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► **To cite this version:**

Laurence Guillot, Carole Le Guyader. Extrapolation of vector fields using the infinity laplacian and with applications to image segmentation. *Communications in Mathematical Sciences*, 2009, 7 (2), pp.87-99. hal-00435898

HAL Id: hal-00435898

<https://hal.science/hal-00435898>

Submitted on 11 Dec 2017

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Extrapolation of Vector Fields Using the Infinity Laplacian and with Applications to Image Segmentation

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Abstract. In this paper, we investigate a new Gradient-Vector-Flow (GVF) ([38])-inspired static external force field for active contour models, deriving from the edge map of a given image and allowing to increase the capture range. Contrary to prior related works, we reduce the number of unknowns to a single one v by assuming that the expected vector field is the gradient field of a scalar function. The model is phrased in terms of a functional minimization problem comprising a data fidelity term and a regularizer based on the super norm of Dv . The minimization is achieved by solving a second order singular degenerate parabolic equation. A comparison principle as well as the existence/uniqueness of a viscosity solution together with regularity results are established. Experimental results for image segmentation with details of the algorithm are also presented.

1 Introduction

1.1 Motivations

Many of the well-known variational segmentation methods require a careful choice of the initial condition. One of the most famous variational methods to process this partition of the image is the active contour model introduced by Kass, Witkin, and Terzopoulos ([30]). It consists in evolving a parameterized curve so that it matches the object boundary. The shape taken by the curve through the process is related to an energy minimization, this energy comprising a data fitting term and a regularizer, and being non-convex. Thereof, we can only expect local minimizers, which, in practice, means that the contour to be deformed must be initialized near the object boundary. Cohen ([18]) has proposed a way to alleviate this constraint by adding an inflating/deflating force in the modelling, defined by $k\mathbf{n}$, \mathbf{n} denoting the unit inward normal to the curve and k , a constant. According to the sign of the constant k , the curve inflates

or deflates. Thereby, in practice, the contour to be deformed is either initialized inside the object, or it encloses the object of interest.

In [38], Xu and Prince address both the problems of initialization and slow and/or poor convergence near boundaries with strong concavities by introducing a new static external force called Gradient Vector Flow (GVF). The initialization constraint is removed, that is, initialization can be made inside, outside or across the object boundaries, and the front evolution is easily handled even in boundary concavities. The main idea behind this model is to increase the capture range of the external edge-map-related force field, and to make the contour evolve toward the desired boundaries, here where classical methods would fail to. Unlike classical active contours, the introduced external force does not derive from a potential function and cannot be computed straightforwardly from the image edge map. More precisely, the model cannot be phrased in terms of a unique functional minimization problem but is defined in two steps. In a first step, the external force (GVF) $\mathbf{w} = (u, v)^T$ is obtained by minimizing an energy functional in a variational framework. The corresponding Euler-Lagrange equations are computed and lead to solve a linear partial differential equation decoupled system by a gradient descent method. The second step consists then in replacing, in the dynamic snake equation, the classical potential force by the newly computed external force \mathbf{w} . This method motivated the following works. In [34], Paragios et al. propose to integrate this boundary spatial diffusion technique to the geodesic active contours ([16]). In [29], Jifeng et al. propose to improve the diffusion properties of the GVF force field. They obtain a new force by replacing the Laplacian operator used in the GVF model by its diffusion term in the normal direction that is, the normalized infinity Laplacian operator. Unlike the GVF model, their new field (called NGVF for GVF in the normal direction) is anisotropic. Furthermore, the NGVF is stable for bigger time steps, improves slightly segmentation results, and allows to detect more quickly long and thin concavities.

Our work is much motivated by [38] and [29]. We wanted to provide, in a rigorous mathematical framework, a new method to generate this external force field. Contrary to these prior works, we propose to reduce the number of unknowns to a single one, by assuming that the sought vector field is the gradient field of a scalar function. Also, the introduced minimization problem contains a data-fitting term related to the original GVF model and a regularizer that penalizes the super norm of the unknown gradient. Thus the problem becomes related to the absolutely minimizing Lipschitz extensions and to the infinity Laplacian.

The absolute minimal Lipschitz extension model was introduced by Aronsson in [2] (see also [1, 3, 4]) in the following way. Given $\Omega \subset \mathbb{R}^n$ a bounded, open and connected domain with sufficiently smooth boundary, and $b \in \mathcal{C}(\partial\Omega)$, solve

$$\inf_{u \in W^{1,\infty}(\Omega), u=b \text{ on } \partial\Omega} \|Du\|_{L^\infty(\Omega)}. \quad (1)$$

A minimizer of (1) is called an absolutely minimizing Lipschitz interpolant of $b|_{\partial\Omega}$ inside Ω . Aronsson proved the existence of an absolute minimal Lipschitz

extension and Jensen proved the uniqueness. Aronsson also derived the Euler-Lagrange equation governing the absolute minimizer in the sense of viscosity solutions:

$$\Delta_\infty u = D^2u(Du, Du) = 0 \quad \text{in } \Omega. \quad (2)$$

We refer to [5, 10, 20] for more details. The operator Δ_∞ is called the infinity Laplacian and solutions of (2) are said to be ∞ -harmonic. Jensen proved a comparison principle and an existence/uniqueness result of (2) for Lipschitz continuous boundary data (see [28]).

Before depicting our model, we briefly make a non-exhaustive review of some prior works related to AMLE and infinity Laplacian in the field of image processing.

1.2 Prior Related Works

As stressed by Caselles et al., the equation $\Delta_\infty u = D^2u(Du, Du) = 0$ was introduced in the field of computer vision as edge detector (see [37], [39]). It earlier appeared in the domain of edge enhancement (see [35]) and served as the basis of Canny edge detection [12]. In [17], Caselles et al. investigate the AMLE and the infinity Laplacian in the field of image processing with applications to the restoration of images. Motivated by prior applications devoted to coding ([13], [14]), they address the issue of interpolating data given on a set of points and/or curves in the plane. Another application, dedicated to shape metamorphism (the process which consists in evolving a source shape into a target shape by intermediate steps) is proposed by Cong et al. in [19] and makes use of the infinity Laplacian. Also, in [31], Mémoli et al. propose a new framework for brain warping using Minimizing Lipschitz Extensions. To finish, in [24], Elion and Vese aim at solving the (BV, G) decomposition model introduced by Meyer in [32]. In that purpose, Elion and Vese focus on an isotropic decomposition of the image $f \approx u + v$ with $v = \Delta P = \text{div}(DP)$ and $DP \in (L^\infty(\Omega))^2$.

The outline of the paper is as follows: Sect. 2 is devoted to the depiction of the model and the derivation of the associated evolution problem. Section 3 is dedicated to the theoretical study of the obtained parabolic problem. We first prove a comparison principle, then prove existence and uniqueness of a viscosity solution. Regularity results of this solution are also given. We conclude the paper with experimental results and integrate this new external force field in a segmentation problem. Details of the algorithm are also provided.

2 Depiction of the Model

Let Ω be a bounded open subset of \mathbb{R}^n , $\partial\Omega$ its boundary and let I be a given bounded image function defined by $I : \bar{\Omega} \rightarrow \mathbb{R}$. For the purpose of illustration we consider $n = 2$. Let g be an edge-detector map. The function g is applied to the norm of the image gradient, and satisfies the following properties: $g : [0, \infty[\rightarrow [0, \infty[$, $g(0) = 1$, g strictly decreasing, and $\lim_{r \rightarrow +\infty} g(r) = 0$. An example of such a function is $g : r \mapsto \frac{1}{1+r^2}$. We denote by $W = (w_1, w_2) = -Dg(\|DI\|)$ the

associated gradient vector field. In homogeneous regions, $\|DI\| \simeq 0$ so $g(\|DI\|)$ is almost equal to 1. On boundaries, $\|DI\|$ is large so $g(\|DI\|)$ is almost zero. Also, in homogeneous regions, W is almost the null vector. Along the boundaries, the vector field W points toward the middle of the edges (see such an example in Fig. 1).

We plan to extrapolate the vector field on the whole image domain in a variational framework. A majority of existing regularization functionals aims at minimizing the global variation of the unknown and thus provides little local control. In this work, we propose to minimize the super norm of the unknown gradient. This choice is also motivated by the fact that the Laplacian operator (that naturally appears in the GVF model) can be decomposed into the sum of the second derivative in the normal direction, and the second derivative in the tangent direction. The former component that is kept in the NGVF model weighs heavily in the extrapolation process and has good properties unlike the later component which proves to be parasitic particularly when dealing with thin and long concavities. Also, unlike prior related works, we reduce the number of unknowns to a single one by assuming that the expected vector field is the gradient vector field of a scalar function. We thus propose to minimize the following functional:

$$\inf_{v \in W^{1,\infty}(\Omega)} \int_{\Omega} \|Dv - W\|^2 \|W\|^2 dx + \mu \|Dv\|_{L^\infty(\Omega)}, \quad (3)$$

where $\mu > 0$ is a tuning parameter.

Remark 1. Functional (3) is defined on $W^{1,\infty}(\Omega)$. The domain Ω being bounded, the inclusion $L^\infty(\Omega) \subset L^2(\Omega)$ holds so $Dv \in L^2(\Omega)$.

Remark 2. If v is a minimizer of (3), so is $v + C$ where C denotes any real constant. This is not a problem since we are interested in the associated gradient vector field. If $v \in W^{1,\infty}(\Omega)$, v is Lipschitz continuous and thus, by Rademacher's theorem, differentiable almost everywhere.

To minimize the above energy, we make use of the absolutely minimizing Lipschitz extensions. Following the results on AMLE recalled in Sect. 1, we obtain the Euler-Lagrange equation satisfied by v if it minimizes (3) and solve it by gradient descent. More precisely, classically, in image processing, the equation is defined on a domain R of \mathbb{R}^2 (e.g., on the square $[0, 1] \times [0, 1]$). In this case, boundary conditions must be defined: Neumann boundary conditions on ∂R are well-suited to the image processing framework since it corresponds to the reflection of the data through the edges. Thus it is no longer necessary to define boundary values. Following [6] and [15], we propose to simplify the problem by working with periodic solutions. The function v , primarily defined on $[0, 1] \times [0, 1]$, is extended to \mathbb{R}^2 . First, by symmetry, we extend it to $[-1, 1] \times [-1, 1]$ and then in all of \mathbb{R}^2 by periodicity (see Sect. 3.3.1 from [6]). We thus obtain that $\forall h \in \mathbb{Z}^2, \forall x \in \mathbb{R}^2$,

$v(x+2h) = v(x)$. Also, we assume that the initial condition v_0 and the functions $x \mapsto w_k(x)$, $k = 1, 2$ are extended to \mathbb{R}^2 with the same periodicity.

Given $T > 0$, we then obtain the following problem:

$$\begin{cases} \frac{\partial v}{\partial t} = 2 \|W\|^2 \Delta v + 2 \langle D \|W\|^2, Dv \rangle - 2 \operatorname{div} (\|W\|^2 W) \\ \quad + \mu D^2 v \left(\frac{Dv}{|Dv|}, \frac{Dv}{|Dv|} \right) \text{ on } \mathbb{R}^2 \times (0, T), \\ = b(x) \Delta v - \langle d(x), Dv \rangle - h(x) + \mu D^2 v \left(\frac{Dv}{|Dv|}, \frac{Dv}{|Dv|} \right), \\ v(x, t = 0) = v_0(x) \text{ in } \mathbb{R}^2, \end{cases} \quad (4)$$

with $b : x \mapsto 2 \|W(x)\|^2$, $d : x \mapsto -2 D \|W\|^2(x)$, $h : x \mapsto 2 \operatorname{div} (\|W\|^2 W)(x)$, and with the assumptions $v_0 \in \mathcal{C}(\mathbb{R}^2) \cap W^{1,\infty}(\mathbb{R}^2)$, $b \in \mathcal{C}(\mathbb{R}^2)$ and bounded by ξ_b , $d \in \mathcal{C}(\mathbb{R}^2) \cap W^{1,\infty}(\mathbb{R}^2)$, bounded by ξ_d and with Lipschitz constant κ_d , $h \in \mathcal{C}(\mathbb{R}^2) \cap W^{1,\infty}(\mathbb{R}^2)$, bounded by ξ_h and with Lipschitz constant κ_h , and with $\langle \cdot, \cdot \rangle$ denoting the euclidean scalar product in \mathbb{R}^2 . We also assume that the mapping $\mathbb{R}^2 \ni x \mapsto b^{1/2}(x)$ is Lipschitz continuous on \mathbb{R}^2 with Lipschitz constant $\kappa_{b^{1/2}}$.

3 Theoretical Results

This problem falls within the framework of the theory of viscosity solutions. Indeed, we obtain a second order singular degenerate parabolic equation. The concept of viscosity solutions has been introduced in 1981 by Crandall and Lions ([22]). This theory was developed to study first-order partial differential equations of nondivergence form, typically, Hamilton-Jacobi equations. Later, the study of viscosity solutions was extended to second-order elliptic and parabolic equations (for a good introduction to the theory of viscosity solutions, we refer to Barles [8, 7], the article of Crandall, Ishii and Lions [21], Crandall, Lions [23], Ishii [26], and Ishii, Lions [27]). We also refer to the related work [9].

In our problem, the evolution equation in (4) can be rewritten in the form:

$$\frac{\partial v}{\partial t} + G(x, Dv, D^2v) = 0,$$

with $G : \mathbb{R}^2 \times \mathbb{R}^2 - \{0_{\mathbb{R}^2}\} \times \mathcal{S}^2$ (\mathcal{S}^2 being the set of symmetric 2×2 matrices equipped with its natural partial order) defined by:

$$\begin{aligned} G(x, p, X) &= \langle d(x), p \rangle + h(x) - b(x) \operatorname{trace}(X) - \mu \frac{p^T}{|p|} X \frac{p}{|p|}, \\ &= \langle d(x), p \rangle + h(x) - b(x) \operatorname{trace}(X) - \mu \operatorname{trace} \left(\frac{p \otimes p}{|p|^2} X \right), \\ &= c(x, p) + E(x, X) + F(p, X), \end{aligned}$$

with the following properties:

1. The operators $G, F : (p, X) \mapsto -\mu \operatorname{trace} \left(\frac{p \otimes p}{|p|^2} X \right)$ and $E : (x, X) \mapsto -b(x) \operatorname{trace}(X)$ are independent of v and are elliptic, i.e., $\forall X, Y \in \mathcal{S}^2, \forall p \in \mathbb{R}^2$,

$$\text{if } X \leq Y \text{ then } F(p, X) \geq F(p, Y). \quad (5)$$

The operators G, E , and F are therefore proper.

2. F is locally bounded on $\mathbb{R}^2 \times \mathcal{S}^2$, continuous on $\mathbb{R}^2 \setminus \{0_{\mathbb{R}^2}\} \times \mathcal{S}^2$, and

$$F^*(0, 0) = F_*(0, 0) = 0, \quad (6)$$

where F^* (resp. F_*) is the upper semicontinuous (usc) envelope (resp. lower semicontinuous (lsc) envelope) of F .

3. $c : \mathbb{R}^2 \times \mathbb{R}^2 \ni (x, p) \mapsto \langle d(x), p \rangle + h(x)$ is locally Lipschitz continuous in space and $\forall x, y \in \mathbb{R}^2 \times \mathbb{R}^2$,

$$|c(x, p) - c(y, p)| \leq (\kappa_d |p| + \kappa_h) |x - y|. \quad (7)$$

We start by proving a comparison principle that will be useful to prove the uniqueness of the viscosity solution of the considered problem.

Theorem 1 (Comparison principle). *Let $u \in USC(\mathbb{R}^2 \times [0, T])$, bounded, periodic (with the same periodicity as the initial condition of (4)), be a subsolution and $v \in LSC(\mathbb{R}^2 \times [0, T])$, bounded, periodic (with the same periodicity as the initial condition of (4)), be a supersolution of (4). Assume that $u_0(x) = u(x, 0) \leq v_0(x) = v(x, 0)$ in \mathbb{R}^2 , then $u \leq v$ in $\mathbb{R}^2 \times [0, T]$.*

Proof. This proof is rather classical. We follow the arguments of [21]. We first observe that for $\lambda > 0$, $\tilde{u} = u - \frac{\lambda}{T-t}$ is also a subsolution of (4) and

$$\tilde{u}_t + G_*(x, D\tilde{u}, D^2\tilde{u}) \leq -\frac{\lambda}{(T-t)^2} \leq -\frac{\lambda}{T^2}.$$

Since $u \leq v$ follows from $\tilde{u} \leq v$ in the limit $\lambda \rightarrow 0$, it will simply suffice to prove the comparison under the additional assumptions:

$$\begin{cases} \text{(i)} & u_t + G_*(x, Du, D^2u) \leq -\frac{\lambda}{T^2}. \\ \text{(ii)} & \lim_{t \rightarrow T} u(x, t) = -\infty. \end{cases} \quad (8)$$

Let us set $M = \sup_{\mathbb{R}^2 \times [0, T]} u(x, t) - v(x, t)$. We aim to show that $M \leq 0$. In this purpose, we argue by contradiction and assume that $M > 0$. We introduce the duplication function $f(x, y, t) = u(x, t) - v(y, t) - (4\varepsilon)^{-1} |x - y|^4$ and consider $M_0 = \sup_{\mathbb{R}^2 \times \mathbb{R}^2 \times [0, T]} \{u(x, t) - v(y, t) - (4\varepsilon)^{-1} |x - y|^4\}$, $\varepsilon > 0$. Obviously, $M_0 \geq M > 0$. Moreover, this supremum is reached owing to the bound above of u and $-v$, the fact that f is such that $\forall h \in \mathbb{Z}^2, f(x + 2h, y + 2h, t) = f(x, y, t)$, and (8)(ii). We denote by $(x_0, y_0, t_0) \in \mathbb{R}^2 \times \mathbb{R}^2 \times [0, T]$ a point of maximum. We first prove that $t_0 > 0$ for ε small enough and then rise a contradiction using Th. 8.3 from [21], which allows to conclude that $M \leq 0$. Consequently, $u \leq v$ in $\mathbb{R}^2 \times [0, T]$. \square

We now give an existence result using the classic Perron's method (see Sect. 4 from [21]).

We start by constructing a subsolution U^- . Let us set $U^- = \inf_{\mathbb{R}^2}(v_0) - Ct$ with $C = \xi_h$. U^- is twice differentiable in space, once differentiable in time, bounded, and periodic with the same periodicity as v_0 and U^- is a subsolution of (4). Similarly, $U^+ = \sup_{\mathbb{R}^2}(v_0) + Ct$ is a supersolution of (4). Obviously, $U^-(x, 0) \leq U^+(x, 0)$. We can define:

$$v = \sup \{w; w \text{ periodic with the same periodicity as } v_0, \\ \text{subsolution such that } U^- \leq w \leq U^+ \}.$$

In that case, Perron's method states that v is a periodic discontinuous solution of (4) with the same periodicity as v_0 . Clearly, the solution is bounded since U^+ is bounded. Also as v is a solution, v^* is a subsolution and v_* a supersolution so from the comparison principle $v^* \leq v_*$. But $v_* \leq v^*$ so $v^* = v_* = v$, which gives that v is continuous on $\mathbb{R}^2 \times [0, T)$.

Conclusion 1. *We have proved the existence and uniqueness of a bounded, periodic, continuous on $\mathbb{R}^2 \times [0, T)$ viscosity solution of (4).*

We now prove that a solution of (4) is Lipschitz continuous in space, and uniformly continuous in time.

Theorem 2 (Regularity results). *Let us assume that $\|Dv_0\|_{L^\infty(\mathbb{R}^2)} \leq B_0$ with $B_0 > 0$. Then the solution of (4) satisfies:*

$$\|Dv(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} \leq B(t),$$

with $B(t) = \kappa_h \frac{e^{\alpha t} - 1}{\alpha} + B_0 e^{\alpha t}$, and with $\alpha = 8\kappa_{b_{1/2}}^2 + \kappa_d$.

Proof. The function v is bounded, continuous on $\mathbb{R}^2 \times [0, T)$, and periodic with the same periodicity as v_0 . We set $\Phi^\varepsilon(x, y, t) = B(t) (|x - y|^2 + \varepsilon^2)^{\frac{1}{2}}$ and aim at proving that $v(x, t) - v(y, t) \leq \Phi^\varepsilon(x, y, t)$.

Let us set $M = \sup_{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2, t \in [0, T)} (v(x, t) - v(y, t) - \Phi^\varepsilon(x, y, t))$. We thus aim to show that $M \leq 0$. Once again, we argue by contradiction and assume that $M > 0$. So we conclude that $v(x, t) - v(y, t) \leq \Phi^\varepsilon(x, y, t)$ and letting ε tend to 0, one obtains:

$$v(x, t) - v(y, t) \leq B(t)|x - y|.$$

Exchanging x and y yields:

$$|v(x, t) - v(y, t)| \leq B(t)|x - y|. \quad \square$$

Theorem 3 (Regularity results). *The solution v is uniformly continuous in time.*

Proof. We proceed like in [25]. In a first time, we assume that v_0 is bounded, periodic, C^2 , and such that there exists C , $\|Dv_0\|_{L^\infty(\mathbb{R}^2)}, \|D^2v_0\|_{L^\infty(\mathbb{R}^2)} \leq C$. Let us set

$$C_1 = \sup_{x \in \mathbb{R}^2} (\zeta + E(x, D^2v_0) + F_*(Dv_0, D^2v_0), \zeta - E(x, D^2v_0) - F^*(Dv_0, D^2v_0))$$

with $\zeta = \xi_d \|Dv_0\|_{L^\infty(\mathbb{R}^2)} + \xi_h$. Let us also set $v^- = v_0 - C_1 t$ and $v^+ = v_0 + C_1 t$. It can be checked that v^- is a subsolution of (4) and v^+ is a supersolution. Then, there exists a unique solution v of (4) and, by the comparison principle, it yields:

$$\forall x \in \mathbb{R}^2, \forall t \in [0, T], |v(x, t) - v_0(x)| \leq C_1 t.$$

Letting $u(x, t) = v(x, t + h)$, we obtain that u is the solution of

$$\begin{cases} \frac{\partial u}{\partial t} + G(x, Du, D^2u) = 0 \\ u(x, t = 0) = v(x, h) \end{cases}.$$

Classical arguments (comparison principle) allow to conclude that $|u(x, t) - v(x, t)| \leq C_1 h$, that is $|v(x, t + h) - v(x, t)| \leq C_1 h$. So v is uniformly continuous in time. Then we assume that v_0 is only bounded, periodic and Lipschitz continuous, and use mollification (see Chap. IV from [11] and Sect. 2.5 from [6]). Using the first step of the proof, we obtain the result and the modulus of continuity of v which depends on B_0 . \square

Conclusion 2. *We have proved the existence and uniqueness of a viscosity solution of problem (4), bounded, periodic, continuous on $\mathbb{R}^2 \times [0, T)$, Lipschitz continuous in space so differentiable almost everywhere, and uniformly continuous in time.*

We now discretize the evolution equation. In the sequel, we set $\Omega \ni x = (x_1, x_2)$.

4 Experimental Results

Let Δx_1 and Δx_2 be the spatial steps, Δt be the time step and $(x_{1i}, x_{2j}) = (i\Delta x_1, j\Delta x_2)$ be the grid points, $1 \leq i \leq M$ and $1 \leq j \leq N$. For a function $\Psi : \Omega \rightarrow \mathbb{R}$, let $\Psi_{i,j}^n = \Psi(i\Delta x_1, j\Delta x_2, n\Delta t)$. To discretize (4), we use an explicit finite difference scheme as follows. Also, the problem is complemented by Neumann boundary conditions. For the discretization of the convection component, we refer to [36] (we have used the usual notations for the finite difference operators and the notation $d = (d_1, d_2)$).

$$\begin{aligned} v_{i,j}^{n+1} &= v_{i,j}^n + \Delta t b_{i,j} (D^{x_1 x_1} v_{i,j}^n + D^{x_2 x_2} v_{i,j}^n) \\ &\quad - \Delta t \left(\max((d_1)_{i,j}, 0) D_-^{x_1} v_{i,j}^n + \min((d_1)_{i,j}, 0) D_+^{x_1} v_{i,j}^n \right. \\ &\quad \left. + \max((d_2)_{i,j}, 0) D_-^{x_2} v_{i,j}^n + \min((d_2)_{i,j}, 0) D_+^{x_2} v_{i,j}^n \right) - \Delta t h_{i,j} \\ &\quad + \Delta t \mu \frac{D^{x_1 x_1} v_{i,j}^n (D^{x_1} v_{i,j}^n)^2 + 2D^{x_1} v_{i,j}^n D^{x_2} v_{i,j}^n D^{x_1, x_2} v_{i,j}^n + D^{x_2 x_2} v_{i,j}^n (D^{x_2} v_{i,j}^n)^2}{(D^{x_1} v_{i,j}^n)^2 + (D^{x_2} v_{i,j}^n)^2 + \varepsilon}. \end{aligned} \tag{9}$$

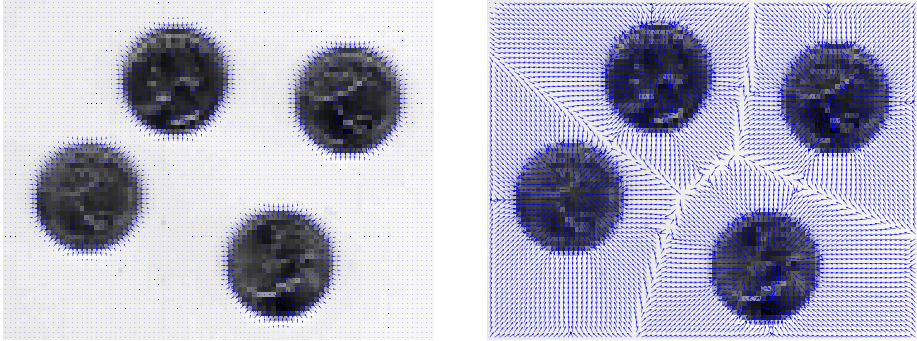


Fig. 1. On the left, depiction of the initial gradient vector field $W = -Dg(\|DI\|)$, on the right, the obtained vector field with our proposed approach ($\mu = 0.05$, $\Delta t = 0.1$)

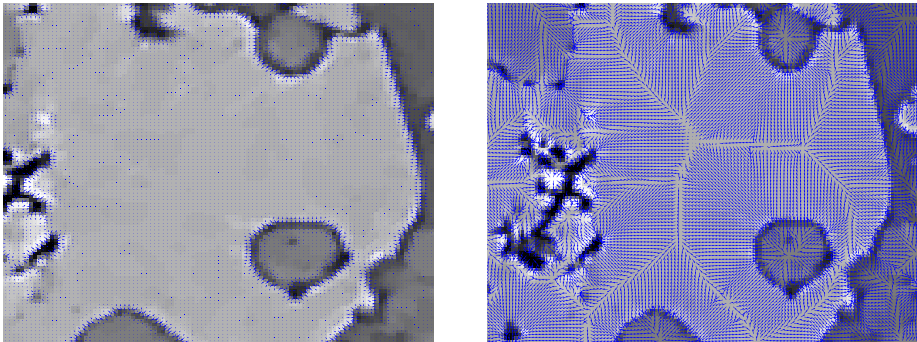


Fig. 2. On the left, depiction of the initial gradient vector field $W = -Dg(\|DI\|)$, on the right, the obtained vector field with our proposed approach ($\mu = 0.1$, $\Delta t = 0.1$)

4.1 Numerical Experimentations of Extrapolation

The experiments have been performed on a 2.21 GHz Athlon with 1.00 GB of RAM. In all our experiments, $\Delta x_1 = \Delta x_2 = 1$. We apply our model to real data and for each test, we provide a view of the initial gradient vector field $-Dg(\|DI\|)$ and a view of the extrapolated vector field. The initialization was made either by setting $v_0 \equiv 0$, or by setting $v_0 \equiv -g(\|DI\|)$. In all the tests we performed, it does not seem to influence the obtained result. The number of iterations as well as the computational time (order of the second) are similar for the three methods (GVF, NGVF and our proposed approach). Our method qualitatively performs in a way similar to the GVF and the NGVF: we increase the capture range of the vector field and we obtain downward components within the boundary concavity. Nevertheless, contrary to the the GVF and NGVF models, the method requires only one unknown. We start with an image taken from the Image Toolbox of Matlab (Fig. 1), and with an image showing a slice of Tuffeau

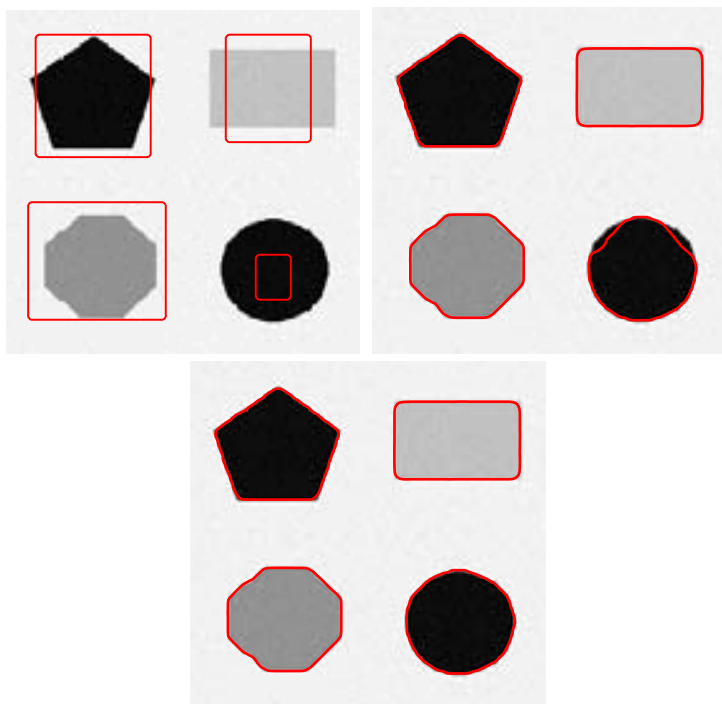


Fig. 3. Steps of the segmentation of the synthetic image taken from [34]

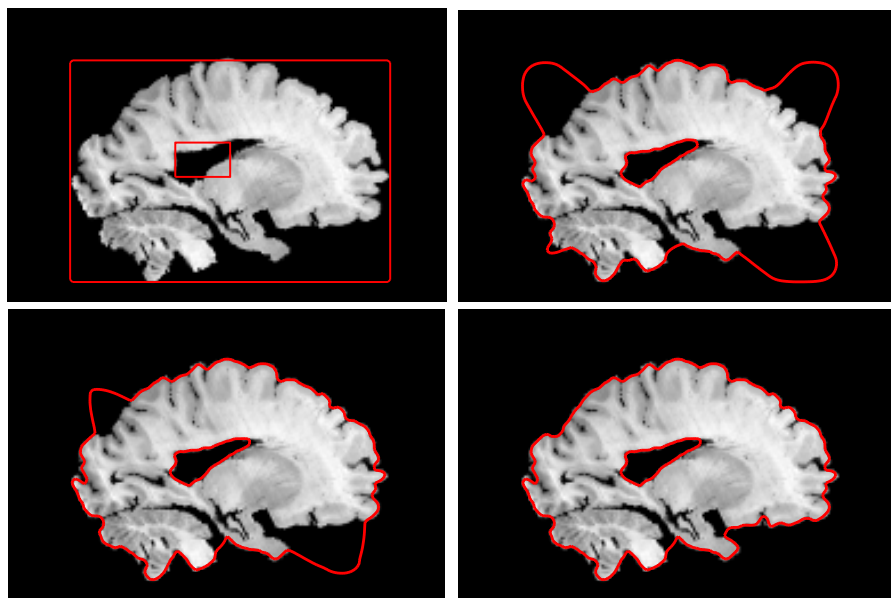


Fig. 4. Steps of the segmentation of the image of the brain

(Fig. 2, Courtesy of ISTO/ESRF). Our proposed approach performs well but seems to be sensitive to the textures of the objects contained in the image.

4.2 Application to Segmentation

This part is dedicated to segmentation and more precisely to the integration of this extrapolated vector field in the geodesic active contour model, in order to alleviate the constraint on the choice of the initial condition.

The geodesic active contour model, introduced by Caselles et al. in [16], is cast in the level set setting developed by Osher and Sethian in [33]. We propose, as done in [34], to replace $W = -Dg(\|DI\|)$ of the geodesic active contour model by the extrapolated vector field obtained with our proposed approach. To illustrate this, we propose an example taken from [34]. It demonstrates that the initial condition can be made of several contours selected inside, outside or across the boundaries of interest, provided the initial curves contain part of the skeleton of the extrapolated vector field. The classical geodesic active contour model does not authorize this flexibility in the initialization step and therefore the method alone would fail to detect all the shapes. Of course, the proposed method cannot detect automatically interior contours but this drawback is overcome, still with the flexibility in the initialization step. We illustrate this remark with Fig. 4 that represents a slice of the brain (Courtesy of the Laboratory Of Neuro Imaging, UCLA).

5 Conclusion

This paper was devoted to the theoretical study of a new method to extrapolate vector fields using the infinity Laplacian and with applications to image processing. Contrary to prior related works, the number of unknowns is reduced to a single one. The problem is phrased in a variational framework and the Euler-Lagrange equation is then derived. It is solved using a gradient descent method, which leads to a parabolic problem that falls within the viscosity solution theory framework. The existence and uniqueness of a viscosity solution continuous in space and time, Lipschitz continuous in space and uniformly continuous in time is established. The theoretical study is complemented by several numerical experimentations, first dedicated to the extrapolation problem, and then extended to the segmentation problem. The experimentations show that the proposed approach performs well, even if in strong concavities the results are slightly less accurate than with the NGVF. The model is sensitive to the geometry of the boundaries and to the textures present in the images. In the segmentation framework, the introduction of this new force field allows to widen the choice of the initial condition.

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