



HAL
open science

Sampling the Fermi statistics and other conditional product measures

Alexandre Gaudilliere, Julien Reygner

► **To cite this version:**

Alexandre Gaudilliere, Julien Reygner. Sampling the Fermi statistics and other conditional product measures. 2009. hal-00434392v1

HAL Id: hal-00434392

<https://hal.science/hal-00434392v1>

Preprint submitted on 23 Nov 2009 (v1), last revised 18 Jan 2011 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Sampling the Fermi statistics and other conditional product measures¹

A. Gaudillière² J. Reygner³

November 23rd 2009

Abstract

Through a Metropolis-like algorithm with single step computational cost of order one, we build a Markov chain that relaxes to the canonical Fermi statistics for k non-interacting particles among m energy levels. Uniformly over the temperature as well as the energy values and degeneracies of the energy levels we give an explicit upper bound with leading term $km \ln k$ for the mixing time of the dynamics. We obtain such construction and upper bound as a special case of a general result on (non-homogeneous) products of ultra log-concave measures (like binomial or Poisson laws) with a global constraint. As a consequence of this general result we also obtain a disorder-independent upper bound on the mixing time of a simple exclusion process on the complete graph with site disorder. This general result is based on an elementary coupling argument and extended to (non-homogeneous) products of log-concave measures.

Key words: Metropolis algorithm, Markov chain, sampling, mixing time, product measure, conservative dynamics.

AMS 2010 classification: 60J10, 82C44.

¹The whole research for this paper was done at the Dipartimento di matematica dell'università Roma Tre, during A.G.'s post-doc and J.R.'s reasearch internship. This work was supported by the European Research Council through the "Advanced Grant" PTRELESS 228032.

²gaudilli@cmi.univ-mrs.fr – LATP, Université de Provence, CNRS, 39 rue F. Joliot Curie, 13013 Marseille, France.

³julien.reygner@polytechnique.edu – CMAP, École Polytechnique, Route de Saclay, 91120 Palaiseau, France.

1 From the Fermi statistics to general conditional products of log-concave measures

1.1 Sampling the Fermi statistics

Given two positive integers k and m , given a non-negative real number β , given m real numbers v_1, \dots, v_m and given m integers n_1, \dots, n_m such that

$$n := \sum_{j=1}^m n_j \geq k, \quad (1.1)$$

the canonical Fermi statistics at *inverse temperature* β for k non-interacting particles among the m *energy levels* $1, \dots, m$, with *energy values* v_1, \dots, v_m and *degeneracies* n_1, \dots, n_m is the conditional probability measure on

$$\mathcal{X}_{k,m} := \{(k_1, \dots, k_m) \in \mathbb{N}^m : k_1 + \dots + k_m = k\} \quad (1.2)$$

given by

$$\nu := \mu(\cdot | \mathcal{X}_{k,m}) \quad (1.3)$$

with μ the product measure on \mathbb{N}^m such that

$$\mu(k_1, \dots, k_m) := \frac{1}{Z} \prod_{j=1}^m \binom{n_j}{k_j} \exp\{-\beta k_j v_j\}, \quad (1.4)$$

$$Z := \sum_{k_1, \dots, k_m} \prod_{j=1}^m \binom{n_j}{k_j} \exp\{-\beta k_j v_j\}. \quad (1.5)$$

In other words, ν is a (non-homogeneous) product of binomial laws in k_1, \dots, k_m with the global constraint

$$k_1 + \dots + k_m = k \quad (1.6)$$

and we can write

$$\nu(k_1, \dots, k_m) = \frac{1}{Q} \prod_{i=1}^m e^{\phi_i(k_i)}, \quad (k_1, \dots, k_m) \in \mathcal{X}_{k,m} \quad (1.7)$$

where the ϕ_j are defined by

$$\phi_j : k_j \in \mathbb{N} \mapsto -\beta k_j v_j + \ln \binom{n_j}{k_j} \in \mathbb{R} \cup \{-\infty\} \quad (1.8)$$

and Q is such that ν is a probability measure.

The first aim of this paper is to describe an algorithm that simulates a sampling according to ν in a time that can be bounded from above by an explicit polynomial in k and m , uniformly over β , $(v_j)_{1 \leq j \leq m}$ and $(n_j)_{1 \leq j \leq m}$.

The reason why we prefer a bound in k and m rather than in the ‘volume’ of the system $n = \sum_j n_j$, will be clarified later.

A first naive (and wrong) idea to do so consists in choosing the position (the energy level) of a first, second, \dots and eventually k^{th} particle in the following way. First choose randomly the position of the first particle according to the exponential weights associated with the ‘free entropies’ of the empty sites, that is choose level j with a probability proportional to $\exp\{-\beta v_j + \ln n_j\}$. Then decrease by 1 the degeneracy of the chosen energy level and repeat the procedure to choose the position of the second, third, \dots and eventually k^{th} particle. It is easy to check that, doing so, the final distribution of the occupation numbers k_1, \dots, k_m associated with the different energy levels, that is of the numbers of particles placed in each level, is in general not given by ν as soon as k is larger than one. But it turns out that this naive idea can be adapted to build an efficient algorithm to perform approximate samplings under the Fermi statistics.

Very classically, the fast sampling performed by the algorithm we will build will be obtained by running a Markov chain X with transition matrix p on $\mathcal{X}_{k,m}$ and with equilibrium measure ν . The efficiency of the algorithm will be measured through the bounds that we will be able to give on the *mixing time* t_ϵ , defined for any positive $\epsilon < 1$ by

$$t_\epsilon := \inf\{t \geq 0 : d(t) \leq \epsilon\} \quad (1.9)$$

$$d(t) := \max_{\eta \in \mathcal{X}_{k,m}} \|p^t(\eta, \cdot) - \nu\|_{\text{TV}} \quad (1.10)$$

where $\|\cdot\|_{\text{TV}}$ stands for the *total variation distance* defined for any probability measures ν_1 and ν_2 on $\mathcal{X}_{k,m}$ by

$$\|\nu_1 - \nu_2\|_{\text{TV}} := \max_{A \subset \mathcal{X}_{k,m}} |\nu_1(A) - \nu_2(A)| = \frac{1}{2} \sum_{\eta \in \mathcal{X}_{k,m}} |\nu_1(\eta) - \nu_2(\eta)|. \quad (1.11)$$

As a consequence, estimating mixing times is not the only one issue of this paper, building a ‘good’ Markov chain is part of the problem.

As far as that part of the problem is concerned, we propose to build a Metropolis-like algorithm that uses the ‘free energies’ of the naive approach to define a conservative dynamics. Assuming that at time $t \in \mathbb{N}$ the system is in some configuration $X_t = \eta$ in $\mathcal{X}_{k,m}$ with $\nu(\eta) > 0$, and defining for any $\eta = (k_1, \dots, k_m)$ and any distinct i and j in $\{1; \dots; m\}$

$$\eta^{ij} := (k'_1, \dots, k'_m) \quad \text{with} \quad k'_s = \begin{cases} k_s & \text{for } s \in \{1; \dots; m\} \setminus \{i; j\} \\ k_i - 1 & \text{if } s = i \\ k_j + 1 & \text{if } s = j \end{cases}, \quad (1.12)$$

the configuration at time $t + 1$ will be decided as follows:

- choose a *particle* with uniform probability (it will stand in a given level i with probability k_i/k),

- choose an *energy level* with uniform probability (a given level j will be chosen with probability $1/m$),
- with i the level where stood the chosen particle and j the chosen energy level, extract a uniform variable U on $[0; 1)$ and set $X_{t+1} = \eta^{ij}$ if $i \neq j$ and

$$U < \exp\{-\beta v_j + \ln(n_j - k_j) + \beta v_i - \ln(n_i - (k_i - 1))\}, \quad (1.13)$$

$X_{t+1} = \eta$ if not.

In other words, denoting by $[a]^+ = (a + |a|)/2$ the positive part of any real number a and with

$$\psi_j : k_j \in \{0; \dots; n_j\} \mapsto -\beta v_j + \ln(n_j - k_j) \in \mathbb{R} \cup \{-\infty\}, \quad j \in \{1; \dots; m\}, \quad (1.14)$$

for any distinct i and j in $\{1; \dots; m\}$

$$P(X_{t+1} = \eta^{ij} | X_t = \eta) = p(\eta, \eta^{ij}) = \frac{k_i}{k} \frac{1}{m} \exp\{-[\psi_i(k_i - 1) - \psi_j(k_j)]^+\}, \quad (1.15)$$

and

$$P(X_{t+1} = \eta | X_t = \eta) = p(\eta, \eta) = 1 - \sum_{i \neq j} p(\eta, \eta^{ij}). \quad (1.16)$$

Remark: In order to avoid any ambiguity in (1.15) in the case $k_i = 0$, we set $\psi_i(-1) = +\infty$ (even though the algorithm we described does not require any convention for $\psi_i(-1)$).

This Markov chain is certainly irreducible and aperiodic. To prove that it relaxes to ν we will check the reversibility of the process with respect to ν . Then we will have to estimate the mixing time of the process. We will carry out both the tasks in a more general setup.

1.2 A general result

For any function $f : \mathbb{N} \rightarrow \mathbb{R}$ we define

$$\nabla_x^+ f := f(x + 1) - f(x), \quad x \in \mathbb{N}, \quad (1.17)$$

$$\nabla_x^- f := f(x - 1) - f(x), \quad x \in \mathbb{N} \setminus \{0\}, \quad (1.18)$$

$$\Delta_x f := \nabla_x^+ f + \nabla_x^- f = -\nabla_x^- \nabla_x^+ f, \quad x \in \mathbb{N} \setminus \{0\}, \quad (1.19)$$

and we say that a measure γ on the integers

$$\gamma : x \in \mathbb{N} \mapsto e^{\phi(x)} \in \mathbb{R}_+, \quad (1.20)$$

with $\phi : \mathbb{N} \rightarrow \mathbb{R} \cup \{-\infty\}$, is log-concave if $\mathbb{N} \setminus \gamma^{-1}(\{0\})$ is an interval of the integers and

$$\gamma(x)^2 \geq \gamma(x - 1)\gamma(x + 1), \quad x \in \mathbb{N} \setminus \{0\}, \quad (1.21)$$

i.e., if $\nabla^+ \phi$ is non-increasing, or, equivalently, $-\Delta \phi$ is non-negative (with the obvious extension of the previous definitions to such a possibly non-finite ϕ). The measure μ defined in (1.4) is a product of log-concave measures and the canonical Fermi statistics is such a product measure normalized over the condition (1.6).

N.B. From now on, and except for explicit mentioning of additional hypotheses, we will only assume that the probability measure ν we want to sample is a product of log-concave measures normalized over the global constraint (1.6), i.e., that ν is a probability on $\mathcal{X}_{k,m}$ that can be written in the form (1.7) with non-increasing $\nabla^+ \phi_j$'s.

In this more general setup we will often refer to the indices j in $\{1; \dots; m\}$ as *sites* rather than energy levels of the system.

Actually the e^{ϕ_j} 's of the Fermi statistics are much more than log-concave measures. They are *ultra log-concave* measures according to the following definition by Pemantle [1] and Liggett [2].

Definition 1.2.1 A measure $\gamma : \mathbb{N} \rightarrow \mathbb{R}_+$ is ultra log-concave if $x \mapsto x! \gamma(x)$ is log-concave.

In other words e^{ϕ_j} is ultra log-concave if and only if

$$\psi_j := \nabla^+ \phi_j + \ln(1 + \cdot) \quad (1.22)$$

is non-increasing (for the Fermi statistics observe that so are the ϕ_j 's and that (1.22) is consistent with (1.14)).

For birth and death processes that are reversible with respect to ultra log-concave measures, Caputo, Dai Pra and Posta [6] proved modified log-Sobolev inequalities and stronger convex entropy decays, both giving good upper bounds on the mixing time of the processes. Johnson [9] proved also easier Poincaré inequalities that give weaker bounds on the mixing times. Our main result directly bounds from above the mixing time of a Markov chain that is reversible with respect to a conditional product of ultra log-concave measures:

Theorem 1 If ν derives from a product of ultra log-concave measures, then the Markov chain with transition matrix p defined by

$$\begin{aligned} p(\eta, \eta^{ij}) &= \frac{k_i}{k} \frac{1}{m} \exp\{-[\psi_i(k_i - 1) - \psi_j(k_j)]^+\}, \\ \eta &= (k_1, \dots, k_m) \in \mathcal{X}_{k,m} \setminus \nu^{-1}\{0\}, \\ & i \neq j \in \{1; \dots; m\}, \end{aligned} \quad (1.23)$$

$$p(\eta, \eta) = 1 - \sum_{i \neq j} p(\eta, \eta^{ij}) \quad (1.24)$$

is reversible with respect to ν and, for any positive $\epsilon < 1$, its mixing time t_ϵ satisfies

$$t_\epsilon \leq km \ln(k/\epsilon). \quad (1.25)$$

Proof: see Section 2.

The most relevant point of Theorem 1 with respect to the previous results we know stands in the uniformity of the upper bound above the disorder of the system (except for the ultra log-concavity hypothesis on e^{ϕ_j} in each j). In particular and as far as the Fermi statistics is concerned, our estimate does not depend on the temperature, and, more generally it is independent from the energy values as well as the level degeneracies.

To illustrate this fact let us start with the case $n_j = 1$ for all j . In this case our dynamic is a simple exclusion process with site disorder. Caputo ([3], [8]) proved Poincaré inequalities for such processes, in their continuous time version, assuming a uniform lower (and upper) bound on general transition rates. Caputo, Dai Pra and Posta [6], looking at particular rates for the process and still assuming moderate disorder, that is, uniform lower and upper bounds on these rates, then proved a modified log-Sobolev inequality giving an upper bound of the same order as that of Theorem 1. It is worth to note that for the particular choice of rates made in [6] such an upper bound on the mixing time could not hold in a strong disorder context (for example with $k = 1$, $m = 3$, $v_1 = v_2 = 0$, $v_3 > 0$ and $\beta \gg 1$). Our uniformity over the disorder of the system depends then strongly on our particular choice for the transition probabilities. As it is often the case with Markov processes on discrete state space, the details of the dynamics are not less important than the properties of its equilibrium measure.

As far as more general conditional non-homogeneous product log-concave measures are concerned, these can be equilibrium measures of zero-range processes. Under a few technical hypotheses implying ultra log-concavity as well as moderate disorder [6] showed modified log-Sobolev inequalities for such processes. Boudou, Caputo, Dai Pra and Posta [5] proved also Poincaré inequalities under an assumption of uniformly increasing rates. Such an assumption still imply moderate disorder (but it only implies log-concavity rather than ultra log-concavity). For our Markov chain we will not prove Poincaré or log-Sobolev inequalities that are the main objects of [3], [5], [6], [8], [9]: we will see in Section 2 that our uniform bound on the mixing time comes from an elementary coupling argument.

1.3 Interpolating between sites and particles

It seems that today available techniques are such that the less log-concavity we have, the more homogeneity we need to control the convergence to equilibrium. Staying to the papers mentioned above, without ultra log-concavity we only have Poincaré inequalities for non-homogeneous product of log-concave measures, and without log-concavity we have modified log-Sobolev inequalities for homogeneous product measures only (see [6]). In addition, the only result we know for a conservative dynamics in (weakly) disordered context and with an equilibrium measure that is a product of measures that are not log-concave is that of Landim and Noronha Neto [4] for the (continuous) Ginzburg-Landau process.

We will see that all the ideas of the proof of Theorem 1 can be extended to deal with a large class of conditional product of log-concave measures that are not ultra log-concave. To do so, let us define

$$\delta := \max \left\{ \lambda \in [0; 1] : \forall j \in \{1; \dots; m\}, \forall k_j > 0, -\Delta_{k_j} \phi_j \geq \lambda \ln \frac{1+k_j}{k_j} \right\}. \quad (1.26)$$

In other words, δ is the largest real number in $[0; 1]$ for which all the

$$\psi_j^{[\delta]} := \nabla^+ \phi_j + \delta \ln(1 + \cdot), \quad j \in \{1; \dots; m\}, \quad (1.27)$$

are non-increasing. Denoting by $a \wedge b$ the minimum of two real numbers a, b and defining

$$l_\delta := k^\delta (k \wedge m)^{1-\delta} \quad (1.28)$$

we will prove:

Theorem 2 *If $\delta > 0$ then the Markov chain with transition matrix p defined by*

$$\begin{aligned} p(\eta, \eta^{ij}) &= \frac{k_i^\delta}{l_\delta} \frac{1}{m} \exp\{-[\psi_i^{[\delta]}(k_i - 1) - \psi_j^{[\delta]}(k_j)]^+\}, \\ \eta &= (k_1, \dots, k_m) \in \mathcal{X}_{k,m} \setminus \nu^{-1}\{0\}, \\ & i \neq j \in \{1; \dots; m\}, \end{aligned} \quad (1.29)$$

$$p(\eta, \eta) = 1 - \sum_{i \neq j} p(\eta, \eta^{ij}) \quad (1.30)$$

is reversible with respect to ν and, for any positive $\epsilon < 1$, its mixing time t_ϵ satisfies

$$t_\epsilon \leq \frac{(k \wedge m)^{1-\delta}}{\delta} km \ln(k/\epsilon). \quad (1.31)$$

Remark 1: By Hölder's inequality, if $\delta > 0$ then

$$\sum_{i=1}^m k_i^\delta = \sum_{i=1}^m k_i^\delta \mathbf{1}_{\{k_i \neq 0\}}^{1-\delta} \leq \left(\sum_{i=1}^m k_i \right)^\delta \left(\sum_{i=1}^m \mathbf{1}_{\{k_i \neq 0\}} \right)^{1-\delta} \leq l_\delta \quad (1.32)$$

and this ensures that (1.29)-(1.30) define a probability matrix.

Remark 2: As far as this can make sense in our discrete setup, we note that the hypothesis $\delta > 0$ is slightly weaker than a “uniform strict log-concavity hypothesis” (see (1.26)).

Proof of Theorem 2: see Section 3.

For $\delta = 1$ the transition matrix represents an algorithm starting with a uniform choice of a particle. For $\delta = 0$ Theorem 2 is empty but (1.29)-(1.30) still define a Markov chain X that can be seen as a particular version of a discrete state space non-homogeneous Ginzburg-Landau process. In this case the transition matrix represents an algorithm that starts with a uniform choice

of a (non-empty) site. The case $0 < \delta < 1$ can be seen as an interpolation between uniform choices of site and particle. More precisely, assuming that at time $t \in \mathbb{N}$ the system is in some configuration $X_t = \eta = (k_1, \dots, k_m)$ in $\mathcal{X}_{k,m} \setminus \nu^{-1}\{0\}$, the configuration at time $t + 1$ can be decided as follows.

- Choose a site i or no site at all with probabilities proportional to k_i^δ and $l_\delta - \sum_i k_i^\delta$.
- If some site i was chosen then proceed as in the previously described algorithm using the functions $\psi_j^{[\delta]}$ instead of the ψ_j 's, if not then set $X_{t+1} = \eta$.

1.4 Last remarks and original motivation

With our definitions $p(\eta, \eta)$ can often be close to one, especially in strong disorder situations or when the right and left hand sides in (1.32) are far from each other. If one would like to use these results to perform practical simulations, then it could be useful to note that the computational time would still be improved by implementing an algorithm that at each step simulates, for a given configuration η on the trajectory of the Markov chain, the elapsed time before the particle reach a *different* configuration (this is a geometric time) and choose this configuration $\eta' \neq \eta$ according to the (easy to compute) associated law. It would then be enough to stop the algorithm as soon as the total simulated time goes beyond the mixing time (and then return the last configuration, that occupied by the system at the mixing time) rather than waiting for the original algorithm to make a step number equal to the mixing time.

Turning back to the first naive and wrong idea, it is interesting to note that it can easily be modified to determine the most probable states of the system, i.e., the configurations $\eta^* = (k_1^*, \dots, k_m^*)$ in $\mathcal{X}_{k,m}$ such that

$$\sum_{j=1}^m \phi_j(k_j^*) = \max_{k_1 + \dots + k_m = k} \sum_{j=1}^m \phi_j(k_j). \quad (1.33)$$

One can prove, using the concavity of the ϕ_j 's, that the most probable configurations for the system with k particles can be obtained from the most probable configurations (k'_1, \dots, k'_m) for the system with $k - 1$ particles simply adding one particle where the corresponding gain in 'free energy' is the highest, that is in j^* such that

$$\nabla_{k_{j^*}}^+ \phi_{j^*} = \max_j \nabla_{k_j}^+ \phi_j. \quad (1.34)$$

As a consequence one can place the particles one by one, each time maximizing this free energy gain, to build the most probable configuration.

Finally, turning back to the Fermi statistics we now explain why we were interested in bounds in k and m rather than in the volume n . Iovanella, Scoppola and Scoppola defined in [7] an algorithm to individuate cliques (i.e., complete subgraphs) with k vertices inside a large Erdős-Reyni random graph with n vertices. Their algorithm requires to perform repeated approximate samplings

of Fermi statistics in volume n , with k particles and $m = 2k + 1$ energy levels. Now, the key observation is that the largest cliques in Erdős-Reyni graphs with n vertices are of order $\ln n$, so that k and m in this problem are logarithmically small with respect to n . Before Theorem 1 the samplings for their algorithm were done by running simple exclusion processes with k particles on the complete graph (with site disorder) of size n . Such processes converge to equilibrium in a time of order $kn \ln k$. Now the samplings are done in a time of order $km \ln k \sim 2k^2 \ln k$, and that was the original motivation of the present work.

2 Proof of Theorem 1

In this section we assume that ν is a measure on $\mathcal{X}_{k,m}$ deriving from a product of ultra log-concave measures, which means that we can write $\nu(k_1, \dots, k_m) = (1/Q) \prod_j e^{\phi_j(k_j)}$ and, for all $j \in \{1; \dots; m\}$, ψ_j defined by (1.22) is non-increasing.

2.1 Reversibility

We first prove that the transition matrix p defined by (1.23) and (1.24) is reversible with respect to the measure ν . Let $\eta = (k_1, \dots, k_m) \in \mathcal{X}_{k,m} \setminus \nu^{-1}\{0\}$ and let $i \neq j \in \{1; \dots; m\}$. We have $p(\eta, \eta^{ij}) \neq 0$ if and only if $\nu(\eta^{ij}) \neq 0$ and, in that case,

$$\frac{\nu(\eta^{ij})}{\nu(\eta)} = \frac{e^{\phi_i(k_i-1)+\phi_j(k_j+1)}}{e^{\phi_i(k_i)+\phi_j(k_j)}} = \exp\{\nabla_{k_i}^- \phi_i + \nabla_{k_j}^+ \phi_j\} \quad (2.1)$$

while

$$\frac{p(\eta^{ij}, \eta)}{p(\eta, \eta^{ij})} = \frac{k_j + 1}{k_i} \exp\{-[\psi_j(k_j) - \psi_i(k_i - 1)]^+ + [\psi_i(k_i - 1) - \psi_j(k_j)]^+\} \quad (2.2)$$

$$= \exp\{\psi_i(k_i - 1) - \psi_j(k_j) + \ln(k_j + 1) - \ln(k_i)\} \quad (2.3)$$

$$= \exp\{\nabla_{k_i-1}^+ \phi_i - \nabla_{k_j}^+ \phi_j\} \quad (2.4)$$

$$= \exp\{-\nabla_{k_i}^- \phi_i - \nabla_{k_j}^+ \phi_j\} \quad (2.5)$$

so that

$$\nu(\eta)p(\eta, \eta^{ij}) = \nu(\eta^{ij})p(\eta^{ij}, \eta). \quad (2.6)$$

2.2 A few words about the coupling method

In order to upper bound the mixing time of the Markov chain with transition matrix p , we will use the coupling method. Given a Markov chain (X^1, X^2) on $\mathcal{X}_{k,m} \times \mathcal{X}_{k,m}$, we say it is a *coupling* for the dynamics if both X^1 and X^2 are Markov chains with transition matrix p . Given such a coupling, we define the

coupling time τ_{couple} as the first (random) time for which the chains meet, that is

$$\tau_{\text{couple}} := \inf\{t \geq 0 : X_t^1 = X_t^2\}. \quad (2.7)$$

In this work, every coupling will also satisfy the condition

$$t \geq \tau_{\text{couple}} \Rightarrow X_t^1 = X_t^2. \quad (2.8)$$

Then, it is a well-known fact that for all $t \geq 0$,

$$d(t) \leq \max_{\eta, \theta \in \mathcal{X}_{k,m}} P(\tau_{\text{couple}} > t | X_0^1 = \eta, X_0^2 = \theta) \quad (2.9)$$

where $d(t)$ is defined by (1.10). A proof of this fact as well as an exhaustive introduction to mixing time theory can be found in [10].

In the proof of both 1 and 2 we will build a coupling for which there exists a function ρ that measures in some sense a ‘distance’ between X_t^1 and X_t^2 and from which we will get a bound on the mixing time thanks to the following proposition.

Proposition 2.2.1 *Let (X^1, X^2) be a coupling for a Markov chain with transition matrix p . We assume that the coupling satisfies the property (2.8). Let $\rho : \mathcal{X}_{k,m} \times \mathcal{X}_{k,m} \rightarrow \mathbb{N}$ such that $\rho(\eta, \theta) = 0$ if and only if $\eta = \theta$. If M is the maximum of ρ and if there exists $\alpha > 1$ such that, for all $t \geq 0$,*

$$E(\rho(X_{t+1}^1, X_{t+1}^2) | X_t^1, X_t^2) \leq \left(1 - \frac{1}{\alpha}\right) \rho(X_t^1, X_t^2), \quad (2.10)$$

then for all $\epsilon > 0$ the mixing time t_ϵ of the dynamics is upper bounded by $\alpha \ln(M/\epsilon)$.

Proof: Taking the expectation in (2.10) we get

$$E(\rho(X_{t+1}^1, X_{t+1}^2)) \leq \left(1 - \frac{1}{\alpha}\right) E(\rho(X_t^1, X_t^2)) \quad (2.11)$$

i.e., $(\rho(X_t^1, X_t^2)(1 - 1/\alpha)^{-t})_{t \in \mathbb{N}}$ is a supermartingale. As a consequence

$$E(\rho(X_t^1, X_t^2)) \leq \left(1 - \frac{1}{\alpha}\right)^t E(\rho(X_0^1, X_0^2)) \leq M e^{-\frac{t}{\alpha}}. \quad (2.12)$$

Since we assume $\rho(\eta, \theta) = 0$ if and only if $\eta = \theta$ and (2.8),

$$P(\tau_{\text{couple}} > t) = P(\rho(X_t^1, X_t^2) > 0) = P(\rho(X_t^1, X_t^2) \geq 1). \quad (2.13)$$

From Markov’s inequality and (2.12) we deduce

$$P(\tau_{\text{couple}} > t) \leq E(\rho(X_t^1, X_t^2)) \leq M e^{-\frac{t}{\alpha}}. \quad (2.14)$$

Since the upper bound in (2.14) is uniform in X_0^1 and X_0^2 , according to (2.9) we get

$$d(t) \leq M e^{-\frac{t}{\alpha}}. \quad (2.15)$$

Thus, given $\epsilon > 0$, if $t \geq \alpha \ln(M/\epsilon)$ then $d(t) \leq \epsilon$, so that $t_\epsilon \leq \alpha \ln(M/\epsilon)$. \square

2.3 A colored coupling

In order to prove Theorem 1, we introduce a dynamics on the set of the possible distributions of the k particles in the m energy levels. Therefore we define the set

$$\Omega := \{\omega : \{1; \dots; k\} \rightarrow \{1; \dots; m\}\} \quad (2.16)$$

and for every such distribution ω , we define $\xi(\omega) = (k_1, \dots, k_m) \in \mathcal{X}_{k,m}$ by

$$\forall i \in \{1; \dots; m\}, k_i := \sum_{x=1}^k \mathbf{1}_{\{\omega(x)=i\}}. \quad (2.17)$$

We will couple two dynamics ω^1 and ω^2 on Ω , then we will work with the coupling $(\xi(\omega^1), \xi(\omega^2))$. We will build the coupling (ω^1, ω^2) thanks to the coloring we now introduce.

At step t , a *red-blue coloring* of (ω_t^1, ω_t^2) is a couple of functions $C^1, C^2 : \{1; \dots; k\} \rightarrow \{\text{blue}; \text{red}\}$ such that for all energy level $i \in \{1; \dots; m\}$, the number of blue particles in the level i is the same in both distributions, i.e., if $|A|$ refers to the cardinality of the set A ,

$$|\{x : C^1(x) = \text{blue}, \omega_t^1(x) = i\}| = |\{x : C^2(x) = \text{blue}, \omega_t^2(x) = i\}| \quad (2.18)$$

and an energy level cannot contain red particles in both distributions

$$C^1(x) = \text{red} \Rightarrow \forall y, (\omega_t^2(y) = \omega_t^1(x) \Rightarrow C^2(y) = \text{blue}), \quad (2.19)$$

$$C^2(x) = \text{red} \Rightarrow \forall y, (\omega_t^1(y) = \omega_t^2(x) \Rightarrow C^1(y) = \text{blue}). \quad (2.20)$$

As a consequence of (2.18), there exists a one-to-one correspondence

$$\Phi : \{x : C^1(x) = \text{blue}\} \rightarrow \{x : C^2(x) = \text{blue}\} \quad (2.21)$$

such that for all x , $\omega_t^2(\Phi(x)) = \omega_t^1(x)$. Since the number of blue particles is the same in both distributions, so is the number of red particles. Moreover, $\xi(\omega_t^1) = \xi(\omega_t^2)$ if and only if all particles are blue. Then we are willing to provide a coupling (ω^1, ω^2) for which the number of red particles is non-increasing. This number does not depend on the red-blue coloring and at any time t we will call it ρ_t . Writing $\xi(\omega_t^1) = (k_1^1, \dots, k_m^1)$ and $\xi(\omega_t^2) = (k_1^2, \dots, k_m^2)$ we have the identities

$$\rho_t = \frac{1}{2} \sum_{i=1}^m |k_i^1 - k_i^2| = \sum_{i=1}^m [k_i^1 - k_i^2]^+ = \sum_{i=1}^m [k_i^2 - k_i^1]^+. \quad (2.22)$$

We are now ready to build our coupling (ω^1, ω^2) . Given (ω_t^1, ω_t^2) and a red-blue coloring (C^1, C^2) at step t (such a coloring certainly exists for any couple of distributions (ω_t^1, ω_t^2)), let x^1 be a uniform random integer in $\{1; \dots; k\}$.

- If $C^1(x^1) = \text{blue}$: then we set $x^2 = \Phi(x^1)$ where Φ is provided by (2.21).
- If $C^1(x^1) = \text{red}$: let x^2 be a uniform random integer in $\{x : C^2(x) = \text{red}\}$.

Lemma 2.3.1 *The random integer x^2 has a uniform distribution on the set $\{1; \dots; k\}$.*

Proof: We write

$$P(x^2 = y) = \sum_{x=1}^k P(x^2 = y | x^1 = x) P(x^1 = x) \quad (2.23)$$

$$= \sum_{x: C^1(x)=\text{blue}} \mathbf{1}_{\{y=\Phi(x)\}} \frac{1}{k} + \sum_{x: C^1(x)=\text{red}} \frac{\mathbf{1}_{\{C^2(y)=\text{red}\}}}{\rho_t} \frac{1}{k} \quad (2.24)$$

$$= \frac{\mathbf{1}_{\{C^2(y)=\text{blue}\}}}{k} + \frac{\mathbf{1}_{\{C^2(y)=\text{red}\}}}{\rho_t} \frac{\rho_t}{k} \quad (2.25)$$

$$= \frac{1}{k}. \quad (2.26)$$

□

We then choose an energy level j uniformly in $\{1; \dots; m\}$ and we write $\xi(\omega_t^1) = (k_1^1, \dots, k_m^1)$ and $\xi(\omega_t^2) = (k_1^2, \dots, k_m^2)$.

- If $C^1(x^1) = \text{blue}$: then $C^2(x^2) = \text{blue}$, and x^1, x^2 are in the same energy level $i := \omega_t^1(x^1) = \omega_t^2(x^2)$. Let $a \neq b \in \{1; 2\}$ such that $\psi_i(k_i^a - 1) - \psi_j(k_j^a) \leq \psi_i(k_i^b - 1) - \psi_j(k_j^b)$. Then,

$$p_a := \exp\{-[\psi_i(k_i^a - 1) - \psi_j(k_j^a)]^+\} \quad (2.27)$$

$$\geq \exp\{-[\psi_i(k_i^b - 1) - \psi_j(k_j^b)]^+\} =: p_b. \quad (2.28)$$

Let U be a uniform random variable on $[0; 1)$.

- If $U < p_b$: then we set $\omega_{t+1}^a(x) = \omega_t^a(x)$ for all $x \neq x^a$, $\omega_{t+1}^a(x^a) = j$, $\omega_{t+1}^b(x) = \omega_t^b(x)$ for all $x \neq x^b$ and $\omega_{t+1}^b(x^b) = j$. Then $\xi(\omega_{t+1}^a) = (\xi(\omega_t^a))^{ij}$ and $\xi(\omega_{t+1}^b) = (\xi(\omega_t^b))^{ij}$, and for any red-blue coloring of $(\omega_{t+1}^1, \omega_{t+1}^2)$ the number of red particles ρ_{t+1} remains the same (since both particles x^1 and x^2 have moved together).
- If $p_b \leq U < p_a$: then we set $\omega_{t+1}^a(x) = \omega_t^a(x)$ for all $x \neq x^a$, $\omega_{t+1}^a(x^a) = j$ and $\omega_{t+1}^b(x) = \omega_t^b(x)$ for all x . Then $\xi(\omega_{t+1}^a) = (\xi(\omega_t^a))^{ij}$ and $\xi(\omega_{t+1}^b) = \xi(\omega_t^b)$. The only situation in which the number of red particles could increase is the following: $k_j^a \geq k_j^b$ and $k_i^a \leq k_i^b$. Since ψ_i and ψ_j are non-increasing, this would imply $p_a \leq p_b$ that contradicts $p_b \leq U < p_a$. Then, $\rho_{t+1} \leq \rho_t$ for any red-blue coloring of $(\omega_{t+1}^1, \omega_{t+1}^2)$.
- If $U \geq p_a$: then we set $\omega_{t+1}^a = \omega_t^a$ and $\omega_{t+1}^b = \omega_t^b$. Then $\xi(\omega_{t+1}^a) = \xi(\omega_t^a)$, $\xi(\omega_{t+1}^b) = \xi(\omega_t^b)$ and $\rho_{t+1} = \rho_t$.

- If $C^1(x^1) = \text{red}$: then $C^2(x^2) = \text{red}$, and we define $i^1 := \omega_t^1(x^1)$ and $i^2 := \omega_t^2(x^2)$. Note that according to (2.19), $i^1 \neq i^2$. Let then V^1, V^2 be two independent uniform random variables on $[0; 1)$.

- If $V^1 < \exp\{-[\psi_{i^1}(k_{i^1}^1 - 1) - \psi_j(k_j^1)]^+\}$ then we set $\omega_{t+1}^1(x) = \omega_t^1(x)$ for all $x \neq x^1$ and $\omega_{t+1}^1(x^1) = j$ and then $\xi(\omega_{t+1}^1) = (\xi(\omega_t^1))^{ij}$; otherwise we leave $\omega_{t+1}^1 = \omega_t^1$.
- If $V^2 < \exp\{-[\psi_{i^2}(k_{i^2}^2 - 1) - \psi_j(k_j^2)]^+\}$ then we set $\omega_{t+1}^2(x) = \omega_t^2(x)$ for all $x \neq x^2$ and $\omega_{t+1}^2(x^2) = j$ and then $\xi(\omega_{t+1}^2) = (\xi(\omega_t^2))^{ij}$; otherwise we leave $\omega_{t+1}^2 = \omega_t^2$.

Whether red particles move or not, the number of red particles cannot increase, so it is clear that $\rho_{t+1} \leq \rho_t$.

We conclude

Proposition 2.3.2 $(\rho_t)_{t \in \mathbb{N}}$ is a non-increasing process.

and claim

Proposition 2.3.3 The process $(\xi(\omega^1), \xi(\omega^2))$ is a coupling for the Markov chain with transition matrix p .

Proof: Writing $\xi(\omega_t^1) = \eta$ and given $i, j \in \{1; \dots; m\}$,

$$\begin{aligned} P(\xi(\omega_{t+1}^1) = \eta^{ij}) &= P(\xi(\omega_{t+1}^1) = \eta^{ij} | C^1(x^1) = \text{blue})P(C^1(x^1) = \text{blue}) \\ &\quad + P(\xi(\omega_{t+1}^1) = \eta^{ij} | C^1(x^1) = \text{red})P(C^1(x^1) = \text{red}). \end{aligned} \quad (2.29)$$

We have

$$P(C^1(x^1) = \text{red}) = 1 - P(C^1(x^1) = \text{blue}) = \frac{\rho_t}{k} \quad (2.30)$$

and

$$\begin{aligned} P(\xi(\omega_{t+1}^1) = \eta^{ij} | C^1(x^1) = \text{blue}) &= P(\omega_t^1(x^1) = i | C^1(x^1) = \text{blue}) \times \frac{1}{m} \times P(U < p_1) \end{aligned} \quad (2.31)$$

$$= \frac{k_i^1 \wedge k_i^2}{(1 - \rho_t)m} \exp\{-[\psi_i(k_i^1 - 1) - \psi_j(k_j^1)]^+\} \quad (2.32)$$

$$\begin{aligned} P(\xi(\omega_{t+1}^1) = \eta^{ij} | C^1(x^1) = \text{red}) &= P(\omega_t^1(x^1) = i | C^1(x^1) = \text{red}) \times \frac{1}{m} \times P(V^1 < p_1) \end{aligned} \quad (2.33)$$

$$= \frac{[k_i^1 - k_i^2]^+}{\rho_t m} \exp\{-[\psi_i(k_i^1 - 1) - \psi_j(k_j^1)]^+\} \quad (2.34)$$

which finally leads to

$$P(\xi(\omega_{t+1}^1) = \eta^{ij}) = \frac{k_i^1}{km} \exp\{-[\psi_i(k_i^1 - 1) - \psi_j(k_j^1)]^+\} = p(\eta, \eta^{ij}). \quad (2.35)$$

Then, $P(\xi(\omega_{t+1}^1) = \eta^{ij} | \omega_t^1, \omega_t^2)$ depends on $\xi(\omega_t^1)$ only, which means that $\xi(\omega^1)$ is a Markov chain. Besides, according to (2.35) its transition matrix is p . Finally, by Lemma 2.3.1, the same is true for $\xi(\omega_2)$. \square

From now on, we will write $X_t^1 := \xi(\omega_t^1)$ and $X_t^2 := \xi(\omega_t^2)$. The previous proposition ensures that (X^1, X^2) is a coupling for the dynamics with transition matrix p .

2.4 Estimating the coupling time

We will use Proposition 2.2.1. Since $(\rho_t)_{t \in \mathbb{N}}$ is non increasing and $\rho_t = 0$ if and only if $X_t^1 = X_t^2$, all we have to do is to estimate from below the probability of the event $\{\rho_{t+1} < \rho_t\}$.

Proposition 2.4.1 *If at step t of the coupled dynamics (ω^1, ω^2) we assume that red particles have been chosen, i.e., $C^1(x^1) = \text{red} = C^2(x^2)$, then there is a choice of $j \in \{i^1; i^2\}$ for which the number of red particles decreases with probability 1.*

Proof: If the inequalities

$$\exp\{-[\psi_{i^1}(k_{i^1}^1 - 1) - \psi_{i^2}(k_{i^2}^1)]^+\} < 1 \quad (2.36)$$

$$\exp\{-[\psi_{i^2}(k_{i^2}^2 - 1) - \psi_{i^1}(k_{i^1}^2)]^+\} < 1 \quad (2.37)$$

holds together, then $\psi_{i^1}(k_{i^1}^1 - 1) > \psi_{i^2}(k_{i^2}^1)$ and $\psi_{i^2}(k_{i^2}^2 - 1) > \psi_{i^1}(k_{i^1}^2)$. Besides, since $C^1(x^1) = \text{red}$, according to (2.19) $k_{i^1}^1 > k_{i^1}^2$ from which we get $k_{i^1}^1 - 1 \geq k_{i^1}^2$ and, since ψ_{i^1} is non-increasing, $\psi_{i^1}(k_{i^1}^1 - 1) \leq \psi_{i^1}(k_{i^1}^2)$. Likewise, since $C^2(x^2) = \text{red}$ we have $\psi_{i^2}(k_{i^2}^2 - 1) \leq \psi_{i^2}(k_{i^2}^1)$. We finally may write

$$\psi_{i^1}(k_{i^1}^1 - 1) > \psi_{i^2}(k_{i^2}^1) \geq \psi_{i^2}(k_{i^2}^2 - 1) > \psi_{i^1}(k_{i^1}^2) \geq \psi_{i^1}(k_{i^1}^1 - 1) \quad (2.38)$$

which is absurd. As a result, either (2.36) or (2.37) is false. For instance let us assume that (2.36) is false, then if $j = i^2$, with probability 1 we have $\omega_{t+1}^1(x) = \omega_t^1(x)$ for all $x \neq x^1$, $\omega_{t+1}^1(x^1) = i^2 \neq \omega_t^1(x^1)$ and $\omega_{t+1}^2 = \omega_t^2$. Then, the number of red particles for any red-blue coloring of $(\omega_{t+1}^1, \omega_{t+1}^2)$ is exactly $\rho_{t+1} = \rho_t - 1$. \square

It follows

Corollary 2.4.2 *At step t , if $\rho_t > 0$, the probability for ρ_{t+1} to be $\rho_t - 1$ is at least ρ_t/km .*

Consequently, and owing to the fact ρ_t cannot increase, we have the inequality $E(\rho_{t+1} - \rho_t | X_t^1, X_t^2) \leq -P(\rho_{t+1} = \rho_t - 1 | X_t^1, X_t^2)$ from which we deduce

$$E(\rho_{t+1} | X_t^1, X_t^2) \leq \left(1 - \frac{1}{km}\right) \rho_t. \quad (2.39)$$

Then we can apply Proposition 2.2.1 to $\rho(X_t^1, X_t^2) = \rho_t$ with $M = k$ and $\alpha = km$, which finally proves Theorem 1.

3 Proof of Theorem 2

We now work with the dynamics defined by (1.29)-(1.30) and we assume $\delta > 0$. First, the reversibility of this dynamics with respect to ν still holds, with exactly the same computation as in Subsection 2.1. However, it is no longer possible to work with an underlying process $\omega_t \in \Omega$ since the factor k_i^δ cannot stand for a number of particles as soon as $\delta < 1$. Therefore we need to adapt the coupling (X^1, X^2) directly on $\mathcal{X}_{k,m}$.

3.1 Generalizing the previous coupling

At step t , let us assume $X_t^1 = (k_1^1, \dots, k_m^1) \in \mathcal{X}_{k,m}$ and $X_t^2 = (k_1^2, \dots, k_m^2) \in \mathcal{X}_{k,m}$. We define the following sets:

$$R_1 := \{i \in \{1; \dots; m\} : k_i^1 > k_i^2\}, \quad (3.1)$$

$$R_2 := \{i \in \{1; \dots; m\} : k_i^2 > k_i^1\}, \quad (3.2)$$

$$B := \{i \in \{1; \dots; m\} : k_i^1 = k_i^2\}, \quad (3.3)$$

and the following quantities:

$$w_1 := \sum_{i \in R_1} (k_i^1)^\delta, \quad w'_1 := \sum_{i \in R_2} (k_i^1)^\delta, \quad (3.4)$$

$$w_2 := \sum_{i \in R_2} (k_i^2)^\delta, \quad w'_2 := \sum_{i \in R_1} (k_i^2)^\delta, \quad (3.5)$$

$$w_B := \sum_{i \in B} (k_i^1)^\delta = \sum_{i \in B} (k_i^2)^\delta. \quad (3.6)$$

Finally we define

$$\rho_t := \frac{1}{2} \sum_{i=1}^m |k_i^1 - k_i^2| = \sum_{i=1}^m [k_i^1 - k_i^2]^+ = \sum_{i=1}^m [k_i^2 - k_i^1]^+. \quad (3.7)$$

Keeping in mind the previous coloring, R_1 (resp. R_2) is the set of sites in which there are red particles for the first (resp. the second) configuration, B is the set of sites in which there are only blue particles or no particles for both configurations, w_1 (resp. w_2) is proportional to the probability to choose a site for the first (resp. the second) configuration in which there are red particles, w'_1 (resp. w'_2) is proportional to the probability to choose a site for the first (resp. the second) configuration in which there are only blue particles while there are red particles in the second (resp. the first) configuration, w_B is proportional to the probability to choose a site in which there are only blue particles for both configurations, and $l_\delta - (w_B + w_1 + w'_1)$ (resp. $l_\delta - (w_B + w_2 + w'_2)$) is proportional to the probability not to choose any site for the first (resp. the second) configuration, and it can be positive as soon as $\delta < 1$. Finally, ρ_t still stands for the number of red particles and it is clear that $\rho_t = 0$ if and only if $X_t^1 = X_t^2$.

N.B. In the remaining part of this subsection, we assume $w_1 + w'_1 \geq w_2 + w'_2$ in order to not overload the notations and not increase the number of cases to investigate. Obviously, the case $w_1 + w'_1 \leq w_2 + w'_2$ is exactly symmetric.

Let I be a uniform random variable on $[0; l_\delta)$.

(i) If $I < w_B$: then there exists a unique $i \in B$ such that

$$\sum_{i' \in B; i' < i} (k_{i'}^1)^\delta \leq I < \sum_{i' \in B; i' \leq i} (k_{i'}^1)^\delta \quad (3.8)$$

and we set $i^1 = i^2 = i$.

Remark: We then have, for all $i \in \{1; \dots; m\}$,

$$P(i^1 = i | (i)) = \mathbf{1}_{\{i \in B\}} \frac{(k_i^1)^\delta}{w_B}, \quad (3.9)$$

$$P(i^2 = i | (i)) = \mathbf{1}_{\{i \in B\}} \frac{(k_i^1)^\delta}{w_B} = \mathbf{1}_{\{i \in B\}} \frac{(k_i^2)^\delta}{w_B}. \quad (3.10)$$

(ii) If $w_B \leq I < w_B + w'_1$: then there exists a unique $i \in R_2$ such that

$$\sum_{i' \in R_2; i' < i} (k_{i'}^1)^\delta \leq I - w_B < \sum_{i' \in R_2; i' \leq i} (k_{i'}^1)^\delta \quad (3.11)$$

and we set $i^1 = i^2 = i$.

Remark: We then have, for all $i \in \{1; \dots; m\}$,

$$P(i^1 = i | (ii)) = \mathbf{1}_{\{i \in R_2\}} \frac{(k_i^1)^\delta}{w'_1}, \quad (3.12)$$

$$P(i^2 = i | (ii)) = \mathbf{1}_{\{i \in R_2\}} \frac{(k_i^1)^\delta}{w'_1}. \quad (3.13)$$

(iii) If $w_B + w'_1 \leq I < w_B + w'_1 + w_1$: then there exists a unique $i \in R_1$ such that

$$\sum_{i' \in R_1; i' < i} (k_{i'}^1)^\delta \leq I - w_B - w'_1 < \sum_{i' \in R_1; i' \leq i} (k_{i'}^1)^\delta. \quad (3.14)$$

We set $i^1 = i$ and we define $u := I - w_B - w'_1 - \sum_{i' \in R_1; i' < i} (k_{i'}^1)^\delta$, so that $0 \leq u < (k_i^1)^\delta$. Notice that since $i \in R_1$, $k_i^1 > k_i^2$. If $u < (k_i^2)^\delta$ then we set $i^2 = i$. Otherwise, for all $i' \in R_2$ we write $v_{i'} := \sum_{\ell \in R_2; \ell < i'} (k_\ell^1)^\delta$ and we denote by T the disjoint union of intervals

$$T := \left\{ \bigcup_{i' \in R_2} [v_{i'} + (k_{i'}^1)^\delta; v_{i'} + (k_{i'}^2)^\delta) \right\} \cup [w_2 + w'_2; w_1 + w'_1). \quad (3.15)$$

Let I' be a uniform random variable on T . If there exists $i' \in R_2$ such that $I' \in [v_{i'} + (k_{i'}^1)^\delta; v_{i'} + (k_{i'}^2)^\delta)$ then we set $i^2 = i'$. Else we do not define i^2 .

Remark: We then have, for all $i \in \{1; \dots; m\}$,

$$P(i^1 = i | \text{(iii)}) = \mathbf{1}_{\{i \in R_1\}} \frac{(k_i^1)^\delta}{w_1} \quad (3.16)$$

and for all $i' \in \{1; \dots; m\}$,

$$\begin{aligned} P(i^2 = i' | \text{(iii)}) &= \mathbf{1}_{\{i' \in R_1\}} \frac{(k_{i'}^2)^\delta}{w_1} \\ &+ \mathbf{1}_{\{i' \in R_2\}} \sum_{i \in R_1} \frac{(k_i^1)^\delta - (k_i^2)^\delta}{w_1} \times \frac{(k_{i'}^2)^\delta - (k_{i'}^1)^\delta}{\lambda(T)} \end{aligned} \quad (3.17)$$

where

$$\lambda(T) := \sum_{i' \in R_2} ((k_{i'}^2)^\delta - (k_{i'}^1)^\delta) + (w_1 + w'_1) - (w_2 + w'_2) \quad (3.18)$$

$$= w_1 - w'_2 \quad (3.19)$$

so that

$$P(i^2 = i' | \text{(iii)}) = \mathbf{1}_{\{i' \in R_1\}} \frac{(k_{i'}^2)^\delta}{w_1} + \mathbf{1}_{\{i' \in R_2\}} \frac{(k_{i'}^2)^\delta - (k_{i'}^1)^\delta}{w_1}. \quad (3.20)$$

(iv) If $I \geq w_B + w_1 + w'_1$ (this case cannot occur when $\delta = 1$): then we do not define i^1 and i^2 .

Before going ahead with the definition of our coupling we note, as a direct consequence of our remarks in (i), (ii), (iii) and of the fact that I has a uniform distribution:

Proposition 3.1.1 *For all $i \in \{1; \dots; m\}$, $P(i^1 = i | X_t^1, X_t^2) = (k_i^1)^\delta / l_\delta$ and $P(i^2 = i | X_t^1, X_t^2) = (k_i^2)^\delta / l_\delta$.*

We then choose an integer $j \in \{1; \dots; m\}$ with uniform law and we distinguish once again between our four previous cases.

(i) If $i^1 \in B$: then $i^2 = i^1$, we just write $i^1 = i^2 = i$. Then $k_i^1 = k_i^2$. Thus, let $a, b \in \{1; 2\}$ such that $a \neq b$ and $k_j^a \leq k_j^b$. Since both $\psi_i^{[\delta]}$ and $\psi_j^{[\delta]}$ are non-increasing,

$$p_a := \exp\{-[\psi_i^{[\delta]}(k_i^a - 1) - \psi_j^{[\delta]}(k_j^a)]^+\} \quad (3.21)$$

$$\geq \exp\{-[\psi_i^{[\delta]}(k_i^b - 1) - \psi_j^{[\delta]}(k_j^b)]^+\} =: p_b. \quad (3.22)$$

Let U be a uniform random variable on $[0; 1)$.

– If $U < p_b$: we set $X_{t+1}^a = (X_t^a)^{ij}$ and $X_{t+1}^b = (X_t^b)^{ij}$.

- If $p_b \leq U < p_a$: we set $X_{t+1}^a = (X_t^a)^{ij}$ and $X_{t+1}^b = X_t^b$.
- If $p_a \leq U$: we set $X_{t+1}^a = X_t^a$ and $X_{t+1}^b = X_t^b$.

In any of these cases, we certainly have $\rho_{t+1} = \rho_t$.

- (ii) If $i^1 \in R_2$: then $i^2 = i^1$, we just write $i^1 = i^2 = i$. Let $a, b \in \{1; 2\}$ such that $a \neq b$ and $\psi_i^{[\delta]}(k_i^a - 1) - \psi_j^{[\delta]}(k_j^a) \leq \psi_i^{[\delta]}(k_i^b - 1) - \psi_j^{[\delta]}(k_j^b)$, so that

$$p_a := \exp\{-[\psi_i^{[\delta]}(k_i^a - 1) - \psi_j^{[\delta]}(k_j^a)]^+\} \quad (3.23)$$

$$\geq \exp\{-[\psi_i^{[\delta]}(k_i^b - 1) - \psi_j^{[\delta]}(k_j^b)]^+\} =: p_b. \quad (3.24)$$

Let U be a uniform random variable on $[0; 1)$.

- If $U < p_b$: we set $X_{t+1}^a = (X_t^a)^{ij}$ and $X_{t+1}^b = (X_t^b)^{ij}$.
- If $p_b \leq U < p_a$: we set $X_{t+1}^a = (X_t^a)^{ij}$ and $X_{t+1}^b = X_t^b$.
- If $p_a \leq U$: we set $X_{t+1}^a = X_t^a$ and $X_{t+1}^b = X_t^b$.

In the last case we obviously have $\rho_{t+1} = \rho_t$. In the first case the particles move together and $\rho_{t+1} = \rho_t$. In the second case the number of red particles could increase only if $k_i^a \leq k_i^b$ and $k_j^a \geq k_j^b$, but, since $\psi_i^{[\delta]}$ and $\psi_j^{[\delta]}$ are non increasing, this would contradict $p_a > p_b$. As a consequence we have $\rho_{t+1} \leq \rho_t$ in all the three cases.

- (iii) If $i^1 \in R_1$: there are three cases for i^2 . Either $i^2 = i^1 = i$ and this case is the symmetric of (ii). Or i^2 is randomly chosen in R_2 , and we define

$$p_1 := \exp\{-[\psi_{i^1}^{[\delta]}(k_{i^1}^1 - 1) - \psi_j^{[\delta]}(k_j^1)]^+\}, \quad (3.25)$$

$$p_2 := \exp\{-[\psi_{i^2}^{[\delta]}(k_{i^2}^2 - 1) - \psi_j^{[\delta]}(k_j^2)]^+\}. \quad (3.26)$$

Or else i^2 is not defined, and we set $p_2 := 0$ still defining p_1 by (3.25). Let then V^1, V^2 be independent uniform random variables on $[0; 1)$.

- If $V^1 < p_1$: we set $X_{t+1}^1 = (X_t^1)^{i^1 j}$, else we set $X_{t+1}^1 = X_t^1$.
- If $V^2 < p_2$: we set $X_{t+1}^2 = (X_t^2)^{i^2 j}$, else we set $X_{t+1}^2 = X_t^2$.

In the first case we have $\rho_{t+1} \leq \rho_t$ as previously. In the last two cases we also have $\rho_{t+1} \leq \rho_t$ since only particles from R_1 in the first configuration and from R_2 in the second configuration can move.

- (iv) If i_1 and i_2 are not defined: then we simply set $(X_{t+1}^1, X_{t+1}^2) = (X_t^1, X_t^2)$ and we have $\rho_{t+1} = \rho_t$.

In any of the previous cases, once i^1, i^2 and j have been defined, the probability for X_{t+1}^1 (resp. X_{t+1}^2) to be $(X_t^1)^{i^1 j}$ (resp. $(X_t^2)^{i^2 j}$) is $\exp\{-[\psi_{i^1}^{[\delta]}(k_{i^1}^1 - 1) - \psi_j^{[\delta]}(k_j^1)]^+\}$ (resp. $\exp\{-[\psi_{i^2}^{[\delta]}(k_{i^2}^2 - 1) - \psi_j^{[\delta]}(k_j^2)]^+\}$). Thus, according to Proposition 3.1.1, the fact that j is uniformly chosen in $\{1; \dots; m\}$ and our study on the variation of ρ we conclude:

Proposition 3.1.2 *The process (X^1, X^2) is a coupling for the dynamics defined by (1.29)-(1.30) and such that $(\rho_t)_{t \in \mathbb{N}}$ is non-increasing.*

3.2 Estimating the coupling time

Similarly to the proof of Theorem 1 we will use Proposition 2.2.1: since $(\rho_t)_{t \in \mathbb{N}}$ is non-increasing it will be enough to give a lower bound for the probability of $\{\rho_{t+1} < \rho_t\}$.

Proposition 3.2.1 *If at step t of the coupled dynamics (X^1, X^2) , we assume that “red particles have been chosen”, i.e., $i^1 \in R_1$ and $i^2 \in R_2$, then, there is a choice of $j \in \{i^1, i^2\}$ for which the number of red particles decreases with probability 1.*

Proof: Assuming $i^1 \in R_1$ and $i^2 \in R_2$ yields $k_{i^1}^1 > k_{i^1}^2$ and $k_{i^2}^1 < k_{i^2}^2$. Using exactly the same argument as for Proposition 2.4.1 we prove that either $\exp\{-[\psi_{i^1}^{[\delta]}(k_{i^1}^1 - 1) - \psi_{i^2}^{[\delta]}(k_{i^2}^1)]^+\} = 1$ or $\exp\{-[\psi_{i^2}^{[\delta]}(k_{i^2}^2 - 1) - \psi_{i^1}^{[\delta]}(k_{i^1}^2)]^+\} = 1$. Eventually, if one red particle in some configuration moves to a site with a red particle in the other configuration, then both particles turn blue and the number of red particles decreases by one. \square

Corollary 3.2.2 *At step t , the probability for ρ_{t+1} to be $\rho_t - 1$ is at least $\delta k^{\delta-1} \rho_t / m l_\delta$.*

Proof: The probability to choose $i^1 \in R_1$ and $i^2 \in R_2$ is

$$\begin{aligned} P(i^1 \in R_1, i^2 \in R_2) &= \sum_{i' \in R_2} \sum_{i \in R_1} P(i^2 = i' | i^1 = i) \times P(i^1 = i) \end{aligned} \quad (3.27)$$

$$= \sum_{i' \in R_2} \sum_{i \in R_1} \frac{(k_i^1)^\delta - (k_i^2)^\delta}{(k_i^1)^\delta} \frac{(k_{i'}^2)^\delta - (k_{i'}^1)^\delta}{\lambda(T)} \times \frac{(k_i^1)^\delta}{l_\delta} \quad (3.28)$$

$$= \frac{1}{l_\delta} \sum_{i' \in R_2} (k_{i'}^2)^\delta - (k_{i'}^1)^\delta. \quad (3.29)$$

Since, for any concave function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ and any $s \in \mathbb{N} \setminus \{0\}$, $(z_1, \dots, z_s) \in \mathbb{R}_+^s \mapsto f(\sum_i z_i) - \sum_i f(z_i)$ is non-increasing in all its s variables (as a consequence of the slope inequalities), by concavity of $z \mapsto z^\delta$ and using the fact that, for all $i' \in R_2$, $k_{i'}^2 > k_{i'}^1$ we get

$$P(i^1 \in R_1, i^2 \in R_2) \geq \frac{1}{l_\delta} \left[\left(\sum_{i' \in R_2} k_{i'}^2 \right)^\delta - \left(\sum_{i' \in R_2} k_{i'}^1 \right)^\delta \right] \quad (3.30)$$

$$= \frac{1}{l_\delta} \left[\left(\rho_t + \sum_{i' \in R_2} k_{i'}^1 \right)^\delta - \left(\sum_{i' \in R_2} k_{i'}^1 \right)^\delta \right]. \quad (3.31)$$

Using the same property of concave functions on \mathbb{R}^+ (with $s = 2$) and the fact that $\sum_{i' \in R_2} k_{i'}^1 \leq k - \rho_t$, then using once again the concavity of $z \mapsto z^\delta$, we write

$$P(i^1 \in R_1, i^2 \in R_2) \geq \frac{1}{l_\delta} [k^\delta - (k - \rho_t)^\delta] \geq \frac{\delta k^{\delta-1} \rho_t}{l_\delta} \quad (3.32)$$

so that, by the previous proposition,

$$P(\rho_{t+1} = \rho_t - 1 | X_t^1, X_t^2) \geq \frac{\delta k^{\delta-1} \rho_t}{ml_\delta}. \quad (3.33)$$

□

As a consequence

$$E(\rho_{t+1} | X_t^1, X_t^2) \leq \left(1 - \frac{\delta}{km(k \wedge m)^{1-\delta}}\right) \rho_t \quad (3.34)$$

and, by Proposition 2.2.1, this finally proves Theorem 2.

Acknowledgment

J.R. is very grateful to everyone who made his stay in Rome possible and pleasant.

References

- [1] R. PEMANTLE (1997) Towards a theory of negative dependence, *J. Math. Phys.* **41**, 1371–1390.
- [2] T. M. LIGGETT (1997) Ultra logconcave sequences and negative dependence, *J. Combin. Theory Ser. A* **79**, 315–325.
- [3] P. CAPUTO (2004) Spectral gap inequalities in product spaces with conservation laws, in: T. Funaki and H. Osada (eds.) *Adv. Studies in Pure Math.*
- [4] C. LANDIM, J. NORONHA NETO (2005) Poincaré and logarithmic Sobolev inequality for Ginzburg-Landau processes in random environment, *Probab. Theory Related Fields* **131**, 229–260
- [5] A.S. BOUDOU, P. CAPUTO, P. DAI PRA, G. POSTA (2006) Spectral gap inequalities for interacting particle systems via a Bochner type inequality, *J. Funct. Anal.* **232**, 222–258.
- [6] P. CAPUTO, P. DAI PRA, G. POSTA (2007) Convex entropy decay via the Bochner-Bakry-Emery approach, *Annales de l'Institut Henri Poincaré Probab. et Stat.* (to appear), arXiv:0712.2578.

- [7] A. IOVANELLA, B. SCOPPOLA, E. SCOPPOLA (2007) Some spin glass ideas applied to the clique problem *Journal of Statistical Physics*, **126**, 895–915.
- [8] P. CAPUTO (2008) On the spectral gap of the Kac walk and other binary collision processes, *ALEA, Latin American Journal Of Probability And Mathematical Statistics* **4**, 205–222.
- [9] O. JOHNSON (2008) Bounds on the Poincaré constant of ultra log-concave random variables, arXiv:0801.2112v1, preprint.
- [10] D.A. LEVIN, Y. PERES, E.L. WILMER (2008) *Markov Chains and mixing times*, American Mathematical Society.