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THE UBIQUITY OF GENERALIZED CLUSTER CATEGORIES

CLAIRE AMIOT, IDUN REITEN, AND GORDANA TODOROV

ABSTRACT. Associated with some finite dimensional algebras of global dimension at most 2, a generalized cluster category was introduced in [Ami08], which was shown to be triangulated and 2-Calabi-Yau when it is Hom-finite. By definition, the cluster categories of [BMR⁺06] are a special case. In this paper we show that a large class of 2-Calabi-Yau triangulated categories, including those associated with elements in Coxeter groups from [BIRS09a], are triangle equivalent to generalized cluster categories. This was already shown for some special elements in [Ami08].

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INTRODUCTION

Throughout the paper k is an algebraically closed field. Let Q be a finite quiver without oriented cycles. In [BMR⁺06], the cluster category \mathcal{C}_Q was defined to be the orbit category $\mathcal{D}^b(kQ)/\tau^-[1]$ where τ is the Auslander-Reiten translation in the bounded derived category $\mathcal{D}^b(kQ)$. The category \mathcal{C}_Q is **Hom**-finite and triangulated [Kel05], and 2-Calabi-Yau (2-CY for short), that is, there is a functorial isomorphism $D\mathrm{Hom}_{\mathcal{C}_Q}(X, Y) \simeq \mathrm{Hom}_{\mathcal{C}_Q}(Y, X[2])$, where $D = \mathrm{Hom}_k(-, k)$. A theory for a special kind of objects, called *cluster-tilting objects*, was developed in [BMR⁺06]. This work was motivated via [MRZ03] by the Fomin-Zelevinsky theory of cluster algebras [FZ02], where the cluster-tilting objects are the analogs of clusters.

Another category where a similar theory was developed is the category $\mathbf{mod} \Lambda$ of finite dimensional modules over a preprojective algebra of Dynkin type [GLS07a], [GLS06]. This category is **Hom**-finite and Frobenius. Moreover it is stably 2-CY, that is, its stable category (which is triangulated) is 2-CY.

Much of the work on cluster categories has been generalized to the setting of 2-CY triangulated categories with cluster-tilting objects, and new results have been proved in the general setting ([IY08], [KR08], [KR07], and others). It is of interest to investigate such categories, both for developing new theory and for providing applications to new classes of cluster algebras. In particular, it is of interest to find new classes of 2-CY triangulated categories with cluster-tilting objects. An important class is the stable categories $\underline{\mathcal{E}}_w$ of the Frobenius categories \mathcal{E}_w associated with reduced words w in Coxeter groups [BIRS09a], [GLS08], [GLS07b]. This class contains both classes of examples discussed above as special cases (see [BIRS09a], [GLS07b]).

A new class of triangulated 2-CY categories was introduced in [Ami08]. They are *generalized cluster categories* $\mathcal{C}_{\bar{A}}$ associated with algebras \bar{A} of global dimension at most 2, rather than global dimension 1. In this case the orbit category $\mathcal{D}^b(\bar{A})/\tau^-[1]$ is not necessarily triangulated, so $\mathcal{C}_{\bar{A}}$ is defined to be its triangulated hull. If $\mathcal{C}_{\bar{A}}$ is **Hom**-finite, then it is triangulated 2-CY and \bar{A} is a cluster-tilting object in $\mathcal{C}_{\bar{A}}$.

A natural question is how the generalized cluster categories are related to the previous classes of triangulated 2-CY categories. It was already shown in [Ami08] that some classes of categories $\underline{\mathcal{E}}_w$ are equivalent to generalized cluster categories, including \mathcal{C}_Q and $\mathbf{mod} \Lambda$. This result is extended to the case of c -sortable words in [AIRT09] with a similar choice for \bar{A} . One of the main results in this paper is: Each category $\underline{\mathcal{E}}_w$ associated with a reduced word is equivalent to a generalized cluster category $\mathcal{C}_{\bar{A}}$ for some algebra \bar{A} of global dimension at most 2 (Theorem 4.4).

Actually, we prove our main result in a more general setting: We start with a Frobenius category \mathcal{E} which we assume to be **Hom**-finite, stably 2-CY and with a cluster-tilting object T . We assume that the endomorphism algebra $\mathrm{End}_{\mathcal{E}}(T)$ is Jacobian and has a grading with certain properties. From these data we construct an algebra \bar{A} of global dimension ≤ 2 and a triangle equivalence $\mathcal{C}_{\bar{A}} \simeq \underline{\mathcal{E}}$ (Theorem 3.1) sending the canonical cluster-tilting object \bar{A} of $\mathcal{C}_{\bar{A}}$ to the cluster-tilting object T . The algebra \bar{A} is constructed as the degree zero part of $\mathrm{End}_{\mathcal{E}}(T)$, and we show $\mathrm{End}_{\mathcal{E}}(T) \simeq \mathrm{End}_{\mathcal{C}_{\bar{A}}}(\bar{A})$ (Proposition 3.5), which is an important step in the proof of the equivalence. It is however not known in general if 2-CY categories are equivalent when they have cluster-tilting objects whose endomorphism algebras are isomorphic. The only general result known of this type is that if the quiver Q of $\mathrm{End}_{\mathcal{C}}(T)$ has no oriented cycles, where T is a cluster-tilting object in an algebraic 2-CY category, then \mathcal{C} is triangle equivalent to the cluster category \mathcal{C}_Q [KR08]. A crucial step in this paper for proving the equivalence $\mathcal{C}_{\bar{A}} \simeq \underline{\mathcal{E}}$ is the construction of a triangle functor $\mathcal{C}_{\bar{A}} \rightarrow \underline{\mathcal{E}}$ sending \bar{A} to T . This is done by constructing a

triangle functor $\mathcal{D}^b(\bar{A}) \rightarrow \underline{\mathcal{E}}$ with the strong use of the Frobenius structure of \mathcal{E} and then using the universal property of the projection $\pi_{\bar{A}} : \mathcal{D}^b(\bar{A}) \rightarrow \mathcal{C}_{\bar{A}}$ from [Kel05] and [Ami08]. It will be important to also deal with the endomorphism algebra $\text{End}_{\mathcal{E}}(T)$ of the cluster-tilting object T in the Frobenius category, which we assume to be a graded frozen Jacobian algebra (see section 1 for definitions) with potential homogeneous of degree 1. Theorem 3.1 applies in particular to the categories \mathcal{E}_w associated to any element of the Coxeter group (see section 4).

The paper is organized as follows. In section 1 we recall some background material on cluster-tilting objects in 2-CY categories, on generalized cluster categories from [Ami08] and on Jacobian algebras from [DWZ08], together with the generalization to frozen Jacobian algebras from [BIRS09b].

In section 2 we construct a special triangle (Proposition 2.8) which is useful for our construction of a functor from $\mathcal{C}_{\bar{A}}$ to $\underline{\mathcal{E}}$.

Section 3 is devoted to the proof of the equivalence from $\mathcal{C}_{\bar{A}}$ to $\underline{\mathcal{E}}$ (Theorem 3.1). We first show that the global dimension of \bar{A} is at most 2. Then we construct our triangle functor from $\mathcal{C}_{\bar{A}}$ to $\underline{\mathcal{E}}$ using the special triangle from section 2, together with a universal property from [Ami08]. Finally we show that our functor is an equivalence by using a criterion from [KR08].

In section 4 we apply the main theorem to prove that for any reduced word w , the 2-CY triangulated category $\underline{\mathcal{E}}_w$ is triangle equivalent to some generalized cluster category, which was our original motivation (Theorem 4.4).

In section 5 we give two examples to illustrate our theorems. The first one is an application of Theorem 4.4. In the second one we use Theorem 3.1 to construct a triangle equivalence from a generalized cluster category $\mathcal{C}_{\bar{A}}$ to a category $\underline{\mathcal{E}}_w$ which sends the canonical cluster-tilting object \bar{A} to a cluster-tilting object T of $\underline{\mathcal{E}}_w$, where T is not associated to a reduced expression of w .

Notations. By triangulated category we mean k -linear triangulated category satisfying the Krull-Schmidt property. For all triangulated categories we will denote the shift functor by $[1]$. By Frobenius category we mean an exact k -category with enough projectives and injectives and where projectives and injectives coincide. For a finite-dimensional k -algebra A , we denote by $\text{mod } A$ the category of finite-dimensional right A -modules. Let D be the usual duality $\text{Hom}_k(?, k)$. The tensor product $- \otimes -$, when not specified, will be over the ground field k . For a quiver Q we will denote by Q_0 its set of vertices, by Q_1 its set of arrows, by s the source map and by t the target map.

1. BACKGROUND

In this section we collect some background material relevant for this paper.

1.1. Cluster-tilting objects. Let \mathcal{C} be a k -category which is **Hom**-finite, that is, has finite dimensional homomorphism spaces. Assume that \mathcal{C} is either Frobenius stably 2-CY (that is, its stable category is 2-CY) or triangulated 2-CY. Then an object T in \mathcal{C} is said to be *cluster-tilting* if

- (i) T is rigid, *i.e.* $\text{Ext}_{\mathcal{C}}^1(T, T) = 0$, and
- (ii) $\text{Ext}_{\mathcal{C}}^1(T, X) = 0$ implies that X is a summand of a finite direct sum of copies of T .

Note that when \mathcal{C} is Frobenius stably 2-CY, then any indecomposable projective-injective module is a summand of every cluster-tilting object. The finite dimensional algebras $\text{End}_{\mathcal{C}}(T)$, where \mathcal{C} is triangulated 2-CY, are called *2-CY-tilted algebras*.

Assume that $T = T_1 \oplus \dots \oplus T_n$ is a cluster-tilting object in a triangulated 2-CY category \mathcal{C} , where the T_i are indecomposable and pairwise non isomorphic. Then for each $i = 1, \dots, n$ there is a unique indecomposable object T_i^* not isomorphic to T_i , such that $T^* = T/T_i \oplus T_i^*$ is a cluster-tilting object [BMR⁺06],[IY08]. The new object T^* is called the *mutation* of T at T_i .

If $T = T_1 \oplus \dots \oplus T_n$ is a cluster-tilting object in a Frobenius stably 2-CY category, we can only mutate the T_i which are not projective-injective.

When T_i^* is defined, there are *exchange sequences* if \mathcal{C} is Frobenius

$$0 \longrightarrow T_i^* \xrightarrow{f} B \xrightarrow{g} T_i \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow T_i \xrightarrow{f'} B' \xrightarrow{g'} T_i^* \longrightarrow 0$$

or *exchange triangles* if \mathcal{C} is triangulated

$$T_i^* \xrightarrow{f} B \xrightarrow{g} T_i \longrightarrow T_i^*[1] \quad \text{and} \quad T_i \xrightarrow{f'} B' \xrightarrow{g'} T_i^* \longrightarrow T_i[1]$$

where f, f' are minimal left $\text{add}(T/T_i)$ -approximations and g, g' are minimal right $\text{add}(T/T_i)$ -approximations. These sequences (or triangles) play an important role in the categorification of cluster algebras.

There is also a related kind of sequences investigated in [IY08].

Proposition 1.1. [Iyama-Yoshino] *Let \mathcal{C} be a Hom-finite Frobenius stably 2-CY category with a cluster-tilting object $T = T_1 \oplus \dots \oplus T_n$. For each $i = 1, \dots, n$, if T_i is not projective-injective, there are exact sequences*

$$0 \longrightarrow T_i^+ \xrightarrow{f} E \xrightarrow{g} T_i \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow T_i \xrightarrow{f'} E' \xrightarrow{g'} T_i^+ \longrightarrow 0$$

for some indecomposable object T_i^+ in \mathcal{C} , such that g (resp. g') is right almost split in $\text{add}(T)$ (resp. in $\text{add}(T/T_i \oplus T_i^+)$) and f' (resp. f) is left almost split in $\text{add}(T)$ (resp. in $\text{add}(T/T_i \oplus T_i^+)$).

The induced sequence $0 \longrightarrow T_i \xrightarrow{f'} E' \xrightarrow{fg'} E \xrightarrow{g} T_i \longrightarrow 0$ is called the *2-almost split sequence* associated with T_i .

There is the corresponding result when \mathcal{C} is triangulated. For any direct summand T_i of a cluster-tilting object T , there are triangles

$$T_i^+ \xrightarrow{f} E \xrightarrow{g} T_i \longrightarrow T_i^+[1] \quad \text{and} \quad T_i \xrightarrow{f'} E' \xrightarrow{g'} T_i^+ \longrightarrow T_i[1]$$

where the maps f, g, f' and g' are almost split. For cluster categories, it was shown in [BMR⁺06] that these triangles coincide with the exchange triangles. More generally, they clearly coincide if and only if there are no loops in the quiver of $\text{End}_{\mathcal{C}}(T)$.

Using the existence of 2-almost split sequences, we can construct minimal projective and injective resolutions of simple modules over $\text{End}_{\mathcal{C}}(T)$.

Proposition 1.2. *Let \mathcal{C} be a Hom-finite Frobenius stably 2-CY category with a cluster-tilting object $T = T_1 \oplus \dots \oplus T_n$. Let $B := \text{End}_{\mathcal{C}}(T)$ be the endomorphism algebra of T , and let Q be the quiver of $B \simeq kQ/I$. For each $i = 1, \dots, n$, such that T_i is not projective-injective, denote by S_i the simple B -module $\text{Hom}_{\mathcal{C}}(T, T_i)/\text{Rad}(\text{Hom}_{\mathcal{C}}(T, T_i))$. Then we have exact sequences in $\text{mod } B$:*

$$0 \longrightarrow e_i B \xrightarrow{(b)} \bigoplus_{b \in Q_1, s(b)=i} e_{t(b)} B \xrightarrow{(rab)} \bigoplus_{a \in Q_1, t(a)=i} e_{s(a)} B \xrightarrow{(a)} e_i B \longrightarrow S_i \longrightarrow 0$$

$$0 \longrightarrow S_i \longrightarrow D(Be_i) \xrightarrow{(b)} \bigoplus_{b \in Q_1, s(b)=i} D(Be_{t(b)}) \xrightarrow{(r'_{ab})} \bigoplus_{a \in Q_1, t(a)=i} D(Be_{s(a)}) \xrightarrow{(a)} D(Be_i) \longrightarrow 0,$$

where the sets $\{ar_{ab} \mid a, b \in Q_1, t(a) = s(b) = i, t(b) = j\}$ and $\{r'_{ab} \mid s(b) = t(a) = i, s(a) = j\}$ are bases of $e_i I e_j$.

Proof. Applying the functor $\mathrm{Hom}_C(T, -)$ to the 2-almost-split sequence

$$0 \longrightarrow T_i \xrightarrow{f'} E' \xrightarrow{fg'} E \xrightarrow{g} T_i \longrightarrow 0$$

we get the following exact sequence of B -modules

$$0 \longrightarrow \mathrm{Hom}_C(T, T_i) \longrightarrow \mathrm{Hom}_C(T, E') \longrightarrow \mathrm{Hom}_C(T, E) \longrightarrow \mathrm{Hom}_C(T, T_i) \longrightarrow S_i \longrightarrow 0$$

which is a minimal projective resolution of the simple B -module S_i .

Let Q be the quiver of B , and $B \simeq kQ/I$. Since g is right almost split in $\mathrm{add}(T)$, we have

$$E \simeq \bigoplus_{a \in Q_1 \mid t(a)=i} T_{s(a)} \quad \text{and} \quad \mathrm{Hom}_C(T, g) \simeq (a)_{\{a \in Q_1 \mid t(a)=i\}}.$$

Since f' is left almost split in $\mathrm{add}(T)$, we have

$$E' \simeq \bigoplus_{b \in Q_1 \mid s(b)=i} T_{t(b)} \quad \text{and} \quad \mathrm{Hom}_C(T, f') \simeq (b)_{\{b \in Q_1 \mid s(b)=i\}}.$$

For $a, b \in Q_1$ with $t(a) = i$ and $s(b) = i$, let $r_{ab} : e_{t(b)}B \rightarrow e_{s(a)}B$ be the map induced by $fg' : \bigoplus_{b \in Q_1 \mid s(b)=i} T_{t(b)} \rightarrow \bigoplus_{a \in Q_1 \mid t(a)=i} T_{s(a)}$. Since the 2-almost split sequence associated to T_i induces a minimal projective resolution of the simple S_i , the set $\{ar_{ab} \mid a, b \in Q_1, t(a) = s(b) = i, t(b) = j\}$ is a basis of the set of relations $e_i I e_j$.

To get the other sequence of the proposition, we apply the functor $D\mathrm{Hom}_C(-, T)$ to the 2-almost split sequence associated to T_i , and we proceed similarly. \square

1.2. Generalized cluster categories. Let Λ be a finite dimensional k -algebra of global dimension ≤ 2 . We denote by $\mathcal{D}^b(\Lambda)$ the bounded derived category of finitely generated Λ -modules. It has a Serre functor that we denote by \mathbb{S} , which coincides with $\tau[1]$.

The category \mathcal{C}_Λ has been defined in [Ami08] as the triangulated hull of the orbit category $\mathcal{D}^b(\Lambda)/\mathbb{S}[-2]$. There is a triangle functor

$$\pi_\Lambda : \mathcal{D}^b(\Lambda) \longrightarrow \mathcal{D}^b(\Lambda)/\mathbb{S}[-2] \longleftarrow \mathcal{C}_\Lambda.$$

When the endomorphism algebra $\mathrm{End}_{\mathcal{C}_\Lambda}(\pi_\Lambda(\Lambda))$ is finite dimensional, the category \mathcal{C}_Λ is called the *generalized cluster category* and we have the following result:

Theorem 1.3 (Theorem 4.10 of [Ami08]). *Let Λ be a finite dimensional algebra of global dimension ≤ 2 , and assume that the endomorphism algebra $\mathrm{End}_{\mathcal{C}_\Lambda}(\pi_\Lambda(\Lambda))$ is finite dimensional. Then \mathcal{C}_Λ is a Hom-finite, 2-Calabi-Yau category and $\pi_\Lambda(\Lambda) \in \mathcal{C}_\Lambda$ is a cluster-tilting object.*

There is the following criterion for constructing triangle functors from a generalized cluster category \mathcal{C}_Λ to some stable category $\underline{\mathcal{E}}$. It can be deduced from the universal property of π_Λ given in subsection 4.1 of [Ami08] (see also section 9 of [Kel05] or appendix of [IO09] for more details). The next proposition is a key-step in the process of constructing the equivalence of the main theorem of this paper.

Here for a Frobenius category \mathcal{E} and an algebra Λ we denote by $\mathcal{D}^b(\Lambda^{op} \otimes \mathcal{E})$ the bounded derived category of $\Lambda^{op} \otimes \mathcal{E}$ as defined in [Kel94]. Objects are bounded complexes of objects in \mathcal{E} with a structure of left Λ -modules.

Proposition 1.4. *Let \mathcal{C}_Λ be a generalized cluster category, where Λ is an algebra of global dimension ≤ 2 . Let \mathcal{E} be a Frobenius category. Let M be an object in \mathcal{E} and assume that M has a structure of left Λ -module. Assume that there is a morphism in $\mathcal{D}^b(\Lambda^{op} \otimes \mathcal{E})$*

$$\alpha : M \longrightarrow R\mathrm{Hom}_\Lambda(D\Lambda, \Lambda) \overset{L}{\otimes}_\Lambda M[2]$$

whose cone lies in $\mathcal{D}^b(\Lambda^{op} \otimes \mathcal{P})$, where \mathcal{P} is the subcategory of \mathcal{E} of projective-injectives. Then there exists a triangle functor $\mathcal{C}_\Lambda \rightarrow \underline{\mathcal{E}}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{D}^b\Lambda & \xrightarrow{-\overset{L}{\otimes}_\Lambda M} & \mathcal{D}^b(\mathcal{E}) . \\ \downarrow \pi_\Lambda & & \downarrow \\ \mathcal{C}_\Lambda & \longrightarrow & \underline{\mathcal{E}} \end{array}$$

Note that the endofunctor $-\overset{L}{\otimes}_\Lambda R\mathrm{Hom}_\Lambda(D\Lambda, \Lambda)[2] \simeq R\mathrm{Hom}_\Lambda(D\Lambda, -)[2]$ of $\mathcal{D}^b(\Lambda)$ is isomorphic to the functor $\mathbb{S}^{-1}[2]$. Hence Proposition 1.4 requires in particular that the images of Λ and of $\mathbb{S}^{-1}\Lambda[2]$ under the composition

$$\mathcal{D}^b(\Lambda) \xrightarrow{-\overset{L}{\otimes}_\Lambda M} \mathcal{D}^b(\mathcal{E}) \longrightarrow \mathcal{D}^b(\mathcal{E})/\mathcal{D}^b(\mathcal{P}) = \underline{\mathcal{E}}$$

are isomorphic. Here the category $\mathcal{D}^b(\mathcal{P})$ is the thick subcategory of $\mathcal{D}^b(\mathcal{E})$ generated by \mathcal{P} . Therefore the localization of $\mathcal{D}^b(\mathcal{E})$ by $\mathcal{D}^b(\mathcal{P})$ is equivalent to the stable category $\underline{\mathcal{E}}$ by [KV87].

1.3. Jacobian algebras and generalizations. Quivers with potentials and their associated Jacobian algebras have been investigated in [DWZ08]. Let Q be a finite quiver. For each arrow a in Q , the *cyclic derivative* ∂_a with respect to a is the unique linear map

$$\partial_a : kQ \rightarrow kQ$$

which takes the class of a path p to the sum $\sum_{p=uv} vu$ taken over all decompositions of the path p (where u and v are possibly idempotent elements e_i associated to the vertex i). A *potential* on Q is any linear combination W of cycles in Q . The associated Jacobian algebra is by definition the algebra

$$\mathrm{Jac}(Q, W) := kQ / \langle \partial_a W; a \in Q_1 \rangle.$$

There is a more general definition given in [DWZ08], dealing with the complete path algebras, and hence there is also a larger class of Jacobian algebras. However in this paper we only consider the Jacobian algebras defined above.

Any finite dimensional Jacobian algebra (in the general sense) is 2-CY-tilted ([Ami08], [Kel09]). As a partial converse, some classes of 2-CY-tilted algebras associated with words are Jacobian ([BIRS09b]). Furthermore, the 2-CY-tilted algebras given by the canonical cluster-tilting object in a generalized cluster category are Jacobian, as stated in the following result obtained from Theorem 6.11 a) of [Kel09].

Theorem 1.5 (Keller). *Let $A = kQ/I$ be an algebra of global dimension ≤ 2 , such that I is generated by a finite set of minimal relations (r_i) . The relation r_i starts at the vertex $s(r_i)$ and ends at the vertex $t(r_i)$. Then we have an isomorphism of algebras:*

$$\text{End}_{\mathcal{C}_A}(A) \simeq \text{Jac}(\tilde{Q}, W_A)$$

where the quiver \tilde{Q} is the quiver Q with additional arrows $a_i : t(r_i) \rightarrow s(r_i)$, and the potential W_A is $\sum_i a_i r_i$.

There is a generalization of quivers with potentials (Q, W) to *frozen quivers with potentials* (Q, W, F) in [BIRS09b], where $F = (F_0, F_1)$ is a pair of a subset F_0 of vertices of Q (called *frozen vertices*) and a subset F_1 of arrows contained in $\{a \in Q_1, s(a) \in F_0 \text{ and } t(a) \in F_0\}$ (called *frozen arrows*). The associated *frozen Jacobian algebra* is by definition the algebra

$$\text{Jac}(Q, W, F) := kQ / \langle \partial_a W, a \notin F_1 \rangle$$

As in [DWZ08], one can define a *reduced* frozen quiver with potential (Q, W, F) such that each term in W has at least one arrow in $Q_1 \setminus F_1$ and has length at least 3.

2. A USEFUL TRIANGLE

This section is devoted to technical results which will be useful in the proof of the main theorem. In particular we construct a triangle which is crucial for constructing the morphism α of Proposition 1.4 and hence the functor from the generalized cluster categories.

Let \mathcal{E} be a Frobenius category which is **Hom**-finite and stably 2-Calabi-Yau. While we are mainly interested in the stable category $\underline{\mathcal{E}}$, it will be important to first consider cluster-tilting objects T in \mathcal{E} , and their endomorphism algebras $B := \text{End}_{\mathcal{E}}(T)$, which we assume to be isomorphic to $\text{Jac}(Q, W, F)$ for some reduced frozen quiver with potential (Q, W, F) . Notice that the quiver of B is Q since the potential W is reduced. We also assume that

(H1) the vertices in F_0 correspond to the projective-injectives in \mathcal{E} ;

(H2) the set $\{\partial_a W, a \notin F_1\}$ forms a basis of the relations of $\text{Jac}(Q, W, F)$, in particular for any a not in F_1 , we have $\partial_a W \neq 0$.

For $i \in Q_0$ we denote by e_i the primitive idempotent of B associated to i . Let us define the idempotent $e_F := \bigoplus_{i \in F_0} e_i$ and the algebra $\bar{B} := B / B e_F B$. Thus we have an isomorphism of algebras $\bar{B} \simeq \text{End}_{\underline{\mathcal{E}}}(T)$.

Let \bar{Q} be the full subquiver of Q with vertices not in F_0 . We have a projection

$$kQ \twoheadrightarrow kQ / kQ e_F kQ \simeq k\bar{Q}.$$

We denote by \bar{W} the image of W under this projection. It is not hard to see that there is an isomorphism $\bar{B} \simeq \text{Jac}(\bar{Q}, \bar{W})$. Indeed, since any arrow a in F_1 satisfies $s(a) \in F_0$ and $t(a) \in F_0$, the partial derivative $\partial_a W$ vanishes for any a in F_1 .

2.1. Description of projective and injective resolutions in $\text{mod}(\bar{B}^{op} \otimes B)$. In this section we describe the minimal projective and injective resolutions of \bar{B} as \bar{B} - B -bimodule. First we give explicitly the projective (resp. injective) resolutions of the simple \bar{B} -modules.

Lemma 2.1. *Let $B \simeq \text{Jac}(Q, W, F)$ be the endomorphism algebra of a cluster-tilting object T in a **Hom**-finite Frobenius stably 2-CY category \mathcal{E} . With the assumptions (H1) and (H2), for any $i \in Q_0 \setminus F_0$, the sequences*

$$0 \longrightarrow e_i B \xrightarrow{(b)} \bigoplus_{b, s(b)=i} e_{t(b)} B \xrightarrow{(a^{-1} \partial_b W)} \bigoplus_{a, t(a)=i} e_{s(a)} B \xrightarrow{(a)} e_i B \longrightarrow S_i \longrightarrow 0$$

and

$$0 \longrightarrow S_i \longrightarrow D(Be_i) \xrightarrow{(b)} \bigoplus_{b,s(b)=i} D(Be_{t(b)}) \xrightarrow{(\partial_a W b^{-1})} \bigoplus_{a,t(a)=i} D(Be_{s(a)}) \xrightarrow{(a)} D(Be_i) \longrightarrow 0$$

are the minimal projective and injective resolutions of the simple B -module S_i . Here for a path v in Q , we use the notation $a^{-1}v = u$ if $v = au$ in kQ and 0 else.

Proof. By hypothesis (H1), if i is not in F_0 , the corresponding summand of T is not projective-injective, hence there exists a 2-almost split sequence associated with T_i by Proposition 1.1. Moreover, since the potential W is reduced, we have $Q_{\text{End}_\varepsilon(T)} = Q$. Thus by Proposition 1.2 we get a minimal projective B -resolution of S_i

$$0 \longrightarrow e_i B \xrightarrow{(b)} \bigoplus_{b,s(b)=i} e_{t(b)} B \xrightarrow{(r_{ab})} \bigoplus_{a,t(a)=i} e_{s(a)} B \xrightarrow{(a)} e_i B \longrightarrow S_i \longrightarrow 0,$$

where $\{ar_{ab} \mid t(a) = i, s(b) = i\}$ is a basis of the space of relations with target i . By hypothesis (H2), the set $\{\partial_b W \mid s(b) = i\}$ is a basis of the same space. Thus a minimal projective resolution of S_i can be written as

$$0 \longrightarrow e_i B \xrightarrow{(b)} \bigoplus_{b,s(b)=i} e_{t(b)} B \xrightarrow{(a^{-1}\partial_b W)} \bigoplus_{a,t(a)=i} e_{s(a)} B \xrightarrow{(a)} e_i B \longrightarrow S_i \longrightarrow 0$$

The proof is similar for the injective resolution. \square

From this lemma, we deduce a description of a minimal projective (resp. injective) resolution of \bar{B} as a \bar{B} - B -bimodule. This is inspired by [Boc08].

Proposition 2.2. *There exist exact sequences in $\text{mod}(\bar{B}^{op} \otimes B)$*

$$(a) \quad 0 \longrightarrow \bigoplus_{i \notin F_0} P_{i,i} \longrightarrow \bigoplus_{a \notin F_1} P_{s(a),t(a)} \longrightarrow \bigoplus_{a \notin F_1} P_{t(a),s(a)} \longrightarrow \bigoplus_{i \notin F_0} P_{i,i} \longrightarrow \bar{B} \longrightarrow 0$$

$$(b) \quad 0 \longrightarrow \bar{B} \longrightarrow \bigoplus_{i \notin F_0} I_{i,i} \longrightarrow \bigoplus_{a \notin F_1} I_{s(a),t(a)} \longrightarrow \bigoplus_{a \notin F_1} I_{t(a),s(a)} \longrightarrow \bigoplus_{i \notin F_0} I_{i,i} \longrightarrow 0$$

where $P_{i,j} := \bar{B}e_i \otimes e_j B$ and $I_{i,j} := \text{Hom}_k(Be_j, \bar{B}e_i)$.

Proof. Denote by $\Pi_{i,j} := Be_i \otimes e_j B$ the projective B -bimodule. Let us define the following sequence

$$(c) \quad \bigoplus_{i \notin F_0} \Pi_{i,i} \xrightarrow{d_2} \bigoplus_{a \notin F_1} \Pi_{s(a),t(a)} \xrightarrow{d_1} \bigoplus_{a \in Q_1} \Pi_{t(a),s(a)} \xrightarrow{d_0} \bigoplus_{i \in Q_0} \Pi_{i,i}.$$

The maps d_0 , d_1 and d_2 are defined as follows:

$$\begin{aligned} d_2(e_i \otimes e_i) &= \sum_{a,t(a)=i} a \otimes e_i - \sum_{b,s(b)=i} e_i \otimes b; \\ d_1(e_{s(a)} \otimes e_{t(a)}) &= \sum_{b \in Q_1} \partial_{a,b} W \text{ where } \partial_{a,b}(apbq) = p \otimes q \in Be_{t(b)} \otimes e_{s(b)} B \\ &\quad \text{for } apbq \text{ a cycle in } Q; \\ d_0(e_{t(a)} \otimes e_{s(a)}) &= a \otimes e_{s(a)} - e_{t(a)} \otimes a. \end{aligned}$$

It is easy to check that this is a complex of B -bimodules, and that $\text{Coker}d_1 = B$. (The map $\bigoplus_{i \in Q_0} \Pi_{i,i} \rightarrow B$ is the multiplication map.)

Applying the functor $\bar{B} \otimes_B -$, we get the complex of \bar{B} - B -bimodules

$$\bigoplus_{i \notin F_0} P_{i,i} \xrightarrow{d_2} \bigoplus_{a \notin F_1} P_{s(a),t(a)} \xrightarrow{d_1} \bigoplus_{a \notin F_1} P_{t(a),s(a)} \xrightarrow{d_0} \bigoplus_{i \notin F_0} P_{i,i}.$$

Indeed, if $i \in F_0$, then $\bar{B}e_i \otimes e_i B = 0$, and if $a \in F_1$, then $\bar{B}e_{t(a)} \otimes e_{s(a)} B = 0$.

Applying the functor $S_i \otimes_{\bar{B}} -$ with $i \notin F_0$, we get the complex

$$0 \longrightarrow e_i B \xrightarrow{(b)} \bigoplus_{b, s(b)=i} e_{t(b)} B \xrightarrow{(\partial_a W b^{-1})} \bigoplus_{a, t(a)=i} e_{s(a)} B \xrightarrow{(a)} e_i B$$

which is exactly the minimal projective resolution of S_i as B -module described in Lemma 2.1. Thus the following sequence in $\text{mod}(\bar{B}^{op} \otimes B)$ is exact:

$$0 \longrightarrow \bigoplus_{i \notin F_0} P_{i,i} \longrightarrow \bigoplus_{a \notin F_1} P_{s(a), t(a)} \longrightarrow \bigoplus_{a \notin F_1} P_{t(a), s(a)} \longrightarrow \bigoplus_{i \notin F_0} P_{i,i} \longrightarrow \bar{B} \longrightarrow 0.$$

We now prove the existence of the sequence (b), where the proof is dual. Let us define the following sequence of B - B -bimodules

$$\bigoplus_{i \in Q_0} \Upsilon_{i,i} \xrightarrow{d^0} \bigoplus_{a \in Q_1} \Upsilon_{s(a), t(a)} \xrightarrow{d^1} \bigoplus_{a \notin F_1} \Upsilon_{t(a), s(a)} \xrightarrow{d^2} \bigoplus_{i \notin F_0} \Upsilon_{i,i}$$

where $\Upsilon_{i,j} := \text{Hom}_k(Be_j, Be_i)$. The maps d^0, d^1, d^2 are the following

$$\begin{aligned} d^0(\phi_i) &= (\sum_{s(a)=i} a \otimes e_i - \sum_{t(b)=i} e_i \otimes b) \cdot (\phi_i) \\ d^1(\phi_a) &= \sum_{b \in Q_1} (\partial_{a,b} W) \cdot (\phi_a) \\ d^2(\phi_a) &= (a \otimes e_i - e_i \otimes a) \cdot (\phi_a) \end{aligned}$$

where $(a \otimes b)(\phi)(-) = \phi(-a)b$; so for instance

$$d^0(\phi_i)(-) = \sum_{s(a)=i} \phi_i(-a) - \sum_{t(b)=i} \phi_i(-)b.$$

The kernel of d^0 is B , the bimodule map $B \rightarrow \bigoplus \Upsilon_{i,i}$ maps 1_B to $(\mathbf{1}_{Be_i})_i$. Using the fact that

$$S_l \otimes_{\bar{B}} \text{Hom}_k(Be_i, \bar{B}e_j) \simeq \delta_{j,l} e_i DB$$

we get the exact sequence (b). □

We have the following direct consequence (see also part (b) of the theorem in section 5.4 of [KR07]) which we include even though it will not be used later in this paper.

Corollary 2.3. *We have an isomorphism in $\mathcal{D}(\bar{B}^{op} \otimes B)$*

$$R\text{Hom}_B(DB, \bar{B})[3] \simeq \bar{B}.$$

Proof. Since $I_{i,j} = \text{Hom}_k(Be_j, e_i \bar{B})$ is injective as a right B -module, then $R\text{Hom}_B(DB, \bar{B})$ is isomorphic in $\mathcal{D}(\bar{B}^{op} \otimes B)$ to the complex

$$\bigoplus_{i \notin F_0} R_{i,i} \longrightarrow \bigoplus_{a \notin F_1} R_{s(a), t(a)} \longrightarrow \bigoplus_{a \notin F_1} R_{t(a), s(a)} \longrightarrow \bigoplus_{i \notin F_0} R_{i,i}$$

where $R_{i,j}$ is the bimodule $\text{Hom}_B(DB, I_{i,j})$. We have the following isomorphisms

$$\begin{aligned} \text{Hom}_B(DB, I_{j,i}) &= \text{Hom}_B(DB, \text{Hom}_k(Be_i, \bar{B}e_j)) \\ &\simeq \text{Hom}_k(DBe_i, \bar{B}e_j) \\ &\simeq \bar{B}e_j \otimes e_i B \\ &= P_{j,i} \end{aligned}$$

in $\text{mod}(\bar{B}^{op} \otimes B)$, from which the result follows immediately. □

Remark 2.1. If the sequences of Lemma 2.1 exist for F_0 being empty, then we get an isomorphism in $\mathcal{D}(B^{op} \otimes B)$ between $R\text{Hom}(DB, B)[3]$ and B (since in this case $\bar{B} = B$). This means by definition that B is bimodule 3-CY. This is exactly what Bocklandt proved in Theorem 4.3 of [Boc08], namely that a Jacobian algebra B is bimodule 3-CY if the sequence (c) is exact. In our setup, since F_0 corresponds to the projective-injectives, it is never empty. Also the algebra B is never Jacobian, and the Jacobian algebra \bar{B} is never of global dimension 3 (indeed it is either hereditary or of infinite global dimension by Corollary of section 2 of [KR07]). However we have an isomorphism in $\mathcal{D}(\bar{B}^{op} \otimes B)$ between \bar{B} and $R\text{Hom}_B(DB, \bar{B})[3]$.

2.2. Grading on the quiver. In order to identify appropriate subalgebras of B which should give rise to the generalized cluster categories we are looking for, it will be convenient to introduce some gradings on the quiver. Let as before $B = \text{End}_{\mathcal{E}}(T)$ be isomorphic to $Jac(Q, W, F)$. We assume that there exists a degree map $\varphi : Q_1 \rightarrow \{0, 1\}$ with the following property:

(H3) the potential W is homogeneous of degree 1.

Since the potential is homogeneous, any relation $\partial_a W$ is homogeneous, thus φ induces a grading on $B = \text{End}_{\mathcal{E}}(T)$.

Define the algebras A and \bar{A} by $A := B_0$ and $\bar{A} := A/Ae_F A$. We have the surjective algebra maps: $B \rightarrow A \rightarrow \bar{A}$.

The purpose of this subsection is to construct a triangle

$$(*) \quad R\text{Hom}_A(DA, \bar{A}) \overset{L}{\otimes}_A B[2] \longrightarrow \bar{A} \overset{L}{\otimes}_A B \longrightarrow \bar{A}_B \longrightarrow R\text{Hom}_A(DA, \bar{A}) \overset{L}{\otimes}_A B[3]$$

in $\mathcal{D}(\bar{A}^{op} \otimes B)$, which will be important for constructing the morphism α of Proposition 1.4.

The proof of the following proposition is exactly the same as the proof of Proposition 2.2. It describes the minimal projective and injective resolutions of \bar{A} as $(\bar{A}^{op} \otimes B)$ -module.

Proposition 2.4. *There exist exact sequences in $\text{mod}(\bar{A}^{op} \otimes B)$*

$$(a') \quad 0 \longrightarrow \bigoplus_{i \notin F_0} P_{i,i} \xrightarrow{d_2} \bigoplus_{a \notin F_1} P_{s(a),t(a)} \xrightarrow{d_1} \bigoplus_{a \notin F_1} P_{t(a),s(a)} \xrightarrow{d_0} \bigoplus_{i \notin F_0} P_{i,i} \longrightarrow \bar{A} \longrightarrow 0$$

$$(b') \quad 0 \longrightarrow \bar{A} \longrightarrow \bigoplus_{i \notin F_0} I_{i,i} \longrightarrow \bigoplus_{a \notin F_1} I_{s(a),t(a)} \longrightarrow \bigoplus_{a \notin F_1} I_{t(a),s(a)} \longrightarrow \bigoplus_{i \notin F_0} I_{i,i} \longrightarrow 0$$

where $P_{i,j} := \bar{A}e_i \otimes e_j B$ and $I_{i,j} := \text{Hom}_k(Be_j, \bar{A}e_i)$.

Let a be an arrow which is not in F_1 and such that $\varphi(a) = 1$. Then by definition $d_1(e_{s(a)} \otimes e_{t(a)}) = \sum_{b \in Q_1} \partial_{a,b} W$. If $apbq$ is a cycle in W , then $\partial_{a,b}(apbq) = p \otimes q$ lies in $\bar{A}e_{t(b)} \otimes e_{s(b)} B$. Since the degree of a is 1, and since the potential W is homogeneous of degree 1, then b is of degree 0. Hence the restriction of d_1

$$\bigoplus_{a \notin F_1, \varphi(a)=1} P_{s(a),t(a)} \longrightarrow \bigoplus_{a \notin F_1, \varphi(a)=1} P_{t(a),s(a)}$$

is zero. Therefore the complex

$$\bigoplus_{i \notin F_0} P_{i,i} \longrightarrow \bigoplus_{a \notin F_1} P_{s(a),t(a)} \longrightarrow \bigoplus_{a \notin F_1} P_{t(a),s(a)} \longrightarrow \bigoplus_{i \notin F_0} P_{i,i}$$

can be viewed as the total complex of the morphism

$$\begin{array}{ccccc}
X := \bigoplus_{i \notin F_0} P_{i,i} & \longrightarrow & \bigoplus_{a \notin F_1, \varphi(a)=0} P_{s(a),t(a)} & \longrightarrow & \bigoplus_{a \notin F_1, \varphi(a)=1} P_{t(a),s(a)} \\
\downarrow & & \downarrow & & \downarrow \\
Y := \bigoplus_{a \notin F_1, \varphi(a)=1} P_{s(a),t(a)} & \longrightarrow & \bigoplus_{a \notin F_1, \varphi(a)=0} P_{t(a),s(a)} & \longrightarrow & \bigoplus_{i \notin F_0} P_{i,i}
\end{array}$$

It is not hard to check that all horizontal maps are homogeneous of degree 0, and all vertical maps are homogeneous of degree 1. Let us call f the map $X \rightarrow Y$. By Proposition 2.4 (a), we then get the following triangle in $\mathcal{D}(\bar{A}^{op} \otimes B)$

$$X \xrightarrow{f} Y \longrightarrow \bar{A} \longrightarrow X[1].$$

Now we will use basic lemmas about complexes of graded modules in order to show that

$$Y \simeq \bar{A} \overset{L}{\otimes}_A B \quad \text{and} \quad X \simeq R\mathrm{Hom}_B(DA, \bar{A}) \overset{L}{\otimes}_A B[2].$$

Lemma 2.5. *Let B be a \mathbb{Z} -graded algebra. Let (X, d_X) and (Y, d_Y) be complexes of graded B -modules such that the differentials d_X and d_Y are homogeneous of degree 0. We denote by (X_p, d_{X_p}) (resp. (Y_p, d_{Y_p})) the part of degree p of the complex (X, d_X) (resp. (Y, d_Y)). These are complexes of B_0 -modules. Let $f : X \rightarrow Y$ be a morphism of complexes homogeneous of degree 1. Denote by $Z = \mathrm{Tot}(f)$ the total complex of $f : X \rightarrow Y$. Then for any integers p, q we have an isomorphism of B_0 -modules:*

$$H^q(Z)_p \simeq H^q(\mathrm{Tot}(f_{p-1} : X_{p-1} \rightarrow Y_p)).$$

Applying the lemma to our situation, we get an isomorphism of $\bar{A}^{op} \otimes A$ -modules (remember that $A = B_0$):

$$H^q(\mathrm{Tot}(f : X \rightarrow Y))_0 \simeq H^q(\mathrm{Tot}(f_{-1} : X_{-1} \rightarrow Y_0)).$$

By the sequence (a') of Proposition 2.4 the left term is zero unless q is 0, and then it is isomorphic to \bar{A} . Since X is only in positive degree, the right hand side is just $H^q(Y_0)$. Thus we get an isomorphism in $\mathcal{D}(\bar{A}^{op} \otimes A)$

$$(1) \quad Y_0 \simeq \bar{A}.$$

Remark 2.2. This implies that any \bar{A} -module has projective dimension 2 when viewed as an A -module.

Dually, using the exact sequence (b') of Proposition 2.4, it is possible to view $\bar{A}[1]$ as the cone of the map

$$\begin{array}{ccccc}
X' := \bigoplus_{i \notin F_0} I_{i,i} & \longrightarrow & \bigoplus_{a \notin F_1, \varphi(a)=0} I_{s(a),t(a)} & \longrightarrow & \bigoplus_{a \notin F_1, \varphi(a)=1} I_{t(a),s(a)} \\
\downarrow & & \downarrow & & \downarrow \\
Y' := \bigoplus_{a \notin F_1, \varphi(a)=1} I_{s(a),t(a)} & \longrightarrow & \bigoplus_{a \notin F_1, \varphi(a)=0} I_{t(a),s(a)} & \longrightarrow & \bigoplus_{i \notin F_0} I_{i,i}
\end{array}$$

where horizontal maps are of degree 0, and vertical maps are of degree -1 . And similarly we get an isomorphism in $\mathcal{D}(\bar{A}^{op} \otimes A)$

$$(2) \quad X'_0 \simeq \bar{A}.$$

Moreover, using the fact that $\mathrm{Hom}_B(DB, I_{i,j}) \simeq P_{i,j}$, it is easy to check that we have an isomorphism in $\mathcal{D}(\bar{A}^{op} \otimes B)$

$$(3) \quad \mathrm{RHom}_B(DB, X')[2] \simeq X.$$

Now we will prove that we have isomorphisms

$$Y \simeq \bar{A} \overset{L}{\otimes}_A B \quad \text{and} \quad X \simeq \mathrm{RHom}_B(DA, \bar{A}) \overset{L}{\otimes}_A B[2]$$

in $\mathcal{D}(\bar{A}^{op} \otimes B)$ in order to get the triangle (*). For this we will use the following lemmas:

Lemma 2.6. *Let B be a \mathbb{Z} -graded algebra and $A := B_0$. Let (P, d) be a complex of graded projective B -modules such that the differential d of P is homogeneous of degree 0. Let P_0 be the degree 0 part of P , then P_0 is a complex over A , and we have an isomorphism*

$$P \simeq P_0 \otimes_A B.$$

Proof. Let $i : A \rightarrow B$ and $p : B \rightarrow A$ be the canonical algebra maps. We get induced functors:

$$i^* = - \otimes_A B : \mathrm{mod} A \rightarrow \mathrm{mod} B, \quad \text{and} \quad p^* = - \otimes_B A : \mathrm{mod} B \rightarrow \mathrm{mod} A$$

Since $B_0 \otimes_A B \simeq B$, then for any graded projective B -module M , we have $M_0 \otimes_A B = i^* \circ p^*(M) \simeq M$. Hence we have

$$P_0 \otimes_A B = (P_0 \otimes_A B, i^* \circ p^*(d)).$$

Since $i \circ p(b) = b$ if and only if b is of degree 0, we get the result. \square

Lemma 2.7. *The functors $(\mathrm{Hom}_B(DB, -))_0$ and $\mathrm{Hom}_A(DA, (-)_0)$ are isomorphic as functors from injective B -modules to projective A -modules.*

Proof. It is enough to check it on DB , and this is clearly true. \square

Since Y is a complex of projective modules, we can apply Lemma 2.6, and using (1), we get an isomorphism in $\mathcal{D}(\bar{A}^{op} \otimes B)$

$$(4) \quad Y \simeq Y_0 \otimes_A B \simeq \bar{A} \overset{L}{\otimes}_A B.$$

Moreover, we have the isomorphisms in $\mathcal{D}(\bar{A}^{op} \otimes B)$

$$\begin{aligned} X &\simeq \mathrm{Hom}_B(DB, X')[2] && \text{by (3)} \\ &\simeq (\mathrm{Hom}_B(DB, X'))_0 \otimes_A B[2] && \text{by Lemma 2.6} \\ &\simeq \mathrm{Hom}_A(DA, X'_0) \otimes_A B[2] && \text{by Lemma 2.7} \\ &\simeq \mathrm{RHom}_A(DA, \bar{A}) \overset{L}{\otimes}_A B[2] && \text{by (2)}. \end{aligned}$$

Hence we have the following.

Proposition 2.8. *Let \mathcal{E} be a Frobenius category which is Hom-finite and stably 2-Calabi-Yau. We assume that there exists a cluster-tilting object T in \mathcal{E} such that its endomorphism algebra is isomorphic to $\mathrm{Jac}(Q, W, F)$ for some frozen quiver with reduced potential (Q, W) . With the assumptions (H1), (H2) and (H3), there exists a triangle*

$$(*) \quad \mathrm{RHom}_A(DA, \bar{A}) \overset{L}{\otimes}_A B[2] \longrightarrow \bar{A} \overset{L}{\otimes}_A B \longrightarrow \bar{A}_B \longrightarrow \mathrm{RHom}_A(DA, \bar{A}) \overset{L}{\otimes}_A B[3]$$

in $\mathcal{D}(\bar{A}^{op} \otimes B)$, where $A = B_0$ and $\bar{A} = A/Ae_F A$.

3. MAIN THEOREM

As in the previous section, \mathcal{E} is a Frobenius category which is Hom-finite and stably 2-Calabi-Yau. We assume that there exists a cluster-tilting object T in \mathcal{E} whose endomorphism algebra is isomorphic to $Jac(Q, W, F)$ for some frozen quiver with reduced potential (Q, W) with assumptions (H1) and (H2).

We assume that there exists a degree map $\varphi : Q_1 \rightarrow \{0, 1\}$ with the following hypothesis:

(H3) the potential W is homogeneous of degree 1;

(H4) if $a : i \rightarrow j$ is in Q_1 with $i \notin F_0$ and $j \in F_0$ then $\varphi(a) = 1$;

(H5) the set $\{\partial_a \bar{W}, \varphi(a) = 1\}$ is linearly independent, where \bar{W} is the image of the potential W under the projection $kQ \rightarrow kQ/kQe_FkQ = k\bar{Q}$. So in particular, $\partial_a \bar{W}$ does not vanish for a in \bar{Q}_1 with $\varphi(a) = 1$.

As before we define the algebras A and \bar{A} as $A := B_0 \subset B$ and $\bar{A} := A/Ae_FA$, where e_F is the idempotent $\bigoplus_{i \in F_0} e_i$.

In this section we prove the following.

Theorem 3.1. *Let \mathcal{E} be a Frobenius category which is Hom-finite and stably 2-CY. Assume that there exists a cluster-tilting object T in \mathcal{E} such that $B := \text{End}_{\mathcal{E}}(T)$ is isomorphic to some graded frozen Jacobian algebra $Jac(Q, W, F)$ which satisfies the conditions (H1)-(H5). Let $A := B_0$ be the subalgebra of degree 0 of B and $\bar{A} := A/Ae_FA$ be the quotient algebra of A where e_F is the idempotent of the frozen vertices. Then*

- (1) *the algebra \bar{A} is of global dimension ≤ 2 ;*
- (2) *there exists a triangle equivalence $\mathcal{C}_{\bar{A}} \simeq \underline{\mathcal{E}}$, where $\mathcal{C}_{\bar{A}}$ is the generalized cluster category associated to the algebra \bar{A} .*

The proof of the theorem is given in the next three subsections: the fact that $gl.dim \bar{A} \leq 2$ is proved in 3.1, the existence of a triangle functor $G : \mathcal{C}_{\bar{A}} \rightarrow \underline{\mathcal{E}}$ is proved in 3.2, and in 3.3 it is proved that G is an equivalence of triangulated categories.

3.1. Global dimension of \bar{A} . We first describe the restriction functor $\mathcal{D}^b(\bar{A}) \longrightarrow \mathcal{D}^b(A)$ induced by the projection $A \longrightarrow \bar{A} = A/Ae_FA$.

Lemma 3.2. *Let \mathcal{E} , $Jac(Q, W, F)$, A and \bar{A} be as in Theorem 3.1. Assume that $Jac(Q, W, F)$ satisfies conditions (H1)-(H4). Let $Res : \mathcal{D}^b(\bar{A}) \longrightarrow \mathcal{D}^b(A)$ be the restriction functor. Then for any $i \notin F_0$:*

- (1) $Res(e_i D\bar{A}) \simeq e_i DA$,
- (2) $Res(e_i \bar{A}) \simeq (P \longrightarrow Q \longrightarrow e_i A)$ where P and Q are in $\text{add}(e_F A)$.

Moreover, the functor Res is fully faithful.

Proof. By hypothesis (H4), the first item is clear.

By Remark 2.2 any \bar{A} -module has projective dimension ≤ 2 when viewed as an A -module, thus combining this with (H4) we get the second item. \square

Proposition 3.3. *Let \mathcal{E} $Jac(Q, W, F)$ and \bar{A} be as in Theorem 3.1. Assume that $Jac(Q, W, F)$ satisfies conditions (H1)-(H4). Then we have*

$$gl.dim \bar{A} \leq 2.$$

Proof. By equation (2) of the previous section \bar{A} is isomorphic to X'_0 as $\bar{A}^{op} \otimes A$ -modules, thus we have an exact sequence in $\text{mod}(\bar{A}^{op} \otimes A)$:

$$0 \longrightarrow \bar{A} \longrightarrow \bigoplus_{i \notin F_0} R_{i,i} \longrightarrow \bigoplus_{a \notin F_1, \varphi(a)=0} R_{t(a),s(a)} \longrightarrow \bigoplus_{a \notin F_1, \varphi(a)=1} R_{s(a),t(a)} \longrightarrow 0$$

where $R_{i,j} := \text{Hom}_k(Ae_i, \bar{A}e_j)$. This gives, for any right \bar{A} -module, an injective resolution of length 3 of its restriction as a right A -module. By (1) of Lemma 3.2, the injective resolution of the restriction of any \bar{A} -module is an injective resolution in $\text{mod} A$. Thus the global dimension of \bar{A} is ≤ 2 . \square

3.2. Construction of a triangle functor. Recall that $A = B_0$ is the subalgebra of degree 0 of $B = \text{End}_{\mathcal{E}}(T)$. Thus T has a structure of left A -module. Hence we have the following diagram:

$$\begin{array}{ccccc} \mathcal{D}^b(\bar{A}) & \xrightarrow{Res} & \mathcal{D}^b(A) & \xrightarrow{-\overset{L}{\otimes}_A T} & \mathcal{D}^b(\mathcal{E}) \longrightarrow \underline{\mathcal{E}}. \\ & & \searrow & & \nearrow \\ & & \mathcal{D}^b(B) & \xleftarrow{-\overset{L}{\otimes}_B T} & \end{array}$$

Applying $-\overset{L}{\otimes}_B T$ to the triangle (*) of Proposition 2.8, we get the following triangle in $\mathcal{D}(\bar{A}^{op} \otimes \mathcal{E})$

$$R\text{Hom}_A(DA, \bar{A}) \overset{L}{\otimes}_A T[2] \longrightarrow \bar{A} \overset{L}{\otimes}_A T \longrightarrow \bar{A} \overset{L}{\otimes}_B T \longrightarrow R\text{Hom}_A(DA, \bar{A}) \overset{L}{\otimes}_A T[3].$$

Using the projective resolution of Lemma 2.1 of the simple B -module S_i for $i \in Q_0 \setminus F_0$, we get that the object $S_i \overset{L}{\otimes}_B T$ is quasi-isomorphic to the complex

$$0 \longrightarrow e_i B \otimes_B T \longrightarrow \bigoplus_{a, s(a)=i} e_{t(a)} B \otimes_B T \longrightarrow \bigoplus_{a, t(a)=i} e_{s(a)} B \otimes_B T \longrightarrow e_i B \otimes_B T,$$

which is the 2-almost split sequence associated with T_i

$$0 \longrightarrow T_i \longrightarrow \bigoplus_{a, s(a)=i} T_{t(a)} \longrightarrow \bigoplus_{a, t(a)=i} T_{s(a)} \longrightarrow T_i \longrightarrow 0.$$

Hence $S_i \overset{L}{\otimes}_B T$ is zero in $\mathcal{D}^b(\mathcal{E})$. Therefore the object $\bar{A} \overset{L}{\otimes}_B T$ is zero in $\mathcal{D}(\bar{A}^{op} \otimes \mathcal{E})$, and hence the morphism

$$(i) \quad R\text{Hom}_A(DA, \bar{A}) \overset{L}{\otimes}_A T[2] \rightarrow \bar{A} \overset{L}{\otimes}_A T$$

is an isomorphism in $\mathcal{D}^b(\bar{A}^{op} \otimes \mathcal{E})$.

Now the exact sequence of A - A -bimodules

$$Ae_F A \twoheadrightarrow A \twoheadrightarrow A/Ae_F A = \bar{A}$$

induces the following triangle in $\mathcal{D}(\bar{A}^{op} \otimes A)$:

$$(**) \quad R\text{Hom}_A(DA, \bar{A}) \overset{L}{\otimes}_A Ae_F A \longrightarrow R\text{Hom}_A(DA, \bar{A}) \longrightarrow R\text{Hom}_A(DA, \bar{A}) \overset{L}{\otimes}_A \bar{A} \longrightarrow \cdot$$

Let Z be an injective resolution of \bar{A} as a right A -module. Then we have

$$R\text{Hom}_A(DA, \bar{A}) \overset{L}{\otimes}_A \bar{A} \simeq \text{Hom}_A(DA, Z) \otimes_A \bar{A}.$$

Now for each A -module M , we define a map

$$G_M : \mathrm{Hom}_A(DA, M) \otimes_A \bar{A} \rightarrow \mathrm{Hom}_A(D\bar{A}, M).$$

Let φ be in $\mathrm{Hom}_A(DA, M)$, $\bar{a} \in \bar{A}$, and $\Phi \in D\bar{A}$. We define

$$G_M(\varphi \otimes \bar{a})(\Phi) := \varphi(\Phi_{\bar{a}}) \in M,$$

where $\Phi_{\bar{a}}(b) = \Phi(b\bar{a})$ for any $b \in A$. Since \bar{A} has finite projective dimension as an A -module, this map is an isomorphism in $\mathrm{mod} A$. Thus we get the isomorphism in $\mathcal{D}(\bar{A}^{op} \otimes A)$

$$R\mathrm{Hom}_A(DA, \bar{A}) \otimes_A^L \bar{A} = \mathrm{Hom}_A(DA, Z) \otimes_A \bar{A} \xrightarrow{G_Z} \mathrm{Hom}_A(D\bar{A}, Z) = R\mathrm{Hom}_A(D\bar{A}, \bar{A}).$$

Moreover by Lemma 3.2 any injective resolution of an \bar{A} -module is an injective resolution of the restriction of the module as an A -module, and the restriction functor is fully faithful. Thus we get isomorphisms in $\mathcal{D}(\bar{A}^{op} \otimes A)$

$$R\mathrm{Hom}_A(D\bar{A}, \bar{A}) = \mathrm{Hom}_A(D\bar{A}, Z) \simeq \mathrm{Hom}_{\bar{A}}(D\bar{A}, Z) = R\mathrm{Hom}_{\bar{A}}(D\bar{A}, \bar{A}).$$

By Lemma 3.2 the A -module $e_i A e_F A$, with $i \notin F_0$, is quasi-isomorphic to a complex of the form $P \longrightarrow Q$ where P and Q are in $\mathrm{add}(e_F A)$. Thus the object $A e_F A \otimes_A^L T$ is in $\mathcal{D}^b(A \otimes \mathcal{P})$, where \mathcal{P} is the full subcategory of \mathcal{E} consisting of projective-injectives. Therefore the object $R\mathrm{Hom}_A(DA, \bar{A}) \otimes_A^L A e_F A \otimes_A^L T$ is in $\mathcal{D}^b(\bar{A}^{op} \otimes \mathcal{P})$.

Hence applying the functor $-\otimes_A^L T$ to the triangle (**), we get that the morphism

$$(ii) \quad R\mathrm{Hom}_A(DA, \bar{A}) \otimes_A^L T \rightarrow R\mathrm{Hom}_{\bar{A}}(D\bar{A}, \bar{A}) \otimes_A^L T$$

has a cone in $\mathcal{D}^b(\bar{A}^{op} \otimes \mathcal{P})$.

Finally, combining (i) and (ii) we have the diagram in $\mathcal{D}(\bar{A}^{op} \otimes \mathcal{E})$

$$\begin{array}{ccc}
 (***) & & R\mathrm{Hom}_A(DA, \bar{A}) \otimes_A^L T[2] \\
 & \swarrow (ii) & \searrow (i) \\
 R\mathrm{Hom}_{\bar{A}}(D\bar{A}, \bar{A}) \otimes_A^L T[2] & & \bar{A} \otimes_A^L T.
 \end{array}$$

The cone of the left morphism is in $\mathcal{D}^b(\bar{A}^{op} \otimes \mathcal{P})$ and the right morphism is an isomorphism in $\mathcal{D}(\bar{A}^{op} \otimes \mathcal{E})$. Therefore we get a morphism

$$\alpha : \bar{A} \otimes_A^L T \rightarrow R\mathrm{Hom}_{\bar{A}}(D\bar{A}, \bar{A}) \otimes_A^L T[2] = R\mathrm{Hom}_{\bar{A}}(D\bar{A}, \bar{A}) \otimes_{\bar{A}}^L (\bar{A} \otimes_A^L T)[2].$$

By Proposition 1.4 (for $M = \bar{A} \otimes_A^L T$) the composition of the functors

$$\begin{array}{ccccc}
 & & \xrightarrow{-\otimes_{\bar{A}}^L (\bar{A} \otimes_A^L T)} & & \\
 \mathcal{D}^b(\bar{A}) & \xrightarrow{Res} & \mathcal{D}^b(A) & \xrightarrow{-\otimes_A^L T} & \mathcal{D}^b(\mathcal{E}) \\
 \downarrow \pi_{\bar{A}} & & & & \downarrow \\
 \mathcal{C}_{\bar{A}} & \xrightarrow{\dots\dots\dots G} & & & \underline{\mathcal{E}}
 \end{array}$$

factors through the generalized cluster category $\mathcal{C}_{\bar{A}}$, and the factorization G is a triangle functor.

3.3. Proof of the equivalence. The aim of this subsection is to show that the triangle functor $G : \mathcal{C}_{\bar{A}} \rightarrow \underline{\mathcal{E}}$ is an equivalence.

We have a triangle

$$Ae_F A \overset{L}{\otimes}_A T \longrightarrow T \longrightarrow \bar{A} \overset{L}{\otimes}_A T \longrightarrow Ae_F A \overset{L}{\otimes}_A T[1]$$

in $\mathcal{D}(\bar{A}^{op} \otimes \mathcal{E})$ and the object $Ae_F A \overset{L}{\otimes}_A T$ is in $\mathcal{D}^b(\bar{A} \otimes \mathcal{P})$. Hence the image of \bar{A} in $\underline{\mathcal{E}}$ is isomorphic to T , so that the triangle functor G sends the cluster-tilting object \bar{A} to the cluster-tilting object T . Then we use Lemma 4.5 of [KR08]:

Proposition 3.4 (Keller-Reiten). *Let \mathcal{C} and \mathcal{C}' be 2-Calabi-Yau triangulated categories. Let T (resp. T') be a cluster-tilting object in \mathcal{C} (resp. \mathcal{C}'). If we have a triangle functor $G : \mathcal{C} \rightarrow \mathcal{C}'$ which sends T to T' and which induces an equivalence between $\text{add}(T)$ and $\text{add}(T')$, then G is an equivalence.*

Thus, in order to prove that G is an equivalence it is enough to prove the following.

Proposition 3.5. *Under assumptions (H1)-(H5) we have an isomorphism of algebras*

$$\text{End}_{\underline{\mathcal{E}}}(T) \simeq \text{End}_{\mathcal{C}_{\bar{A}}}(\bar{A}).$$

Proof. The algebra $\bar{B} = B/Be_F B$ is isomorphic to the Jacobian algebra $\text{Jac}(\bar{Q}, \bar{W})$ (cf section 2), where \bar{Q} is the full subquiver of Q whose vertices are not in F_0 and where \bar{W} is the image of W under the projection $kQ \twoheadrightarrow kQ/kQe_F kQ \simeq k\bar{Q}$.

Let Q' be the subquiver of Q defined by:

- $Q'_0 := Q_0$;
- $Q'_1 := \{a \in Q_1, \varphi(a) = 0\}$.

By definition we have

$$A \simeq kQ' / \langle \partial_a W, a \notin F_1 \text{ and } \varphi(a) = 1 \rangle.$$

Let \bar{Q}' be the full subquiver of Q' with vertices $\bar{Q}'_0 = Q'_0 \setminus F_0$. Thus we get

$$\bar{A} \simeq k\bar{Q}' / \langle \partial_a \bar{W}, \varphi(a) = 1 \rangle.$$

By condition (H5) the set $\{\partial_a \bar{W}, \varphi(a) = 1\}$ is a basis for the ideal of relations $\langle \partial_a \bar{W}, \varphi(a) = 1 \rangle$. By Theorem 1.5, using the fact that $\{\partial_a \bar{W}, \varphi(a) = 1\}$ is a basis, the endomorphism algebra $\text{End}_{\mathcal{C}_{\bar{A}}}(\bar{A})$ is isomorphic to the Jacobian algebra $\text{Jac}(\widetilde{\bar{Q}'}, W_{\bar{A}})$ where $W_{\bar{A}}$ is the potential

$$W_{\bar{A}} = \sum_{a, \varphi(a)=1} a \partial_a \bar{W}.$$

By condition (H3) the potential \bar{W} is homogeneous of degree 1 and the degree map φ has non-negative values. Thus any term in the potential \bar{W} contains exactly one arrow of degree 1. Hence we have $\widetilde{\bar{Q}'} = \bar{Q}$ and $\bar{W} = W_{\bar{A}}$. Therefore we have

$$\text{Jac}(\widetilde{\bar{Q}'}, W_{\bar{A}}) \simeq \text{Jac}(\bar{Q}, \bar{W})$$

which gives the desired isomorphism. □

Hence we have finished the proof of Theorem 3.1.

4. APPLICATION TO 2-CALABI-YAU CATEGORIES ASSOCIATED WITH ELEMENTS IN THE COXETER GROUP

In this section we apply Theorem 3.1 to the categories associated with elements in the Coxeter group introduced in [BIRS09a].

4.1. Results of [BIRS09a] and [BIRS09b]. Let Q be a finite quiver without oriented cycles. We denote by $Q_0 = \{1, \dots, n\}$ the set of vertices and as usual by Q_1 the set of arrows. The preprojective algebra associated to Q is the algebra

$$k\overline{Q}/\langle \sum_{a \in Q_1} aa^* - a^*a \rangle$$

where \overline{Q} is the double quiver of Q , which is obtained from Q by adding to each arrow $a : i \rightarrow j$ in Q_1 an arrow $a^* : j \leftarrow i$ pointing in the opposite direction. We denote by Λ the completion of the preprojective algebra associated to Q and by $\text{f.l.}\Lambda$ the category of right Λ -modules of finite length.

Let C_Q be the Coxeter group associated to Q . It is defined by the generators s_i where $i \in Q_0$ and by the relations

- $s_i^2 = 1$,
- $s_i s_j = s_j s_i$ if there is no arrows between i and j ,
- $s_i s_j s_i = s_j s_i s_j$ if there is exactly one arrow between i and j .

A *reduced expression* $w = s_{u_1} \dots s_{u_l}$ of an element of C_Q is an expression of w with l as small as possible. When $s_{u_1} \dots s_{u_l}$ is a reduced expression of w , the integer $l(w) := l$ is then called the length of w .

For a vertex i in Q_0 we denote by \mathcal{I}_i the two-sided ideal $\Lambda(1 - e_i)\Lambda$. Let $w = s_{u_1} \dots s_{u_l}$ be a reduced expression of an element in C_Q . For $p \leq l$ we denote by \mathcal{I}_{w_p} the two-sided ideal $\mathcal{I}_{u_p} \mathcal{I}_{u_{p-1}} \dots \mathcal{I}_{u_1}$. We denote by Λ_w the algebra $\Lambda/\mathcal{I}_{w_l}$ and by $\mathcal{E}_w := \text{Sub } \Lambda_w$ the subcategory of $\text{f.l.}\Lambda$ consisting of submodules of finite direct sums of copies of Λ_w . The ideal \mathcal{I}_w depends on the element $w \in C_Q$ and not on the choice of the reduced expression.

Let us recall Theorem III.2.8 of [BIRS09a].

Theorem 4.1 (Buan-Iyama-Reiten-Scott). *Let w be an element in the Coxeter group C_Q . Then the category \mathcal{E}_w is a Hom-finite Frobenius stably 2-CY category.*

Moreover for any reduced expression $s_{u_1} \dots s_{u_l}$ of w , the object $T_w = \bigoplus_{p=1}^l e_{u_p}(\Lambda/\mathcal{I}_{w_p})$ is a cluster-tilting object. The projective-injective indecomposable objects are $e_{u_{t_i}}(\Lambda/\mathcal{I}_{w_{t_i}})$ where t_i is the maximal integer such that $u_{t_i} = i$ for $i \in Q_0$.

The cluster-tilting object T_w depends on the choice of the reduced expression of w . We refer to a cluster-tilting object of this form as a *standard cluster-tilting object*. Note that by mutation we may get other cluster-tilting objects which are not standard.

We now define a quiver Q_w associated with a reduced expression $w = s_{u_1} \dots s_{u_l}$ as follows:

- vertices: $1, \dots, l(w)$.
- for each $i \in Q_0$, one arrow $t \leftarrow s$ if t and s are two consecutive vertices of type i (*i.e.* $u_s = u_t = i$) and $t < s$ (we call these arrows *arrows going to the left*);
- for each $a : i \rightarrow j \in Q_1$, put $a : t \rightarrow s$ if t is a vertex of type i , s of type j , and if there is no vertex of type i between t and s and if s is the last vertex of type j before the next vertex of type i in the expression $w = s_{u_1} \dots s_{u_l}$ (we call these arrows the *Q -arrows*)

- for each $a : i \rightarrow j \in Q_1$, put $a^* : t \rightarrow s$ if t is of type j , s is of type i , if there is no vertex of type j between t and s and if s is the last vertex of type i before the next vertex of type j in the expression of w (we call these arrows the Q^* -arrows).

For each Q -arrow $a : t \rightarrow s$ in Q_w , we denote by W_a the composition aa^*p if there is a (unique) Q^* -arrow $a^* : r \rightarrow t$ in Q_w where $u_r = u_s$ and where p is the composition of arrows going to the left $r \leftarrow \dots \leftarrow s$. Otherwise we put $W_a = 0$. For each Q^* -arrow $a^* : t \rightarrow r$ in Q_w , we denote by W_{a^*} the composition a^*ap if there exists a (unique) Q -arrow $a : s \rightarrow t$ with $u_s = u_r$ in Q_w and where p is the composition of arrows going to the left $s \leftarrow \dots \leftarrow r$. Otherwise we put $W_{a^*} = 0$. Then let W_w be the sum

$$W_w = \sum_{a \text{ } Q\text{-arrow}} W_a - \sum_{a^* \text{ } Q^*\text{-arrow}} W_{a^*}.$$

Let us recall Theorem 6.8 of [BIRS09b].

Theorem 4.2 (Buan-Iyama-Reiten-Smith). *Let $w = s_{u_1} \dots s_{u_l}$ be a reduced expression of an element of the Coxeter group C_Q . Let T_w be the standard cluster-tilting object of the category \mathcal{E}_w associated to this reduced expression. Then we have an isomorphism*

$$\text{End}_{\mathcal{E}_w}(T_w) \simeq \text{Jac}(Q_w, W_w, F)$$

where $F_0 := \{t_1, \dots, t_n\}$ and $F_1 := \{a \in Q_1, s(a) \in F_0 \text{ and } t(a) \in F_1\}$.

4.2. Description of the grading. The algebra Λ and the Coxeter group do not depend on the orientation of Q . For any reduced expression $w = s_{u_1} \dots s_{u_l}$, the category \mathcal{E}_w and the cluster-tilting object T_w do not depend on the orientation of Q . From now on we assume that the orientation of Q satisfies the property

$$\text{if there exists } i \rightarrow j, \text{ then } t_i < t_j,$$

where t_i is the maximal integer satisfying $u_{t_i} = i$.

We define a grading on the quiver Q_w as follows:

- $\varphi(b) = 1$ if b is a Q^* -arrow;
- $\varphi(b) = 0$ if b is a Q -arrow or an arrow going to the left.

Lemma 4.3. *The graded Jacobian algebra $(\text{Jac}(Q_w, W_w, F), \varphi)$ satisfies the conditions (H1)-(H5) of Theorem 3.1.*

Proof. (H1) This holds by Theorems 4.1 and 4.2.

(H2) The potential $W_w = W$ is reduced. Moreover two different terms of the potential W differ by at least two arrows. Thus the set $\{\partial_a W \mid a \notin F_1, \partial_a W \neq 0\}$ is a basis. Let us show that for any arrow $a \notin F_1$, the derivative $\partial_a W$ does not vanish. Assume that a Q -arrow or a Q^* -arrow $a : r \rightarrow s$ does not appear in the potential, where $u_r = i$, $u_s = j$ and there is an arrow between i and j in the quiver Q . The fact that a does not appear in the potential implies that there is no u_t of type i with $t > r$. Thus we have $r = t_i$. Then we must have $s = t_j$. Therefore $a : r = t_i \rightarrow s = t_j$ is in F_1 . Now let $p : r \leftarrow s$ be an arrow going to the left, where r and s are two consecutive vertices of type i . Since the expression $s_{u_1} \dots s_{u_l}$ is reduced, there exists t with $r < t < s$ such that u_t is of type j and there is (at least) one arrow between i and j in the quiver Q . Let t be the maximal integer with this property. Then there is an arrow $a : r \rightarrow t$, and there is also an arrow $a^* : t \rightarrow v$ where $v \geq s$ and u_v is of type i . Thus p appears in the potential W . Therefore for any arrow a of Q_w which is not in F_1 , the derivative $\partial_a W$ is not zero, and we have (H2).

(H3) By the definition of the potential in Theorem 4.2, this follows immediately.

(H4) By the choice of the orientation of Q , any arrow in F_1 is a Q -arrow, hence of degree 0. Thus all arrows with target in F_0 and source not in F_0 are Q^* -arrows, hence of degree 1, and all arrows with source in F_0 and target not in F_0 are arrows going to the left, hence of degree 0. This implies condition (H4).

(H5) Let \bar{Q}_w be the full subquiver of Q_w with vertices not in F_0 . Let $b : r \rightarrow s$ be an arrow of \bar{Q}_w of degree 1, thus we have $r \neq t_i$ and $s \neq t_j$. Hence, there exists an arrow $a : s \rightarrow t$. Since s is not t_j , we have $t \neq t_i$. Thus the arrow a is in \bar{Q}_w . Therefore the arrow b appears in the potential \bar{W} . Moreover two different summands of the potential \bar{W} differ by at least two arrows. Thus condition (H5) holds. \square

Now we get the following direct consequence, which is one of our main results.

Theorem 4.4. *The stable category $\underline{\mathcal{E}}_w$ is a generalized cluster category.*

4.3. Meaning of the grading. We show that the grading φ on $\text{End}_{\mathcal{E}_w}(T)$ defined in the previous section is induced by a natural grading on the preprojective algebra.

Let $w = s_{u_1} \dots s_{u_l}$, be a reduced expression of an element of the Coxeter group C_Q . Assume that the orientation of Q satisfies the property

$$\text{if there exists } i \rightarrow j, \text{ then } t_i < t_j,$$

where t_i is the maximal integer satisfying $u_{t_i} = i$.

Let us define a grading on the double quiver \bar{Q} :

- $\text{deg}(a) = 0$ if a is an arrow of Q ;
- $\text{deg}(a^*) = 1$ if $a^* : j \rightarrow i$ is an arrow pointing in the opposite direction of an arrow $a : i \rightarrow j$ of Q .

The ideal of relations $(\sum_{a \in Q_1} aa^* - a^*a)$ is homogeneous of degree 1, thus the grading on the double quiver \bar{Q} induces a grading on the preprojective algebra Λ .

For any i in Q_0 , the Λ -module $e_i\Lambda$ can be seen as a graded Λ -module with top in degree 0. Then the ideal $\mathcal{I}_i = \Lambda(1 - e_i)\Lambda$ is a graded ideal. Hence for $p \leq l$ the ideal $\mathcal{I}_{w_p} = \mathcal{I}_{u_p} \dots \mathcal{I}_{u_1}$ is a graded ideal and the module $T_p = e_{u_p}(\Lambda/\mathcal{I}_{w_p})$ is a finite length graded Λ -module.

Therefore the cluster-tilting object $T_w = T_1 \oplus \dots \oplus T_l$ is a graded Λ -module. Thus its endomorphism algebra $\text{End}_{\Lambda}(T_w)$ is naturally graded.

We have the following connection with the previous grading.

Proposition 4.5. *The isomorphism of algebras $\text{End}_{\Lambda}(T_w) \simeq \text{Jac}(Q_w, W_w, F)$ of Theorem 4.2 is an isomorphism of graded algebras*

$$(\text{End}_{\Lambda}(T_w), \text{deg}) \simeq \text{Jac}(Q_w, W_w, F), \varphi$$

where deg is induced by the grading deg on the preprojective algebra Λ , and φ is the grading on Q_w defined in the previous section.

Proof. Each $a : i \rightarrow j$ in Q_1 gives maps $e_i(\Lambda/\mathcal{I}_{w_r}) \rightarrow e_j(\Lambda/\mathcal{I}_{w_s})$, where $u_r = i$ and $u_s = j$. These maps are obviously of degree 0 since they are induced by the degree zero map $a : e_i\Lambda \rightarrow e_j\Lambda$.

Each $a : i \rightarrow j$ in Q induces maps $e_j(\Lambda/\mathcal{I}_{w_t}) \rightarrow e_i(\Lambda/\mathcal{I}_{w_s})$, where $u_t = j$ and $u_s = i$. They are induced by the degree 1 map $a^* : e_j\Lambda \rightarrow e_i\Lambda$, thus they are maps of degree 1.

For any i in Q_0 , there are surjective maps $e_i(\Lambda/\mathcal{I}_{w_t}) \rightarrow e_i(\Lambda/\mathcal{I}_{w_r})$, where $u_t = u_r = i$ and $t > r$. They are induced by the identity $e_i\Lambda \rightarrow e_i\Lambda$, thus they are maps of degree 0.

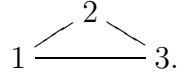
Hence we have the same grading as φ defined in the previous section. \square

Remark 4.1. Note that the summands of Λ_w are all graded Λ -modules, but this does not imply that all the objects in \mathcal{E}_w are gradable. In the proof of Proposition 5.2 of [GS05], Geiss and Schröer describe explicitly a non gradable module over the preprojective algebra associated to the Dynkin graph A_6 .

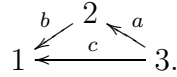
5. EXAMPLES

In this section we illustrate the previous theory through two examples. The first one is an example covered by Theorem 4.2, given by a standard cluster-tilting object in the category \mathcal{E}_w for some reduced word w . The second example shows that Theorem 3.1 also may apply for cluster-tilting objects in \mathcal{E}_w which are not standard.

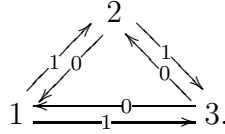
5.1. Canonical cluster-tilting object associated to a word. Let Q be the following graph



Let w be the reduced word $s_1s_2s_3s_1s_3s_2s_1$ in the Coxeter group C_Q . An admissible orientation of Q as defined in section 4.2 is



Let us put the following grading for the preprojective algebra Λ .



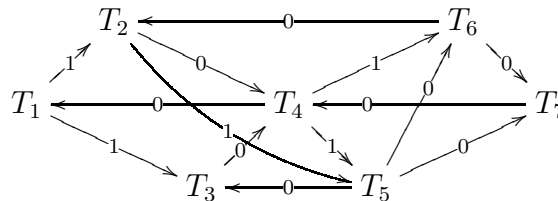
Then the canonical cluster-tilting object T_w of the Frobenius category \mathcal{E}_w has the following indecomposable summands:

$$\begin{aligned} T_1 = T_{(1,1)} &= 1, & T_2 = T_{(2,1)} &= \begin{array}{c} 2 \\ \downarrow \\ 1 \end{array}, & T_3 = T_{(3,1)} &= \begin{array}{c} 3 \\ \downarrow \\ 2 \\ \downarrow \\ 1 \end{array}, & T_4 = T_{(1,2)} &= \begin{array}{c} 2 \\ \downarrow \\ 1 \end{array} \begin{array}{c} 3 \\ \downarrow \\ 2 \\ \downarrow \\ 1 \end{array}, \\ T_5 = T_{(3,2)} &= \begin{array}{c} 3 \\ \downarrow \\ 2 \\ \downarrow \\ 1 \end{array} \begin{array}{c} 2 \\ \downarrow \\ 1 \end{array}, & T_6 = T_{(2,2)} &= \begin{array}{c} 3 \\ \downarrow \\ 2 \\ \downarrow \\ 1 \end{array} \begin{array}{c} 2 \\ \downarrow \\ 1 \end{array} \begin{array}{c} 3 \\ \downarrow \\ 2 \\ \downarrow \\ 1 \end{array} \end{aligned}$$

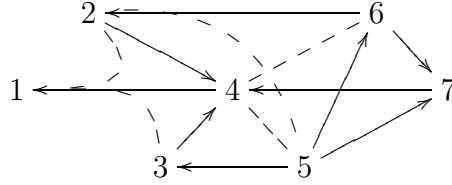
(Note: In the original image, the last term is written as $T_7 = T_{(1,3)} = \dots$ with a different arrangement of summands.)

The indecomposable projective injectives are T_5 , T_6 and T_7 . As Λ_0 -module ($=kQ$ -module) T_6 is isomorphic to the direct sum $\frac{2}{3} \oplus \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{1}{3} \oplus 1 \oplus 2 \oplus 1$.

By [BIRS09a] and [BIRS09b], we know the shape of the quiver of $B = \text{End}_{\mathcal{E}_w}(T)$. Its grading coming from the grading of Λ is the following.



The algebra $A = B_0$ is then given by the quiver with relations.



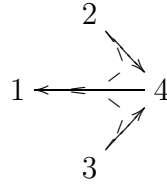
The indecomposable projective A -modules are

$$\frac{1}{4}, \quad \frac{2}{6}, \quad \frac{3}{5}, \quad \frac{3}{5} \frac{4}{7} \frac{2}{6}, \quad 5, \quad \frac{6}{5}, \quad \text{and} \quad \frac{1}{5} \frac{6}{7} \frac{5}{5}$$

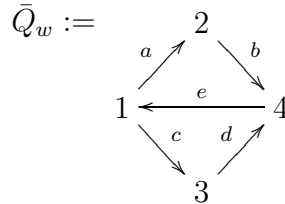
and the indecomposable injectives are

$$1, \quad \frac{4}{2}, \quad \frac{4}{3}, \quad \frac{1}{4}, \quad \frac{3}{5} \frac{4}{7} \frac{6}{6} \frac{7}{7}, \quad \frac{7}{6} \frac{4}{2}, \quad \text{and} \quad \frac{1}{7}$$

The algebra \bar{A} is given by the quiver with relations:



It is an algebra of global dimension 2. We have $\text{End}_{\mathcal{C}_A}(\bar{A}) \simeq \text{Jac}(\bar{Q}_w, \bar{W}_w)$ where



and $\bar{W}_w := bae + dce$. It is isomorphic to the algebra \bar{B} .

We denote by G the composition

$$G : \mathcal{D}^b(\bar{A}) \xrightarrow{\text{Res}} \mathcal{D}^b(A) \xrightarrow{-\overset{L}{\otimes}_A B} \mathcal{D}^b(B) \xrightarrow{-\overset{L}{\otimes}_B T} \mathcal{D}^b(\mathcal{E}_w) \longrightarrow \mathcal{D}^b(\mathcal{E}_w)/\mathcal{D}^b(\mathcal{P}) \simeq \underline{\mathcal{E}}_w.$$

Let S_2 be the simple \bar{A} -module associated to the vertex 2. We will show that $G(S_2)$ and $G \circ S^{-1}(S_2)[2]$ are isomorphic as objects in $\underline{\mathcal{E}}_w$.

The restriction of S_2 in the category $\mathcal{D}^b(A)$ is quasi-isomorphic to the complex

$$\cdots \longrightarrow 0 \longrightarrow e_5 A \longrightarrow e_6 A \longrightarrow e_2 A \longrightarrow 0 \longrightarrow \cdots$$

Tensoring it with B over A we get

$$\cdots \longrightarrow 0 \longrightarrow e_5 B \longrightarrow e_6 B \longrightarrow e_2 B \longrightarrow 0 \longrightarrow \cdots.$$

Therefore $G(S_2)$ is the complex

$$G(S_2) = (T_5 \longrightarrow T_6 \longrightarrow T_2).$$

Now the simple S_2 is quasi-isomorphic in $\mathcal{D}^b(\bar{A})$ to the complex

$$\cdots \longrightarrow 0 \longrightarrow e_2(D\bar{A}) \longrightarrow e_4(D\bar{A}) \longrightarrow e_1(D\bar{A}) \longrightarrow 0 \longrightarrow \cdots$$

Hence the restriction of $S_2 \otimes_{\bar{A}}^L R\mathrm{Hom}_{\bar{A}}(D\bar{A}, \bar{A})[2] = \mathbb{S}^{-1}(S_2)[2]$ in $\mathcal{D}^b A$ is isomorphic to

$$\cdots \longrightarrow 0 \longrightarrow 2 \longrightarrow 2 \begin{smallmatrix} 4 \\ 3 \end{smallmatrix} \longrightarrow \frac{1}{4} \longrightarrow 0 \longrightarrow \cdots$$

which is quasi-isomorphic to the complex

$$\cdots \longrightarrow 0 \longrightarrow 5 \oplus \frac{2}{6} \longrightarrow 3 \begin{smallmatrix} 4 \\ 5 \\ 6 \end{smallmatrix} 2 \longrightarrow \frac{1}{7} \longrightarrow 0 \longrightarrow \cdots$$

that is, to the complex

$$\cdots \longrightarrow 0 \longrightarrow e_5 A \oplus e_2 A \longrightarrow e_4 A \longrightarrow e_1 A \longrightarrow 0 \longrightarrow \cdots$$

Therefore the object $G(\mathbb{S}^{-1}S_2[2])$ is the complex

$$G(\mathbb{S}^{-1}S_2[2]) = (T_5 \oplus T_2 \longrightarrow T_4 \longrightarrow T_1) .$$

Now the simple S_2 is also quasi-isomorphic to the complex

$$\cdots \longrightarrow 0 \longrightarrow e_2(DA) \longrightarrow e_4(DA) \longrightarrow e_1(DA) \longrightarrow 0 \longrightarrow \cdots$$

Hence the complex $S_2 \otimes_{\bar{A}}^L R\mathrm{Hom}_A(DA, \bar{A}) \otimes_A^L B[2] \simeq R\mathrm{Hom}_A(DA, S_2) \otimes_A^L B[2]$ is the complex

$$\cdots \longrightarrow 0 \longrightarrow e_2 B \xrightarrow{a_1} e_4 B \xrightarrow{q_1^1} e_1 B \longrightarrow \cdots .$$

We have a morphism in the category $\mathcal{D}^b(B)$:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & e_2 B & \longrightarrow & e_4 B & \longrightarrow & e_1 B & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & e_5 B & \longrightarrow & e_6 B & \longrightarrow & e_2 B & \longrightarrow & \cdots \end{array}$$

whose cone is

$$\cdots \longrightarrow 0 \longrightarrow e_2 B \longrightarrow e_5 B \oplus e_4 B \longrightarrow e_6 B \oplus e_1 B \longrightarrow e_2 B \longrightarrow 0 \longrightarrow \cdots .$$

One can compute that it is quasi-isomorphic to the simple B -module S_2 . Hence we get a triangle in $\mathcal{D}^b(B)$:

$$S_2 \otimes_{\bar{A}}^L R\mathrm{Hom}_A(DA, \bar{A}) \otimes_A^L B[2] \longrightarrow S_2 \otimes_A^L B \longrightarrow S_2 \longrightarrow S_2 \otimes_A^L R\mathrm{Hom}_A(DA, \bar{A}) \otimes_{\bar{A}}^L B[3] .$$

Note that if we apply the triangle functor $S_2 \otimes_{\bar{A}}^L -$ to the triangle (*) of Proposition 2.8, we get the same triangle.

The object $S_2 \otimes_B^L T$ of $\mathcal{D}^b(\mathcal{E}_w)$ is then the complex

$$\cdots \longrightarrow 0 \longrightarrow T_2 \longrightarrow T_5 \oplus T_4 \longrightarrow T_6 \oplus T_1 \longrightarrow T_2 \longrightarrow 0 \longrightarrow \cdots .$$

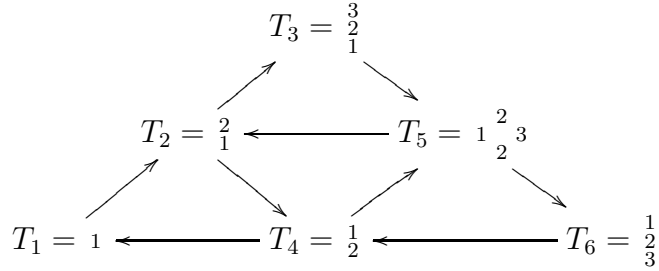
A direct computation shows that it is acyclic. Indeed it is the 2-almost split sequence associated with T_2 .

Finally we have morphisms

$$\begin{array}{ccc}
 & (T_2 \rightarrow T_4 \rightarrow T_1) & \\
 & \xleftarrow{(i)} & \xrightarrow{(ii)} \\
 (***) & & \\
 G(S_2) = (T_5 \rightarrow T_6 \rightarrow T_2) & & (T_5 \oplus T_2 \rightarrow T_4 \rightarrow T_1) = G(\mathbb{S}^{-1}S_2[2]) \\
 \downarrow & & \downarrow \\
 0 \simeq (T_2 \rightarrow T_5 \oplus T_4 \rightarrow T_1 \oplus T_6 \rightarrow T_2) & & (T_5) \in \mathcal{D}^b(\mathcal{P})
 \end{array}$$

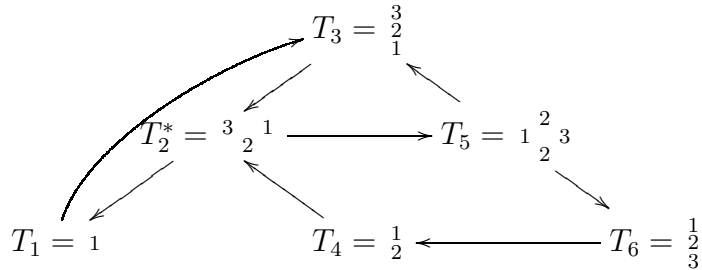
The cone of the morphism (i) is acyclic, and the cone of the morphism (ii) is $T_5[-2]$ which is perfect. Thus in $\underline{\mathcal{E}}_w = \mathcal{D}^b(\mathcal{E}_w)/\mathcal{D}^b(\mathcal{P})$, the objects $G(S_2)$ and $G(\mathbb{S}^{-1}S_2[2])$ are isomorphic.

5.2. Example which is not associated with a word. Let \mathcal{E} be the category $\text{mod } \Lambda$ where Λ is the preprojective algebra of type A_3 , which is one of the cases investigated by Geiss, Leclerc and Schröer in [GLS06]. This is a Frobenius category which is stably 2-Calabi-Yau and of the form \mathcal{E}_w where w is the element in the Coxeter group of maximal length. Corresponding to the word $s_1s_2s_3s_1s_2s_1$ is the canonical cluster-tilting object:

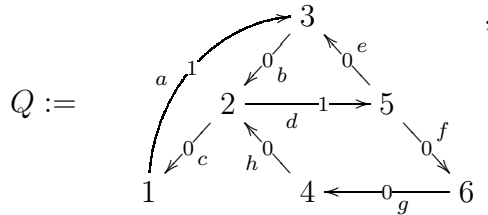


The endomorphism algebra is a frozen Jacobian algebra.

Let us do the mutation of the object $T_2 = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$. Its complement is $T_2^* = \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}$. The new cluster-tilting object T^* is given by



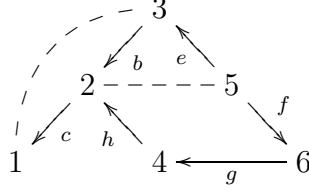
One can easily check that the endomorphism algebra is isomorphic to the frozen Jacobian algebra $B = \text{Jac}(Q, W, F)$, where



$W := acb + dbe + dhgf$, $F_0 := \{3, 5, 6\}$ and $F_1 := \{e, f\}$.

If we put $\varphi(a) = \varphi(d) = 1$ and $\varphi(b) = \varphi(c) = \varphi(e) = \varphi(f) = \varphi(h) = 0$, we obtain a grading satisfying hypothesis (H1)-(H5) of Theorem 3.1.

The algebra A is then given by the quiver



with relations $cb = 0$ and $be = hgf$. The algebra \bar{A} is the hereditary algebra with quiver

$$1 \xleftarrow{c} 2 \xleftarrow{h} 4.$$

One can check that the image of S_2 under the functor $G : \mathcal{D}^b(\bar{A}) \xrightarrow{Res} \mathcal{D}^b(A) \xrightarrow{-\otimes_A^L T_2^*} \mathcal{D}^b(\mathcal{E})$ is

$$(T_5 \xrightarrow{\begin{pmatrix} e \\ fg \end{pmatrix}} T_4 \oplus T_3 \xrightarrow{(b \ h)} T_2^*).$$

The object $\mathbb{S}^{-1}(S_2)[2]$ is quasi-isomorphic to the complex $\frac{2}{4} \longrightarrow \frac{1}{4}$, thus its restriction in $\mathcal{D}^b(A)$ is quasi-isomorphic to the complex

$$\cdots \longrightarrow 0 \longrightarrow e_3 A \xrightarrow{b} e_2 A \xrightarrow{c} e_1 A \longrightarrow 0 \longrightarrow \cdots.$$

Hence the complex $G(\mathbb{S}^{-1}S_2[2])$ is

$$\cdots \longrightarrow 0 \longrightarrow T_3 \xrightarrow{b} T_2^* \xrightarrow{c} T_1 \longrightarrow 0 \longrightarrow \cdots.$$

We have morphisms in $\mathcal{D}^b(\mathcal{E})$

$$\begin{array}{ccc} & (***) & (T_2^* \rightarrow T_1) \\ & \swarrow (i) & \searrow (ii) \\ G(S_2) = (T_5 \rightarrow T_4 \oplus T_3 \rightarrow T_2^*) & & (T_3 \rightarrow T_2^* \rightarrow T_1) = G \circ \mathbb{S}(S_2)[2] \\ \downarrow & & \downarrow \\ 0 \simeq (T_2^* \rightarrow T_1 \oplus T_5 \rightarrow T_3 \oplus T_4 \rightarrow T_2^*) & & (T_3) \in \mathcal{D}^b(\mathcal{P}) \end{array}$$

The cone of the morphism (i) is

$$T_2^* \xrightarrow{\begin{pmatrix} c \\ d \end{pmatrix}} T_1 \oplus T_5 \xrightarrow{\begin{pmatrix} a & e \\ 0 & fg \end{pmatrix}} T_3 \oplus T_4 \xrightarrow{(b \ h)} T_2^*$$

which is the 2-almost-split sequence associated to T_2^* , hence an acyclic complex. The cone of the morphism (ii) is $T_3[-2]$ which is perfect. Thus in $\mathcal{D}^b(\mathcal{E})/\mathcal{D}^b(\mathcal{P})$ the objects $G(S_2)$ and $G(\mathbb{S}^{-1}S_2[2])$ are isomorphic.

This example gives some hope that Theorem 3.1 can be applied to stably 2-CY categories other than those coming from an element of the Coxeter group.

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