(U - 1)$, so that

$$UP = P + KPUP. \quad (4)$$

The operator K is called the *reduced evolution operator* by Lindgren and collaborators²⁰. If we define $\omega = P + QKP$, eq. (4) becomes

$$UP = P + (\omega - P + PKP)PUP = P + \omega PUP - PUP + PKPUP.$$

This equation can be simplified by using eq. (4) again

$$UP = P + \omega PUP - PP = \omega PUP.$$

Thus, $\omega = \Omega$ and eq. (3) is satisfied. □

Despite its elegance, eq. (3) is not immediately usable. To illustrate this point, consider the third-order term

$$\Omega_3 = iQ \int_{t_0}^t dt_1 \int_{t_0}^t dt_2 \int_{t_0}^t dt_3 H(t_1) \theta_P(t_1 - t_2) H(t_2) \theta_P(t_2 - t_3) H(t_3) P.$$

If we expand $\theta_P(t) = \theta(t)Q - \theta(-t)P$, we obtain four terms

$$\begin{aligned} & iQH(t_1)QH(t_2)QH(t_3)P \quad \text{for } t_1 \geq t_2 \text{ and } t_2 \geq t_3, \\ & -iQH(t_1)QH(t_2)PH(t_3)P \quad \text{for } t_1 \geq t_2 \text{ and } t_2 \leq t_3, \\ & -iQH(t_1)PH(t_2)QH(t_3)P \quad \text{for } t_1 \leq t_2 \text{ and } t_2 \geq t_3, \\ & iQH(t_1)PH(t_2)PH(t_3)P \quad \text{for } t_1 \leq t_2 \text{ and } t_2 \leq t_3. \end{aligned}$$

The first and last terms have integration range $t_1 \geq t_2 \geq t_3$ and $t_3 \geq t_2 \geq t_1$, respectively and give rise to iterated integrals. The integration range of the second term is $t_1 \geq t_2$ and $t_2 \leq t_3$. Such an integration range is not convenient because the relative position of t_1 and t_3 is not specified. The integration range has to be split into the two subranges $t_1 \geq t_3 \geq t_2$ and $t_3 \geq t_1 \geq t_2$. Each subrange defines now an iterated integral. For example $t_1 \geq t_3 \geq t_2$ gives

$$-i \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_3 \int_{t_0}^{t_3} dt_2 QH(t_1)QH(t_2)PH(t_3)P.$$

Similarly, the integration range of the third term ($t_1 \leq t_2$ and $t_2 \geq t_3$) is the union of $t_2 \geq t_1 \geq t_3$ and $t_2 \geq t_3 \geq t_1$. We see that Ω_3 is sum of six iterated integrals corresponding to the six possible orderings of t_1, t_2 and t_3 .

B. Ω in terms of permutations

We consider again the previous example, and we change variables to have a fixed integration range $s_1 \geq s_2 \geq s_3$. If we sum over all time orderings, we obtain

$$\begin{aligned} \Omega_3 = & i \int_{t_0}^t ds_1 \int_{t_0}^{s_1} ds_2 \int_{t_0}^{s_2} ds_3 \left(QH(s_1)QH(s_2)QH(s_3)P - QH(s_1)QH(s_3)PH(s_2)P - QH(s_2)QH(s_3)PH(s_1)P \right. \\ & \left. - QH(s_2)PH(s_1)QH(s_3)P - QH(s_3)PH(s_1)QH(s_2)P + QH(s_3)PH(s_2)PH(s_1)P \right). \end{aligned} \quad (5)$$

In ref. 21, we showed that this result can be generalized to all orders and that Ω_n is a sum of $n!$ iterated integrals corresponding to all the orderings of t_1, \dots, t_n . More precisely, we obtained the series expansion for Ω

$$\Omega(t, t_0) = P + \sum_{n=1}^{\infty} \sum_{\sigma \in \mathcal{S}_n} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n X(t_{\sigma(1)}) \dots X(t_{\sigma(n)}), \quad (6)$$

where \mathcal{S}_n is the group of permutations of n elements. The operators X are defined, for $n = 1$, by $X(t) = -iQH(t)P$ and, for $n > 1$ and any $\sigma \in \mathcal{S}_n$, by $X(t_{\sigma(1)}) = -iQH(t_{\sigma(1)})$ and

$$\begin{aligned} X(t_{\sigma(p)}) &= -iQH(t_{\sigma(p)}) \text{ if } 1 < p < n \text{ and } \sigma(p) > \sigma(p-1), \\ X(t_{\sigma(p)}) &= iPH(t_{\sigma(p)}) \text{ if } 1 < p < n \text{ and } \sigma(p) < \sigma(p-1), \\ X(t_{\sigma(n)}) &= -iQH(t_{\sigma(n)})P \text{ if } \sigma(n) > \sigma(n-1), \\ X(t_{\sigma(n)}) &= iPH(t_{\sigma(n)})P \text{ if } \sigma(n) < \sigma(n-1). \end{aligned}$$

Each term of eq. (6) is now written as an iterated integral. However, the expansion (6) is still not optimal because some of its terms can be summed to get a simpler expression. As an example, consider the fourth and fifth terms of eq. (5), where we replace s_i by t_i .

$$Z = -\imath \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \left(QH(t_2)PH(t_1)QH(t_3)P + QH(t_3)PH(t_1)QH(t_2)P \right).$$

The first and second terms of the right hand side of this equation are denoted by Z_1 and Z_2 , respectively. We transform Z_2 by exchanging variables t_2 and t_3 .

$$Z_2 = -\imath \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_3 \int_{t_0}^{t_3} dt_2 QH(t_2)PH(t_1)QH(t_3)P = -\imath \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_2}^{t_1} dt_3 QH(t_2)PH(t_1)QH(t_3)P,$$

where we also exchanged the order of the integrations over t_3 and t_2 . This can be added to Z_1 and we obtain the simpler expression

$$Z = -\imath \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_1} dt_3 QH(t_2)PH(t_1)QH(t_3)P.$$

Such a simplification is not possible for the other terms of Ω_3 . In the next section, we determine how this simplification can be extended to the general term of Ω .

Before closing this section, we need to specify more precisely the relation between the permutations σ and the series of P and Q in the expansion of eq. (3). When we expand all the $\theta_P(t_i - t_{i+1})$ in eq. (3), we obtain an integrand of the form $(-\imath)^n QH(t_1)R_1H(t_2)\dots R_{n-1}H(t_n)P$, multiplied by a product of Heaviside functions, where R_i takes the value $-P$ or Q . We aim to determine the relation between the sequence $R_1\dots R_{n-1}$ and the permutations σ in eq. (6). From the definition of $X(t_{\sigma(i)})$, it appears that $R_i = -P$ if $\sigma(i) > \sigma(i+1)$ and $R_i = Q$ if $\sigma(i) < \sigma(i+1)$. The set of indices i such that $\sigma(i) > \sigma(i+1)$ is called the *descent set* of σ , denoted by D_σ . It is also called the *shape* of the permutation²². For instance, the descent set of the permutations (213) and (312) is $\{1\}$, corresponding to $(R_1, R_2) = (-P, Q)$.

C. Permutations and trees

In many-body physics, the kind of simplification that we encountered in the previous section was observed in several special cases^{1,18,23,24,25,26}. However, as far as we know, it was never stated in full generality. The first difficulty is to find the proper combinatorial object to represent the sets of permutations that lead to simplified sums. We shall find it after a short algebraic detour meant to recall the notion of tree and its relation to permutations^{13,22}. The trees we consider have received various names in the literature: binary trees in ref. 22, but also (to quote only a few occurrences) plane rooted complete binary tree²⁷, extended binary trees²⁸ or planar binary trees²⁹. Since they are the only trees we will consider in the present article, no confusion can arise and we shall simply call them *trees*.

For any integer n , the set of trees Y_n is defined recursively by $Y_0 = \{|\}$ and, for $n > 0$, $Y_n = \{T_1 \vee T_2 : T_1 \in Y_k, T_2 \in Y_{n-k-1}, k = 0, \dots, n-1\}$, where $T_1 \vee T_2$ is the grafting of the two trees T_1 and T_2 , by which the trunks of T_1 and T_2 are brought together and a new trunk is grown from their juncture. For example, $|\vee| = \vee$, $|\vee\vee = \vee\vee$, $\vee\vee\vee = \vee\vee\vee$. We have $Y_1 = \{\vee\}$, $Y_2 = \{\vee\vee, \vee\vee\}$, $Y_3 = \{\vee\vee\vee, \vee\vee\vee, \vee\vee\vee, \vee\vee\vee, \vee\vee\vee\}$.

If C_n denotes the number of elements of Y_n , the recursive definition of Y_n implies that $C_0 = 1$ and $C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$, so that $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the famous Catalan number. For $n=0$ to 10, $C_n=1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796$.

The Catalan numbers enumerate a large number of (sometimes quite different) combinatorial objects³⁰. There are also many ways to encode trees by sequences of integers³¹. The main practical interest of binary trees with respect to other combinatorial interpretations is that the recursive definition is very easy to implement. The lowest point of the trunk of a tree is a vertex called the *root*, its *inner vertices* are the vertices to which three edges are incident, its *leaves* are its vertices to which a single edge is incident and which are different from the root. Thus, each tree of Y_n has a single root, n inner vertices and $n+1$ leaves. The order $|T|$ of a tree T is the number of its inner vertices.

There are several ways to associate a tree to a permutation. Most of them are equivalent, in that they can be deduced from each other by applying some of the standard automorphisms of the symmetric groups (inversion, conjugation by a cycle of maximal length, and so on). A very convenient map from permutations to trees (probably the most classical one) belongs to “the ABC’s of classical enumeration”³² and is historically one of the founding tools

of modern bijective and algebraic combinatorics^{13,22,33}. Quite surprisingly, as we show below, the same map shows up very naturally when trying to regroup systematically the expansions of time-dependent perturbation theory. The key technical reason is certainly our algebro-analytic lemma 12. We will advocate in the Appendix (to which we refer for details and references) that combinatorial Hopf algebraic structures allow to deepen this understanding and form a “Lie theory” suited for the study of effective Hamiltonians, very much as the classical theory of free Lie algebras is the right framework to understand the usual expansions of linear differential equations.

1. From permutations to trees

We first map any n -tuple $I = (i_1, \dots, i_n)$ of distinct integers to a tree $T \in Y_n$. The mapping ϕ is defined recursively as follows: if $I = (i)$ contains a single integer, then $\phi(I) = \text{Y}$; otherwise, pick up the smallest element i_k of I ,

$$\begin{aligned}\phi(I) &= | \vee \phi((i_2, \dots, i_n)), \text{ if } k = 1, \\ \phi(I) &= \phi((i_1, \dots, i_{n-1})) \vee |, \text{ if } k = n, \\ \phi(I) &= \phi((i_1, \dots, i_{k-1})) \vee \phi((i_{k+1}, \dots, i_n)), \text{ otherwise.}\end{aligned}$$

In the following, we frequently abuse notation by writing $\phi(i_1, \dots, i_n)$ or even $\phi(i_1 \dots i_n)$ instead of $\phi((i_1, \dots, i_n))$. Then, for a permutation $\sigma \in \mathcal{S}_n$, the corresponding tree is $\phi(\sigma) = \phi(I)$, where $I = (\sigma(1), \dots, \sigma(n))$. For example

$$\begin{aligned}\phi(1) &= \text{Y}, \\ \phi(12) &= \text{Y} \text{Y}, \quad \phi(21) = \text{Y} \text{Y}, \\ \phi(123) &= \text{Y} \text{Y} \text{Y}, \quad \phi(132) = \text{Y} \text{Y} \text{Y}, \quad \phi(213) = \text{Y} \text{Y} \text{Y}, \\ \phi(231) &= \text{Y} \text{Y} \text{Y}, \quad \phi(312) = \text{Y} \text{Y} \text{Y}, \quad \phi(321) = \text{Y} \text{Y} \text{Y}.\end{aligned}$$

Note that the two permutations (213) and (312) correspond to the same tree, and they are also the two permutations that add up to a simple sum in the calculation of Z at the end of section II B. This is not a coincidence.

To simplify the proofs, we embrace the three cases of the definition of ϕ into a single one as follows. We first define the *concatenation product* of two tuples $I = (i_1, \dots, i_n)$ and $J = (j_1, \dots, j_m)$ as $I \cdot J = (i_1, \dots, i_n, j_1, \dots, j_m)$. We extend this definition to the case of the zero-tuple $I_0 = \emptyset$ by $I_0 \cdot I = I \cdot I_0 = I$. Then, for any n -tuple I of distinct integers, we define $\phi(I)$ by $\phi(I) = |$ if $n = 0$ and $\phi(I) = \phi(I_1) \vee \phi(I_2)$ otherwise, where I_1 and I_2 are determined by $I = I_1 \cdot (\min I) \cdot I_2$. Note that I_1 or I_2 can be the zero-tuple.

We show that two n -tuples $I = (i_1, \dots, i_n)$ and $J = (j_1, \dots, j_n)$ of distinct integers have the same image by ϕ when (but not only when) the elements of I and J have the same ordering, in other words when $i_k < i_l$ if and only if $j_k < j_l$ for all k and l in $\{1, \dots, n\}$. The proof is by induction. If $n = 0$, then $I = J = \emptyset$ and $\phi(I) = \phi(J) = |$. Assume that the property is true for k -tuples of distinct integers up to $k = n - 1$ and take two n -tuples I and J having the same ordering. Then, the minimum element of both is at the same position k (i.e. $\min I = i_k$ and $\min J = j_k$) and $I = I_1 \cdot (i_k) \cdot I_2$, $J = J_1 \cdot (j_k) \cdot J_2$, where I_1 and J_1 (I_2 and J_2 , respectively) are two $(k - 1)$ -tuples ($(n - k)$ -tuples, respectively) of distinct integers have the same ordering. By the recursion hypothesis, we have $\phi(I_1) = \phi(J_1)$ and $\phi(I_2) = \phi(J_2)$ and the definition of ϕ gives us $\phi(I) = \phi(I_1) \vee \phi(I_2) = \phi(J_1) \vee \phi(J_2) = \phi(J)$.

As a useful particular case, we consider the situation where J describes the ordering of I . More precisely, we order the elements of $I = (i_1, \dots, i_n)$ increasingly as $i_{k_1} < \dots < i_{k_n}$. Then j_l is the position of i_l in this ordering. More formally, if we define the permutation $\tau = (k_1, \dots, k_n)$, then $J = (\tau^{-1}(1), \dots, \tau^{-1}(n))$. The n -tuple J is called the *standardization* of I and it is denoted by $st(I)$. If we take the example of $I = (5, 8, 2)$, the position of 5, 8 and 2 in the ordering $2 < 5 < 8$ is 2, 3 and 1, respectively. Thus, $st(5, 8, 2) = (2, 3, 1)$. By construction, I and $st(I)$ have the same ordering and $\phi(I) = \phi(st(I))$. Of course, we extend the standardization to the case of $I = \emptyset$ by $st(\emptyset) = \emptyset$.

2. From trees to permutations

Conversely, we shall need to know the permutations corresponding to a given tree: $S_T = \{\sigma \in \mathcal{S}_{|T|} : \phi(\sigma) = T\}$ (we extend this definition to the case of $T = |$ by defining the zero-element permutation group $\mathcal{S}_0 = \{\emptyset\}$). The solution of this problem is known^{28,29} and we express it by using the standardization map defined in the previous section. The simplest examples are $S_T = \{\emptyset\}$ for $T = |$ and $S_T = \{(1)\}$ for $T = \text{Y}$. Now take $T = T_1 \vee T_2$, where $|T_1| = n$

and $|T_2| = m$ (n or m can be zero) and consider a permutation σ in S_T . By the definition of ϕ , the minimum of the tuple $I = (\sigma(1), \dots, \sigma(n+m+1))$ is $\sigma(n+1) = 1$ and $I = I_1 \cdot (1) \cdot I_2$, where $\phi(I_1) = T_1$ and $\phi(I_2) = T_2$. We saw in the previous section that $\phi(I_1) = \phi(st(I_1))$. By definition $st(I_1)$ is a permutation of S_n . Therefore, $st(I_1)$ belongs to S_{T_1} and, similarly, $st(I_2)$ belongs to S_{T_2} . This provides a recursive construction of S_T : pick up a permutation α in S_{T_1} and a permutation β in S_{T_2} . Pick up n elements in the set $\{2, \dots, n+m+1\}$ and build I_1 by ordering these elements according to α (i.e. so that $st(I_1) = \alpha$), then build I_2 by ordering the remaining elements according to β . Then, $I = I_1 \cdot (1) \cdot I_2$ defines a permutation of S_T . This construction allows us to recursively determine the number of elements of S_T , denoted by $|S_T|$, by $|S_T| = 1$ for $T = |$ and $T = \vee$ and, for $T = T_1 \vee T_2$,

$$|S_T| = \binom{|T|-1}{|T_1|} |S_{T_1}| |S_{T_2}|. \quad (7)$$

See ref. 28 for an alternative proof.

Example: Consider the tree $T = T_1 \vee T_2$, with $T_1 = \vee$ and $T_2 = \vee\vee$, so that $n = 1$ and $m = 3$. S_{T_1} contains the single permutation $\alpha = (1)$ and, according to the examples given in the previous section, the two permutations of S_{T_2} are $\beta_1 = (213)$ and $\beta_2 = (312)$. We choose the permutations α and β_1 , we pick up $n = 1$ element (for example 3) in the set $J = \{2, 3, 4, 5\}$, so that $I_1 = (3)$ and we order the remaining elements $\{2, 4, 5\}$ according to β_1 , so that $I_2 = (4, 2, 5)$. This gives us $\sigma = (31425)$. If we pick up the other elements of J to build I_1 we obtain (21435) , (41325) and (51324) . We add the elements obtained by choosing β_2 and we obtain eight permutations:

$$S_T = \{(21435), (21534), (31425), (31524), (41325), (41523), (51324), (51423)\}.$$

We can check that eq. (7) holds and that $|S_T| = 8$.

D. Recursion formula

The permutations corresponding to a tree can be used to make a partial summation of the terms of the Picard-Dyson expansion.

Definition 2 For any tree T , we define $\Omega_T(t, t_0)$ by $\Omega_T(t, t_0) = P$ if $T = |$ and

$$\Omega_T(t, t_0) = \sum_{\sigma \in S_T} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n X(t_{\sigma(1)}) \dots X(t_{\sigma(n)}),$$

otherwise, where $n = |T|$.

With this notation we have obviously $\Omega(t, t_0) = \sum_T \Omega_T(t, t_0) = \sum_{\sigma} \Omega_{\sigma}(t, t_0)$, with the notation $\Omega_{\sigma}(t, t_0) = \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n X(t_{\sigma(1)}) \dots X(t_{\sigma(n)})$ and where σ runs over all permutations (of all orders). Apart from $T = |$, the simplest term is

$$\Omega_T(t, t_0) = -i \int_{t_0}^t ds Q H(s) P,$$

for $T = \vee$. The other terms enjoy a remarkably simple recurrence relation:

Theorem 3 If $|T| > 1$, then $\Omega_T(t, t_0)$ can be expressed recursively by

$$\begin{aligned} \Omega_T(t, t_0) &= -i \int_{t_0}^t ds Q H(s) \Omega_{T_2}(s, t_0), \text{ if } T = | \vee T_2, \\ \Omega_T(t, t_0) &= i \int_{t_0}^t ds \Omega_{T_1}(s, t_0) H(s) P, \text{ if } T = T_1 \vee |, \\ \Omega_T(t, t_0) &= i \int_{t_0}^t ds \Omega_{T_1}(s, t_0) H(s) \Omega_{T_2}(s, t_0), \text{ if } T = T_1 \vee T_2, \end{aligned}$$

where $T_1 \neq |$ and $T_2 \neq |$.

A. Nonperturbative proof

The nonperturbative proof is important in this context because its range of validity is wider than the series expansion (no convergence criterion for the series is required). Moreover, the proof that the wave operator indeed leads to an effective Hamiltonian is much easier to give in the nonperturbative setting.

The first condition required in the nonperturbative setting is that the perturbation V must be relatively bounded with respect to H_0 with a bound strictly smaller than 1. This condition is satisfied for the Hamiltonian describing nuclei and electrons interacting through a Coulomb potential³⁸. Before stating the second condition, we define the time independent Hamiltonian $h(\lambda) = H_0 + \lambda V$, its eigenvalues $E_j(\lambda)$ and its eigenprojectors $P_j(\lambda)$, such that $h(\lambda)P_j(\lambda) = E_j(\lambda)P_j(\lambda)$. The second condition is that the eigenvalues $E_j(\lambda)$ coming from the eigenstates of the model space (i.e. such that $P_j(0)P = P_j(0)$) are separated by a finite gap from the rest of the spectrum. According to Kato³⁸, the eigenvalues and eigenprojectors can be chosen analytic in λ . Then, a recent version of the adiabatic theorem^{39,40} shows that there exists a unitary operator A , independent of ϵ , such that

$$\lim_{\epsilon \rightarrow 0} \|U_\epsilon(0, -\infty)P_j(0) - e^{i\theta_j/\epsilon}AP_j(0)\| = 0,$$

where $U_\epsilon(t, t_0)$ is the evolution operator for the Hamiltonian $H(t) = e^{-\epsilon|t|}e^{iH_0t}Ve^{-iH_0t}$ and

$$\theta_j = \int_0^1 \frac{E_j(0) - E_j(\lambda)}{\lambda} d\lambda.$$

In other words, the singularity of $U_\epsilon(0, -\infty)P_j(0)$ is entirely described by the factor $e^{i\theta_j/\epsilon}$. The operator A satisfies the intertwining property $AP_j(0) = P_j(1)A$.

To connect this result with the case that we are considering in this paper, we have to choose a model space M that satisfies the following condition: there is a set I of indices j such that $P = \sum_{j \in I} P_j(0)$, where P is the projector onto M . We first define $A_{jk} = P_j(0)AP_k(0)$, for j and k in I . Then an inverse B of PAP is defined by $\sum_{k \in I} A_{jk}B_{kl} = \delta_{jl}P_j(0)$ where $B_{jk} = P_j(0)BP_k(0)$. Then, $(PU_\epsilon(0, -\infty)P)^{-1} \simeq \sum_{jk} e^{-i\theta_j/\epsilon} B_{jk}$ and

$$\Omega(0, -\infty) = U_\epsilon(0, -\infty)P(PU_\epsilon(0, -\infty)P)^{-1} \simeq \sum_{jk} AP_j(0)B_{jk}.$$

Since the right hand side does not depend on ϵ , then $\Omega(0, -\infty)$ has no singularity and

$$\overline{\Omega} = \lim_{\epsilon \rightarrow 0} \Omega(0, -\infty) = \sum_{jk} AP_j(0)B_{jk}.$$

There are states $|\tilde{\varphi}_j\rangle$ in the model space such that $\overline{\Omega}|\tilde{\varphi}_j\rangle$ is an eigenstate of H with eigenvalue $E_j(1)$. Moreover, these states can be written $|\tilde{\varphi}_j\rangle = PA|\varphi_j\rangle$, where $|\varphi_j\rangle$ is an eigenstate of $P_j(0)$: $P_j(0)|\varphi_j\rangle = |\varphi_j\rangle$.

Indeed, we have

$$\overline{\Omega}|\tilde{\varphi}_j\rangle = \overline{\Omega}PAP_j(0)|\varphi_j\rangle = \sum_{km} AP_k(0)B_{km}A_{mj}|\varphi_j\rangle = AP_j(0)|\varphi_j\rangle = P_j(1)A|\varphi_j\rangle,$$

where we used the intertwining property in the last equation. Thus, $\overline{\Omega}|\tilde{\varphi}_j\rangle$ is an eigenstate of H with eigenvalue $E_j(1)$. This is obvious from the previous equation but we check it again

$$H\overline{\Omega}|\tilde{\varphi}_j\rangle = h(1)P_j(1)A|\varphi_j\rangle = E_j(1)P_j(1)A|\varphi_j\rangle = E_j(1)AP_j(0)|\varphi_j\rangle = E_j(1)\overline{\Omega}|\tilde{\varphi}_j\rangle. \quad (9)$$

Moreover, if we define the effective Hamiltonian $H_{\text{eff}} = PH\overline{\Omega}$, then eq. (9) becomes by left multiplication with P

$$H_{\text{eff}}|\tilde{\varphi}_j\rangle = E_j(1)P\overline{\Omega}|\tilde{\varphi}_j\rangle = E_j(1)|\tilde{\varphi}_j\rangle,$$

because $P\overline{\Omega} = P$ and $P|\tilde{\varphi}_j\rangle = |\tilde{\varphi}_j\rangle$. Thus, the eigenvalues of H_{eff} are eigenvalues of the full Hamiltonian H . This is exactly what is expected from an effective Hamiltonian. In practice, the operator A is not known and the states $|\tilde{\varphi}_j\rangle$ are obtained by diagonalizing H_{eff} .

B. Series expansion

We consider again the series expansion in terms of permutations. A straightforward calculation^{41,42} of the Picard-Dyson series gives us

$$U(0, -\infty)|\Phi_0\rangle = |\Phi_0\rangle + \sum_{n=1}^{\infty} \sum_{i_1 \dots i_n} \frac{|\Phi_{i_1}\rangle \langle \Phi_{i_1}| V |\Phi_{i_2}\rangle \dots \langle \Phi_{i_{n-1}}| V |\Phi_{i_n}\rangle \langle \Phi_{i_n}| V |\Phi_0\rangle}{(E_0 - E_{i_1} + n i \epsilon)(E_0 - E_{i_2} + (n-1) i \epsilon) \dots (E_0 - E_{i_n} + i \epsilon)},$$

where we used the completeness relation $1 = \sum_i |\Phi_i\rangle \langle \Phi_i|$. This expression clearly shows that the terms of the expansion (and the evolution operator) are divergent as $\epsilon \rightarrow 0$ when any E_{i_k} is equal to E_0 .

For $\sigma \in \mathcal{S}_n$, we set $\Omega_\sigma(t) = \Omega_\sigma(t, -\infty)$. We then have

$$\Omega_\sigma(t) = (-i)^n \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 \dots \int_{-\infty}^{t_{n-1}} dt_n Q e^{(\epsilon + i H_0) t_{\sigma(1)}} V e^{-i H_0 t_{\sigma(1)}} R_\sigma^1 e^{(\epsilon + i H_0) t_{\sigma(2)}} V e^{-i H_0 t_{\sigma(2)}} R_\sigma^2 \dots R_\sigma^{n-1} e^{(\epsilon + i H_0) t_{\sigma(n)}} V e^{-i H_0 t_{\sigma(n)}} P,$$

where $R_\sigma^k = Q$ if $\sigma(k+1) > \sigma(k)$ and $R_\sigma^k = -P$ if $\sigma(k+1) < \sigma(k)$. We replace R_σ^k by $\pm \sum_{\alpha_{k+1}} |\alpha_{k+1}\rangle \langle \alpha_{k+1}|$ where, if $R_\sigma^k = Q$, then $\pm = +$ and the sum is over the image of Q , and if $R_\sigma^k = -P$, then $\pm = -$ and the sum is over the image of P . Thus

$$\Omega_\sigma(t) = (-i)^n (-1)^d \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 \dots \int_{-\infty}^{t_{n-1}} dt_n e^{(\epsilon + i F_1 - i F_2) t_{\sigma(1)}} e^{(\epsilon + i F_2 - i F_3) t_{\sigma(2)}} \dots e^{(\epsilon + i F_n - i F_{n+1}) t_{\sigma(n)}} \sum_{\alpha_1 \dots \alpha_{n+1}} |\alpha_1\rangle \langle \alpha_1| V |\alpha_2\rangle \dots \langle \alpha_n| V |\alpha_{n+1}\rangle \langle \alpha_{n+1}|, \quad (10)$$

where d is the number of elements of the descent set of σ , F_i is the energy of α_i and where the sum over α_1 is over the image of Q , the sum over α_{n+1} is over the image of P and the sum over α_k for $1 < k < n+1$ is over the image of Q if $\sigma(k) > \sigma(k-1)$ and over the image of P otherwise. Consider now the time integral

$$\begin{aligned} f_\sigma(t) &= \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 \dots \int_{-\infty}^{t_{n-1}} dt_n e^{(\epsilon + i F_1 - i F_2) t_{\sigma(1)}} e^{(\epsilon + i F_2 - i F_3) t_{\sigma(2)}} \dots e^{(\epsilon + i F_n - i F_{n+1}) t_{\sigma(n)}} \\ &= \int_{-\infty}^t ds_{\tau(1)} \int_{-\infty}^{s_{\tau(1)}} ds_{\tau(2)} \dots \int_{-\infty}^{s_{\tau(n-1)}} ds_{\tau(n)} e^{(\epsilon + i F_1 - i F_2) s_1} e^{(\epsilon + i F_2 - i F_3) s_2} \dots e^{(\epsilon + i F_n - i F_{n+1}) s_n}, \end{aligned}$$

where $\tau = \sigma^{-1}$. The integral over $s_{\tau(n)}$ is

$$\int_{-\infty}^{s_{\tau(n-1)}} ds_{\tau(n)} e^{(\epsilon + i F_{\tau(n)} - i F_{\tau(n)+1}) s_{\tau(n)}} = \frac{e^{(\epsilon + i F_{\tau(n)} - i F_{\tau(n)+1}) s_{\tau(n-1)}}}{(\epsilon + i F_{\tau(n)} - i F_{\tau(n)+1})}.$$

The integrand of the integral over $s_{\tau(n-1)}$ becomes

$$\frac{e^{(2\epsilon + i(F_{\tau(n)} + F_{\tau(n-1)} - F_{\tau(n)+1} - F_{\tau(n-1)+1}) s_{\tau(n-1)})}}{(\epsilon + i F_{\tau(n)} - i F_{\tau(n)+1})}.$$

A straightforward recursive argument shows that

$$f_\sigma(t) = \frac{e^{X_\sigma(n)t}}{X_\sigma(1) \dots X_\sigma(n)},$$

where

$$X_\sigma(k) = k\epsilon + i(F_{\sigma^{-1}(n)} + \dots + F_{\sigma^{-1}(n-k+1)} - F_{\sigma^{-1}(n)+1} - \dots - F_{\sigma^{-1}(n-k+1)+1}), \quad (11)$$

and where $X_\sigma(k)$ depends on $\alpha_1, \dots, \alpha_{n+1}$, although we did not emphasize this dependency notationally.

Therefore

$$\Omega_\sigma(t) = \sum_{\alpha_1 \dots \alpha_{n+1}} \frac{(-i)^n (-1)^d e^{X_\sigma(n)t}}{X_\sigma(1) \dots X_\sigma(n)} |\alpha_1\rangle \langle \alpha_1| V |\alpha_2\rangle \dots \langle \alpha_n| V |\alpha_{n+1}\rangle \langle \alpha_{n+1}|. \quad (12)$$

C. Examples

We give now a few examples of $\Omega_\sigma(0)$.

$$\begin{aligned}\Omega_{(1)}(0) &= (-\imath) \sum_{\Phi_i^Q \Phi_j^P} \frac{|\Phi_i^Q\rangle\langle\Phi_i^Q|V|\Phi_j^P\rangle\langle\Phi_j^P|}{\epsilon + \imath(E_i^Q - E_j^P)}, \\ \Omega_{(12)}(0) &= (-\imath)^2 \sum_{\Phi_i^Q \Phi_j^Q \Phi_k^P} \frac{|\Phi_i^Q\rangle\langle\Phi_i^Q|V|\Phi_j^Q\rangle\langle\Phi_j^Q|V|\Phi_k^P\rangle\langle\Phi_k^P|}{(\epsilon + \imath(E_j^Q - E_k^P))(2\epsilon + \imath(E_i^Q - E_k^P))}, \\ \Omega_{(21)}(0) &= -(-\imath)^2 \sum_{\Phi_i^Q \Phi_j^P \Phi_k^P} \frac{|\Phi_i^Q\rangle\langle\Phi_i^Q|V|\Phi_j^P\rangle\langle\Phi_j^P|V|\Phi_k^P\rangle\langle\Phi_k^P|}{(\epsilon + \imath(E_i^Q - E_j^P))(2\epsilon + \imath(E_i^Q - E_k^P))}.\end{aligned}$$

Finally we consider two examples that will prove useful:

$$\begin{aligned}\Omega_{(213)}(0) &= -(-\imath)^3 \sum_{\Phi_i^Q \Phi_j^P \Phi_k^Q \Phi_l^P} \frac{|\Phi_i^Q\rangle\langle\Phi_i^Q|V|\Phi_j^P\rangle\langle\Phi_j^P|V|\Phi_k^Q\rangle\langle\Phi_k^Q|V|\Phi_l^P\rangle\langle\Phi_l^P|}{(\epsilon + \imath(E_k^Q - E_l^P))(2\epsilon + \imath(E_k^Q + E_i^Q - E_l^P - E_j^P))(3\epsilon + \imath(E_i^Q - E_l^P))}, \\ \Omega_{(312)}(0) &= -(-\imath)^3 \sum_{\Phi_i^Q \Phi_j^P \Phi_k^Q \Phi_l^P} \frac{|\Phi_i^Q\rangle\langle\Phi_i^Q|V|\Phi_j^P\rangle\langle\Phi_j^P|V|\Phi_k^Q\rangle\langle\Phi_k^Q|V|\Phi_l^P\rangle\langle\Phi_l^P|}{(\epsilon + \imath(E_i^Q - E_j^P))(2\epsilon + \imath(E_k^Q + E_i^Q - E_l^P - E_j^P))(3\epsilon + \imath(E_i^Q - E_l^P))}.\end{aligned}$$

By adding these two terms, we obtain a denominator involving only the difference of two energies.

$$\Omega_{(213)}(0) + \Omega_{(312)}(0) = -(-\imath)^3 \sum_{\Phi_i^Q \Phi_j^P \Phi_k^Q \Phi_l^P} \frac{|\Phi_i^Q\rangle\langle\Phi_i^Q|V|\Phi_j^P\rangle\langle\Phi_j^P|V|\Phi_k^Q\rangle\langle\Phi_k^Q|V|\Phi_l^P\rangle\langle\Phi_l^P|}{(\epsilon + \imath(E_i^Q - E_j^P))(\epsilon + \imath(E_k^Q - E_l^P))(3\epsilon + \imath(E_i^Q - E_l^P))}.$$

Note that the sum is simpler than either $\Omega_{(213)}(0)$ or $\Omega_{(312)}(0)$. This is a general statement and the simplification becomes spectacular at higher orders. For the example of $n = 7$, there is a single tree T which is the sum of 80 permutations, and the denominator of Ω_T is simpler than the denominator of Ω_σ for any of the 80 permutations σ . This will be proved in section IV. Note also that, if we assume that the image of P is separated from the image of Q by a finite gap δ , so that $E_i^Q - E_j^P \geq \delta$, then the denominators of all the examples are non-zero when $\epsilon \rightarrow 0$. In other words, the limit $\lim_{\epsilon \rightarrow 0} \Omega_\sigma(0)$ exists for all the examples. In the next section, we show that this result is true for all permutations σ .

D. Convergence of $\Omega_\sigma(t)$

Definition (11) is convenient for a computer implementation but it does not make it clear that $X_\sigma(k)$ is nonzero if $\epsilon = 0$. For that purpose, we need an alternative expression for $X_\sigma(k)$. This expression is essentially a corrected version of the graphical rule given by Michels and Suttorp¹⁸. We first extend any permutation $\sigma \in \mathcal{S}_n$ to the sequence of $n + 2$ integers $\bar{\sigma} = (\bar{\sigma}(1), \dots, \bar{\sigma}(n + 2)) = (0, \sigma(1), \dots, \sigma(n), 0)$. Then, for $k \in \{1, \dots, n\}$, we define the two sets

$$\begin{aligned}S_\sigma^<(k) &= \{i \mid 1 \leq i \leq n + 1 \text{ and } \bar{\sigma}(i) < k \leq \bar{\sigma}(i + 1)\}, \\ S_\sigma^>(k) &= \{i \mid 1 \leq i \leq n + 1 \text{ and } \bar{\sigma}(i) \geq k > \bar{\sigma}(i + 1)\}.\end{aligned}$$

For example, if $\sigma = (41325)$, then $S_\sigma^<(1) = \{1\}$, $S_\sigma^<(2) = \{1, 3\}$, $S_\sigma^<(3) = \{1, 3, 5\}$, $S_\sigma^<(4) = \{1, 5\}$, $S_\sigma^<(5) = \{5\}$ and $S_\sigma^>(1) = \{6\}$, $S_\sigma^>(2) = \{2, 6\}$, $S_\sigma^>(3) = \{2, 4, 6\}$, $S_\sigma^>(4) = \{2, 6\}$, $S_\sigma^>(5) = \{6\}$. The graphical meaning of these sets is illustrated in Figure 1. Notice that the vertical axis is oriented downwards in order to reflect the time-ordering in the integrals $S_\sigma(t)$.

Lemma 4 (i) $S_\sigma^<(k)$ cannot be empty and (ii) $S_\sigma^<(k)$ and $S_\sigma^>(k)$ have the same number of elements.

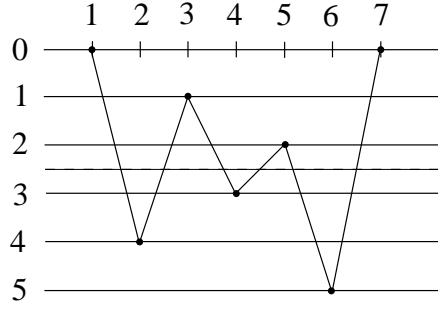


FIG. 1: Construction of $S_\sigma^<(3)$ for $\sigma = (41325)$. We build $\bar{\sigma} = (0, 4, 1, 3, 2, 5, 0)$, we draw a continuous line L starting from $(1, \bar{\sigma}(1)) = (1, 0)$ to $(2, \bar{\sigma}(2)) = (2, 4)$, to $(3, \bar{\sigma}(3)) = (3, 1)$, ..., up to $(7, \bar{\sigma}(7)) = (7, 0)$. To determine $S_\sigma^<(3)$, we draw a horizontal dashed line just above 3 and we gather the segments of L crossing the dashed line from above. In our case the segments are $((1, 0), (2, 4))$, $((3, 1), (4, 3))$ and $((5, 2), (6, 5))$. $S_\sigma^<(3)$ is the set of abscissae of the first point of each segment: $S_\sigma^<(3) = \{1, 3, 5\}$. Similarly, $S_\sigma^>(3)$ is obtained from the segments that cross the dashed line from below: $S_\sigma^>(3) = \{2, 4, 6\}$.

The lemma follows from the graphical interpretation of the construction of the $S_\sigma^>(k)$ and $S_\sigma^<(k)$. The graph of σ (constructed as in Figure 1) is a sequence of edges connecting the points $(i, \bar{\sigma}(i))$. Since the graph starts from $(0, 0)$ and since there exists one point with ordinate n , any horizontal line with non integer ordinate y , $0 < y < n$, will be crossed from above by a segment (remember the vertical axis is oriented downwards). A similar elementary topological argument shows that such a horizontal line is always crossed successively from above and below by segments, the series of crossings starting from above and ending from below, which implies $|S_\sigma^<(k)| = |S_\sigma^>(k)|$.

The key step in the proof of convergence is the

Lemma 5 For $\sigma \in \mathcal{S}_n$ and a given sequence $\alpha_1, \dots, \alpha_{n+1}$, compatible with σ (see eq. (10)), we have

$$X_\sigma(k) = k\epsilon + \sum_{j \in S_\sigma^<(n-k+1)} \iota F_j^Q - \sum_{j \in S_\sigma^>(n-k+1)} \iota F_j^P, \quad (13)$$

where we write $F_i^Q = F_i$ (resp. $F_i^P = F_i$) when α_i belongs to the image of Q (resp. of P).

Proof. We first show that it is true for $k = 1$. Indeed, $X_\sigma(1) = \epsilon + \iota(F_{\sigma^{-1}(n)} - F_{\sigma^{-1}(n)+1})$. Let $j = \sigma^{-1}(n)$, we have $\sigma(j) = n$. Then, either $j = 1$ and $|\alpha_1\rangle$ is in the image of Q , or $j > 1$ and $\sigma \in \mathcal{S}_n$ implies $\sigma(j) = n > \sigma(j-1)$, so that $|\alpha_j\rangle$ is in the image of Q . Thus, in all cases, $F_{\sigma^{-1}(n)} = F_{\sigma^{-1}(n)}^Q$. Consider now F_{j+1} . Either $j = n$ and $|\alpha_{j+1}\rangle = |\alpha_{n+1}\rangle$ is in the image of P , or $j < n$ and $\sigma(j+1) < \sigma(j) = n$, so that $|\alpha_{j+1}\rangle$ is in the image of P . Thus, in all cases, $F_{\sigma^{-1}(n)+1} = F_{\sigma^{-1}(n)+1}^P$. Therefore, $X_\sigma(1) = \epsilon + \iota(F_{\sigma^{-1}(n)}^Q - F_{\sigma^{-1}(n)+1}^P)$. On the other hand, $S_\sigma^<(n) = \{\sigma^{-1}(n)\}$ and $S_\sigma^>(n) = \{\sigma^{-1}(n) + 1\}$ since (j, n) is the only point of the graph with ordinate n . Thus, the two members of eq. (13) are equal for $k = 1$.

Assume now that eq. (13) holds for all the $X_\sigma(i)$ with $i = 1, \dots, k$, $k < n$, and consider the equation (which is true by definition of the $X_\sigma(i)$ s):

$$X_\sigma(k+1) = X_\sigma(k) + \epsilon + \iota(F_j - F_{j+1}), \quad (14)$$

where $j = \sigma^{-1}(n-k)$. We first treat the case $1 < j < n$. Four possible situations can arise: (i) $\sigma(j-1) > \sigma(j) > \sigma(j+1)$, (ii) $\sigma(j-1) < \sigma(j) > \sigma(j+1)$, (iii) $\sigma(j-1) > \sigma(j) < \sigma(j+1)$ and (iv) $\sigma(j-1) < \sigma(j) < \sigma(j+1)$. In case (i), we have $F_j = F_j^P$ and $F_{j+1} = F_{j+1}^P$. On the other hand, condition (i) implies $\bar{\sigma}(j) > \bar{\sigma}(j+1) > \bar{\sigma}(j+2)$, so that $S_\sigma^>(n-k) = S_\sigma^>(n-k+1)$ and $S_\sigma^<(n-k)$ is obtained from $S_\sigma^<(n-k+1)$ by removing $\{j\}$ and adding $\{j+1\}$. eq. (14) together with the hypothesis that eq. (13) holds for $X_\sigma(k)$ imply that eq. (13) holds for $X_\sigma(k+1)$. Case (ii) implies $F_j = F_j^Q$ and $F_{j+1} = F_{j+1}^P$, case (iii) implies $F_j = F_j^P$ and $F_{j+1} = F_{j+1}^Q$, case (iv) implies $F_j = F_j^Q$ and $F_{j+1} = F_{j+1}^Q$. In all cases, these identities imply that the two expressions (14) and (13) for $X_\sigma(k+1)$ do agree.

It remains to treat the boundary cases. If $j = 1$, then $F_j = F_1^Q$ and we have either (i) $\sigma(1) < \sigma(2)$ and $F_2 = F_2^Q$ or (ii) $\sigma(1) > \sigma(2)$ and $F_2 = F_2^P$. We know that $\bar{\sigma}(1) = 0$, thus, case (i) corresponds to $\bar{\sigma}(1) < \bar{\sigma}(2) < \bar{\sigma}(3)$, so that according to eq. (13),

$$X_\sigma(k+1) - X_\sigma(k) = \iota(F_1^Q - F_2^Q),$$

in agreement with eq. (14). In case (ii) we have $\bar{\sigma}(1) < \bar{\sigma}(2) > \bar{\sigma}(3)$, which amounts to add a $\imath F_1^Q$ and remove a $\imath F_2^P$. Again, eq. (13) holds for $k+1$. Finally, if $j = n$, then $F_{j+1} = F_{n+1}^P$ and we have (i) $\sigma(n-1) < \sigma(n)$ and $F_n = F_n^Q$ or (ii) $\sigma(n-1) > \sigma(n)$ and $F_n = F_n^P$. Case (i) corresponds to $\bar{\sigma}(n) < \bar{\sigma}(n+1) > \bar{\sigma}(n+2)$, case (ii) corresponds to $\bar{\sigma}(n) > \bar{\sigma}(n+1) > \bar{\sigma}(n+2)$. In all cases, the relation given by eq. (13) is satisfied for $k+1$ and the induction proof is complete. \square

We are now ready to prove the convergence of $\Omega_\sigma(t)$ as $\epsilon \rightarrow 0$. Consider eq. (13). The sets $S_\sigma^<(n-k+1)$ and $S_\sigma^>(n-k+1)$ have the same number of elements, say n_k and, by the gap hypothesis, we have $F_j^Q - F_i^P \geq \delta$ for any i and j . Therefore, $|X_\sigma(k)|^2 \geq k^2 \epsilon^2 + n_k^2 \delta^2 \geq \delta^2$, since $n_k \geq 1$. Thus, the denominator remains away from zero by a finite amount for any $\epsilon \geq 0$ and the limit of $1/X_\sigma(k)$ for $\epsilon \rightarrow 0$ is well-defined. As a consequence, the limit

$$\bar{\Omega}_\sigma = \lim_{\epsilon \rightarrow 0} \Omega_\sigma(0), \quad (15)$$

is well-defined.

IV. TREES

We showed that, for each permutation σ , the wave operator $\Omega_\sigma(t, -\infty)$ has a well-defined limit as $\epsilon \rightarrow 0$. However, giving a detailed proof was rather lengthy and the final expression for $X_\sigma(k)$ suggests physically the simultaneous occurrence of transitions from lower to higher energy states. The convergence is actually much easier to show in terms of trees, and the expressions showing up in the expansion are simpler, mathematically and physically, each factor of the denominator corresponding to a unique transition from a low to a high energy state.

In this section, if N is the dimension of the model space, we write $i \in Q$ for $i > N$ and $j \in P$ for $1 \leq j \leq N$, both for notational simplicity, and to emphasize the meaning of the indices (that correspond respectively to eigenstates in the image of P and Q).

Proposition 6 *If $T = T_1 \vee T_2$, then, for $t \leq 0$,*

$$\Omega_T(t) = \Omega_T(t, -\infty) = \sum_{i \in Q, j \in P} e^{(\imath E_i^Q - \imath E_j^P + |T|\epsilon)t} \Omega_T^{ij} |\Phi_i\rangle \langle \Phi_j|, \quad (16)$$

where Ω_T^{ij} is obtained recursively by:

$$\text{For } T = \Upsilon, \Omega_\Upsilon^{ij} = -\imath \frac{\langle \Phi_i | V | \Phi_j \rangle}{\imath E_i^Q - \imath E_j^P + \epsilon}.$$

$$\text{For } T_1 = |, T_2 \neq | : \Omega_T^{ij} = -\imath \sum_{k \in Q} \frac{\langle \Phi_i | V | \Phi_k \rangle \Omega_{T_2}^{kj}}{\imath E_i^Q - \imath E_j^P + |T|\epsilon}.$$

$$\text{For } T_1 \neq |, T_2 = | : \Omega_T^{ij} = \imath \sum_{k \in P} \frac{\Omega_{T_1}^{ik} \langle \Phi_k | V | \Phi_j \rangle}{\imath E_i^Q - \imath E_j^P + |T|\epsilon}.$$

$$\text{For } T_1 \neq |, T_2 \neq | : \Omega_T^{ij} = \imath \sum_{k \in P, l \in Q} \frac{\Omega_{T_1}^{ik} \langle \Phi_k | V | \Phi_l \rangle \Omega_{T_2}^{lj}}{\imath E_i^Q - \imath E_j^P + |T|\epsilon}.$$

Proof. The computation of Ω_Υ^{ij} follows from eq. (12). Let us consider for example the case $T_1 = |, T_2 \neq |$. Then, applying theorem 3:

$$\Omega_T(t) = -\imath \sum_{i \in Q, j \in P} \int_{-\infty}^t ds Q e^{\epsilon s} e^{-\imath E_i^Q s} e^{(\imath E_i^Q - \imath E_j^P + |T_2|\epsilon)s} \Omega_{T_2}^{ij} e^{i H_0 s} V |\Phi_i\rangle \langle \Phi_j|.$$

We replace Q by $\sum_{k \in Q} |\Phi_k\rangle \langle \Phi_k|$ and obtain, by using $|T| = |T_2| + 1$,

$$\begin{aligned} \Omega_T(t) &= -\imath \sum_{k, i \in Q, j \in P} \int_{-\infty}^t ds e^{(\imath E_k^Q - \imath E_j^P + |T|\epsilon)s} \langle \Phi_k | V | \Phi_i \rangle \Omega_{T_2}^{ij} |\Phi_k\rangle \langle \Phi_j| \\ &= -\imath \sum_{k, i \in Q, j \in P} \frac{e^{(\imath E_k^Q - \imath E_j^P + |T|\epsilon)t}}{\imath E_k^Q - \imath E_j^P + |T|\epsilon} \langle \Phi_k | V | \Phi_i \rangle \Omega_{T_2}^{ij} |\Phi_k\rangle \langle \Phi_j|, \end{aligned}$$

The two other cases can be treated similarly. \square

Since, for arbitrary i and j , $E_i^Q - E_j^P \geq \delta$, we get:

Corollary 7 *The limit $\bar{\Omega}_T(t) = \lim_{\epsilon \rightarrow 0} \Omega_T(t)$ is well defined.*

A. Relation with the Rayleigh-Schrödinger perturbation theory

Kvasnička⁷ and Lindgren⁶ independently obtained an equation for the time-independent Rayleigh-Schrödinger perturbation theory of possibly degenerate systems:

$$[\bar{\omega}, H_0]P = V\bar{\omega}P - \bar{\omega}PV\bar{\omega}P, \quad (17)$$

where the time-independent wave operator $\bar{\omega}P$ transforms eigenstates $|\Phi_0\rangle$ of H_0 into eigenstates $\bar{\omega}P|\Phi_0\rangle$ of $H_0 + V$ and where $P\bar{\omega}P = P$ (see ref. 43 p. 202 for details).

Equation (17) is an important generalization of Bloch's classical results⁴⁴ because it is valid even if the eigenstates of H_0 in the model space have different energies.

The relation between time-dependent and time-independent perturbation theory is established by the following proposition:

Proposition 8 *We have $\bar{\omega}P = \bar{\Omega} = \lim_{\epsilon \rightarrow 0} \Omega(0)$.*

Proof. We take the derivative of eq. (8) with respect to time and we substitute $\chi(t, t_0) = \Omega(t, t_0) - P$. This gives us

$$\imath \frac{d}{dt} \Omega(t, t_0) = H(t)\Omega(t, t_0) - \Omega(t, t_0)H(t)\Omega(t, t_0).$$

If we take $t = 0$ and $t_0 = -\infty$, we obtain by continuity

$$\imath \frac{d}{dt} \Omega(t)|_{t=0} = V\Omega(0) - \Omega(0)V\Omega(0).$$

When we compare this equation with eq. (17), we see that $\bar{\omega}P$ and $\bar{\Omega}$ satisfy the same equation if

$$\imath \lim_{\epsilon \rightarrow 0} \frac{d\Omega(t)}{dt}|_{t=0} = [\lim_{\epsilon \rightarrow 0} \Omega(0), H_0]. \quad (18)$$

To show this, we prove it for each term Ω_T . Indeed, eq. (16) gives us

$$\imath \frac{d\Omega_T(t)}{dt} = \sum_{i \in Q, j \in P} (E_j^P - E_i^Q + \imath|T|\epsilon) e^{(\imath E_i^Q - \imath E_j^P + |T|\epsilon)t} \Omega_T^{ij} |\Phi_i\rangle \langle \Phi_j|,$$

and

$$[\Omega_T(t), H_0] = \sum_{i \in Q, j \in P} (E_j^P - E_i^Q) e^{(\imath E_i^Q - \imath E_j^P + |T|\epsilon)t} \Omega_T^{ij} |\Phi_i\rangle \langle \Phi_j|.$$

By continuity in ϵ , these two expressions are identical for all t if $\epsilon \rightarrow 0$. If we take $t = 0$ and we sum over all trees T , then we recover eq. (18). Therefore, $\bar{\omega}P$ and $\bar{\Omega}$ satisfy the same equation. It remains to show that they have the same boundary conditions: $\bar{\Omega}P = \bar{\Omega}$ and $P\bar{\Omega} = P$. By eq. (3), these two equations are true for $\Omega(t)$ with any value of t and ϵ . \square

As a corollary, proposition 6 provides a recursive construction of the wave operator $\bar{\Omega}$.

B. An explicit formula for Ω_T

In this section, we show how $\Omega_T(t)$ can be obtained non-recursively from the knowledge of T . The key idea is to replace T by another combinatorial object, better suited to that particular computation. We write therefore γ_T for the smallest permutation for the lexicographical ordering in S_T (we view a permutation as a word to make sense of the lexicographical ordering: to (35421) corresponds the word 35421, so that e.g. (35421) < (54231)). Since S_T is always non empty, the map $\gamma : T \mapsto \gamma_T$ is well-defined and an injection from the set of trees to the set of permutations. These permutations are called Catalan permutations²⁸, 312-avoiding permutations (i.e. permutations for which there does not exist $i < j < k$ such that $\sigma(j) < \sigma(k) < \sigma(i)$, see ref. 30 p. 224), Kempf elements⁴⁵ or stack words⁴⁶.

Various elementary manipulations can be done to understand such a map. We list briefly some obvious properties and introduce some notation that will be useful in our forthcoming developments. If $I = (a_1, \dots, a_k)$ is a sequence of integers, we write $I[n]$ for the shifted sequence $(a_1 + n, \dots, a_k + n)$.

Then, let $T = T_1 \vee T_2$ be a tree. The permutation $\gamma(T)$ (that we identify with the corresponding word or sequence) can be constructed recursively as $\gamma(|) = \emptyset$, $\gamma(\Upsilon) = (1)$ and

$$\gamma(T) = (\gamma(T_1)[1], 1, \gamma(T_2)[|T_1| + 1]).$$

The left inverse of γ , say \mathcal{T} , is also easily described recursively as $\mathcal{T}(\emptyset) = |$, $\mathcal{T}(1) = \Upsilon$ and

$$\mathcal{T}(\sigma) = \mathcal{T}((\sigma(1), \dots, \sigma(k))[-1]) \vee \mathcal{T}((\sigma(k+2), \dots, \sigma(n))[-k-1]),$$

where $\sigma \in \mathcal{S}_n$, $\sigma = (\sigma(1), \dots, \sigma(k), 1, \sigma(k+2), \dots, \sigma(n))$ is in the image of γ .

Permutations in the image of γ can be characterized recursively similarly: with the same notation as in the previous paragraph, σ is in the image of γ if and only if $(\sigma(1), \dots, \sigma(k))[-1]$ and $(\sigma(k+2), \dots, \sigma(n))[-k-1]$ are in the image of γ , where k is the integer such that $\sigma(k) = 1$,

We are now in a position to compute $\Omega_T(t)$. Let us write \mathcal{A}_T for all the sequences $\alpha = (\alpha_1, \dots, \alpha_{n+1})$ associated to $\gamma(T)$ as in equation (10). We write, as usual, F_i for the eigenvalue associated to α_i . Recall that $\alpha_1 \in Q$, $\alpha_{n+1} \in P$ whereas $\alpha_i, i \neq 1, n+1$ is in Q if $\sigma(i) > \sigma(i-1)$ and in P otherwise.

These sequences are actually common to the expansions of all the $\Omega_\sigma(t)$, $\sigma \in S_T$ (this is because they depend only on the positions of descents in the permutations $\sigma \in S_T$, as discussed in the proof of thm. 3). They appear therefore in the expansion of $\Omega_T(t) = \sum_{\sigma \in S_T} \Omega_\sigma(t)$. They actually also correspond exactly to the sequences of eigenvectors that show up in the recursive expansion of $\Omega_T(t)$ (proposition 16) (this should be clear from our previous remarks, but can be checked directly from the definition of the recursive expansion). We can refine the recursion of proposition 6 accordingly:

Lemma 9 *We have: $\Omega_T(t) = \sum_{\alpha} e^{(\imath F_1 - \imath F_{n+1} + |T|\epsilon)t} \Omega_T^\alpha |\alpha_1\rangle \langle \alpha_{n+1}|$, where $\Omega_T^\alpha(t)$ is defined recursively by:*

$$\Omega_{\Upsilon}^{\Phi_i, \Phi_j} = -\imath \frac{\langle \Phi_i | V | \Phi_j \rangle}{\imath E_i^Q - \imath E_j^P + \epsilon},$$

$$\Omega_T^\alpha = \imath \frac{\Omega_{T_1}^{(\alpha_1, \dots, \alpha_k)} \langle \alpha_k | V | \alpha_{k+1} \rangle \Omega_{T_2}^{(\alpha_{k+1}, \dots, \alpha_n, \alpha_{n+1})}}{\imath F_1 - \imath F_{n+1} + |T|\epsilon},$$

where $T = T_1 \vee T_2$ and $k = |T_1|$. For $T_1 = |$, we have $\Omega_{T_1}^{(\alpha_1)} = -1$ and for $T_2 = |$, we have $\Omega_{T_2}^{(\alpha_{n+1})} = 1$.

Let us now consider γ_T . For $i = 1, \dots, n$, we set: $l(i) = \inf\{j \leq i | \forall k, j \leq k \leq i, \gamma_T(k) \geq \gamma_T(i)\}$ and $r(i) = \sup\{j \geq i | \forall l, j \geq l \geq i, \gamma_T(l) \geq \gamma_T(i)\}$. In words, $l(i)$ is defined as follows: consider all the consecutive positions k on the left of position i , such that the value of the permutation $\gamma_T(k)$ is larger than $\gamma_T(i)$. Then, $l(i)$ is the leftmost of these positions k . Similarly, $r(i)$ is the rightmost position j such that, on all positions k between i and j , the permutation $\gamma_T(k)$ is larger than $\gamma_T(i)$.

Theorem 10 *We have:*

$$\Omega_T^\alpha = (-\imath)^n (-1)^{d-1} \langle \alpha_1 | V | \alpha_2 \rangle \dots \langle \alpha_n | V | \alpha_{n+1} \rangle \prod_{i=1}^n \frac{1}{\imath F_{l(i)} - \imath F_{r(i)+1} + (r(i) - l(i) + 1)\epsilon},$$

or, equivalently,

$$\Omega_T(t) = (-\imath)^n (-1)^{d-1} \sum_{\alpha} |\alpha_1\rangle \langle \alpha_1 | V | \alpha_2 \rangle \dots \langle \alpha_n | V | \alpha_{n+1} \rangle \langle \alpha_{n+1} | e^{(\imath F_1 - \imath F_{n+1} + |T|\epsilon)t} \prod_{i=1}^n \frac{1}{\imath F_{l(i)} - \imath F_{r(i)+1} + (r(i) - l(i) + 1)\epsilon},$$

where d is the number of leaves of T pointing to the right.

For example, if $T = \begin{array}{c} \diagup \diagdown \\ | \end{array}$, then $\gamma_T = (213)$, $l = (113)$, $r = (133)$ and

$$\Omega_T^{\Phi_i \Phi_j \Phi_k \Phi_l} = -\imath \sum_{\Phi_i \Phi_j \Phi_k \Phi_l} \frac{|\Phi_i\rangle \langle \Phi_i| V |\Phi_j\rangle \langle \Phi_j| V |\Phi_k\rangle \langle \Phi_k| V |\Phi_l\rangle \langle \Phi_l|}{(\epsilon + \imath(E_i^Q - E_j^P))(\epsilon + \imath(E_k^Q - E_l^P))(3\epsilon + \imath(E_i^Q - E_l^P))}.$$

Proof. We show that Ω_T^α as defined in theorem 10 satisfies the recursion relation of lemma 9. Let us first consider that $T_1 \neq |$ and $T_2 \neq |$. Let i_0 denote the index such that $\gamma_T(i_0) = 1$, with $1 < i_0 < n$. The term of the numerator corresponding to i_0 is $\langle \alpha_{i_0} | V | \alpha_{i_0+1} \rangle$, which is the central term of the recursion relation. We have $l(i_0) = 1$ and $r(i_0) = n$. Thus, the denominator is $\imath F_{l(i_0)} - \imath F_{r(i_0)+1} + (r(i_0) - l(i_0) + 1)\epsilon = \imath F_1 - \imath F_{n+1} + n\epsilon$, which is the denominator of the recursion relation. Now we check that the product of terms for $i < i_0$ in theorem 10 is $\Omega_{T_1}^{(\alpha_1, \dots, \alpha_{i_0-1})}$. The matrix elements $\langle \alpha_1 | V | \alpha_2 \rangle \dots \langle \alpha_{i_0-1} | V | \alpha_{i_0} \rangle$ obviously agree, so we must check that the denominators agree. Thus, we verify that, for $1 \leq i < i_0$, $l_T(i) = l_{T_1}(i)$ and $r_T(i) = r_{T_1}(i)$, where l_T and r_T denote the l and r vectors for tree T . We know that $\gamma_T(i) = \gamma_{T_1}(i) + 1$ for $1 \leq i < i_0$. Thus, for $k \leq i$, $\gamma_T(k) \geq \gamma_T(i)$ if and only if $\gamma_{T_1}(k) \geq \gamma_{T_1}(i)$ and $l_T(i) = l_{T_1}(i)$. For r_T , we notice that for $l = i_0$ we have $\gamma_T(l) = 1 < \gamma_T(i)$ and the relation $\gamma_T(l) \geq \gamma_T(i)$ does not hold. Therefore, $r_T(i) < i_0$ and $r_T(i) = r_{T_1}(i)$ by the same argument as $l_T(i) = l_{T_1}(i)$. The same reasoning holds for T_2 and the recursion relation is satisfied. The cases $T_1 = |$ or $T_2 = |$ are proved similarly. \square

We conclude this section with a geometrical translation of the previous theorem. Consider a tree T with $|T| = n$ and number its leaves from 1 for the leftmost leaf to $n+1$ for the rightmost one. For each inner vertex v of T , take the subtree T_v for which v is just above the root. In other words, T_v is obtained by chopping the edge below v and considering the edge dangling from v as the root of T_v . For each tree T_v , build the pair (l_v, r_v) which are the indices of the leftmost and rightmost leaves of T_v . Recall that $|\alpha_i\rangle$ belongs to the image of Q (resp. P) if leaf i points to the left (resp. right): this implies in particular that $F_{l_v} = F_{l_v}^Q$ and $F_{r_v} = F_{r_v}^P$. Then, the set of pairs (l_v, r_v) where v runs over the inner vertices of T is the same as the set of pairs $(l(i), r(i) + 1)$ of the theorem. The formula for Ω_T^α can be rewritten accordingly and this geometrical version can be proved recursively as theorem 10. Conversely, it can be used to determine the tree corresponding to a given denominator.

V. CONCLUSION

We considered three expansions of the wave operator and we proved their adiabatic convergence. We proposed to expand the wave operator over trees, and proved that this expansion reduced the number of terms of the expansion with respect to usual (tractable) ones, simplified the denominators of the expansion into a product of the difference of two energies and lead to powerful formulas and recursive computational methods.

We then showed that this simplification is closely related to the algebraic structure of the linear span of permutations and of a certain convolution subalgebra of trees. However, these algebras are also Hopf algebras. The role of the coalgebraic structure of this algebra will be presented in a forthcoming publication.

As far as the many-body problem is concerned, we showed that the simplification of diagrams is not due to the details of the Hamiltonian but to the general structure of the wave operator. When the eigenstates and Hamiltonian are expressed in terms of creation and annihilation operators and quantum fields, the algebra of trees mixes with the Hopf algebraic structure of fields^{47,48}. It would be interesting to investigate the interplay of these algebraic structures. Another interesting issue is the possibility to use the tree structure to sum an infinite number of contributions.

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APPENDIX A: A CRUCIAL LEMMA

In this appendix we state and prove a lemma that is crucial to demonstrate theorem 3, which is one of the main results of our paper. We first need a noncommutative analogue of Chen's formulas for products of iterated integrals. This analogue, proved in refs. 49 and 21, provides a systematic link between the theory of iterated integrals, the combinatorics of descents, free Lie algebras and noncommutative symmetric functions (see ref. 21 and Appendix B of the present article for further details).

Let $L = (L_1, \dots, L_n)$ be an arbitrary sequence of time-dependent operators $L_i(t)$, satisfying the same regularity conditions as $H(t)$ in section II. Let σ be a permutation in \mathcal{S}_n and define:

$$\Omega_\sigma^L(t, t_0) = \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n L_1(t_{\sigma(1)}) \dots L_n(t_{\sigma(n)}).$$

The notation is extended linearly to combinations of permutations, so that for $\mu = \sum_{\sigma \in \mathcal{S}_n} \mu_\sigma \sigma$, with $\mu_\sigma \in \mathbb{C}$. Then $\Omega_\mu^L(t, t_0)$ is defined as the linear combination $\sum_{\sigma \in \mathcal{S}_n} \mu_\sigma \Omega_\sigma^L(t, t_0)$. For $K = (K_1, \dots, K_m)$ another sequence of time-dependent operators, we write $L \cdot K$ for the concatenation product $(L_1, \dots, L_n, K_1, \dots, K_m)$.

We also need to define the convolution product of two permutations. If $\alpha \in \mathcal{S}_n$ and $\beta \in \mathcal{S}_m$, then $\alpha * \beta$ is the sum of the $\binom{n+m}{n}$ permutations $\gamma \in \mathcal{S}_{n+m}$ such that $st(\gamma(1), \dots, \gamma(n)) = (\alpha(1), \dots, \alpha(n))$ and $st(\gamma(n+1), \dots, \gamma(n+m)) = (\beta(1), \dots, \beta(m))$. Here, st is the standardization map defined in section II C 1. For instance,

$$\begin{aligned} (2, 3, 1) * (1) &= (2, 3, 1, 4) + (2, 4, 1, 3) + (3, 4, 1, 2) + (3, 4, 2, 1), \\ (1, 2) * (2, 1) &= (1, 2, 4, 3) + (1, 3, 4, 2) + (1, 4, 3, 2) + (2, 3, 4, 1) + (2, 4, 3, 1) + (3, 4, 2, 1). \end{aligned}$$

In words, the product of two permutations $\alpha \in \mathcal{S}_n$ and $\beta \in \mathcal{S}_m$ is the sum of all permutations σ of \mathcal{S}_{n+m} such that the elements of the sequence $(\sigma(1), \dots, \sigma(n))$ are ordered as the elements of $(\alpha(1), \dots, \alpha(n))$, in the sense that $\alpha(i) > \alpha(j)$ if and only if $\sigma(i) > \sigma(j)$ and the elements of $(\sigma(n+1), \dots, \sigma(n+m))$ are ordered as the elements of $(\beta(1), \dots, \beta(m))$.

Now, we can state the noncommutative Chen formula⁴⁹ (see also remark 3.3, p. 4111 of ref. 21)

Lemma 11 *We have:*

$$\Omega_\alpha^L(t, t_0) \Omega_\beta^K(t, t_0) = \Omega_{\alpha * \beta}^{L \cdot K}(t, t_0).$$

The following lemma can be proven similarly:

Lemma 12 *We have, for L and K as above and J a time-dependent operator:*

$$\int_{t_0}^t ds \Omega_\alpha^L(s, t_0) J(s) \Omega_\beta^K(s, t_0) = \sum_\gamma \Omega_\gamma^{L \cdot (J) \cdot K}(t, t_0),$$

where γ runs over the permutations in \mathcal{S}_{n+m+1} with $\gamma(n+1) = 1$, $st(\gamma(1), \dots, \gamma(n)) = \alpha$, $st(\gamma(n+2), \dots, \gamma(n+m+1)) = \beta$.

Proof. We expand Ω_α^L and Ω_β^K

$$\begin{aligned} \int_{t_0}^t ds \Omega_\alpha^L(s, t_0) J(s) \Omega_\beta^K(s, t_0) &= \int_{t_0}^t ds \int_{t_0}^s du_1 \dots \int_{t_0}^{u_{n-1}} du_n \int_{t_0}^s dv_1 \dots \int_{t_0}^{v_{m-1}} dv_m \\ &\quad L_1(u_{\alpha(1)}) \dots L_n(u_{\alpha(n)}) J(s) K_1(v_{\beta(1)}) \dots K_m(v_{\beta(m)}) \end{aligned}$$

This is the same formula as for the expansion of $\Omega_\alpha^L(s, t_0) \Omega_\beta^K(s, t_0)$, up to the term $J(s)$ and the integration $\int_{t_0}^t ds$ that however do not change the underlying combinatorics. Therefore, the lemma 11 holds in the form

$$\begin{aligned} \int_{t_0}^t ds \Omega_\alpha^L(s, t_0) J(s) \Omega_\beta^K(s, t_0) &= \sum_\sigma \int_{t_0}^t ds \int_{t_0}^s ds_1 \dots \int_{t_0}^{s_{n+m-1}} ds_{n+m} \\ &\quad L_1(s_{\sigma(1)}) \dots L_n(s_{\sigma(n)}) J(s) K_1(s_{\sigma(n+1)}) \dots K_m(s_{\sigma(n+m)}), \end{aligned}$$

where, by the definition of $\alpha * \beta$, the sum over σ is over all the permutations of \mathcal{S}_{n+m} such that $st(\sigma(1), \dots, \sigma(n)) = \alpha$ and $st(\sigma(n+1), \dots, \sigma(n+m)) = \beta$. Now, we change variables to $t_1 = s$, $t_{i+1} = s_i$ for $i = 1, \dots, n+m$. The permutation γ of t_1, \dots, t_{n+m+1} corresponding to the permutation σ of s_1, \dots, s_{n+m} is characterized by $\gamma(n+1) = 1$ (because $s \geq s_i$ for all i), $\gamma(i) = \sigma(i) + 1$ for $1 \leq i \leq n$ and $\gamma(i+1) = \sigma(i) + 1$ for $n+1 \leq i \leq n+m$. Therefore,

$$\begin{aligned} \int_{t_0}^t ds \Omega_\alpha^L(s, t_0) J(s) \Omega_\beta^K(s, t_0) &= \sum_\gamma \int_{t_0}^t dt_1 \dots \int_{t_0}^{t_{n+m}} dt_{n+m+1} \\ &\quad L_1(t_{\gamma(1)}) \dots L_n(t_{\gamma(n)}) J(t_{\gamma(n+1)}) K_1(t_{\gamma(n+2)}) \dots K_m(t_{\gamma(n+m+1)}), \end{aligned}$$

where γ satisfies $\gamma(n+1) = 1$, $st(\gamma(1), \dots, \gamma(n)) = \alpha$, $st(\gamma(n+2), \dots, \gamma(n+m+1)) = \beta$. The lemma is proved. \square

Notice that the same process would allow to derive combinatorial formulas for arbitrary products of iterated integrals and for integrals with integrands involving iterated integrals, provided these products and integrands have expressions similar to the ones considered in the two lemmas.

APPENDIX B: THE ALGEBRAIC STRUCTURE OF TREE-SHAPED ITERATED INTEGRALS

Our exposition of tree-parametrized time-dependent perturbation theory has focussed on the derivation of convergence results and explicit formulas for the time-dependent wave operator. However, the reasons why such an approach is possible and efficient are grounded into various algebraic and combinatorial properties of trees, descents and similar objects.

These properties suggest that the theory of effective Hamiltonians is grounded into a new “Lie theory” generalizing the usual Lie theory (or, more precisely, generalizing the part of the classical Lie theory that is relevant to the study of the solutions of differential equations). First indications that such a theory exists were already pointed out in our ref. 21. Indeed, we showed in this article that descent algebras of hyperoctahedral groups and generalizations thereof are relevant to the time-dependent perturbation theory. Applications included an extension of the Magnus expansion for the time-dependent wave operator. Our results below provide complementary insights on the subject and further evidence that algebraic structures underly many-body theories.

1. The algebra structure

We know from ref. 21 and Sect IIB that the family of integrals $\Omega_\sigma^L(t, t_0)$ is closed under the product. This closure property is reflected into the convolution product of permutations. This result is a natural noncommutative generalization of Chen’s formula for the product of iterated integrals. We show, in the present section, that the same result holds for integrals parametrized by trees. We will explain, in the next sections, why such a result -which may seem surprising from the analytical point of view- could be expected from the modern theory of combinatorial Hopf algebras.

For $L = (L_1, \dots, L_n)$ a family of time-dependent operators (with the usual regularity conditions), and T a tree with n internal vertices, we write

$$\Omega_T^L(t, t_0) = \sum_{\sigma \in S_T} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n L_1(t_{\sigma(1)}) L_2(t_{\sigma(2)}) \dots L_n(t_{\sigma(n)}).$$

This notation is extended to linear combinations of trees, so that e.g. if $Z = T + 2T'$, where T and T' are two arbitrary trees with the same number of vertices, then $\Omega_Z^L(t, t_0) = \Omega_T^L(t, t_0) + 2\Omega_{T'}^L(t, t_0)$. For $i \leq n$, we write $L_{\leq i} = (L_1, \dots, L_i)$, $L_{\geq i} = (L_i, \dots, L_n)$.

Proposition 13 *For $L = (L_1, \dots, L_n)$ and $K = (K_1, \dots, K_m)$ two families of time-dependent operators and $T = T_1 \vee T_2$, $U = U_1 \vee U_2$ two trees, $|T| = n$, $|T_1| = p$, $|T_2| = q$, $|U| = m$, $|U_1| = l$, $|U_2| = k$, we have:*

$$\Omega_T^L(t, t_0) \Omega_U^K(t, t_0) = \int_{t_0}^t ds \Omega_{T_1}^{L_{\leq p}}(s, t_0) \Omega_{U_1}^{K_{\leq l}}(s, t_0) K_{l+1}(s) \Omega_{U_2}^{K_{\geq l+2}}(s, t_0) + \int_{t_0}^t ds \Omega_{T_1}^{L_{\leq p}}(s, t_0) L_{p+1}(s) \Omega_{T_2}^{L_{\geq p+2}}(s, t_0) \Omega_U^K(s, t_0).$$

In the formula, one or several of the trees T_1, T_2, U_1, U_2 may be the trivial tree $|$.

Proof. Recall the integration by parts formula. For any integrable functions f and g we define $F(t) = \int_{t_0}^t ds f(s)$, $G(t) = \int_{t_0}^t ds g(s)$ and $H(t) = F(t)G(t)$. Then

$$H(t) = \int_{t_0}^t ds \frac{dH(s)}{ds} = \int_{t_0}^t ds f(s)G(s) + \int_{t_0}^t ds F(s)g(s).$$

Now, we use this identity with $F(t) = \Omega_{T_1}^L(t, t_0)$. It follows from the proof of theorem 3 that $f(s)$ is given by

$$f(s) = \Omega_{T_1}^{L_{\leq p}}(s, t_0) L_{p+1}(s) \Omega_{T_2}^{L_{\geq p+2}}(s, t_0),$$

with a similar identity for $G(t) = \Omega_U^K(t, t_0)$. The proposition follows. \square

In particular, it is a consequence of the proposition and a straightforward recursion argument that the linear span of the integrals $\Omega_T^L(t, t_0)$ is closed under products.

2. Hopf algebras and Lie theory

This result may be formalized algebraically. Let us write \mathcal{T} for the set of formal power series with complex coefficients over the set of trees. Proposition 13 enables us to define a product on trees, denoted by $*$, such that $\Omega_T^L(t, t_0)\Omega_U^K(t, t_0) = \Omega_{T*U}^{L\cdot K}(t, t_0)$. This product is defined recursively by the equation²⁹

$$T * U = (T * U_1) \vee U_2 + T_1 \vee (T_2 * U).$$

Proof. The root $|$ is the unit for the product $*$. Assume that $T * U$ is defined and satisfies the recursive relation for all trees such that $|T| + |U| < n$, and consider two trees T and U with $|T| + |U| = n$. The first term on the right hand side of proposition 13 is $\int_{t_0}^t ds \Omega_T^L(s, t_0) \Omega_{U_1}^{K \leq l}(s, t_0) K_{l+1}(s) \Omega_{U_2}^{K \geq l+2}(s, t_0)$. By the recursive relation, we have

$$\Omega_T^L(s, t_0) \Omega_{U_1}^{K \leq l}(s, t_0) = \Omega_{T*U_1}^{L\cdot K \leq l}.$$

Thus, the whole term can be written

$$\int_{t_0}^t ds \Omega_T^L(s, t_0) \Omega_{U_1}^{K \leq l}(s, t_0) K_{l+1}(s) \Omega_{U_2}^{K \geq l+2}(s, t_0) = \Omega_{(T*U_1) \vee U_2}^{L\cdot K}.$$

The second term of proposition 13 is treated similarly and we obtain

$$\Omega_T^L(t, t_0) \Omega_U^K(t, t_0) = \Omega_{(T*U_1) \vee U_2}^{L\cdot K} + \Omega_{T_1 \vee (T_2*U)}^{L\cdot K}$$

Therefore, the relation $\Omega_T^L(t, t_0) \Omega_U^K(t, t_0) = \Omega_{T*U}^{L\cdot K}(t, t_0)$ gives us

$$T * U = (T * U_1) \vee U_2 + T_1 \vee (T_2 * U).$$

□

Corollary 14 *The product provides \mathcal{T} with the structure of an associative algebra.*

The corollary is a by-product of the associativity of the product of operators and of proposition 13.

Of course, although the analysis of tree-shaped iterated integrals leads to a straightforward proof, the associativity property is a purely combinatorial phenomenon that originates ultimately in the associativity of the shuffle product. See ref. 21 for the connexions between the noncommutative Chen formula and shuffle products, see also section 4 of ref. 50 for Schützenberger’s classical (but rarely quoted) analysis of the formal properties of the shuffle product -in fact, the splitting of the convolution product of trees reflects the classical splitting of shuffle products into left and right half-shuffle products that had appeared in the study of Lie polynomials and was first encoded combinatorially in ref. 50. The associativity can also be checked directly or deduced from the associativity of the convolution product $*$ of permutations, since one may verify easily, using e.g. our description of the permutations in S_T that the convolution product as introduced above in nothing but the restriction to \mathcal{T} of the convolution product on the algebra $\mathcal{S} = \prod_{n \in \mathbb{N}} \mathbb{C}[\mathcal{S}_n]$.

This fact that the linear span of trees (often written **PBT**) defines a subalgebra of \mathcal{S} for the convolution product (and even a Hopf subalgebra, referred to as the Hopf algebra of planar binary trees, whereas \mathcal{S} is referred to as the Malvenuto-Reutenauer or Hopf algebra of free quasi-symmetric functions in the litterature) is well-known. It was first observed in ref. 29 and further investigated in a series of papers^{34,36,37,51}. There is however a slight subtlety here, since the embedding of **PBT** in \mathcal{S} considered e.g. in ref. 29 is not the one we consider (another projection map from permutations to trees is used) so that **PBT** and \mathcal{T} , although isomorphic as algebras (the product rule in **PBT** is the same as the one in \mathcal{T} , see e.g. proposition 3.2 of ref. 29, where **PBT** is written $\mathbb{C}[Y_\infty]$) do not agree as subalgebras of \mathcal{S} .

However, a corollary of this isomorphism is that the structures existing on **PBT** carry over to the analysis of the algebraic properties of tree-shaped iterated integrals. The existence of a Hopf algebra structure seems particularly meaningful since the classical applications of the theory of free Lie algebras to the analysis of differential equations can be rewritten using the framework of Hopf algebras (see e.g. the accounts in refs. 52,53).

The same observation holds for the direct sum of the hyperoctahedral group algebras, that also carries naturally a Hopf algebra structure^{54,55}, and which applications to time-dependent perturbative Hamiltonians were studied in ref. 21. We leave for further research the investigation of the possible role of these Hopf algebra structures in the context of many-body theories.

3. Permutations, trees and descents

Another meaningful observation, along the same lines, is that the three expansions we derived, based respectively on sequences of P and Q (as in the first expression for $\Omega(t, t_0)$), on trees (third expression) and on permutations (second expression), reflect at the analytical level the existence of projection maps and embeddings between hypercubes (or the descent algebra), planar binary trees, and permutations. Following Viennot²², these maps have been at the origin of modern enumerative combinatorics. We refer to refs. 29,34,36,37 for a detailed study of these maps that emphasizes the existence of underlying geometrical structures that go beyond the Hopf algebraic ones.

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