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DOUBLE AFFINE HECKE ALGEBRAS AND AFFINE FLAG MANIFOLDS, I

M. Varagnolo, E. Vasserot

0. Introduction

This paper is the first of a series of papers reviewing the geometric construction of the double affine Hecke algebra via affine flag manifolds. The aim of this work is to explain the main results in [V], [VV], but also to give a simpler approach to some of them, and to give the proof of some 'folklore' related statements whose proofs are not available in the published literature. This work should be therefore be viewed as a companion to loc. cit., and is by no means a logically independent treatment of the theory from the very begining. In order that the length of each paper remains reasonable, we have split the whole exposition into several parts. This one concerns the most basic facts of the theory : the geometric construction of the double affine Hecke algebra via the equivariant, algebraic K-theory and the classification of the simple modules of the category $\boldsymbol{\mathcal{O}}$ of the double affine Hecke algebra. It is our hope that by providing a detailed explanation of some of the difficult aspects of the foundations, this theory will be better understood by a wider audience.

This paper contains three chapters. The first one is a reminder on $\mathcal{O}$-modules over non Noetherian schemes and over ind-schemes. The second one deals with affine flag manifolds. The last chapter concerns the classification of simple modules in the category $\boldsymbol{\mathcal{O}}$ of the double affine Hecke algebra. Let us review these parts in more details.

In the second chapter of the paper we use two different versions of the affine flag manifold. The first one is an ind-scheme of ind-finite type, while the second one is a pro-smooth, coherent, separated, non Noetherian, and non quasi-compact scheme. Thus, in the first chapter we recall some basic fact on $\mathcal{O}$-modules over coherent schemes, pro-schemes, and ind-schemes. The first section is a reminder on pro-objects and ind-objects in an arbitrary category. We give the definition of direct and inverse 2-limits of categories. Next we recall the definition of the K-homology of a scheme. We'll use non quasi-compact non Noetherian scheme. Also, it is convenient to consider a quite general setting involving unbounded derived categories, pseudo-coherent complexes and perfect complexes. Fortunately, since all the schemes we'll consider are coherent the definition of the K-theory remains quite close to the usual one. To simplify the exposition it is convenient to introduce the derived direct image of a morphism of non Noetherian schemes, its derived inverse image, and the derived tensor product in the unbounded derived categories of $\mathcal{O}$-modules. Finally we consider the special case of pro-schemes (compact schemes, pro-smooth schemes, etc) and of ind-schemes. They are important tools in this work. This section finishes with equivariant $\mathcal{O}$-modules and some basic tools in equivariant

K-theory (induction, reduction of the group action, the Thom isomorphism, and the Thomason concentration theorem).

The second chapter begins with the definition of the affine flag manifold, which is an ind-scheme of ind-finite type, and with the definition of the Kashiwara affine flag manifold, which is a non quasi-compact coherent scheme. This leads us in Section 2.3.6 to the definition of an associative multiplication on a group of equivariant K-theory $\mathbf{K}^{\boldsymbol{I}}(\mathfrak{N})$. Here $\mathfrak{N}$ is an ind-scheme which can be regarded as the affine analogue of the Steinberg variety for reductive groups. Then, in section 2.4.1, we define an affine analogue of the concentration map for convolution rings in K-theory used in [CG]. It is a ring homomorphism relates $\mathbf{K}^{\boldsymbol{I}}(\mathfrak{N})$ to the K-theory of the fixed points subset for a torus action. This concentration map is new, and it simplifies the proofs in [V]. The double affine Hecke algebra is introduced in section 2.5.1 and its geometric realization is proved in Theorem 2.5.6. We use here an approach similar to the one in [BFM], where a degenerate version of the double affine Hecke algebra is constructed geometrically. Compare also [GG], where the regular representation of the double affine Hecke algebra is constructed geometrically. The proof we give uses a reduction to the fixed points of a torus acting on the affine analogue of the Steinberg variety, and the concentration map in K-theory.

The third chapter is a review of the classification of the simple modules in the category $\boldsymbol{\mathcal{O}}$ of the double affine Hecke algebra. The main theorem was proved in [V]. The proof we give here is simpler than in loc. cit. because it uses the concentration map. The first section contains generalities on convolution algebras in the cohomology of schemes of infinite type which are locally of finite type. The proof of the classification is given in the second section.

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1. Schemes and ind-schemes

1.1. Categories and Grothendieck groups.

1.1.1. Ind-objects and pro-objects in a category. A standard reference for the material in this section is [SGA4, sec. 8], [KS1], [KS2].

Let $\boldsymbol{Set}$ be the category of sets. Given a category $\boldsymbol{\mathcal{C}}$ let $\boldsymbol{\mathcal{C}}^\circ$ be the opposite category. The category $\boldsymbol{\mathcal{C}}^\wedge$ of *presheaves over* $\boldsymbol{\mathcal{C}}$ is the category of functors $\boldsymbol{\mathcal{C}}^\circ \to \boldsymbol{Set}$. We'll abbreviate $\boldsymbol{\mathcal{C}}^\vee$ for the category $((\boldsymbol{\mathcal{C}}^\circ)^\wedge)^\circ$. Yoneda's lemma yields fully faithful functors

$$\boldsymbol{\mathcal{C}} \to \boldsymbol{\mathcal{C}}^\wedge,\ X \mapsto \mathrm{Hom}_{\boldsymbol{\mathcal{C}}}(\cdot, X), \quad \boldsymbol{\mathcal{C}} \to \boldsymbol{\mathcal{C}}^\vee,\ X \mapsto \mathrm{Hom}_{\boldsymbol{\mathcal{C}}}(X, \cdot).$$

Let $\boldsymbol{\mathcal{A}}$ be a category and $\alpha \mapsto X_\alpha$ be a functor $\boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{C}}$ or $\boldsymbol{\mathcal{A}}^\circ \to \boldsymbol{\mathcal{C}}$ (also called a *system* in $\boldsymbol{\mathcal{C}}$ indexed by $\boldsymbol{\mathcal{A}}$ or $\boldsymbol{\mathcal{A}}^\circ$). Let $\mathrm{colim}_\alpha X_\alpha$ and $\lim_\alpha X_\alpha$ denote the colimit or the limit of this system whenever it is well-defined. If the category $\boldsymbol{\mathcal{A}}$ is small or filtrant the colimit and the limit are said to be *small* or *filtrant*. A poset $\boldsymbol{\mathcal{A}} = (A, \leqslant)$ may be viewed as a category, with A as the set of objects and with a morphism $\alpha \to \beta$ whenever $\alpha \leqslant \beta$. A direct set is a poset $\boldsymbol{\mathcal{A}}$ which is filtrant as a category. A *direct system* over $\boldsymbol{\mathcal{C}}$ is a functor $\boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{C}}$ and an *inverse system* over $\boldsymbol{\mathcal{C}}$ is a functor $\boldsymbol{\mathcal{A}}^\circ \to \boldsymbol{\mathcal{C}}$, where $\boldsymbol{\mathcal{A}}$ is a direct set. A *direct colimit* (also called *inductive limit*) is the colimit of a direct system. An *inverse limit* (also called *projective limit*) is the limit of an inverse system. Both are small and filtrant.

A *complete* or *cocomplete* category is one that has all small limits or all small colimits. A *Grothendieck category* is a cocomplete Abelian category with a generator such that the small filtrant colimits are exact.

Given a direct system or an inverse system in $\boldsymbol{\mathcal{C}}$ we define the following functors

$$\text{“}\mathrm{colim}_\alpha\text{”}\, X_\alpha :\ \boldsymbol{\mathcal{C}}^\circ \to \boldsymbol{Set}, \quad Y \mapsto \mathrm{colim}_\alpha \mathrm{Hom}_{\boldsymbol{\mathcal{C}}}(Y, X_\alpha),$$
$$\text{“}\lim_\alpha\text{”}\, X_\alpha :\ \boldsymbol{\mathcal{C}} \to \boldsymbol{Set}, \quad Y \mapsto \mathrm{colim}_\alpha \mathrm{Hom}_{\boldsymbol{\mathcal{C}}}(X_\alpha, Y).$$

The categories of *ind-objects* of $\boldsymbol{\mathcal{C}}$ and the category of *pro-objects* of $\boldsymbol{\mathcal{C}}$ are the full subcategory $\boldsymbol{Ind}(\boldsymbol{\mathcal{C}})$ of $\boldsymbol{\mathcal{C}}^\wedge$ and the full subcategory $\boldsymbol{Pro}(\boldsymbol{\mathcal{C}})$ of $\boldsymbol{\mathcal{C}}^\vee$ consisting of objects isomorphic to some “colim_α” X_α and “$\lim_\alpha$” X_α respectively. Note that we have $\boldsymbol{Pro}(\boldsymbol{\mathcal{C}}) = \boldsymbol{Ind}(\boldsymbol{\mathcal{C}}^\circ)^\circ$. By the Yoneda functor we may look upon $\boldsymbol{\mathcal{C}}$ as a full subcategory of $\boldsymbol{Ind}(\boldsymbol{\mathcal{C}})$ or $\boldsymbol{Pro}(\boldsymbol{\mathcal{C}})$. We'll say that an ind-object or a pro-object is *representable* if it is isomorphic to an object in $\boldsymbol{\mathcal{C}}$. Note that, for each object Y of $\boldsymbol{\mathcal{C}}$ we have the following formulas, see, e. g., [KS1, sec. 1.11], [KS2, sec. 2.6]

$$\mathrm{Hom}_{\boldsymbol{Ind}(\boldsymbol{\mathcal{C}})}(Y, \text{“}\mathrm{colim}_\alpha\text{”}\, X_\alpha) = \mathrm{Hom}_{\boldsymbol{\mathcal{C}}^\wedge}(Y, \text{“}\mathrm{colim}_\alpha\text{”}\, X_\alpha) = \mathrm{colim}_\alpha \mathrm{Hom}_{\boldsymbol{\mathcal{C}}}(Y, X_\alpha),$$
$$\mathrm{Hom}_{\boldsymbol{Pro}(\boldsymbol{\mathcal{C}})}(\text{“}\lim_\alpha\text{”}\, X_\alpha, Y) = \mathrm{Hom}_{\boldsymbol{\mathcal{C}}^\vee}(\text{“}\lim_\alpha\text{”}\, X_\alpha, Y) = \mathrm{colim}_\alpha \mathrm{Hom}_{\boldsymbol{\mathcal{C}}}(X_\alpha, Y).$$

1.1.2. Direct and inverse 2-limits. Let $\boldsymbol{\mathcal{A}} = (A, \leqslant)$ be a directed set. Given a direct system of categories $(\boldsymbol{\mathcal{C}}_\alpha, i_{\alpha\beta} : \boldsymbol{\mathcal{C}}_\alpha \to \boldsymbol{\mathcal{C}}_\beta)$ the *2-limit* (also called the *2-colimit*) of this system is the category $\boldsymbol{\mathcal{C}} = 2\mathrm{colim}_\alpha \boldsymbol{\mathcal{C}}_\alpha$ whose objects are the pairs (α, X_α) with X_α an object of $\boldsymbol{\mathcal{C}}_\alpha$. The morphisms are given by

$$\mathrm{Hom}_{\boldsymbol{\mathcal{C}}}((\alpha, X_\alpha), (\beta, X_\beta)) = \mathrm{colim}_{\gamma \geqslant \alpha, \beta} \mathrm{Hom}_{\boldsymbol{\mathcal{C}}_\gamma}(i_{\alpha\gamma}(X_\alpha), i_{\beta\gamma}(X_\beta)).$$

Given an inverse system of categories $(\boldsymbol{\mathcal{C}}_\alpha, i_{\alpha\beta} : \boldsymbol{\mathcal{C}}_\beta \to \boldsymbol{\mathcal{C}}_\alpha)$ the *2-limit* of this system is the category $\boldsymbol{\mathcal{C}} = 2\lim_\alpha \boldsymbol{\mathcal{C}}_\alpha$ whose objects are the families of objects X_α of $\boldsymbol{\mathcal{C}}_\alpha$ and of isomorphisms $i_{\alpha\beta}(X_\beta) \simeq X_\alpha$ satisfying the obvious composition rules. The morphisms are defined in the obvious way. See [W, app. A] for more details on 2-colimits and 2-limits.

1.1.3. Grothendieck groups and derived categories. Given an Abelian category $\boldsymbol{\mathcal{A}}$ let $\boldsymbol{\mathcal{C}}(\boldsymbol{\mathcal{A}})$ be the category of complexes of objects of $\boldsymbol{\mathcal{A}}$ with differential of degree $+1$ and chain maps as morphisms, let $\boldsymbol{\mathcal{D}}(\boldsymbol{\mathcal{A}})$ be the corresponding (unbounded) derived category, let $\boldsymbol{\mathcal{D}}(\boldsymbol{\mathcal{A}})^-$ be the full subcategory of complexes bounded above, let $\boldsymbol{\mathcal{D}}(\boldsymbol{\mathcal{A}})^b$ be the full subcategory of bounded complexes. Finally let $[\boldsymbol{\mathcal{A}}]$ the Grothendieck group of $\boldsymbol{\mathcal{A}}$.

The Grothendieck group $[\boldsymbol{\mathcal{T}}]$ of a triangulated category $\boldsymbol{\mathcal{T}}$ is the quotient of the free Abelian group with one generator for each object X of $\boldsymbol{\mathcal{T}}$ modulo the relations $X = X' + X''$ for each distinguished triangle

$$X' \to X \to X'' \to X'[1].$$

Here the symbol [1] stands for the shift functor in the triangulated category $\boldsymbol{\mathcal{T}}$. Throughout, we'll use the same symbol for an object of $\boldsymbol{\mathcal{T}}$ and is class in $[\boldsymbol{\mathcal{T}}]$.

Recall that the Grothendieck group of $\boldsymbol{\mathcal{D}}(\boldsymbol{\mathcal{A}})^b$ is canonically isomorphic to $[\boldsymbol{\mathcal{A}}]$, and that two quasi-isomorphic complexes of $\boldsymbol{\mathcal{C}}(\boldsymbol{\mathcal{A}})$ have the same class in $[\boldsymbol{\mathcal{A}}]$.

1.1.4. Proposition. *Let $(\boldsymbol{\mathcal{C}}_\alpha)$ be a direct system of Abelian categories (resp. of triangulated categories) and exact functors. Then the direct 2-limit $\boldsymbol{\mathcal{C}}$ of $(\boldsymbol{\mathcal{C}}_\alpha)$ is also an Abelian category (resp. a triangulated category) and we have a canonical group isomorphism $[\boldsymbol{\mathcal{C}}] = \mathrm{colim}_\alpha[\boldsymbol{\mathcal{C}}_\alpha]$.*

1.2. K-theory of schemes.

This section is a recollection of standard results from [SGA6], [TT] on the K-theory of schemes, possibly of infinite type.

1.2.1. Background. For any Abelian category $\boldsymbol{\mathcal{A}}$ a complex in $\boldsymbol{\mathcal{C}}(\boldsymbol{\mathcal{A}})$ is cohomologically bounded if the cohomology sheaves vanish except for a finite number of them. The canonical functor yields an equivalence from $\boldsymbol{\mathcal{D}}(\boldsymbol{\mathcal{A}})^b$ to the full subcategory of $\boldsymbol{\mathcal{D}}(\boldsymbol{\mathcal{A}})$ consisting of cohomologically bounded complexes [KS1, p. 45].

A quasi-compact scheme is a scheme that has a finite covering by affine open subschemes (e.g., a Noetherian scheme or a scheme of finite type is quasi-compact) and a quasi-separated scheme is a scheme such that the intersection of any two affine open subschemes is quasi-compact (e.g., a separated scheme is quasi-separated). More generally, a scheme homomorphism $f : X \to Y$ is said to be quasi-compact, resp. quasi-separated, if for every affine open $U \subset Y$ the inverse image of U is quasi-compact, resp. quasi-separated. Elementary properties of quasi-compact and quasi-separated morphisms can be found in [GD, chap. I, sec. 6.1]. For instance quasi-compact and quasi-separated morphisms are stable under composition and pullback, and if $f : X \to Y$ is a scheme homomorphism with Y quasi-compact and quasi-separated then X is quasi-compact and separated iff f is quasi-compact and quasi-separated. *Throughout, by the word scheme we'll always mean a separated $\mathbb{C}$-scheme and by the word scheme homomorphism we'll always mean a morphism of separated $\mathbb{C}$-schemes.* In particular a scheme homomorphism will always be separated (hence quasi-separated) [GD, chap. I, sec. 5.3].

Given a scheme X, the word $\mathcal{O}_X$-module will mean a sheaf on the scheme X which is a sheaf of modules over the sheaf of rings $\mathcal{O}_X$. Unless otherwise stated, modules are left modules. This applies also to $\mathcal{O}_X$-modules. Since $\mathcal{O}_X$ is commutative this specification is indeed irrelevant. Let $\boldsymbol{\mathcal{O}}(X)$ be the Abelian category

of all $\mathcal{O}_X$-modules. Given a closed subscheme $Y \subset X$ let $\boldsymbol{\mathcal{O}}(X \text{ on } Y)$ be the full subcategory of $\mathcal{O}_X$-modules supported on Y.

Let $\boldsymbol{\mathcal{C}oh}(X)$, $\boldsymbol{\mathcal{Q}coh}(X)$ be the categories of coherent and quasi-coherent $\mathcal{O}_X$-modules. They are Abelian subcategories of $\boldsymbol{\mathcal{O}}(X)$ which are stable under extensions. Quasi-coherent sheaves are preserved by tensor products, by arbitrary colimits, and by inverse images [GD, chap. I, sec. 2.2]. They are well-behaved on quasi-compact (quasi-separated) schemes : under this assumption quasi-coherent $\mathcal{O}_X$-modules are preserved by direct images and any quasi-coherent $\mathcal{O}_X$-module is the limit of a direct system of finitely presented $\mathcal{O}_X$-modules. Further, if X is quasi-compact (quasi-separated) the category $\boldsymbol{\mathcal{Q}coh}(X)$ is a Grothendieck category. In particular for any such X there are enough injective objects in $\boldsymbol{\mathcal{Q}coh}(X)$ [GD, chap. I, sec. 6.7, 6.9], [TT, sec. B.3]. Given a closed subscheme $Y \subset X$ let $\boldsymbol{\mathcal{C}oh}(X \text{ on } Y)$, $\boldsymbol{\mathcal{Q}coh}(X \text{ on } Y)$ be the full subcategories of sheaves supported on Y.

We'll abbreviate $\boldsymbol{\mathcal{C}}(X) = \boldsymbol{\mathcal{C}}(\boldsymbol{\mathcal{O}}(X))$ and $\boldsymbol{\mathcal{D}}(X) = \boldsymbol{\mathcal{D}}(\boldsymbol{\mathcal{O}}(X))$. Let $\boldsymbol{\mathcal{D}}(X)_{qc}$ be the full subcategory of $\boldsymbol{\mathcal{D}}(X)$ of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology.

1.2.2. Remark. Bökstedt and Neeman proved that if X is quasi-compact (separated) then the canonical functor is an equivalence

$$\boldsymbol{\mathcal{D}}(\boldsymbol{\mathcal{Q}coh}(X)) \to \boldsymbol{\mathcal{D}}(X)_{qc} \tag{1.2.1}$$

[BN, cor. 5.5], [Li, prop. 3.9.6]. Further, the standard derived functors in 1.2.10-12 below, evaluated on quasi-coherent sheaves, are the same taken in $\boldsymbol{\mathcal{O}}(X)$ or in $\boldsymbol{\mathcal{Q}coh}(X)$, see e.g., [TT, cor. B.9]. So from now on we'll identify the categories $\boldsymbol{\mathcal{D}}(\boldsymbol{\mathcal{Q}coh}(X))$ and $\boldsymbol{\mathcal{D}}(X)_{qc}$.

A commutative ring R is *coherent* iff it is coherent as a R-module, or, equivalently, if every finitely generated ideal of R is finitely presented. For instance a Noetherian ring is coherent, the quotient of a coherent ring by a finitely generated ideal is a coherent ring and the localization of a coherent ring is again coherent.

1.2.3. Definitions. Let X be any scheme. We say that

(a) X is *coherent* if its structure ring $\mathcal{O}_X$ is coherent,

(b) X is *locally of countable type* if the $\mathbb{C}$-algebra $\mathcal{O}_X(U)$ is generated by a countable number of elements for any affine open subset $U \subset X$,

(c) a closed subscheme $Y \subset X$ is *good* if the ideal of Y in $\mathcal{O}_X(U)$ is finitely generated for any affine open subset $U \subset X$.

If the scheme X is coherent then an $\mathcal{O}_X$-module is coherent iff it is finitely presented, and we have $f^*(\boldsymbol{\mathcal{C}oh}(Y)) \subset \boldsymbol{\mathcal{C}oh}(X)$ for any morphism $f : X \to Y$. If X is quasi-compact and coherent then any quasi-coherent $\mathcal{O}_X$-module is the direct colimit of a system of coherent $\mathcal{O}_X$-modules. Finally a good subscheme Y of a coherent scheme X is again coherent and the direct image of $\mathcal{O}_Y$ is a coherent $\mathcal{O}_X$-module. See [EGAI, chap. 0, sec. 5.3] for details.

1.2.4. K-theory of a quasi-compact coherent scheme. For an arbitrary scheme X the K-homology group (=K-theory) may differ from $[\boldsymbol{\mathcal{C}oh}(X)]$, one reason being that $\mathcal{O}_X$ may not be an object of $\boldsymbol{\mathcal{C}oh}(X)$. Let us recall briefly some relevant definitions and results concerning pseudo-coherence. Details can be found in [SGA6, chap. I], [TT] and [Li, sec. 4.3]. We'll assume that X is quasi-compact and coherent.

1.2.5. Definition-Lemma. (a) A complex of quasi-coherent $\mathcal{O}_X$-modules $\mathcal{E}$ is *pseudo-coherent* if it is locally quasi-isomorphic to a bounded above complex of vector bundles. Since X is coherent, this simply means that $\mathcal{E}$ has coherent cohomology sheaves vanishing in all sufficiently large degrees [SGA6, cor. I.3.5(iii)]. In particular any coherent $\mathcal{O}_X$-module is a pseudo-coherent complex.

(b) Let $\boldsymbol{\mathcal{P}coh}(X)$ be the full subcategory of $\boldsymbol{\mathcal{D}}(X)_{qc}$ consisting of the cohomologically bounded pseudo-coherent complexes. Given a closed subscheme $Y \subset X$ the full subcategory of complexes which are acyclic on $X - Y$ is $\boldsymbol{\mathcal{P}coh}(X \text{ on } Y)$. It is a triangulated category.

Note that for a general scheme the equivalence of categories (1.2.1) does not hold and a pseudo-coherent complex may consist of non quasi-coherent $\mathcal{O}_X$-modules. The K-homology group of the pair (X, Y) is [SGA6, def. IV.2.2]

$$\mathbf{K}(X \text{ on } Y) = [\boldsymbol{\mathcal{P}coh}(X \text{ on } Y)], \quad \mathbf{K}(X) = \mathbf{K}(X \text{ on } X).$$

By 1.2.5 the K-homology groups are well-behaved on quasi-compact coherent schemes. More precisely we have the following.

1.2.6. Proposition. *Assume that X is quasi-compact and coherent. We have $\mathbf{K}(X \text{ on } Y) = [\boldsymbol{\mathcal{C}oh}(X \text{ on } Y)]$ for any closed subscheme $Y \subset X$. If $Y \subset X$ is good there is a canonical isomorphism $\mathbf{K}(Y) \to \mathbf{K}(X \text{ on } Y)$.*

If X is coherent but not quasi-compact we define the group $\mathbf{K}(X)$ as follows. Fix a covering $X = \bigcup_w X^w$ by quasi-compact open subsets. The restrictions yield a inverse system of categories with a functor

$$\boldsymbol{\mathcal{C}oh}(X) \to 2\lim_w \boldsymbol{\mathcal{C}oh}(X^w).$$

By functoriality of the K-theory we have also an inverse system of Abelian groups. We define

$$\mathbf{K}(X) = \lim_w \mathbf{K}(X^w) = \lim_w [\boldsymbol{\mathcal{C}oh}(X^w)].$$

The group $\mathbf{K}(X)$ does not depend on the choice of the open covering. It may be regarded as a completion of the K-homology group of X, as defined in [SGA6].

1.2.7. Remark. Let X be a quasi-compact scheme. A *perfect complex* over X is a complex of quasi-coherent $\mathcal{O}_X$-modules which is locally quasi-isomorphic to a bounded complex of vector bundles. The K-cohomology groups of X is the Grothendieck group of the the full subcategory of $\boldsymbol{\mathcal{D}}(X)_{qc}$ of perfect complexes. We'll not use it.

1.2.8. Basic properties of the K-theory of a coherent quasi-compact scheme. Recall that for any $\mathcal{O}_X$-modules $\mathcal{E}, \mathcal{F}$ the *local hypertor* is the $\mathcal{O}_X$-module $\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ whose stalk at a point x is $\mathrm{Tor}_i^{\mathcal{O}_{X,x}}(\mathcal{E}_x, \mathcal{F}_x)$.

1.2.9. Definitions. (a) An $\mathcal{O}_X$-module $\mathcal{E}$ has a *finite tor-dimension* if there is an integer n such that $\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) = 0$ for each $i > n$ and each $\mathcal{F} \in \boldsymbol{\mathcal{Q}coh}(X)$.

(b) A scheme homomorphism $f : X \to Y$ has *finite tor-dimension* if there is an integer n such that $\mathcal{T}or_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{E}) = 0$ for each $i > n$ and each $\mathcal{E} \in \boldsymbol{\mathcal{Q}coh}(Y)$. Equivalently, f has finite tor-dimension if there is an integer n such that for each $x \in X$ there is an exact sequence of $\mathcal{O}_{Y,f(x)}$-modules

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to \mathcal{O}_{X,x} \to 0$$

with P_i flat over $\mathcal{O}_{Y,f(x)}$.

(c) A scheme X *satisfies the Poincaré duality* if any quasi-coherent $\mathcal{O}_X$-module has a finite tor-dimension.

Poincaré duality is a local property. Note that, since taking a local hypertor commutes with direct colimits [EGAIII, prop. 6.5.6], a coherent scheme satisfies the Poincaré duality iff any coherent $\mathcal{O}_X$-module has a finite tor-dimension. Note that if X satisfies the Poincaré duality then any cohomologically bounded pseudo-coherent complex is perfect [TT, thm. 3.21].

Now, let us recall a few basic properties of direct/inverse image of complexes of $\mathcal{O}$-modules. We'll use derived functors in the unbounded derived category of $\mathcal{O}$-modules. This simplifies the exposition. Their definition and properties can be found in [Li]. To simplify, in Sections 1.2.10 to 1.2.15 we'll also assume that all schemes are quasi-compact and coherent. Therefore, all morphisms will also be quasi-compact.

1.2.10. Derived inverse image. For any morphism $f : Z \to X$ the inverse image functor Lf^* maps $\boldsymbol{\mathcal{D}}(X)_{qc}$, $\boldsymbol{\mathcal{D}}(X)^-$ into $\boldsymbol{\mathcal{D}}(Z)_{qc}$, $\boldsymbol{\mathcal{D}}(Z)^-$ respectively. It preserves pseudo-coherent and perfect complexes [Li, prop. 3.9.1], [TT, sec. 2.5.1]. Further if $\mathcal{E}$ is a pseudo-coherent complex then the complex $Lf^*(\mathcal{E})$ is cohomologically bounded if the map f has finite tor-dimension. Under this assumption, for each closed subscheme $Y \subset X$ the functor Lf^* yields a group homomorphism

$$Lf^* : \mathbf{K}(X \text{ on } Y) \to \mathbf{K}(Z \text{ on } f^{-1}(Y)).$$

If the schemes X, Z satisfy the Poincaré duality then f has a finite tor-dimension.

1.2.11. Derived tensor product. Let $\otimes_X$ denote the tensor product of $\mathcal{O}$-modules on any scheme X. The standard theory of the derived tensor product of $\mathcal{O}$-modules applies to complexes in $\boldsymbol{\mathcal{D}}(X)^-$, see e.g., [Hr, p.93]. Following Spaltenstein [Sp] we can extend the theory to arbitrary complexes in $\boldsymbol{\mathcal{D}}(X)$, see also [Li, sec. 2.5]. This yields a functor

$$\overset{L}{\otimes}_X : \boldsymbol{\mathcal{D}}(X) \times \boldsymbol{\mathcal{D}}(X) \to \boldsymbol{\mathcal{D}}(X)$$

which maps $\boldsymbol{\mathcal{D}}(X)_{qc} \times \boldsymbol{\mathcal{D}}(X)_{qc}$, $\boldsymbol{\mathcal{D}}(X)^- \times \boldsymbol{\mathcal{D}}(X)^-$ to $\boldsymbol{\mathcal{D}}(X)_{qc}$, $\boldsymbol{\mathcal{D}}(X)^-$ respectively. It preserves pseudo-coherent complexes [TT, sec. 2.5.1]. If $\mathcal{E}$, $\mathcal{F}$ are pseudo-coherent complexes their derived tensor product is cohomologically bounded if either $\mathcal{E}$ is perfect or $\mathcal{F}$ is perfect [TT, sec. 3.15]. Recall that if X satisfies the Poincaré duality then any cohomologically bounded pseudo-coherent complex is perfect. Under this assumption, for each closed subschemes $Y, Z \subset X$ there is a (derived) tensor product

$$\overset{L}{\otimes}_X : \mathbf{K}(X \text{ on } Y) \times \mathbf{K}(X \text{ on } Z) \to \mathbf{K}(X \text{ on } Y \cap Z).$$

Given a map f as in 1.2.10 there is a functorial isomorphism in $\boldsymbol{\mathcal{D}}(X)$ [Li, prop. 3.2.4]

$$Lf^*(\mathcal{E} \overset{L}{\otimes}_X \mathcal{F}) = Lf^*(\mathcal{E}) \overset{L}{\otimes}_Z Lf^*(\mathcal{F})$$

We'll refer to this relation by saying that the derived tensor product commutes with Lf^*.

1.2.12. Derived direct image. For any $f : X \to Z$ the direct image functor Rf_* is right adjoint to Lf^* and it maps $\boldsymbol{\mathcal{D}}(X)_{qc}$, $\boldsymbol{\mathcal{D}}(X)^b$ into $\boldsymbol{\mathcal{D}}(Z)_{qc}$, $\boldsymbol{\mathcal{D}}(Z)^b$ respectively [Li, prop. 3.9.2]. We say that the map f is *pseudo-coherent* if it factors, locally on X, as $f = p \circ i$ where i is a closed embedding with $i_*\mathcal{O}_X$ coherent and p is smooth. Kiehl's finiteness theorem insures that if f is proper and pseudo-coherent then Rf_* preserves pseudo-coherent complexes [Li, cor. 4.3.3.2]. Therefore if f is proper and pseudo-coherent, for any closed subscheme $Y \subset X$, the functor Rf_* yields a group homomorphism

$$Rf_* : \mathbf{K}(X \text{ on } Y) \to \mathbf{K}(Z \text{ on } f(Y)).$$

1.2.13. Example. A good embedding is proper and pseudo-coherent. In this case we have indeed an exact functor $f_* : \boldsymbol{\mathcal{C}oh}(X) \to \boldsymbol{\mathcal{C}oh}(Z)$. It yields the isomorphism $\mathbf{K}(X) \to \mathbf{K}(Z \text{ on } X)$ in 1.2.5. Note that a closed embedding $X \subset Z$ with $Z = \mathbb{A}^{\mathbb{N}}$ and X of finite type is not pseudo-coherent. Here $\mathbb{A}^{\mathbb{N}} = \mathrm{Spec}(\mathbb{C}[x_i; i \in \mathbb{N}])$ is a coherent scheme.

1.2.14. Projection formula. For any $f : X \to Z$ there is a canonical isomorphism called the *projection formula* [Li, prop. 3.9.4]

$$Rf_*(\mathcal{E} \overset{L}{\otimes}_X Lf^*(\mathcal{F})) = Rf_*(\mathcal{E}) \overset{L}{\otimes}_Z \mathcal{F}, \quad \forall \mathcal{E} \in \boldsymbol{\mathcal{D}}(X)_{qc},\ \mathcal{F} \in \boldsymbol{\mathcal{D}}(Z)_{qc}.$$

1.2.15. Base change. Consider the following Cartesian square

$$\begin{tikzcd} X' \arrow[d, "g"'] & Y' \arrow[l, "f'"'] \arrow[d, "g'"] \\ X & Y. \arrow[l, "f"'] \end{tikzcd}$$

Assume that it is *tor-independent*, i.e., assume that we have

$$\mathrm{Tor}_i^{\mathcal{O}_{X,x}}(\mathcal{O}_{X',x'}, \mathcal{O}_{Y,y}) = 0, \quad \forall i > 0,\ \forall x \in X, \forall x' \in X', \forall y \in Y,\ x = g(x') = f(y).$$

Then we have a functorial base-change isomorphism [Li, thm. 3.10.3]

$$Lg^* Rf_*(\mathcal{E}) \simeq Rf'_* L(g')^*(\mathcal{E}), \quad \forall \mathcal{E} \in \boldsymbol{\mathcal{D}}(Y)_{qc}.$$

1.2.16. Compact schemes. A simple way to produce quasi-compact schemes of infinite type is to use pro-schemes. Let us explain this.

1.2.17. Lemma-Definition. A scheme is *compact* if it is the limit of an inverse system of finite type schemes with affine morphisms. A scheme is compact iff it is quasi-compact [TT, thm. C.9].

1.2.18. Remarks. Let X be a compact scheme and (X^α) be an inverse system of schemes as above. Then the canonical maps $p_\alpha : X \to X^\alpha$ are affine. Further the following hold.

(a) If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module there is an α and a coherent $\mathcal{O}_{X^\alpha}$-module $\mathcal{F}^\alpha$ such that $\mathcal{F} = (p_\alpha)^*(\mathcal{F}^\alpha)$. Given two coherent $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{G}$ and two coherent

$\mathcal{O}_{X^\alpha}$-modules $\mathcal{F}^\alpha$, $\mathcal{G}^\alpha$ as above we set $\mathcal{F}^\beta = (p_{\alpha\beta})^*(\mathcal{F}^\alpha)$ and $\mathcal{G}^\beta = (p_{\alpha\beta})^*(\mathcal{G}^\alpha)$ for each $\beta \geqslant \alpha$. Then we have [TT, sec. C4], [EGAIV, sec. 8.5]

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \mathrm{colim}_{\beta \geqslant \alpha} \mathrm{Hom}_{\mathcal{O}_{X^\beta}}(\mathcal{F}^\beta, \mathcal{G}^\beta).$$

(b) If $f : Y \to X$ is a scheme finitely presented over X then there is an $\alpha \in A$ and a finitely presented $f_\alpha : Y^\alpha \to X^\alpha$ such that [TT, sec. C.3]

$$f = f_\alpha \times \mathrm{id}, \quad Y = Y^\alpha \times_{X^\alpha} X = \lim_\beta Y^\beta, \quad Y^\beta = Y^\alpha \times_{X^\alpha} X^\beta, \quad \beta \geqslant \alpha.$$

1.2.19. Definition. A compact scheme $X =$ "$\lim_\alpha$" X^α *satisfies the property* (S) if $A = \mathbb{N}$ and $(X^\alpha)_{\alpha \in A}$ is an inverse system of smooth schemes of finite type with smooth affine morphisms. A scheme is *pro-smooth* if it is covered by a finite number of open subsets satisfying (S).

1.2.20. Proposition. *A pro-smooth scheme is quasi-compact, coherent, and it satisfies the Poincaré duality.*

Proof : A pro-smooth scheme X is coherent by [K1, prop. 1.1.6]. Let us prove that X satisfies the Poincaré duality. Let $\mathcal{F} \in \boldsymbol{\mathcal{C}oh}(X)$. We must prove that $\mathrm{Tor}_i^{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{E}_x) = 0$ for each $i \gg 0$, each $x \in X$, and each $\mathcal{E} \in \boldsymbol{\mathcal{Q}coh}(X)$. Since the question is local around x we can assume that X is a compact scheme satisfying the property (S). By 1.2.18(a) there is an $\alpha \in A$ and $\mathcal{F}^\alpha \in \boldsymbol{\mathcal{C}oh}(X^\alpha)$ such that $\mathcal{F} = (p_\alpha)^*(\mathcal{F}^\alpha)$. Write again $x = p_\alpha(x)$. Since X^α is smooth of finite type the $\mathcal{O}_{X^\alpha,x}$-module $\mathcal{F}^\alpha_x$ has finite tor-dimension. Since the map p_α is affine we have

$$\mathrm{Tor}_i^{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{E}_x) = \mathrm{Tor}_i^{\mathcal{O}_{X^\alpha,x}}(\mathcal{F}^\alpha_x, (p_\alpha)_*(\mathcal{E}_x)).$$

Since $(p_\alpha)_*(\mathcal{E})$ is quasi-coherent and the scheme X^α is smooth of finite type, and since taking the Tor's commutes with direct colimits, the rhs vanishes for large i. $\square$

1.2.21. Remarks. (a) Let $\boldsymbol{\mathcal{S}ch}$ be the category of schemes and $\boldsymbol{\mathcal{S}ch}^{\mathrm{ft}}$ be the full subcategory of schemes of finite type. The category of compact schemes can be identified with a full subcategory in $\boldsymbol{\mathcal{P}ro}(\boldsymbol{\mathcal{S}ch}^{\mathrm{ft}})$ via the assignment $\lim_\alpha X^\alpha \mapsto$ "$\lim_\alpha$" X^α. From now on we'll omit the quotation marks for compact schemes.

(b) Let X be a quasi-compact coherent scheme and $Y \subset X$ be a good subscheme. Since X is a compact scheme we can fix an inverse system of finite type schemes (X^α) with affine morphisms $p_{\alpha\beta} : X^\beta \to X^\alpha$ such that $X = \lim_\alpha X^\alpha$. Since the scheme X is coherent and since Y is a good subscheme, the inclusion $Y \subset X$ is finitely presented. Thus, by 1.2.18(b) there is an $\alpha \in A$ and a closed subscheme $Y^\alpha \subset X^\alpha$ such that $Y = p_\alpha^{-1}(Y^\alpha)$. Setting $Y^\beta = p_{\alpha\beta}^{-1}(Y^\alpha)$ for each $\beta \geqslant \alpha$ we get a direct system of categories $\boldsymbol{\mathcal{C}oh}(X^\alpha \text{ on } Y^\alpha)$ with functors $(p_{\alpha\beta})^*$. Now, assume that the pro-object $X = \lim_\alpha X^\alpha$ satisfies the property (S). The pull-back by the canonical map $p_\alpha : X \to X^\alpha$ yields an equivalence of categories [EGAIV, thm. 8.5.2]

$$2\mathrm{colim}_\alpha \boldsymbol{\mathcal{C}oh}(X^\alpha \text{ on } Y^\alpha) \to \boldsymbol{\mathcal{C}oh}(X \text{ on } Y),$$

and we have a group isomorphism

$$\mathrm{colim}_\alpha \mathbf{K}(X^\alpha \text{ on } Y^\alpha) = \mathbf{K}(X \text{ on } Y).$$

See also [SGA6, sec. IV.3.2.2], [TT, prop. 3.20].

1.2.22. Pro-finite-dimensional vector bundles. An important particular case of compact schemes is given by pro-finite-dimensional vector bundles.

1.2.23. Definition. A *pro-finite-dimensional vector bundle* $\pi : X \to Y$ is a scheme homomorphism which is represented as the inverse limit of a system of vector bundle $\pi_n : X^n \to Y$ (of finite rank) with n an integer $\geqslant 0$, such that the morphism $X^m \to X^n$ is a vector bundle homomorphism for each $m \geqslant n$.

1.2.24. Proposition. *A pro-finite-dimensional vector bundle $\pi : X \to Y$ is flat. If Y is compact then X is compact. If Y is pro-smooth then X is pro-smooth. If Y is coherent then X is coherent.*

Proof : The first claim is obvious. By 1.2.18(b) any vector bundle over Y is, locally over Y, pulled-back from a vector bundle over some Y^α where (Y^α) is a inverse system as in 1.2.19. This implies the second and the third claim. The last one follows from [K1, prop. 1.1.6].

□

1.3. K-theory of ind-coherent ind-schemes.

1.3.1. Spaces and ind-schemes. Let $\boldsymbol{\mathcal{A}lg}$ be the category of associative, commutative $\mathbb{C}$-algebras with 1. The category of *spaces* is the category $\boldsymbol{\mathcal{S}pace}$ of functors $\boldsymbol{\mathcal{A}lg} \to \boldsymbol{\mathcal{S}et}$. By Yoneda's lemma $\boldsymbol{\mathcal{S}ch}$ can be considered as a full subcategory in the category $\boldsymbol{\mathcal{S}ch}^\wedge$ of presheaves on $\boldsymbol{\mathcal{S}ch}$. It can be as well realized as a full subcategory in $\boldsymbol{\mathcal{S}pace}$ via the functor

$$\boldsymbol{\mathcal{S}ch} \to \boldsymbol{\mathcal{S}pace}, \quad X \mapsto \mathrm{Hom}_{\boldsymbol{\mathcal{S}ch}}(\mathrm{Spec}(\cdot), X).$$

By a *subspace* we mean a subfunctor. A subspace $Y \subset X$ is said to be *closed, open* if for every scheme Z and every $Z \to X$ the subspace $Z \times_X Y \subset Z$ is a closed, open subscheme.

1.3.2. Definitions. (a) An *ind-scheme* is an ind-object X of $\boldsymbol{\mathcal{S}ch}$ represented as $X =$ "colim_α"X_α where $A = \mathbb{N}$ and $(X_\alpha)_{\alpha \in A}$ is a direct system of quasi-compact schemes with closed embeddings $i_{\alpha\beta} : X_\alpha \to X_\beta$ for each $\alpha \leqslant \beta$.

(b) A *closed ind-subscheme* Y of the ind-scheme X is a closed subspace of X. An *open ind-subscheme* Y of the ind-scheme X is an ind-scheme which is an open subspace of X.

Since direct colimits exist in the category $\boldsymbol{\mathcal{S}pace}$ we may regard $\boldsymbol{\mathcal{I}sch}$ as a full subcategory of $\boldsymbol{\mathcal{S}pace}$. Hence, to unburden the notation we'll omit the quotation marks for ind-schemes.

1.3.3. Remarks. (a) A closed subscheme of an ind-scheme is always quasi-compact.

(b) We may consider ind-objects of $\boldsymbol{\mathcal{S}ch}$ which are representented by a direct system of non quasi-compact schemes X_α with closed embeddings. To avoid any confusion we'll call them ind$'$-schemes.

(c) Given a closed ind-subscheme $Y \subset X$, for each $\alpha \in A$ the closed immersion $X_\alpha \subset X$ yields a closed subscheme $Y_\alpha = X_\alpha \times_X Y \subset X_\alpha$. Further the closed immersion $X_\alpha \subset X_\beta$, $\alpha \leqslant \beta$, factors to a closed immersion $Y_\alpha \subset Y_\beta$. The ind-scheme Y is represented as $Y = \mathrm{colim}_\alpha Y_\alpha$.

(d) For each ind-scheme X and each quasi-compact scheme Y we have

$$\mathrm{Hom}_{\boldsymbol{\mathcal{I}sch}}(Y, X) = \mathrm{colim}_\alpha \mathrm{Hom}_{\boldsymbol{\mathcal{S}ch}}(Y, X_\alpha).$$

1.3.4. Definitions. (a) An ind-scheme X is *ind-proper* or of *ind-finite type* if it can be represented as the direct colimit of system of proper schemes or of finite type schemes respectively with closed embeddings.

(b) An ind-scheme X is *ind-coherent* if it can be represented as the direct colimit of a system of coherent quasi-compact schemes with good embeddings.

1.3.5. Coherent and quasi-coherent $\mathcal{O}$-modules over ind-coherent ind-schemes. Let X be an ind-coherent ind-scheme and $Y \subset X$ be a closed ind-subscheme. Given X_α, Y_α as in 1.3.3(c) we have a direct system of Abelian categories $\boldsymbol{\mathcal{C}oh}(X_\alpha \text{ on } Y_\alpha)$ with exact functors $(i_{\alpha\beta})_*$. We define the following Abelian categories

$$\boldsymbol{\mathcal{C}oh}(X \text{ on } Y) = 2\mathrm{colim}_\alpha \boldsymbol{\mathcal{C}oh}(X_\alpha \text{ on } Y_\alpha), \quad \boldsymbol{\mathcal{C}oh}(X) = \boldsymbol{\mathcal{C}oh}(X \text{ on } X).$$

These categories do not depend on the direct system (X_α) up to canonical equivalences. An object of $\boldsymbol{\mathcal{C}oh}(X)$ is called a coherent $\mathcal{O}_X$-module.

We define also quasi-coherent $\mathcal{O}_X$-modules in the following way [BD, sec. 7.11.3], [D, sec. 6.3.2]. We have a inverse system of categories $\boldsymbol{\mathcal{Q}coh}(X_\alpha)$ with functors $(i_{\alpha\beta})^*$. We set

$$\boldsymbol{\mathcal{Q}coh}(X) = 2\lim_\alpha \boldsymbol{\mathcal{Q}coh}(X_\alpha).$$

The category $\boldsymbol{\mathcal{Q}coh}(X)$ is a tensor category, but it need not be Abelian. It is independent on the choice of the system (X_α) up to canonical equivalences of categories. A quasi-coherent $\mathcal{O}_X$-module can be regarded as a rule that assigns to each scheme Z with a morphism $Z \to X$ a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{E}_Z$, and to each scheme homomorphism $f : W \to Z$ an isomorphism $f^*\mathcal{E}_Z \simeq \mathcal{E}_W$ satisfying the obvious composition rules.

Finally we define the Grothendieck group of the pair (X, Y) by

$$\mathbf{K}(X \text{ on } Y) = [\boldsymbol{\mathcal{C}oh}(X \text{ on } Y)].$$

Note that we have $\mathbf{K}(X \text{ on } Y) = \mathrm{colim}_\alpha \mathbf{K}(X_\alpha \text{ on } Y_\alpha)$ where $\mathbf{K}(X_\alpha \text{ on } Y_\alpha) = [\boldsymbol{\mathcal{C}oh}(X_\alpha \text{ on } Y_\alpha)]$ for each α.

1.3.6. Remarks. (a) There is another notion of quasi-coherent $\mathcal{O}_X$-modules on an ind-scheme, called $\mathcal{O}_X^!$-modules in [BD, sec. 7.11.4]. They form an Abelian category. We'll not need this.

(b) Any morphism of ind-coherent ind-schemes $f : X \to Y$ yields a functor $f^* : \boldsymbol{\mathcal{Q}coh}(Y) \to \boldsymbol{\mathcal{Q}coh}(X)$. If f is an open embedding the base change yields an exact functor $f^* : \boldsymbol{\mathcal{C}oh}(Y) \to \boldsymbol{\mathcal{C}oh}(X)$ and a group homomorphism $f^* : \mathbf{K}(Y) \to \mathbf{K}(X)$.

1.3.7. Definition. Let X be an ind-scheme. A closed ind-subscheme $Y \subset X$ is *good* if for every scheme $Z \to X$ the closed subscheme $Z \times_X Y \subset Z$ is good.

Note that if X is ind-coherent and $Y \subset X$ is a good ind-subscheme then Y is again an ind-coherent ind-scheme. If $f : Y \to X$ is an ind-proper homomorphism of ind-schemes of ind-finite type, or a good ind-subscheme of an ind-coherent ind-scheme, then there is a functor $f_* : \boldsymbol{\mathcal{C}oh}(Y) \to \boldsymbol{\mathcal{C}oh}(X)$ and a group homomorphism $Rf_* : \mathbf{K}(Y) \to \mathbf{K}(X)$.

1.4. Group actions on ind-schemes.

1.4.1. Ind-groups and group-schemes. Let $\mathcal{G}rp$ be the category of groups. A group-scheme is a scheme representing a functor $\mathcal{A}lg \to \mathcal{G}rp$. An ind-group is an ind-scheme representing a functor $\mathcal{A}lg \to \mathcal{G}rp$.

1.4.2. Definition. We abbreviate *linear group* for linear algebraic group. A *pro-linear group* G is a compact, affine, group-scheme which is represented as the inverse limit of a system of linear groups $G = \lim_n G^n$ with n any integer $\geqslant 0$, such that the morphism $G^m \to G^n$ is a group-scheme homomorphism for each $m \geqslant n$.

1.4.3. Examples. Let G be a linear group. For each $\mathbb{C}$-algebra R the set of R-points of G is $G(R) = \mathrm{Hom}_{\mathcal{S}ch}(\mathrm{Spec}(R), G)$.

(a) The algebraic group $G(\mathbb{C}[\varpi]/(\varpi^n))$ represents the functor $R \mapsto G(R[\varpi]/(\varpi^n))$. The functor $R \mapsto G(R[[\varpi]])$ is represented by a group-scheme, denoted by $K = G(\mathbb{C}[[\varpi]])$. The group-scheme K is a pro-linear group, since it is the limit of the inverse system of linear groups $G(\mathbb{C}[\varpi]/(\varpi^n))$ with $n \geqslant 0$.

(b) The functor $R \mapsto G(R[\varpi^{-1}])$ is represented by an ind-group, denoted by $G(\mathbb{C}[\varpi^{-1}])$.

(c) The functor $R \mapsto G(R((\varpi)))$ is represented by an ind-group, denoted by $G(\mathbb{C}((\varpi)))$.

Throughout we'll use the same symbol for an ind-scheme X and the set of $\mathbb{C}$-points $X(\mathbb{C})$. For instance the symbol K will denote both the functor above and the group of $\mathbb{C}[[\varpi]]$-points of the linear group G.

1.4.4. Group actions on an ind-scheme. Let G be an ind-group and X be an ind-scheme. We'll say that G acts on X if there is a morphism of functors $G \times X \to X$ satisfying the obvious composition rules. A *G-equivariant ind-scheme* is an ind-scheme with a (given) G-action. We'll abbreviate G-ind-scheme for G-equivariant ind-scheme. We'll also call ind-G-scheme a G-ind-scheme which is represented as the direct colimit of a system of quasi-compact G-schemes (X_α) as in 1.3.2.

1.4.5. Definition. Let $G = \lim_n G^n$ be a pro-linear group.

(a) A (compact) G-scheme X is *admissible* if it is represented as the inverse limit of a system of G-schemes of finite type with affine morphisms (X^α) such that, for each α, the G-action on X^α factors through a G^n-action if $n \geqslant n_\alpha$ for some integer n_α.

(b) A morphism of admissible G-schemes $f : X \to Y$ is *admissible* if there are inverse systems of (X^α), (Y^α) as above such that f is the limit of a morphism of systems of G-schemes $(f^\alpha) : (X^\alpha) \to (Y^\alpha)$ and the following square is Cartesian for each $\alpha \leqslant \beta$

$$\begin{tikzcd} X^\beta \arrow[r, "f_\beta"] \arrow[d] & Y^\beta \arrow[d] \\ X^\alpha \arrow[r, "f_\alpha"] & Y^\alpha. \end{tikzcd}$$

(c) An ind-G-scheme X is *admissible* if it is the direct colimit of a system of compact admissible G-schemes with admissible closed embeddings.

1.4.6. Remarks. (a) Let X be a G-torsor, with $G = \lim_n G^n$ a pro-linear group. For each n let G_n be the kernel of the canonical morphism $G \to G^n$. If the quotient scheme X/G is of finite type then the G-scheme X is admissible. Indeed X is the inverse limit of the system of G-schemes (X/G_n), and the G-action on X/G_n factors through a G^n-action.

(b) If $f : Y \to X$ is a finitely presented morphism of G-schemes with X admissible then Y and f are also admissible [TT, sec. C.3].

1.5. Equivariant K-theory of ind-schemes.

To simplify, in this section we'll assume that all schemes are quasi-compact.

1.5.1. Equivariant quasi-coherent $\mathcal{O}$-modules over a scheme. Let G be a group-scheme and X be a G-scheme. Let $a, p : G \times X \to X$ be the action and the obvious projection.

1.5.2. Definition. A *G-equivariant quasi-coherent $\mathcal{O}_X$-module* is a quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$ with an isomorphism $\theta : a^*(\mathcal{E}) \to p^*(\mathcal{E})$. The obvious cocycle condition is to hold. Let $\boldsymbol{\mathcal{Q}coh}^G(X)$ be the category of G-equivariant quasi-coherent $\mathcal{O}_X$-modules. Given a closed subset $Y \subset X$ we define the category $\boldsymbol{\mathcal{Q}coh}^G(X \text{ on } Y)$ of G-equivariant quasi-coherent $\mathcal{O}_X$-modules supported on Y in the obvious way.

The category $\boldsymbol{\mathcal{Q}coh}^G(X)$ is Abelian. The forgetful functor

$$for : \boldsymbol{\mathcal{Q}coh}^G(X) \to \boldsymbol{\mathcal{Q}coh}(X)$$

is exact and it *reflects exactness*, i.e., whenever a sequence in $\boldsymbol{\mathcal{Q}coh}^G(X)$ is exact in $\boldsymbol{\mathcal{Q}coh}(X)$ it is also exact in $\boldsymbol{\mathcal{Q}coh}^G(X)$. A G-equivariant quasi-coherent $\mathcal{O}_X$-module is said to be *coherent*, *of finite type* or *finitely presented* if it is coherent, of finite type or finitely presented as an $\mathcal{O}_X$-module. We define the categories $\boldsymbol{\mathcal{C}oh}^G(X \text{ on } Y)$ and $\boldsymbol{\mathcal{C}oh}^G(X)$ in the obvious way.

Let $\boldsymbol{\mathcal{C}}^G(X)_{qc}$ be the category of complexes of G-equivariant quasi-coherent $\mathcal{O}_X$-modules, and let $\boldsymbol{\mathcal{D}}^G(X)_{qc}$ be the derived category of G-equivariant quasi-coherent $\mathcal{O}_X$-modules. Note that this notation may be confusing. We do not claim that $\boldsymbol{\mathcal{D}}^G(X)_{qc}$ is the same as the derived category of G-equivariant $\mathcal{O}_X$-modules with quasi-coherent cohomology. For a coherent quasi-compact G-scheme X we set

$$\mathbf{K}^G(X \text{ on } Y) = [\boldsymbol{\mathcal{C}oh}^G(X \text{ on } Y)], \quad \mathbf{K}^G(X) = \mathbf{K}^G(X \text{ on } X).$$

The representation ring of G is defined by $\mathbf{R}^G = \mathbf{K}^G(pt)$. It acts on the group $\mathbf{K}^G(X \text{ on } Y)$ by tensor product.

To define the standard derived functors for equivariant sheaves we need more material. There are a number of foundational issues to be addressed in translating the theory of derived functors of quasi-coherent sheaves from the non equivariant setting to the equivariant one. Here we briefly consider the issues that are relevant to the present paper.

1.5.3. Definitions. (a) An *ample family of line bundles* on X is a family of line bundles $\{\mathcal{L}_i\}$ such that for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$ the evaluation map yields an epimorphism

$$\bigoplus_i \bigoplus_{n>0} \Gamma(X, \mathcal{E} \otimes_X \mathcal{L}_i^{\otimes n}) \otimes \mathcal{L}_i^{-\otimes n} \to \mathcal{E}.$$

We'll say that X *satisfies the property* (A_G) if it has an ample family of G-equivariant line bundles.

(b) We say that X *satisfies the (resolution) property* (R_G) if for every G-equivariant quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$ there is a G-equivariant, flat, quasi-coherent $\mathcal{O}_X$-module $\mathcal{P}$ and a surjection of G-equivariant $\mathcal{O}_X$-modules $f : \mathcal{P} \to \mathcal{E}$. We'll also demand that we can choose $\mathcal{P}$ and f in a functorial way with respect to $\mathcal{E}$.

(c) We say that a G-equivariant complex of quasi-coherent $\mathcal{O}_X$-modules $\mathcal{E}$ admits a *K-flat resolution* if there is a G-equivariant quasi-isomorphism $\mathcal{P} \to \mathcal{E}$ with $\mathcal{P}$ a G-equivariant complex of quasi-coherent $\mathcal{O}_X$-modules such that $\mathcal{P} \otimes_X \mathcal{F}$ is acyclic for every acyclic complex $\mathcal{F}$ in $\boldsymbol{\mathcal{C}}^G(X)_{qc}$, see [Sp, def. 5.1].

(d) We say that a G-equivariant complex of quasi-coherent $\mathcal{O}_X$-modules $\mathcal{E}$ admits a *K-injective resolution* if there is a G-equivariant quasi-isomorphism $\mathcal{E} \to \mathcal{I}$ with $\mathcal{I}$ a G-equivariant complex of quasi-coherent $\mathcal{O}_X$-modules such that the complex of chain homomorphisms $\mathcal{F} \to \mathcal{I}$ in $\boldsymbol{\mathcal{C}}^G(X)_{qc}$ is acyclic for every acyclic complex $\mathcal{F}$ in $\boldsymbol{\mathcal{C}}^G(X)_{qc}$, see [Sp, def. 1.1].

If G is the trivial group we'll abbreviate $(A) = (A_G)$ and $(R) = (R_G)$.

1.5.4. Remarks. (a) The property (A_G) implies the property (R_G). It implies also that any G-equivariant quasi-coherent $\mathcal{O}_X$-module of finite type is the quotient of a G-equivariant vector bundle, because X is quasi-compact [GD, chap. 0, (5.2.3)].

(b) If G is linear and X is Noetherian, normal, and satisfies the property (A), then X satisfies also the property (A_G) [T3, lem. 2.10 and sec. 2.2]. Since any quasi-projective scheme satisfies (A), we recover the well-known fact that X satisfies the property (R_G) if it is quasi-projective and normal and if G is linear.

(c) If G is linear and X is Noetherian and regular, then X satisfies (A_G) by part (b), because it satisfies (A) [SGA6, II.2.2.7.1].

(d) Let X be an admissible G-scheme represented as the inverse limit of a system of G-schemes (X^α) as in 1.4.5(a). If X^α satisfies (A_G) for some α then X satisfies also (A_G), as well as X^β for each $\beta \geqslant \alpha$ [TT, ex. 2.1.2(g)]. Thus if X is an admissible G-scheme which satisfies the property (S) in 1.2.19 then it satisfies also the property (A_G) (as well as X^α for each α) by part (c) above.

The G-equivariant quasi-coherent sheaves are well-behaved on quasi-compact schemes satisfying the property (A_G). *In the rest of Section 1.5 all G-schemes are assumed to be quasi-compact and to satisfy* (A_G).

1.5.5. Lemma. *Assume that X is coherent. Then any G-equivariant quasi-coherent $\mathcal{O}_X$-module is the direct colimit of a system of G-equivariant coherent $\mathcal{O}_X$-modules.*

Proof : For any G-equivariant quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$ the property (A_G) yields an epimorphism in $\boldsymbol{\mathcal{Q}coh}^G(X)$

$$\mathcal{F} = \bigoplus_i \bigoplus_{n>0} \Gamma(X, \mathcal{E} \otimes_X \mathcal{L}_i^{\otimes n}) \otimes \mathcal{L}_i^{-\otimes n} \to \mathcal{E}.$$

Any (rational) G-module is locally finite, see e.g., [J, sec. I.2.13]. Choose a finite number of i's and n's and a finite dimensional G-submodule of $\Gamma(X, \mathcal{E} \otimes_X \mathcal{L}_i^{\otimes n})$ for each i and each n in these finite sets. Then $\mathcal{F}$ is represented as the union

of a system of G-equivariant locally free $\mathcal{O}_X$-submodules of finite type $\mathcal{F}_\alpha \subset \mathcal{F}$. Taking the image under the epimorphism above we can represent $\mathcal{E}$ as the direct colimit of a system of G-equivariant $\mathcal{O}_X$-submodules of finite type $\mathcal{E}_\alpha$ with surjective maps $\phi_\alpha : \mathcal{F}_\alpha \to \mathcal{E}_\alpha$. The kernel of ϕ_α is again a G-equivariant quasi-coherent $\mathcal{O}_X$-module. Considering its finitely generated G-equivariant quasi-coherent $\mathcal{O}_X$-submodules, we prove as in [GD, chap. I, cor. 6.9.12] that $\mathcal{E}$ is the direct colimit of a system of G-equivariant finitely presented $\mathcal{O}_X$-modules. Since X is a coherent scheme, any finitely presented $\mathcal{O}_X$-module is coherent.

□

1.5.6. Proposition. *(a) Any complex in $\boldsymbol{\mathcal{C}}^G(X)_{qc}$ admits a K-flat resolution.*

(b) There is a left derived tensor product $\boldsymbol{\mathcal{D}}^G(X)_{qc} \times \boldsymbol{\mathcal{D}}^G(X)_{qc} \to \boldsymbol{\mathcal{D}}^G(X)_{qc}$.

(c) If $f : X \to Y$ is a morphism of G-schemes there is a left derived functor $Lf^* : \boldsymbol{\mathcal{D}}^G(Y)_{qc} \to \boldsymbol{\mathcal{D}}^G(X)_{qc}$.

Proof : The non equivariant case is treated in [Sp]. The equivariant case is very similar and is left to the reader. For instance, part (a) is proved as in [Li, sec. 2.5], while parts (b), (c) follow from (a) and the general theory of derived functors [Li, sec. 2.5, 3.1], [KS1], [KS2].

□

1.5.7. Proposition. *(a) The category $\boldsymbol{\mathcal{Q}coh}^G(X)$ is a Grothendieck category. It has enough injective objects. Any complex of $\boldsymbol{\mathcal{C}}^G(X)_{qc}$ has a K-injective resolution.*

(b) If $f : X \to Y$ is a morphism of G-schemes there is a right derived functor $Rf_* : \boldsymbol{\mathcal{D}}^G(X)_{qc} \to \boldsymbol{\mathcal{D}}^G(Y)_{qc}$.

Proof : Part (b) follows from (a) and the general theory of derived functors [Li], [KS1], [KS2]. Let us concentrate on part (a). The second claim is a well-known consequence of the first one. The third claim follows also from the first one by [S]. See also [AJS, thm. 5.4], [KS2, sec. 14]. So we must check that $\boldsymbol{\mathcal{Q}coh}^G(X)$ is a Grothendieck category. To do so we must prove that it has a generator, that it is cocomplete, and that direct colimits are exact. Fix a small category $\boldsymbol{\mathcal{A}}$ and a functor $\boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{Q}coh}^G(X)$, $\alpha \mapsto \mathcal{E}_\alpha$. Composing it with the forgetful functor we get a functor $\boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{Q}coh}(X)$ with a colimit

$$\mathcal{E} = \mathrm{colim}_\alpha for(\mathcal{E}_\alpha),$$

because the category $\boldsymbol{\mathcal{Q}coh}(X)$ is cocomplete. For the same reason we have also the following colimits

$$\mathrm{colim}_\alpha a^*(for(\mathcal{E}_\alpha)), \quad \mathrm{colim}_\alpha p^*(for(\mathcal{E}_\alpha)).$$

Since the functors a^*, p^* have right adjoints, a general result yields

$$a^*(\mathcal{E}) = \mathrm{colim}_\alpha a^*(for(\mathcal{E}_\alpha)), \quad p^*(\mathcal{E}) = \mathrm{colim}_\alpha p^*(for(\mathcal{E}_\alpha)).$$

Next, since $(\mathcal{E}_\alpha)$ is a system of G-equivariant quasi-coherent sheaves we have an isomorphism of systems $a^*(for(\mathcal{E}_\alpha)) \to p^*(for(\mathcal{E}_\alpha))$. Taking the colimit we get an isomorphism of quasi-coherent sheaves $a^*(\mathcal{E}) \to p^*(\mathcal{E})$. This isomorphism yields a G-equivariant structure on $\mathcal{E}$. The resulting G-equivariant sheaf is a colimit

in $\boldsymbol{\mathcal{Q}coh}^G(X)$. Thus the category $\boldsymbol{\mathcal{Q}coh}^G(X)$ is cocomplete and the functor for preserves colimits. Since for reflects exactness and $\boldsymbol{\mathcal{Q}coh}(X)$ is a Grothendieck category we obtain that the direct colimit is an exact functor in $\boldsymbol{\mathcal{Q}coh}^G(X)$. Finally we must prove that the Abelian category $\boldsymbol{\mathcal{Q}coh}^G(X)$ has a generator. This is obvious, because the proof of 1.5.5 implies that the tensor powers of the $\mathcal{L}_i$'s generate the category $\boldsymbol{\mathcal{Q}coh}^G(X)$.

□

1.5.8. Compatibility of the derived functors. It seems to be unknown whether for any quasi-compact G-scheme satisfying the property (A_G) the derived tensor product, the derived pull-back and the derived direct image satisfy the equivariant analogue of the properties in 1.2.10-1.2.13. Here we briefly discuss a weaker version of those which is enough for the present paper.

First Rf_* is right adjoint to Lf^*, next Rf_* preserves the cohomologically bounded complexes, and finally Lf^* commutes with the derived tensor product. These three properties are proved as in the non equivariant case, see e.g., [Li, prop. 3.2, 3.9], [KS2, sec. 14,18]. For instance the second one follows from the spectral sequence $R^pf_* \circ H^q \Rightarrow R^{p+q}f_*$, where $R^pf_* = H^p \circ Rf_*$, and the third one from the fact that both derived functors can be computed via K-flat resolutions.

Next Lf^* and the derived tensor product both commute with the forgetful functor for because they can be computed via K-flat resolutions in both the equivariant and the non equivariant cases, and because for takes flat $\mathcal{O}_X$-modules in $\boldsymbol{\mathcal{Q}coh}^G(X)$ to flat $\mathcal{O}_X$-modules in $\boldsymbol{\mathcal{Q}coh}(X)$.

The remaining properties require some work and more hypothesis. We'll say that a quasi-coherent $\mathcal{O}_X$-module is *f_*-acyclic* if it is annihilated by R^pf_* for each $p > 0$. By the general theory of derived functors, for any G-equivariant quasi-coherent sheaf $\mathcal{E}$ the complex $Rf_*(\mathcal{E})$ can be computed using a G-equivariant resolution of $\mathcal{E}$ by f_*-acyclic sheaves. Assume that G is a pro-linear group. The following lemma is standard.

1.5.9. Lemma. *Let $f : X \to Y$ be a morphism of G-schemes. If X is normal and quasi-projective then any G-equivariant quasi-coherent sheaf has a G-equivariant right resolution by f_*-acyclic quasi-coherent sheaves.*

Proof : By Sumihiro's theorem there is a G-equivariant ample line bundle $\mathcal{L}$ on X, see e.g., [CG, sec. 5.1]. For a large enough integer $n > 0$ the sheaf $\mathcal{L}^n$ is generated by its global sections, and we have a G-equivariant inclusion $\mathcal{O}_X \subset \mathcal{G}_n = \mathcal{L}^n \otimes V_n$, where V_n is a finite dimensional rational G-module such that the cokernel is a locally free $\mathcal{O}_X$-module. For any G-equivariant coherent sheaf $\mathcal{E}$ we have an inclusion $\mathcal{E} \subset \mathcal{E} \otimes_X \mathcal{G}_n$ such that $\mathcal{E} \otimes_X \mathcal{G}_n$ is f_*-acyclic (because $\mathcal{L}$ is ample). If $\mathcal{E}$ is a G-equivariant quasi-coherent sheaf we can represent it as the direct limit $\mathcal{E} = \mathrm{colim}_\alpha \mathcal{E}_\alpha$ of a system of G-equivariant coherent sheaves. Choose integers n_α such that $\mathcal{E}_\alpha \otimes_X \mathcal{G}_{n_\alpha}$ is f_*-acyclic for each α and $\mathcal{G}_{n_\alpha} \subset \mathcal{G}_{n_\beta}$ for $\alpha \leqslant \beta$. We have an inclusion of G-equivariant quasi-coherent sheaves $\mathcal{E} \subset \mathcal{E} \otimes_X \mathcal{G}$ where $\mathcal{G} = \mathrm{colim}_\alpha \mathcal{G}_{n_\alpha}$ and $\mathcal{E} \otimes_X \mathcal{G}$ is f_*-acyclic. The cokernel $(\mathcal{E} \otimes_X \mathcal{G})/\mathcal{E}$ is again a G-equivariant quasi-coherent sheaf. By induction we get a resolution of $\mathcal{E}$.

□

We'll say that a coherent G-scheme X is *almost-quasi-projective* if it is represented as the inverse limit of a system of normal quasi-projective G-schemes X^α

with affine morphisms such that the G-action on X^α factors through G^{n_α} for some integer n_α, compare 1.4.5(a).

Let $f : X \to Y$ be a morphism of almost-quasi-projective G-schemes. Then any G-equivariant quasi-coherent sheaf $\mathcal{E}$ over X has a G-equivariant right resolution by f_*-acyclic quasi-coherent sheaves by 1.2.18. Therefore the general theory of derived functors implies that $Rf_*(for(\mathcal{E})) = for(Rf_*(\mathcal{E}))$ in $\boldsymbol{\mathcal{D}}(Y)_{qc}$. Further, for any G-equivariant quasi-coherent sheaves $\mathcal{E}$, $\mathcal{F}$ over X, Y respectively the projection formula holds, i.e., we have

$$Rf_*(\mathcal{E})\overset{L}{\otimes}_Y\mathcal{F} = Rf_*(\mathcal{E}\overset{L}{\otimes}_X Lf^*(\mathcal{F})).$$

Indeed, by adjunction we have a natural projection map from the lhs to the rhs. To prove that this map is invertible it is enough to observe that it is invertible in the non equivariant case, because the forgetful functor commutes with Rf_*, Lf^* and the derived tensor product. The details are left to the reader. Note that, since the projection formula and the base change hold for coherent sheaves, they hold a fortiori in G-equivariant K-theory. This is enough for this paper.

Recall that we have assumed that all G-schemes are quasi-compact and satisfy the property (A_G). This insures that the standard derived functors are well-defined. *In the rest of Section 1.5 we'll also assume that the standard derived functors satisfy the properties in 1.5.8.*

1.5.10. Equivariant coherent sheaves over an ind-coherent ind-scheme. Let G be a group-scheme, X be an ind-coherent ind-G-scheme, and $Y \subset X$ be a closed ind-subscheme which is preserved by the G-action. Let (Y_α), (X_α) be systems of quasi-compact G-schemes as in 1.4.4, such that the inclusion $Y \subset X$ is represented by a system of inclusions $Y_\alpha \subset X_\alpha$. Since the maps $i_{\alpha\beta} : X_\alpha \to X_\beta$ are good G-subschemes we have a direct system of Abelian categories $\boldsymbol{\mathcal{C}oh}^G(X_\alpha \text{ on } Y_\alpha)$ and exact functors $(i_{\alpha\beta})_*$. We set

$$\begin{aligned} \boldsymbol{\mathcal{C}oh}^G(X \text{ on } Y) &= 2\mathrm{colim}_\alpha \boldsymbol{\mathcal{C}oh}^G(X_\alpha \text{ on } Y_\alpha), \\ \mathbf{K}^G(X \text{ on } Y) &= [\boldsymbol{\mathcal{C}oh}^G(X \text{ on } Y)] = \mathrm{colim}_\alpha \mathbf{K}^G(X_\alpha \text{ on } Y_\alpha). \end{aligned} \tag{1.5.1}$$

1.5.11. Proposition. *The category $\boldsymbol{\mathcal{C}oh}^G(X \text{ on } Y)$ is independent of the choice of the system (X_α), up to canonical equivalence. The group $\mathbf{K}^G(X \text{ on } Y)$ is independent of the choice of the system (X_α), up to canonical isomorphism.*

Proof : Let $(\tilde{X}_{\tilde{\alpha}})$ be another direct system of closed subschemes of X representing X. So we have $X = \mathrm{colim}_\alpha X_\alpha$ and $X = \mathrm{colim}_{\tilde{\alpha}} \tilde{X}_{\tilde{\alpha}}$. The second equality means that each X_α is included into some $\tilde{X}_{\tilde{\alpha}}$ as a closed subset and vice-versa. Therefore the 2-limits of both systems are identified. □

Once again we write $\boldsymbol{\mathcal{C}oh}^G(X) = \boldsymbol{\mathcal{C}oh}^G(X \text{ on } X)$ and $\mathbf{K}^G(X) = \mathbf{K}^G(X \text{ on } X)$. Note that the tensor product $\otimes_X$ yields an action of the ring $\mathbf{R}^G$ on the category $\boldsymbol{\mathcal{C}oh}^G(X \text{ on } Y)$ and on the Abelian group $\mathbf{K}^G(X \text{ on } Y)$.

1.5.12. Admissible ind-coherent ind-schemes and reduction of the group action. Let G be a pro-linear group. Fix a system (G^n) as in 1.4.2. For each integer $n \geqslant 0$ let G_n be the kernel of the canonical map $G \to G^n$. Let X be an

admissible coherent G-scheme. Let X^α, n_α be as in 1.4.5(a). We have a direct system of categories $\boldsymbol{Coh}^{G^n}(X^\alpha)$, $n \geqslant n_\alpha$, with exact functors. The pull-back by the canonical map $p_\alpha : X \to X^\alpha$ yields a functor

$$2\mathrm{colim}_\alpha 2\mathrm{colim}_{n\geqslant n_\alpha}\boldsymbol{Coh}^{G^n}(X^\alpha) \to \boldsymbol{Coh}^G(X). \tag{1.5.2}$$

1.5.13. Proposition. (a) *Assume that the pro-object $X = \lim_\alpha X^\alpha$ satisfies the property (S). Then the functor (1.5.2) is an equivalence of Abelian categories, and it yields a group isomorphism*

$$\mathrm{colim}_\alpha \mathrm{colim}_{n\geqslant n_\alpha}\mathbf{K}^{G^n}(X^\alpha) \to \mathbf{K}^G(X).$$

(b) *If G_0 is pro-unipotent, the canonical map $\mathbf{K}^{G^0}(X) \to \mathbf{K}^G(X)$ is invertible.*

Proof : The proof of (b) is standard, see e.g., [CG]. Let us concentrate on part (a). The functor (1.5.2) is fully faithful by 1.2.18(a). Let us check that it is essentially surjective. To do so, fix a G-equivariant coherent $\mathcal{O}_X$-module $\mathcal{E}$. By 1.2.18(a) there is an α and a coherent $\mathcal{O}_{X^\alpha}$-module $\mathcal{E}^\alpha$ such that $\mathcal{E} = p_\alpha^*(\mathcal{E}^\alpha)$. We must check that we can choose $\mathcal{E}^\alpha$ such that the G-action on $\mathcal{E}$ factors to a G^n-action on $\mathcal{E}^\alpha$ for some $n \geqslant n_\alpha$. The unit of the adjoint pair of functors $(p_\alpha^*, (p_\alpha)_*)$ yields an inclusion of quasi-coherent $\mathcal{O}_{X^\alpha}$-modules

$$\mathcal{E}^\alpha \subset (p_\alpha)_* p_\alpha^*(\mathcal{E}^\alpha) = (p_\alpha)_*(\mathcal{E}).$$

Since X^α is a Noetherian G-scheme and $(p_\alpha)_*(\mathcal{E})$ is a quasi-coherent G-equivariant $\mathcal{O}_{X^\alpha}$-module, we know that $(p_\alpha)_*(\mathcal{E})$ is the union of all its G-equivariant coherent subsheaves. Fix a G-equivariant coherent $\mathcal{O}_{X^\alpha}$-module $\mathcal{F}^\alpha$ containing $\mathcal{E}^\alpha$. The G-action on $\mathcal{F}^\alpha$ factors to an action of the linear group G^n for some $n \geqslant n_\alpha$. Let $\mathcal{G}^\alpha \subset \mathcal{F}^\alpha$ be the G^n-equivariant quasi-coherent subsheaf generated by $\mathcal{E}^\alpha$. It is again a coherent $\mathcal{O}_{X^\alpha}$-module, because $\mathcal{F}^\alpha$ is coherent and X^α is Noetherian. Since $\mathcal{E}$ is already G-equivariant the inclusion

$$\mathcal{E} = p_\alpha^*(\mathcal{E}^\alpha) \subset p_\alpha^*(\mathcal{G}^\alpha)$$

is indeed an equality of $\mathcal{O}_X$-modules $\mathcal{E} \simeq p_\alpha^*(\mathcal{G}^\alpha)$.

□

Now, let X be an admissible ind-coherent ind-G-scheme represented as the direct colimit of a system of admissible G-schemes (X_α) as in 1.4.5(c). By (1.5.1) we have

$$\boldsymbol{Coh}^G(X) = 2\mathrm{colim}_\alpha \boldsymbol{Coh}^G(X_\alpha), \quad \mathbf{K}^G(X) = \mathrm{colim}_\alpha \mathbf{K}^G(X_\alpha).$$

If G_0 is pro-unipotent then 1.5.13 yields isomorphisms

$$\mathbf{K}^{G^0}(X) = \mathbf{K}^G(X), \quad \mathbf{R}^{G^0} = \mathbf{R}^G.$$

This is called the *reduction of the group action.*

1.5.14. Thom isomorphism and pro-finite-dimensional vector bundles over ind-schemes. A *vector bundle* over the ind-scheme Y is an ind-scheme homomorphism $X \to Y$ which is represented as the direct colimit of a system of vector bundles $X_\alpha \to Y_\alpha$. More precisely, we require that $X = \mathrm{colim}_\alpha X_\alpha$, $Y = \mathrm{colim}_\alpha Y_\alpha$, and for $\alpha \leqslant \beta$ we have a Cartesian square

$$\begin{tikzcd} X_\alpha \arrow[r] \arrow[d] & X_\beta \arrow[d] \\ Y_\alpha \arrow[r] & Y_\beta \end{tikzcd}$$

such that the vertical maps are vector-bundles and the upper horizontal map is a morphism of vector bundles. To any vector bundle over an ind-scheme X we can associate its sheaf of sections which is a quasi-coherent $\mathcal{O}_X$-module, see 1.3.5.

A *pro-finite-dimensional vector bundle* over the ind-scheme Y is defined in the same way by replacing everywhere vector bundles by pro-finite-dimensional vector bundles, see 1.2.23. In other words, it is an ind-scheme homomorphism which is represented as the "double limit" of a system of vector-bundles $X^n_\alpha \to Y_\alpha$ with n an integer $\geqslant 0$. Further, for each α and each $m \geqslant n$ we have a vector-bundle homomorphism $X^m_\alpha \to X^n_\alpha$ over Y_α, and for each n and each $\alpha \leqslant \beta$ we have an isomorphism of vector-bundles $X^n_\alpha \to X^n_\beta \times_{Y_\beta} Y_\alpha$. We require that these data satisfy the obvious composition rules. In particular for each $m \geqslant n$ and each $\beta \geqslant \alpha$ the following square is Cartesian

$$\begin{tikzcd} X^m_\alpha \arrow[r] \arrow[d] & X^m_\beta \arrow[d] \\ X^n_\alpha \arrow[r] & X^n_\beta. \end{tikzcd}$$

Note that a pro-finite-dimensional vector bundle over an ind-coherent ind-scheme is again an ind-coherent ind-scheme.

Let $\pi : X \to Y$ be an admissible G-equivariant pro-finite-dimensional vector bundle over an admissible ind-coherent ind-G-scheme Y. From 1.5.10 and base change we get an exact functor $\pi^* : \boldsymbol{Coh}^G(Y) \to \boldsymbol{Coh}^G(X)$. It factors to a group homomorphism $\pi^* : \mathbf{K}^G(Y) \to \mathbf{K}^G(X)$. The *Thom isomorphism* implies that π^* is invertible.

1.5.15. Descent and torsors over ind-schemes. Fix a pro-linear group $G = \lim_n G^n$. For each integer $n \geqslant 0$ let G_n be the kernel of the canonical map $G \to G^n$.

Let Y be a scheme. A *G-torsor* over Y is a scheme homomorphism $P \to Y$ which is represented as the inverse limit of a system consisting of a G^n-torsor $P^n \to Y$ for each integer $n \geqslant 0$ such that the morphism of Y-schemes $P^m \to P^n$, $m \geqslant n$, intertwines the G^m-action on the lhs and the G^n-action on the rhs, via the canonical group-scheme homomorphism $G^m \to G^n$.

Now, assume that $Y = \mathrm{colim}_\alpha Y_\alpha$ is an ind-scheme. A *G-torsor* over Y is an ind-scheme homomorphism $P \to Y$ which is represented as the direct colimit of a system of G-torsors $P_\alpha \to Y_\alpha$. More precisely, we require that $P = \mathrm{colim}_\alpha P_\alpha$, that for each α we have a system of G^n-torsors $P^n_\alpha \to Y_\alpha$ representing the G-torsor

$P_\alpha \to Y_\alpha$, and that for each n and each $\beta \geqslant \alpha$ we have an isomorphism of G^n-torsors $P^n_\alpha \to P^n_\beta \times_{Y_\beta} Y_\alpha$. In particular, for each $m \geqslant n$ and each $\beta \geqslant \alpha$ we have a Cartesian square

$$\begin{array}{ccc} P^m_\alpha & \longrightarrow & P^m_\beta \\ \downarrow & & \downarrow \\ P^n_\alpha & \longrightarrow & P^n_\beta. \end{array}$$

Note that a G-torsor over a pro-smooth scheme is again a pro-smooth scheme, and that a G-torsor over an ind-coherent ind-scheme is again an ind-coherent ind-scheme.

Now, assume that Y is a scheme and let $P \to Y$ be a G-torsor. Note that for each integer $n \geqslant 0$ we have a G^n-torsor $P/G_n \to Y$. In the rest of this subsection we consider the *induction functors*.

Let X be an admissible G-scheme and let X^α, n_α be as in 1.4.5(a). For each α and each integer $n \geqslant n_\alpha$ the quotient space $(X^\alpha)_Y = P \times_G X^\alpha$ is equal to the Y-scheme $(P/G_n) \times_{G^n} X^\alpha$. Further, if $\beta \geqslant \alpha$ the canonical map $X^\beta \to X^\alpha$ yields a Y-scheme homomorphism $(X^\beta)_Y \to (X^\alpha)_Y$. Thus the quotient space $X_Y = P \times_G X$ is a Y-scheme which is represented as the inverse limit $X_Y = \lim_\alpha (X^\alpha)_Y$. By (1.5.2) we have an equivalence of categories

$$2\mathrm{colim}_\alpha 2\mathrm{colim}_{n \geqslant n_\alpha} \boldsymbol{\mathcal{C}oh}^{G^n}(X^\alpha) \to \boldsymbol{\mathcal{C}oh}^G(X).$$

By faithfully flat descent we have a functor

$$\boldsymbol{\mathcal{C}oh}^{G^n}(X^\alpha) \to \boldsymbol{\mathcal{C}oh}((P/G_n) \times_{G^n} X^\alpha) = \boldsymbol{\mathcal{C}oh}((X^\alpha)_Y).$$

This yields a functor (called *induction* functor)

$$\boldsymbol{\mathcal{C}oh}^G(X) \to \boldsymbol{\mathcal{C}oh}(X_Y).$$

Next, let Z be an admissible ind-G-scheme which is represented as the direct colimit of a system of admissible G-schemes Z_α as in 1.4.5(c). Then the quotient space $Z_Y = P \times_G Z$ is an ind-scheme over Y which is represented as the direct colimit $Z_Y = \lim_\alpha (Z_\alpha)_Y$, and $(Z_\alpha)_Y$ is defined as above for each α. If Z is incoherent and Y is coherent then the ind-scheme Z_Y is again ind-coherent and (1.5.1) yields a functor (called *induction* functor)

$$\boldsymbol{\mathcal{C}oh}^G(Z) \to \boldsymbol{\mathcal{C}oh}(Z_Y).$$

The induction functor is defined in a similar way if Y is an ind-scheme. The details of the construction are left to the reader.

1.5.16. Remark. We define in a similar way induction functors for quasi-coherent sheaves. Next, let H be a group-scheme acting on the G-torsor $P \to Y$, i.e., the group H acts on P, Y and the action commutes with the G-action and with the projection $P \to Y$. Then there is a H-action on Z_Y and the induction yields a functor $\boldsymbol{\mathcal{C}oh}^G(Z) \to \boldsymbol{\mathcal{C}oh}^H(Z_Y)$.

1.5.17. Complements on the concentration map. Now, assume that $G = S$ is a diagonalizable linear group. Let $\mathbf{X}^S$ be the group of characters of S. Each $\lambda \in \mathbf{X}^S$ defines a one-dimensional representation θ_λ of S. Let θ_λ denote also its class in $\mathbf{R}^S$. The ring $\mathbf{R}^S$ is spanned by the elements θ_λ with $\lambda \in \mathbf{X}^S$. So any element of $\mathbf{R}^S$ may be viewed as a function on S. For any $\Sigma \subset S$ let $\mathbf{R}^S_\Sigma$ be the ring of quotients of $\mathbf{R}^S$ with respect to the multiplicative set of the functions in $\mathbf{R}^S$ which do not vanish identically on Σ.

Now, let X be an ind-coherent ind-S-scheme. We'll say that Σ is *X-regular* if the fixed points subsets X^Σ, X^S are equal. Write

$$\mathbf{K}^S(X)_\Sigma = \mathbf{R}^S_\Sigma \otimes_{\mathbf{R}^S} \mathbf{K}^S(X).$$

The *Thomason concentration theorem* says that the map

$$\mathbf{K}^S(X^S)_\Sigma \to \mathbf{K}^S(X)_\Sigma$$

given by the direct image by the canonical inclusion $X^S \subset X$ is invertible if X is a scheme of finite type and Σ is X-regular [T1], [T2]. We'll use some form of the concentration theorem in some more general situation, which we consider below. In each case, the proof of the concentration theorem can be reduced to the original statement of Thomason using the discussion above. It is left to the reader.

1.5.18. Let X be a pro-smooth admissible S-scheme. It is easy to check that the fixed-points subset $X^S \subset X$ is a closed subscheme which is again pro-smooth. Thus the obvious inclusion $j : X^S \to X$ has a finite tor-dimension by 1.2.10, 1.2.20. Hence it yields a $\mathbf{R}^S$-linear map $Lj^* : \mathbf{K}^S(X) \to \mathbf{K}^S(X^S)$. This map can be viewed as follows. Any coherent $\mathcal{O}_X$-module $\mathcal{E}$ has locally a finite resolution by locally free modules of finite rank. Hence the p-th left derived functor $L_p j^*\mathcal{E} = H^{-p}(Lj^*\mathcal{E})$ vanishes for $p \gg 0$. We have $Lj^*(\mathcal{E}) = \sum_{p\geqslant 0}(-1)^p L_p j^*(\mathcal{E})$. If Σ is X-regular we get a group isomorphism

$$Lj^* : \mathbf{K}^S(X)_\Sigma \to \mathbf{K}^S(X^S)_\Sigma.$$

1.5.19. Let X be an admissible ind-S-scheme of ind-finite type. The inclusion of the fixed points subset $i : X^S \to X$ is a good ind-subscheme. Thus the direct image yields a $\mathbf{R}^S$-linear map $i_* : \mathbf{K}^S(X^S) \to \mathbf{K}^S(X)$. If Σ is X-regular we get a group isomorphism

$$i_* : \mathbf{K}^S(X^S)_\Sigma \to \mathbf{K}^S(X)_\Sigma.$$

1.5.20. Let X be a pro-smooth admissible S-scheme and $f : Y \to X$ be an admissible ind-S-scheme over X. We'll assume that the map f is locally trivial in the following sense : there is an admissible ind-S-scheme F of ind-finite type and a S-equivariant finite affine open cover $X = \bigcup_w X^w$ such that over each X^w the map f is isomorphic to the obvious projection $X^w \times F \to X^w$, where the group S acts diagonally on the lhs. The ind-scheme Y is ind-coherent by 1.2.20. Further, the fixed points subset X^S is again pro-smooth. Setting $Y' = X^S \times_X Y$ we get the following diagram

$$\begin{tikzcd} Y^S \arrow[r, "i"] \arrow[d] & Y' \arrow[r, "j"] \arrow[d] & Y \arrow[d, "f"] \\ X^S \arrow[r, "="] & X^S \arrow[r] & X. \end{tikzcd}$$

Over the open set X^w the map j is isomorphic to the obvious inclusion

$$(X^w)^S \times F \subset X^w \times F.$$

The inclusion $(X^w)^S \subset X^w$ has a finite tor-dimension by 1.2.10, 1.2.20. Thus the map j has also a finite tor-dimension. By base change we have a $\mathbf{R}^S$-linear map $Lj^* : \mathbf{K}^S(Y) \to \mathbf{K}^S(Y')$. Since i is the inclusion of a good ind-subscheme the direct image gives a map $i_* : \mathbf{K}^S(Y^S) \to \mathbf{K}^S(Y')$. If Σ is Y-regular we get a group isomorphism

$$(i_*)^{-1} \circ Lj^* : \mathbf{K}^S(Y)_\Sigma \to \mathbf{K}^S(Y^S)_\Sigma.$$

2. Affine flag manifolds

2.1. Notation relative to the loop group.

2.1.1. Let G be a simple, connected and simply connected linear group over $\mathbb{C}$ with the Lie algebra $\mathfrak{g}$. Let $T \subset G$ be a Cartan subgroup and W be the Weyl group of the pair (G, T). Recall that $\mathbf{X}^T$ is the Abelian group of characters of T and that $\mathbf{Y}^T$ is the Abelian group of cocharacters of T. Let $\mathfrak{t}$ be the Lie algebra of T and $\mathfrak{t}^*$ be the set of linear forms on $\mathfrak{t}$. We'll view $\mathbf{X}^T$, $\mathbf{Y}^T$ as lattices in $\mathfrak{t}^*$, $\mathfrak{t}$ in the usual way. Note that, since G is simply connected, the lattice $\mathbf{X}^T$ is spanned by the fundamental weight and the lattice $\mathbf{Y}^T$ is spanned by the simple coroots. Let $\mathbf{X}^T_+ \subset \mathbf{X}^T$ and $\mathbf{Y}^T_+ \subset \mathbf{Y}^T$ denote the monoids of dominant characters and cocharacters.

Fix a Borel subgroup $B \subset G$. Let Δ be the set of roots of (G, T) and $\Pi \subset \Delta$ be the subset of simple roots associated to B. Let $\Delta^\vee$ be the set of coroots. Let θ be the highest root and $\check{\theta}$ be the corresponding coroot. Let

$$\langle\ ,\ \rangle : \mathbf{X}^T \times \mathbf{Y}^T \to \mathbb{Z}, \quad (\ ,\ ) : \mathfrak{t}^* \times \mathfrak{t}^* \to \mathbb{C}$$

be the canonical perfect pairing and the nondegenerate W-invariant bilinear form normalized by $(\theta, \theta) = 2$. We'll denote by κ the corresponding homomorphism $\mathfrak{t} \to \mathfrak{t}^*$ and we'll abbreviate $(\check{\lambda}, \check{\mu}) = (\kappa(\check{\lambda}), \kappa(\check{\mu}))$ for each $\check{\lambda}, \check{\mu} \in \mathfrak{t}$.

Let $\tilde{\Delta}$ be the set of affine roots, $\tilde{\Delta}_e$ be the subset of positive affine roots and $\tilde{\Pi}$ be the subset of simple affine roots. Let $\alpha_0 \in \tilde{\Pi}$ be the unique simple affine root which does not belong to Π. We have $\tilde{\Pi} = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$ where n is the rank of G. Let $\tilde{\Delta}^\vee$ be the set of affine coroots. We have $\tilde{\Pi}^\vee = \{\check{\alpha}_0, \check{\alpha}_1, \dots, \check{\alpha}_n\}$ where $\check{\alpha}_i$ is the affine coroot associated to the simple affine root α_i for each i.

Let $\tilde{W} = W \ltimes \mathbf{Y}^T$ be the affine Weyl group of G. For any affine real root α let $s_\alpha \in \tilde{W}$ be the corresponding affine reflection. We'll abbreviate $s_i = s_{\alpha_i}$ for each i. Since G is simply connected the group $\tilde{W}$ is a Coxeter group with simple reflections the s_i's.

We'll abbreviate $w = (w, 0)$ and $\xi_{\check{\lambda}} = (e, \check{\lambda})$ for each $(w, \check{\lambda}) \in \tilde{W}$. In particular we'll regard to W as a subgroup of $\tilde{W}$ in the obvious way. Here e denotes the unit, both in W and in $\tilde{W}$.

2.1.2. We'll fix a decreasing sequence of subsets $\tilde{\Delta}_l \subset \tilde{\Delta}_e$, with $l \in \mathbb{N}$, such that

$$(\tilde{\Delta}_l + \tilde{\Delta}_e) \cap \tilde{\Delta}_e \subset \tilde{\Delta}_l, \quad \sharp(\tilde{\Delta}_e \setminus \tilde{\Delta}_l) < \infty, \quad \bigcap_l \Delta_l = \emptyset.$$

For instance we may set $\tilde{\Delta}_l = l\delta + \tilde{\Delta}_e$ where δ is the smallest positive imaginary root. Put also $\tilde{\Delta}_l^\circ = -\tilde{\Delta}_l$.

2.1.3. We'll abbreviate $G((\varpi)) = G(\mathbb{C}((\varpi)))$, $\mathfrak{g}((\varpi)) = \mathfrak{g}(\mathbb{C}((\varpi)))$, etc. Recall that $K = G(\mathbb{C}[[\varpi]])$. Let $I \subset K$ be the standard Iwahori subgroup and $I^\circ \subset K^\circ = G(\mathbb{C}[[\varpi^{-1}]])$ be the opposite Iwahori subgroup. Let N, N° be the pro-unipotent radicals of I, I° respectively. The groups I, I°, N, N° are compact.

2.1.4. Let $\mathfrak{n}$, $\mathfrak{n}^\circ$, $\mathfrak{i}$, $\mathfrak{k}$ be the Lie algebras of N, N°, I, K. For any integer $l \geqslant 0$ let $\mathfrak{n}_l \subset \mathfrak{n}$ and $\mathfrak{n}_l^\circ \subset \mathfrak{n}^\circ$ be the product of all weight subspaces associated to the roots in $\tilde{\Delta}_l$, $\tilde{\Delta}_l^\circ$ respectively. Put also

$$\mathfrak{n}^l = \mathfrak{n}/\mathfrak{n}_l, \quad \mathfrak{n}^{\circ,l} = \mathfrak{n}^\circ/\mathfrak{n}_l^\circ, \quad \mathfrak{n}_w = w(\mathfrak{n}) \cap \mathfrak{n}, \quad \mathfrak{n}_w^\circ = w(\mathfrak{n}^\circ) \cap \mathfrak{n}, \quad w \in \tilde{W}.$$

Let N_l, N_l°, etc, be the groups associated with the Lie algebras $\mathfrak{n}_l$, $\mathfrak{n}_l^\circ$, etc. We'll write $\tilde{\Delta}_w$, $\tilde{\Delta}_w^\circ$ for the set of roots of $\mathfrak{n}_w$, $\mathfrak{n}_w^\circ$. Note that $\mathfrak{n}$, $\mathfrak{n}_l$, $\mathfrak{n}_w$ have a natural structure of admissible I-equivariant compact coherent schemes and that $\mathfrak{n}_l$, $\mathfrak{n}_w$ are good subschemes of $\mathfrak{n}$. Further the quotient $\mathfrak{n}^l$ is finite dimensional and $\mathfrak{n}_w^\circ$ has a natural structure of I-scheme of finite type. The I-action is the adjoint one. We'll call an Iwahori Lie subalgebra of $\mathfrak{g}((\varpi))$ any Lie subalgebra which is $G((\varpi))$-conjugate to $\mathfrak{i}$.

2.1.5. The group $\mathbb{C}^\times$ acts on $\mathbb{C}((\varpi))$ by loop rotations, i.e., a complex number $z \in \mathbb{C}^\times$ takes a formal series $f(\varpi)$ to $f(z\varpi)$. This yields $\mathbb{C}^\times$-actions on $G((\varpi))$, I and $\mathfrak{g}((\varpi))$. Consider the semi-direct products

$$\hat{G} = \mathbb{C}^\times \ltimes G((\varpi)), \quad \hat{I} = \mathbb{C}^\times \ltimes I, \quad \hat{I}^\circ = \mathbb{C}^\times \ltimes I^\circ, \quad \hat{T} = \mathbb{C}^\times \times T.$$

The group $\hat{G}$ acts again on $\mathfrak{g}((\varpi))$.

2.1.6. Let $\tilde{G}$ be the maximal, "simply connected", Kac-Moody group associated to G defined by Garland [G]. It is a group ind-scheme which is a central extension

$$1 \to \mathbb{C}^\times \to \tilde{G} \to \hat{G} \to 1.$$

See [K, sec. 13.2] for details. Let $\tilde{I}$, $\tilde{T}$, $\tilde{K}$ be the corresponding Iwahori, Cartan and maximal compact subgroup. Note that $\tilde{K} = \hat{K} \times \mathbb{C}^\times$, $\tilde{I} = \hat{I} \times \mathbb{C}^\times$ and $\tilde{T} = \hat{T} \times \mathbb{C}^\times$, i.e., the central extension splits. We define also the opposite Iwahori group $\tilde{I}^\circ = \hat{I}^\circ \times \mathbb{C}^\times$. Let $\tilde{\mathfrak{g}}$, $\tilde{\mathfrak{i}}$, $\tilde{\mathfrak{k}}$ be the Lie algebras of $\tilde{G}$, $\tilde{I}$, $\tilde{K}$. The group $\tilde{G}$ acts on $\mathfrak{g}((\varpi))$ and $\tilde{\mathfrak{g}}$ by the adjoint action. By an Iwahori Lie subalgebra of $\tilde{\mathfrak{g}}$ we simply mean a Lie subalgebra which is $\tilde{G}$-conjugate to $\tilde{\mathfrak{i}}$.

2.1.7. We'll also use the groups

$$\boldsymbol{G} = \tilde{G} \times \mathbb{C}^\times, \quad \boldsymbol{I} = \tilde{I} \times \mathbb{C}^\times, \quad \boldsymbol{T} = \tilde{T} \times \mathbb{C}^\times.$$

The group $\boldsymbol{G}$ acts also on $\tilde{\mathfrak{g}}$. We simply require that an element $z \in \mathbb{C}^\times$ acts by multiplication by z (=by dilatations). Note that $\boldsymbol{T} = T \times (\mathbb{C}^\times)^3$. To distinguish the different copies of $\mathbb{C}^\times$ we may use the following notation : $\mathbb{C}^\times_{\mathrm{rot}}$ corresponds to loop rotation, $\mathbb{C}^\times_{\mathrm{cen}}$ to the central extension, and $\mathbb{C}^\times_{\mathrm{qua}}$ to dilatations. Thus we have

$$\hat{T} = T \times \mathbb{C}^\times_{\mathrm{rot}}, \quad \tilde{T} = \hat{T} \times \mathbb{C}^\times_{\mathrm{cen}}, \quad \boldsymbol{T} = \tilde{T} \times \mathbb{C}^\times_{\mathrm{qua}}.$$

We'll also write $T_{\mathrm{cen}} = T \times \mathbb{C}^\times_{\mathrm{cen}}$.

2.1.8. We'll abbreviate $\tilde{\mathbf{X}} = \mathbf{X}^{\tilde{T}}$, $\mathbf{X} = \mathbf{X}^{\boldsymbol{T}}$, $\tilde{\mathbf{Y}} = \mathbf{Y}^{\tilde{T}}$ and $\mathbf{Y} = \mathbf{Y}^{\boldsymbol{T}}$. The pairing $\langle\ ,\ \rangle$ extends to the canonical pairing

$$\langle\ ,\ \rangle : \tilde{\mathbf{X}} \times \tilde{\mathbf{Y}} \to \mathbb{Z}.$$

Let d, c be the canonical generators of $\mathbf{Y}^{\mathbb{C}^\times_{\mathrm{rot}}}$, $\mathbf{Y}^{\mathbb{C}^\times_{\mathrm{cen}}}$. We have

$$\tilde{\mathbf{Y}} = \mathbf{Y}^T \oplus \mathbb{Z}d \oplus \mathbb{Z}c = \mathbb{Z}d \oplus \bigoplus_{i=0}^{n} \mathbb{Z}\check{\alpha}_i, \quad \check{\alpha}_0 = c - \check{\theta}.$$

The affine fundamental weights are the unique elements $\omega_i \in \tilde{\mathbf{X}}$, $i = 0, 1, \ldots, n$, such that $\langle \omega_i, \check{\alpha}_j \rangle = \delta_{i,j}$ for each i, j. We have

$$\tilde{\mathbf{X}} = \mathbf{X}^T \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\omega_0 = \mathbb{Z}\delta \oplus \bigoplus_{i=0}^{n} \mathbb{Z}\omega_i, \quad \mathbf{X} = \tilde{\mathbf{X}} \oplus \mathbb{Z}t,$$

where δ is the smallest positive imaginary root. Recall that $\alpha_0 = \delta - \theta$. Then δ, ω_0, t are the canonical generators of $\mathbf{X}^{\mathbb{C}^\times_{\mathrm{rot}}}$, $\mathbf{X}^{\mathbb{C}^\times_{\mathrm{cen}}}$, and $\mathbf{X}^{\mathbb{C}^\times_{\mathrm{qua}}}$ respectively.

2.1.9. There is a $\tilde{W}$-action on $\tilde{\mathbf{X}}$, $\tilde{\mathbf{Y}}$ such that the natural pairing is $\tilde{W}$-invariant. It is given by :

- W fixes the elements δ, ω, d, c and it acts in the usual way on $\mathbf{X}^T$, $\mathbf{Y}^T$,
- the element $\xi_{\check{\lambda}}$, $\check{\lambda} \in \mathbf{Y}^T$, maps $\mu \in \tilde{\mathbf{X}}$ to

$$\mu + \langle \mu, c \rangle\, \kappa(\check{\lambda}) - \big(\langle \mu, \check{\lambda} \rangle + (\check{\lambda}, \check{\lambda}) \langle \mu, c \rangle / 2\big)\delta,$$

- the element $\xi_{\check{\lambda}}$, $\check{\lambda} \in \mathbf{Y}^T$, maps $\check{\mu} \in \tilde{\mathbf{Y}}$ to

$$\check{\mu} + \langle \delta, \check{\mu} \rangle \check{\lambda} - \big(\langle \kappa(\check{\lambda}), \check{\mu} \rangle + (\check{\lambda}, \check{\lambda}) \langle \delta, \check{\mu} \rangle / 2\big)c.$$

This action is denoted by ${}^w\mu$, ${}^w\check{\mu}$ for each $w \in \tilde{W}$, $\mu \in \tilde{\mathbf{X}}$, and $\check{\mu} \in \tilde{\mathbf{Y}}$.

2.1.10. There is also a $\tilde{W}$-action on $\tilde{T}$. It is given by :

- W fixes $\mathbb{C}^\times_{\mathrm{rot}}$, $\mathbb{C}^\times_{\mathrm{cen}}$ and it acts on the usual way on T,
- the element $\xi_{\check{\lambda}}$, $\check{\lambda} \in \mathbf{Y}^T$, maps the pair (s, τ) with $s \in T_{\mathrm{cen}}$ and $\tau \in \mathbb{C}^\times_{\mathrm{rot}}$ to the pair

$$\big(s\check{\lambda}(\tau)c(\kappa(\check{\lambda})(sh))^{-1}, \tau\big) \text{ with } h^2 = \check{\lambda}(\tau).$$

  Here we regard $\check{\lambda}, c$ as group homomorphisms $\mathbb{C}^\times \to T_{\mathrm{cen}}$ and $\kappa(\check{\lambda})$ as a group homomorphism $T_{\mathrm{cen}} \to \mathbb{C}^\times$.

2.1.11. Since I is a group-scheme the ring $\mathbf{R}^I$ is well-defined. By devissage we have $\mathbf{R}^I = \mathbf{R}^T$. Recall that $\mathbf{R}^T = \sum_{\lambda \in \mathbf{X}^T} \mathbb{Z}\theta_\lambda$ is the group algebra of $\mathbf{X}^T$, see 1.5.17. We'll abbreviate $q = \theta_\delta$, $t = \theta_t$ in $\mathbf{R}^{\boldsymbol{T}}$. We may also use the following $\mathbb{Z}_t$-algebras

$$\mathbb{Z}_t\tilde{\mathbf{X}} = \sum_{\lambda \in \tilde{\mathbf{X}}} \mathbb{Z}_t\theta_\lambda, \quad \mathbb{Z}_t\tilde{\mathbf{Y}} = \sum_{\check{\lambda} \in \tilde{\mathbf{Y}}} \mathbb{Z}_t\theta_{\check{\lambda}}.$$

Note that $\mathbb{Z}_t\tilde{\mathbf{X}} = \mathbf{R}^{\boldsymbol{T}}$.

2.2. Reminder on the affine flag manifold.

2.2.1. The affine flag manifold. Let $\mathfrak{F} = \mathfrak{F}_G$ be the affine flag manifold of G. It is an ind-proper ind-scheme of ind-finite type whose set of $\mathbb{C}$-points is

$$\mathfrak{F} = G((\varpi))/I = \{\text{Iwahori Lie subalgebra of } \mathfrak{g}((\varpi))\}.$$

The space $\mathfrak{F}$ can be viewed as the sheaf for the fppf topology over the flat affine site over $\mathbb{C}$, associated with the quotient pre-sheaf $\tilde{G}/\tilde{I}$. In particular there is a canonical ind-scheme homomorphism $\tilde{G} \to \mathfrak{F}$ which is a $\tilde{I}$-torsor as in 1.5.15. The set of $\mathbb{C}$-points of $\mathfrak{F}$ is simply the quotient set $\tilde{G}/\tilde{I}$. It will be convenient to regard an element of this set as an Iwahori Lie subalgebra of $\tilde{\mathfrak{g}}$. The ind-group $\tilde{G}$ acts on itself by left multiplication. This action yields a $\tilde{G}$-action on $\mathfrak{F}$. The group-scheme $\tilde{I}$ acts also on $\mathfrak{F}$, and the later has the structure of an admissible ind-$\tilde{I}$-scheme. The $\tilde{I}$-orbits are numbered by the elements of $\tilde{W}$

$$\mathfrak{F} = \bigsqcup_{w \in \tilde{W}} \overset{\circ}{\mathfrak{F}}_w.$$

Let $\leqslant$ be the Bruhat order on $\tilde{W}$. We have

$$\mathfrak{F} = \mathrm{colim}_w \mathfrak{F}_w, \quad \mathfrak{F}_w = \bigsqcup_{v \leqslant w} \overset{\circ}{\mathfrak{F}}_v.$$

Further $\mathfrak{F}_w$ is a projective, normal, $\tilde{I}$-scheme for every w. We have

$$\mathbf{K}(\mathfrak{F}) = \mathrm{colim}_w \mathbf{K}(\mathfrak{F}_w), \quad \mathbf{K}(\mathfrak{F}_w) = [\boldsymbol{\mathcal{C}oh}(\mathfrak{F}_w)].$$

For a future use, we'll abbreviate $\mathfrak{D} = \mathfrak{F} \times \mathfrak{F}$. For each v, w let $\mathfrak{D}_{v,w} = \mathfrak{F}_v \times \mathfrak{F}_w$.

2.2.2. The Kashiwara affine flag manifold. We'll also use the Kashiwara flag manifold $\mathfrak{X}$. See [K], [KT1], [KT2] for details. It is a coherent, pro-smooth (non quasi-compact) scheme locally of countable type with a left $\tilde{I}^\circ$-action. Recall that a G-scheme X is *locally free* if any point of X has a G-stable open neighborhood which is isomorphic, as a G-scheme, to $G \times Y$ for some scheme Y. In this case the quotient X/G is representable by a scheme. The Kashiwara flag manifold is constructed as a quotient $\mathfrak{X} = \tilde{G}_\infty/\tilde{I}$, where $\tilde{G}_\infty$ is a coherent scheme with a locally free left action of $\tilde{I}^\circ$ and a locally free right action of $\tilde{I}$. In particular there is a canonical scheme homomorphism $\tilde{G}_\infty \to \mathfrak{X}$ which is a $\tilde{I}$-torsor as in 1.5.15. There is a $\tilde{I}^\circ$-orbit decomposition

$$\mathfrak{X} = \bigsqcup_{w \in \tilde{W}} \overset{\circ}{\mathfrak{X}}{}^w$$

where $\overset{\circ}{\mathfrak{X}}{}^w$ is a locally closed subscheme of codimension $l(w)$ (=the length of w in $\tilde{W}$) which is isomorphic to the infinite-dimensional affine space $\mathbb{A}^{\mathbb{N}}$. The Zariski closure of $\overset{\circ}{\mathfrak{X}}{}^w$ is $\bigsqcup_{v \geqslant w} \overset{\circ}{\mathfrak{X}}{}^v$. The scheme $\mathfrak{X}$ is covered by the following open subsets

$$\mathfrak{X}^w = \bigsqcup_{v \leqslant w} \overset{\circ}{\mathfrak{X}}{}^v.$$

Note that $\mathfrak{X}^w$ is a $\tilde{I}^\circ$-stable finite union of translations of the big cell $\mathfrak{X}^e$ and that $\mathfrak{X}^e \simeq \mathbb{A}^{\mathbb{N}}$. Thus $\mathfrak{X}^w$ is quasi-compact and pro-smooth. Since $\mathfrak{X}$ is not quasi-compact, we have

$$\mathbf{K}(\mathfrak{X}) = \lim_w \mathbf{K}(\mathfrak{X}^w), \quad \mathbf{K}(\mathfrak{X}^w) = [\boldsymbol{\mathcal{C}oh}(\mathfrak{X}^w)].$$

For each w there is a closed immersion $\mathfrak{F}_w \subset \mathfrak{X}^w$, see [KT2, prop. 1.3.2]. Therefore the restriction of $\mathcal{O}$-modules yields a functor

$$\boldsymbol{\mathcal{Q}coh}(\mathfrak{X}) \to \boldsymbol{\mathcal{Q}coh}(\mathfrak{F}).$$

The tensor product of quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules yields a functor

$$\otimes_{\mathfrak{X}} : \ \boldsymbol{\mathcal{C}oh}(\mathfrak{F}) \times \boldsymbol{\mathcal{C}oh}(\mathfrak{X}) \to \boldsymbol{\mathcal{C}oh}(\mathfrak{F}).$$

Since $\mathfrak{X}^w$ is pro-smooth we have also a group homomorphism

$$\otimes^L_{\mathfrak{X}} : \ \mathbf{K}(\mathfrak{F}) \otimes \mathbf{K}(\mathfrak{X}) \to \mathbf{K}(\mathfrak{F}).$$

Finally, we have the following important property.

2.2.3. Proposition. *The $\tilde{I}^\circ$-scheme $\mathfrak{X}^w$ is admissible and it satisfies $(A_{\tilde{I}^\circ})$.*

Proof : The admissibility follows from 1.4.6. Given an integer $l \geqslant 0$ we consider the quotients

$$\mathfrak{X}^{w,l} = N_l^\circ \setminus \mathfrak{X}^w, \quad \tilde{I}^{\circ,l} = \tilde{I}^\circ / N_l^\circ.$$

Note that $\tilde{I}^{\circ,l}$ is a linear group, that $\mathfrak{X}^{w,l}$ is a smooth $\tilde{I}^{\circ,l}$-scheme and that the canonical map $\mathfrak{X}^w \to \mathfrak{X}^{w,l}$ is a $\tilde{I}^\circ$-equivariant N_l°-torsor [KT2, lem. 2.2.1]. A priori $\mathfrak{X}^{w,l}$ could be not separated. See the remark after [KT2, lem. 2.2.1]. The separatedness is proved in [VV, sec. A.6]. See also 2.2.4 below. Since the $\tilde{I}^{\circ,l}$-scheme $\mathfrak{X}^{w,l}$ is Noetherian and regular it satisfies the property $(A_{\tilde{I}^{\circ,l}})$. Then $\mathfrak{X}^w$ satisfies also the property $(A_{\tilde{I}^\circ})$ by 1.5.4.

□

2.2.4. Remarks. (a) The scheme $\mathfrak{X}^{w,l}$ above is separated if l is large enough, even in the more general case of Kac-Moody groups considered in [KT2]. This follows from [TT, prop. C.7] and the fact that $\mathfrak{X}^w$ is a separated scheme.

(b) The $\tilde{T}$-fixed points subsets in $\overset{\circ}{\mathfrak{F}}_w$ and $\overset{\circ}{\mathfrak{X}}{}^w$ are reduced to the same single point. We'll denote it by $\mathfrak{b}_w$. Note that $\mathfrak{b}_e$ is identified with the Iwahori Lie algebra $\mathfrak{i}$ (or $\tilde{\mathfrak{i}}$) for e the unit element of $\tilde{W}$.

2.2.5. Pro-finite-dimensional vector-bundles over $\mathfrak{F}$. Consider the ind-coherent ind-scheme of ind-infinite type

$$\tilde{\mathfrak{g}} \times \mathfrak{F} = \mathrm{colim}_{w,l}(\tilde{\mathfrak{g}}_l \times \mathfrak{F}_w),$$

where $l \geqslant 0$ and $\tilde{\mathfrak{g}}_l \subset \tilde{\mathfrak{g}}$ is the sum of all weight subspaces which do not belong to $\tilde{\Delta}_l^\circ$. Given a Lie subalgebra $\mathfrak{b} \subset \tilde{\mathfrak{g}}$ let $\mathfrak{b}_{\mathrm{nil}}$ denote its pro-nilpotent radical. Set

$$\dot{\mathfrak{n}} = \{(x, \mathfrak{b}) \in \tilde{\mathfrak{g}} \times \mathfrak{F}; x \in \mathfrak{b}_{\mathrm{nil}}\}.$$

It is a pro-finite-dimensional vector bundle over $\mathfrak{F}$. Thus it is an ind-coherent ind-schemes such that

$$\dot{\mathfrak{n}} = \mathrm{colim}_w \dot{\mathfrak{n}}_w, \quad \dot{\mathfrak{n}}_w = \dot{\mathfrak{n}} \cap (\tilde{\mathfrak{g}} \times \mathfrak{F}_w),$$

where $\dot{\mathfrak{n}}_w$ is a compact coherent scheme for each w. Define also

$$\mathfrak{N} = \dot{\mathfrak{n}} \cap (\mathfrak{n} \times \mathfrak{F}).$$

It is an ind-coherent admissible ind-$\tilde{I}$-scheme such that

$$\mathfrak{N} = \mathrm{colim}_w \mathfrak{N}_w, \quad \mathfrak{N}_w = \dot{\mathfrak{n}}_w \cap (\mathfrak{n} \times \mathfrak{F}_w).$$

Note that the $\tilde{I}$-scheme $\dot{\mathfrak{n}}_w$ satisfies the property $(A_{\tilde{I}})$ because the canonical map $\dot{\mathfrak{n}}_w \to \mathfrak{F}_w$ is $\tilde{I}$-equivariant, affine, and $\mathfrak{F}_w$ is normal and projective, see 1.5.4 and 2.2.1.

2.2.6. Group actions on flag varieties and related objects. Recall that the ind-group $\tilde{G}$ acts on the ind-scheme $\mathfrak{F}$ by left multiplication. It acts also on $\mathfrak{D} = \mathfrak{F} \times \mathfrak{F}$ diagonally, on $\tilde{\mathfrak{g}}$ by conjugation, and on $\tilde{\mathfrak{g}} \times \mathfrak{F}$ diagonally. For each $w \in \tilde{W}$ let $\mathfrak{D}_w \subset \mathfrak{D}$ be the smallest $\tilde{G}$-stable subset containing the pair $(\mathfrak{b}_e, \mathfrak{b}_v)$ for each $v \leqslant w$.

Similarly, the group $\boldsymbol{G}$ acts also on $\mathfrak{F}$, $\mathfrak{D}$, $\tilde{\mathfrak{g}}$, and $\tilde{\mathfrak{g}} \times \mathfrak{F}$. We simply require that an element $z \in \mathbb{C}^\times_{\mathrm{qua}}$ acts trivially on $\mathfrak{F}$ and that z acts by multiplication by z on $\tilde{\mathfrak{g}}$. This action preserves $\mathfrak{n} \times \mathfrak{F}$ and $\dot{\mathfrak{n}}$, and it restricts to an admissible $\boldsymbol{I}$-action on both of them. Note that $\mathfrak{F}$, $\tilde{\mathfrak{g}} \times \mathfrak{F}$, $\mathfrak{n} \times \mathfrak{F}$ and $\dot{\mathfrak{n}}$ are admissible ind-coherent ind-$\boldsymbol{I}$-schemes. We also equip $\mathfrak{X}$ with the canonical $\boldsymbol{I}^\circ$-action such that $\mathbb{C}^\times_{\mathrm{qua}}$ acts trivially.

For a future use let us introduce the following notation. Given $\lambda \in \mathbf{X}$ we can view θ_λ as a one-dimensional representation of $\boldsymbol{I}$. Then for each $\boldsymbol{I}$-scheme X and each $\boldsymbol{I}$-equivariant $\mathcal{O}_X$-module $\mathcal{E}$ we'll write $\mathcal{E}\langle\lambda\rangle$ for the $\boldsymbol{I}$-equivariant $\mathcal{O}_X$-module

$$\mathcal{E}\langle\lambda\rangle = \theta_\lambda \otimes \mathcal{E}.$$

2.3. K-theory and the affine flag manifold.

2.3.1. Induction of ind-schemes. Recall that the Kashiwara flag manifold is equipped with a canonical $\tilde{I}$-torsor $\tilde{G}_\infty \to \mathfrak{X}$, where $\tilde{G}_\infty$ is a coherent scheme with a $\tilde{I}^\circ \times \tilde{I}$-action. For any admissible ind-$\boldsymbol{I}$-scheme Z we equip the quotient

$$Z_{\mathfrak{X}} = \tilde{G}_\infty \times_{\tilde{I}} Z$$

with the $\boldsymbol{I}^\circ$-action such that the subgroup $\tilde{I}^\circ$ acts by left multiplication on $\tilde{G}_\infty$ and $\mathbb{C}^\times_{\mathrm{qua}}$ through its action on Z. We can regard $Z_{\mathfrak{X}}$ as a bundle over $\mathfrak{X}$. For any subspace $X \subset \mathfrak{X}$ let Z_X be the restriction of $Z_{\mathfrak{X}}$ to X. We'll abbreviate $Z^{(w)} = Z_{\mathfrak{X}^w}$ and $Z_{(w)} = Z_{\mathfrak{F}_w}$ for each $w \in \tilde{W}$. The discussion in Section 1.5.15 yields the following.

2.3.2. Proposition. *Let Z be an ind-coherent admissible ind-$\boldsymbol{I}$-scheme. Then $Z_{(w)}$ and $Z_{\mathfrak{F}}$ are ind-coherent admissible ind-$\boldsymbol{I}$-schemes, and $Z^{(w)}$ is an ind-coherent admissible ind-$\boldsymbol{I}^\circ$-scheme.*

Note that $Z_{\mathfrak{X}}$ is only a ind$'$-scheme, because $\mathfrak{X}$ is not quasi-compact. Now, we discuss a few examples which are important for us.

2.3.3. Examples. (a) Set $Z = \mathfrak{F}$. Consider the natural projection $p : \mathfrak{F}_{\mathfrak{F}} \to \mathfrak{F}$ and the action map $a : \mathfrak{F}_{\mathfrak{F}} \to \mathfrak{F}$. The pair (p, a) gives an ind-$\boldsymbol{I}$-scheme isomorphism

$$\mathfrak{F}_{\mathfrak{F}} \to \mathfrak{D} = \mathfrak{F} \times \mathfrak{F}, \quad (g, \mathfrak{b}) \bmod \tilde{I} \mapsto (g(\mathfrak{i}), g(\mathfrak{b})),$$

where $\tilde{I}$ acts diagonally on $\mathfrak{D}$. Under this isomorphism the maps p, a are identified with the projections $\mathfrak{D} = \mathfrak{F} \times \mathfrak{F} \to \mathfrak{F}$ to the first and the second factors respectively. Further, the ind-subscheme $(\mathfrak{F}_w)_{\mathfrak{F}}$ is taken to the ind-subscheme $\mathfrak{D}_w \subset \mathfrak{D}$.

(b) Taking $Z = \mathfrak{n} \times \dot{\mathfrak{n}}$ the induction yields an ind-scheme which is canonically isomorphic to $\dot{\mathfrak{n}} \times \dot{\mathfrak{n}}$.

(c) Taking $Z = \mathfrak{N}$ the induction yields the ind-scheme $\mathfrak{N}_{\mathfrak{F}}$. We'll abbreviate $\mathfrak{M} = \mathfrak{N}_{\mathfrak{F}}$. By 2.3.3$(a)$ we can view $\mathfrak{M}$ as the admissible $\boldsymbol{I}$-equivariant pro-finite-dimensional vector bundle over $\mathfrak{D}$ whose total space is

$$\{(x, \mathfrak{b}, \mathfrak{b}') \in \tilde{\mathfrak{g}} \times \mathfrak{D}; x \in \mathfrak{b}_{\mathrm{nil}} \cap \mathfrak{b}'_{\mathrm{nil}}\}.$$

The $\boldsymbol{I}$-action is the diagonal one. In 2.2.5 we have defined $\mathfrak{N}$ as an ind-subscheme of $\dot{\mathfrak{n}}$ and of $\mathfrak{n} \times \mathfrak{F}$. We may also regard it as an ind-subscheme of $\mathfrak{n} \times \dot{\mathfrak{n}}$ by taking a pair $(x, \mathfrak{b}) \in \mathfrak{N}$ to the pair $(x, (x, \mathfrak{b})) \in \mathfrak{n} \times \dot{\mathfrak{n}}$. Hence we have an inclusion $\mathfrak{M} \subset \dot{\mathfrak{n}} \times \dot{\mathfrak{n}}$ which takes a triple $(x, \mathfrak{b}, \mathfrak{b}')$ to the pair $((x, \mathfrak{b}), (x, \mathfrak{b}'))$. Composing this inclusion with the obvious projections

$$q : \dot{\mathfrak{n}} \times \dot{\mathfrak{n}} \to \dot{\mathfrak{n}} \times \mathfrak{F}, \quad p : \dot{\mathfrak{n}} \times \dot{\mathfrak{n}} \to \mathfrak{F} \times \dot{\mathfrak{n}}$$

we can also view $\mathfrak{M}$ as a good ind-subscheme either of $\dot{\mathfrak{n}} \times \mathfrak{F}$ or of $\mathfrak{F} \times \dot{\mathfrak{n}}$. For each v, w we'll write

$$\mathfrak{M}_v = \{(x, \mathfrak{b}, \mathfrak{b}') \in \mathfrak{M}; (\mathfrak{b}, \mathfrak{b}') \in \mathfrak{D}_v\}, \quad \mathfrak{M}_{w,u} = \{(x, \mathfrak{b}, \mathfrak{b}') \in \mathfrak{M}; \mathfrak{b} \in \mathfrak{F}_w, \mathfrak{b}' \in \mathfrak{F}_u\}.$$

Note that $(\mathfrak{N}_v)_{\mathfrak{F}} \simeq \mathfrak{M}_v$ and that $\mathfrak{M} = \mathrm{colim}_{w,u} \mathfrak{M}_{w,u}$.

(d) Taking $Z = \mathfrak{n}$ the induction yields a pro-finite-dimensional vector bundle $\mathfrak{n}^{(w)}$ over $\mathfrak{X}^w$, and a pro-finite-dimensional vector bundle $\mathfrak{n}_{(w)}$ over $\mathfrak{F}_w$ for each w. Note that we have $\dot{\mathfrak{n}}_w = \mathfrak{n}_{(w)}$, see 2.2.5. For any integer $l \geqslant 0$ we'll set $\dot{\mathfrak{n}}^l_w = (\mathfrak{n}^l)_{(w)}$. The canonical projection $\mathfrak{n} \to \mathfrak{n}^l$ yields a smooth affine morphism $\dot{\mathfrak{n}}_w \to \dot{\mathfrak{n}}^l_w$. Both maps are denoted by the symbol p.

2.3.4. Induction of $\boldsymbol{I}$-equivariant sheaves. Fix an admissible ind-coherent ind-$\boldsymbol{I}$-scheme Z. Consider the induced ind-scheme $Z^{(w)}$ over $\mathfrak{X}^w$ for each $w \in \tilde{W}$. For any elements $v, w \in \tilde{W}$ such that $v \leqslant w$ the open embedding $\mathfrak{X}^v \subset \mathfrak{X}^w$ yields an open embedding of ind-schemes $Z^{(v)} \subset Z^{(w)}$. Fix a closed subgroup $S \subset \boldsymbol{T}$. We obtain an inverse system of categories $(\boldsymbol{\mathcal{Q}coh}^S(Z^{(w)}))$, an inverse system of categories $(\boldsymbol{\mathcal{C}oh}^S(Z^{(w)}))$, and an inverse system of $\mathbf{R}^S$-modules $\mathbf{K}^S(Z^{(w)})$, see 1.3.6. We define

$$\boldsymbol{\mathcal{Q}coh}^S(Z_{\mathfrak{X}}) = 2\lim_w \boldsymbol{\mathcal{Q}coh}^S(Z^{(w)}),$$
$$\boldsymbol{\mathcal{C}oh}^S(Z_{\mathfrak{X}}) = 2\lim_w \boldsymbol{\mathcal{C}oh}^S(Z^{(w)}),$$
$$\mathbf{K}^S(Z_{\mathfrak{X}}) = \lim_w \mathbf{K}^S(Z^{(w)}).$$

The discussion in Section 1.5.15 implies the following

- for each $w \in \tilde{W}$ the induction yields exact functors $\boldsymbol{\mathcal{Q}coh}^{\boldsymbol{I}}(Z) \to \boldsymbol{\mathcal{Q}coh}^{S}(Z^{(w)})$ and $\boldsymbol{\mathcal{C}oh}^{\boldsymbol{I}}(Z) \to \boldsymbol{\mathcal{C}oh}^{S}(Z^{(w)})$ which commute with tensor products and a group homomorphism $\mathbf{K}^{\boldsymbol{I}}(Z) \to \mathbf{K}^{S}(Z^{w})$,
- taking the inverse limit over all w's we get functors $\boldsymbol{\mathcal{Q}coh}^{\boldsymbol{I}}(Z) \to \boldsymbol{\mathcal{Q}coh}^{S}(Z_{\mathfrak{X}})$, $\boldsymbol{\mathcal{C}oh}^{\boldsymbol{I}}(Z) \to \boldsymbol{\mathcal{C}oh}^{S}(Z_{\mathfrak{X}})$ which commute with tensor products and a group homomorphism $\mathbf{K}^{\boldsymbol{I}}(Z) \to \mathbf{K}^{S}(Z_{\mathfrak{X}})$.
- For each $w \in \tilde{W}$ and each $\boldsymbol{I}$-equivariant quasi-coherent $\mathcal{O}_Z$-module $\mathcal{E}$ the restriction of the induced $\mathcal{O}_{Z^{(w)}}$-module to the ind-scheme $Z_{(w)}$ is naturally $\boldsymbol{I}$-equivariant. Hence the induction yields also functors $\boldsymbol{\mathcal{Q}coh}^{\boldsymbol{I}}(Z) \to \boldsymbol{\mathcal{Q}coh}^{\boldsymbol{I}}(Z_{(w)})$, $\boldsymbol{\mathcal{C}oh}^{\boldsymbol{I}}(Z) \to \boldsymbol{\mathcal{C}oh}^{\boldsymbol{I}}(Z_{(w)})$, and a group homomorphism $\mathbf{K}^{\boldsymbol{I}}(Z) \to \mathbf{K}^{\boldsymbol{I}}(Z_{(w)})$.

For each $\mathcal{E} \in \boldsymbol{\mathcal{Q}coh}^{\boldsymbol{I}}(Z)$ we'll write $\mathcal{E}_{\mathfrak{X}}$, $\mathcal{E}_{(w)}$, and $\mathcal{E}^{(w)}$ for the induced $\mathcal{O}$-modules over $Z_{\mathfrak{X}}$, $Z_{(w)}$ and $Z^{(w)}$ respectively.

2.3.5. Examples. (a) For each $\lambda \in \mathbf{X}$ let $\mathcal{O}_{\mathfrak{X}}(\lambda)$ be the line bundle over $\mathfrak{X}$ induced from the character θ_λ. The local sections of $\mathcal{O}_{\mathfrak{X}}(\lambda)$ are the regular functions $f : \tilde{G}_\infty \to \mathbb{C}$ such that $f(xb) = \lambda(b)f(x)$ for each $x \in \tilde{G}_\infty$ and $b \in \tilde{I}$. Note that $\mathcal{O}_{\mathfrak{X}}(t) = \mathcal{O}_{\mathfrak{X}}\langle t\rangle$, where t is as in 2.1.8.

Restricting $\mathcal{O}_{\mathfrak{X}}(\lambda)$ to $\mathfrak{F}$ we get also a line bundle $\mathcal{O}_{\mathfrak{F}}(\lambda)$ over the ind-scheme $\mathfrak{F}$. We'll write $\mathcal{O}_X(\lambda) = f^*\mathcal{O}_{\mathfrak{X}}(\lambda)$ or $f^*\mathcal{O}_{\mathfrak{F}}(\lambda)$ for any map $f : X \to \mathfrak{X}$ or $f : X \to \mathfrak{F}$. For instance we have the line bundles $\mathcal{O}_{\mathfrak{n}_{\mathfrak{X}}}(\lambda)$, $\mathcal{O}_{\dot{\mathfrak{n}}}(\lambda)$, and $\mathcal{O}_{\mathfrak{N}}(\lambda)$. For any $\mathcal{O}_X$-module $\mathcal{E}$ we'll abbreviate

$$\mathcal{E}(\lambda) = \mathcal{E} \otimes_X \mathcal{O}_X(\lambda).$$

(b) Given $\lambda, \mu \in \mathbf{X}$ we can consider the $\boldsymbol{I}$-equivariant line bundle $\mathcal{O}_{\mathfrak{F}}(\mu)\langle\lambda\rangle$ over $\mathfrak{F}$. By induction and 2.3.3(a) it yields an $\boldsymbol{I}$-equivariant line bundle over $\mathfrak{D}$. Recall that the $\boldsymbol{I}$-action on $\mathfrak{D}$ is the diagonal one. Restricting the induced bundle $(\mathcal{O}_{\mathfrak{F}}(\mu)\langle\lambda\rangle)_{\mathfrak{X}}$ over $\mathfrak{F}_{\mathfrak{X}}$ to $\mathfrak{F}_{\mathfrak{F}} \simeq \mathfrak{D}$ we get the line bundle

$$\mathcal{O}_{\mathfrak{D}}(\lambda, \mu) = \mathcal{O}_{\mathfrak{F}}(\lambda) \boxtimes \mathcal{O}_{\mathfrak{F}}(\mu).$$

For any map $Z \to \mathfrak{D}$ we'll write $\mathcal{O}_Z(\lambda, \mu) = f^*\mathcal{O}_{\mathfrak{D}}(\lambda, \mu)$. We write also $\mathcal{O}_{\mathfrak{N}_{\mathfrak{X}}}(\lambda, \mu) = (\mathcal{O}_{\mathfrak{N}}(\mu)\langle\lambda\rangle)_{\mathfrak{X}}$.

2.3.6. Convolution product on $\mathbf{K}^{\boldsymbol{I}}(\mathfrak{N})$. The purpose of this section is to define an associative multiplication

$$\star : \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}) \otimes \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}) \to \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}).$$

Fix $\mathcal{E}, \mathcal{F} \in \boldsymbol{\mathcal{C}oh}^{\boldsymbol{I}}(\mathfrak{N})$. Recall that

$$\boldsymbol{\mathcal{C}oh}^{\boldsymbol{I}}(\mathfrak{N}) = 2\mathrm{colim}_w \boldsymbol{\mathcal{C}oh}^{\boldsymbol{I}}(\mathfrak{N}_w).$$

Choose $v, w \in \tilde{W}$ such that $\mathcal{E} \in \boldsymbol{\mathcal{C}oh}^{\boldsymbol{I}}(\mathfrak{N}_w)$ and $\mathcal{F} \in \boldsymbol{\mathcal{C}oh}^{\boldsymbol{I}}(\mathfrak{N}_v)$. We can regard $\mathcal{E}$ as a coherent $\mathcal{O}_{\dot{\mathfrak{n}}_w}$-module and $\mathcal{F}$ as a quasi-coherent $\mathcal{O}_{\mathfrak{n}\times\dot{\mathfrak{n}}_v}$-module. Note that the closed embedding $\mathfrak{N}_v \subset \mathfrak{n} \times \dot{\mathfrak{n}}_v$ is not good. Fix $u \in \tilde{W}$ such that the isomorphism $(\mathfrak{n} \times \dot{\mathfrak{n}})_{\mathfrak{F}} = \dot{\mathfrak{n}} \times \dot{\mathfrak{n}}$ in 2.3.3(b) factors to a good embedding

$$\nu : (\mathfrak{n} \times \dot{\mathfrak{n}}_v)_{(w)} \to \dot{\mathfrak{n}}_w \times \dot{\mathfrak{n}}_u. \tag{2.3.2}$$

Consider the obvious projections

$$\dot{\mathfrak{n}}_w \xleftarrow{f_2} \dot{\mathfrak{n}}_w \times \dot{\mathfrak{n}}_u \xrightarrow{f_1} \dot{\mathfrak{n}}_u.$$

Then $f_2^*(\mathcal{E})$ and $\nu_*(\mathcal{F}_{(w)})$ are both $\boldsymbol{I}$-equivariant quasi-coherent $\mathcal{O}$-modules over $\dot{\mathfrak{n}}_w \times \dot{\mathfrak{n}}_u$, and we can define the following complex in $\boldsymbol{\mathcal{D}}^{\boldsymbol{I}}(\dot{\mathfrak{n}}_w \times \dot{\mathfrak{n}}_u)_{qc}$

$$\mathcal{G} = f_2^*(\mathcal{E}) \overset{L}{\otimes}_{\dot{\mathfrak{n}}_w \times \dot{\mathfrak{n}}_u} \nu_*(\mathcal{F}_{(w)}). \tag{2.3.3}$$

We'll view it as a complex of $\boldsymbol{I}$-equivariant quasi-coherent $\mathcal{O}$-modules over the ind-scheme $\dot{\mathfrak{n}} \times \dot{\mathfrak{n}}$ supported on the subscheme $\dot{\mathfrak{n}}_w \times \dot{\mathfrak{n}}_u$. We want to consider its direct image by the map f_1. Since the schemes $\dot{\mathfrak{n}}_w$, $\dot{\mathfrak{n}}_u$ are not of finite type, this requires some work.

2.3.7. Proposition. *The complex of $\mathcal{O}$-modules $\mathcal{G}$ over $\dot{\mathfrak{n}} \times \dot{\mathfrak{n}}$ does not depend on the choices of u, v, w up to quasi-isomorphisms. It is cohomologically bounded. Its direct image $R(f_1)_*(\mathcal{G})$ is a cohomologically bounded pseudo-coherent complex over $\dot{\mathfrak{n}}_u$ with cohomology sheaves supported on $\mathfrak{N}_u$. The assignment $\mathcal{E} \otimes \mathcal{F} \mapsto R(f_1)_*(\mathcal{G})$ yields a group homomorphism $\star : \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}) \otimes \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}) \to \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N})$.*

Proof : We'll abbreviate

$$T = (\mathfrak{n} \times \dot{\mathfrak{n}}_v)_{(w)}, \quad Y = \dot{\mathfrak{n}}_w \times \dot{\mathfrak{n}}_u, \quad \phi_2 = f_2 \circ \nu, \quad \phi_1 = f_1 \circ \nu.$$

Thus we have the following diagram

$$\begin{tikzcd} \dot{\mathfrak{n}}_w & Y \arrow[l, "f_2"'] \arrow[r, "f_1"] & \dot{\mathfrak{n}}_u \\ & T. \arrow[ul, "\phi_2"] \arrow[u, "\nu"'] \arrow[ur, "\phi_1"'] & \end{tikzcd} \tag{2.3.4}$$

The map ϕ_2 is flat. We claim that the complex of $\boldsymbol{I}$-equivariant quasi-coherent $\mathcal{O}_T$-modules

$$\mathcal{H} = \phi_2^*(\mathcal{E}) \overset{L}{\otimes}_T \mathcal{F}_{(w)}$$

is cohomologically bounded. It is enough to prove that the complex $for(\mathcal{H})$ is cohomologically bounded. Since the derived tensor product commutes with the forgetful functor we may forget the $\boldsymbol{I}$-action everywhere. Hence we can use base change and the projection formula in full generality. To unburden the notation in the rest of the proof we'll omit the functor for.

Now, for an integer $l \geqslant 0$ we have the maps in 2.3.3(d)

$$p : \mathfrak{n} \to \mathfrak{n}^l, \quad p : \dot{\mathfrak{n}}_w \to \dot{\mathfrak{n}}_w^l. \tag{2.3.5}$$

Since $\mathcal{E}$ is an object of $\boldsymbol{\mathcal{C}oh}(\dot{\mathfrak{n}}_w)$, by 1.5.13 there is an l and an object $\mathcal{E}^l$ of $\boldsymbol{\mathcal{C}oh}(\dot{\mathfrak{n}}_w^l)$ such that $\mathcal{E} = p^*(\mathcal{E}^l)$. Next, recall that $\mathcal{F}$ is an object of $\boldsymbol{\mathcal{Q}coh}(\mathfrak{n} \times \dot{\mathfrak{n}}_v)$. In the commutative diagram

$$\begin{tikzcd} \mathfrak{n} \times \dot{\mathfrak{n}}_v \arrow[r, "p \times 1"] & \mathfrak{n}^l \times \dot{\mathfrak{n}}_v \arrow[r, "1 \times p"] & \mathfrak{n}^l \times \dot{\mathfrak{n}}_v^l \\ & \mathfrak{N}_v \arrow[ul] \arrow[u] & \end{tikzcd}$$

the right vertical map is a good inclusion. Thus the $\mathcal{O}$-module $(p \times 1)_*(\mathcal{F})$ over $\mathfrak{n}^l \times \dot{\mathfrak{n}}_v$ is coherent. So if l is large enough there is a coherent sheaf $\mathcal{F}^l$ over $\mathfrak{n}^l \times \dot{\mathfrak{n}}_v^l$ such that

$$(p \times 1)_*(\mathcal{F}) = (1 \times p)^*(\mathcal{F}^l).$$

We'll abbreviate $T^l = (\mathfrak{n}^l \times \dot{\mathfrak{n}}_v^l)_{(w)}$. Set $\phi_{2,l} = f_{2,l} \circ \nu_l$ and $\phi_{1,l} = f_{1,l} \circ \nu_l$, where ν_l, $f_{2,l}$ and $f_{1,l}$ are the obvious inclusion and projections in the diagram

$$\begin{tikzcd}
\dot{\mathfrak{n}}_w^l & \dot{\mathfrak{n}}_w^l \times \dot{\mathfrak{n}}_u^l \arrow[l, "f_{2,l}"'] \arrow[r, "f_{1,l}"] & \dot{\mathfrak{n}}_u^l \\
& T^l. \arrow[ul, "\phi_{2,l}"] \arrow[u, "\nu_l"] \arrow[ur, "\phi_{1,l}"'] &
\end{tikzcd}$$

Let us consider the complex $\mathcal{H}^l = \phi_{2,l}^*(\mathcal{E}^l) \overset{L}{\otimes}_{T^l} \mathcal{F}^l_{(w)}$ over T^l. The projection p in (2.3.5) gives a chain of maps

$$T \xrightarrow{q} (\mathfrak{n}^l \times \dot{\mathfrak{n}}_v)_{(w)} \xrightarrow{r} T^l \ .$$

Note that $Rq_* = q_*$ and $Lr^* = r^*$ because q is affine and r is flat. Thus a short computation using base change and the projection formula implies that

$$q_*(\mathcal{H}) = r^*(\mathcal{H}^l).$$

So to prove that $\mathcal{H}$ is cohomologically bounded it is enough to prove that $\mathcal{H}^l$ itself is cohomologically bounded. This can be proved using the Kashiwara affine flag manifold as follows. Write $X^l = \dot{\mathfrak{n}}_w^l$ and

$$T' = (\mathfrak{n}^l \times \dot{\mathfrak{n}}_v^l)^{(w)}, \quad X' = (\mathfrak{n}^l)^{(w)}.$$

Consider the Cartesian square

$$\begin{tikzcd}
T' \arrow[r, "\phi_2'"] & X' \\
T^l \arrow[u, "i"] \arrow[r, "\phi_{2,l}"] & X^l, \arrow[u, "i"']
\end{tikzcd}$$

where the vertical maps are the embeddings induced by the inclusion $\mathfrak{F}_w \subset \mathfrak{X}^w$. Recall that $\mathcal{E}^l$ is a coherent $\mathcal{O}_{X^l}$-module and that $\mathcal{F}^l_{(w)}$ is the restriction to T^l of the coherent $\mathcal{O}_{T'}$-module $\mathcal{F}' = (\mathcal{F}^l)^{(w)}$. Since the scheme $\mathfrak{X}^w$ satisfies the property (S), by 1.2.18 there is also a Cartesian square

$$\begin{tikzcd}
T^\alpha \arrow[r, "\phi_{2,\alpha}"] & X^\alpha \\
T' \arrow[u, "p_\alpha"] \arrow[r, "\phi_2'"] & X' \arrow[u, "p_\alpha"']
\end{tikzcd}$$

where the vertical maps are smooth and affine, X^α is smooth of finite type and the composed maps $j = p_\alpha \circ i$ are closed embeddings. Further we can assume that $\mathcal{F}' = (p_\alpha)^*(\mathcal{F}^\alpha)$ for some coherent $\mathcal{O}_{T^\alpha}$-module $\mathcal{F}^\alpha$. We have

$$i_*(\mathcal{H}^l) = (\phi_2')^* i_*(\mathcal{E}^l) \overset{L}{\otimes}_{T'} \mathcal{F}'.$$

Thus we have also

$$j_*(\mathcal{H}^l) = (\phi_{2,\alpha})^* j_*(\mathcal{E}^l) \overset{L}{\otimes}_{T^\alpha} \mathcal{F}^\alpha.$$

Now $(\phi_{2,\alpha})^* j_*(\mathcal{E}^l)$ and $\mathcal{F}^\alpha$ are both coherent $\mathcal{O}_{T^\alpha}$-modules and $j_*(\mathcal{E}^l)$ is perfect because X^α is smooth, see 1.2.11. Hence the complex $j_*(\mathcal{H}^l)$ is pseudo-coherent and cohomologically bounded. Thus the complex $\mathcal{H}^l$ is also pseudo-coherent and cohomologically bounded, because j is a closed embedding. So $\mathcal{H}$ is also cohomologically bounded.

Now we can prove that $\mathcal{G}$ and $R(f_1)_*(\mathcal{G})$ are cohomologically bounded. Once again we can omit the $\boldsymbol{I}$-action. Since $R\nu_* = \nu_*$, using the projection formula we get $\mathcal{G} = \nu_*(\mathcal{H})$. Thus the complex $\mathcal{G}$ is cohomologically bounded. Hence $R(f_1)_*(\mathcal{G})$ is also cohomologically bounded because the derived direct image preserves cohomologically bounded complexes.

To prove that the complex $R(f_1)_*(\mathcal{G})$ is pseudo-coherent it is enough to observe that we have $R(f_1)_*(\mathcal{G}) = p^* R(\phi_{1,l})_*(\mathcal{H}^l)$ and that $\mathcal{H}^l$ is pseudo-coherent.

The first claim of the proposition is obvious and is left to the reader. For instance, since $\mathcal{G} = \nu_*(\mathcal{H})$ the complex of $\mathcal{O}$-modules $\mathcal{G}$ over the ind-scheme $\dot{\mathfrak{n}} \times \dot{\mathfrak{n}}$ does not depend on the choice of u. The independence on v, w is proved in a similar way.

□

The following proposition will be proved in 2.4.9 below.

2.3.8. Proposition. *The map $\star$ equips $\mathbf{K}^{\boldsymbol{I}}(\mathfrak{N})$ with a ring structure.*

2.3.9. Remarks. (a) The map $\star$ is an affine analogue of the convolution product used in [CG]. It is $\mathbf{R}^{\boldsymbol{I}}$-linear in the first variable (see part (c) below) but not in the second one. The definition of $\star$ we have given here is inspired from [BFM, sec. 7.2]. Observe, however, that the complex $\mathcal{G}$ is not a complex of coherent sheaves over $\dot{\mathfrak{n}} \times \dot{\mathfrak{n}}$, contrarily to what is claimed in loc. cit. (in a slightly different setting).

(b) Since $\mathfrak{N}_e \subset \mathfrak{N}$ is a good subscheme, for each $\lambda \in \mathbf{X}$ we have the $\boldsymbol{I}$-equivariant coherent $\mathcal{O}_{\mathfrak{N}}$-module $\mathcal{O}_{\mathfrak{N}_e}(\lambda) = \mathcal{O}_{\mathfrak{N}_e}\langle\lambda\rangle$. Consider the diagram

$$\begin{tikzcd} \dot{\mathfrak{n}}_w & \dot{\mathfrak{n}}_w \times \dot{\mathfrak{n}}_w \arrow[l, "f_2"'] \arrow[r, "f_1"] & \dot{\mathfrak{n}}_w \\ & \dot{\mathfrak{n}}_w. \arrow[u, "\delta"'] & \end{tikzcd}$$

where f_1, f_2 are the obvious projections and δ is the diagonal inclusion. Given $\lambda \in \mathbf{X}$ and an object $\mathcal{E}$ in $\boldsymbol{Coh}^{\boldsymbol{I}}(\mathfrak{N}_w)$ let $\mathcal{E}(\lambda)$ be the "twisted" sheaf defined in 2.3.5(a). We have

$$\mathcal{E} \star \mathcal{O}_{\mathfrak{N}_e}(\lambda) = R(f_1)_*\big(f_2^*(\mathcal{E}) \otimes_{\dot{\mathfrak{n}}_w \times \dot{\mathfrak{n}}_w} \delta_* \mathcal{O}_{\dot{\mathfrak{n}}_w}(\lambda)\big) = \mathcal{E}(\lambda).$$

Thus the associativity of $\star$ yields

$$\mathcal{E}\star\mathcal{F}(\lambda)=(\mathcal{E}\star\mathcal{F})(\lambda),\quad \forall\,\mathcal{E},\mathcal{F}\in\boldsymbol{\mathcal{C}oh^I}(\mathfrak{N}).$$

(c) Consider the diagram

$$\begin{tikzcd} \mathfrak{n} & \mathfrak{n}\times\dot{\mathfrak{n}}_v \arrow[l, "f_2"'] \arrow[r, "f_1"] & \dot{\mathfrak{n}}_v \\ & \mathfrak{N}_v \arrow[u, "\delta"'] & \end{tikzcd}$$

where f_1, f_2 are the obvious projections and δ is the diagonal inclusion. Given $\lambda\in\mathbf{X}$ and an object $\mathcal{F}$ in $\boldsymbol{\mathcal{C}oh^I}(\mathfrak{N}_v)$ let $\mathcal{F}\langle\lambda\rangle$ be the "twisted" sheaf defined in 2.2.6. We have

$$\mathcal{O}_{\mathfrak{N}_e}\langle\lambda\rangle\star\mathcal{F}=R(f_1)_*\big(f_2^*(\mathcal{O}_{\mathfrak{n}}\langle\lambda\rangle)\otimes_{\mathfrak{n}\times\dot{\mathfrak{n}}_v}\delta_*(\mathcal{F})\big)=\mathcal{F}\langle\lambda\rangle.$$

Thus the associativity of $\star$ yields

$$\mathcal{E}\langle\lambda\rangle\star\mathcal{F}=(\mathcal{E}\star\mathcal{F})\langle\lambda\rangle,\quad \forall\,\mathcal{E},\mathcal{F}\in\boldsymbol{\mathcal{C}oh^I}(\mathfrak{N}).$$

2.4. Complements on the concentration in K-theory.

2.4.1. Definition of the concentration map $\mathbf{r}_\Sigma$. Let $S\subset\boldsymbol{T}$ be a closed subgroup. We'll say that S is *regular* if the schemes $\mathfrak{n}^S$, $\mathfrak{X}^S$ are both locally of finite type. Note that if S is regular then we have $\mathfrak{X}^S=\mathfrak{F}^S$ (as sets, because the lhs is a scheme of infinite type and locally finite type while the rhs an ind-scheme of ind-finite type). Next, we'll say that a subset $\Sigma\subset S$ is *regular* if we have $\mathfrak{n}^S=\mathfrak{n}^\Sigma$ and $\mathfrak{X}^S=\mathfrak{X}^\Sigma$. In this subsection we'll assume that S and Σ are both regular. Let $\mathfrak{F}(\alpha)$, $\alpha\in A$, be the connected components of $\mathfrak{F}^S$. We have

$$\dot{\mathfrak{n}}^S=\bigsqcup_{\alpha\in A}\dot{\mathfrak{n}}(\alpha),$$

where $\dot{\mathfrak{n}}(\alpha)$ is a vector bundle over $\mathfrak{F}(\alpha)$ for each α. Since S is regular we have

$$\mathfrak{N}^S_{\mathfrak{X}}=\mathfrak{M}^S=\bigsqcup_{\alpha,\beta}\mathfrak{M}(\alpha,\beta),\quad \mathfrak{M}(\alpha,\beta)=\mathfrak{M}\cap(\dot{\mathfrak{n}}(\alpha)\times\dot{\mathfrak{n}}(\beta)),$$

where $\mathfrak{M}$ is as in 2.3.3. Here we have abbreviated $\mathfrak{N}^S_{\mathfrak{X}}=(\mathfrak{N}_{\mathfrak{X}})^S$. We define

$$\mathbf{K}^S(\mathfrak{N}^S_{\mathfrak{X}})=\lim\nolimits_w\mathrm{colim}_v\mathbf{K}^S\big((\mathfrak{N}_v)^{(w),S}\big)=\prod_\alpha\bigoplus_\beta\mathbf{K}^S(\mathfrak{M}(\alpha,\beta)).$$

Observe that in 2.3.4 we have defined the group $\mathbf{K}^S(\mathfrak{N}_{\mathfrak{X}})$ in a similar way by setting

$$\mathbf{K}^S(\mathfrak{N}_{\mathfrak{X}})=\lim\nolimits_w\mathrm{colim}_v\mathbf{K}^S\big((\mathfrak{N}_v)^{(w)}\big).$$

Now we can define the concentration map. Consider the closed embeddings

$$(\mathfrak{n}\times\mathfrak{F})^S_{\mathfrak{X}}\xrightarrow{\ i\ }\mathfrak{n}^S_{\mathfrak{X}}\times_{\mathfrak{X}}\mathfrak{F}_{\mathfrak{X}}\xrightarrow{\ j\ }(\mathfrak{n}\times\mathfrak{F})_{\mathfrak{X}}=\mathfrak{n}_{\mathfrak{X}}\times_{\mathfrak{X}}\mathfrak{F}_{\mathfrak{X}}. \tag{2.4.1}$$

The scheme $\mathfrak{n}_{\mathfrak{X}}$ is pro-smooth and the inclusion $\mathfrak{N}\subset\mathfrak{n}\times\mathfrak{F}$ is good. Thus 1.5.20 yields a group homomorphism

$$\boldsymbol{\gamma}_\Sigma=(i_*)^{-1}\circ Lj^*:\mathbf{K}^S(\mathfrak{N}_{\mathfrak{X}})\to\mathbf{K}^S(\mathfrak{N}^S_{\mathfrak{X}})_\Sigma.$$

Composing it with the induction $\Gamma:\mathcal{E}\mapsto\mathcal{E}_{\mathfrak{X}}$ yields a group homomorphism

$$\mathbf{r}_\Sigma:\ \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N})\longrightarrow\mathbf{K}^S(\mathfrak{N}_{\mathfrak{X}})\longrightarrow\mathbf{K}^S(\mathfrak{N}^S_{\mathfrak{X}})_\Sigma\ . \tag{2.4.2}$$

The map $\mathbf{r}_\Sigma$ is called the *concentration map.*

2.4.2. Remark. The map $\mathbf{r}_\Sigma$ is an affine analogue of the concentration map defined in [CG, thm. 5.11.10]. It can also be described in the following way. Let $\mathcal{E}$ be an $\boldsymbol{I}$-equivariant coherent $\mathcal{O}$-module over $\mathfrak{N}$. Fix $v \in \tilde{W}$ such that $\mathcal{E} \in \boldsymbol{\mathcal{C}oh^I}(\mathfrak{N}_v)$. Given any $w \in \tilde{W}$ we fix ν, u as in (2.3.2). Under the direct image by ν we can view the S-equivariant coherent $\mathcal{O}$-module $\mathcal{E}_{(w)}$ as a S-equivariant quasi-coherent $\mathcal{O}$-module over $\dot{\mathfrak{n}}_w \times \dot{\mathfrak{n}}_u$ supported on $\mathfrak{M}_v \cap \mathfrak{M}_{w,u}$. The obvious projection

$$q : \dot{\mathfrak{n}}_w \times \dot{\mathfrak{n}}_u \to \dot{\mathfrak{n}}_w \times \mathfrak{F}_u$$

yields a good inclusion $\mathfrak{M}_{w,u} \subset \dot{\mathfrak{n}}_w \times \mathfrak{F}_u$. Thus, under the direct image by q we can also regard $\mathcal{E}_{(w)}$ as a S-equivariant coherent $\mathcal{O}$-module over $\dot{\mathfrak{n}}_w \times \mathfrak{F}_u$. Then we consider the following chain of inclusions

$$\dot{\mathfrak{n}}_w^S \times \mathfrak{F}_u^S \xrightarrow{\ i\ } \dot{\mathfrak{n}}_w^S \times \mathfrak{F}_u \xrightarrow{\ j\ } \dot{\mathfrak{n}}_w \times \mathfrak{F}_u. \tag{2.4.3}$$

Since $\mathcal{E}_{(w)}$ is flat over $\mathfrak{F}_w$ and since $\dot{\mathfrak{n}}_w \to \mathfrak{F}_w$ is a pro-finite-dimensional vector bundle, we have a cohomologically bounded pseudo-coherent complex $\mathcal{E}' = Lj^*(\mathcal{E}_{(w)})$ over $\dot{\mathfrak{n}}_w^S \times \mathfrak{F}_u$. Next, the Thomason theorem yields an invertible map

$$i_* : \mathbf{K}^S(\dot{\mathfrak{n}}_w^S \times \mathfrak{F}_u^S)_\Sigma \to \mathbf{K}^S(\dot{\mathfrak{n}}_w^S \times \mathfrak{F}_u)_\Sigma.$$

Thus we have a well-defined element $\mathcal{E}'' = (i_*)^{-1}(\mathcal{E}')$. It can be regarded as an element of $\mathbf{K}^S(\mathfrak{M}_{u,w}^S)_\Sigma$ for a reason of supports. If w, u are large enough then $\mathfrak{M}(\alpha, \beta)$ is a closed and open subset of $\mathfrak{M}_{u,w}^S$. The component of $\mathbf{r}_\Sigma(\mathcal{E})$ in $\mathbf{K}^S(\mathfrak{M}(\alpha, \beta))_\Sigma$ is the restriction of $\mathcal{E}''$ to $\mathfrak{M}(\alpha, \beta)$.

2.4.3. Proposition. *If $S = \Sigma = \boldsymbol{T}$ then $\mathbf{r}_S$ is an injective map.*

Proof : We have $\mathfrak{X}^e = N^\circ = \tilde{I}^\circ/\tilde{T}$ as a $\tilde{I}^\circ$-scheme. Thus we have $\mathfrak{N}^{(e)} = N^\circ \times \mathfrak{N}$. The induction yields an inclusion

$$\mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}) \to \mathbf{K}^{\boldsymbol{T}}(\mathfrak{N}^{(e)}),\ \mathcal{E} \mapsto \mathcal{E}^{(e)}. \tag{2.4.4}$$

Therefore the induction map $\mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}) \to \mathbf{K}^{\boldsymbol{T}}(\mathfrak{N}_{\mathfrak{X}})$ is also injective, because composing it with the canonical map

$$\mathbf{K}^{\boldsymbol{T}}(\mathfrak{N}_{\mathfrak{X}}) = \lim_w \mathbf{K}^{\boldsymbol{T}}(\mathfrak{N}^{(w)}) \to \mathbf{K}^{\boldsymbol{T}}(\mathfrak{N}^{(e)})$$

yields (2.4.4). Thus, to prove that $\mathbf{r}_{\boldsymbol{T}}$ is injective it is enough to check that the canonical map $\mathbf{K}^{\boldsymbol{T}}(\mathfrak{N}^{(e)}) \to \mathbf{K}^{\boldsymbol{T}}(\mathfrak{N}^{(e)})_{\boldsymbol{T}}$ is injective. This is obvious because the $\mathbf{R}^{\boldsymbol{T}}$-module $\mathbf{K}^{\boldsymbol{T}}(\mathfrak{N}^{(e)})$ is torsion-free (use an affine cell decomposition of $\mathfrak{N}$).

□

2.4.4. Concentration of $\mathcal{O}$-modules supported on $\mathfrak{N}_e$. Let $\mathcal{E}$ be an $\boldsymbol{I}$-equivariant vector bundle over $\mathfrak{N}_e$. Since the inclusion $\mathfrak{N}_e \subset \mathfrak{n} \times \mathfrak{F}$ is good we may view $\mathcal{E}$ as an object of $\boldsymbol{\mathcal{C}oh^I}(\mathfrak{n} \times \mathfrak{F})$. Now we consider the diagrams (2.4.1) and (2.4.2). The induced coherent sheaf $\Gamma(\mathcal{E}) = \mathcal{E}_{\mathfrak{X}}$ is flat over $\mathfrak{n}_{\mathfrak{X}}$. Thus we have $Lj^*(\mathcal{E}_{\mathfrak{X}}) = j^*(\mathcal{E}_{\mathfrak{X}})$. Thus we obtain

$$\mathbf{r}_\Sigma(\mathcal{E}) = (i_*)^{-1} j^*(\mathcal{E}_{\mathfrak{X}}).$$

Next we have $j^{-1}((\mathfrak{N}_e)_{\mathfrak{X}}) = i((\mathfrak{N}_e)_{\mathfrak{X}}^S)$. This implies that

$$\mathbf{r}_\Sigma(\mathcal{E}) = j^*(\mathcal{E}_{\mathfrak{X}}).$$

Therefore we have proved the following.

2.4.5. Proposition. *If $\mathcal{E}$ is an $\boldsymbol{I}$-equivariant vector bundle over $\mathfrak{N}_e$ then $\mathbf{r}_\Sigma(\mathcal{E})$ is the restriction of the coherent sheaf $\mathcal{E}_{\mathfrak{X}}$ over $\mathfrak{N}_{\mathfrak{X}}$ to the fixed-points subscheme $\mathfrak{N}_{\mathfrak{X}}^S$.*

2.4.6. Concentration of $\mathcal{O}$-modules supported on $\mathfrak{N}'_{s_\alpha}$. Fix a simple affine root $\alpha \in \tilde{\Pi}$. Recall that s_α is the corresponding simple reflection and that $\mathfrak{n}^\circ_{s_\alpha}$ is a 1-dimensional $\boldsymbol{T}$-module whose class in the ring $\mathbf{R}^{\boldsymbol{T}}$ is $\theta_{t+\alpha}$. Recall also that $\mathfrak{N} \subset \mathfrak{n} \times \mathfrak{F}$ and that $\mathfrak{n}_{s_\alpha} \subset \mathfrak{b}_{\mathrm{nil}}$ are good inclusions for each $\mathfrak{b} \in \mathfrak{F}_{s_\alpha}$. So we have a good $\boldsymbol{I}$-equivariant subscheme $\mathfrak{N}'_{s_\alpha} \subset \mathfrak{N}$ given by

$$\mathfrak{N}'_{s_\alpha} = \mathfrak{n}_{s_\alpha} \times \mathfrak{F}_{s_\alpha}.$$

By a good subscheme we means that $\mathfrak{N}'_{s_\alpha}$ is a good ind-subscheme, as in 1.3.7, which is a scheme. Note that $\mathfrak{N}'_{s_\alpha}$ is pro-smooth, because it is a pro-finite-dimensional vector bundle over the smooth scheme $\mathfrak{F}_{s_\alpha}$. Let $\mathcal{E}$ be an $\boldsymbol{I}$-equivariant vector bundle over $\mathfrak{N}'_{s_\alpha}$. We'll view it as a $\boldsymbol{I}$-equivariant coherent $\mathcal{O}$-module over $\mathfrak{N}$ or $\mathfrak{n} \times \mathfrak{F}$. The purpose of this section is to compute the element $\mathbf{r}_\Sigma(\mathcal{E})$.

First we assume that $S = \boldsymbol{T}$. Consider the diagrams (2.4.1) and (2.4.2). The coherent sheaf $\Gamma(\mathcal{E}) = \mathcal{E}_{\mathfrak{X}}$ is flat over $(\mathfrak{N}'_{s_\alpha})_{\mathfrak{X}}$. So it is also flat over $(\mathfrak{n}_{s_\alpha})_{\mathfrak{X}}$. However it is not flat over $\mathfrak{n}_{\mathfrak{X}}$. To compute $Lj^*(\mathcal{E}_{\mathfrak{X}})$ we need a resolution of $\mathcal{E}_{\mathfrak{X}}$ by flat $\mathcal{O}_{\mathfrak{n}_{\mathfrak{X}}}$-modules. For this it is enough to construct a resolution of $\mathcal{E}$ by flat $\mathcal{O}_{\mathfrak{n}}$-modules, and to apply induction to it. We have a closed immersion

$$\mathfrak{N}'_{s_\alpha} \subset \mathfrak{N}''_{s_\alpha}, \quad \mathfrak{N}''_{s_\alpha} = \mathfrak{n} \times \mathfrak{F}_{s_\alpha}.$$

The Koszul resolution of $\mathcal{O}_{\mathfrak{N}'_{s_\alpha}}$ by locally-free $\mathcal{O}_{\mathfrak{N}''_{s_\alpha}}$-modules is the complex

$$\Lambda_{\mathfrak{N}''_{s_\alpha}}(\alpha) = \left\{ \mathcal{O}_{\mathfrak{N}''_{s_\alpha}}\langle t+\alpha\rangle \to \mathcal{O}_{\mathfrak{N}''_{s_\alpha}} \right\}$$

situated in degrees $[-1, 0]$. We may assume that

$$\mathcal{E} = \mathcal{O}_{\mathfrak{n}_{s_\alpha}} \boxtimes \mathcal{F},$$

where $\mathcal{F}$ is a $\boldsymbol{I}$-equivariant locally free $\mathcal{O}_{\mathfrak{F}_{s_\alpha}}$-module. Set

$$\mathcal{E}' = \mathcal{O}_{\mathfrak{n}} \boxtimes \mathcal{F}.$$

It is a $\boldsymbol{I}$-equivariant locally free $\mathcal{O}_{\mathfrak{N}''_{s_\alpha}}$-module whose restriction to $\mathfrak{N}'_{s_\alpha}$ is equal to $\mathcal{E}$. We have

$$\begin{aligned}\mathbf{r}_\Sigma(\mathcal{E}) &= (i_*)^{-1} Lj^*\Gamma(\mathcal{E}' \otimes_{\mathcal{O}_{\mathfrak{N}''_{s_\alpha}}} \Lambda_{\mathfrak{N}''_{s_\alpha}}(\alpha)),\\ &= (i_*)^{-1} j^*\Gamma(\mathcal{E}') - (i_*)^{-1} j^*\Gamma(\mathcal{E}'\langle t+\alpha\rangle).\end{aligned}$$

Since $S = \boldsymbol{T}$ we have $j^{-1}((\mathfrak{N}'_{s_\alpha})_{\mathfrak{X}}) = j^{-1}((\mathfrak{N}''_{s_\alpha})_{\mathfrak{X}})$. Thus, for each $\mathcal{O}_{\mathfrak{N}''_{s_\alpha}}$-module $\mathcal{F}$ we have $j^*\Gamma(\mathcal{F}) = j^*\Gamma(\mathcal{F}|_{\mathfrak{N}'_{s_\alpha}})$. This implies that

$$\mathbf{r}_\Sigma(\mathcal{E}) = (i_*)^{-1} j^*\Gamma(\mathcal{E}) - (i_*)^{-1} j^*\Gamma(\mathcal{E}\langle t+\alpha\rangle).$$

Next, observe that $\mathfrak{N}_{\mathfrak{X}}^S = \mathfrak{D}^S$. Thus the map

$$i : (\mathfrak{N}'_{s_\alpha})_{\mathfrak{X}}^S \to j^{-1}((\mathfrak{N}'_{s_\alpha})_{\mathfrak{X}})$$

is equal to the obvious inclusion

$$\mathfrak{D}^S_{s_\alpha} \subset \mathfrak{D}'_{s_\alpha}, \quad \mathfrak{D}'_{s_\alpha} = \mathfrak{D}_{s_\alpha} \cap (\mathfrak{F}^S \times \mathfrak{F}).$$

Now, we have an exact sequence of $\mathcal{O}_{\mathfrak{D}'_{s_\alpha}}$-modules

$$0 \to \mathcal{O}_{\mathfrak{D}'_{s_\alpha}}(0,-\alpha) \to \mathcal{O}_{\mathfrak{D}'_{s_\alpha}} \to \mathcal{O}_{\mathfrak{D}^S_{s_\alpha}} \to 0.$$

Therefore we have

$$\mathbf{r}_\Sigma(\mathcal{E}) = (1 - \mathcal{O}_{\mathfrak{N}^S_{\mathfrak{X}}}(\alpha+t,0))\,(1 - \mathcal{O}_{\mathfrak{N}^S_{\mathfrak{X}}}(0,-\alpha))^{-1}\mathcal{E}_{\mathfrak{X}}|_{\mathfrak{N}^S_{\mathfrak{X}}},$$

where $\mathcal{E}_{\mathfrak{X}}|_{\mathfrak{N}^S_{\mathfrak{X}}}$ is the restriction to $\mathfrak{N}^S_{\mathfrak{X}}$ of the induced sheaf $\mathcal{E}_{\mathfrak{X}}$ over $\mathfrak{N}_{\mathfrak{X}}$.

For any $S \subset \boldsymbol{T}$ we obtain in the same way the following formula, compare [VV, (2.4.6)].

2.4.7. Proposition. *We have*

$$\mathbf{r}_\Sigma(\mathcal{E}) = \begin{cases} (1 - \mathcal{O}_{\mathfrak{N}^S_{\mathfrak{X}}}(\alpha+t,0))\,(1 - \mathcal{O}_{\mathfrak{N}^S_{\mathfrak{X}}}(0,-\alpha))^{-1}\mathcal{E}_{\mathfrak{X}}|_{\mathfrak{N}^S_{\mathfrak{X}}} & \text{if } \theta_\alpha \neq 1, t, \\ (1 - \mathcal{O}_{\mathfrak{N}^S_{\mathfrak{X}}}(\alpha+t,0))\,\mathcal{E}_{\mathfrak{X}}|_{\mathfrak{N}^S_{\mathfrak{X}}} & \text{if } \theta_\alpha = 1 \neq t, \\ (1 - \mathcal{O}_{\mathfrak{N}^S_{\mathfrak{X}}}(0,-\alpha))^{-1}\mathcal{E}_{\mathfrak{X}}|_{\mathfrak{N}^S_{\mathfrak{X}}} & \text{if } \theta_\alpha = t \neq 1, \\ \mathcal{E}_{\mathfrak{X}}|_{\mathfrak{N}^S_{\mathfrak{X}}} & \text{if } \theta_\alpha = t = 1. \end{cases}$$

2.4.8. Multiplicativity of $\mathbf{r}_\Sigma$. Let $S \subset \boldsymbol{T}$ be a regular closed subgroup. We have

$$\mathfrak{M}^S = \mathrm{colim}_{w,u}\mathfrak{M}^S_{w,u}, \quad \mathbf{K}(\mathfrak{M}^S) = \mathrm{colim}_{w,u}\mathbf{K}(\mathfrak{M}^S_{w,u}) = \bigoplus_{\alpha,\beta}\mathbf{K}(\mathfrak{M}(\alpha,\beta)),$$

where $\mathfrak{M}(\alpha,\beta)$ is as in 2.4.1. We have also

$$\mathbf{K}(\mathfrak{N}^S_{\mathfrak{X}}) = \prod_\alpha \bigoplus_\beta \mathbf{K}(\mathfrak{M}(\alpha,\beta)).$$

Therefore the group $\mathbf{K}(\mathfrak{N}^S_{\mathfrak{X}})$ can be regarded as the completion of $\mathbf{K}(\mathfrak{M}^S)$. Note that $\mathfrak{M}(\alpha,\beta)$ is a closed subscheme of $\dot{\mathfrak{n}}(\alpha) \times \dot{\mathfrak{n}}(\beta)$ and the later is smooth and of finite type because S is regular. So $\mathbf{K}(\mathfrak{M}^S)$, $\mathbf{K}(\mathfrak{N}^S_{\mathfrak{X}})$ are both equipped with an associative convolution product. See Section 3.1 and the proof of the proposition below for details.

2.4.9. Proposition. *The map $\star$ yields a ring structure on $\mathbf{K}^{\boldsymbol{I}}(\mathfrak{N})$. If the group S is regular then the map $\mathbf{r}_\Sigma : \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}) \to \mathbf{K}^S(\mathfrak{N}^S_{\mathfrak{X}})_\Sigma$ is a ring homomorphism.*

Proof : Since the group S acts trivially on $\mathfrak{N}^S_{\mathfrak{X}}$ we have

$$\mathbf{K}^S(\mathfrak{N}^S_{\mathfrak{X}})_\Sigma = \mathbf{K}(\mathfrak{N}^S_{\mathfrak{X}}) \otimes \mathbf{R}^S_\Sigma.$$

The multiplication on the lhs is deduced by base change from the product on $\mathbf{K}(\mathfrak{N}^S_{\mathfrak{X}})$ mentioned above. It is enough to check that we have

$$\mathbf{r}_\Sigma(x \star y) = \mathbf{r}_\Sigma(x) \star \mathbf{r}_\Sigma(y), \quad \forall x, y.$$

Indeed, setting $S = \Sigma = \boldsymbol{T}$, this relation and 2.4.3 imply that $\mathbf{K}^{\boldsymbol{I}}(\mathfrak{N})$ is a subring of $\mathbf{K}(\mathfrak{N}^S_{\mathfrak{X}}) \otimes \mathbf{R}^S_S$.

Fix $v, w \in \tilde{W}$. Let $u \in \tilde{W}$ be as in 2.3.6. Fix $\mathcal{E} \in \boldsymbol{\mathcal{C}oh}^{\boldsymbol{I}}(\mathfrak{N}_w)$ and $\mathcal{F} \in \boldsymbol{\mathcal{C}oh}^{\boldsymbol{I}}(\mathfrak{N}_v)$. Recall that $\mathcal{E}$, $\mathcal{F}$ denote also the corresponding classes in $\mathbf{K}^{\boldsymbol{I}}(\mathfrak{N})$ and that $\mathcal{E} \star \mathcal{F}$ is the class of an $\boldsymbol{I}$-equivariant cohomologically bounded pseudo-coherent complex over $\mathfrak{N}_u$. Let us recall the construction of this complex. We'll regard $\mathcal{E}$ as an $\boldsymbol{I}$-equivariant coherent $\mathcal{O}_{\dot{\mathfrak{n}}_w}$-module and $\mathcal{F}$ as an $\boldsymbol{I}$-equivariant quasi-coherent $\mathcal{O}_{\mathfrak{n}\times\dot{\mathfrak{n}}_v}$-module. Consider the diagram (2.3.4) that we reproduce below for the comfort of the reader

$$\begin{tikzcd} \dot{\mathfrak{n}}_w & Y \arrow[l, "f_2"'] \arrow[r, "f_1"] & \dot{\mathfrak{n}}_u \\ & T. \arrow[u, "\nu"'] & \end{tikzcd}$$

The map f_2 is flat and we have

$$\mathcal{E} \star \mathcal{F} = R(f_1)_*(\mathcal{G}), \quad \mathcal{G} = f_2^*(\mathcal{E}) \overset{L}{\otimes}_Y \nu_*(\mathcal{F}_{(w)}).$$

We want to compute $\mathbf{r}_\Sigma(\mathcal{E} \star \mathcal{F})$. Fix an element $x \in \tilde{W}$. First, we consider the induced complex $(\mathcal{E} \star \mathcal{F})_{(x)}$ over $(\dot{\mathfrak{n}}_u)_{(x)}$. Under induction the maps f_1, f_2 yield flat morphisms

$$(\dot{\mathfrak{n}}_w)_{(x)} \xleftarrow{f_{2,(x)}} Y_{(x)} \xrightarrow{f_{1,(x)}} (\dot{\mathfrak{n}}_u)_{(x)}.$$

The induction is exact and it commutes with tensor products. Thus we have

$$(\mathcal{E} \star \mathcal{F})_{(x)} = R(f_{1,(x)})_*\big(f^*_{2,(x)}(\mathcal{E}_{(x)}) \overset{L}{\otimes}_{Y_{(x)}} (\nu_*\mathcal{F}_{(w)})_{(x)}\big).$$

Fix $y, z \in \tilde{W}$ such that the canonical isomorphisms $\dot{\mathfrak{n}}_{\mathfrak{F}} = \mathfrak{F} \times \dot{\mathfrak{n}}$ and $(\mathfrak{n} \times \dot{\mathfrak{n}})_{\mathfrak{F}} = \dot{\mathfrak{n}} \times \dot{\mathfrak{n}}$ yield inclusions

$$\lambda : (\dot{\mathfrak{n}}_w)_{(x)} \to \mathfrak{F}_x \times \dot{\mathfrak{n}}_y, \quad \mu : (\dot{\mathfrak{n}}_u)_{(x)} \to \mathfrak{F}_x \times \dot{\mathfrak{n}}_z, \quad \nu : (\mathfrak{n} \times \dot{\mathfrak{n}}_v)_{(y)} \to \dot{\mathfrak{n}}_y \times \dot{\mathfrak{n}}_z.$$

We put

$$\mathcal{G}' = \mu_*((\mathcal{E} \star \mathcal{F})_{(x)}), \quad \mathcal{E}' = \lambda_*(\mathcal{E}_{(x)}), \quad \mathcal{F}' = \nu_*(\mathcal{F}_{(y)}).$$

We have

$$\mathcal{G}' = R(\pi_2)_*\big(\pi_3^*(\mathcal{E}') \overset{L}{\otimes}_{Y'} \pi_1^*(\mathcal{F}')\big), \tag{2.4.5}$$

where $Y' = \mathfrak{F}_x \times \dot{\mathfrak{n}}_y \times \dot{\mathfrak{n}}_z$ and π_1, π_2, π_3 are the obvious projections

$$\begin{tikzcd} \dot{\mathfrak{n}}_y \times \dot{\mathfrak{n}}_z & Y' \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\pi_3"'] & \mathfrak{F}_x \times \dot{\mathfrak{n}}_z \\ & \mathfrak{F}_x \times \dot{\mathfrak{n}}_y. & \end{tikzcd}$$

As explained in 2.4.2 we can regard the complex $\mathcal{G}'$, which is supported on $\mathfrak{M}_{x,z}$, as a complex over $\dot{\mathfrak{n}}_x \times \mathfrak{F}_z$. Let $\mathcal{G}''$ denote the later. We have

$$\mathbf{r}_\Sigma(\mathcal{E} \star \mathcal{F}) = \boldsymbol{\gamma}_\Sigma(\mathcal{G}'').$$

Now we compute $\mathcal{G}''$. Applying the base change formula to (2.4.5) we obtain the following equality in $\mathbf{K}^S(\dot{\mathfrak{n}}_x \times \mathfrak{F}_z)$

$$\mathcal{G}'' = R(p_2)_*\big(p_3^*(\mathcal{E}'') \overset{L}{\otimes}_{Y''} p_1^*(\mathcal{F}'')\big). \tag{2.4.6}$$

Here $Y'' = \dot{\mathfrak{n}}_x \times \dot{\mathfrak{n}}_y \times \mathfrak{F}_z$ and p_1, p_2, p_3 are the projections

$$\begin{tikzcd} \dot{\mathfrak{n}}_y \times \mathfrak{F}_z & Y'' \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "p_3"'] & \dot{\mathfrak{n}}_x \times \mathfrak{F}_z \\ & \dot{\mathfrak{n}}_x \times \dot{\mathfrak{n}}_y. & \end{tikzcd}$$

Further $\mathcal{E}''$, $\mathcal{F}''$ are S-equivariant quasi-coherent $\mathcal{O}$-modules over $\dot{\mathfrak{n}}_x \times \dot{\mathfrak{n}}_y$, $\dot{\mathfrak{n}}_y \times \mathfrak{F}_z$ respectively which are characterized by the following properties

$$p_*(\mathcal{E}'') = \mathcal{E}', \quad \mathcal{F}'' = q_*(\mathcal{F}'), \quad \mathcal{E}'' \text{ is supported on } \mathfrak{M}_{x,y},$$

where p, q are the obvious maps

$$\mathfrak{F}_w \times \dot{\mathfrak{n}}_u \xleftarrow{\ p\ } \dot{\mathfrak{n}}_w \times \dot{\mathfrak{n}}_u \xrightarrow{\ q\ } \dot{\mathfrak{n}}_w \times \mathfrak{F}_u\ .$$

Now, recall that we must prove that the following formula holds in $\mathbf{K}^S(\mathfrak{N}^S_{\mathfrak{X}})_\Sigma$

$$\mathbf{r}_\Sigma(\mathcal{E} \star \mathcal{F}) = \mathbf{r}_\Sigma(\mathcal{E}) \star \mathbf{r}_\Sigma(\mathcal{F}).$$

Let i, j be as in 2.4.2. The lhs is

$$\mathbf{r}_\Sigma(\mathcal{E} \star \mathcal{F}) = \boldsymbol{\gamma}_\Sigma(\mathcal{G}'') = (i_*)^{-1} L j^*(\mathcal{G}''). \tag{2.4.7}$$

Let us describe the rhs. We'll abbreviate

$$N = \dot{\mathfrak{n}}^S, \quad M = \mathfrak{M}^S.$$

Both are regarded as ind-schemes of ind-finite type. Thus we have

$$\mathbf{K}(N) = \bigoplus_\alpha \mathbf{K}(\dot{\mathfrak{n}}(\alpha)), \quad \mathbf{K}(M) = \bigoplus_{\alpha,\beta} \mathbf{K}(\mathfrak{M}(\alpha,\beta)).$$

Note that $\mathbf{K}(M) = \mathbf{K}(N^2 \text{ on } M)$ for the inclusion $M \subset N^2$ given by

$$(x, \mathfrak{b}, \mathfrak{b}') \mapsto \big((x, \mathfrak{b}), (x, \mathfrak{b}')\big).$$

Given $a = 1, 2, 3$ let $q_a : N^3 \to N^2$ be the projection along the a-th factor. We define the convolution product on $\mathbf{K}(M)$ by

$$x \star y = R(q_2)_*\big(q_3^*(x) \overset{L}{\otimes}_{N^3} q_1^*(y)\big), \quad \forall x, y \in \mathbf{K}(M). \tag{2.4.8}$$

Note that N is a disjoint union of smooth schemes of finite type and that q_1, q_3 are flat maps. We'll use another expression for $\star$. For this, we write

$$F = \mathfrak{F}^S, \quad NF = N \times F, \quad N^2F = N \times N \times F.$$

The obvious projections below are flat morphisms

$$\begin{tikzcd} NF & N^2F \arrow[l, "\rho_1"'] \arrow[r, "\rho_2"] \arrow[d, "\rho_3"'] & NF \\ & N^2. & \end{tikzcd}$$

We have also $\mathbf{K}(M) = \mathbf{K}(NF \text{ on } M)$ for the inclusion $M \subset NF$ given by

$$(x, \mathfrak{b}, \mathfrak{b}') \mapsto \big(\mathfrak{b}, (x, \mathfrak{b}')\big).$$

The projection formula yields

$$x \star y = R(\rho_2)_*\big(\rho_3^*(x) \overset{L}{\otimes}_{N^2F} \rho_1^*(y)\big). \tag{2.4.9}$$

Note that N^2F is also a disjoint union of smooth schemes of finite type. Finally we must compute $\mathbf{r}_\Sigma(\mathcal{E})$ and $\mathbf{r}_\Sigma(\mathcal{F})$. Once again, as explained in 2.4.2, we must first regard $\mathcal{E}'$, $\mathcal{F}'$ as complexes of $\mathcal{O}$-modules over $\dot{\mathfrak{n}}_x \times \mathfrak{F}_y$, $\dot{\mathfrak{n}}_y \times \mathfrak{F}_z$ respectively, and then we apply the map $(i_*)^{-1} \circ Lj^*$ to their class in K-theory.

Now, by (2.4.6), (2.4.7) and (2.4.9) we are reduced to prove the following equality

$$\begin{aligned} &(i_*)^{-1} Lj^* R(p_2)_*\big(p_3^*(\mathcal{E}'') \overset{L}{\otimes}_{Y''} p_1^*(\mathcal{F}'')\big) = \\ &= R(\rho_2)_*\big(\rho_3^*(i'_*)^{-1} L(j')^*(\mathcal{E}'') \overset{L}{\otimes}_{N^2F} \rho_1^*(i_*)^{-1} Lj^*(\mathcal{F}'')\big). \end{aligned}$$

Here i, i', j and j' are the obvious inclusions in the following diagram

$$\begin{tikzcd} \dot{\mathfrak{n}}^S \times \mathfrak{F}^S \arrow[r, "i"] & \dot{\mathfrak{n}}^S \times \mathfrak{F} \arrow[r, "j"] & \dot{\mathfrak{n}} \times \mathfrak{F} \\ \dot{\mathfrak{n}}^S \times \dot{\mathfrak{n}}^S \arrow[r, "i'"] \arrow[u, "q"] & \dot{\mathfrak{n}}^S \times \dot{\mathfrak{n}} \arrow[r, "j'"] \arrow[u, "q"] & \dot{\mathfrak{n}} \times \dot{\mathfrak{n}}. \arrow[u, "q"] \end{tikzcd}$$

This is an easy consequence of the base change and of the projection formula.

□

2.5. Double affine Hecke algebras.

2.5.1. Definitions. First, let us introduce the following notation : given any commutative ring $\mathbf{A}$ we'll write $\mathbf{A}_t = \mathbf{A}[t, t^{-1}]$ and $\mathbf{A}_{q,t} = \mathbf{A}[q, q^{-1}, t, t^{-1}]$. Recall that G is a simple, connected and simply connected linear group over $\mathbb{C}$. The double affine Hecke algebra (=DAHA) associated to G is the associative $\mathbb{Z}_{q,t}$-algebra $\mathbf{H}$ with 1 generated by the symbols T_w, X_λ with $w \in \tilde{W}$, $\lambda \in \tilde{\mathbf{X}}$ such that the T_w's satisfy the braid relations of $\tilde{W}$ and such that

$$\begin{aligned} &X_\delta = q, \quad X_\mu X_\lambda = X_{\lambda+\mu}, \quad (T_{s_\alpha} - t)(T_{s_\alpha} + 1) = 0, \\ &X_\lambda T_{s_\alpha} - T_{s_\alpha} X_{\lambda - r\alpha} = (t-1) X_\lambda (1 + X_{-\alpha} + ... + X_{-\alpha}^{r-1}) \quad \text{if } \langle \lambda, \check{\alpha} \rangle = r \geq 0. \end{aligned}$$

Here α is any simple affine root. Let $\mathbf{H}^f \subset \mathbf{H}$ be the subring generated by t and the T_w's with $w \in W$. Let $\mathbf{R} \subset \mathbf{H}$ be the subring generated by $\{X_\lambda\,;\,\lambda \in \tilde{\mathbf{X}}\}$. To avoid confusions we may write $\mathbf{R}_X = \mathbf{R}$. Finally let $\mathbf{R}_Y \subset \mathbf{H}$ be the subring generated by $\{Y_{\check\lambda}\,;\,\check\lambda \in \tilde{\mathbf{Y}}\}$, where $Y_{\check\lambda} = T_{\xi_{\check\lambda_1}} T_{\xi_{\check\lambda_2}}^{-1}$ with $\check\lambda = \check\lambda_1 - \check\lambda_2$ and $\check\lambda_1$, $\check\lambda_2$ dominant. The following fundamental result has been proved by Cherednik. We'll refer to it as the PBW theorem for $\mathbf{H}$.

2.5.2. Proposition. *The multiplication in* $\mathbf{H}$ *yields* $\mathbb{Z}_{q,t}$*-isomorphisms*

$$\mathbf{R} \otimes \mathbf{H}^f \otimes \mathbf{R}_Y \to \mathbf{H}, \quad \mathbf{R}_Y \otimes \mathbf{H}^f \otimes \mathbf{R} \to \mathbf{H}.$$

The $\mathbb{Z}_{q,t}$*-algebra* $\mathbf{H}^f$ *is isomorphic to the Iwahori-Hecke algebra (over the commutative ring* $\mathbb{Z}_t$*) associated to the Weyl group* W*. The rings* $\mathbf{R}$*,* $\mathbf{R}_Y$ *are the group-rings associated to the lattices* $\tilde{\mathbf{X}}$*,* $\tilde{\mathbf{Y}}$ *respectively.*

2.5.3. Remark. The algebra $\mathbf{H}$ is the one considered in [V]. It is denoted by the symbol $\hat{\mathbf{H}}$ in [VV]. Note that we have $X_{\omega_0} T_{s_0} X_{\omega_0}^{-1} = X_{\alpha_0} T_{s_0}^{-1}$. Thus $\mathbf{H}$ is a semidirect product $\mathbb{C}[X_{\omega_0}^{\pm 1}] \ltimes \mathcal{H}(\tilde{W}, \mathbf{X} \oplus \mathbb{Z}\delta)$ with the notation in [H, sec. 5]. Changing the lattices in the definition of $\mathbf{H}$ yields different versions of the DAHA whose representation theory is closely related to the representation theory of $\mathbf{H}$. These different algebras are said to be *isogeneous*. In this paper we'll only consider the case of $\mathbf{H}$ to simplify the exposition. For more details the reader may consult [VV, sec. 2.5].

Let $\boldsymbol{\mathcal{O}}(\mathbf{H})$ be the category of all right $\mathbb{C}\mathbf{H}$-modules which are finitely generated, locally finite over $\mathbf{R}$ (i.e., for each element m the $\mathbb{C}$-vector space $m\mathbf{R}$ is finite dimensional), and such that q, t act by multiplication by a complex number. It is an Abelian category. Any object has a finite length. For any module M in $\boldsymbol{\mathcal{O}}(\mathbf{H})$ we have

$$M = \bigoplus_{h \in \tilde{T}} M_h, \quad M_h = \bigcap_{\lambda \in \tilde{\mathbf{X}}} \bigcup_{r \geqslant 0} \{m \in M; m(X_\lambda - \lambda(h))^r = 0\}.$$

We'll call M_h the *h-weight subspace*. It is finite dimensional. Next, we set

$$\widehat{M} = \prod_h M_h.$$

The vector space $\widehat{M}$ is equipped with the product topology, the M_h's being equipped with the discrete topology. Note that $M \subset \widehat{M}$ is a dense subset. The $\mathbb{C}\mathbf{H}$-action on M extends uniquely to a continuous $\mathbb{C}\mathbf{H}$-action on $\widehat{M}$.

Fix an element $h = (s, \tau)$ of $\tilde{T}$, i.e., we let $s \in T_{\mathrm{cen}}$ and $\tau \in \mathbb{C}^\times_{\mathrm{rot}}$. For each $\zeta \in \mathbb{C}^\times_{\mathrm{qua}}$ we can form the corresponding tuple $(h, \zeta) \in \boldsymbol{T}$. Let $\boldsymbol{\mathcal{O}}_{h,\zeta}(\mathbf{H})$ be the full subcategory of $\boldsymbol{\mathcal{O}}(\mathbf{H})$ consisting of the modules M such that $q = \tau$, $t = \zeta$ and $M_{h'} = 0$ if h' is not in the orbit of h relatively to the $\tilde{W}$-action on $\tilde{T}$ in 2.1.10. Let $\tilde{W}$ act on $\boldsymbol{T}$ so that it acts on $\tilde{T}$ as in 2.1.10 and it acts trivially on $\mathbb{C}^\times_{\mathrm{qua}}$. We have

$$\boldsymbol{\mathcal{O}}(\mathbf{H}) = \bigoplus_{h,\zeta} \boldsymbol{\mathcal{O}}_{h,\zeta}(\mathbf{H}),$$

where (h, ζ) varies in a set of representatives of the $\tilde{W}$-orbits [VV, lem. 2.1.3, 2.1.6].

2.5.4. Geometric construction of the DAHA. We can now give a geometric construction of the $\mathbb{Z}_{q,t}$-algebra $\mathbf{H}$. First, let us introduced a few more notations. For each $\lambda \in \mathbf{X}$ we consider the following element of $\mathbf{K}^{\boldsymbol{I}}(\mathfrak{N})$

$$x_\lambda = \mathcal{O}_{\mathfrak{N}_e}(\lambda) = \mathcal{O}_{\mathfrak{N}_e}\langle\lambda\rangle.$$

Next, given a simple affine root $\alpha \in \tilde{\Pi}$ we have the good $\boldsymbol{I}$-equivariant subscheme $\mathfrak{N}'_{s_\alpha} \subset \mathfrak{N}$ introduced in 2.4.6. For each weights $\lambda, \mu \in \mathbf{X}$ we define the $\boldsymbol{I}$-equivariant coherent sheaf $\mathcal{O}_{\mathfrak{N}'_{s_\alpha}}(\lambda,\mu)$ over $\mathfrak{N}$ as the direct image of the $\boldsymbol{I}$-equivariant vector bundle $\mathcal{O}_{\mathfrak{N}'_{s_\alpha}}(\mu)\langle\lambda\rangle$ over $\mathcal{O}_{\mathfrak{N}'_{s_\alpha}}$, see 2.3.5($b$). Assume further that

$$\lambda + \mu = -\alpha, \quad \langle\lambda, \check{\alpha}\rangle = \langle\mu, \check{\alpha}\rangle = -1. \tag{2.5.1}$$

Then we consider the following element of $\mathbf{K}^{\boldsymbol{I}}(\mathfrak{N})$ given by

$$t_{s_\alpha} = -1 - \mathcal{O}_{\mathfrak{N}'_{s_\alpha}}(\lambda,\mu).$$

2.5.5. Lemma. *The element t_{s_α} is independent of the choice of λ, μ as above.*

Proof : It is enough to observe that if $\langle\lambda', \check{\alpha}\rangle = 0$ then the $\boldsymbol{I}$-equivariant line bundle $\mathcal{O}_{\mathfrak{N}'_{s_\alpha}}(\lambda', -\lambda')$ is trivial.

□

The assignment $\theta_\lambda \mapsto X_\lambda$ identifies $\mathbf{R}^{\tilde{T}}$ with the ring $\mathbf{R} = \mathbf{R}_X$, and $\mathbf{R}_t = \mathbf{R}^{\boldsymbol{T}}$ with the subring of $\mathbf{H}$ generated by t and $\mathbf{R}$. Now we can prove the main result of this section.

2.5.6. Theorem. *There is an unique ring isomorphism $\Phi : \mathbf{H} \to \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N})$ such that $T_{s_\alpha} \mapsto t_{s_\alpha}$ and $X_\lambda \mapsto x_\lambda$ for each $\alpha \in \tilde{\Pi}$, $\lambda \in \tilde{\mathbf{X}}$. Under Φ and the forgetting map $\mathbf{R}^{\boldsymbol{I}} = \mathbf{R}^{\boldsymbol{T}}$, the canonical (left) $\mathbf{R}^{\boldsymbol{I}}$-action on $\mathbf{K}^{\boldsymbol{I}}(\mathfrak{N})$ is identified with the canonical (left) $\mathbf{R}_t$-action on $\mathbf{H}$.*

Proof : First we prove that the assignment

$$T_{s_\alpha} \mapsto t_{s_\alpha}, \quad X_\lambda \mapsto x_\lambda, \quad \forall \alpha \in \tilde{\Pi}, \lambda \in \tilde{\mathbf{X}}$$

yields a $\mathbb{Z}_{q,t}$-algebra homomorphism

$$\Phi : \mathbf{H} \to \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}).$$

We must check that the elements t_{s_α}, x_λ satisfy the defining relations of $\mathbf{H}$. To do so let $S = \Sigma = \boldsymbol{T}$ and consider the group homomorphism

$$\mathbf{r}_S : \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}) \to \mathbf{K}^S(\mathfrak{N}^S_{\mathfrak{X}})_S.$$

Note that $\mathfrak{N}^S_{\mathfrak{X}} = \mathfrak{D}^S$, because $S = \boldsymbol{T}$. Thus we have a $\mathbf{R}^S$-module isomorphism

$$\begin{aligned} \mathbf{K}^S(\mathfrak{N}^S_{\mathfrak{X}}) &= \lim_w \mathrm{colim}_v \mathbf{K}^S\big((\mathfrak{D}_v)^S_{(w)}\big) \\ &= \lim_w \mathrm{colim}_u \mathbf{K}^S(\mathfrak{D}^S_{w,u}) \\ &= \prod_w \bigoplus_u \mathbf{R}^S \mathbf{x}_{w,u}. \end{aligned} \tag{2.5.2}$$

Here the symbol $\mathbf{x}_{w,u}$ stands for the fundamental class of the fixed point $(\mathfrak{b}_w, \mathfrak{b}_u)$, see 2.2.4($b$). The convolution product is $\mathbf{R}^S$-linear and is given by

$$\mathbf{x}_{v,w} \star \mathbf{x}_{y,z} = \begin{cases} \mathbf{x}_{v,z} & \text{if } w = y, \\ 0 & \text{else.} \end{cases}$$

Let λ, μ be as in (2.5.1). Under the isomorphism above we have

$$\begin{aligned} \mathcal{O}_{\mathfrak{N}'_{s_\alpha}}(\lambda,\mu)\mathfrak{x}|_{\mathfrak{D}^S} &= \mathcal{O}_{\mathfrak{D}^S_{s_\alpha}}(\lambda,\mu), \\ &= \sum_w^\infty (\theta_{w\lambda+w\mu}\mathbf{x}_{w,w} + \theta_{w\lambda+ws_\alpha\mu}\mathbf{x}_{w,ws_\alpha}), \\ &= \sum_w^\infty (\theta_{-w\alpha}\mathbf{x}_{w,w} + \mathbf{x}_{w,ws_\alpha}). \end{aligned}$$

Here the symbol $\sum^\infty$ denotes an infinite sum. Thus 2.4.7 yields

$$\begin{aligned} \mathbf{r}_S(1+t_{s_\alpha}) &= -\mathbf{r}_S(\mathcal{O}_{\mathfrak{N}'_{s_\alpha}}(\lambda,\mu)), \\ &= -(1-\mathcal{O}_{\mathfrak{D}^S_{s_\alpha}}(\alpha+t,0))(1-\mathcal{O}_{\mathfrak{D}^S_{s_\alpha}}(0,-\alpha))^{-1}\mathcal{O}_{\mathfrak{D}^S_{s_\alpha}}(\lambda,\mu), \\ &= \sum_w^\infty \frac{1-t\theta_{w\alpha}}{1-\theta_{w\alpha}}(\mathbf{x}_{w,w} - \mathbf{x}_{w,ws_\alpha}). \end{aligned} \tag{2.5.3}$$

By 2.4.4 we have also

$$\mathbf{r}_S(x_\lambda) = \mathbf{r}_S(\mathcal{O}_{\mathfrak{N}_e}\langle\lambda\rangle) = \sum_w^\infty \theta_{w\lambda}\mathbf{x}_{w,w}. \tag{2.5.4}$$

Using (2.5.3) and (2.5.4) the relations are reduced to a simple linear algebra computation which is left to the reader.

Next we prove that Φ is surjective. First note that $\mathbf{R}_t = \mathbf{R}^{\boldsymbol{T}} = \mathbf{R}^{\boldsymbol{I}}$. We have

$$\mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}) = \mathrm{colim}_w \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}_w), \quad \mathfrak{N}_w = \bigsqcup_{v\leqslant w} \mathring{\mathfrak{N}}_v, \quad \mathring{\mathfrak{N}}_v = \mathfrak{N} \cap (\mathfrak{n} \times \mathring{\mathfrak{F}}_v).$$

Further, we have $\mathbf{T}$-scheme isomorphisms

$$\mathring{\mathfrak{F}}_v = \mathfrak{n}^\circ_v, \quad \mathring{\mathfrak{N}}_v = \mathfrak{n}_v \times \mathring{\mathfrak{F}}_v = \mathfrak{n}.$$

In particular $\mathring{\mathfrak{N}}_v$ is an affine space. Let $\mathfrak{N}'_v$ be the Zariski closure of $\mathring{\mathfrak{N}}_v$ in $\mathfrak{N}$ and let $\mathbf{g}_v = \mathcal{O}_{\mathfrak{N}'_v}$, regarded as an element of $\mathbf{K}^{\boldsymbol{I}}(\mathfrak{N})$. The direct image by the inclusion $\mathfrak{N}_w \subset \mathfrak{N}$ identifies the $\mathbf{R}_t$-module $\mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}_w)$ with the direct summand

$$\bigoplus_{v\leqslant w} \mathbf{R}_t\, \mathbf{g}_v \subset \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}).$$

See [CG, sec. 7.6] for a similar argument for non-affine flags. On the other hand the PBW theorem for $\mathbf{H}$ implies that $\mathbf{H} = \bigoplus_{w\in\tilde{W}} \mathbf{R}_t T_w$ as a left $\mathbf{R}_t$-module. Set

$\mathbf{H}_w = \bigoplus_{v\leqslant w} \mathbf{R}_t T_v$. We must prove that Φ restricts to a surjective $\mathbf{R}_t$-module homomorphism $\mathbf{H}_w \to \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}_w)$ for each w. This is proved by induction on the length $l(w)$ of w. More precisely this is obvious if $l(w)=1$ and we know that

$$l(vw) = l(v) + l(w) \;\Rightarrow\; \mathbf{H}_v\,\mathbf{H}_w = \mathbf{H}_{vw},\; \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}_v) \star \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}_w) \subset \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}_{vw}).$$

Therefore we are reduced to prove that under the previous assumption we have

$$\mathbf{g}_v \star \mathbf{g}_w \equiv a_{v,w}\,\mathbf{g}_{vw},$$

with $a_{v,w}$ a unit of $\mathbf{R}_t$. Here the symbol $\equiv$ means an equality modulo lower terms for the Bruhat order. To do that we fix $S = \boldsymbol{T}$ and we consider the image of $\mathbf{g}_w$ by $\mathbf{r}_\Sigma$. It is, of course, to complicated to compute the whole expression, but we only need the terms $g^{(z)}_{y,yz}$ with $l(yz) = l(y) + l(z)$ in the sum below

$$\mathbf{r}_\Sigma(\mathbf{g}_x) = \sum_{y,z}^{\infty} g^{(x)}_{y,z}\,\mathbf{x}_{y,z},$$

because the coefficient a above is given by the following relation

$$g^{(w)}_{v,vw}\,g^{(v)}_{e,v} = a\,g^{(vw)}_{e,vw}.$$

The same computation as in 2.4.6 shows that

$$g^{(z)}_{y,yz} = \prod_{\alpha\in\tilde{\Delta}^\circ_z} \frac{1-t\theta_{y\alpha}}{1-\theta_{-y\alpha}}.$$

Now, recall that

$$l(vw) = l(v) + l(w) \;\Rightarrow\; \tilde{\Delta}^\circ_{vw} = \tilde{\Delta}^\circ_v \sqcup v(\tilde{\Delta}^\circ_w).$$

Thus we have $a_{v,w} = 1$.

Finally, since Φ restricts to a surjective $\mathbf{R}_t$-module homomorphism $\mathbf{H}_w \to \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}_w)$ for each w and both sides are free $\mathbf{R}_t$-modules of rank $l(w)$ necessarily Φ is injective.

The last claim of the theorem follows from 2.3.9(c).

□

2.5.7. Remark. By 2.3.9 the convolution product

$$\star : \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}) \otimes \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}) \to \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N})$$

is $\mathbf{R}^{\boldsymbol{I}}$-linear in the first variable. Recall that forgetting the group action yields an isomorphism $\mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}) \to \mathbf{K}^{\boldsymbol{T}}(\mathfrak{N})$. Further, since $\mathfrak{N}$ has a partition into affine cell a standard argument implies that the forgetting map gives an isomorphism

$$\mathbf{R}^S \otimes_{\mathbf{R}^{\boldsymbol{T}}} \mathbf{K}^{\boldsymbol{T}}(\mathfrak{N}) = \mathbf{K}^S(\mathfrak{N})$$

for each closed subgroup $S \subset \boldsymbol{T}$. Thus the map $\star$ factors to a group homomorphism

$$\star : \mathbf{K}^S(\mathfrak{N}) \otimes \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}) \to \mathbf{K}^S(\mathfrak{N}).$$

The assignment $\theta_\lambda \mapsto X_\lambda$ identifies $\mathbf{R}^{\boldsymbol{T}}$ with the subring $\mathbf{R}_t \subset \mathbf{H}$ generated by t and the X_λ's. By 2.5.6 the group homomorphism above is identified, via the map Φ, with the right multiplication of $\mathbf{H}$ on $\mathbf{R}^S \otimes_{\mathbf{R}_t} \mathbf{H}$.

3. Classification of the simple admissible modules of the Double affine Hecke algebra

3.1. Constructible sheaves and convolution algebras.

The purpose of this section is to revisit the sheaf-theoretic analysis of convolution algebras in [CG, sec. 8.6] in a more general setting including the case of schemes locally of finite type. It is an expanded version of [V, sec. 6, app. B].

3.1.1. Convolution algebras and schemes locally of finite type. Let $N = \bigsqcup_{\alpha\in A} N(\alpha)$ be a disjoint union of smooth quasi-projective connected schemes. We'll assume that the set A is countable. We'll view N as an ind-scheme, by setting

$$N = \mathrm{colim}_{B\subset A} N(B), \quad N(B) = \bigsqcup_{\alpha\in B} N(\alpha),$$

where B is any finite subset of A. Let C be a quasi-projective scheme (possibly singular) and $\pi : N \to C$ be an ind-proper map. For each $\alpha, \beta \in A$ we set

$$M(\alpha,\beta) = N(\alpha) \times_C N(\beta)$$

(the reduced fiber product). It is a closed subscheme of $N(\alpha) \times N(\beta)$. The fiber product means indeed the reduced fiber product. Note that $N(\alpha)$, $M(\alpha,\beta)$ are complex varieties which can be equipped with their transcendental topology. The symbol $\mathbf{H}_*(.,\mathbb{C})$ will denote the Borel-Moore homology with complex coefficients. We'll view $M = \bigsqcup_{\alpha,\beta} M(\alpha,\beta)$ as an ind-scheme in the obvious way. We set

$$\mathbf{H}_*(M,\mathbb{C}) = \bigoplus_{\alpha,\beta} \mathbf{H}_*(M(\alpha,\beta),\mathbb{C}), \quad \widehat{\mathbf{H}}_*(M,\mathbb{C}) = \prod_{\alpha}\bigoplus_{\beta} \mathbf{H}_*(M(\alpha,\beta),\mathbb{C}).$$

We'll view $\widehat{\mathbf{H}}_*(M,\mathbb{C})$ as a topological $\mathbb{C}$-vector space in the following way

- $\bigoplus_\beta \mathbf{H}_*(M(\alpha,\beta),\mathbb{C})$ is given the discrete topology for each α,
- $\widehat{\mathbf{H}}_*(M,\mathbb{C})$ is given the product topology.

We also equip $\mathbf{H}_*(M,\mathbb{C})$ with a convolution product $\star$ as in [CG, sec. 8]. The following is immediate.

3.1.2. Lemma. *The multiplication on $\mathbf{H}_*(M,\mathbb{C})$ is bicontinuous and yields the structure of a topological ring on $\widehat{\mathbf{H}}_*(M,\mathbb{C})$.*

3.1.3. Remark. We may also consider the K-theory rather than the Borel-Moore homology. Since M, N are ind-schemes of ind-finite type we have

$$\mathbf{K}(N) = \bigoplus_{\alpha} \mathbf{K}(N(\alpha)), \quad \mathbf{K}(M) = \bigoplus_{\alpha,\beta} \mathbf{K}(M(\alpha,\beta)).$$

We'll also set

$$\widehat{\mathbf{K}}(M) = \prod_{\alpha}\bigoplus_{\beta} \mathbf{K}(M(\alpha,\beta)).$$

Thus $\widehat{\mathbf{K}}(M)$ is again a topological ring. The multiplication in $\mathbf{K}(M)$, $\widehat{\mathbf{K}}(M)$ is the convolution product associated with the inclusion $M \subset N^2$. It is defined as in (2.4.8). By [CG, thm. 5.11.11] the bivariant Riemann-Roch map yields a topological ring homomorphism

$$RR : \mathbb{C}\widehat{\mathbf{K}}(M) \to \widehat{\mathbf{H}}_*(M,\mathbb{C})$$

which maps $\mathbb{C}\widehat{\mathbf{K}}(M(\alpha,\beta))$ to $\mathbf{H}_*(M(\alpha,\beta),\mathbb{C})$ for each α, β. It is invertible if all $\mathbf{H}_*(M(\alpha,\beta),\mathbb{C})$'s are spanned by algebraic cycles.

3.1.4. Admissible modules over the convolution algebra. Let $\boldsymbol{\mathcal{D}}(C)^b_{\mathbb{C}\text{-c}}$ be the derived category of bounded complexes of constructible sheaves of $\mathbb{C}$-vector spaces over the quasi-projective scheme C. Given two complexes $\mathcal{L}$, $\mathcal{L}'$ in $\boldsymbol{\mathcal{D}}(C)^b_{\mathbb{C}\text{-c}}$ we'll abbreviate

$$\mathrm{Ext}^n(\mathcal{L},\mathcal{L}') = \mathrm{Hom}(\mathcal{L},\mathcal{L}'[n]), \quad \mathrm{Ext}(\mathcal{L},\mathcal{L}') = \bigoplus_{n\in\mathbb{Z}} \mathrm{Ext}^n(\mathcal{L},\mathcal{L}'),$$

where the homomorphisms are computed in the category $\boldsymbol{\mathcal{D}}(C)^b_{\mathbb{C}\text{-c}}$. Now, we set

$$\mathcal{C}_\alpha = \mathbb{C}_{N(\alpha)}[\dim(N(\alpha))], \quad \mathcal{L}_\alpha = \pi_*(\mathcal{C}_\alpha), \quad \forall \alpha \in A.$$

Each $\mathcal{L}_\alpha$ is a semi-simple complex by the decomposition theorem. Assume that there is a finite set $\mathcal{X}$ of irreducible perverse sheaves over C such that

$$\mathcal{L}_\alpha \simeq \bigoplus_{n\in\mathbb{Z}} \bigoplus_{\mathcal{S}\in\mathcal{X}} L_{\mathcal{S},\alpha,n} \otimes \mathcal{S}[n],$$

where $L_{\mathcal{S},\alpha,n}$ are finite-dimensional $\mathbb{C}$-vector spaces. We set

$$L_{\mathcal{S},\alpha} = \bigoplus_{n\in\mathbb{Z}} L_{\mathcal{S},\alpha,n}, \quad L_{\mathcal{S}} = \bigoplus_{\alpha\in A} L_{\mathcal{S},\alpha}, \quad L = \bigoplus_{\mathcal{S}\in\mathcal{X}} L_{\mathcal{S}}.$$

For each complexes $\mathcal{L}, \mathcal{L}', \mathcal{L}''$ the Yoneda product is a bilinear map

$$\mathrm{Ext}(\mathcal{L},\mathcal{L}') \times \mathrm{Ext}(\mathcal{L}',\mathcal{L}'') \to \mathrm{Ext}(\mathcal{L},\mathcal{L}'').$$

By [CG, lem. 8.6.1, 8.9.1] we have an algebra isomorphism

$$\widehat{\mathbf{H}}_*(M,\mathbb{C}) = \prod_\alpha \bigoplus_\beta \mathrm{Ext}(\mathcal{L}_\alpha,\mathcal{L}_\beta),$$

where the rhs is given the Yoneda product. We have the following decomposition as $\mathbb{C}$-vector spaces $\widehat{\mathbf{H}}_*(M,\mathbb{C}) \simeq R \oplus J$, where

$$R = \bigoplus_{\mathcal{S}\in\mathcal{X}} \mathrm{End}(L_{\mathcal{S}}), \quad J = \bigoplus_{\mathcal{S},\mathcal{T}\in\mathcal{X}} \bigoplus_{n>0} \mathrm{Hom}(L_{\mathcal{T}}, L_{\mathcal{S}}) \otimes \mathrm{Ext}^n(\mathcal{T},\mathcal{S}).$$

Further J is a nilpotent two-sided ideal of $\widehat{\mathbf{H}}_*(M,\mathbb{C})$ and the $\mathbb{C}$-algebra structure on $\widehat{\mathbf{H}}_*(M,\mathbb{C})/J$ is the obvious $\mathbb{C}$-algebra structure on R. Before to explain what is the topology on R recall the following basic fact.

3.1.5. Definition. (a) Let $\mathbf{A}$ be any ring and let M, N be $\mathbf{A}$-modules. The *finite topology* on $\mathrm{Hom}_{\mathbf{A}}(M,N)$ is the linear topology for which a basis of open neighborhoods for 0 is given by the annihilator of M', for all finite set $M' \subset M$. This is actually the topology induced on $\mathrm{Hom}_{\mathbf{A}}(M,N)$ from N^M (a product of topological spaces where N has the discrete topology).

(b) If $\mathbf{A}$ is a topological ring we'll say that a right $\mathbf{A}$-module is *admissible* (or *smooth*) if for each element m the subset $\{x \in \mathbf{A}; mx = 0\}$ is open.

Now we can formulate the following lemma.

3.1.6. Lemma. *The two-sided ideal $J \subset \widehat{\mathbf{H}}_*(M,\mathbb{C})$ is closed. The quotient topology on $\widehat{\mathbf{H}}_*(M,\mathbb{C})/J$ coincides with the finite topology on R.*

Therefore, the Jacobson density theorem implies that the set of simple admissible right representations of R is $\{L_{\mathcal{S}}; \mathcal{S} \in \mathcal{X}\}$, see e.g., [V, sec. B]. This yields the following.

3.1.7. Proposition. *The set of the simple admissible right $\widehat{\mathbf{H}}_*(M,\mathbb{C})$-modules is canonically identified the set $\{L_{\mathcal{S}};\ \mathcal{S} \in \mathcal{X}\}$.*

3.2. Simple modules in the category $\boldsymbol{\mathcal{O}}$.

This section reviews the classification of the simple modules in $\boldsymbol{\mathcal{O}}(\mathbf{H})$ from [V]. The main arguments are the same as in loc. cit., but the use of the concentration map simplifies the exposition. Note that $\boldsymbol{\mathcal{O}}(\mathbf{H})$ consists of *right* $\mathbb{C}\mathbf{H}$-modules. This specification is indeed irrelevant because the $\mathbb{Z}_{q,t}$-algebra $\mathbf{H}$ is isomorphic to its opposit algebra, see e.g., [C, thm. 1.4.4].

3.2.1. From $\boldsymbol{\mathcal{O}}(\mathbf{H})$ to modules over the convolution algebra of $\mathfrak{M}$. In this section we apply the construction from Section 3.1 in the following setting. Fix a regular closed subgroup $S \subset \boldsymbol{T}$. Following [KL] we define the set of the *topologically nilpotent elements* in $\tilde{\mathfrak{g}}$ by

$$\mathfrak{Nil} = \bigcup_{\mathfrak{b}\in\mathfrak{F}} \mathfrak{b}_{\mathrm{nil}}.$$

Let $N = \dot{\mathfrak{n}}^S$, $C = \mathfrak{Nil}^S$, and let $\pi : N \to C$ be the obvious projection. The ind-scheme M in 3.1.1 is given by $M = \mathfrak{M}^S$. It is an ind-scheme of ind-finite type. We'll use the notation from 2.4.8. Recall that

$$\mathbf{K}(\mathfrak{M}^S) = \bigoplus_{\alpha,\beta} \mathbf{K}(\mathfrak{M}(\alpha,\beta)), \quad \widehat{\mathbf{K}}(\mathfrak{M}^S) = \prod_{\alpha} \bigoplus_{\beta} \mathbf{K}(\mathfrak{M}(\alpha,\beta)).$$

Now we fix an element $(h,\zeta) = (s,\tau,\zeta)$ in $\boldsymbol{T}$, i.e., we have $h = (s,\tau) \in \tilde{T}$, $s \in T \times \mathbb{C}^\times_{\mathrm{cen}}$, $\tau \in \mathbb{C}^\times_{\mathrm{rot}}$ and $\zeta \in \mathbb{C}^\times_{\mathrm{qua}}$. Assume that $S = \langle (h,\zeta)\rangle$, i.e., we assume that S is the closed subgroup of $\boldsymbol{T}$ generated by the element (h,ζ). Let $\tilde{G}^h$ be the centralizer of the element h in the group $\tilde{G}$.

3.2.2. Definition. We'll say that the pair (τ,ζ) is *regular* if τ is not a root of 1 and $\tau^k \neq \zeta^m$ for each $m,k > 0$.

For each set X with a $\boldsymbol{T}$-action we'll abbreviate $X^{h,\zeta}$ for the fixed points subset $X^{(h,\zeta)}$. We have the following [V, lem. 2.13], [VV, lem. 2.4.1-2].

3.2.3. Proposition. *Assume that the pair (τ,ζ) is regular. The group $\langle (h,\zeta)\rangle$ is regular. The group $\tilde{G}^h$ is reductive and connected. The scheme $\mathfrak{Nil}^{h,\zeta}$ is of finite type and it consists of nilpotent elements of $\tilde{\mathfrak{g}}$. Further $\mathfrak{Nil}^{h,\zeta}$ contains only a finite number of $\tilde{G}^h$-orbits.*

This proposition is essentially straightforward, except for the connexity of the reductive group $\tilde{G}^h$. This is an affine analogue of a well-known result of Steinberg which says that the centralizer of a semi-simple element in a connected reductive group with simply connected derived subgroup is again connected. The proof of the connexity relies on a theorem of Kac and Peterson [KP] which says that a reductive

subgroup of $\tilde{G}$ is always conjugated to a subgroup of a proper Lévi subgroup of $\tilde{G}$. Since the proper Lévi subgroups of $\tilde{G}$ are reductive with simply connected derived subgroup, because $\tilde{G}$ is the maximal affine Kac-Moody group, the claim is reduced to the Steinberg theorem.

Therefore, if (τ, ζ) is regular then the scheme $\mathfrak{Nil}^{h,\zeta}$ is of finite type, the scheme $\mathfrak{M}^{h,\zeta}$ is locally of finite type, the homology group $\widehat{\mathbf{H}}_*(\mathfrak{M}^{h,\zeta}, \mathbb{C})$ is a topological ring by 3.1.1, and the simple admissible right $\widehat{\mathbf{H}}_*(\mathfrak{M}^{h,\zeta}, \mathbb{C})$-modules are labeled by the set of irreducible perverse sheaves over $\mathfrak{Nil}^{h,\zeta}$ which occur as a shift of a direct summand of the complex $\pi_*(\mathbb{C}_{\dot{\mathfrak{n}}^{h,\zeta}})$.

Set $\Sigma = \{(h, \zeta)\}$ and $S = \langle (h, \zeta) \rangle$. We'll abbreviate

$$\mathbf{r}_{h,\zeta} = \mathbf{r}_\Sigma, \quad \mathbf{R}_{h,\zeta} = \mathbf{R}_\Sigma^S.$$

Composing Φ, $\mathbf{r}_{h,\zeta}$ and the tensor product by the character

$$\chi_{h,\zeta} : \mathbf{R}_{h,\zeta} \to \mathbb{C}, \quad f \mapsto f(h, \zeta),$$

we get a $\mathbb{C}$-algebra homomorphism

$$\Phi_{h,\zeta} : \mathbb{C}\mathbf{H} \to \mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta}).$$

Note that $\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta})$ is a topological ring by 3.1.3 and that the bivariant Riemann-Roch map yields a topological ring homomorphism

$$RR : \mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta}) \to \widehat{\mathbf{H}}_*(\mathfrak{M}^{h,\zeta}, \mathbb{C}).$$

We'll write

$$\Psi_{h,\zeta} = RR \circ \Phi_{h,\zeta} : \mathbb{C}\mathbf{H} \to \widehat{\mathbf{H}}_*(\mathfrak{M}^{h,\zeta}, \mathbb{C}).$$

Throughout we'll use the following notation : for any ring homomorphism

$$\phi : \mathbf{A} \to \mathbf{B}$$

and for any (left or right) $\mathbf{B}$-module M let $\phi^\bullet(M)$ be the corresponding $\mathbf{A}$-module.

3.2.4. Proposition. *Assume that the pair (τ, ζ) is regular.*

(a) The map $\Phi_{h,\zeta} : \mathbb{C}\mathbf{H} \to \mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta})$ has a dense image.

(b) The map $RR : \mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta}) \to \widehat{\mathbf{H}}_*(\mathfrak{M}^{h,\zeta}, \mathbb{C})$ is an isomorphism.

(c) The pull-back by the composed map $\Psi_{h,\zeta} = RR \circ \Phi_{h,\zeta}$ gives a bijection between the set of simple right $\mathbb{C}\mathbf{H}$-modules in $\boldsymbol{\mathcal{O}}_{h,\zeta}(\mathbf{H})$ and the set of simple admissible right $\widehat{\mathbf{H}}_*(\mathfrak{M}^{h,\zeta}, \mathbb{C})$-modules.

The proof of 3.2.4 is given in 3.2.7 below. Before this we need more material.

3.2.5. The regular representation of H. First we define a right representation of $\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta})$ on $\mathbf{K}(\mathfrak{N}^{h,\zeta})$. We'll use the same notation as in the previous subsection. In particular $S = \langle (h, \zeta) \rangle$ is a regular closed subgroup of $\boldsymbol{T}$. Recall that $\dot{\mathfrak{n}}^{h,\zeta}$ and $\mathfrak{M}^{h,\zeta}$ are both ind-scheme of ind-finite type, that $\dot{\mathfrak{n}}^{h,\zeta}$ is a disjoint union of smooth quasi-projective varieties, and that $\mathfrak{M}^{h,\zeta}$ is regarded as a closed subset of $(\dot{\mathfrak{n}}^{h,\zeta})^2$. The convolution product on $\mathbf{K}(\mathfrak{M}^{h,\zeta})$ is given by

$$x \star y = R(q_2)_*\big(q_3^*(x) \overset{L}{\otimes}_{(\dot{\mathfrak{n}}^{h,\zeta})^3} q_1^*(y)\big), \quad \forall x, y \in \mathbf{K}(\mathfrak{M}^{h,\zeta}),$$

where $q_a : (\dot{\mathfrak{n}}^{h,\zeta})^3 \to (\dot{\mathfrak{n}}^{h,\zeta})^2$ is the projection along the a-th factor for $a = 1, 2, 3$. The inclusion $\mathfrak{N} \subset \dot{\mathfrak{n}}$ yields an inclusion of ind-schemes $\mathfrak{N}^{h,\zeta} \subset \dot{\mathfrak{n}}^{h,\zeta}$. For each $x \in \mathbf{K}(\mathfrak{N}^{h,\zeta})$ and each $y \in \mathbf{K}(\mathfrak{M}^{h,\zeta})$ we define the following element in $\mathbf{K}(\mathfrak{N}^{h,\zeta})$

$$x \star y = R(p_1)_*(p_2^*(x) \overset{L}{\otimes}_{(\dot{\mathfrak{n}}^{h,\zeta})^2} y), \tag{3.2.1}$$

where $p_a : (\dot{\mathfrak{n}}^{h,\zeta})^2 \to \dot{\mathfrak{n}}^{h,\zeta}$ is the projection along the a-th factor for $a = 1, 2$. It is well-known that the map (3.2.1) defines a right representation of $\mathbf{K}(\mathfrak{M}^{h,\zeta})$ on $\mathbf{K}(\mathfrak{N}^{h,\zeta})$, see e.g., [CG].

3.2.6. Lemma. (a) *The right representation of* $\mathbf{K}(\mathfrak{M}^{h,\zeta})$ *on* $\mathbf{K}(\mathfrak{N}^{h,\zeta})$ *extends uniquely to an admissible right representation of* $\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta})$ *on* $\mathbf{K}(\mathfrak{N}^{h,\zeta})$.

(b) *The right* $\mathbb{C}\mathbf{H}$*-module* $\chi_{h,\zeta} \otimes_{\mathbf{R}_t} \mathbf{H}$ *belongs to* $\boldsymbol{\mathcal{O}}_{h,\zeta}(\mathbf{H})$.

(c) *There is an isomorphism of right* $\mathbf{H}$*-modules* $\chi_{h,\zeta} \otimes_{\mathbf{R}_t} \mathbf{H} \simeq \Phi^{\bullet}_{h,\zeta}(\mathbb{C}\mathbf{K}(\mathfrak{N}^{h,\zeta}))$.

Proof : The first claim is obvious, because we have

$$\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta}) = \prod_{\alpha} \bigoplus_{\beta} \mathbf{K}(\mathfrak{M}(\alpha, \beta)), \quad \mathbf{K}(\mathfrak{N}^{h,\zeta}) = \bigoplus_{\alpha} \mathbf{K}(\mathfrak{N}(\alpha)),$$
$$\mathbf{K}(\mathfrak{N}(\alpha)) \star \mathbf{K}(\mathfrak{M}(\alpha, \beta)) \subset \mathbf{K}(\mathfrak{N}(\beta)),$$

where $\mathfrak{N}(\alpha) = \mathfrak{N} \cap \dot{\mathfrak{n}}(\alpha)$. Part (b) is a standard computation, see e.g., [V]. Let us concentrate on part (c). Composing the map $\chi_{h,\zeta} : \mathbf{R}_{h,\zeta} \to \mathbb{C}$ with the canonical map $\mathbf{R}^{\boldsymbol{T}} \to \mathbf{R}_{h,\zeta}$ we may regard $\chi_{h,\zeta}$ as the one-dimensional $\mathbf{R}^{\boldsymbol{T}}$-module given by $f \mapsto f(h, \zeta)$. Recall that $\mathbf{R}_t = \mathbf{R}^{\boldsymbol{T}}$, see 2.5.7. The vector space $\chi_{h,\zeta} \otimes_{\mathbf{R}_t} \mathbf{H}$ has an obvious structure of right $\mathbf{H}$-module. The isomorphism 2.5.6 factors to a right $\mathbf{H}$-module isomorphism

$$\chi_{h,\zeta} \otimes_{\mathbf{R}_t} \mathbf{H} \to \chi_{h,\zeta} \otimes_{\mathbf{R}^{\boldsymbol{T}}} \mathbf{K}^{\boldsymbol{T}}(\mathfrak{N}).$$

We claim that there is a right $\mathbf{H}$-module isomorphism

$$\chi_{h,\zeta} \otimes_{\mathbf{R}^{\boldsymbol{T}}} \mathbf{K}^{\boldsymbol{T}}(\mathfrak{N}) \to \Phi^{\bullet}_{h,\zeta}(\mathbb{C}\mathbf{K}(\mathfrak{N}^{h,\zeta})).$$

To prove this, recall that composing the maps $\mathbf{r}_{h,\zeta}$ and $\chi_{h,\zeta}$ yields an algebra homomorphism

$$\mathbf{K}^{\boldsymbol{T}}(\mathfrak{N}) \to \mathbb{C}\mathbf{K}(\mathfrak{N}^{h,\zeta}_{\mathfrak{X}}) = \mathbb{C}\mathbf{K}(\mathfrak{M}^{h,\zeta}).$$

Thus we must construct a map $\mathbf{r} : \mathbf{K}^{\boldsymbol{T}}(\mathfrak{N}) \to \mathbb{C}\mathbf{K}(\mathfrak{N}^{h,\zeta})$ which intertwines the right $\star$-product of $\mathbf{K}^{\boldsymbol{T}}(\mathfrak{N})$ on itself, see 2.3.7, with the right $\star$-product of $\mathbb{C}\mathbf{K}(\mathfrak{M}^{h,\zeta})$ on $\mathbb{C}\mathbf{K}(\mathfrak{N}^{h,\zeta})$, see (3.2.1), relatively to the ring homomorphism

$$\chi_{h,\zeta} \circ \mathbf{r}_{h,\zeta} : \mathbf{K}^{\boldsymbol{T}}(\mathfrak{N}) \to \mathbb{C}\mathbf{K}(\mathfrak{M}^{h,\zeta}).$$

Further the map $\mathbf{r}$ should factor to an isomorphism

$$\chi_{h,\zeta} \otimes_{\mathbf{R}^{\boldsymbol{T}}} \mathbf{K}^{\boldsymbol{T}}(\mathfrak{N}) \to \mathbb{C}\mathbf{K}(\mathfrak{N}^{h,\zeta}).$$

Consider the following chain of inclusions

$$\mathfrak{n}^{h,\zeta} \times \mathfrak{F}^{h,\zeta} \xrightarrow{\ i\ } \mathfrak{n}^{h,\zeta} \times \mathfrak{F} \xrightarrow{\ j\ } \mathfrak{n} \times \mathfrak{F}. \tag{3.2.2}$$

Since $\mathfrak{n}$ is pro-smooth we can consider the map Lj^* in K-theory, see 1.5.18. Since $\mathfrak{F}$ is an ind-S-scheme of ind-finite type we can consider the map i_* in K-theory, see 1.5.19. Both maps are invertible, and the composed map is an isomorphism

$$(i_*)^{-1} \circ Lj^* : \mathbf{K}^S(\mathfrak{n} \times \mathfrak{F})_\Sigma \to \mathbf{K}^S(\mathfrak{n}^{h,\zeta} \times \mathfrak{F}^{h,\zeta})_\Sigma, \quad \Sigma = \{(h,\zeta)\}.$$

Now, recall that we have a good embedding $\mathfrak{N} \subset \mathfrak{n} \times \mathfrak{F}$. Thus, we obtain also in this way an isomorphism $\mathbf{K}^S(\mathfrak{N})_\Sigma \to \mathbb{C}\mathbf{K}(\mathfrak{N}^{h,\zeta})$. Composing it with the obvious map $\mathbf{K}^{\boldsymbol{T}}(\mathfrak{N}) \to \mathbf{K}^S(\mathfrak{N})_\Sigma$ it yields a map

$$\mathbf{r} : \mathbf{K}^{\boldsymbol{T}}(\mathfrak{N}) \to \mathbb{C}\mathbf{K}(\mathfrak{N}^{h,\zeta}).$$

We must check that the map $\mathbf{r}$ is compatible with the right $\star$-product, in the above sense. This is left to the reader. The proof is the same as the proof of 2.4.9. Compare (2.3.3), (2.4.3) with (3.2.1), (3.2.2).

□

3.2.7. Proof of 3.2.4. (a) The map $\Phi : \mathbf{H} \to \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N})$ is invertible by 2.5.6. The composed map

$$\mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}) \xrightarrow{\mathbf{r}_{h,\zeta}} \mathbf{K}^S(\mathfrak{M}^{h,\zeta})_{h,\zeta} = \mathbf{R}_{h,\zeta} \otimes \mathbf{K}(\mathfrak{M}^{h,\zeta}) \xrightarrow{\chi_{h,\zeta}} \mathbb{C}\mathbf{K}(\mathfrak{M}^{h,\zeta}) \subset \mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta})$$

has a dense image, because the image contains $\mathbb{C}\mathbf{K}(\mathfrak{M}(\alpha,\beta))$ for each α, β. Composing both maps we get $\Phi_{h,\zeta}$. Thus $\Phi_{h,\zeta}$ has a dense image.

(b) It is easy to see that $\mathbf{H}_*(\mathfrak{M}(\alpha,\beta),\mathbb{C})$ is spanned by algebraic cycles for all α, β. Therefore we have $\mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta}) \simeq \widehat{\mathbf{H}}_*(\mathfrak{M},\mathbb{C})$.

(c) By part (b) it is enough to check that the map $\Phi^\bullet_{h,\zeta}$ yields a bijection between the set of simple objects in $\boldsymbol{\mathcal{O}}_{h,\zeta}(\mathbf{H})$ and the set of simple admissible right representations of the topological $\mathbb{C}$-algebra $\mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta})$. Our proof uses the following lemma, which will be checked later on.

3.2.8. Lemma. *(a) For each $\lambda \in \tilde{\mathbf{X}}$ the operator of right multiplication by $\Phi_{h,\zeta}(X_\lambda)$ in any admissible right $\mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta})$-module is locally finite and its spectrum belongs to the set $\{{}^w\lambda(h); w \in \tilde{W}\}$.*

(b) If the elements $h, h' \in \tilde{T}$ are $\tilde{W}$-conjugate then the topological rings $\mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta})$, $\mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h',\zeta})$ and the homomorphisms $\Phi_{h,\zeta}$, $\Phi_{h',\zeta}$ are canonically identified.

The claim 3.2.4(c) is a corollary of 3.2.6 and 3.2.8. First, let M be a simple admissible right $\mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta})$-module. The right $\mathbf{H}$-module $\Phi^\bullet_{h,\zeta}(M)$ belongs to $\boldsymbol{\mathcal{O}}_{h,\zeta}(\mathbf{H})$ by 3.2.8(a). Further $\Phi^\bullet_{h,\zeta}(M)$ is a simple right $\mathbf{H}$-module. Indeed, since $\Phi_{h,\zeta}(\mathbb{C}\mathbf{H})$ is dense in $\mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta})$ by 3.2.4(a) and since M is admissible and simple as a right $\mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta})$-module, we have

$$x \star \Phi_{h,\zeta}(\mathbb{C}\mathbf{H}) = x \star \mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta}) = M, \quad \forall\, 0 \neq x \in M.$$

Thus M is a simple object of $\boldsymbol{\mathcal{O}}_{h,\zeta}(\mathbf{H})$.

Next, let L be a simple object of $\boldsymbol{\mathcal{O}}_{h,\zeta}(\mathbf{H})$. We claim that there is a simple admissible right $\mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta})$-module M such that $L \simeq \Phi^\bullet_{h,\zeta}(M)$. Indeed, since L

belongs to $\boldsymbol{\mathcal{O}}_{h,\zeta}(\mathbf{H})$ there is an element $h' \in \tilde{W} \cdot h$ such that the h'-weight subspace $L_{h'}$ is non-zero. Since L is simple, it is therefore a quotient of the right $\mathbb{C}\mathbf{H}$-module $\chi_{h',\zeta} \otimes_{\mathbf{R}_t} \mathbf{H}$. The later is isomorphic to $\Phi^{\bullet}_{h',\zeta}(\mathbb{C}\mathbf{K}(\mathfrak{N}^{h',\zeta}))$ by 3.2.6. Let J be the kernel of the quotient map $\Phi^{\bullet}_{h',\zeta}(\mathbb{C}\mathbf{K}(\mathfrak{N}^{h',\zeta})) \to L$. Hence, J is a right $\Phi_{h',\zeta}(\mathbb{C}\mathbf{H})$-submodule of $\mathbb{C}\mathbf{K}(\mathfrak{N}^{h',\zeta})$. Hence it is also a right $\mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h',\zeta})$-module because $\Phi_{h',\zeta}(\mathbb{C}\mathbf{H}) \subset \mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h',\zeta})$ is dense and $\mathbb{C}\mathbf{K}(\mathfrak{N}^{h',\zeta})$ is admissible. By 3.2.8(b) we can regard $\mathbb{C}\mathbf{K}(\mathfrak{N}^{h',\zeta})$ as a right $\mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta})$-module and J as a right $\mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta})$-submodule of $\mathbb{C}\mathbf{K}(\mathfrak{N}^{h',\zeta})$. Then the quotient $\mathbb{C}\mathbf{K}(\mathfrak{N}^{h',\zeta})/J$ is again a right $\mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta})$-submodule and we have

$$L \simeq \Phi^{\bullet}_{h,\zeta}(\mathbb{C}\mathbf{K}(\mathfrak{N}^{h',\zeta})/J)$$

as right $\mathbf{H}$-modules. Further, since L is a simple right $\mathbb{C}\mathbf{H}$-module the quotient $\mathbb{C}\mathbf{K}(\mathfrak{N}^{h',\zeta})/J$ is a simple admissible right $\mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta})$-module.

Finally if M, M' are admissible right $\mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta})$-modules such that $\Phi^{\bullet}_{h,\zeta}(M)$, $\Phi^{\bullet}_{h,\zeta}(M')$ are isomorphic as right $\mathbf{H}$-modules then M, M' are isomorphic as right $\mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta})$-modules, because they are isomorphic as right $\Phi_{h,\zeta}(\mathbb{C}\mathbf{H})$-modules and $\Phi_{h,\zeta}(\mathbb{C}\mathbf{H})$ is a dense subring of the topological ring $\mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta})$.

□

Proof of 3.2.8 : (a) For each $\lambda \in \tilde{\mathbf{X}}$ we have

$$\Phi_{h,\zeta}(X_\lambda) = x_\lambda = \mathcal{O}_{\mathfrak{N}_e}\langle\lambda\rangle = \mathcal{O}_{\mathfrak{N}_e}(\lambda) \in \mathbf{K}^{\boldsymbol{I}}(\mathfrak{N}).$$

Note that the set $\mathfrak{M}_e(\alpha,\beta) = \mathfrak{M}_e \cap \mathfrak{M}(\alpha,\beta)$ is empty if $\alpha \neq \beta$ and that it is the diagonal of $\dot{\mathfrak{n}}(\alpha)$ else. Recall that $S = \langle(h,\zeta)\rangle$. For each α let $\lambda_\alpha : S \to \mathbb{C}^\times$ be the character of the group S such that any element $g \in S$ acts on the equivariant line bundle $\mathcal{O}_{\mathfrak{F}(\alpha)}(\lambda)$ by fiberwise multiplication by the scalar $\lambda_\alpha(g)$. It is well-known that for each α there is an element $w \in \tilde{W}$ such that $\lambda_\alpha = ({}^w\lambda)|_S$. By 2.4.5 we have

$$\Phi_{h,\zeta}(X_\lambda) = \sum_\alpha^\infty \lambda_\alpha(h)\, \mathcal{O}_{\mathfrak{M}_e(\alpha,\alpha)}(\lambda,0) = \sum_\alpha^\infty \lambda_\alpha(h)\, \mathcal{O}_{\mathfrak{M}_e(\alpha,\alpha)}(0,\lambda) \in \mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta}).$$

Thus the operator of multiplication by $\Phi_{h,\zeta}(X_\lambda)$ in any admissible $\mathbb{C}\widehat{\mathbf{K}}(\mathfrak{M}^{h,\zeta})$-module is locally finite and its spectrum belongs to the set $\{{}^w\lambda(h); w \in \tilde{W}\}$. See [V, lem. 4.8] for details.

(b) Since h and h' are $\tilde{W}$-conjugate they are also $\tilde{G}$-conjugate. The group $\tilde{G}$ acts on $\mathfrak{M}$. This yields an ind-scheme isomorphism $\mathfrak{M}^{h,\zeta} \simeq \mathfrak{M}^{h',\zeta}$. The rest of the claim is obvious.

□

3.2.9. The classification theorem. We can now compose 3.1.7 with 3.2.4(c). We get the following theorem [V, thm. 7.6], [VV, prop. 2.5.1] whose proof uses the connexity of the reductive group $\tilde{G}^h$ in 3.2.3. To state the theorem we need more material. Assume that the pair (τ,ζ) is regular. As above, we'll write $S = \langle(h,\zeta)\rangle$.

Let $\mathcal{X}_{h,\zeta}$ be the set of irreducible perverse sheaves over $\mathfrak{Nil}^{h,\zeta}$ which are direct summand (up to some shift) of the complex

$$\bigoplus_{\alpha}(\pi_{h,\zeta})_*\mathbb{C}_{\dot{\mathfrak{n}}(\alpha)}, \quad \pi_{h,\zeta} : \dot{\mathfrak{n}}^{h,\zeta} = \bigsqcup_{\alpha}\dot{\mathfrak{n}}(\alpha) \to \mathfrak{Nil}^{h,\zeta}.$$

Here the map $\pi_{h,\zeta}$ is the obvious projection. There is a finite number of $\tilde{G}^h$-orbits in $\mathfrak{Nil}^{h,\zeta}$. For each closed point $x \in \mathfrak{Nil}^{h,\zeta}$ let $A(h,\zeta,x)$ be the group of connected components of the isotropy subgroup of x in $\tilde{G}^h$. The group $A(h,\zeta,x)$ acts in an obvious way on the homology space

$$H_*(\pi_{h,\zeta}^{-1}(x),\mathbb{C}) = \bigoplus_{\alpha} H_*(\pi_{h,\zeta}^{-1}(x)\cap\dot{\mathfrak{n}}(\alpha),\mathbb{C}).$$

Let $\mathrm{Irr}(A(h,\zeta,x))$ be the set of irreducible representations of the finite group $A(h,\zeta,x)$. Each representation in $\mathrm{Irr}(A(h,\zeta,x))$ can be regarded as a $\tilde{G}^h$-equivariant irreducible local system over the $\tilde{G}^h$-orbit O of x. Therefore we may regard $\mathcal{X}_{h,\zeta}$ as a set of pairs (x,χ) in $\bigsqcup_x \mathrm{Irr}(A(h,\zeta,x))$.

3.2.10. Theorem. *Assume that (τ,ζ) is regular.*

(a) The set $\{\Psi^{\bullet}_{h,\zeta}(L_{\mathcal{S}}); \mathcal{S}\in\mathcal{X}_{h,\zeta}\}$ is the set of all simple objects in $\boldsymbol{\mathcal{O}}_{h,\zeta}(\mathbf{H})$.

(b) The set $\mathcal{X}_{h,\zeta}$ is identified with the set of pairs (x,χ) such that $\chi\in\mathrm{Irr}(A(h,\zeta,x))$ is a Jordan-Hölder factor of the $A(h,\zeta,x)$-module $H_*(\pi_{h,\zeta}^{-1}(x),\mathbb{C})$.

(c) The simple right $\mathbf{H}$-modules $\Psi^{\bullet}_{h,\zeta}(L_{x,\chi})$ and $\Psi^{\bullet}_{h',\zeta}(L_{x',\chi'})$ are isomorphic iff the triplets (h,x,χ) and (h',x',χ') are $\tilde{G}$-conjugate.

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