



**HAL**  
open science

# Regularity of the Exercise Boundary for American Put Options on Assets with Discrete Dividends

Benjamin Jourdain, Michel Vellekoop

► **To cite this version:**

Benjamin Jourdain, Michel Vellekoop. Regularity of the Exercise Boundary for American Put Options on Assets with Discrete Dividends. 2009. hal-00436327v1

**HAL Id: hal-00436327**

**<https://hal.science/hal-00436327v1>**

Preprint submitted on 26 Nov 2009 (v1), last revised 3 Jun 2010 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Regularity of the Exercise Boundary for American Put Options on Assets with Discrete Dividends

B. Jourdain\* and M. Vellekoop†

November 26, 2009

## Abstract

We analyze the regularity of the optimal exercise boundary for the American Put option when the underlying asset pays a discrete dividend at a known time  $t_d$  during the lifetime of the option. The ex-dividend asset price process is assumed to follow Black-Scholes dynamics and the dividend amount is a deterministic function of the ex-dividend asset price just before the dividend date. The solution to the associated optimal stopping problem can be characterised in terms of an optimal exercise boundary which, in contrast to the case when there are no dividends, is no longer monotone. In this paper we prove that when the dividend function is positive and concave, then the boundary tends to 0 as time tends to  $t_d^-$  and is non-increasing in a left-hand neighbourhood of  $t_d$ . We also show that the exercise boundary is continuous and a high contact principle holds in such a neighbourhood when the dividend function is moreover linear in a neighbourhood of 0.

## Introduction

We consider the American Put option with strike  $K > 0$  and maturity  $T > 0$  on an underlying stock. We assume that the stochastic dynamics of the ex-dividend price process of this stock can be modelled by the Black-Scholes model and that at a given time  $t_d \in (0, T)$  a discrete dividend is paid. The value of this dividend is a function  $D : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  of the ex-dividend asset price  $S_{t_d^-}$ . This means that

$$dS_u = \sigma S_u dW_u + r S_u du - D(S_{u^-}) d1_{\{u \geq t_d\}} \quad (0.1)$$

for an initial price  $S_0$ , interest rate  $r$  and volatility  $\sigma$  which are assumed to be positive and with  $W$  a standard Brownian Motion.

Throughout the paper we assume that the dividend function  $D$  is non-negative and non-decreasing and such that  $x \in \mathbb{R}_+ \mapsto x - D(x)$  is non-negative and non-decreasing. We will pay particular attention to the following special cases :

- $D(x) = (1 - \rho)x$  where  $\rho \in (0, 1)$ , which we will call the *proportional* dividend case,

---

\*Université Paris-Est, CERMICS, Projet MathFi ENPC-INRIA-UMLV, 6 et 8 avenue Blaise Pascal, 77455 Marne La Vallée, Cedex 2, France , e-mail : jourdain@cermics.enpc.fr. This research benefited from the support of the “Chair Risques Financiers”, Fondation du Risque

†University of Amsterdam, Faculty of Economics and Business, Department of Quantitative Economics, Section Actuarial Science, Roetersstraat 11, 1018 WB Amsterdam, e-mail : m.h.vellekoop@uva.nl

- $D(x) = D \wedge x$  with  $D > 0$ , which we will call the *constant* dividend case.

For  $t \in [0, T]$ , let

$$U_t = \text{ess. sup}_{\tau \in \mathcal{T}_{[t, T]}} \mathbb{E}[e^{-r(\tau-t)}(K - S_\tau)^+ | \mathcal{F}_t] \quad (0.2)$$

where  $\mathcal{T}_{[t, T]}$  is the set of stopping times with respect to the filtration  $\mathcal{F}_t \stackrel{\text{def}}{=} \sigma(W_s, 0 \leq s \leq t)$  taking values in  $[t, T]$  denote the price at time  $t$  of the American Put option.

The solution to this optimal stopping problem for the case without dividends goes back to the work of McKean [15] and Van Moerbeke [20]. The optimal stopping time is the first time that the asset price process falls below a time-dependent value (the so-called exercise boundary which we will denote by  $\bar{c}$ ), and McKean derived a free-boundary problem involving both the pricing function  $\bar{u}$  such that  $U_t = \bar{u}(t, S_t)$  and  $\bar{c}$ . Van Moerbeke derived an integral equation which involves both  $\bar{c}$  and its derivative, but in later work by Kim [13], Jacka [11] and Carr, Jarrow and Myneni [2] an integral equation was derived which only involves  $\bar{c}$  itself. The regularity and uniqueness of solutions to this equation was left as an open problem in those papers. Uniqueness was proven by Peskir [18], using his change-of-variable formula with local time on curves [17]. It is known that the optimal exercise boundary is convex [4, 5] and its asymptotic behaviour at maturity is given in [14]. But although it was claimed in several papers (for example [16]) that it is  $C^1$  at all points prior to maturity, a complete proof has been given only recently by Chen and Chadam [3]. In fact, in that paper it was actually shown that it is  $C^\infty$  in all those points and a later paper by Bayraktar and Xing [1] shows that this remains true if the underlying asset pays continuous dividends at a fixed rate.

In practice, continuous dividends are not a satisfying model since dividends are paid once a year or quarterly. That is why we are interested in discrete dividends. To begin with, we deal in this paper with the simplest situation where there is only one dividend time  $t_d$  before the maturity  $T$  of the Put option. When we assume discrete dividend payments such as the proportional or fixed dividend payments mentioned above, the optimal exercise boundary will become discontinuous at the dividend date and before the dividend date it may not be monotone (see Figure 1). Integral formulas for the exercise boundary which are similar to the ones in [2] have been derived under the assumption that the boundary is Lipschitz continuous (see Göttsche and Vellekoop [9]) or locally monotonic (Vellekoop & Nieuwenhuis [22]). In this paper we therefore study conditions under which such regularity properties of the optimal exercise boundary under discrete dividend payments can be proven.

In the first Section, we introduce the pricing function  $u$  of the American Put option in the model (0.1) and the associated exercise boundary  $c$ . We also explain that, on the time-interval  $[0, t_d)$ , the American Put price is equal to the price of an American option with maturity  $t_d$ , Put payoff  $x \mapsto (K - x)^+$  when exercised early and a modified payoff  $x \mapsto \bar{u}(t_d, x - D(x))$  when exercised at maturity  $t_d$  in the Black-Scholes model with no dividends. Last, we study properties of this function  $x \mapsto \bar{u}(t_d, x - D(x))$ . In the second Section, we prove that when the dividend function is positive and concave, then the boundary tends to 0 as time tends to  $t_d^-$  and is non-increasing in a left-hand neighbourhood of  $t_d$ . In the third Section we assume moreover that the dividend function is linear in a neighbourhood of 0, a condition satisfied in both the *proportional* and the *constant* dividend cases. Then we show that the exercise boundary is continuous and a high contact principle holds in a left-hand neighbourhood of  $t_d$ .

## Notations and definitions :

- For  $t \in [0, T]$  and  $x \geq 0$ , we use the notation  $\bar{S}_t^x = xe^{\sigma W_t + (r - \frac{\sigma^2}{2})t}$  for the stock price at time  $t$  when the initial price is equal to  $x$  for the case where there is no dividend (i.e.  $D \equiv 0$ ). We also denote by  $L_t^y(\bar{S}^x)$  the local time at level  $y > 0$  and time  $t$  of the process  $\bar{S}^x$  and by  $p(t, y) = \frac{1_{\{y>0\}}}{\sigma y \sqrt{2\pi t}} \exp\left(-\frac{(\log(y/x) - (r - \frac{\sigma^2}{2})t)^2}{2\sigma^2 t}\right)$  the density of  $\bar{S}_t^x$  with respect to the Lebesgue measure when  $t, x > 0$ .
- Let  $\mathcal{A}$  denote the infinitesimal generator of the Black-Scholes model without dividends :  $\mathcal{A}f(x) = \frac{\sigma^2 x^2}{2} f''(x) + rx f'(x) - rf(x)$ .
- If  $(t, x) \in [0, T] \times \mathbb{R}_+$ , we write  $S_u^{x,t}$  for the solution to (0.1) for  $u \geq t$  under the initial condition that  $S_t^{x,t} = x$ .
- Let  $N(y) = \int_{-\infty}^y e^{-z^2/2} \frac{dz}{\sqrt{2\pi}}$  be the cumulative distribution function of the standard normal law.
- Let  $C$  denote a constant with may change from line to line.
- We say that  $D$  is positive when  $\forall x > 0, D(x) > 0$ .
- By a left-hand neighbourhood of  $x \in \mathbb{R}$ , we mean an open interval  $(x - \varepsilon, x)$  for some  $\varepsilon > 0$ .

## 1 Preliminary results

The following results, which have been proven in [6, 7, 10, 19], provide an optimal stopping time in (0.2).

**Proposition 1.1** *Let  $\{G_t, t \in [0, T]\}$  be an  $(\mathcal{F}_t)$ -adapted right-continuous upper-semicontinuous process with  $\mathbb{E}(\sup_{t \in [0, T]} |G_t|) < \infty$ .*

*Then the càdlàg version of the Snell envelope  $U_t = \text{ess. sup}_{\tau \in \mathcal{T}_{[t, T]}} \mathbb{E}(G_\tau | \mathcal{F}_t)$  is continuous on  $[0, T]$  and the stopping time  $\tau = \inf\{s \geq t : U_s = G_s\}$  is optimal :  $U_t = \mathbb{E}(G_\tau | \mathcal{F}_t)$ .*

The conditions for this result are satisfied by  $G_t = e^{-rt}(K - S_t)^+$  since for all  $t \in [0, T]$  we have  $|G_t| \leq K$  and  $G_t$  is right-continuous and upper semicontinuous for all  $t \in [0, T]$  since the jump size of  $S_t$  at  $t = t_d$  is non-positive. According to [7], there exists a pricing function  $u$  such that  $U_t = u(t, S_t)$  :

**Proposition 1.2** *The Snell envelop  $U$  of  $\{G_t = e^{-rt}(K - S_t)^+, t \in [0, T]\}$  is such that  $U_t = e^{-rt}u(t, S_t)$  where*

$$\forall (t, x) \in [0, T] \times \mathbb{R}_+, u(t, x) \stackrel{\text{def}}{=} \sup_{\tau \in \mathcal{T}_{[t, T]}} \mathbb{E}(e^{-r(\tau-t)}(K - S_\tau^{x,t})^+).$$

*Moreover the previous supremum is attained for  $\tau = \inf\{s \geq t : u(s, S_s^{x,t}) = (K - S_s^{x,t})^+\}$ .*

Let us now derive some properties of the pricing function  $u$  which ensure existence of the exercise boundary.

**Lemma 1.3** *Let the dividend function  $D$  be non-negative, non-decreasing and such that  $x \in \mathbb{R}_+ \mapsto x - D(x)$  is non-negative and non-decreasing. Then we have*

$$\forall t \in [0, T], \forall x > y \geq 0, 0 \leq u(t, y) - u(t, x) \leq x - y. \quad (1.1)$$

For  $t \in [0, T]$ , let

$$c(t) = \inf\{x > 0 : u(t, x) > (K - x)^+\}.$$

Then we have that  $\{x \geq 0 : u(t, x) = (K - x)^+\} = [0, c(t)]$  and the function  $c$  cannot vanish on an interval.

Figure 1 plots the exercise boundary  $t \mapsto c(t)$  of the Put option with strike  $K = 100$  and maturity  $T = 4$  in the model (0.1) with  $r = 0.04$ ,  $\sigma = 0.3$ ,  $t_d = 3.5$  and *proportional* dividends with  $\rho = 0.95$ . This exercise boundary was computed by a binomial tree method (see [21]).

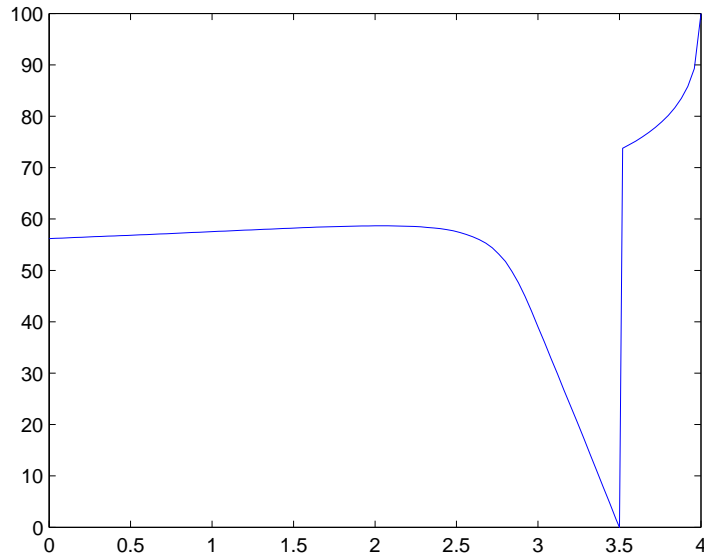


Figure 1: Exercise boundary  $t \mapsto c(t)$  ( $K = 100$ ,  $T = 4$ ,  $t_d = 3.5$ ,  $r = 0.04$ ,  $\sigma = 0.3$ , *proportional* dividends :  $\rho = 0.05$ ) obtained by a binomial tree method

**Proof .** For the first part, we use a similar proof as in [9]. For a fixed  $t \in [0, T]$  take  $x > y \geq 0$  which, with the monotonicity of  $z \mapsto z - D(z)$  implies that  $S_v^{x,t} \geq S_v^{y,t}$  for all  $v \in [t, T]$ . For  $\tau_x \in \mathcal{T}_{[t,T]}$  such that  $u(t, x) = \mathbb{E}[e^{-r(\tau_x-t)}(K - S_{\tau_x}^{x,t})^+]$ , since  $\tau_x$  need not be optimal for the case where  $S_t = y$ , we deduce

$$u(t, x) - u(t, y) \leq \mathbb{E}[e^{-r(\tau_x-t)}((K - S_{\tau_x}^{x,t})^+ - (K - S_{\tau_x}^{y,t})^+)] \leq 0.$$

For  $\tau_y \in \mathcal{T}_{[t,T]}$  such that  $u(t, y) = \mathbb{E}[e^{-r(\tau_y-t)}(K - S_{\tau_y}^{y,t})^+]$ ,

$$\begin{aligned} u(t, y) - u(t, x) &\leq \mathbb{E}[e^{-r(\tau_y-t)}(K - S_{\tau_y}^{y,t})^+] - \mathbb{E}[e^{-r(\tau_y-t)}(K - S_{\tau_y}^{x,t})^+] \\ &\leq \mathbb{E}[e^{-r(\tau_y-t)}(S_{\tau_y}^{x,t} - S_{\tau_y}^{y,t})] \\ &= x - y - \mathbb{E}[e^{-r(\tau_y-t)}1_{\{\tau_y \geq t_d\}}(D(S_{\tau_y}^{x,t}) - D(S_{\tau_y}^{y,t}))S_{\tau_y}^{1,t_d}] \leq x - y \end{aligned}$$

because of our assumption that  $D$  is non-decreasing.

Since  $u(t, x) \geq (K - x)^+$  for all  $t \in [0, T]$  and  $x \geq 0$ , the definition of  $c(t)$  implies that  $u(t, x) = (K - x)^+$  for  $x \in [0, c(t))$  and by the continuity of  $x \rightarrow u(t, x) - (K - x)^+$  this must then be true for  $x = c(t)$  as well when  $c(t) > 0$ . When  $c(t) = 0$ ,  $u(t, c(t)) = K = (K - c(t))^+$ . If  $x > c(t)$  then, by definition of  $c(t)$  there exists  $y \in (c(t), x]$  such that  $u(t, y) > (K - y)^+$  and  $u(t, x) \geq u(t, y) + y - x > K - x$ . Since  $u(t, x) \geq \mathbb{E}(e^{-r(T-t)}(K - S_T^{x,t})^+) > 0$ , one deduces that  $u(t, x) > (K - x)^+$  for  $x > c(t)$ .

Assume that there exists an interval  $[t_1, t_2]$  with  $0 \leq t_1 \leq t_2 \leq T$  such that  $c$  is zero in every point of this interval, and for  $x > 0$ , let  $\tau_x \in \mathcal{T}_{[t_1, T]}$  be such that we have that  $u(t_1, x) = \mathbb{E}[e^{-r(\tau_x - t_1)}(K - S_{\tau_x}^{x, t_1})^+]$ . Then  $\tau_x \geq t_2$  so  $K e^{-r(t_2 - t_1)} \geq K \mathbb{E}[e^{-r(\tau_x - t_1)}] \geq u(t_1, x) \geq (K - x)^+$ . Letting  $x \rightarrow 0^+$ , one deduces that  $t_2 = t_1$ .  $\blacksquare$

Let us now prove some regularity properties of the pricing function  $u$ .

**Lemma 1.4** *Under the assumptions of Lemma 1.3, the function  $u$  is continuous on  $[0, t_d) \times \mathbb{R}_+$  and for all  $x$  outside the at most countable set of discontinuities of  $D$ , the limit  $\lim_{t \rightarrow t_d^-} u(t, x)$  exists and is equal to  $u(t_d, x - D(x))$ .*

*Moreover, for all  $t \in (0, t_d)$  and  $x > c(t)$  the partial derivatives  $\partial_t u(t, x)$ ,  $\partial_x u(t, x)$  and  $\partial_{xx} u(t, x)$  exist and  $\mathcal{A}u(t, \cdot)(x) + \partial_t u(t, x) = 0$ .*

**Proof .** Let us check the behaviour of  $u$  as  $t \rightarrow t_d^-$ , the continuity of this function on  $[0, t_d) \times \mathbb{R}_+$  following from a similar but easier argument.

Since  $S_{t_d} = S_{t_d^-} - D(S_{t_d^-})$ , one has, using (1.1) for the inequality,

$$|u(t, S_{t_d^-}) - u(t_d, S_{t_d^-} - D(S_{t_d^-}))| = |u(t, S_{t_d^-}) - u(t_d, S_{t_d^-})| \leq |S_t - S_{t_d^-}| + |u(t, S_t) - u(t_d, S_{t_d^-})|.$$

By continuity of the process  $(u(t, S_t))_{t \in [0, T]}$  ensured by Propositions 1.1 and 1.2, one deduces that a.s.,  $\lim_{t \rightarrow t_d^-} u(t, S_{t_d^-}) = u(t_d, S_{t_d^-} - D(S_{t_d^-}))$ . Since  $S_{t_d^-}$  admits a positive density w.r.t. the Lebesgue measure on  $(0, +\infty)$ ,  $dx$  a.e.  $\lim_{t \rightarrow t_d^-} u(t, x) = u(t_d, x - D(x))$ . By continuity of  $x \mapsto u(t_d, x)$ , the function  $x \mapsto u(t_d, x - D(x))$  is continuous outside the at most countable set of discontinuities of the non-decreasing function  $D$ . With (1.1), one concludes that for all  $x$  outside this set,  $\lim_{t \rightarrow t_d^-} u(t, x) = u(t_d, x - D(x))$ .

By continuity of  $u$  on  $[0, t_d) \times \mathbb{R}_+$ ,  $\{(t, x) \in [0, t_d) \times \mathbb{R}_+ : x > c(t)\}$  which by Lemma 1.3 is equal to  $\{(t, x) \in [0, t_d) \times \mathbb{R}_+ : u(t, x) > (K - x)^+\}$  is an open subset of  $[0, t_d) \times \mathbb{R}_+$ . Let  $t \in (0, t_d)$ ,  $x > c(t)$  and  $B$  be an open neighbourhood of  $(t, x)$  with regular boundary  $\partial B$  such that  $y > c(s)$  and  $s < t_d$  for all  $(s, y) \in B$ . Define the stopping times  $\tau = \inf\{v \geq t : S_v^{x,t} \leq c(v)\}$  and  $\tau_{B^c} = \inf\{v \geq t : S_v^{x,t} \in B^c\} < \tau$ . The flow property for the Black-Scholes model without dividends implies that for  $v \geq \tau_{B^c}$ ,  $S_v^{x,t} = S_v^{S_{\tau_{B^c}}^{x,t}, \tau_{B^c}}$  and  $\tau = \inf\{v \geq \tau_{B^c} : S_v^{S_{\tau_{B^c}}^{x,t}, \tau_{B^c}} \leq c(v)\}$ . Using the strong Markov property for the third equality, one deduces

$$\begin{aligned} u(t, x) &= \mathbb{E}[e^{-r(\tau-t)}(K - S_{\tau}^{x,t})^+] = \mathbb{E}[e^{-r(\tau_{B^c}-t)} \mathbb{E}[e^{-r(\tau-\tau_{B^c})}(K - S_{\tau}^{S_{\tau_{B^c}}^{x,t}, \tau_{B^c}})^+ | \mathcal{F}_{\tau_{B^c}}]] \\ &= \mathbb{E}[e^{-r(\tau_{B^c}-t)} u(\tau_{B^c}, S_{\tau_{B^c}}^{x,t})]. \end{aligned} \quad (1.2)$$

Let  $f(s, x)$  be a solution to the Dirichlet problem where  $\partial_s f + \mathcal{A}f = 0$  on  $B$  and  $f = u$  on  $\partial B$ . By Theorem 3.6.3. in [8] this function  $f$  is  $C^{1,2}$  in  $B$  and continuous on  $\bar{B}$ . But then  $u(t, x) = \mathbb{E}[e^{-r(\tau_{B^c}-t)} u(\tau_{B^c}, S_{\tau_{B^c}}^{x,t})] = \mathbb{E}[e^{-r(\tau_{B^c}-t)} f(\tau_{B^c}, S_{\tau_{B^c}}^{x,t})] = f(t, x) + \mathbb{E} \int_t^{\tau_{B^c}} (\partial_s f + \mathcal{A}f)(s, S_s^{x,t}) ds = f(t, x)$  by optional sampling so  $u = f$  on  $B$  and therefore its partial derivatives exist in  $(t, x)$  and satisfy  $\partial_t u(t, x) + \mathcal{A}u(t, \cdot)(x) = 0$ .  $\blacksquare$

The interpretation of the restriction of  $u$  to  $[0, t_d] \times \mathbb{R}_+^*$  as the pricing function of an American option in the Black-Scholes model with no dividends stated in the next Proposition is the key of the study of the exercise boundary  $c(t)$  performed in the following sections.

**Proposition 1.5** *Under the assumptions of Lemma 1.3,*

$$\forall (t, x) \in [0, t_d] \times \mathbb{R}_+, u(t, x) = \sup_{\tau \in \mathcal{T}_{[0, t_d-t]}} \mathbb{E}[e^{-r\tau} ((K - \bar{S}_\tau^x)^+ 1_{\{\tau < t_d-t\}} + g(\bar{S}_{t_d}^x) 1_{\{\tau = t_d-t\}})].$$

where  $g(x) \stackrel{\text{def}}{=} u(t_d, x - D(x))$  and the supremum is attained for  $\tau = \inf\{s \in [0, t_d - t] : \bar{S}_s^x \leq c(t + s)\} \wedge t_d - t$  (convention  $\inf \emptyset = +\infty$ ). Moreover, the restriction of  $u$  to  $[t_d, T] \times \mathbb{R}_+$  coincides with the one of the pricing function  $\bar{u}$  of the American Put option with maturity  $T$  in the Black-Scholes model without dividends. In particular,  $\forall x \geq 0, g(x) = \bar{u}(t_d, x - D(x))$ .

**Proof .** The second statement is obvious since no dividend is payed on the time interval  $[t_d, T]$ . Let  $(t, x) \in [0, t_d] \times \mathbb{R}_+$  and  $\tau_x = \inf\{v \geq t : S_v^{x,t} \leq c(v)\}$ . Arguing like in the derivation of (1.2), one easily checks that

$$\mathbb{E} \left[ e^{-r(\tau_x-t)} (K - S_{\tau_x}^{x,t})^+ 1_{\{\tau_x \geq t_d\}} \right] = \mathbb{E} \left[ e^{-r(t_d-t)} u(t_d, S_{t_d}^{x,t}) 1_{\{\tau_x \geq t_d\}} \right] = \mathbb{E} \left[ e^{-r(t_d-t)} g(S_{t_d}^{x,t}) 1_{\{\tau_x \geq t_d\}} \right]$$

and deduces that

$$\begin{aligned} u(t, x) &= \mathbb{E} \left[ e^{-r(\tau_x-t)} (K - S_{\tau_x}^{x,t})^+ 1_{\{\tau_x < t_d\}} + e^{-r(t_d-t)} g(S_{t_d}^{x,t}) 1_{\{\tau_x \geq t_d\}} \right] \\ &= \mathbb{E} \left[ e^{-r\tau} (K - \bar{S}_\tau^x)^+ 1_{\{\tau < t_d-t\}} + e^{-r(t_d-t)} g(\bar{S}_{t_d}^x) 1_{\{\tau = t_d-t\}} \right], \end{aligned}$$

when  $\tau = \inf\{s \in [0, t_d - t] : \bar{S}_s^x \leq c(t + s)\} \wedge t_d - t$ .

Let now  $\tau$  be any stopping time in  $\mathcal{T}_{[0, t_d-t]}$ . For  $f : C([0, t_d - t], \mathbb{R}) \rightarrow [0, t_d]$  such that  $\tau = f(W_s, 0 \leq s \leq t_d - t)$ ,

$$\tau_x \stackrel{\text{def}}{=} \begin{cases} t + f(W_s - W_t, t \leq s \leq t_d) & \text{if } t + f(W_s - W_t, t \leq s \leq t_d) < t_d \\ \inf\{s \geq t_d : S_s^{x,t} \leq c(s)\} & \text{otherwise} \end{cases}$$

belongs to  $\mathcal{T}_{[t, T]}$  and is such that

$$\begin{aligned} &\mathbb{E} \left[ e^{-r\tau} (K - \bar{S}_\tau^x)^+ 1_{\{\tau < t_d-t\}} + e^{-r(t_d-t)} g(\bar{S}_{t_d}^x) 1_{\{\tau = t_d-t\}} \right] \\ &= \mathbb{E} \left[ e^{-r(\tau_x-t)} (K - S_{\tau_x}^{x,t})^+ 1_{\{\tau_x < t_d\}} + e^{-r(t_d-t)} u(t_d, S_{t_d}^{x,t}) 1_{\{\tau_x \geq t_d\}} \right] \\ &= \mathbb{E} \left[ e^{-r(\tau_x-t)} (K - S_{\tau_x}^{x,t})^+ \right] \leq u(t, x). \end{aligned}$$

■

We now derive some properties of the function  $g(x) = \bar{u}(t_d, x - D(x))$ . We will write  $\bar{c}$  for the optimal exercise boundary of the American Put when there are no dividends. Obviously,  $c(t) = \bar{c}(t)$  for  $t \in [t_d, T]$ .

**Lemma 1.6** *Assume that  $D$  is a non-negative concave function such that  $x - D(x)$  is non-negative. Then  $D$  is continuous, non-decreasing and such that  $x - D(x)$  is non-decreasing. Let*

$D'_-(x)$  and  $D''(dx)$  respectively denote the left-hand derivative of  $D$  and the non-positive Radon measure equal to the second order distribution derivative of  $D$  on  $(0, +\infty)$ . The function  $g$  is continuous, non-increasing and  $g(x) \geq (K - x)^+$  for all  $x \geq 0$ . The function

$$\gamma(x) \stackrel{\text{def}}{=} \frac{\sigma^2 x^2}{2} (1 - D'_-(x))^2 \partial_{22} \bar{u}(t_d, x - D(x)) + rx(1 - D'_-(x)) \partial_2 \bar{u}(t_d, x - D(x)) - r \bar{u}(t_d, x - D(x))$$

where, by convention,  $\partial_{22} \bar{u}(t_d, \bar{c}(t_d)) = 0$ , is not greater than  $-rK$  on  $(0, x^*)$  where  $x^* \stackrel{\text{def}}{=} \sup\{x : x - D(x) < \bar{c}(t_d)\} > 0$ , and globally bounded.

If  $g$  is convex, then there is a constant  $\rho \in [0, 1]$  such that  $g(x) = K - \rho x$  and  $D(x) = (1 - \rho)x$  for  $x < x^*$ , the second order distribution derivative of  $g$  admits a density  $g''$  w.r.t. the Lebesgue measure and  $\mathcal{A}g(x)$  is equal to  $-rK$  on  $(0, x^*)$  and  $dx$  a.e. on  $(x^*, +\infty)$ ,  $\mathcal{A}g(x) \geq -rK$ .

To prove this lemma, we need the following properties of the pricing function  $\bar{u}$  in the model without dividends.

**Lemma 1.7** *For the case without dividends we have that the partial derivatives  $\partial_t \bar{u}(t, x)$ ,  $\partial_x \bar{u}(t, x)$  and  $\partial_{xx} \bar{u}(t, x)$  exist and  $\partial_t \bar{u}(t, x) + \mathcal{A}\bar{u}(t, \cdot)(x) = 0$  for all  $t \in [0, T]$  and  $x > \bar{c}(t)$ . Moreover,  $\forall t \in [0, T]$ ,  $x \mapsto \bar{u}(t, x)$  is convex and  $C^1$  on  $\mathbb{R}_+$ . Last,*

$$\forall t \in [0, T], \forall x > \bar{c}(t), \partial_t \bar{u}(t, x) \geq -\frac{e^{-r(T-t)} \sigma^2 K}{2\sigma \sqrt{2\pi(T-t)}} \exp\left(-\frac{(\log(K/x) - (r - \frac{\sigma^2}{2})(T-t))^2}{2\sigma^2(T-t)}\right).$$

Before proving these Lemmas, let us give some examples of functions  $g$  obtained for different choices of the dividend function  $D$ .

### Examples of functions $g$ :

- In the *constant* dividend case,  $x^* = \bar{c}(t_d) + D$  and the function  $g$  is equal to  $K$  on  $[0, D]$  and to  $K + D - x$  for  $x \in (D, x^*)$ ,  $C^1$  on  $[0, D] \cup (D, +\infty)$  with  $g'$  taking its values in  $[-1, 0]$ ,  $C^2$  on  $[0, D] \cup (D, x^*) \cup (x^*, +\infty)$  and such that  $\mathcal{A}g(dx) = \gamma(x)dx - \frac{\sigma^2 D^2}{2} \delta_D(dx)$  where  $\gamma$  is equal to  $-rK$  on  $(0, D)$  and to  $-r(K + D)$  on  $(D, x^*)$ .
- In the *proportional* dividend case,  $x^* = \bar{c}(t_d)/\rho$  and  $g(x) = \bar{u}(t_d, \rho x)$  is convex,  $C^1$  with  $g'$  taking its values in  $[-\rho, 0]$  and  $C^2$  on  $[0, x^*) \cup (x^*, +\infty)$ .
- The *proportional* dividend case provides an example of a non-negative concave function  $D$  such that  $x - D(x)$  is non-negative which leads to a convex function  $g$ . This example is not unique. For instance, let  $\rho \in (0, 1)$ . The function  $y \mapsto \bar{u}(t_d, y)$  is convex positive nonincreasing and such that  $\lim_{y \rightarrow +\infty} \bar{u}(t_d, y) = 0$ . So it is continuous and decreasing and admits an inverse  $V(t_d, \cdot) : (0, K] \rightarrow [0, +\infty)$ . For  $x \in (\bar{c}(t_d)/\rho, K/\rho)$ , we set  $d(x) = x - V(t_d, K - \rho x)$ . The continuous function  $d'(x) = 1 + \rho/\partial_2 \bar{u}(t_d, V(t_d, K - \rho x))$  is non-increasing on  $(\bar{c}(t_d)/\rho, K/\rho)$  by the non-increasing property of both  $V(t_d, \cdot)$  and  $-\partial_2 \bar{u}(t_d, \cdot)$  and the positivity of this last function. It tends respectively to  $1 - \rho$  and  $-\infty$  as  $x \rightarrow \bar{c}(t_d)/\rho$  and  $x \rightarrow K/\rho$ . Let  $x_0 = \sup\{x \in (\bar{c}(t_d)/\rho, K/\rho) : d'(x) \geq 0\}$ . One has  $d'(x_0) = 0$  which also writes  $\partial_2 \bar{u}(t_d, x_0 - d(x_0)) = -\rho$ . The function

$$D(x) = \begin{cases} (1 - \rho)x & \text{for } x \in [0, \bar{c}(t_d)/\rho] \\ d(x \wedge x_0) & \text{for } x > \bar{c}(t_d)/\rho \end{cases}$$



is non-negative, concave and such that  $x - D(x)$  is non-negative. The convexity of  $x \mapsto \bar{u}(t_d, x)$  combined with the equality  $\partial_2 \bar{u}(t_d, x_0 - d(x_0)) = -\rho$  implies that

$$g(x) = \begin{cases} K - \rho x & \text{for } x \in [0, x_0] \\ \bar{u}(t_d, x - d(x_0)) & \text{for } x > x_0 \end{cases}$$

is convex.

Figure 2 illustrates the construction of the function  $g$  from  $x \mapsto \bar{u}(t_d, x)$  on the three previous examples of dividend functions.

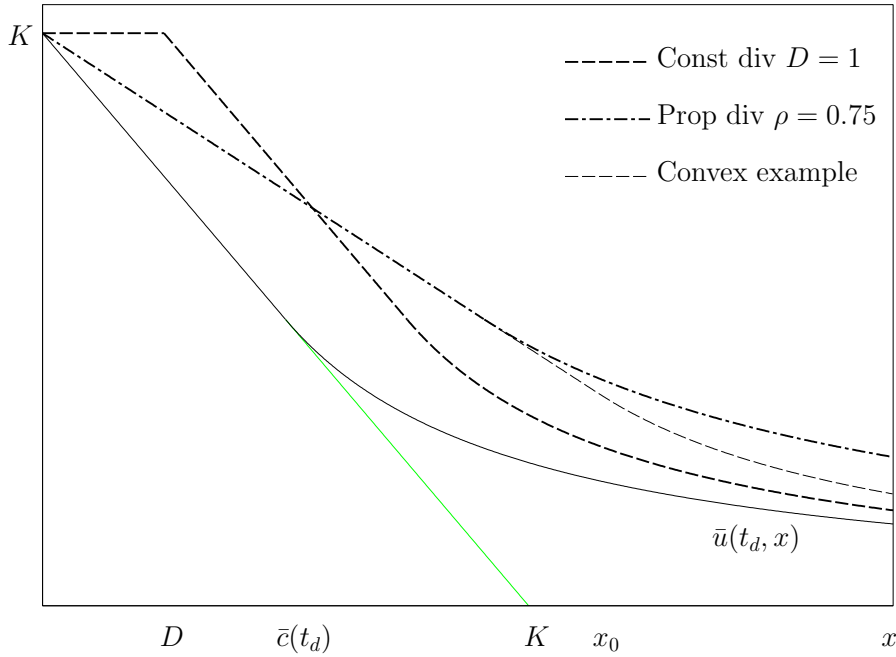


Figure 2: Examples of functions  $g$

**Proof of Lemma 1.6.** Since the concave function  $D$  is non-negative, it is continuous and non-decreasing. And since  $x - D(x)$  is non-negative,  $D(0) = 0$ . The convex function  $x - D(x)$  being non-negative and equal to 0 for  $x = 0$ , is non-decreasing. Since  $x \mapsto \bar{u}(t_d, x)$  is continuous, non-increasing and not smaller than  $(K - x)^+$ , the same properties hold for  $g$ .

For  $x \in (0, x^*)$ ,  $\gamma(x) = rx(D'_-(x) - 1) - r(K - x + D(x)) = -rK - r(D(x) - xD'_-(x))$ . By concavity of  $D$ ,

$$\forall x > 0, D(x) - xD'_-(x) \geq D(0) = 0. \quad (1.3)$$

So  $\gamma$  is not greater than  $-rK$  on  $(0, x^*)$ . The constant  $x^*$  is infinite if and only if  $D$  is the identity function and then  $\gamma$  is constant and equal to  $-rK$ . When  $x^* < +\infty$ ,  $\gamma$  is bounded from below by  $-r(K + D(x^*))$  on  $(0, x^*)$ . Moreover, since  $D$  is concave, continuous and  $D(0) = 0$ ,

$$\forall x > x^*, \frac{D(x)}{x} \leq \frac{D(x^*)}{x^*} = \frac{x^* - \bar{c}(t_d)}{x^*} \text{ and } x - D(x) \geq \frac{x\bar{c}(t_d)}{x^*} > \bar{c}(t_d). \quad (1.4)$$

One has

$$\begin{aligned} \gamma(x) - \mathcal{A}\bar{u}(t_d, \cdot)(x - D(x)) &= \frac{\sigma^2}{2} \partial_{22}\bar{u}(t_d, x - D(x)) [x^2(1 - D'_-(x))^2 - (x - D(x))^2] \\ &\quad + r(D(x) - xD'_-(x)) \partial_2 \bar{u}(t_d, x - D(x)) \end{aligned} \quad (1.5)$$

where the last term is non-positive by (1.3) and since  $\partial_2 \bar{u} \leq 0$ . Define  $M = \sup_{x > \bar{c}(t_d)} \mathcal{A}\bar{u}(t_d, \cdot)(x)$  which is finite by Lemma 1.7. Since  $\bar{u}(t_d, x) - x \partial_x \bar{u}(t_d, x)$  is non-increasing by convexity of  $x \mapsto \bar{u}(t_d, x)$  and equal to  $K$  on  $[0, \bar{c}(t_d))$ , one deduces

$$\forall x > \bar{c}(t_d), \partial_{xx} \bar{u}(t_d, x) \leq \frac{2(M + rK)}{\sigma^2 x^2}. \quad (1.6)$$

With  $x - D(x)$ , which is larger than  $\bar{c}(t_d)$ , substituted in (1.6), and using (1.4) and  $D'_-(x) \in [0, 1]$ , one concludes that when  $x^* < +\infty$ ,

$$\forall x > x^*, \gamma(x) \leq M + (M + rK) \frac{x^{*2} - \bar{c}(t_d)^2}{\bar{c}(t_d)^2}.$$

For  $x > x^*$ , since  $x D'_-(x) \partial_2 \bar{u}(t_d, x - D(x))$  and  $\partial_{22} \bar{u}(t_d, x - D(x)) [x^2(1 - D'_-(x))^2 - (x - D(x))^2]$  are non-negative and  $\mathcal{A}\bar{u}(t_d, \cdot)(x - D(x)) = -\partial_t \bar{u}(t_d, x - D(x)) > 0$ , we have by (1.5),

$$\begin{aligned} \gamma(x) &\geq r D(x) \partial_2 \bar{u}(t_d, x - D(x)) \geq r \frac{x^* - \bar{c}(t_d)}{\bar{c}(t_d)} (x - D(x)) \partial_2 \bar{u}(t_d, x - D(x)) \\ &= r \frac{x^* - \bar{c}(t_d)}{\bar{c}(t_d)} \left( -K + \int_{\bar{c}(t_d)}^{x-D(x)} y \partial_{22} \bar{u}(t_d, y) dy + \bar{u}(t_d, x - D(x)) \right) \geq -rK \frac{x^* - \bar{c}(t_d)}{\bar{c}(t_d)}, \end{aligned}$$

where we used that  $D(x) \leq (x - D(x))(x^* - \bar{c}(t_d))/\bar{c}(t_d)$  by (1.4) for the second inequality and the smooth fit property  $\partial_2 \bar{u}(t_d, \bar{c}(t_d)) = -1$  and a partial integration for the equality. Last, if  $g$  is convex, then the left-hand derivative  $g'_-(x) = \partial_2 \bar{u}(t_d, x - D(x))(1 - D'_-(x))$  is non-decreasing. But  $g'_-(x) - g'_-(x^-) = -\partial_2 \bar{u}(t_d, x - D(x))(D'_-(x) - D'_-(x^-))$  and since  $\partial_2 \bar{u}$  is negative and  $D'_-$  non-increasing, the right-hand-side of this equality is non-positive and the left-hand-side is non-negative so both are zero and the function  $g'_-$  is continuous. So  $g$  and  $D$  are  $C^1$  with  $g'(x) = \partial_2 \bar{u}(t_d, x - D(x))(1 - D'(x))$ . The first factor in the right-hand-side being globally continuous and  $C^1$  on  $(0, x^*) \cup (x^*, +\infty)$ , one deduces that the distribution derivative of  $g'$  is equal to  $\partial_{22} \bar{u}(t_d, x - D(x))(1 - D'(x))^2 dx - \partial_2 \bar{u}(t_d, x - D(x)) D''(dx)$ . This measure being non-negative by convexity of  $g$ ,  $D''$  is absolutely continuous with respect to the Lebesgue measure and so is the second order distribution derivative of  $g$ . For  $x < x^*$ ,  $g'(x) = D'(x) - 1$  where the left-hand-side is non-decreasing and the right-hand-side non-increasing. So there is a constant  $\rho \in [0, 1]$  such that  $g(x) = K - \rho x$  and  $D(x) = (1 - \rho)x$  for  $x < x^*$ . As a consequence  $x^* = \bar{c}(t_d)/\rho$  and  $\mathcal{A}g(x) = rxg'(x) - rg(x) = -rK$  on  $(0, x^*)$ . The convexity of  $g$  implies that  $rxg'(x) - rg(x)$  is non-decreasing and therefore that  $dx$  a.e. on  $(x^*, +\infty)$ ,  $\mathcal{A}g(x) = \frac{\sigma^2 x^2}{2} g''(x) + rxg'(x) - rg(x) \geq -rK$ .  $\blacksquare$

**Proof of Lemma 1.7.** The proof of the first statement is similar to the one of Lemma 1.4. Moreover,  $x \mapsto \bar{u}(t, x) = \sup_{\tau \in \mathcal{T}_{[0, T-t]}} \mathbb{E} \left( e^{-r\tau} (K - x e^{\sigma W_\tau + (r - \frac{\sigma^2}{2})\tau})^+ \right)$  is convex as the supremum of convex functions. We refer for instance to Lemma 7.8 in Section 2.6 [12] for the continuous differentiability property of this function.

Let  $0 \leq s \leq t \leq T$ ,  $x > 0$ , and take  $\tau \in \mathcal{T}_{[0, T-s]}$  such that  $\bar{u}(s, x) = \mathbb{E}(e^{-r\tau} (K - \bar{S}_\tau^x)^+)$  and  $\tilde{\tau} = \tau \wedge (T - t)$ . One has

$$\bar{u}(t, x) \geq \mathbb{E} \left( e^{-r\tilde{\tau}} (K - \bar{S}_{\tilde{\tau}}^x)^+ \right) = \bar{u}(s, x) - \mathbb{E} \left( 1_{\{\tau > T-t\}} \left( e^{-r\tau} (K - \bar{S}_\tau^x)^+ - e^{-r(T-t)} (K - \bar{S}_{T-t}^x)^+ \right) \right)$$

By Tanaka's formula, when  $\tau > T - t$ ,

$$(K - \bar{S}_\tau^x)^+ = (K - \bar{S}_{T-t}^x)^+ - \int_{T-t}^\tau 1_{\{\bar{S}_v^x \leq K\}} (\sigma \bar{S}_v^x dW_v + r \bar{S}_v^x dv) + \frac{1}{2} (L_\tau^K(\bar{S}^x) - L_{T-t}^K(\bar{S}^x)).$$

One deduces that

$$\bar{u}(t, x) \geq \bar{u}(s, x) - \frac{e^{-r(T-t)}}{2} \mathbb{E}(L_{T-s}^K(\bar{S}^x) - L_{T-t}^K(\bar{S}^x)) = \bar{u}(s, x) - \frac{e^{-r(T-t)} \sigma^2 K^2}{2} \int_s^t p(T-v, K) dv.$$

■

## 2 Limit behaviour and monotonicity of the exercise boundary as $t \rightarrow t_d^-$

Using the results in the previous section, we first check that  $c(t)$  tends to 0 as  $t \rightarrow t_d^-$  if  $D$  is positive (i.e.  $\forall x > 0, D(x) > 0$ ).

**Lemma 2.1** *Let  $D$  be a non-negative and non-decreasing function s.t.  $x \mapsto x - D(x)$  is non-negative and non-decreasing. Then  $t \mapsto c(t)$  is upper-semicontinuous on  $[0, t_d)$ .*

Assume moreover that  $D$  is positive, then we have  $\lim_{t \rightarrow t_d^-} c(t) = 0$ , and

- if  $D$  is concave, then for all  $y > 0$ , there is a left-hand neighbourhood of  $t_d$  in which  $c(t) \leq \frac{rKy}{D(y)}(t_d - t) + o(t_d - t)$ ,
- if  $D$  is concave and  $g$  is convex then  $\forall t \in [0, t_d)$ ,  $c(t) < \frac{1 - e^{-r(t_d-t)}}{1 - \rho} K$  where  $\rho \in [0, 1)$  is the constant such that, according to Lemma 1.6,  $\forall x \in (0, x^*)$ ,  $\bar{D}(x) = (1 - \rho)x$ .

**Proof .** By Lemma 1.4,  $\{(t, x) \in [0, t_d) \times \mathbb{R}_+ : u(t, x) = (K - x)^+\}$  which is equal to  $\{(t, x) \in [0, t_d) \times \mathbb{R}_+ : 0 \leq x \leq c(t)\}$  by Lemma 1.3 is a closed subset of  $[0, t_d) \times \mathbb{R}_+$ . As a consequence  $t \mapsto c(t)$  is upper-semicontinuous on  $[0, t_d)$ .

To prove that  $\lim_{t \rightarrow t_d^-} c(t) = 0$  assume that there exists a sequence  $(t_n)_{n \in \mathbb{N}}$  such that  $t_n \uparrow t_d$  with  $c(t_n) > y$  for some  $y > 0$  which we may take smaller than  $K$  and such that  $y$  is not one of the countably many discontinuity points of  $D$ . Then  $u(t_n, y) = K - y$  for all  $t_n$  and taking the limit and applying Lemma 1.4 gives that  $u(t_d, y - D(y)) = K - y$  but  $u(t_d, y - D(y)) \geq (K - y + D(y))^+ = K - y + D(y)$  which contradicts the assumption that  $D$  is positive.

Assume that  $D$  is concave and positive and let  $y > 0$ . Then  $\forall z \in (0, y)$ ,  $D(z) \geq \frac{zD(y)}{y}$  and for  $(t, x) \in [0, t_d) \times \mathbb{R}_+^*$ ,

$$\begin{aligned} u(t, x) &\geq \mathbb{E}(e^{-r(t_d-t)} g(\bar{S}_{t_d-t}^x)) \geq \mathbb{E}(e^{-r(t_d-t)} (K - \bar{S}_{t_d-t}^x + D(\bar{S}_{t_d-t}^x))) \\ &\geq e^{-r(t_d-t)} K - x + \frac{x D(y)}{y} \mathbb{E} \left( e^{-r(t_d-t)} \bar{S}_{t_d-t}^1 1_{\{\bar{S}_{t_d-t}^x \leq y\}} \right) \\ &= e^{-r(t_d-t)} K - x + \frac{x D(y)}{y} N \left( \frac{\log(y/x) - (r + \frac{\sigma^2}{2})(t_d - t)}{\sigma \sqrt{t_d - t}} \right). \end{aligned}$$

For  $x \leq \frac{y}{3}$  and  $t_d - t \leq \frac{\log(3)}{2(r + \frac{\sigma^2}{2})}$ , one has  $\frac{\log(y/x) - (r + \frac{\sigma^2}{2})(t_d - t)}{\sigma\sqrt{t_d - t}} \geq \frac{\log(3)}{2\sigma\sqrt{t_d - t}}$  and

$$u(t, x) \geq Ke^{-r(t_d - t)} - x + \frac{x D(y)}{y} \left( 1 - \frac{2\sigma\sqrt{t_d - t}}{\sqrt{2\pi} \log(3)} e^{-\frac{\log^2(3)}{8\sigma^2(t_d - t)}} \right)$$

which ensures that  $u(t, x) > K - x$  when  $x > (y/D(y))K(1 - e^{-r(t_d - t)})(1 - \frac{2\sigma\sqrt{t_d - t}}{\sqrt{2\pi} \log(3)} e^{-\frac{\log^2(3)}{8\sigma^2(t_d - t)}})^{-1}$ . The upper-bound for  $c(t)$  easily follows.

When  $g$  is also convex, according to Lemma 1.6, either  $D$  is the identity function and  $g$  is constant and equal to  $K$  or there is a constant  $\rho \in (0, 1)$  such that  $D(x) = (1 - \rho)x$  for  $x \in (0, \bar{c}(t_d)/\rho]$ . In the latter case, one has  $g(x) = K - \rho x$  for  $x \in (0, \bar{c}(t_d)/\rho]$  and  $g(x) \geq (K - \rho x)^+$  for  $x > \bar{c}(t_d)/\rho$ . As a consequence,  $\mathbb{E}(e^{-r(t_d - t)}g(\bar{S}_{t_d - t}^x)) > \mathbb{E}(e^{-r(t_d - t)}(K - \rho\bar{S}_{t_d - t}^x)) = e^{-r(t_d - t)}K - \rho x$ . One deduces that when  $x \geq \frac{1 - e^{-r(t_d - t)}}{1 - \rho}K$ ,  $u(t, x) > K - x$  which implies that  $c(t) < \frac{1 - e^{-r(t_d - t)}}{1 - \rho}K$ . This obviously still holds with  $\rho = 0$  when  $D$  is the identity function.  $\blacksquare$

We now obtain monotonicity of the exercise boundary in a left-hand neighbourhood of the dividend date  $t_d$ .

**Proposition 2.2** *If  $D$  is a positive concave function such that  $x - D(x)$  is non-negative, there exists a constant  $\varepsilon > 0$  such that for  $x \in (0, \varepsilon)$ ,  $t \mapsto u(t, x)$  is non-decreasing on  $(t_d - \varepsilon, t_d)$ . Moreover, we have for all  $t \in [0, t_d)$  and all  $x > c(t)$  that*

$$\partial_t u(t, x) \geq -e^{-r(t_d - t)} \sup_{y > 0} \gamma(y) \quad (2.1)$$

$$\frac{\sigma^2 x^2}{2} \partial_{xx} u(t, x) \leq e^{-r(t_d - t)} \sup_{y > 0} \gamma(y) + r(x + K). \quad (2.2)$$

Last, for any  $t \in [0, t_d)$  such that  $c(t) > 0$ ,  $\forall x > c(t)$ ,  $\int_{c(t)}^x |\partial_{xx} u(t, y)| dy < +\infty$  and  $x \mapsto \partial_x u(t, x)$  admits a right-hand limit  $\partial_x u(t, c(t)^+) \in [-1, 0]$  as  $x \rightarrow c(t)^+$ .

One easily deduces the following Corollary, where the positivity is a consequence of the monotonicity and the fact that the function  $c(\cdot)$  cannot vanish on an interval according to Lemma 1.3 and the left continuity follows from the monotonicity and the upper-semicontinuity.

**Corollary 2.3** *If  $D$  is a positive concave function such that  $x - D(x)$  is non-negative, then  $t \mapsto c(t)$  is non-increasing, positive and left-continuous on  $(t_d - \varepsilon, t_d)$ .*

**Proof of Proposition 2.2.** Let  $0 \leq t \leq s < t_d$ ,  $x > 0$  and  $\tau \in \mathcal{T}_{[0, t_d - t]}$  be such that  $u(t, x) = \mathbb{E}(e^{-r\tau}(K - \bar{S}_\tau^x)^+ 1_{\{\tau < t_d - t\}} + e^{-r(t_d - t)}g(\bar{S}_{t_d - t}^x) 1_{\{\tau = t_d - t\}})$ . Since by Lemma 1.6,  $\forall x > 0$ ,  $g(x) \geq (K - x)^+$ ,

$$\begin{aligned} u(t, x) &\leq \mathbb{E}(e^{-r\tau}(K - \bar{S}_\tau^x)^+ 1_{\{\tau < t_d - s\}} + e^{-r\tau}g(\bar{S}_\tau^x) 1_{\{\tau \geq t_d - s\}}) \\ &= \mathbb{E}\left(e^{-r\tau}(K - \bar{S}_\tau^x)^+ 1_{\{\tau < t_d - s\}} + e^{-r(t_d - s)}g(\bar{S}_{t_d - s}^x) 1_{\{\tau \geq t_d - s\}}\right) \\ &\quad + \mathbb{E}\left(1_{\{\tau > t_d - s\}}\left(e^{-r\tau}g(\bar{S}_\tau^x) - e^{-r(t_d - s)}g(\bar{S}_{t_d - s}^x)\right)\right). \end{aligned} \quad (2.3)$$

By Tanaka's formula,

$$d(\bar{S}_v^x - D(\bar{S}_v^x)) = (1 - D'_-(\bar{S}_v^x))d\bar{S}_v^x - \frac{1}{2} \int_0^{+\infty} D''(da)dL_v^a(\bar{S}_v^x).$$

In particular  $d\langle \bar{S}_v^x - D(\bar{S}_v^x) \rangle_v = (\sigma \bar{S}_v^x (1 - D'_-(\bar{S}_v^x)))^2 dv$ . The function  $x \mapsto \bar{u}(t_d, x)$  is convex and  $C^1$  on  $[0, +\infty)$  and  $C^2$  on  $[0, \bar{c}(t_d))$  and  $(\bar{c}(t_d), +\infty)$ . Hence its second order distribution derivative is equal to  $\partial_{22}\bar{u}(t_d, x)dx$  where, by convention,  $\partial_{22}\bar{u}(t_d, \bar{c}(t_d)) = 0$ . Applying again Tanaka's formula and the occupation times formula, one deduces that

$$dg(\bar{S}_v^x) = \partial_2 \bar{u}(t_d, \bar{S}_v^x - D(\bar{S}_v^x))d(\bar{S}_v^x - D(\bar{S}_v^x)) + \frac{\sigma^2}{2} \partial_{22} \bar{u}(t_d, \bar{S}_v^x - D(\bar{S}_v^x))((1 - D'_-(\bar{S}_v^x))\bar{S}_v^x)^2 dv.$$

One deduces that for  $\gamma$  defined in Lemma 1.6,

$$d(e^{-rv}g(\bar{S}_v^x)) = e^{-rv} \left( \partial_2 \bar{u}(t_d, \bar{S}_v^x - D(\bar{S}_v^x)) \left[ (1 - D'_-(\bar{S}_v^x))\sigma \bar{S}_v^x dW_v - \frac{1}{2} \int_0^{+\infty} D''(da)dL_v^a(\bar{S}_v^x) \right] + \gamma(\bar{S}_v^x)dv \right). \quad (2.4)$$

The process  $(\int_0^v e^{-rw} \sigma \bar{S}_w^x \partial_2 \bar{u}(t_d, \bar{S}_w^x - D(\bar{S}_w^x))(1 - D'_-(\bar{S}_w^x))dW_w)_v$  is a martingale since  $\partial_2 \bar{u} \in [-1, 0]$  by (1.1) and  $(1 - D'_-) \in [0, 1]$  according to Lemma 1.6. With (2.3), one deduces that

$$u(s, x) - u(t, x) \geq -\mathbb{E} \left( \mathbf{1}_{\{\tau > t_d - s\}} \int_{t_d - s}^{\tau} e^{-rv} \gamma(\bar{S}_v^x) dv \right) = -\mathbb{E} \left( \int_{t_d - s}^{t_d - t} \mathbf{1}_{\{\tau > v\}} e^{-rv} \gamma(\bar{S}_v^x) dv \right). \quad (2.5)$$

One easily deduces (2.1) and, since by Lemma 1.6,  $C \stackrel{\text{def}}{=} \sup_{x>0} \gamma(x) < +\infty$  and  $\gamma(x)$  is not greater than  $-rK$  for  $x < x^*$ ,

$$u(s, x) \geq u(t, x) + \int_{t_d - s}^{t_d - t} e^{-rv} (rK \mathbb{P}(\tau > v, \bar{S}_v^x < x^*) - C \mathbb{P}(\tau > v, \bar{S}_v^x \geq x^*)) dv. \quad (2.6)$$

Define  $\hat{c}(s) = \sup_{v \in [t_d - s, t_d]} c(v)$  and let  $\alpha \in (0, t_d]$  be such that  $\hat{c}(\alpha) < x^*$ . The existence of  $\alpha$  is ensured by Lemma 2.1 which applies since, according to the proof of Lemma 1.6, the function  $D$  is continuous and both  $D$  and  $x - D(x)$  are non-decreasing. We now choose  $t \in [t_d - \alpha, t_d)$  and  $x \in (c(t), y)$  where  $y \in (\hat{c}(\alpha), x^*)$ . One has  $\tau = \inf\{v \in [0, t_d - t) : \bar{S}_v^x \leq c(t + v)\}$  with convention  $\inf \emptyset = t_d - t$ . Let  $\tau_y = \inf\{v \geq 0 : \bar{S}_v^x = y\}$ . For  $v \in [0, t_d - t)$ , by the Markov property, one has

$$\mathbb{P}(\tau > v, \bar{S}_v^x \geq x^*) = \mathbb{P}(\tau > v, \tau_y \leq v, \bar{S}_v^x \geq x^*) \leq \mathbb{P}(\tau_y \leq v, \tau > \tau_y) \mathbb{P} \left( \max_{w \in [0, v]} \bar{S}_w^1 \geq x^*/y \right).$$

In the same time,

$$\mathbb{P}(\tau > v) \geq \mathbb{P}(\tau_y \leq v, \tau > v) \geq \mathbb{P}(\tau_y \leq v, \tau > \tau_y) \mathbb{P} \left( \min_{w \in [0, v]} \bar{S}_w^1 > \hat{c}(\alpha)/y \right).$$

Combining both inequalities, one obtains

$$\mathbb{P}(\tau > v, \bar{S}_v^x \geq x^*) \leq \mathbb{P}(\tau > v) \frac{\mathbb{P}(\max_{w \in [0, \alpha]} \bar{S}_w^1 \geq x^*/y)}{\mathbb{P}(\min_{w \in [0, \alpha]} \bar{S}_w^1 > \hat{c}(\alpha)/y)}.$$

The ratio  $\frac{\mathbb{P}(\max_{w \in [0, \beta]} \bar{S}_w^1 \geq z)}{\mathbb{P}(\min_{w \in [0, \beta]} \bar{S}_w^1 > \eta)}$  equals

$$\frac{N\left(\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\beta - \frac{\log z}{\sigma}\right) + e^{\frac{2\log z}{\sigma}\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)} N\left(-\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\beta - \frac{\log z}{\sigma}\right)}{1 - N\left(\frac{\log \eta}{\sigma} - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\beta\right) - e^{\frac{2\log \eta}{\sigma}\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)} N\left(\frac{\log \eta}{\sigma} + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\beta\right)}$$

and for  $\beta > 0$  and  $z > 1 > \eta > 0$  this converges to 0 as  $\beta$  and  $\eta$  go to  $0^+$  while  $z$  goes to  $+\infty$ . Since by Lemma 2.1,  $\hat{c}(\alpha)$  converges to 0 as  $\alpha$  goes to  $0^+$ , one may choose positive constants  $y, \alpha$  such that  $y \in (\hat{c}(\alpha), x^*)$  and

$$\frac{\mathbb{P}(\max_{w \in [0, \alpha]} \bar{S}_w^1 \geq x^*/y)}{\mathbb{P}(\min_{w \in [0, \alpha]} \bar{S}_w^1 > \hat{c}(\alpha)/y)} \leq \frac{rK}{rK + C}.$$

With  $\mathbb{P}(\tau > v, \bar{S}_v^x < x^*) = \mathbb{P}(\tau > v) - \mathbb{P}(\tau > v, \bar{S}_v^x \geq x^*)$  and (2.6), we conclude that

$$\forall t_d - \alpha \leq t \leq s < t_d, \forall x \in (0, y), u(t, x) \leq u(s, x).$$

Since for  $t \in (0, t_d)$  and  $x > c(t)$ ,  $\frac{\sigma^2 x^2}{2} \partial_{xx} u(t, x) = -\partial_t u(t, x) - rx \partial_x u(t, x) + ru(t, x)$  with  $\partial_x u \in [-1, 0]$  according to (1.1) and  $u \leq K$ , (2.2) easily follows from (2.1). Let  $t \in [0, t_d)$  be such that  $c(t) > 0$ . For  $z \geq x > c(t)$ , one has  $\partial_x u(t, x) = \partial_x u(t, z) - \int_x^z \partial_{xx} u(t, y) dy$ . By (1.1),  $\partial_x u(t, x) \in [-1, 0]$ . With (2.2), one deduces that  $y \mapsto \partial_{xx} u(t, y)$  is integrable on  $[c(t), z]$  and the right-hand limit  $\partial_x u(t, c(t)^+)$  makes sense.  $\blacksquare$

**Remark 2.4** When  $T = +\infty$  i.e. when the Put option is perpetual,

$$u(t_d, x) = \begin{cases} K - x & \text{if } x < \bar{c}(t_d) = \frac{-K\alpha}{1-\alpha} \\ (K - \bar{c}(t_d))(x/\bar{c}(t_d))^\alpha & \text{otherwise} \end{cases}, \text{ where } \alpha = -\frac{2r}{\sigma^2}.$$

In the proportional dividend case,  $\gamma(x) = -rK1_{\{x < \bar{c}(t_d)/\rho\}}$  since  $\mathcal{A}f(x) = 0$  for  $f(x) = x^\alpha$ . With (2.5), one deduces that for any  $x > 0$ ,  $t \mapsto u(t, x)$  is non-decreasing on  $[0, t_d)$ .

In the constant dividend case,

$$\gamma(x) = \begin{cases} -rK & \text{if } x \in (0, D) \\ -r(K + D) & \text{if } x \in (D, \bar{c}(t_d) + D) \\ -\alpha(K - \bar{c}(t_d))\bar{c}(t_d)^{-\alpha} D(r\alpha + \frac{\sigma^2}{2}(2x - D))(x - D)^{\alpha-2} & \text{if } x > \bar{c}(t_d) + D \end{cases}$$

is positive on  $(\bar{c}(t_d) + D, +\infty)$ .

### 3 Continuity of the exercise boundary and high contact principle

We can now state our main result concerning the continuity of the exercise boundary  $c(t)$ . Note that it applies to both the *proportional* and the *constant* dividend cases.

**Proposition 3.1** Assume that  $D$  is a positive concave function such that  $x - D(x)$  is non-negative and let  $t_{ni} = \inf\{t \in [0, t_d) : v \mapsto c(v)$  is non-increasing on  $[t, t_d)\}$ .

Then for  $t \in (t_{ni}, t_d)$ ,  $\lim_{s \rightarrow t^+} \partial_x u(s, c(s)^+) = -1$  and, when  $c$  is continuous at  $t$ , the smooth

contact property  $\partial_x u(t, c(t)^+) = -1$  holds.

If  $g$  is convex, then  $t \mapsto c(t)$  is continuous on  $(t_{ni}, t_d)$ . More generally, if  $D$  is such that

$$\exists x_0 > 0, \exists \rho \in [0, 1), \forall x \in (0, x_0), D(x) = (1 - \rho)x, \quad (3.1)$$

then there exists an  $\varepsilon \in (0, t_d]$  such that  $t \mapsto c(t)$  is continuous on  $(t_d - \varepsilon, t_d)$ .

In order to prove the Proposition, we will need the following estimations of the first order time derivative and the second order spatial derivative of the pricing function  $u$  in the continuation region.

**Lemma 3.2** *Assume that  $D$  is a non-negative concave function such that  $x - D(x)$  is non-negative. Then*

$$\forall t \in [0, t_d), \forall x > c(t), \partial_t u(t, x) \leq -e^{-r(t_d-t)} \inf_{y>0} \gamma(y) + \frac{\sigma x}{2\sqrt{2\pi}(t_d-t)} \quad (3.2)$$

$$\text{and } \frac{\sigma^2 x^2}{2} \partial_{xx} u(t, x) \geq e^{-r(t_d-t)} \inf_{y>0} \gamma(y) - \frac{\sigma x}{2\sqrt{2\pi}(t_d-t)} + r(K - x)^+. \quad (3.3)$$

If  $g$  is convex, then for  $(t, x) \in [0, t_d) \times \mathbb{R}_+^*$  such that  $x > c(t)$ ,  $\partial_t u(t, x) \leq rK e^{-r(t_d-t)}$  and  $\partial_{xx} u(t, x) \geq 0$ .

More generally, under (3.1), there exists  $\varepsilon \in (0, t_d]$  such that for all  $t \in (t_d - \varepsilon, t_d)$  and for all  $x \in (c(t), c(t) + \varepsilon)$  we have  $\partial_t u(t, x) \leq rK \frac{1+e^{-r(t_d-t)}}{2}$ .

**Proof of Proposition 3.1.** For  $t \in (0, t_d)$  such that  $c(t) > 0$ , which is true for  $t \in (t_{ni}, t_d)$ , by Proposition 2.2, the following Taylor expansion makes sense

$$\forall x \geq c(t), u(t, x) = (K - c(t)) + (x - c(t)) \partial_x u(t, c(t)^+) + \int_{c(t)}^x (x - y) \partial_{xx} u(t, y) dy. \quad (3.4)$$

Let  $t_0 \in (t_{ni}, t_d)$  be such that  $c$  is continuous at  $t_0$ . To prove that  $\partial_x u(t_0, c(t_0)^+) = -1$ , we are first going to check that  $\lim_{t \rightarrow t_0^+} \partial_x u(t, c(t)^+) = \partial_x u(t_0, c(t_0)^+)$ . Let  $x \in (c(t_0), 2c(t_0))$ .

Substituting  $c(t_0^+)$  for  $x$  in (3.4) and subtracting the result from (3.4) itself gives

$$\partial_x u(t, c(t)^+) = \frac{u(t, x) - u(t, c(t_0^+))}{x - c(t_0^+)} - \int_{c(t)}^{c(t_0^+)} \partial_{xx} u(t, y) dy - \frac{1}{x - c(t_0^+)} \int_{c(t_0^+)}^x (x - y) \partial_{xx} u(t, y) dy. \quad (3.5)$$

Computing  $\partial_x u(t_0, c(t_0)^+)$  from (3.4) written with  $t_0$  replacing  $t$ , one deduces

$$\begin{aligned} \partial_x u(t_0, c(t_0)^+) - \partial_x u(t, c(t)^+) &= \frac{1}{x - c(t_0)} \left( u(t_0, x) - u(t, x) + u(t, c(t_0)) - u(t_0, c(t_0)) \right) \\ &\quad + \frac{1}{x - c(t_0)} \int_{c(t_0)}^x (x - y) (\partial_{xx} u(t, y) - \partial_{xx} u(t_0, y)) dy \\ &\quad + \int_{c(t)}^{c(t_0)} \partial_{xx} u(t, y) dy. \end{aligned} \quad (3.6)$$

By (2.1) and (3.2) one checks that for fixed  $x \in (c(t_0), 2c(t_0))$ , the first term in the r.h.s. of (3.6) converges to 0 as  $t \rightarrow t_0^+$ . Moreover, (2.2) and (3.3) ensure that the second term in the

r.h.s. of (3.6) is arbitrarily small uniformly for  $t < (t_0 + t_d)/2$  when  $x$  is close enough to  $c(t_0)$ . Last, with the continuity of  $c$  at  $t_0$ , the third term converges to 0 as  $t \rightarrow t_0^+$ , which ensures the desired right-continuity property.

Let us now assume that  $\partial_x u(t_0, c(t_0)^+) > -1$  and obtain a contradiction. Let  $t \in (t_0, \frac{t_0+t_d}{2})$ . According to (3.2) and (3.3), there exists a constant  $C \in (0, +\infty)$  such that  $u(t, c(t_0)) \leq K - c(t_0) + C(t - t_0)$  and  $\int_{c(t)}^{c(t_0)} (c(t_0) - y) \partial_{xx} u(t, y) dy \geq -C \frac{(c(t_0) - c(t))^2}{c(t)^2}$ . Writing (3.4) for  $x = c(t_0)$ , one deduces that

$$\left(1 + \partial_x u(t, c(t)^+) - C \frac{c(t_0) - c(t)}{c(t)^2}\right) (c(t_0) - c(t)) \leq C(t - t_0).$$

Since  $\partial_x u(t, c(t)^+)$  tends to  $\partial_x u(t_0, c(t_0)^+) > -1$  as  $t \rightarrow t_0^+$  and  $c$  is continuous at  $t_0$ , one deduces the existence of  $\varepsilon \in (0, t_d - t_0)$  such that

$$\forall t \in [t_0, t_0 + \varepsilon], c(t) - c(t_0) \geq -\frac{2C(t - t_0)}{1 + \partial_x u(t_0, c(t_0)^+)}. \quad (3.7)$$

For  $x > c(t_0)$ , let  $\tau_x = \inf\{s > 0 : \bar{S}_s^x \leq c(t_0 + s)\} \wedge (t_d - t_0)$  denote the stopping time such that

$$u(t_0, x) = \mathbb{E} \left( e^{-r\tau_x} (K - \bar{S}_{\tau_x}^x)^+ 1_{\{\tau_x < t_d - t_0\}} + e^{-r(t_d - t_0)} g(\bar{S}_{\tau_x}^x) 1_{\{\tau_x = t_d - t_0\}} \right).$$

One has  $u(t_0, c(t_0)) \geq \mathbb{E} \left( e^{-r\tau_x} (K - \bar{S}_{\tau_x}^{c(t_0)})^+ 1_{\{\tau_x < t_d - t_0\}} + e^{-r(t_d - t_0)} g(\bar{S}_{\tau_x}^{c(t_0)}) 1_{\{\tau_x = t_d - t_0\}} \right)$ . Computing the difference, using the monotonicity of  $g$  and the Lipschitz continuity of  $y \mapsto (K - y)^+$  one deduces that

$$\frac{u(t_0, x) - u(t_0, c(t_0))}{x - c(t_0)} \leq -\mathbb{E} \left( e^{-r\tau_x} \bar{S}_{\tau_x}^1 1_{\{\tau_x < t_d - t_0\}} \right). \quad (3.8)$$

By (3.7),  $\tau_x \leq \tilde{\tau}_x \stackrel{\text{def}}{=} \inf\{s \in (0, \varepsilon] : \bar{S}_s^x \leq c(t_0) - 2Cs/(1 + \partial_x u(t_0, c(t_0)^+))\} \wedge (t_d - t_0)$ . When  $x$  tends to  $c(t_0)^+$ ,  $\tilde{\tau}_x$  converges a.s. to  $\inf\{s \in (0, \varepsilon] : \bar{S}_s^1 < 1 - 2Cs/(c(t_0)(1 + \partial_x u(t_0, c(t_0)^+))\} \wedge (t_d - t_0)$  which is equal to 0 according to the iterated logarithm law satisfied by the Brownian motion  $W$ . Hence  $\tau_x$  converge a.s. to 0 as  $x \rightarrow c(t_0)^+$ . Since  $\mathbb{E}(\sup_{s \in [0, t_d - t_0]} \bar{S}_s^1) < +\infty$ , by Lebesgue's theorem, the right-hand-side of (3.8) converges to  $-1$  as  $x \rightarrow c(t_0)^+$  which implies the desired contradiction :  $\partial_x u(t_0, c(t_0)^+) \leq -1$ .

Combination of the two first steps of the proof ensures that if  $c$  is continuous at  $t_0 \in (t_{ni}, t_d)$ ,

$$\lim_{t \rightarrow t_0^+} \partial_x u(t, c(t)^+) = -1. \quad (3.9)$$

Let now  $t_0 \in (t_{ni}, t_d)$  be such that  $c(t_0^+) < c(t_0)$ . We are going to prove (3.9) before obtaining a contradiction when  $g$  is convex or  $t_0$  is close to  $t_d$  under (3.1). Let  $x \in (c(t_0^+), c(t_0))$  and  $t \in (t_0, \frac{t_0+t_d}{2})$ . The left-hand-side of (3.5) is not smaller than  $-1$ . When  $t$  tends to  $t_0^+$ , by continuity of  $u$ , the first term in the right-hand-side tends to  $\frac{K - x - (K - c(t_0^+))}{x - c(t_0^+)} = -1$ . The second term converges to 0 according to (3.3) and (2.2). Moreover, by (3.3), there is a constant  $C \in (0, +\infty)$  such that

$$\frac{1}{x - c(t_0^+)} \int_{c(t_0^+)}^x (x - y) \partial_{xx} u(t, y) dy \geq \frac{1}{x - c(t_0^+)} \int_{c(t_0^+)}^x (x - y) (-2C/c(t_0^+)^2) dy = -\frac{C(x - c(t_0^+))}{c(t_0^+)^2}.$$

Hence

$$\limsup_{t \rightarrow t_0^+} \partial_x u(t, c(t)^+) \leq -1 + \frac{C(x - c(t_0^+))}{c(t_0^+)^2}.$$



Letting  $x$  decrease to  $c(t_0^+)$ , one concludes that (3.9) holds.

By (3.4) and Proposition 2.2,

$$\begin{aligned} \forall x > c(t), \int_{c(t)}^x y \partial_{xx} u(t, y) dy &= x \partial_x u(t, x) - c(t) \partial_x u(t, c(t)^+) - u(t, x) + u(t, c(t)) \\ &= x \partial_x u(t, x) - u(t, x) + K - c(t) (1 + \partial_x u(t, c(t)^+)). \end{aligned}$$

With the equality  $\partial_t u(t, x) + \mathcal{A}u(t, x) = 0$  and Lemma 3.2, one deduces that for  $t_0$  close to  $t_d$  under (3.1) and with no restriction in the convex case,

$$\begin{aligned} \forall x \in (c(t), c(t_0)), \frac{\sigma^2 x^2}{2} \partial_{xx} u(t, x) + r \int_{c(t)}^x y \partial_{xx} u(t, y) dy &= rK - \partial_t u(t, x) - rc(t)(1 + \partial_x u(t, c(t)^+)) \\ &\geq \frac{rK(1 - e^{-r(t_d-t)})}{2} - rc(t)(1 + \partial_x u(t, c(t)^+)). \end{aligned} \quad (3.10)$$

According to (2.2), there is a finite constant  $C$  such that  $\forall t \in [0, t_d], \forall x > c(t), y \partial_{xx} u(t, y) \leq \frac{C}{y}$  so  $r \int_{c(t)}^x y \partial_{xx} u(t, y) dy \leq rK(1 - e^{-r(t_d-t)})/8$  if we take  $x/c(t) \leq e^{\frac{K(1-e^{-r(t_d-t)})}{8C}}$ . With (3.9) and (3.10), one deduces the existence of  $\eta \in (0, t_d - t_0)$  such that for  $t_0$  close to  $t_d$  under (3.1) and with no restriction in the convex case,

$$\begin{aligned} \forall x \in \left( c(t_0^+), c(t_0) \wedge c(t_0^+) e^{\frac{K(1-e^{-r(t_d-t_0)})}{16C}} \right), \forall t \in (t_0, t_0 + \eta), \frac{\sigma^2 x^2}{2} \partial_{xx} u(t, x) &\geq \frac{rK(1 - e^{-r(t_d-t)})}{4} \\ \text{and } \frac{1}{x - c(t_0^+)} \int_{c(t_0^+)}^x (x - y) \partial_{xx} u(t, y) dy &\geq \frac{rK(1 - e^{-r(t_d-t)})}{4\sigma^2 x^2} (x - c(t_0^+)). \end{aligned}$$

Taking the limit  $t \rightarrow t_0^+$  in (3.5), we now obtain  $\limsup_{t \rightarrow t_0^+} \partial_x u(t, c(t)^+) < -1$ , which contradicts (3.9).  $\blacksquare$

**Proof of Lemma 3.2.** Let  $t \in [0, t_d]$ . When  $g$  is convex, since  $x \mapsto (K - x)^+$  is also convex, for each stopping time  $\tau \in \mathcal{T}_{[0, t_d-t]}$ ,  $x \mapsto \mathbb{E}(e^{-r\tau}(K - \bar{S}_\tau^x)^+ 1_{\{\tau < t_d-t\}} + e^{-r(t_d-t)} g(\bar{S}_{t_d-t}^x) 1_{\{\tau = t_d-t\}})$  is convex. So  $x \mapsto u(t, x)$  which is equal to the supremum over  $\tau$  of the previous functions is convex.

Let now  $0 \leq t \leq s < t_d$ ,  $x > 0$  and  $\tau \in \mathcal{T}_{[0, t_d-s]}$  be such that

$$u(s, x) = \mathbb{E} \left( e^{-r\tau}(K - \bar{S}_\tau^x)^+ 1_{\{\tau < t_d-s\}} + e^{-r(t_d-s)} g(\bar{S}_{t_d-s}^x) 1_{\{\tau = t_d-s\}} \right).$$

Since  $u(t, x) \geq \mathbb{E} \left( e^{-r\tau}(K - \bar{S}_\tau^x)^+ 1_{\{\tau < t_d-s\}} + e^{-r(t_d-t)} g(\bar{S}_{t_d-t}^x) 1_{\{\tau = t_d-s\}} \right)$ , one has

$$u(t, x) - u(s, x) \geq \mathbb{E} \left( 1_{\{\tau = t_d-s\}} \left( e^{-r(t_d-t)} g(\bar{S}_{t_d-t}^x) - e^{-r(t_d-s)} g(\bar{S}_{t_d-s}^x) \right) \right).$$

When  $g$  is convex, according to Lemma 1.6,  $\mathcal{A}g$  is a function bounded from below by  $-rK$ , the right-hand-side is equal to  $\mathbb{E} \left( 1_{\{\tau = t_d-s\}} \int_{t_d-s}^{t_d-t} e^{-rv} \mathcal{A}g(\bar{S}_v^x) dv \right)$ , so one easily concludes. In general, by (2.4) and the martingale property of the process  $(\int_0^v e^{-rw} \sigma \bar{S}_w^x \partial_2 \bar{u}(t_d, \bar{S}_w^x - D(\bar{S}_w^x))(1 - D'_-(\bar{S}_w^x)) dW_w)_v$ , the previous inequality writes

$$\begin{aligned} u(t, x) - u(s, x) &\geq \mathbb{E} \left( 1_{\{\tau = t_d-s\}} \int_{t_d-s}^{t_d-t} e^{-rv} \left[ \gamma(\bar{S}_v^x) dv - \frac{\partial_2 \bar{u}(t_d, \bar{S}_v^x - D(\bar{S}_v^x))}{2} \int_0^\infty D''(da) dL_v^a(\bar{S}^x) \right] \right). \end{aligned} \quad (3.11)$$

Since  $\partial_2 \bar{u}(t_d, y) \geq -1$ , using the occupation times formula, one deduces that

$$u(s, x) - u(t, x) \leq \int_{t_d-s}^{t_d-t} e^{-rv} \left( -\inf_{y>0} \gamma(y) - \int_0^{+\infty} \frac{\sigma^2 a^2}{2} p(v, a) D''(da) \right) dv.$$

Since  $D(x)$  and  $x - D(x)$  are both non-decreasing,  $D''((0, +\infty)) \geq -1$ . Using moreover

$$\forall v \in [0, t_d - t], \forall a > 0, a^2 p(v, a) = \frac{x e^{rv}}{\sigma \sqrt{2\pi v}} e^{-\frac{(\log(a/x) - (r + \frac{\sigma^2}{2})v)^2}{2\sigma^2 v}} \leq \frac{x e^{rv}}{\sigma \sqrt{2\pi v}},$$

one deduces (3.2). The inequality (3.3) follows since for  $x > c(t)$  we have  $\frac{\sigma^2 x^2}{2} \partial_{xx} u(t, x) = -\partial_t u(t, x) - rx \partial_x u(t, x) + ru(t, x) \geq -\partial_t u(t, x) + r(K - x)^+$ .

Assume (3.1). Then  $\gamma$  is equal to  $-rK$  on  $(0, x_0 \wedge x^*)$ ,  $D''((0, x_0)) = 0$  and (3.11) implies that

$$u(s, x) - u(t, x) \leq \int_{t_d-s}^{t_d-t} e^{-rv} \left( rK - \left( \inf_{y>0} \gamma(y) + rK \right) \mathbb{P}(\bar{S}_v^x \geq x_0 \wedge x^*) - \int_{x_0}^{+\infty} \frac{\sigma^2 a^2}{2} p(v, a) D''(da) \right) dv.$$

For  $x \in (0, x_0 e^{-(r + \frac{\sigma^2}{2})(t_d-t)}]$ , one has  $\forall v \in [0, t_d-t], \forall a \geq x_0, a^2 p(v, a) \leq \frac{x e^{rv}}{\sigma \sqrt{2\pi v}} e^{-\frac{(\log(x_0/x) - (r + \frac{\sigma^2}{2})v)^2}{2\sigma^2 v}}$ .

For  $t$  close enough to  $t_d$  we have that  $c(t) < x_0 e^{-(r + \frac{\sigma^2}{2})(t_d-t)}$  by Lemma 2.1 and for  $x \in (c(t), x_0 e^{-(r + \frac{\sigma^2}{2})(t_d-t)})$ ,

$$\begin{aligned} \partial_t u(t, x) &\leq e^{-r(t_d-t)} \left( rK - \left( \inf_{y>0} \gamma(y) + rK \right) N \left( \frac{\log(x/(x_0 \wedge x^*)) + (r - \frac{\sigma^2}{2})(t_d-t)}{\sigma \sqrt{t_d-t}} \right) \right) \\ &\quad + \frac{\sigma x}{2\sqrt{2\pi}(t_d-t)} e^{-\frac{(\log(x_0/x) - (r + \frac{\sigma^2}{2})(t_d-t))^2}{2\sigma^2(t_d-t)}}. \end{aligned}$$

Bounding from above the two last terms like in the derivation of the upper-bound for  $c(t)$  in the proof of Lemma 2.1, one deduces the last assertion.  $\blacksquare$

## 4 Conclusions and Further Research

We have proven local results concerning the regularity of the exercise boundary for a dividend-paying asset. Even in the simplest case of *proportional dividends*, it would be of great interest to check the following feature observed in numerical simulations : when  $t_d$  is large, the exercise boundary is non-decreasing for small times and monotonicity seems to change only once before  $t_d$ .

Different dividend models have been considered in the present paper, but in all cases the dividend could be written as a fixed function  $D$  of the ex-dividend stock price. In an alternative model for dividends, known as the *Escrowed Dividend Model*, the dividend payment consists of a deterministic amount  $D > 0$  and the stock price dynamics are given by

$$S_t = (S_0 - D e^{-rt_d}) e^{\sigma W_t + (r - \frac{1}{2}\sigma^2)t} + D e^{-r(t_d-t)} \mathbf{1}_{\{t < t_d\}}. \quad (4.1)$$

Establishing the properties of the optimal exercise boundary for the American Put option under these stock dynamics would be an interesting topic for further research.

## References

- [1] E. Bayraktar and H. Xing. An efficient method for pricing American options for jump diffusions. *CoRR*, abs/0706.2331, 2007.
- [2] P. Carr, R. Jarrow, and R. Myneni. Alternative characterizations of American puts. *Mathematical Finance*, 2:87–106, 1992.
- [3] X. Chen and J. Chadam. Mathematical analysis of the optimal exercise boundary for American put options. *SIAM Journal on Mathematical Analysis*, 38(5):1613–1641, 2006/07.
- [4] X. Chen, J. Chadam, L. Jiang, and W. Zheng. Convexity of the exercise boundary of the American put option on a zero dividend asset. *Mathematical Finance*, 18:185–197, 2008.
- [5] E. Ekström. Convexity of the optimal stopping boundary for the American put option. *Journal of Mathematical Analysis and Applications*, 229(1):147–156, 2004.
- [6] N. El Karoui. *Les aspects probabilistes du contrôle stochastique, Lecture Notes in Mathematics 876*. Springer-Verlag, Berlin, 1981.
- [7] N. El Karoui, J.-P. Lepeltier, and A. Millet. A probabilistic approach of the reduite. *Probability and Mathematical Statistics*, 13:97–121, 1992.
- [8] A. Friedman. *Stochastic Differential Equations and Applications: I*. Academic Press, 1975.
- [9] O. Götsche and M.H. Vellekoop. The early exercise premium for the American put under discrete dividends. To appear in *Mathematical Finance*, 2009.
- [10] S. Hamadène and J.-P. Lepeltier. Reflected BSDEs and mixed game problem. *Stochastic Processes and their Applications*, 85:177–188, 2000.
- [11] S.D. Jacka. Local times, optimal stopping and semimartingales. *Annals of Probability*, 21:329–339, 1993.
- [12] I. Karatzas and S.E. Shreve. *Methods of mathematical finance*. Applications of Mathematics, 39, Springer-Verlag, 1998.
- [13] I.J. Kim. The analytical valuation of American options. *Review of Financial Studies*, 3:547–472, 1990.
- [14] D. Lamberton. Critical price for an American option near maturity. Seminar on Stochastic Analysis, Random Fields and Applications (Ascona, 1993), *Progr. Probab.*, 36:353–358, Birkhäuser, 1995.
- [15] H.P. McKean. Appendix: a free boundary problem for the heat equation arising from a problem of mathematical economics. *Ind. Management Rev.*, 6:32–39, 1965.
- [16] R. Myneni. The pricing of the American option. *Annals of Applied Probability*, 2(1):1–28, 1992.
- [17] G. Peskir. A change-of-variable formula with local times on curves. *Journal of Theoretical Probability*, 18:499–535, 2005.
- [18] G. Peskir. On the American option problem. *Mathematical Finance*, 15:169–181, 2005.
- [19] G. Peskir and A. Shiryaev. *Optimal Stopping and Free-Boundary Problems*. Birkhäuser, Basel, 2006.

- [20] P. Van Moerbeke. On optimal stopping and free boundary problems. *Arch. Ration. Mech. Anal.*, 60:101–148, 1976.
- [21] M.H. Vellekoop and J.W. Nieuwenhuis. Efficient Pricing of Derivatives on Assets with Discrete Dividends. *Applied Mathematical Finance*, 13(3):265–284, 2006.
- [22] M.H. Vellekoop and J.W. Nieuwenhuis. The early exercise premium for American put options on stocks with dividends. Working Paper, 2009.