

THE UBIQUITY OF GENERALIZED CLUSTER CATEGORIES

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ABSTRACT. Associated with a finite dimensional algebra of global dimension at most 2, a generalized cluster category was introduced in [Ami08]. It was shown to be triangulated, and 2-Calabi-Yau when it is Hom-finite. By definition, the cluster categories of [BMR⁺06] are a special case. In this paper we show that a large class of 2-Calabi-Yau triangulated categories, including those associated with elements in Coxeter groups from [BIRS09a], are triangle equivalent to generalized cluster categories. This was already shown for some special elements in [Ami08] and then more generally for c -sortable elements in [AIRT10].

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INTRODUCTION

Throughout this paper k is an algebraically closed field. Let Q be a finite quiver without oriented cycles. In [BMR⁺06], the cluster category \mathcal{C}_Q was defined to be the orbit category $\mathcal{D}^b(kQ)/\tau^{-}[1]$, where τ is the Auslander-Reiten translation in the bounded derived category $\mathcal{D}^b(kQ)$. The category \mathcal{C}_Q is **Hom**-finite, triangulated [Kel05], and 2-Calabi-Yau (2-CY for short), that is, there is a functorial isomorphism $D\mathbf{Hom}_{\mathcal{C}_Q}(X, Y) \simeq \mathbf{Hom}_{\mathcal{C}_Q}(Y, X[2])$, where $D = \mathbf{Hom}_k(-, k)$. A theory for a special kind of objects, called *cluster-tilting objects*, was developed in [BMR⁺06]. This work was motivated via [MRZ03] by the Fomin-Zelevinsky theory of cluster algebras [FZ02], where the cluster-tilting objects are the analogs of clusters.

Another category where a similar theory was developed is the category $\mathbf{mod} \Lambda$ of finite dimensional modules over a preprojective algebra Λ of Dynkin type [GLS06, GLS07a]. This category is **Hom**-finite and Frobenius. Moreover, it is stably 2-CY, that is, its stable category $\underline{\mathbf{mod}} \Lambda$ (which is triangulated) is 2-CY.

Much of the work on cluster categories from [BMR⁺06, BMR07, BMR08] has been generalized to the setting of 2-CY triangulated categories with cluster-tilting objects, and new results have been proved in the general setting ([IY08, KR08, KR07], and others). It is of interest to investigate such categories, both for developing new theory and for providing applications to new classes of cluster algebras. In particular, it is of interest to find classes of 2-CY triangulated categories with cluster-tilting objects. An important class is the stable categories $\underline{\mathcal{E}}_w$ of the Frobenius categories \mathcal{E}_w associated with elements w in Coxeter groups [BIRS09a], (see [GLS07b] for independent work when w is adaptable). This class contains both the cluster categories \mathcal{C}_Q and $\mathbf{mod} \Lambda$ discussed above as special cases (see [BIRS09a], [GLS07b]). In \mathcal{E}_w and $\underline{\mathcal{E}}_w$, there are *standard* cluster-tilting objects $T_{\mathbf{w}}$ associated with any reduced expression \mathbf{w} of w .

A new class of triangulated 2-CY categories was introduced in [Ami08]. They are *generalized cluster categories* $\mathcal{C}_{\bar{A}}$ associated with algebras \bar{A} of global dimension at most 2, rather than global dimension 1. In this case the orbit category $\mathcal{D}^b(\bar{A})/\tau^{-}[1]$ is not necessarily triangulated, so $\mathcal{C}_{\bar{A}}$ is defined to be its triangulated hull. If $\mathcal{C}_{\bar{A}}$ is **Hom**-finite, then it is triangulated 2-CY and \bar{A} is a cluster-tilting object in $\mathcal{C}_{\bar{A}}$.

A natural question is how the generalized cluster categories are related to the previous classes of **Hom**-finite triangulated 2-CY categories. It was already shown in [Ami08] that some classes of categories $\underline{\mathcal{E}}_w$ are equivalent to generalized cluster categories, including \mathcal{C}_Q and $\mathbf{mod} \Lambda$, where Λ is preprojective of Dynkin type. This result is extended to the case of c -sortable words in [AIRT10], with a similar choice for \bar{A} . One of the main results in this paper is the following: Each category $\underline{\mathcal{E}}_w$ associated with an element w in a Coxeter group is equivalent to a generalized cluster category $\mathcal{C}_{\bar{A}}$ for some algebra \bar{A} of global dimension at most 2 (Theorem 4.4).

Actually, we prove our main result in a more general setting: We start with a Frobenius category \mathcal{E} , which we assume to be **Hom**-finite, stably 2-CY and which has a cluster-tilting object T . We assume that the endomorphism algebra $\mathbf{End}_{\mathcal{E}}(T)$ is Jacobian and has a grading with certain properties. From these data we construct an algebra \bar{A} of global dimension at most 2 and a triangle equivalence $\mathcal{C}_{\bar{A}} \simeq \underline{\mathcal{E}}$ (Theorem 3.1) sending the canonical cluster-tilting object \bar{A} of $\mathcal{C}_{\bar{A}}$ to the cluster-tilting object T in \mathcal{E} . The algebra \bar{A} is constructed as the degree zero part of $\mathbf{End}_{\mathcal{E}}(T)$, and we show that $\mathbf{End}_{\mathcal{E}}(T)$ and $\mathbf{End}_{\mathcal{C}_{\bar{A}}}(\bar{A})$ are isomorphic algebras (Proposition 3.12). This is an important step in the proof of the equivalence. It is however not known in general if 2-CY categories are equivalent when they have cluster-tilting objects whose endomorphism algebras are isomorphic. The only general result known of this type is that if the quiver Q of $\mathbf{End}_{\mathcal{C}}(T)$ has no oriented cycles, where T is a cluster-tilting object in an algebraic 2-CY

category \mathcal{C} , then \mathcal{C} is triangle equivalent to the cluster category \mathcal{C}_Q [KR08]. A crucial step in this paper for proving the equivalence $\mathcal{C}_{\bar{A}} \simeq \underline{\mathcal{E}}$ is the construction of a triangle functor from $\mathcal{C}_{\bar{A}}$ to $\underline{\mathcal{E}}$, sending \bar{A} to T . This is done by first constructing a triangle functor from $\mathcal{D}^b(\bar{A})$ to $\underline{\mathcal{E}}$, with strong use of the Frobenius structure of \mathcal{E} .

It is also important to deal with the endomorphism algebra $\text{End}_{\mathcal{E}}(T)$ of the cluster-tilting object T in the Frobenius category \mathcal{E} , rather than only $\underline{\text{End}}_{\mathcal{E}}(T)$. We assume that this algebra is a graded frozen Jacobian algebra (see section 1 for definitions), with potential homogeneous of degree 1. Theorem 3.1 applies in particular to the categories \mathcal{E}_w associated to any element in a Coxeter group associated with a finite quiver without oriented cycles (see section 4).

The paper is organized as follows. In section 1 we recall some background material on cluster-tilting objects in 2-CY categories, on generalized cluster categories from [Ami08] and on Jacobian algebras from [DWZ08], together with the generalization to frozen Jacobian algebras given in [BIRS09b].

In section 2 we construct a special triangle (Proposition 2.7), which is useful for our construction of a functor from $\mathcal{C}_{\bar{A}}$ to $\underline{\mathcal{E}}$.

Section 3 is devoted to the proof of the triangle equivalence from $\mathcal{C}_{\bar{A}}$ to $\underline{\mathcal{E}}$ (Theorem 3.1). We first show that the global dimension of \bar{A} is at most 2. Then we construct our triangle functor from $\mathcal{C}_{\bar{A}}$ to $\underline{\mathcal{E}}$ using the special triangle from section 2, together with a universal property from [Kel05, Ami08]. Finally we show that our functor is an equivalence by using a criterion from [KR08].

In section 4 we apply the main theorem to prove that for any element w in a Coxeter group, the 2-CY triangulated category $\underline{\mathcal{E}}_w$ is triangle equivalent to some generalized cluster category, which was our original motivation (Theorem 4.4).

In section 5 we give two examples to illustrate our results. The first one is an illustration of Theorem 4.4. In the second one we use Theorem 3.1 to construct a triangle equivalence from a generalized cluster category $\mathcal{C}_{\bar{A}}$ to a category $\underline{\mathcal{E}}_w$ which sends the canonical cluster-tilting object \bar{A} to a cluster-tilting object T in $\underline{\mathcal{E}}_w$, where T is not associated to a reduced expression of w .

Notations. By a triangulated category we mean a k -linear triangulated category satisfying the Krull-Schmidt property. For all triangulated categories we will denote the shift functor by $[1]$. By a Frobenius category we mean an exact k -category with enough projectives and injectives, where the projectives and the injectives coincide. If \mathcal{E} is Frobenius, then the associated stable category $\underline{\mathcal{E}}$ is triangulated [Hap88] and is by definition algebraic. For an object T in an additive k -category, we denote by $\text{add}(T)$ the additive closure of T . For a k -algebra A , we denote by $\text{Mod } A$ the category of right A -modules and by $\text{mod } A$ the category of finitely presented right A -modules. We also denote by $\mathcal{D}(A)$ the derived category $\mathcal{D}(\text{Mod } A)$ and by $\mathcal{D}^b A$ the bounded derived category $\mathcal{D}^b(\text{mod } A)$. Let D be the usual duality $\text{Hom}_k(?, k)$. The tensor product $- \otimes -$, when not specified, is over the ground field k . For a quiver Q we denote by Q_0 its set of vertices, by Q_1 its set of arrows, by s the source map and by t the target map.

1. BACKGROUND

In this section we collect some background material relevant for this paper.

1.1. Cluster-tilting objects. Let \mathcal{C} be a k -category which is Hom-finite, that is, has finite dimensional homomorphism spaces over k . Assume that \mathcal{C} is either Frobenius stably 2-CY

(that is, its stable category is 2-CY) or triangulated 2-CY. Then an object T in \mathcal{C} is said to be *cluster-tilting* if

- (i) T is rigid, *i.e.* $\text{Ext}_{\mathcal{C}}^1(T, T) = 0$, and
- (ii) $\text{Ext}_{\mathcal{C}}^1(T, X) = 0$ implies that X is a summand of a finite direct sum of copies of T .

Note that when \mathcal{C} is Frobenius stably 2-CY, then any indecomposable projective-injective module is a summand of every cluster-tilting object. The finite dimensional algebras $\text{End}_{\mathcal{C}}(T)$, where \mathcal{C} is triangulated 2-CY, are called *2-CY-tilted algebras*.

Assume that $T = T_1 \oplus \dots \oplus T_n$ is a cluster-tilting object in a triangulated 2-CY category \mathcal{C} , where the T_i are indecomposable and pairwise not isomorphic. Then for each $i = 1, \dots, n$ there is a unique indecomposable object T_i^* not isomorphic to T_i , such that $T^* = (T/T_i) \oplus T_i^*$ is a cluster-tilting object [BMR⁺06], [IY08]. The new object T^* is called the *mutation* of T at T_i .

If $T = T_1 \oplus \dots \oplus T_n$ is a cluster-tilting object in a Frobenius stably 2-CY category, we can only mutate at the T_i which are not projective-injective.

When T_i^* is defined, there are *exchange sequences* if \mathcal{C} is Frobenius

$$0 \longrightarrow T_i^* \xrightarrow{f} B \xrightarrow{g} T_i \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow T_i \xrightarrow{f'} B' \xrightarrow{g'} T_i^* \longrightarrow 0$$

or *exchange triangles* if \mathcal{C} is triangulated

$$T_i^* \xrightarrow{f} B \xrightarrow{g} T_i \longrightarrow T_i^*[1] \quad \text{and} \quad T_i \xrightarrow{f'} B' \xrightarrow{g'} T_i^* \longrightarrow T_i[1]$$

where f, f' are minimal left $\text{add}(T/T_i)$ -approximations and g, g' are minimal right $\text{add}(T/T_i)$ -approximations. These sequences (or triangles) play an important role in the categorification of cluster algebras.

There is also a related kind of sequences investigated in [IY08].

Proposition 1.1. [Iyama-Yoshino] *Let as before \mathcal{C} be a Hom-finite Frobenius stably 2-CY category with a cluster-tilting object $T = T_1 \oplus \dots \oplus T_n$. For each $i = 1, \dots, n$, if T_i is not projective-injective, there are exact sequences*

$$0 \longrightarrow T_i^+ \xrightarrow{f} E \xrightarrow{g} T_i \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow T_i \xrightarrow{f'} E' \xrightarrow{g'} T_i^+ \longrightarrow 0$$

for some indecomposable object T_i^+ in \mathcal{C} , such that g (resp. g') is right almost split in $\text{add}(T)$ (resp. in $\text{add}((T/T_i) \oplus T_i^+)$) and f' (resp. f) is left almost split in $\text{add}(T)$ (resp. in $\text{add}((T/T_i) \oplus T_i^+)$).

The induced sequence $0 \longrightarrow T_i \xrightarrow{f'} E' \xrightarrow{fg'} E \xrightarrow{g} T_i \longrightarrow 0$ is called the *2-almost split sequence* associated with T_i .

There is a corresponding result when \mathcal{C} is triangulated. For any indecomposable direct summand T_i of a cluster-tilting object T , there are triangles

$$T_i^+ \xrightarrow{f} E \xrightarrow{g} T_i \longrightarrow T_i^+[1] \quad \text{and} \quad T_i \xrightarrow{f'} E' \xrightarrow{g'} T_i^+ \longrightarrow T_i[1]$$

where the maps f, f' are left almost split and g, g' are right almost split. For cluster categories, it was shown in [BMR⁺06] that these triangles coincide with the exchange triangles. More generally, they clearly coincide with the exchange triangles if and only if there are no loops in the quiver of $\text{End}_{\mathcal{C}}(T)$.

Using the existence of 2-almost split sequences in a **Hom**-finite Frobenius category, we can construct minimal projective and injective resolutions of simple modules over $\text{End}_{\mathcal{C}}(T)$. Note that it already follows from [Iya07, 2.5.3] that $\text{End}_{\mathcal{C}}(T)$ has global dimension at most 3.

Proposition 1.2. *Let \mathcal{C} be a **Hom**-finite Frobenius stably 2-CY category with a cluster-tilting object $T = T_1 \oplus \dots \oplus T_n$, where the T_i are indecomposable and pairwise non isomorphic. Let $B := \text{End}_{\mathcal{C}}(T)$ be the endomorphism algebra of T , and let Q be the quiver of B , which is then isomorphic to kQ/I for some admissible ideal I in kQ . For each $i = 1, \dots, n$, such that T_i is not projective-injective, denote by S_i the simple B -module $\text{Hom}_{\mathcal{C}}(T, T_i)/\text{Rad}(\text{Hom}_{\mathcal{C}}(T, T_i))$. Then the minimal projective and injective resolutions of the simple B -module S_i are of the form:*

$$0 \longrightarrow e_i B \xrightarrow{(b)} \bigoplus_{b \in Q_1, s(b)=i} e_{t(b)} B \xrightarrow{(r_{ab})} \bigoplus_{a \in Q_1, t(a)=i} e_{s(a)} B \xrightarrow{(a)} e_i B \longrightarrow S_i \longrightarrow 0$$

$$0 \longrightarrow S_i \longrightarrow D(Be_i) \xrightarrow{(b)} \bigoplus_{b \in Q_1, s(b)=i} D(Be_{t(b)}) \xrightarrow{(r'_{ab})} \bigoplus_{a \in Q_1, t(a)=i} D(Be_{s(a)}) \xrightarrow{(a)} D(Be_i) \longrightarrow 0,$$

for some maps r_{ab} , where the sets $\{ar_{ab} \mid a, b \in Q_1, t(a) = s(b) = i, t(b) = j\}$ and $\{r'_{ab} \mid s(b) = t(a) = i, s(a) = j\}$ are bases of $e_i I e_j$.

Proof. Applying the functor $\text{Hom}_{\mathcal{C}}(T, -)$ to the 2-almost-split sequence

$$0 \longrightarrow T_i \xrightarrow{f'} E' \xrightarrow{fg'} E \xrightarrow{g} T_i \longrightarrow 0$$

we get the following exact sequence of B -modules

$$0 \longrightarrow \text{Hom}_{\mathcal{C}}(T, T_i) \longrightarrow \text{Hom}_{\mathcal{C}}(T, E') \longrightarrow \text{Hom}_{\mathcal{C}}(T, E) \longrightarrow \text{Hom}_{\mathcal{C}}(T, T_i) \longrightarrow S_i \longrightarrow 0$$

which is a minimal projective resolution of the simple B -module S_i .

Let Q be the quiver of B , and $B \simeq kQ/I$. Since g is minimal right almost split in $\text{add}(T)$, we have

$$E \simeq \bigoplus_{a \in Q_1 \mid t(a)=i} T_{s(a)} \quad \text{and} \quad \text{Hom}_{\mathcal{C}}(T, g) \simeq (a)_{\{a \in Q_1 \mid t(a)=i\}}.$$

Since f' is minimal left almost split in $\text{add}(T)$, we have

$$E' \simeq \bigoplus_{b \in Q_1 \mid s(b)=i} T_{t(b)} \quad \text{and} \quad \text{Hom}_{\mathcal{C}}(T, f') \simeq (b)_{\{b \in Q_1 \mid s(b)=i\}}.$$

For $a, b \in Q_1$ with $t(a) = i$ and $s(b) = i$, let $r_{ab} : e_{t(b)} B \rightarrow e_{s(a)} B$ be the map induced by $fg' : \bigoplus_{b \in Q_1 \mid s(b)=i} T_{t(b)} \rightarrow \bigoplus_{a \in Q_1 \mid t(a)=i} T_{s(a)}$. Since the 2-almost split sequence associated to T_i induces a minimal projective resolution of the simple B -module S_i , the set $\{ar_{ab} \mid a, b \in Q_1, t(a) = s(b) = i, t(b) = j\}$ is a basis of the set of relations $e_i I e_j$.

To get the other sequence of the proposition, we apply the functor $D\text{Hom}_{\mathcal{C}}(-, T)$ to the 2-almost split sequence associated to T_i , and we proceed similarly. \square

1.2. Generalized cluster categories. Let Λ be a finite dimensional k -algebra of global dimension at most 2. We denote by $\mathcal{D}^b(\Lambda)$ the bounded derived category of finitely generated (right) Λ -modules. It has a Serre functor that we denote by \mathbb{S} , which coincides with $\tau[1]$.

The *generalized cluster category* \mathcal{C}_{Λ} is defined in [Ami08] as the triangulated hull $(\mathcal{D}^b(\Lambda)/\mathbb{S}[-2])_{\Delta}$ in the sense of [Kel05] of the orbit category $\mathcal{D}^b(\Lambda)/\mathbb{S}[-2]$. The composition of the functors

$$\pi_{\Lambda} : \mathcal{D}^b(\Lambda) \longrightarrow \mathcal{D}^b(\Lambda)/\mathbb{S}[-2] \hookrightarrow \mathcal{C}_{\Lambda}$$

is a triangle functor.

The following definition and theorem give a more explicit description of the generalized cluster category. This is added here for the convenience of the reader but will not be used later.

Definition 1.3. [Kel09] Denote by Θ_2 a cofibrant resolution of the complex of Λ -bimodules $\mathbf{RHom}_\Lambda(D\Lambda, \Lambda)[2]$, that is a complex of projective Λ -bimodules which is quasi-isomorphic to $\mathbf{RHom}_\Lambda(D\Lambda, \Lambda)[2]$. Then the *derived 3-preprojective algebra* is defined as the tensor DG algebra:

$$\mathbf{\Pi}_3(\Lambda) := T_\Lambda \Theta_2 = \Lambda \oplus \Theta_2 \oplus (\Theta_2 \otimes_\Lambda \Theta_2) \oplus \dots$$

We set $\Pi_3(\Lambda) := H^0(\mathbf{\Pi}_3(\Lambda))$. It is called the *3-preprojective algebra*.

Theorem 1.4. [Kel05, Ami08] *Let Λ be a finite dimensional algebra of global dimension at most 2. Then there exists a triangle equivalence*

$$\mathcal{C}_\Lambda := (\mathcal{D}^b \Lambda / \mathbb{S}[-2])_\Delta \simeq \text{per } \mathbf{\Pi}_3(\Lambda) / \mathcal{D}^b \mathbf{\Pi}_3(\Lambda)$$

where $\text{per } \mathbf{\Pi}_3(\Lambda)$ is the thick subcategory of $\mathcal{D}\mathbf{\Pi}_3(\Lambda)$ generated by $\mathbf{\Pi}_3(\Lambda)$, and $\mathcal{D}^b \mathbf{\Pi}_3(\Lambda)$ is the thick subcategory of $\mathcal{D}\mathbf{\Pi}_3(\Lambda)$ formed by the objects having finite dimensional total cohomology.

There is a useful criterion for constructing triangle functors from a generalized cluster category \mathcal{C}_Λ to some stable category $\underline{\mathcal{E}}$. It can be deduced from the universal property of π_Λ given in subsection 4.1 of [Ami08] (see also section 9 of [Kel05] or the appendix of [IO09] for more details). This criterion, which is given in the next proposition, is a key step for proving the equivalence of the main theorem of this paper.

For a Frobenius category \mathcal{E} and an algebra Λ , we here denote by $\mathcal{D}^b(\Lambda^{\text{op}} \otimes \mathcal{E})$ the bounded derived category of the exact category $(\text{mod } \Lambda^{\text{op}}) \otimes \mathcal{E}$ whose objects are objects in \mathcal{E} having a structure of finitely generated left Λ -module (see for example [Kel06, sections 2 and 3] for precise definitions of tensor products of k -categories and derived categories).

Proposition 1.5. *Let \mathcal{C}_Λ be a generalized cluster category, where Λ is a finite dimensional algebra of global dimension at most 2. Let \mathcal{E} be a Frobenius category. Let M be an object in \mathcal{E} and assume that M has a left Λ -module structure. Assume that there is a morphism in $\mathcal{D}^b(\Lambda^{\text{op}} \otimes \mathcal{E})$*

$$\alpha : M \longrightarrow \mathbf{RHom}_\Lambda(D\Lambda, \Lambda) \overset{\mathbf{L}}{\otimes}_\Lambda M[2]$$

whose cone lies in $\mathcal{D}^b(\Lambda^{\text{op}} \otimes \mathcal{P})$, where \mathcal{P} is the full subcategory of \mathcal{E} of projective-injectives. Then there exists a triangle functor $\mathcal{C}_\Lambda \rightarrow \underline{\mathcal{E}}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{D}^b(\Lambda) & \xrightarrow{-\overset{\mathbf{L}}{\otimes}_\Lambda M} & \mathcal{D}^b(\mathcal{E}) \\ \downarrow \pi_\Lambda & & \downarrow \\ \mathcal{C}_\Lambda & \longrightarrow & \underline{\mathcal{E}} \end{array}$$

Note that the endofunctor $-\overset{\mathbf{L}}{\otimes}_\Lambda \mathbf{RHom}_\Lambda(D\Lambda, \Lambda)[2] \simeq \mathbf{RHom}_\Lambda(D\Lambda, -)[2]$ of $\mathcal{D}^b(\Lambda)$ is isomorphic to the functor $\mathbb{S}^{-1}[2]$. Hence Proposition 1.5 requires in particular that for any X in $\mathcal{D}^b(\Lambda)$, there is a morphism $X \overset{\mathbf{L}}{\otimes}_\Lambda M \rightarrow X \overset{\mathbf{L}}{\otimes}_\Lambda \mathbf{RHom}_\Lambda(D\Lambda, \Lambda) \overset{\mathbf{L}}{\otimes}_\Lambda M[2] \simeq \mathbb{S}^{-1}X[2] \overset{\mathbf{L}}{\otimes}_\Lambda M$ in $\mathcal{D}^b(\mathcal{E})$ whose cone is in $\mathcal{D}^b(\mathcal{P})$. In other words, the images of X and of $\mathbb{S}^{-1}X[2]$ under the composition

$$\mathcal{D}^b(\Lambda) \xrightarrow{-\overset{\mathbf{L}}{\otimes}_\Lambda M} \mathcal{D}^b(\mathcal{E}) \longrightarrow \mathcal{D}^b(\mathcal{E}) / \mathcal{D}^b(\mathcal{P}) = \underline{\mathcal{E}}$$

are isomorphic. Here the category $\mathcal{D}^b(\mathcal{P})$ is the thick subcategory of $\mathcal{D}^b(\mathcal{E})$ generated by \mathcal{P} . Thus the localization of $\mathcal{D}^b(\mathcal{E})$ by $\mathcal{D}^b(\mathcal{P})$ is equivalent to the stable category $\underline{\mathcal{E}}$ by [KV87], and this localization gives us the right vertical map of this diagram. This implies immediately that the functor $\bigotimes_{\Lambda}^{\mathbf{L}} M : \mathcal{D}^b(\Lambda) \rightarrow \underline{\mathcal{E}}$ factors through the orbit category $\mathcal{D}^b(\Lambda)/\mathbb{S}[-2]$. However the proof that it factors through the generalized cluster category as a triangle functor is non trivial and uses the universal property of triangulated orbit categories [Kel05, IO09].

The case when the 3-preprojective algebra of Λ is finite dimensional is especially nice.

Theorem 1.6. [Ami08, Theorem 4.10] *Let Λ be a finite dimensional algebra of global dimension at most 2, and assume that the algebra $\Pi_3(\Lambda)$ is finite dimensional. Then the category \mathcal{C}_{Λ} is a Hom-finite 2-Calabi-Yau category, the object $\pi_{\Lambda}(\Lambda) \in \mathcal{C}_{\Lambda}$ is a cluster-tilting object and we have an isomorphism*

$$\mathrm{End}_{\mathcal{C}_{\Lambda}}(\pi_{\Lambda}(\Lambda)) \simeq \Pi_3(\Lambda).$$

1.3. Jacobian algebras and generalizations. Quivers with potentials and their associated Jacobian algebras have been investigated in [DWZ08]. Let Q be a finite quiver. For each arrow a in Q , the *cyclic derivative* ∂_a with respect to a is the unique linear map $\partial_a : kQ \rightarrow kQ$ which takes the class of a path p to the sum $\sum_{p=uv} vu$ taken over all decompositions of the path p (where u and v are possibly idempotent elements e_i associated to the vertex i). A *potential* on Q is any linear combination W of cycles in Q . The associated Jacobian algebra is by definition the algebra

$$\mathrm{Jac}(Q, W) := kQ / \langle \partial_a W; a \in Q_1 \rangle.$$

There is a more general definition given in [DWZ08], dealing with the complete path algebras, and hence there is also a larger class of Jacobian algebras. However, in this paper we only consider the Jacobian algebras defined above.

Any finite dimensional Jacobian algebra (in the general sense) is 2-CY-tilted ([Ami08, Kel09]). As a partial converse, some classes of 2-CY-tilted algebras associated with elements in Coxeter groups are Jacobian ([BIRS09b]). Furthermore, the 2-CY-tilted algebras given by the canonical cluster-tilting object in a generalized cluster category are Jacobian, as stated in the following result [Kel09, Theorem 6.11 a)].

Theorem 1.7 (Keller). *Let $A = kQ/I$ be an algebra of global dimension at most 2, such that I is generated by a finite set of minimal relations (r_i) . The relation r_i starts at the vertex $s(r_i)$ and ends at the vertex $t(r_i)$. Let \tilde{Q} be the quiver obtained from Q by adding additional arrows $a_i : t(r_i) \rightarrow s(r_i)$ for each minimal relation r_i , and let W_A be the potential $\sum_i a_i r_i$. Then there is an isomorphism of algebras:*

$$\mathrm{End}_{\mathcal{C}_A}(A) \simeq \mathrm{Jac}(\tilde{Q}, W_A).$$

There is a generalization of quivers with potentials (Q, W) to *frozen quivers with potentials* (Q, W, F) in [BIRS09b]. Here $F = (F_0, F_1)$ is a pair consisting of a subset F_0 of vertices of Q (called *frozen vertices*) and the subset $F_1 = \{a \in Q_1, s(a) \in F_0 \text{ and } t(a) \in F_0\}$ of arrows (called *frozen arrows*). The associated *frozen Jacobian algebra* is by definition the algebra

$$\mathrm{Jac}(Q, W, F) := kQ / \langle \partial_a W, a \notin F_1 \rangle$$

As for ordinary quivers with potential in [DWZ08], one can define a *reduced frozen quiver* with potential (Q, W, F) by requiring that each term in W has length at least 3 and has at least one arrow in $Q_1 \setminus F_1$.

2. A USEFUL TRIANGLE

In this section we study the Calabi-Yau property of a frozen Jacobian algebra B , when B is the endomorphism algebra of a cluster-tilting object in a Frobenius stably 2-Calabi-Yau category (subsection 2.2). Then we assume that B has a special grading and construct some algebras A and \bar{A} related to B (subsection 2.3). Using the grading on B and the Calabi-Yau property we construct a triangle in the category $\mathcal{D}^b(\bar{A}^{\text{op}} \otimes B)$, which will be crucial for the proof of our main result (subsections 2.4 and 2.5).

2.1. Basic setup. Let \mathcal{E} be a Frobenius category which is Hom-finite and stably 2-Calabi-Yau. While we are mainly interested in the stable category $\underline{\mathcal{E}}$, it will be important to first consider a cluster-tilting object T in the Frobenius category \mathcal{E} , and its endomorphism algebra $B := \text{End}_{\mathcal{E}}(T)$. We assume that this algebra B is isomorphic to $\text{Jac}(Q, W, F)$ for some reduced frozen quiver with potential (Q, W, F) . Notice that the quiver of B is Q since the potential W is reduced. We also assume the following:

(H1) The vertices in F_0 correspond to the isoclasses of projective-injective indecomposables in the Frobenius category \mathcal{E} .

(H2) The set $\{\partial_a W, a \notin F_1\}$, which by definition generates the ideal of relations of the algebra $\text{Jac}(Q, W, F)$, forms a k -basis for this ideal. In particular, for any $a \in Q_1 \setminus F_1$, we have $\partial_a W \neq 0$.

For $i \in Q_0$ we denote by e_i the primitive idempotent of B associated to i . Denote by e_F the idempotent $\sum_{i \in F_0} e_i$ and consider the factor algebra $\bar{B} := B/B e_F B$. By (H1) we have an isomorphism of algebras $\bar{B} \simeq \text{End}_{\mathcal{E}}(T)$.

Let \bar{Q} be the full subquiver of Q obtained from Q by deleting the vertices in F_0 . We then have a projection $kQ \twoheadrightarrow kQ/kQ e_F kQ \simeq k\bar{Q}$. We denote by \bar{W} the image of W under this projection. It is not hard to see that there is an isomorphism $\bar{B} \simeq \text{Jac}(\bar{Q}, \bar{W})$. Indeed, since any arrow a in F_1 satisfies $s(a) \in F_0$ and $t(a) \in F_0$, the partial derivative $\partial_a \bar{W}$ vanishes for any a in F_1 .

2.2. Calabi-Yau property for B . In this subsection we describe the minimal projective and injective resolutions of \bar{B} as \bar{B} - B -bimodule and deduce a Calabi-Yau property linking these two algebras. We start with giving explicit projective and injective resolutions over B of the simple \bar{B} -modules.

Lemma 2.1. *Let $B \simeq \text{Jac}(Q, W, F)$ be the endomorphism algebra of a cluster-tilting object T in a Hom-finite Frobenius stably 2-CY category \mathcal{E} . Assume that (H1) and (H2) hold. Then for any $i \in Q_0 \setminus F_0$, the sequences*

$$0 \longrightarrow e_i B \xrightarrow{(b)} \bigoplus_{b, s(b)=i} e_{t(b)} B \xrightarrow{(a^{-1}(\partial_b W))} \bigoplus_{a, t(a)=i} e_{s(a)} B \xrightarrow{(a)} e_i B \longrightarrow S_i \longrightarrow 0$$

and

$$0 \longrightarrow S_i \longrightarrow D(Be_i) \xrightarrow{(b)} \bigoplus_{b, s(b)=i} D(Be_{t(b)}) \xrightarrow{((\partial_a W)b^{-1})} \bigoplus_{a, t(a)=i} D(Be_{s(a)}) \xrightarrow{(a)} D(Be_i) \longrightarrow 0$$

are minimal projective and injective resolutions of the simple B -module S_i . When v is a path in Q , we write $a^{-1}v = u$ if $v = au$ in kQ and $a^{-1}v = 0$ otherwise.

Proof. By (H1), if i is not in F_0 , the corresponding summand T_i of T is not projective-injective. Hence by Proposition 1.1, there exists a 2-almost split sequence associated with T_i . Moreover, since the potential W is reduced, then Q is the quiver of $\text{End}_{\mathcal{E}}(T)$. Thus by Proposition 1.2 we obtain a minimal projective B -resolution of S_i

$$0 \longrightarrow e_i B \xrightarrow{(b)} \bigoplus_{b, s(b)=i} e_{t(b)} B \xrightarrow{(r_{ab})} \bigoplus_{a, t(a)=i} e_{s(a)} B \xrightarrow{(a)} e_i B \longrightarrow S_i \longrightarrow 0,$$

where $\{ar_{ab} \mid t(a) = i, s(b) = i\}$ is a basis for the space of relations with target i . By (H2) the set $\{\partial_b W \mid s(b) = i\}$ is a basis for the same space. Thus a minimal projective resolution of S_i can be written as

$$0 \longrightarrow e_i B \xrightarrow{(b)} \bigoplus_{b, s(b)=i} e_{t(b)} B \xrightarrow{(a^{-1}(\partial_b W))} \bigoplus_{a, t(a)=i} e_{s(a)} B \xrightarrow{(a)} e_i B \longrightarrow S_i \longrightarrow 0$$

which proves the first claim. The proof is similar for the injective resolution. \square

We use Lemma 2.1 to obtain exact sequences of \bar{B} - B -bimodules which permit to get the functorial minimal projective and injective resolutions of any \bar{B} -module when viewed as a B -module. This is inspired by [Boc08].

Proposition 2.2. *There exist exact sequences in $\text{mod}(\bar{B}^{\text{op}} \otimes B)$*

$$(a) \quad 0 \longrightarrow \bigoplus_{i \notin F_0} P_{i,i} \longrightarrow \bigoplus_{a \notin F_1} P_{s(a), t(a)} \longrightarrow \bigoplus_{a \notin F_1} P_{t(a), s(a)} \longrightarrow \bigoplus_{i \notin F_0} P_{i,i} \longrightarrow \bar{B} \longrightarrow 0$$

$$(b) \quad 0 \longrightarrow \bar{B} \longrightarrow \bigoplus_{i \notin F_0} I_{i,i} \longrightarrow \bigoplus_{a \notin F_1} I_{s(a), t(a)} \longrightarrow \bigoplus_{a \notin F_1} I_{t(a), s(a)} \longrightarrow \bigoplus_{i \notin F_0} I_{i,i} \longrightarrow 0$$

where $P_{i,j} := \bar{B}e_i \otimes e_j B$ and $I_{i,j} := \text{Hom}_k(Be_j, \bar{B}e_i)$ for $(i, j) \in Q_0 \times Q_0$.

Remark 2.3. For $M \in \text{mod } \bar{B}$, if we apply the functor $M \otimes_{\bar{B}} -$ to the sequence (a) we get an exact sequence in $\text{mod } B$ since $P_{i,j}$ is projective as left \bar{B} -module. This sequence is the minimal projective resolution of M when viewed as a B -module since $P_{i,j}$ is projective as right B -module.

Similarly, if we apply the functor $M \otimes_{\bar{B}} -$ to the sequence (b) we get an exact sequence in $\text{mod } B$ since $I_{i,j}$ is projective as left \bar{B} -module. This sequence is the minimal injective resolution of M when viewed as a B -module since $I_{i,j}$ is injective as right B -module.

Proof. For $(i, j) \in Q_0 \times Q_0$ denote by $\Pi_{i,j}$ the projective B -bimodule $Be_i \otimes e_j B$. Then consider the following sequence

$$(c) \quad \bigoplus_{i \notin F_0} \Pi_{i,i} \xrightarrow{d_2} \bigoplus_{a \notin F_1} \Pi_{s(a), t(a)} \xrightarrow{d_1} \bigoplus_{a \in Q_1} \Pi_{t(a), s(a)} \xrightarrow{d_0} \bigoplus_{i \in Q_0} \Pi_{i,i}.$$

where the maps d_0 , d_1 and d_2 are defined as follows:

$$\begin{aligned} d_2(e_i \otimes e_i) &= \sum_{a, t(a)=i} a \otimes e_i - \sum_{b, s(b)=i} e_i \otimes b; \\ d_1(e_{s(a)} \otimes e_{t(a)}) &= \sum_{b \in Q_1} \partial_{a,b} W \text{ where } \partial_{a,b}(apbq) = p \otimes q \in Be_{t(b)} \otimes e_{s(b)} B \\ &\quad \text{for a cycle } apbq \text{ in } Q; \\ d_0(e_{t(a)} \otimes e_{s(a)}) &= a \otimes e_{s(a)} - e_{t(a)} \otimes a. \end{aligned}$$

It is easy to check that this is a complex of B -bimodules, and that $\text{Coker } d_0 = B$. (The map $\bigoplus_{i \in Q_0} \Pi_{i,i} \rightarrow B$ is the multiplication map.)

Applying the functor $\bar{B} \otimes_B -$, we get the complex of \bar{B} - B -bimodules

$$\bigoplus_{i \notin F_0} P_{i,i} \xrightarrow{d_2} \bigoplus_{a \notin F_1} P_{s(a),t(a)} \xrightarrow{d_1} \bigoplus_{a \notin F_1} P_{t(a),s(a)} \xrightarrow{d_0} \bigoplus_{i \notin F_0} P_{i,i}.$$

Indeed, if $i \in F_0$, then $\bar{B}e_i \otimes e_i B = 0$, and if $a \in F_1$, then $\bar{B}e_{t(a)} \otimes e_{s(a)} B = 0$.

Applying the functor $S_i \otimes_{\bar{B}} -$ with $i \notin F_0$, we get the complex

$$0 \longrightarrow e_i B \xrightarrow{(b)} \bigoplus_{b, s(b)=i} e_{t(b)} B \xrightarrow{((\partial_a W)^{b^{-1}})} \bigoplus_{a, t(a)=i} e_{s(a)} B \xrightarrow{(a)} e_i B$$

which is exactly the minimal projective resolution of S_i as B -module described in Lemma 2.1. Thus the following sequence in $\text{mod}(\bar{B}^{\text{op}} \otimes B)$ is exact:

$$0 \longrightarrow \bigoplus_{i \notin F_0} P_{i,i} \longrightarrow \bigoplus_{a \notin F_1} P_{s(a),t(a)} \longrightarrow \bigoplus_{a \notin F_1} P_{t(a),s(a)} \longrightarrow \bigoplus_{i \notin F_0} P_{i,i} \longrightarrow \bar{B} \longrightarrow 0,$$

so that (a) follows.

We now prove the existence of the sequence (b), where the proof is dual. Let us define the sequence of B -bimodules

$$\bigoplus_{i \in Q_0} \Upsilon_{i,i} \xrightarrow{d^0} \bigoplus_{a \in Q_1} \Upsilon_{s(a),t(a)} \xrightarrow{d^1} \bigoplus_{a \notin F_1} \Upsilon_{t(a),s(a)} \xrightarrow{d^2} \bigoplus_{i \notin F_0} \Upsilon_{i,i}$$

where $\Upsilon_{i,j} := \text{Hom}_k(Be_j, Be_i)$ for $(i, j) \in Q_0 \times Q_0$. The maps d^0, d^1, d^2 are the following

$$\begin{aligned} d^0(\phi_i) &= (\sum_{s(a)=i} a \otimes e_i - \sum_{t(b)=i} e_i \otimes b) \cdot (\phi_i) \\ d^1(\phi_a) &= \sum_{b \in Q_1} (\partial_{a,b} W) \cdot (\phi_a) \\ d^2(\phi_a) &= (a \otimes e_i - e_i \otimes a) \cdot (\phi_a) \end{aligned}$$

where $(a \otimes b)(\phi)(-) = \phi(-a)b$. For instance we have

$$d^0(\phi_i)(-) = \sum_{s(a)=i} \phi_i(-a) - \sum_{t(b)=i} \phi_i(-b).$$

The kernel of d^0 is B , the bimodule map $B \rightarrow \bigoplus \Upsilon_{i,i}$ maps 1_B to $(\mathbf{1}_{Be_i})_i$. Using the fact that $S_l \otimes_{\bar{B}} \text{Hom}_k(Be_i, \bar{B}e_j) \simeq \delta_{j,l} e_i DB$ we get the exact sequence (b). \square

We have the following direct consequence (see also [KR07, 5.4, Thm (b)]) which we include even though it will not be used later in this paper.

Corollary 2.4. *There is an isomorphism $\mathbf{R}\text{Hom}_B(DB, \bar{B})[3] \simeq \bar{B}$ in $\mathcal{D}^b(\bar{B}^{\text{op}} \otimes B)$.*

Proof. Since $I_{i,j} = \text{Hom}_k(Be_j, e_i \bar{B})$ is injective as a right B -module, then using the exact sequence (b) of Proposition 2.2, we obtain that $\mathbf{R}\text{Hom}_B(DB, \bar{B})$ is isomorphic in $\mathcal{D}^b(\bar{B}^{\text{op}} \otimes B)$ to the complex

$$\bigoplus_{i \notin F_0} R_{i,i} \longrightarrow \bigoplus_{a \notin F_1} R_{s(a),t(a)} \longrightarrow \bigoplus_{a \notin F_1} R_{t(a),s(a)} \longrightarrow \bigoplus_{i \notin F_0} R_{i,i}$$

where $R_{i,j}$ is the bimodule $\text{Hom}_B(DB, I_{i,j})$. Moreover we have the following isomorphisms

$$\begin{aligned} \text{Hom}_B(DB, I_{i,j}) &= \text{Hom}_B(DB, \text{Hom}_k(Be_j, \bar{B}e_i)) \\ &\simeq \text{Hom}_k(DBe_j, \bar{B}e_i) \\ &\simeq \bar{B}e_i \otimes e_j B \\ &= P_{i,j} \end{aligned}$$

in $\text{mod}(\bar{B}^{\text{op}} \otimes B)$. Then the claim follows from the exact sequence (a) of Proposition 2.2. \square

Remark 2.5. If there exists an algebra $B = \text{Jac}(Q, W, F)$ with $F_0 = \emptyset$ such that the sequences in Lemma 2.1 are exact, then we get an isomorphism in $\mathcal{D}(B^{\text{op}} \otimes B)$ between $\mathbf{R}\text{Hom}(DB, B)[3]$ and B (since in this case $\bar{B} = B$). This means by definition that B is bimodule 3-CY. This is exactly what Bocklandt proved in [Boc08, Theorem 4.3], namely that a Jacobian algebra B is bimodule 3-CY if the sequence (c) is exact. In our setup, F_0 is never empty since it corresponds to the projective-injectives. Also the algebra B is not Jacobian in general, and the Jacobian algebra \bar{B} is never of global dimension 3 (indeed it is either hereditary or of infinite global dimension [KR07, section 2 Corollary]). However we have an isomorphism between \bar{B} and $\mathbf{R}\text{Hom}_B(DB, \bar{B})[3]$ in $\mathcal{D}^b(\bar{B}^{\text{op}} \otimes B)$.

Remark 2.6. In the next subsections, we will assume that moreover the algebra B has a special grading that induces an isomorphism $\mathbf{R}\text{Hom}_B(DB, \bar{B})[3](-1) \simeq \bar{B}$ in $\mathcal{D}^b(\text{gr}(\bar{B}^{\text{op}} \otimes B))$, where $\text{gr}(\bar{B}^{\text{op}} \otimes B)$ is the category of finite dimensional graded $\bar{B}^{\text{op}} \otimes B$ -modules and where (1) is the degree-shift in the category of graded B -modules. However, since Proposition 2.2 and Corollary 2.4 hold without any grading hypothesis, we have separated subsection 2.2 from subsections 2.3, 2.4 and 2.5.

2.3. Construction of the algebras A and \bar{A} . In order to identify appropriate subalgebras of B which should give rise to the generalized cluster categories we are looking for, it will be convenient to introduce some special gradings on the quiver. Assume as before that $B = \text{End}_{\mathcal{E}}(T)$ is isomorphic to some frozen Jacobian algebra $\text{Jac}(Q, W, F)$. Then we assume that there exists a degree map $\varphi : Q_1 \rightarrow \{0, 1\}$ with the following property:

(H3) The potential W is homogeneous of degree 1.

Since the potential is homogeneous, any relation $\partial_a W$ is homogeneous. Hence φ induces a grading on B . Define the algebras A and \bar{A} by $A := B_0$ and $\bar{A} := A/Ae_F A$. We have surjective algebra maps: $B \rightarrow A \rightarrow \bar{A}$. We want to show the following, which is the main result of the section.

Proposition 2.7. *Let \mathcal{E} be a Frobenius category which is Hom-finite and stably 2-Calabi-Yau. We assume that there exists a cluster-tilting object T in \mathcal{E} such that its endomorphism algebra is isomorphic to $\text{Jac}(Q, W, F)$ for some reduced frozen quiver with potential (Q, W, F) . With the assumptions (H1), (H2) and (H3), there exists a triangle*

$$(*) \quad \mathbf{R}\text{Hom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A B[2] \longrightarrow \bar{A} \overset{\mathbf{L}}{\otimes}_A B \longrightarrow \bar{A} \longrightarrow \mathbf{R}\text{Hom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A B[3]$$

in $\mathcal{D}^b(\bar{A}^{\text{op}} \otimes B)$, where $A = B_0$ and $\bar{A} = A/Ae_F A$.

The proof is given in the next subsections. In subsection 2.4 we construct a triangle $X \rightarrow Y \rightarrow \bar{A} \rightarrow X[1]$ in $\mathcal{D}^b(\bar{A}^{\text{op}} \otimes B)$. In subsection 2.5, we show that this triangle is the triangle (*) of Proposition 2.7.

2.4. A triangle $X \rightarrow Y \rightarrow \bar{A} \rightarrow X[1]$. The following is proved the same way as Proposition 2.2. We describe exact sequences of \bar{A} - B -bimodules, which give the functorial minimal projective and injective resolutions of the \bar{A} -modules when viewed as B -modules (cf Remark 2.3).

Proposition 2.8. *There exist exact sequences in $\text{mod}(\bar{A}^{\text{op}} \otimes B)$*

$$(a') \quad 0 \longrightarrow \bigoplus_{i \notin F_0} P_{i,i} \xrightarrow{d_2} \bigoplus_{a \notin F_1} P_{s(a),t(a)} \xrightarrow{d_1} \bigoplus_{a \notin F_1} P_{t(a),s(a)} \xrightarrow{d_0} \bigoplus_{i \notin F_0} P_{i,i} \longrightarrow \bar{A} \longrightarrow 0$$

$$(b') \quad 0 \longrightarrow \bar{A} \longrightarrow \bigoplus_{i \notin F_0} I_{i,i} \longrightarrow \bigoplus_{a \notin F_1} I_{s(a),t(a)} \longrightarrow \bigoplus_{a \notin F_1} I_{t(a),s(a)} \longrightarrow \bigoplus_{i \notin F_0} I_{i,i} \longrightarrow 0$$

where $P_{i,j} := \bar{A}e_i \otimes e_j B$, $I_{i,j} := \text{Hom}_k(Be_j, \bar{A}e_i)$ and the d_i are defined as in Proposition 2.2.

Let $a \in Q_1 \setminus F_1$ be an arrow with $\varphi(a) = 1$. By definition $d_1(e_{s(a)} \otimes e_{t(a)}) = \sum_{b \in Q_1} \partial_{a,b} W$. If $apbq$ is a cycle in W , then $\partial_{a,b}(apbq) = p \otimes q$ lies in $\bar{A}e_{t(b)} \otimes e_{s(b)} B$. Since the degree of a is 1, and since we have assumed that the potential W is homogeneous of degree 1, then b is of degree 0. Hence the restriction of d_1

$$\bigoplus_{a \notin F_1, \varphi(a)=1} P_{s(a),t(a)} \longrightarrow \bigoplus_{a \notin F_1, \varphi(a)=1} P_{t(a),s(a)}$$

is zero. Therefore the complex

$$\bigoplus_{i \notin F_0} P_{i,i} \longrightarrow \bigoplus_{a \notin F_1} P_{s(a),t(a)} \longrightarrow \bigoplus_{a \notin F_1} P_{t(a),s(a)} \longrightarrow \bigoplus_{i \notin F_0} P_{i,i}$$

is isomorphic to the mapping cone of a complex morphism

$$\begin{array}{ccccc} X := \bigoplus_{i \notin F_0} P_{i,i} & \longrightarrow & \bigoplus_{a \notin F_1, \varphi(a)=0} P_{s(a),t(a)} & \longrightarrow & \bigoplus_{a \notin F_1, \varphi(a)=1} P_{t(a),s(a)} \\ \downarrow & & \downarrow & & \downarrow \\ Y := \bigoplus_{a \notin F_1, \varphi(a)=1} P_{s(a),t(a)} & \longrightarrow & \bigoplus_{a \notin F_1, \varphi(a)=0} P_{t(a),s(a)} & \longrightarrow & \bigoplus_{i \notin F_0} P_{i,i} \end{array}$$

It is not hard to check that all horizontal maps are homogeneous of degree 0, and all vertical maps are homogeneous of degree 1. Denote by f the map $X \rightarrow Y$. By Proposition 2.8 (a'), we obtain the triangle $X \xrightarrow{f} Y \longrightarrow \bar{A} \longrightarrow X[1]$ in $\mathcal{D}(\bar{A}^{\text{op}} \otimes B)$.

Dually, using the exact sequence (b') in Proposition 2.8, it is possible to view $\bar{A}[1]$ as the mapping cone of a morphism $g : X' \rightarrow Y'$, where horizontal maps are of degree 0, and vertical maps are of degree -1 .

$$\begin{array}{ccccc} X' := \bigoplus_{i \notin F_0} I_{i,i} & \longrightarrow & \bigoplus_{a \notin F_1, \varphi(a)=0} I_{s(a),t(a)} & \longrightarrow & \bigoplus_{a \notin F_1, \varphi(a)=1} I_{t(a),s(a)} \\ \downarrow & & \downarrow & & \downarrow \\ Y' := \bigoplus_{a \notin F_1, \varphi(a)=1} I_{s(a),t(a)} & \longrightarrow & \bigoplus_{a \notin F_1, \varphi(a)=0} I_{t(a),s(a)} & \longrightarrow & \bigoplus_{i \notin F_0} I_{i,i} \end{array}$$

Thus we get a triangle $\bar{A} \longrightarrow X' \xrightarrow{g} Y' \longrightarrow \bar{A}[1]$ in $\mathcal{D}(\bar{A}^{\text{op}} \otimes B)$.

2.5. Interpretation of X and Y . The aim of this subsection is to construct isomorphisms

$$Y \simeq \bar{A} \overset{\mathbf{L}}{\otimes}_A B \quad \text{and} \quad X \simeq \mathbf{R}\text{Hom}_B(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A B[2] \quad \text{in } \mathcal{D}^b(\bar{A}^{\text{op}} \otimes B),$$

in order to prove Proposition 2.7.

We will first show the following

Lemma 2.9. *In the setup of subsection 2.4, we have isomorphisms*

$$Y_0 \simeq \bar{A} \quad \text{and} \quad X'_0 \simeq \bar{A} \quad \text{in } \mathcal{D}^b(\bar{A}^{\text{op}} \otimes A)$$

The proof of this result uses the next lemma.

Lemma 2.10. *Let B be a \mathbb{Z} -graded algebra. Let (X, d_X) and (Y, d_Y) be complexes of graded B -modules such that the differentials d_X and d_Y are homogeneous of degree 0. We denote by (X_p, d_{X_p}) (resp. (Y_p, d_{Y_p})) the part of degree p of the complex (X, d_X) (resp. (Y, d_Y)). They are complexes of B_0 -modules. Let $f : X \rightarrow Y$ be a morphism homogeneous of degree d , that is the part f_p of degree p of f is a morphism of B_0 -complexes $X_p \rightarrow Y_{p+d}$. Denote by $Z = \text{Cone}(f)$ the mapping cone of $f : X \rightarrow Y$. Then for any integers p, q we have an isomorphism of B_0 -modules:*

$$H^q(Z)_p \simeq H^q(\text{Cone}(f_{p-d} : X_{p-d} \rightarrow Y_p)).$$

Proof. The mapping cone of a morphism f homogeneous of degree d in the category of complexes of graded modules with differential homogeneous of degree 0 is still a complex of graded modules with differential homogeneous of degree 0, and we have

$$\text{Cone}(f) \simeq \bigoplus_{p \in \mathbb{Z}} \text{Cone}(f_{p-d} : X_{p-d} \rightarrow Y_p).$$

Then one can check the isomorphisms

$$H^q(\text{Cone}(f))_p \simeq H^q((\text{Cone}(f))_p) \simeq H^q(\text{Cone}(f_{p-d})).$$

□

Proof of Lemma 2.9. Applying Lemma 2.10 to the morphism $f : X \rightarrow Y$ defined in subsection 2.4, we get an isomorphism of $(\bar{A}^{\text{op}} \otimes A)$ -modules (remember that $A = B_0$):

$$H^q(\text{Cone}(f : X \rightarrow Y))_0 \simeq H^q(\text{Cone}(f_{-1} : X_{-1} \rightarrow Y_0)).$$

By the sequence (a') of Proposition 2.8 the left term is zero unless q is 0, and when q is 0, it is isomorphic to \bar{A} . Since X is non zero only in positive degrees, the right hand side is just $H^q(Y_0)$. Thus we get an isomorphism $Y_0 \simeq \bar{A}$ in $\mathcal{D}^b(\bar{A}^{\text{op}} \otimes A)$.

Using the triangle $\bar{A} \longrightarrow X' \xrightarrow{g} Y' \longrightarrow \bar{A}[1]$ similarly, we get an isomorphism $X'_0 \simeq \bar{A}$ in $\mathcal{D}(\bar{A}^{\text{op}} \otimes A)$. □

The complex Y_0 is a complex of projective $(\bar{A}^{\text{op}} \otimes A)$ -modules. Thus for any \bar{A} -module M , $M \otimes_{\bar{A}}^{\mathbf{L}} Y_0$ is a complex of projective A -modules. By Lemma 2.9, it is quasi-isomorphic to M viewed as an A -module. Hence we have the following.

Corollary 2.11. *Any \bar{A} -module has projective dimension at most 2 when viewed as an A -module.*

We can now prove our desired isomorphisms.

Lemma 2.12. *For complexes X and Y defined as in subsection 2.3, there are isomorphisms*

$$Y \simeq \bar{A} \otimes_A^{\mathbf{L}} B \quad \text{and} \quad X \simeq \mathbf{RHom}_B(DA, \bar{A}) \otimes_A^{\mathbf{L}} B[2] \quad \text{in } \mathcal{D}^b(\bar{A}^{\text{op}} \otimes B).$$

For the proof of this lemma, we need some basic results:

Lemma 2.13. *Let B be a \mathbb{Z} -graded algebra and $A := B_0$. Let $P = (P^j, d)$ be a complex of graded projective B -modules such that the differential d of P is homogeneous of degree 0. Let P_0 be the degree 0 part of P . Then P_0 is a complex over A , and we have an isomorphism of complexes*

$$P \simeq P_0 \otimes_A B.$$

Proof. Let $i : A \rightarrow B$ and $p : B \rightarrow A$ be the canonical algebra maps. We get induced functors:

$$i^* = - \otimes_A B : \mathbf{mod} A \rightarrow \mathbf{mod} B, \text{ and } p^* = - \otimes_B A : \mathbf{mod} B \rightarrow \mathbf{mod} A$$

Since $B_0 \otimes_A B \simeq B$, then we have $P_0^j \otimes_A B = i^* \circ p^*(P^j) \simeq P^j$ since P^j is a graded projective B -module. Hence, since P is a complex of graded projective modules, we get

$$P_0 \otimes_A B = (P_0^j \otimes_A B, i^* \circ p^*(d)).$$

Since $i \circ p(b) = b$ if and only if b is of degree 0, we get $i^* \circ p^*(d) = d$. \square

Lemma 2.14. *The functors $(\mathbf{Hom}_B(DB, -))_0$ and $\mathbf{Hom}_A(DA, (-)_0)$ are isomorphic as functors from injective B -modules to projective A -modules.*

Proof. We have $\mathbf{Hom}_B(DB, DB)_0 \simeq B_0 = A$ and $\mathbf{Hom}_A(DA, (DB)_0) \simeq \mathbf{Hom}_A(DA, DA) \simeq A$. The rest is easy to check. It is enough to check it on DB , and this is clearly true. \square

Proof of Lemma 2.12. Since Y is a complex of projective modules, we can apply Lemmas 2.13 and 2.9 to get an isomorphism

$$Y \simeq Y_0 \otimes_A B \simeq \bar{A} \overset{\mathbf{L}}{\otimes}_A B \quad \text{in } \mathcal{D}(\bar{A}^{\text{op}} \otimes B).$$

Using the fact that $\mathbf{Hom}_B(DB, I_{i,j}) \simeq P_{i,j}$, we get an isomorphism

$$\mathbf{RHom}_B(DB, X')[2] \simeq X \quad \text{in } \mathcal{D}(\bar{A}^{\text{op}} \otimes B).$$

Hence we obtain the following isomorphisms in $\mathcal{D}(\bar{A}^{\text{op}} \otimes B)$

$$\begin{aligned} X &\simeq \mathbf{Hom}_B(DB, X')[2] \\ &\simeq (\mathbf{Hom}_B(DB, X'))_0 \otimes_A B[2] \quad \text{by Lemma 2.13} \\ &\simeq \mathbf{Hom}_A(DA, X'_0) \otimes_A B[2] \quad \text{by Lemma 2.14} \\ &\simeq \mathbf{RHom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A B[2] \quad \text{by Lemma 2.9.} \end{aligned}$$

\square

Proposition 2.7 is a direct consequence of the construction in subsection 2.4 of the triangle $X \xrightarrow{f} Y \longrightarrow \bar{A} \longrightarrow X[1]$ in $\mathcal{D}(\bar{A}^{\text{op}} \otimes B)$ and of Lemma 2.12.

3. MAIN THEOREM

As in the previous section, \mathcal{E} is a Frobenius category which is \mathbf{Hom} -finite and stably 2-Calabi-Yau. We assume that there exists a cluster-tilting object T in \mathcal{E} whose endomorphism algebra is isomorphic to $\mathbf{Jac}(Q, W, F)$ for some reduced frozen quiver with potential (Q, W) such that (H1), (H2) and (H3) are satisfied. Under an additional assumption (H4), we show in this section that the stable category $\underline{\mathcal{E}}$ is triangle equivalent to a generalized cluster category.

3.1. Statement of the main result. In addition to the above assumptions, we assume that we have the following, where φ is the degree map required for (H3):

(H4) If $a : i \rightarrow j$ is in Q_1 with $i \notin F_0$ and $j \in F_0$ then $\varphi(a) = 1$.

As before we define the algebras A and \bar{A} as $A := B_0 \subset B$ and $\bar{A} := A/Ae_F A$, where e_F is the idempotent $\bigoplus_{i \in F_0} e_i$. The aim of this section is to prove the following.

Theorem 3.1. *Let \mathcal{E} be a Frobenius category which is Hom-finite and stably 2-CY. Assume that there exists a cluster-tilting object T in \mathcal{E} such that $B := \text{End}_{\mathcal{E}}(T)$ is isomorphic to some graded frozen Jacobian algebra $\text{Jac}(Q, W, F)$ which satisfies the conditions (H1)-(H4). Let $A := B_0$ be the subalgebra of degree 0 of B and $\bar{A} := A/Ae_F A$. Then we have the following*

- (a) *The algebra \bar{A} is of global dimension at most 2;*
- (b) *There exists a triangle equivalence $\mathcal{C}_{\bar{A}} \simeq \underline{\mathcal{E}}$, where $\mathcal{C}_{\bar{A}}$ is the generalized cluster category associated to the algebra \bar{A} .*

The proof of the theorem is given in the next three subsections. The fact that $gl.dim \bar{A} \leq 2$ is proved in subsection 3.2, the existence of a triangle functor $G : \mathcal{C}_{\bar{A}} \rightarrow \underline{\mathcal{E}}$ is proved in subsection 3.3. Finally it is proved that G is an equivalence of triangulated categories in subsection 3.4.

3.2. Global dimension of \bar{A} . We start with describing the restriction functor $R : \mathcal{D}^b(\bar{A}) \rightarrow \mathcal{D}^b(A)$ induced by the projection $A \rightarrow \bar{A} = A/Ae_F A$.

Lemma 3.2. *Let \mathcal{E} , $\text{Jac}(Q, W, F)$, A and \bar{A} be as in Theorem 3.1. Assume that $\text{Jac}(Q, W, F)$ satisfies conditions (H1)-(H4). Then for any $i \notin F_0$, we have the following:*

- (a) $R(e_i D \bar{A}) \simeq e_i D A$,
- (b) $R(e_i \bar{A}) \simeq (P \rightarrow Q \rightarrow e_i A)$ where P and Q are in $\text{add}(e_F A)$.
- (c) *The functor R is fully faithful.*

Proof. Part (a) follows directly from (H4). By Corollary 2.11 any \bar{A} -module has projective dimension at most 2 when viewed as an A -module. Then (b) follows from (H4). Part (c) follows directly. \square

Proposition 3.3. *Let \mathcal{E} , $\text{Jac}(Q, W, F)$ and \bar{A} be as in Theorem 3.1, and assume that $\text{Jac}(Q, W, F)$ satisfies (H1)-(H4). Then the global dimension of \bar{A} is at most 2.*

Proof. The complex X' , defined in subsection 2.4 by

$$X' := (\bigoplus_{i \notin F_0} I_{i,i} \rightarrow \bigoplus_{a \notin F_1, \varphi(a)=0} I_{s(a),t(a)} \rightarrow \bigoplus_{a \notin F_1, \varphi(a)=1} I_{t(a),s(a)}),$$

is a complex of graded $(\bar{A}^{\text{op}} \otimes B)$ -modules with differential homogeneous of degree 0. The part X'_0 of degree 0 is the complex

$$X'_0 = (\bigoplus_{i \notin F_0} J_{i,i} \rightarrow \bigoplus_{a \notin F_1, \varphi(a)=0} J_{s(a),t(a)} \rightarrow \bigoplus_{a \notin F_1, \varphi(a)=1} J_{t(a),s(a)})$$

where $J_{(i,j)} := (I_{(i,j)})_0 = \text{Hom}_k(Be_j, \bar{A}e_i)_0 \simeq \text{Hom}_k(Ae_j, \bar{A}e_i)$. By Lemma 2.9, the complex X'_0 is quasi-isomorphic to \bar{A} . Hence there exists an exact sequence in $\text{mod}(\bar{A}^{\text{op}} \otimes A)$

$$0 \rightarrow \bar{A} \rightarrow \bigoplus_{i \notin F_0} J_{i,i} \rightarrow \bigoplus_{a \notin F_1, \varphi(a)=0} J_{s(a),t(a)} \rightarrow \bigoplus_{a \notin F_1, \varphi(a)=1} J_{t(a),s(a)} \rightarrow 0$$

Since $J_{i,j}$ is projective as left \bar{A} -module and injective as right A -module, any right \bar{A} -module, when viewed as an A -module, has injective dimension 2 (cf Remark 2.3). By Lemma 3.2 (a) the injective \bar{A} -modules are injective when viewed as A -modules. Thus the global dimension of \bar{A} is at most 2. \square

3.3. Construction of a triangle functor. Recall that $A = B_0$ is the subalgebra of degree 0 of $B = \text{End}_{\mathcal{E}}(T)$. Thus T has a left A -module structure. Hence we have the following diagram:

$$\begin{array}{ccccc}
 & & \xrightarrow{-\mathbf{L}_{\bar{A}}(\bar{A} \otimes_A T)} & & \\
 \mathcal{D}^b(\bar{A}) & \xrightarrow{R} & \mathcal{D}^b(A) & \xrightarrow{-\mathbf{L}_A T} & \mathcal{D}^b(\mathcal{E}) \\
 \downarrow \pi_{\bar{A}} & & & & \downarrow \\
 \mathcal{C}_{\bar{A}} & \xrightarrow{\quad G \quad} & & & \underline{\mathcal{E}}.
 \end{array}$$

In this subsection, we prove the existence of a triangle functor $G : \mathcal{C}_{\bar{A}} \rightarrow \underline{\mathcal{E}}$ making the above diagram commute. This follows from Proposition 1.5 (for $M = \bar{A} \otimes_A T$) together with the following.

Proposition 3.4. *In the setup of Theorem 3.1, there exists a morphism*

$$\alpha : \bar{A} \otimes_A T \longrightarrow \mathbf{RHom}_{\bar{A}}(D\bar{A}, \bar{A}) \otimes_{\bar{A}} (\bar{A} \otimes_A T)[2] \quad \text{in } \mathcal{D}^b(\bar{A}^{\text{op}} \otimes \mathcal{E})$$

whose cone is in $\mathcal{D}^b(\bar{A}^{\text{op}} \otimes \mathcal{P})$, where \mathcal{P} is the subcategory of \mathcal{E} consisting of the projective-injectives.

We divide the proof of this result into three lemmas.

Lemma 3.5. *There exists an isomorphism*

$$\mathbf{RHom}_A(DA, \bar{A}) \otimes_A T[2] \xrightarrow{\sim} \bar{A} \otimes_A T \quad \text{in } \mathcal{D}^b(\bar{A}^{\text{op}} \otimes \mathcal{E}).$$

Proof. Applying $-\otimes_B T$ to the triangle (*) in Proposition 2.7, we get the following triangle in $\mathcal{D}(\bar{A}^{\text{op}} \otimes \mathcal{E})$:

$$\mathbf{RHom}_A(DA, \bar{A}) \otimes_A T[2] \longrightarrow \bar{A} \otimes_A T \longrightarrow \bar{A} \otimes_B T \longrightarrow \mathbf{RHom}_A(DA, \bar{A}) \otimes_A T[3].$$

Using the projective resolution from Lemma 2.1 of the simple B -module S_i for $i \in Q_0 \setminus F_0$, we conclude that the object $S_i \otimes_B T$ is quasi-isomorphic to the complex

$$0 \longrightarrow e_i B \otimes_B T \longrightarrow \bigoplus_{a, s(a)=i} e_{t(a)} B \otimes_B T \longrightarrow \bigoplus_{a, t(a)=i} e_{s(a)} B \otimes_B T \longrightarrow e_i B \otimes_B T,$$

which is the 2-almost split sequence associated with T_i

$$0 \longrightarrow T_i \longrightarrow \bigoplus_{a, s(a)=i} T_{t(a)} \longrightarrow \bigoplus_{a, t(a)=i} T_{s(a)} \longrightarrow T_i \longrightarrow 0.$$

Hence $S_i \otimes_B T$ is zero in $\mathcal{D}^b(\mathcal{E})$. Therefore for each $M \in \mathbf{mod} B$ whose support is in $Q_0 \setminus F_0$, the object $M \otimes_B T$ is zero in $\mathcal{D}^b(\mathcal{E})$. Consequently the object $\bar{A} \otimes_B T$ is zero in $\mathcal{D}(\bar{A}^{\text{op}} \otimes \mathcal{E})$, and the morphism

$$\mathbf{RHom}_A(DA, \bar{A}) \otimes_A T[2] \longrightarrow \bar{A} \otimes_A T$$

is an isomorphism in $\mathcal{D}^b(\bar{A}^{\text{op}} \otimes \mathcal{E})$. □

Lemma 3.6. *There exists a morphism*

$$\alpha : \mathbf{RHom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A T \longrightarrow \mathbf{RHom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A (\bar{A} \overset{\mathbf{L}}{\otimes}_A T) \quad \text{in } \mathcal{D}^b(\bar{A}^{\text{op}} \otimes \mathcal{E})$$

whose cone is in $\mathcal{D}^b(\bar{A}^{\text{op}} \otimes \mathcal{P})$.

Proof. Applying the functor $\mathbf{RHom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A - \overset{\mathbf{L}}{\otimes}_A T$ to the exact sequence

$$Ae_FA \twoheadrightarrow A \twoheadrightarrow A/Ae_FA = \bar{A}$$

of A -bimodules, we get the following triangle

$$\mathbf{RHom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A e_FT \longrightarrow \mathbf{RHom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A T \longrightarrow \mathbf{RHom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A (\bar{A} \overset{\mathbf{L}}{\otimes}_A T) \longrightarrow \cdot$$

Since A is finite dimensional and of finite global dimension, $\mathbf{RHom}_A(DA, \bar{A})$ is in $\text{thick}(A)$, the thick subcategory of $\mathcal{D}(A)$ generated by A . Thus $\mathbf{RHom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A e_FT$ is in $\text{thick}(e_FT)$, the thick subcategory of $\mathcal{D}(\mathcal{E})$ generated by e_FT . By (H1), e_FT is projective injective, and therefore it is in \mathcal{P} . Hence the cone of the morphism

$$\mathbf{RHom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A T \longrightarrow \mathbf{RHom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A (\bar{A} \overset{\mathbf{L}}{\otimes}_A T)$$

is in $\mathcal{D}^b(\bar{A}^{\text{op}} \otimes \mathcal{P})$. □

Lemma 3.7. *There is an isomorphism*

$$\mathbf{RHom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A \bar{A} \xrightarrow{\sim} \mathbf{RHom}_{\bar{A}}(D\bar{A}, \bar{A}) \quad \text{in } \mathcal{D}^b(\bar{A}^{\text{op}} \otimes A).$$

Proof. Let Z be an injective resolution of \bar{A} as a right A -module. Then we have

$$\mathbf{RHom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A \bar{A} \simeq \mathbf{Hom}_A(DA, Z) \otimes_A \bar{A}$$

using the fact that $\mathbf{Hom}_A(DA, Z)$ is a complex of projectives.

For each A -module M , we define a morphism $H_M : \mathbf{Hom}_A(DA, M) \otimes_A \bar{A} \rightarrow \mathbf{Hom}_A(D\bar{A}, M)$ as follows. Let φ be in $\mathbf{Hom}_A(DA, M)$, $\bar{a} \in \bar{A}$, and $\Phi \in D\bar{A}$. We define $H_M(\varphi \otimes \bar{a})(\Phi) := \varphi(\Phi_{\bar{a}}) \in M$, where $\Phi_{\bar{a}}(b) = \Phi(b\bar{a})$ for any $b \in A$. The morphism H_M is functorial in M . An easy computation shows that the map H_{DA} is an isomorphism. Since Z is a bounded complex of injective A -modules, we get the following isomorphism in $\mathcal{D}^b(\bar{A}^{\text{op}} \otimes A)$

$$\mathbf{RHom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A \bar{A} = \mathbf{Hom}_A(DA, Z) \otimes_A \bar{A} \xrightarrow{H_Z} \mathbf{Hom}_A(D\bar{A}, Z) = \mathbf{RHom}_A(D\bar{A}, \bar{A}).$$

Moreover by Lemma 3.2 any injective resolution of an \bar{A} -module X is an injective resolution of X viewed as an A -module, and the restriction functor is fully faithful. Thus we get isomorphisms in $\mathcal{D}^b(\bar{A}^{\text{op}} \otimes A)$

$$\mathbf{RHom}_A(D\bar{A}, \bar{A}) = \mathbf{Hom}_A(D\bar{A}, Z) \simeq \mathbf{Hom}_{\bar{A}}(D\bar{A}, Z) = \mathbf{RHom}_{\bar{A}}(D\bar{A}, \bar{A}).$$

□

Proof of Proposition 3.4. Combining Lemmas 3.5, 3.6 and 3.7, we get the following morphisms in $\mathcal{D}^b(\bar{A}^{\text{op}} \otimes \mathcal{E})$

$$\begin{array}{ccc}
\bar{A} \overset{\mathbf{L}}{\otimes}_A T & & \\
\wr \Big| \text{Lemma 3.5} & & \\
\mathbf{RHom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A T[2] & \xrightarrow[\text{Lemma 3.6}]{\alpha} & \mathbf{RHom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A (\bar{A} \overset{\mathbf{L}}{\otimes}_A T)[2] \\
& & \wr \Big| \text{Lemma 3.7} \\
& & \mathbf{RHom}_{\bar{A}}(D\bar{A}, \bar{A}) \overset{\mathbf{L}}{\otimes}_A T[2] \\
& & \wr \Big| \\
& & \mathbf{RHom}_{\bar{A}}(D\bar{A}, \bar{A}) \overset{\mathbf{L}}{\otimes}_{\bar{A}} (\bar{A} \overset{\mathbf{L}}{\otimes}_A T)[2]
\end{array}$$

By Lemma 3.6, the cone of the horizontal morphism is in $\mathcal{D}^b(\bar{A}^{\text{op}} \otimes \mathcal{P})$. Hence we get Proposition 3.4. \square

Using Proposition 3.4, we can apply Proposition 1.5 for $\Lambda = \bar{A}$ and $M = \bar{A} \overset{\mathbf{L}}{\otimes}_A T$ and we obtain the following.

Corollary 3.8. *In the setup of Theorem 3.1, there exists a triangle functor $G : \mathcal{C}_{\bar{A}} \rightarrow \underline{\mathcal{E}}$ such that the following diagram commute:*

$$\begin{array}{ccccc}
& & \bar{A} \overset{\mathbf{L}}{\otimes}_A T & & \\
& \xrightarrow{\quad -\otimes_{\bar{A}}(\bar{A} \overset{\mathbf{L}}{\otimes}_A T) \quad} & & \xrightarrow{\quad -\otimes_A T \quad} & \\
\mathcal{D}^b(\bar{A}) & \xrightarrow{R} & \mathcal{D}^b(A) & \xrightarrow{\quad -\otimes_A T \quad} & \mathcal{D}^b(\mathcal{E}) \\
\downarrow \pi_{\bar{A}} & & & & \downarrow \\
\mathcal{C}_{\bar{A}} & \xrightarrow{G} & & & \underline{\mathcal{E}}.
\end{array}$$

3.4. Proof of the equivalence. In this subsection we show that the triangle functor $G : \mathcal{C}_{\bar{A}} \rightarrow \underline{\mathcal{E}}$ of Corollary 3.8 is an equivalence. The proof is separated into four steps. First we show that the functor G sends the cluster-tilting object \bar{A} to the cluster-tilting object $T \in \underline{\mathcal{E}}$. The second step (Proposition 3.10) consists of proving that G induces an injective map from the endomorphism algebra of $\bar{A} \in \mathcal{C}_{\bar{A}}$ to the endomorphism algebra of $T \in \underline{\mathcal{E}}$. In Proposition 3.12, we prove that these two endomorphism algebras are isomorphic. The last step follows from [KR08] (Proposition 3.13) and finishes the proof of Theorem 3.1.

We have a triangle

$$Ae_F A \overset{\mathbf{L}}{\otimes}_A T \longrightarrow T \longrightarrow \bar{A} \overset{\mathbf{L}}{\otimes}_A T \longrightarrow Ae_F A \overset{\mathbf{L}}{\otimes}_A T[1] \quad \text{in } \mathcal{D}(\bar{A}^{\text{op}} \otimes \mathcal{E})$$

and the object $Ae_F A \overset{\mathbf{L}}{\otimes}_A T = e_F T$ is in \mathcal{P} . Hence $\bar{A} \overset{\mathbf{L}}{\otimes}_A T$ is isomorphic to T in $\underline{\mathcal{E}}$. Therefore the triangle functor G sends the cluster-tilting object \bar{A} to the cluster-tilting object T .

In order to prove Proposition 3.10 which is the second step, we first prove the following.

Lemma 3.9. *Let M be an object in $\mathcal{D}^b(\bar{A})$ such that its homology is concentrated in non positive degrees, and let $f \in \text{Hom}_{\mathcal{D}^b(\bar{A})}(\bar{A}, M)$ be a non zero morphism. Then the morphism $f \otimes_A^{\mathbf{L}} T \in \text{Hom}_{\mathcal{D}^b(\mathcal{E})}(\bar{A} \otimes_A^{\mathbf{L}} T, M \otimes_A^{\mathbf{L}} T)$ is non zero viewed as a morphism in $\underline{\mathcal{E}}$.*

Proof. By Lemma 3.2 (c), the restriction functor $\mathcal{D}^b(\bar{A}) \rightarrow \mathcal{D}^b(A)$ is fully faithful. We first show that the morphism $f \otimes_A^{\mathbf{L}} T$ is non zero as a morphism in $\mathcal{D}^b(\mathcal{E})$.

Now by Lemma 3.2 (a), \bar{A} is quasi-isomorphic to some complex in $\mathcal{D}^b(A)$ of the form:

$$\bar{A} = \cdots \longrightarrow 0 \longrightarrow Q^{-2} \longrightarrow Q^{-1} \longrightarrow (1 - e_F)A \longrightarrow 0 \longrightarrow \cdots$$

with Q^{-1} and Q^{-2} in $\text{add}(e_F A)$. Since M is concentrated in non positive degrees, M is quasi isomorphic to some bounded complex in $\mathcal{D}^b(A)$ of the form

$$M = \cdots \longrightarrow 0 \longrightarrow \cdots \longrightarrow e_{-3}A \longrightarrow e_{-2}A \longrightarrow e_{-1}A \longrightarrow e_0A \longrightarrow 0 \longrightarrow \cdots$$

Hence f is isomorphic to a morphism of complexes of the form

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & Q^{-2} & \longrightarrow & Q^{-1} & \longrightarrow & (1 - e_F)A & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow f^{-2} & & \downarrow f^{-1} & & \downarrow f^0 & & \downarrow & & \\ \cdots & \longrightarrow & e_{-3}A & \xrightarrow{d^{-3}} & e_{-2}A & \xrightarrow{d^{-2}} & e_{-1}A & \xrightarrow{d^{-1}} & e_0A & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Hence f is isomorphic to $f \circ p$, where $p : (1 - e_F)A \rightarrow \bar{A}$ is the projective cover. Hence f^0 is a non zero morphism.

By definition the morphism $f^0 \otimes_A^{\mathbf{L}} T$ is the morphism of complexes:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & (1 - e_F)T & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow f^0 \otimes_A T & & \downarrow & & \\ \cdots & \longrightarrow & e_{-3}T & \xrightarrow{d^{-3}} & e_{-2}T & \xrightarrow{d^{-2}} & e_{-1}T & \xrightarrow{d^{-1}} & e_0T & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Note that $f^0 \otimes_A^{\mathbf{L}} T = f^0 \otimes_A T$ since $(1 - e_F)A$ and e_0A are projective A -modules. In this diagram, the maps d^i and $f^0 \otimes_A T$ are morphisms of degree 0, since they come from morphisms in $\text{add}(A)$.

Denote by $P := \cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0$ a projective resolution in \mathcal{E} of the object $(1 - e_F)T$ and denote by p' the map $p' : P \rightarrow (1 - e_F)T$. Note that the P^i are also injective since \mathcal{E} is a Frobenius category. Assume that the morphism $f^0 \otimes_A^{\mathbf{L}} T$ vanishes in $\mathcal{D}^b(\mathcal{E})$. It implies that the morphism $(f^0 \otimes_A T) \circ p'$ is homotopic to zero.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P^{-3} & \longrightarrow & P^{-2} & \longrightarrow & P^{-1} & \longrightarrow & P^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow p' & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & (1 - e_F)T & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow f^0 \otimes_A T & & \downarrow & & \\ \cdots & \longrightarrow & e_{-3}T & \xrightarrow{d^{-3}} & e_{-2}T & \xrightarrow{d^{-2}} & e_{-1}T & \xrightarrow{d^{-1}} & e_0T & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

(Note: In the original image, there are diagonal arrows from P^{-3} to $e_{-3}T$, P^{-2} to $e_{-2}T$, P^{-1} to $e_{-1}T$, and P^0 to e_0T . A dotted arrow labeled x points from $e_{-1}T$ to e_0T .)

Since the complex $M \otimes_A^{\mathbf{L}} T$ is quasi isomorphic to a bounded complex with components in $\text{add}(T)$, an easy induction shows that the map $f^0 \otimes_A T = d^{-1}x$ is homotopic to zero. Since

$f^0 \otimes_A T$ and d^{-1} are homogeneous of degree 0, then we can assume that $x : (1 - e_F)T \rightarrow e_{-1}T$ is homogenous of degree 0. Therefore x can be written $y \otimes_A T$ for a map $y \in \mathbf{Hom}_{\mathcal{D}^b(A)}((1 - e_F)A, e_0A)$. Therefore $f^0 = d^{-1}y$ is homotopic to zero, hence f^0 vanishes as a map in $\mathcal{D}^b(A)$. Consequently, if f^0 is not zero in $\mathcal{D}^b A$, then the map $f \overset{\mathbf{L}}{\otimes}_A T$ is not zero in $\mathcal{D}^b(\mathcal{E})$, since $f^0 \overset{\mathbf{L}}{\otimes}_A T = (f \overset{\mathbf{L}}{\otimes}_A T) \circ (p \overset{\mathbf{L}}{\otimes}_A T)$.

Assume now that $f \overset{\mathbf{L}}{\otimes}_A T$ is zero in $\underline{\mathcal{E}}$. Since $p \overset{\mathbf{L}}{\otimes}_A T : (1 - e_F)T \rightarrow \bar{A} \overset{\mathbf{L}}{\otimes}_A T$ is an isomorphism in $\underline{\mathcal{E}}$, then $f \overset{\mathbf{L}}{\otimes}_A T$ vanishes in $\underline{\mathcal{E}}$ if and only if $f^0 \overset{\mathbf{L}}{\otimes}_A T$ vanishes in $\underline{\mathcal{E}}$. Thus $f^0 \overset{\mathbf{L}}{\otimes}_A T$ factors through an object P in $\mathcal{D}^b(\mathcal{P})$. For $i \in \mathbb{Z}$ we denote by $P^{\geq i}$ and $P^{\leq i}$ the positive and negative truncations

$$P^{\geq i} := \dots 0 \longrightarrow P^i \longrightarrow P^{i+1} \longrightarrow P^{i+2} \longrightarrow \dots \quad P^{\leq i} := \dots \longrightarrow P^{i-2} \longrightarrow P^{i-1} \longrightarrow P^i \longrightarrow 0 \dots$$

For any $i \in \mathbb{Z}$ the object P^i is projective and injective, thus the complex P and all its truncations are fibrant and cofibrant. Therefore the space $\mathbf{Hom}_{\mathcal{D}^b(\mathcal{E})}((1 - e_F)T, P^{\leq -1})$ vanishes because $(1 - e_F)T$ is concentrated in degree 0. And the space $\mathbf{Hom}_{\mathcal{D}^b(\mathcal{E})}(P^{\geq 1}, M \overset{\mathbf{L}}{\otimes}_A T)$ vanishes since M (hence $M \overset{\mathbf{L}}{\otimes}_A T$) is concentrated in non positive degrees. Consequently we can assume that $P = P^0$ is a stalk complex with $P \in \mathbf{add}(e_F T) = \mathcal{P}$. We write $f^0 \overset{\mathbf{L}}{\otimes}_A T = g \circ h$ with $g \in \mathbf{Hom}_{\mathcal{D}^b(\mathcal{E})}(P^0, e_0 T)$ and $h \in \mathbf{Hom}_{\mathcal{D}^b(\mathcal{E})}((1 - e_F)T, P^0)$. Now by definition $f^0 \overset{\mathbf{L}}{\otimes}_A T \in \mathbf{Hom}_{\mathcal{D}^b(\mathcal{E})}((1 - e_F)T, e_0 T)$ is of degree 0. By hypothesis (H4), the morphism h is in non negative degrees. Since $B = \mathbf{End}_{\mathcal{D}^b(\mathcal{E})}(T)$ is only in positive degree we get a contradiction. \square

Using this fundamental lemma, we can prove the following.

Proposition 3.10. *The functor G constructed in subsection 3.3 induces an injective map*

$$\mathbf{End}_{\mathcal{C}_{\bar{A}}}(\bar{A}) \longrightarrow \mathbf{End}_{\mathcal{E}}(T).$$

Proof. The two algebras $\mathbf{End}_{\mathcal{C}_{\bar{A}}}(\bar{A}) = \bigoplus_{i \geq 0} \mathbf{Hom}_{\mathcal{D}^b(\bar{A})}(\bar{A}, \mathbb{S}^{-i} \bar{A}[2i])$ and $\bar{B} = \mathbf{End}_{\mathcal{E}}(T)$ are graded algebras. The first part of the proof consists of showing that the functor $G : \mathcal{D}^b(\bar{A}) \rightarrow \underline{\mathcal{E}}$, which sends $\bar{A} \in \mathcal{C}_{\bar{A}}$ to $T \in \underline{\mathcal{E}}$, induces a morphism of graded algebras $\mathbf{End}_{\mathcal{C}_{\bar{A}}}(\bar{A}) \longrightarrow \mathbf{End}_{\mathcal{E}}(T)$ and then that this morphism is injective using Lemma 3.9.

Let $i \geq 0$ and $f^i \in \mathbf{Hom}_{\mathcal{D}^b(\bar{A})}(\bar{A}, \mathbb{S}^{-i} \bar{A}[2i])$ be a non zero morphism. We define $\bar{\theta}^p$ for $p \geq 0$ by induction by $\bar{\theta}^1 = \bar{\theta} = \mathbf{RHom}_{\bar{A}}(D\bar{A}, \bar{A})[2] = \mathbb{S}_2^{-1} \bar{A}[2]$ and $\bar{\theta}^p := \bar{\theta} \overset{\mathbf{L}}{\otimes}_{\bar{A}} \bar{\theta}^{p-1}$. Hence $\bar{\theta}^p = \mathbb{S}^{-p}(\bar{A})[2p]$ and f^i is a non zero map in $\mathbf{Hom}_{\mathcal{D}^b(\bar{A})}(\bar{A}, \bar{\theta}^i)$.

We first show that the triangle of Proposition 2.7 is in fact a triangle

$$\mathbf{RHom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A B(-1)[2] \longrightarrow \bar{A} \overset{\mathbf{L}}{\otimes}_A B \longrightarrow \bar{A} \longrightarrow \mathbf{RHom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A B(-1)[3]$$

in the category $\mathcal{D}^b(\mathbf{gr}(\bar{A} \otimes B))$, where (1) is the degree shift of the graded algebra B . Indeed the map $f : X \rightarrow Y$ constructed in subsection 2.4 is homogeneous of degree 1. Therefore by Lemma 3.5, we obtain an isomorphism

$$\bar{A} \overset{\mathbf{L}}{\otimes}_A B(1) \overset{\mathbf{L}}{\otimes}_B T \simeq \mathbf{RHom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A T[2].$$

Since the maps constructed in Lemmas 3.6 and 3.7 come from maps in $\mathcal{D}^b(A)$ they are of degree 0. Consequently the triangle of Proposition 2.7 is a triangle

$$\mathbf{RHom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A B(-1)[2] \longrightarrow \bar{A} \overset{\mathbf{L}}{\otimes}_A B \longrightarrow \bar{A} \longrightarrow \mathbf{RHom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A B(-1)[3]$$

in $\mathcal{D}^b(\mathbf{gr}(\bar{A} \otimes B))$. Hence by Proposition 3.4 we get a map $\alpha : \bar{A} \overset{\mathbf{L}}{\otimes}_A B(1) \overset{\mathbf{L}}{\otimes}_B T \rightarrow \bar{\theta} \overset{\mathbf{L}}{\otimes}_A T$ in $\mathcal{D}^b(\bar{A}^{\text{op}} \otimes \mathcal{E})$ whose cone is in $\mathcal{D}^b(\mathcal{P})$. We denote by α^i the composition:

$$\alpha^i : \bar{A} \overset{\mathbf{L}}{\otimes}_A B(i) \overset{\mathbf{L}}{\otimes}_B T \xrightarrow{\alpha} \bar{\theta} \overset{\mathbf{L}}{\otimes}_A B(i-1) \overset{\mathbf{L}}{\otimes}_B T \xrightarrow{\bar{\theta}^0 \overset{\mathbf{L}}{\otimes}_A \alpha} \bar{\theta}^2 \overset{\mathbf{L}}{\otimes}_A B(i-2) \overset{\mathbf{L}}{\otimes}_B T \longrightarrow \cdots \longrightarrow \bar{\theta}^i \overset{\mathbf{L}}{\otimes}_A T.$$

The image of $f^i \overset{\mathbf{L}}{\otimes}_A T$ in $\underline{\mathcal{E}}$ is isomorphic (as a morphism in $\underline{\mathcal{E}}$) to the left fraction $(\alpha^i)^{-1} \circ (f^i \overset{\mathbf{L}}{\otimes}_A T)$:

$$\begin{array}{ccc} \bar{A} \overset{\mathbf{L}}{\otimes}_A T & & \bar{A} \overset{\mathbf{L}}{\otimes}_A B(i) \overset{\mathbf{L}}{\otimes}_B T \\ & \searrow f^i \overset{\mathbf{L}}{\otimes}_A T & \swarrow \alpha^i \\ & \bar{\theta}^i \overset{\mathbf{L}}{\otimes}_A T & \end{array}$$

Thus the image of $f^i \overset{\mathbf{L}}{\otimes}_A T$ in $\underline{\mathcal{E}}$ is a map of degree i and G induces a morphism of graded algebras $\text{End}_{\mathcal{C}_{\bar{A}}}(\bar{A}) \longrightarrow \underline{\text{End}}_{\mathcal{E}}(T)$

The functor $- \overset{\mathbf{L}}{\otimes}_A T : \mathcal{D}^b \bar{A} \rightarrow \underline{\mathcal{E}}$ induces a map $\text{Hom}_{\mathcal{D}^b(\bar{A})}(\bar{A}, \bar{\theta}^i) \rightarrow \bar{B}$. Since the object $\bar{\theta}^i \in \mathcal{D}^b(\bar{A})$ has its homology concentrated in non positive degrees, this map is injective by Lemma 3.9. By the remark above, if $f^i \in \text{Hom}_{\mathcal{D}^b \bar{A}}(\bar{A}, \bar{\theta}^i)$, then $f^i \overset{\mathbf{L}}{\otimes}_A T$ is a morphism of degree i in the graded algebra $\underline{\text{End}}_{\mathcal{E}}(T) = \bar{B}$. Hence there is an injective map $\text{Hom}_{\mathcal{D}^b(\bar{A})}(\bar{A}, \bar{\theta}^i) \rightarrow \bar{B}_i$, where \bar{B}_i is the degree i part of the graded algebra $\bar{B} = \underline{\text{End}}_{\mathcal{E}}(T)$. Therefore the functor $G : \mathcal{C}_{\bar{A}} \rightarrow \underline{\mathcal{E}}$ induces a morphism of graded algebras

$$\bigoplus_{i \geq 0} \text{Hom}_{\mathcal{D}^b(\bar{A})}(\bar{A}, \mathbb{S}^{-i} \bar{A}[2i]) = \bigoplus_{i \geq 0} \text{Hom}_{\mathcal{D}^b(\bar{A})}(\bar{A}, \bar{\theta}^i) \longrightarrow \bar{B} = \bigoplus_{i \geq 0} \bar{B}_i,$$

which is an injection $\text{Hom}_{\mathcal{D}^b(\bar{A})}(\bar{A}, \bar{\theta}^i) \rightarrow \bar{B}_i$ for any $i \geq 0$. Consequently it is an injective algebra morphism. □

The next result will be used in the proof of Proposition 3.12 which is the third step.

Lemma 3.11. *Let $\text{Jac}(Q, W, F)$ be a frozen Jacobian algebra with W reduced. Assume there is a grading $\varphi : Q_1 \rightarrow \{0, 1\}$ satisfying the hypotheses (H2), (H3) and (H4). We denote by \bar{Q} the full subquiver of Q with set of vertices $\bar{Q}_0 := Q_0 \setminus F_0$, and by \bar{W} the image of W under the projection $kQ \rightarrow k\bar{Q}$. Then the set $\{\partial_a \bar{W}, \varphi(a) = 1\}$ is linearly independent. In particular, $\partial_a \bar{W}$ does not vanish for a in \bar{Q}_1 with $\varphi(a) = 1$.*

Proof. Let $a \in \bar{Q}_1$ be an arrow with $\varphi(a) = 1$. By condition (H3) the potential \bar{W} is homogeneous of degree 1, and by assumption the degree map φ has non-negative values. Thus any term in the potential \bar{W} contains exactly one arrow of degree 1. Consequently a cycle containing a which is a summand of W does not contain any other arrow of degree 1. Then by (H4) it

follows that this cycle does not pass through vertices in F_0 . Therefore we have $\partial_a W = \partial_a \bar{W}$. Now by (H2) the set $\{\partial_a W, \varphi(a) = 1\}$ is linearly independent, therefore we get the result. \square

Proposition 3.12. *Under the assumptions (H1)-(H4) there is an isomorphism of algebras*

$$\text{End}_{\underline{\mathcal{E}}}(T) \simeq \text{End}_{\mathcal{C}_{\bar{A}}}(\bar{A}).$$

Proof. The algebra $\bar{B} = B/Be_F B$ is isomorphic to the Jacobian algebra $\text{Jac}(\bar{Q}, \bar{W})$ (cf section 2), where \bar{Q} is the full subquiver of Q whose vertices are not in F_0 and where \bar{W} is the image of W under the projection $kQ \twoheadrightarrow kQ/kQe_F kQ \simeq k\bar{Q}$.

Let Q' be the subquiver of Q defined by:

- $Q'_0 := Q_0$;
- $Q'_1 := \{a \in Q_1, \varphi(a) = 0\}$.

By definition we have $A \simeq kQ'/\langle \partial_a W, a \notin F_1 \text{ and } \varphi(a) = 1 \rangle$. Let \bar{Q}' be the full subquiver of Q' with vertices $\bar{Q}'_0 = Q'_0 \setminus F_0$. Thus we get $\bar{A} \simeq k\bar{Q}'/\langle \partial_a \bar{W}, \varphi(a) = 1 \rangle$.

By Theorem 1.7, the endomorphism algebra $\text{End}_{\mathcal{C}_{\bar{A}}}(\bar{A})$ is isomorphic to the Jacobian algebra $\text{Jac}(\widetilde{\bar{Q}'}, W_{\bar{A}})$ where $\widetilde{\bar{Q}'}$ and $W_{\bar{A}}$ are defined in Theorem 1.7. By the previous lemma, the set $\{\partial_a \bar{W}, \varphi(a) = 1\}$ is a basis for the ideal of relations of \bar{A} . Hence we immediately see that $\widetilde{\bar{Q}'} = \bar{Q}$ and that $W_{\bar{A}}$ is the potential

$$W_{\bar{A}} = \sum_{a, \varphi(a)=1} a \partial_a \bar{W}.$$

Therefore the potential \bar{W} is cyclically equivalent to the potential $W_{\bar{A}}$. Thus we have an isomorphism $\text{Jac}(\widetilde{\bar{Q}'}, W_{\bar{A}}) \simeq \text{Jac}(\bar{Q}, \bar{W})$ which gives the desired isomorphism. \square

Now we finish the proof of Theorem 3.1. The triangle functor $G : \mathcal{C}_{\bar{A}} \rightarrow \underline{\mathcal{E}}$ constructed in subsection 3.3 sends the cluster-tilting object $\bar{A} \in \mathcal{C}_{\bar{A}}$ to the cluster-tilting object $T \in \underline{\mathcal{E}}$. By Proposition 3.10, it induces an injective map $\text{End}_{\mathcal{C}_{\bar{A}}}(\bar{A}) \rightarrow \underline{\text{End}}_{\underline{\mathcal{E}}}(T)$. These two algebras are finite dimensional algebras which are isomorphic by Proposition 3.12. Therefore G induces a bijection $\text{End}_{\mathcal{C}_{\bar{A}}}(\bar{A}) \simeq \underline{\text{End}}_{\underline{\mathcal{E}}}(T)$. Then we conclude the proof of Theorem 3.1 using the following.

Proposition 3.13 ([KR08] Lemma 4.5). *Let \mathcal{C} and \mathcal{C}' be Hom-finite 2-Calabi-Yau triangulated categories. Let T (resp. T') be a cluster-tilting object in \mathcal{C} (resp. \mathcal{C}'). If we have a triangle functor $G : \mathcal{C} \rightarrow \mathcal{C}'$ which sends T to T' and which induces an isomorphism between $\text{End}_{\mathcal{C}}(T)$ and $\text{End}_{\mathcal{C}'}(T')$, then G is an equivalence.*

4. 2-CALABI-YAU CATEGORIES ASSOCIATED WITH ELEMENTS IN THE COXETER GROUP

In this section we apply Theorem 3.1 to the categories associated with elements in the Coxeter group introduced in [BIRS09a].

4.1. Results of [BIRS09a] and [BIRS09b]. Let Q be a finite quiver without oriented cycles. We denote as usual by $Q_0 = \{1, \dots, n\}$ the set of vertices and by Q_1 the set of arrows. The preprojective algebra associated to Q is the algebra

$$k\bar{Q}/\langle \sum_{a \in Q_1} aa^* - a^*a \rangle$$

where \overline{Q} is the double quiver of Q , which by definition is obtained from Q by adding to each arrow $a : i \rightarrow j$ in Q_1 an arrow $a^* : i \leftarrow j$ pointing in the opposite direction. We denote by Λ the completion of the preprojective algebra associated to Q , and by $\text{f.l.}\Lambda$ the category of right Λ -modules of finite length.

Let C_Q be the Coxeter group associated to Q . It is defined by the generators s_i where $i \in Q_0$ and by the relations

- $s_i^2 = 1$,
- $s_i s_j = s_j s_i$ if there is no arrow between i and j ,
- $s_i s_j s_i = s_j s_i s_j$ if there is exactly one arrow between i and j .

A *reduced expression* $\mathbf{w} = s_{u_1} \dots s_{u_l}$ of an element w in C_Q is an expression with l smallest possible. When $\mathbf{w} = s_{u_1} \dots s_{u_l}$ is a reduced expression of w , the integer $l(w) := l$ is called the length of w .

For a vertex i in Q_0 we denote by \mathcal{I}_i the ideal $\Lambda(1 - e_i)\Lambda$. Let $\mathbf{w} = s_{u_1} \dots s_{u_l}$ be a reduced expression of an element w in C_Q . For $p \leq l$ we denote by $\mathcal{I}_{\mathbf{w}_p}$ the ideal $\mathcal{I}_{u_p} \mathcal{I}_{u_{p-1}} \dots \mathcal{I}_{u_1}$. The ideal $\mathcal{I}_{\mathbf{w}} := \mathcal{I}_{\mathbf{w}_l}$ depends only on the element $w \in C_Q$ and not on the choice of the reduced expression \mathbf{w} . Therefore we denote by Λ_w the algebra $\Lambda/\mathcal{I}_{\mathbf{w}}$ and by $\mathcal{E}_w := \text{Sub } \Lambda_w$ the subcategory of $\text{f.l.}\Lambda$ consisting of submodules of finite direct sums of copies of Λ_w .

We have the following [BIRS09a, Theorem III.2.8].

Theorem 4.1 (Buan-Iyama-Reiten-Scott). *Let w be an element in the Coxeter group C_Q . Then we have the following.*

- (a) *The category \mathcal{E}_w is a Hom-finite Frobenius stably 2-CY category.*
- (b) *For any reduced expression $\mathbf{w} = s_{u_1} \dots s_{u_l}$ of w , the object $T_{\mathbf{w}} = \bigoplus_{p=1}^l e_{u_p}(\Lambda/\mathcal{I}_{\mathbf{w}_p})$ is a cluster-tilting object.*
- (c) *The projective-injective indecomposable objects are $e_{u_{t_i}}(\Lambda/\mathcal{I}_{\mathbf{w}_{t_i}})$ where t_i is the maximal integer such that $u_{t_i} = i$ for $i \in Q_0$.*

The cluster-tilting object $T_{\mathbf{w}}$ depends on the choice of the reduced expression of w . We refer to a cluster-tilting object of this form as a *standard cluster-tilting object*. Note that by mutation we may get other cluster-tilting objects which are not standard.

We now define a quiver $Q_{\mathbf{w}}$ associated with a reduced expression $\mathbf{w} = s_{u_1} \dots s_{u_l}$ of an element $w \in C_Q$ as follows:

- vertices: $1, \dots, l(w)$.
- for each $i \in Q_0$, one arrow $t \leftarrow s$ if t and s are two consecutive vertices of type i (i.e. $u_s = u_t = i$) and $t < s$ (we call these arrows *arrows going to the left*);
- for each $a : i \rightarrow j \in Q_1$, put $a : t \rightarrow s$ if t is a vertex of type i , s of type j , and if there is no vertex of type i between t and s , and if s is the last vertex of type j before the next vertex of type i in the expression $\mathbf{w} = s_{u_1} \dots s_{u_l}$ (we call these arrows the *Q-arrows*);
- for each $a : i \rightarrow j \in Q_1$, put $a^* : t \rightarrow s$ if t is of type j , s is of type i , if there is no vertex of type j between t and s and if s is the last vertex of type i before the next vertex of type j in the expression \mathbf{w} (we call these arrows the *Q*-arrows*).

For each Q -arrow $a : t \rightarrow s$ in $Q_{\mathbf{w}}$, we denote by W_a the composition aa^*p if there is a (unique) Q^* -arrow $a^* : r \rightarrow t$ in $Q_{\mathbf{w}}$ where $u_r = u_s$ and where p is the composition of arrows going to the left $r \leftarrow \dots \leftarrow s$. Otherwise we put $W_a = 0$. For each Q^* -arrow $a^* : t \rightarrow r$ in $Q_{\mathbf{w}}$, we denote by W_{a^*} the composition a^*ap if there exists a (unique) Q -arrow $a : s \rightarrow t$ with $u_s = u_r$ in $Q_{\mathbf{w}}$ and where p is the composition of arrows going to the left $s \leftarrow \dots \leftarrow r$.

Otherwise we put $W_{a^*} = 0$. Then let $W_{\mathbf{w}}$ be the sum

$$W_{\mathbf{w}} = \sum_{a \text{ } Q\text{-arrow}} W_a - \sum_{a^* \text{ } Q^*\text{-arrow}} W_{a^*}.$$

We have the following [BIRS09b, Theorem 6.8].

Theorem 4.2 (Buan-Iyama-Reiten-Smith). *Let $\mathbf{w} = s_{u_1} \dots s_{u_l}$ be a reduced expression of an element w of the Coxeter group C_Q . Let $T_{\mathbf{w}}$ be the associated standard cluster-tilting object in the category \mathcal{E}_w . Then we have an isomorphism*

$$\text{End}_{\mathcal{E}_w}(T_{\mathbf{w}}) \simeq \text{Jac}(Q_{\mathbf{w}}, W_{\mathbf{w}}, F)$$

where $F_0 := \{t_1, \dots, t_n\}$ consists of the last vertex of each kind in the reduced expression \mathbf{w} and $F_1 := \{a \in Q_1, s(a) \in F_0 \text{ and } t(a) \in F_0\}$.

4.2. Definition of the grading. The algebra Λ and the associated Coxeter group do not depend on the orientation of Q . For any reduced expression $\mathbf{w} = s_{u_1} \dots s_{u_l}$ of $w \in C_Q$, the category \mathcal{E}_w and the cluster-tilting object $T_{\mathbf{w}}$ do not depend on the orientation of Q . From now on we assume that the orientation of Q satisfies the property that $t_i < t_j$ if there exists an arrow $i \rightarrow j$, where t_i is the maximal integer satisfying $u_{t_i} = i$.

We define a grading on the quiver $Q_{\mathbf{w}}$ as follows:

- $\varphi(b) = 1$ if b is a Q^* -arrow;
- $\varphi(b) = 0$ if b is a Q -arrow or an arrow going to the left.

With this choice of grading we show that our axioms are satisfied.

Lemma 4.3. *The graded Jacobian algebra $(\text{Jac}(Q_{\mathbf{w}}, W_{\mathbf{w}}, F), \varphi)$ satisfies the conditions (H1)-(H4) of Theorem 3.1.*

Proof. (H1) This holds by Theorems 4.1 and 4.2.

(H2) The potential $W_{\mathbf{w}} = W$ is reduced. Moreover it is easy to see that two different terms of the potential W differ by at least two arrows. It follows that the set $\{\partial_a W \mid a \notin F_1, \partial_a W \neq 0\}$ is a basis for the ideal of relations. We next show that for any arrow $a \notin F_1$, the derivative $\partial_a W$ does not vanish. Assume that some Q -arrow or Q^* -arrow $a : r \rightarrow s$ does not appear in the potential, where $u_r = i$, $u_s = j$ and there is an arrow between i and j in the quiver Q . Then there is no u_t of type i with $t > r$. Thus we have $r = t_i$. Then we must have $s = t_j$. Therefore $a : r = t_i \rightarrow s = t_j$ is in F_1 . Now let $p : r \leftarrow s$ be an arrow going to the left, where r and s are two consecutive vertices of type i . Since the expression $s_{u_1} \dots s_{u_l}$ is reduced, there exists t with $r < t < s$ such that u_t is of type j and there is (at least) one arrow between i and j in the quiver Q . Let t be the maximal integer with this property. Then there is an arrow $a : r \rightarrow t$, and there is also an arrow $a^* : t \rightarrow v$ where $v \geq s$ and u_v is of type i . Thus p appears in the potential W . Therefore for any arrow a of $Q_{\mathbf{w}}$ which is not in F_1 , the derivative $\partial_a W$ is not zero, and we have (H2).

(H3) By the definition of the potential in Theorem 4.2, this follows immediately.

(H4) By the choice of the orientation of Q , any arrow in F_1 is a Q -arrow, hence of degree 0. Thus all arrows with target in F_0 and source not in F_0 are Q^* -arrows, hence of degree 1, and all arrows with source in F_0 and target not in F_0 are arrows going to the left, hence of degree 0. This implies condition (H4). □

The following direct consequence is one of our main results.

Theorem 4.4. *The stable category $\underline{\mathcal{E}}_w$ is a generalized cluster category.*

4.3. Meaning of the grading. We show that the grading φ on $\text{End}_{\mathcal{E}_w}(T)$ defined in the previous subsection is induced by a natural grading on the preprojective algebra.

Let $\mathbf{w} = s_{u_1} \dots s_{u_l}$ be a reduced expression of an element w of the Coxeter group C_Q . Assume that the orientation of Q satisfies the property that $t_i < t_j$ if there exists an arrow $i \rightarrow j$, where t_i is the maximal integer satisfying $u_{t_i} = i$.

We define a grading on the double quiver \overline{Q} as follows:

- $\deg(a) = 0$ if a is an arrow of Q ;
- $\deg(a^*) = 1$ if $a^* : j \rightarrow i$ is an arrow pointing in the opposite direction of an arrow $a : i \rightarrow j$ in Q .

The ideal of relations $(\sum_{a \in Q_1} aa^* - a^*a)$ is homogeneous of degree 1, thus the grading on the double quiver \overline{Q} induces a grading on the preprojective algebra Λ .

For any i in Q_0 , the Λ -module $e_i\Lambda$ can be seen as a graded Λ -module with top in degree 0. Then the ideal $\mathcal{I}_i = \Lambda(1 - e_i)\Lambda$ is a graded ideal. Hence for $p \leq l$ the ideal $\mathcal{I}_{\mathbf{w}_p} = \mathcal{I}_{u_p} \dots \mathcal{I}_{u_1}$ is a graded ideal and the module $T_p = e_{u_p}(\Lambda/\mathcal{I}_{\mathbf{w}_p})$ is a finite length graded Λ -module. Therefore the standard cluster-tilting object $T_{\mathbf{w}} = T_1 \oplus \dots \oplus T_l$ is a graded Λ -module. Thus its endomorphism algebra $\text{End}_{\Lambda}(T_{\mathbf{w}})$ is naturally graded. We have the following connection with the previous grading.

Proposition 4.5. *The isomorphism of algebras $\text{End}_{\Lambda}(T_{\mathbf{w}}) \simeq \text{Jac}(Q_{\mathbf{w}}, W_{\mathbf{w}}, F)$ of Theorem 4.2 is an isomorphism of graded algebras*

$$(\text{End}_{\Lambda}(T_{\mathbf{w}}), \deg) \simeq (\text{Jac}(Q_{\mathbf{w}}, W_{\mathbf{w}}, F), \varphi)$$

where \deg is induced by the grading \deg on the preprojective algebra Λ , and φ is the grading on $Q_{\mathbf{w}}$ defined in the previous section.

Proof. Each $a : i \rightarrow j$ in Q_1 gives maps $e_i(\Lambda/\mathcal{I}_{\mathbf{w}_r}) \rightarrow e_j(\Lambda/\mathcal{I}_{\mathbf{w}_s})$, where $u_r = i$ and $u_s = j$. These maps are obviously of degree 0 since they are induced by the degree zero map $a : e_i\Lambda \rightarrow e_j\Lambda$.

Each $a : i \rightarrow j$ in Q induces maps $e_j(\Lambda/\mathcal{I}_{\mathbf{w}_t}) \rightarrow e_i(\Lambda/\mathcal{I}_{\mathbf{w}_s})$, where $u_t = j$ and $u_s = i$. They are induced by the degree 1 map $a^* : e_j\Lambda \rightarrow e_i\Lambda$, thus they are maps of degree 1.

For any i in Q_0 , there are surjective maps $e_i(\Lambda/\mathcal{I}_{\mathbf{w}_t}) \rightarrow e_i(\Lambda/\mathcal{I}_{\mathbf{w}_r})$, where $u_t = u_r = i$ and $t > r$. They are induced by the identity $e_i\Lambda \rightarrow e_i\Lambda$, thus they are maps of degree 0.

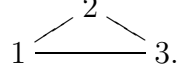
Hence we get the grading φ defined in the previous section. \square

Remark 4.6. Note that the summands of Λ_w are all graded Λ -modules, but this does not imply that all the objects in \mathcal{E}_w are gradable. In the proof of [GS05, Proposition 5.2], Geiss and Schröer describe explicitly a non gradable module over the preprojective algebra associated to the Dynkin graph A_6 .

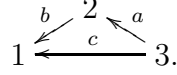
5. EXAMPLES

In this section we illustrate the previous theory through two examples. The first one is an example covered by Theorem 4.2. It is given by a standard cluster-tilting object in the category \mathcal{E}_w for some reduced word \mathbf{w} . The second example shows that Theorem 3.1 may also apply for cluster-tilting objects in \mathcal{E}_w which are not standard.

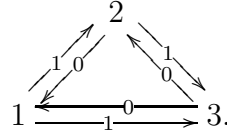
5.1. Standard cluster-tilting object associated to a reduced word. Let Q be the following graph



Let \mathbf{w} be the reduced word $s_1 s_2 s_3 s_1 s_3 s_2 s_1$ in the Coxeter group C_Q . An admissible orientation of Q as defined in section 4.2 is



Let us put the following grading for the preprojective algebra Λ .

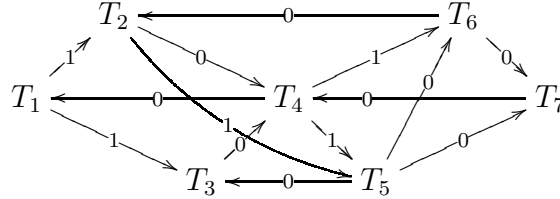


Then the canonical cluster-tilting object $T_{\mathbf{w}}$ of the Frobenius category $\mathcal{E}_{\mathbf{w}}$ has the following indecomposable summands:

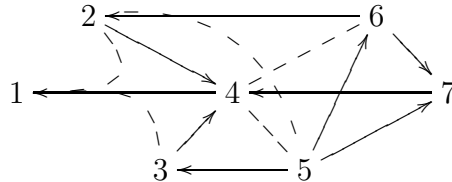
$$\begin{aligned} T_1 &= 1, & T_2 &= \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, & T_3 &= \begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, & T_4 &= \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}, \\ T_5 &= \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, & T_6 &= \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \text{ and } T_7 &= \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \end{aligned}$$

The indecomposable projective injectives are T_5 , T_6 and T_7 . As Λ_0 -module ($=kQ$ -module) T_6 is isomorphic to the direct sum $\begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \oplus 1 \oplus 2 \oplus 1$.

By [BIRS09a] and [BIRS09b], we know the shape of the quiver of $B = \text{End}_{\mathcal{E}_{\mathbf{w}}}(T)$. Its grading coming from the grading of Λ is the following.



The algebra $A = B_0$ is then given by the quiver with relations.



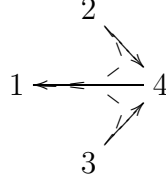
The indecomposable projective A -modules are

$$\begin{smallmatrix} 1 \\ 4 \\ 7 \end{smallmatrix}, \quad \begin{smallmatrix} 2 \\ 6 \end{smallmatrix}, \quad \begin{smallmatrix} 3 \\ 5 \end{smallmatrix}, \quad \begin{smallmatrix} 3 \\ 5 \end{smallmatrix} \begin{smallmatrix} 4 \\ 7 \\ 6 \end{smallmatrix} \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}, \quad 5, \quad \begin{smallmatrix} 6 \\ 5 \end{smallmatrix}, \quad \text{and} \quad \begin{smallmatrix} 6 \\ 5 \end{smallmatrix} \begin{smallmatrix} 7 \\ 5 \end{smallmatrix}$$

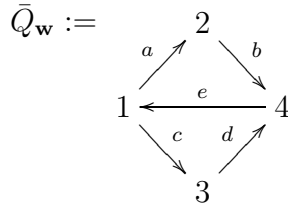
and the indecomposable injectives are

$$1, \quad \frac{4}{2}, \quad \frac{4}{3}, \quad \frac{1}{4}, \quad 3 \frac{4}{7} \frac{6}{5} \frac{7}{6}, \quad \frac{7}{6} \frac{4}{2}, \quad \text{and} \quad \frac{1}{4} \frac{7}{7}$$

The algebra \bar{A} is given by the quiver with relations:



It is an algebra of global dimension 2. We have $\text{End}_{\mathcal{C}_{\bar{A}}}(\bar{A}) \simeq \text{Jac}(\bar{Q}_{\mathbf{w}}, \bar{W}_{\mathbf{w}})$ where



and $\bar{W}_{\mathbf{w}} := bae + dce$. It is isomorphic to the algebra \bar{B} .

We denote by G the composition

$$G : \mathcal{D}^b(\bar{A}) \xrightarrow{\text{Res}} \mathcal{D}^b(A) \xrightarrow{-\overset{\mathbf{L}}{\otimes}_A B} \mathcal{D}^b(B) \xrightarrow{-\overset{\mathbf{L}}{\otimes}_B T} \mathcal{D}^b(\mathcal{E}_w) \longrightarrow \mathcal{D}^b(\mathcal{E}_w)/\mathcal{D}^b(\mathcal{P}) \simeq \underline{\mathcal{E}}_w.$$

Let S_2 be the simple \bar{A} -module associated to the vertex 2. We will show that $G(S_2)$ and $G \circ \mathbb{S}^{-1}(S_2)[2]$ are isomorphic as objects in $\underline{\mathcal{E}}_w$, where \mathbb{S} denotes the Serre functor of $\mathcal{D}^b \bar{A}$.

The restriction of S_2 in the category $\mathcal{D}^b(\bar{A})$ is quasi-isomorphic to the complex

$$\cdots \longrightarrow 0 \longrightarrow e_5 A \longrightarrow e_6 A \longrightarrow e_2 A \longrightarrow 0 \longrightarrow \cdots$$

Tensoring with B over A we get the complex

$$\cdots \longrightarrow 0 \longrightarrow e_5 B \longrightarrow e_6 B \longrightarrow e_2 B \longrightarrow 0 \longrightarrow \cdots.$$

Therefore $G(S_2)$ is the complex

$$G(S_2) = (T_5 \longrightarrow T_6 \longrightarrow T_2).$$

Now the simple \bar{A} -module S_2 is quasi-isomorphic in $\mathcal{D}^b(\bar{A})$ to the complex

$$\cdots \longrightarrow 0 \longrightarrow e_2(D\bar{A}) \longrightarrow e_4(D\bar{A}) \longrightarrow e_1(D\bar{A}) \longrightarrow 0 \longrightarrow \cdots$$

Hence the restriction of $S_2 \overset{\mathbf{L}}{\otimes}_{\bar{A}} \mathbf{R}\text{Hom}_{\bar{A}}(D\bar{A}, \bar{A})[2] = \mathbb{S}^{-1}(S_2)[2]$ in $\mathcal{D}^b A$ is isomorphic to

$$\cdots \longrightarrow 0 \longrightarrow 2 \longrightarrow 2 \frac{4}{7} \frac{6}{5} \frac{7}{6} \longrightarrow \frac{1}{4} \longrightarrow 0 \longrightarrow \cdots$$

which is quasi-isomorphic to the complex

$$\cdots \longrightarrow 0 \longrightarrow 5 \oplus \frac{2}{6} \longrightarrow 3 \frac{4}{7} \frac{6}{5} \frac{7}{6} \longrightarrow \frac{1}{4} \longrightarrow 0 \longrightarrow \cdots$$

that is, to the complex

$$\cdots \longrightarrow 0 \longrightarrow e_5 A \oplus e_2 A \longrightarrow e_4 A \longrightarrow e_1 A \longrightarrow 0 \longrightarrow \cdots$$

Therefore the object $G(\mathbb{S}^{-1}S_2[2])$ is the complex

$$G(\mathbb{S}^{-1}S_2[2]) = (T_5 \oplus T_2 \longrightarrow T_4 \longrightarrow T_1) .$$

Now the simple \bar{A} -module S_2 is also quasi-isomorphic to the complex

$$\cdots \longrightarrow 0 \longrightarrow e_2(DA) \longrightarrow e_4(DA) \longrightarrow e_1(DA) \longrightarrow 0 \longrightarrow \cdots$$

Hence the complex $S_2 \overset{\mathbf{L}}{\otimes}_{\bar{A}} \mathbf{RHom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A B[2] \simeq \mathbf{RHom}_A(DA, S_2) \overset{\mathbf{L}}{\otimes}_A B[2]$ is the complex

$$\cdots \longrightarrow 0 \longrightarrow e_2B \longrightarrow e_4B \longrightarrow e_1B \longrightarrow \cdots .$$

We have a morphism in the category $\mathcal{D}^b(B)$:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & e_2B & \longrightarrow & e_4B \longrightarrow e_1B \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & 0 & \longrightarrow & e_5B & \longrightarrow & e_6B \longrightarrow e_2B \longrightarrow \cdots \end{array}$$

whose cone is

$$\cdots \longrightarrow 0 \longrightarrow e_2B \longrightarrow e_5B \oplus e_4B \longrightarrow e_6B \oplus e_1B \longrightarrow e_2B \longrightarrow 0 \longrightarrow \cdots .$$

An easy computation shows that it is quasi-isomorphic to the simple B -module S_2 . Hence we get the following triangle in $\mathcal{D}^b(B)$:

$$S_2 \overset{\mathbf{L}}{\otimes}_{\bar{A}} \mathbf{RHom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A B[2] \longrightarrow S_2 \overset{\mathbf{L}}{\otimes}_A B \longrightarrow S_2 \longrightarrow S_2 \overset{\mathbf{L}}{\otimes}_A \mathbf{RHom}_A(DA, \bar{A}) \overset{\mathbf{L}}{\otimes}_A B[3] .$$

Note that this triangle is nothing but $S_2 \overset{\mathbf{L}}{\otimes}_{\bar{A}} (*)$ where $(*)$ is the triangle defined in Proposition 2.7. The object $S_2 \overset{\mathbf{L}}{\otimes}_B T$ in $\mathcal{D}^b(\mathcal{E}_w)$ is then the complex

$$\cdots \longrightarrow 0 \longrightarrow T_2 \longrightarrow T_5 \oplus T_4 \longrightarrow T_6 \oplus T_1 \longrightarrow T_2 \longrightarrow 0 \longrightarrow \cdots .$$

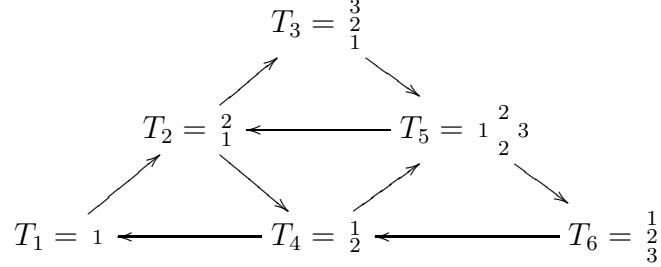
A direct computation shows that it is acyclic. Indeed it is the 2-almost split sequence associated with T_2 . Finally we have morphisms

$$\begin{array}{ccc} & (T_2 \rightarrow T_4 \rightarrow T_1) & \\ \swarrow (i) & & \searrow (ii) \\ G(S_2) = (T_5 \rightarrow T_6 \rightarrow T_2) & & (T_5 \oplus T_2 \rightarrow T_4 \rightarrow T_1) = G(\mathbb{S}^{-1}S_2[2]) \\ \downarrow & & \downarrow \\ 0 \simeq (T_2 \rightarrow T_5 \oplus T_4 \rightarrow T_1 \oplus T_6 \rightarrow T_2) & & (T_5) \in \mathcal{D}^b(\mathcal{P}) \end{array}$$

The cone of the morphism (i) is acyclic, and the cone of the morphism (ii) is $T_5[-2]$, which is perfect. Thus in $\underline{\mathcal{E}}_w = \mathcal{D}^b(\mathcal{E}_w)/\mathcal{D}^b(\mathcal{P})$, the objects $G(S_2)$ and $G(\mathbb{S}^{-1}S_2[2])$ are isomorphic.

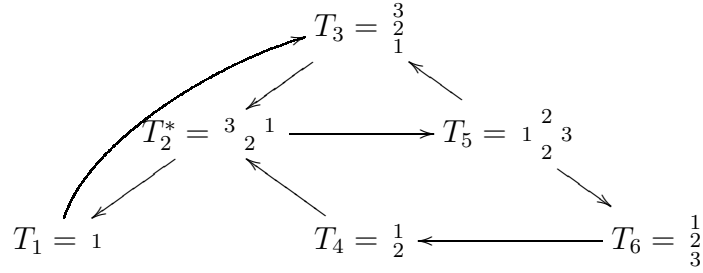
5.2. Example which is not associated with a word. Let \mathcal{E} be the category $\mathbf{mod} \Lambda$ where Λ is the preprojective algebra of type A_3 , which is one of the cases investigated by Geiss, Leclerc and Schröer in [GLS06]. This is a Frobenius category which is stably 2-Calabi-Yau and of the

form \mathcal{E}_w where w is the element in the Coxeter group of maximal length. Corresponding to the reduced expression $s_1 s_2 s_3 s_1 s_2 s_1$ is the standard cluster-tilting object:

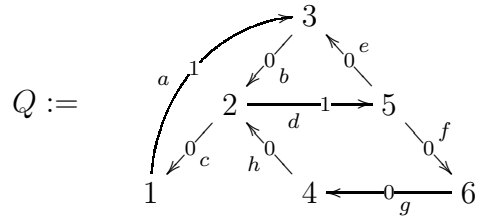


The endomorphism algebra is a frozen Jacobian algebra.

Let us mutate the object $T_2 = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$. Its complement is $T_2^* = \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \begin{smallmatrix} 1 \\ \end{smallmatrix}$. The new cluster-tilting object T^* is given by

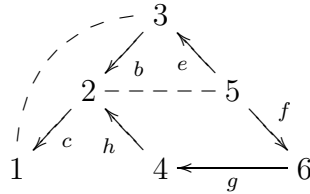


One can easily check that the endomorphism algebra is isomorphic to the frozen Jacobian algebra $B = \text{Jac}(Q, W, F)$, where

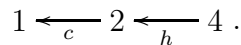


$W := acb + dbe + dhgf$, $F_0 := \{3, 5, 6\}$ and $F_1 := \{e, f\}$.

If we put $\varphi(a) = \varphi(d) = 1$ and $\varphi(b) = \varphi(c) = \varphi(e) = \varphi(f) = \varphi(h) = 0$, we obtain a grading satisfying hypothesis (H1)-(H4) of Theorem 3.1. The algebra A is then given by the quiver



with relations $cb = 0$ and $be = hgf$. The algebra \bar{A} is the hereditary algebra with quiver



One can check that the image of S_2 under the functor $G : \mathcal{D}^b(\bar{A}) \xrightarrow{\text{Res}} \mathcal{D}^b(A) \xrightarrow{-\otimes_A T^*} \mathcal{D}^b(\mathcal{E})$ is

$$(T_5 \xrightarrow{\begin{pmatrix} e \\ fg \end{pmatrix}} T_4 \oplus T_3 \xrightarrow{\begin{pmatrix} b & h \end{pmatrix}} T_2^*).$$

The object $\mathbb{S}^{-1}(S_2)[2]$ is quasi-isomorphic to the complex $\frac{2}{4} \longrightarrow \frac{1}{4}$, thus its restriction in $\mathcal{D}^b(A)$ is quasi-isomorphic to the complex

$$\cdots \longrightarrow 0 \longrightarrow e_3 A \xrightarrow{b} e_2 A \xrightarrow{c} e_1 A \longrightarrow 0 \longrightarrow \cdots.$$

Hence the complex $G(\mathbb{S}^{-1}S_2[2])$ is

$$\cdots \longrightarrow 0 \longrightarrow T_3 \xrightarrow{b} T_2^* \xrightarrow{c} T_1 \longrightarrow 0 \longrightarrow \cdots.$$

We have morphisms in $\mathcal{D}^b(\mathcal{E})$

$$\begin{array}{ccc} & (T_2^* \rightarrow T_1) & \\ (i) \swarrow & & \searrow (ii) \\ G(S_2) = (T_5 \rightarrow T_4 \oplus T_3 \rightarrow T_2^*) & & (T_3 \rightarrow T_2^* \rightarrow T_1) = G \circ \mathbb{S}(S_2)[2] \\ \downarrow & & \downarrow \\ 0 \simeq (T_2^* \rightarrow T_1 \oplus T_5 \rightarrow T_3 \oplus T_4 \rightarrow T_2^*) & & (T_3) \in \mathcal{D}^b(\mathcal{P}) \end{array}$$

The cone of the morphism (i) is

$$T_2^* \xrightarrow{\begin{pmatrix} c \\ d \end{pmatrix}} T_1 \oplus T_5 \xrightarrow{\begin{pmatrix} a & e \\ 0 & fg \end{pmatrix}} T_3 \oplus T_4 \xrightarrow{\begin{pmatrix} b & h \end{pmatrix}} T_2^*$$

which is the 2-almost-split sequence associated to T_2^* , hence an acyclic complex. The cone of the morphism (ii) is $T_3[-2]$, which is perfect. Thus in $\mathcal{D}^b(\mathcal{E})/\mathcal{D}^b(\mathcal{P})$ the objects $G(S_2)$ and $G(\mathbb{S}^{-1}S_2[2])$ are isomorphic.

This example gives some hope that Theorem 3.1 can be applied to stably 2-CY categories other than those coming from an element of a Coxeter group.

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