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A kinetic scheme for transient mixed flows in non uniform closed pipes: a global manner to upwind all the source terms

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Abstract

We present a numerical kinetic scheme for an unsteady mixed pressurised and free surface model. This model has a source term depending on both the space variable and the unknown U of the system. The source term is composed by a topography, a section variation, a curvature (also called corrective) and a friction term. Using the Finite Volume and Kinetic (FVK) framework, we propose an approximation of the source terms following the principle of interfacial upwind with a kinetic interpretation: the source term is not treated as a volumic term, but included in the numerical fluxes. Then, several numerical tests are presented.

Keywords : Finite volume scheme, Kinetic scheme, conservative source terms, non-conservative source terms, friction

1 Introduction

In this paper, we study a way to upwind the source terms of a mixed flows in non uniform closed water pipes, in a one dimensional framework. The pressurised and free surface model is a numerical approximation of a model for unsteady water flows in pipes. In the case of free surface incompressible flows, the model is called **FS**-model and it is an extension of more classical shallow water models. When the pipe is full, we introduce the pressurised model, called **P**-model, which describes the evolution of a compressible inviscid flow and is close to gas dynamics equation in a nozzle. In order to cope with the transition between a free surface and a pressurised model (that is compressible), we use a mixed model called **PFS**-model. The

PFS-model is based on balance laws and provides an hyperbolic system with source terms corresponding to the topography, the section variation and the friction term.

Several ways to compute the numerical approximation of conservation laws with source terms have already been investigated. The main difficulty is to preserve numerically the properties satisfied by the continuous model; the invariant domain, the well-balanced property for instance. The Finite Volume methods [6, 7, 17] are largely used since they present the remarkable property to be domain invariant (for instance for shallow water equations, to be water height conservative). We have also some Well-balanced Finite Volume scheme to preserve steady states initially introduced by Greenberg *et al* [10, 11]. All these methods are based on two principles: firstly, the conservative quantities are cell-centered as usual finite volume schemes, and secondly the source terms are upwinded at the cell interfaces (Roe [16]). The non-conservative source terms [10, 8] are defined by means of non-conservative DLM theory [13], both for hyperbolic scalar and systems of PDE [9].

In this paper, we consider a class of particular Finite Volume-Kinetic schemes to compute the **PFS**-model. These schemes based on the classical kinetic interpretation [15, 12, 1, 14, 2, 5] of the system.

The source terms appearing in the **PFS**-model are either conservative, non-conservative or else. All conservative terms are upwinded at the cell interfaces and, we use the definition of DLM theory to define the non-conservative products. The particular case of the friction term which is neither conservative nor non-conservative will be “upwinded” using the notion of static and dynamic slope. We take into account the source term directly in the definition of the numerical fluxes where the fluxes are computed from the microscopic ones provided by the concept of potential barrier and its kinetic interpretation.

The paper is organized as follows. In the second section, we describe the **PFS**-model and emphasizes our attention on the source terms. The detailed description of the method used (see [4]) to deal with the transition points (when a change of state occurs) is not presented. We state some theoretical properties of the system. In the third section, we give the kinetic formulation of the **PFS**-System with the corresponding kinetic scheme. Finally, several numerical tests are provided in the last section.

2 A model for unsteady water flows in pipes

The **PFS**-model [4] is a mixed model of a pressurised (compressible) and free surface (incompressible) flows in a one dimensional rigid pipes with variable cross-section. The pressurised parts of the flow corresponds to a

full pipe whereas for the free surface flow the section is not completely filled. The **PFS**-System is governed by the mixed hydrostatic and “acoustic” laws (1) and the conservative variables $A = \frac{\rho}{\rho_0} S$ and $Q = Au$ where S is the surface of the cross-section Ω . A and Q represents respectively the *equivalent* wet area and the discharge where $\rho = \rho(t, x)$ is the density of the water, ρ_0 the density of the water at atmospheric pressure p_0 and u the mean speed of the water over Ω .

In order to deal with transition points (that is, when a change of state occurs), we introduce a state variable E which is equal to 1 if the state is pressurised and to 0 if the state is free surface. We define the *physical wet area* by:

$$\mathbf{S} = \mathbf{S}(A, E) = \begin{cases} S & \text{if } E = 1 (\rho \neq \rho_0) \\ A & \text{if } E = 0 (\rho = \rho_0) \end{cases} .$$

The pressure law given by a mixed “hydrostatic” (for the free surface part of the flow) and “acoustic” type (for the pressurised part of the flow) as follows:

$$p(x, A, E) = c^2(A - \mathbf{S}) + gI_1(x, \mathbf{S}) \cos \theta \quad (1)$$

where g is the gravity constant, c the sonic speed of the water and θ the inclination of the pipe. The term I_1 is the hydrostatic pressure:

$$I_1(x, \mathbf{S}) = \int_{-R}^{\mathcal{H}(\mathbf{S})} (\mathcal{H}(\mathbf{S}) - z) \sigma dz$$

where $\sigma(x, z)$ is the width of the cross-section, $R = R(x)$ the radius of the cross-section and $\mathcal{H}(\mathbf{S})$ is the z -coordinate of the free surface over the main axis.

Thus, the defined pressure (1) is continuous throughout transition points and allows us to write the **PFS**-model by:

$$\begin{cases} \partial_t(A) + \partial_x(Q) & = 0 \\ \partial_t(Q) + \partial_x \left(\frac{Q^2}{A} + p(x, A, E) \right) & = -g A Z' + Pr(x, A, E) \\ & -G(x, A, E) \\ & -K(x, A, E) \frac{Q|Q|}{A} \end{cases} \quad (2)$$

where $Z = Z(x)$ is the altitude of the main axis. The terms Pr , G and K denote respectively the pressure source term, a corrective term and the friction term defined as follows:

$$\begin{aligned} Pr(x, A, E) &= c^2 \left(\frac{A}{\mathbf{S}} - 1 \right) S' + g I_2(x, \mathbf{S}) \cos \theta, \\ G(x, A, E) &= g A \frac{Z'}{1}(\mathbf{S}) = g A (\mathcal{H}(\mathbf{S}) - I_1(x, \mathbf{S})/\mathbf{S}) (\cos \theta)', \\ K(x, A, E) &= \frac{1}{K_s^2 R_h(\mathbf{S})^{4/3}} \end{aligned}$$

where we have used the notation f' to denote the derivative with respect to the space variable x of any function $f(x)$. The term I_2 is the hydrostatic pressure source term defined by:

$$I_2(x, \mathbf{S}) = \int_{-R}^{\mathcal{H}(\mathbf{S})} (\mathcal{H}(\mathbf{S}) - z) \partial_x \sigma \, dz .$$

The term $K_s > 0$ is the Strickler coefficient depending on the material and $R_h(\mathbf{S})$ is the hydraulic radius.

Then, System (2) has the following properties:

Theorem 2.1

1. System (2) is strictly hyperbolic on $\{A(t, x) > 0\}$.
2. For smooth solutions, the mean velocity $u = Q/A$ satisfies

$$\begin{aligned} \partial_t u + \partial_x \left(\frac{u^2}{2} + c^2 \ln(A/\mathbf{S}) + g\mathcal{H}(\mathbf{S}) \cos \theta + gZ \right) \\ = -gK(x, A, E)u|u| \leq 0. \end{aligned} \quad (3)$$

3. The still water steady state for $u = 0$ reads:

$$c^2 \ln(A/\mathbf{S}) + g\mathcal{H}(\mathbf{S}) \cos \theta + gZ = 0. \quad (4)$$

4. System (2) admits a mathematical entropy

$$\mathcal{E}(A, Q, E) = \frac{Q^2}{2A} + c^2 A \ln(A/\mathbf{S}) + c^2 S + gA\bar{Z}(x, \mathbf{S}) \cos \theta + gAZ$$

which satisfies the entropy relation for smooth solutions

$$\partial_t \mathcal{E} + \partial_X ((\mathcal{E} + p(x, A, E))U) = -gAK(x, A, E)u^2|u| \leq 0. \quad (5)$$

In what follows, when no confusion is possible, the term $K(x, A, E)$ will be noted simply $K(x, A)$ for free surface and $K(x, S)$ for pressurised states.

3 The Kinetic scheme

The kinetic formulation (3.1) is a microscopic description of the **PFS**-System. The kinetic description of the system is provided by a given density function χ satisfying the following three properties:

$$\chi(\omega) = \chi(-\omega) \geq 0, \quad \int_{\mathbb{R}} \chi(\omega) d\omega = 1, \quad \int_{\mathbb{R}} \omega^2 \chi(\omega) d\omega = 1,$$

which permits to define a maxwellian, also called Gibbs equilibrium, function \mathcal{M} by $\mathcal{M}(\xi, A, E, u) = \frac{A}{c(A, E)} \chi\left(\frac{\xi - u}{c(A, E)}\right)$ where

$$c(A, E) = \begin{cases} \sqrt{g \frac{I_1(x, A)}{A} \cos \theta} & \text{if } E = 0 \\ \sqrt{g \frac{I_1(x, S)}{A} \cos \theta + c^2} & \text{if } E = 1 \end{cases} .$$

The microscopic quantity \mathcal{M} is related to the **PFS**-System by the classical *macro-microscopic* kinetic relations:

$$A = \int_{\mathbb{R}} \mathcal{M}(\xi, A, E, u) d\xi, \quad (6)$$

$$Q = \int_{\mathbb{R}} \xi \mathcal{M}(\xi, A, E, u) d\xi, \quad (7)$$

$$\frac{Q^2}{A} + Ac(A, E)^2 = \int_{\mathbb{R}} \xi^2 \mathcal{M}(\xi, A, E, u) d\xi. \quad (8)$$

As a result by (6-7-8), the non linear **PFS**-System can be viewed as a single linear equation involving a non linear quantity \mathcal{M} , which takes the form:

Theorem 3.1 (Kinetic Formulation of the PFS-model) (A, Q) is a strong solution of System (2) if and only if \mathcal{M} satisfies the kinetic transport equation:

$$\partial_t \mathcal{M} + \xi \cdot \partial_x \mathcal{M} - g\phi \partial_\xi \mathcal{M} = K(t, x, \xi) \quad (9)$$

for some collision term $K(t, x, \xi,)$ which satisfies for (t, x) a.e.

$$\int_{\mathbb{R}} K d\xi = 0, \quad \int_{\mathbb{R}} \xi K d\xi = 0.$$

The source terms are defined as:

$$\phi(x, \mathbf{W}) = \mathbf{B}(x, \mathbf{W}) \cdot \partial_x \mathbf{W} \quad (10)$$

with $\mathbf{W} = (Z + \int_x K(x, A)u|u| dx, S, \cos \theta)$

$$\text{and } \mathbf{B} = \begin{cases} \left(1, -\frac{c^2}{g} \left(\frac{A-S}{AS}\right) - \frac{\gamma(x, S) \cos \theta}{A}, \bar{Z}(x, S)\right) & \text{if } E = 1 \\ \left(1, -\frac{\gamma(x, A) \cos \theta}{A}, \bar{Z}(x, A)\right) & \text{if } E = 0 \end{cases}$$

where $I_2(x, \mathbf{S})$ reads $\gamma(x, \mathbf{S})S'$ for some function γ (depending on the geometry of the pipe).

Remark 3.1 The term $\frac{d}{dx} \left(Z + \int_x K(x, A)u|u| dx \right)$ is called *dynamic slope* since it is time and space variable dependent contrary to the *static slope* Z which only x -dependent.

Denoting by \mathbf{W}_l and \mathbf{W}_r the left and the right states of the cell interface, and using the "straight lines" paths

$$\Psi(s, \mathbf{W}_l, \mathbf{W}_r) = s\mathbf{W}_r + (1-s)\mathbf{W}_l, s \in [0, 1]$$

(see e.g. [9, 13]) we are able to give a sense to the non-conservative products ϕ by:

$$[[\mathbf{W}]] \cdot \int_0^1 \mathbf{B}(s, \Psi(s, \mathbf{W}_l, \mathbf{W}_r)) ds \quad (11)$$

where $[[\mathbf{W}]]$ denotes the jump $\mathbf{W}_r - \mathbf{W}_l$ across the discontinuity localized at the cell interface. As the first component of \mathbf{B} is 1, we recover the classical interfacial upwinding for the topography term (appearing e.g. in shallow water equations) since it is a conservative term.

Based on the kinetic formulation, we construct easily a Finite Volume scheme where the conservative quantities are cell-centered and source terms are included into the numerical fluxes by a standard kinetic scheme with reflections [14]. To this end, let us consider a uniform mesh on \mathbb{R} where cells are denoted for every $i \in \mathbb{Z}$ by $m_i = (x_{i-1/2}, x_{i+1/2})$, with $x_i = \frac{x_{i-1/2} + x_{i+1/2}}{2}$ and $\Delta x = x_{i+1/2} - x_{i-1/2}$ the spacestep. We also consider a time discretisation t^n defined by $t^{n+1} = t^n + \Delta t^n$ with Δt^n the timestep. We consider then $A_i^n, Q_i^n, \mathcal{M}_i^n$ the cell-centered approximation of A, Q and \mathcal{M} on the cell m_i at time t^n .

The source term is given by its piecewise constant representation, that is $\mathbf{W} = \mathbf{W}_i \mathbb{1}_{m_i}(x)$ with the approximation \mathbf{W}_i defined by (16). It follows that the kinetic transport equation (9) simply reads on the cell m_i :

$$\begin{cases} \frac{\partial}{\partial t} f + \xi \cdot \frac{\partial}{\partial x} f = 0 \\ f(t_n, x, \xi) = \mathcal{M}(t_n, x, \xi) \end{cases} \quad (12)$$

and thus it may be discretised as follows:

$$\mathcal{M}_i^{n+1}(\xi) = \mathcal{M}_i^n(\xi) + \frac{\Delta t^n}{\Delta x} \xi (\mathcal{M}_{i+\frac{1}{2}}^-(\xi) - \mathcal{M}_{i-\frac{1}{2}}^+(\xi)) \quad (13)$$

where the contribution of the source term is included into the microscopic numerical fluxes $\mathcal{M}_{i\pm 1/2}^\pm$. This is the principle of interfacial source upwind.

Using the *macro-microscopic* relations (6-7-8) and integrating Equation (13) against ξ and ξ^2 , we obtain the Finite Volume scheme:

$$U_i^{n+1} = U_i^n + \frac{\Delta t^n}{\Delta x} (F_{i+\frac{1}{2}}^- - F_{i-\frac{1}{2}}^+) \quad (14)$$

where the numerical fluxes are computed by :

$$F_{i+\frac{1}{2}}^\pm = \int_{\mathbb{R}} \begin{pmatrix} \xi \\ \xi^2 \end{pmatrix} \mathcal{M}_{i+\frac{1}{2}}^\pm(\xi) d\xi.$$

U denotes the unknown vector (A, Q) . At the cell interfaces which are not transition points between free surface and pressurised states, the microscopic fluxes are given by (see [14, 2, 3] for instance):

$$\begin{aligned} \mathcal{M}_{i+1/2}^-(\xi) &= \overbrace{\mathbb{1}_{\xi>0} \mathcal{M}_i^n(\xi)}^{\text{positive transmission}} + \overbrace{\mathbb{1}_{\xi<0, \xi^2 - 2g\Delta\phi_{i+1/2} < 0} \mathcal{M}_i^n(-\xi)}^{\text{reflection}} \\ &+ \underbrace{\mathbb{1}_{\xi<0, \xi^2 - 2g\Delta\phi_{i+1/2} > 0} \mathcal{M}_{i+1}^n\left(-\sqrt{\xi^2 - 2g\Delta\phi_{i+1/2}}\right)}_{\text{negative transmission}} \\ \mathcal{M}_{i+1/2}^+(\xi) &= \overbrace{\mathbb{1}_{\xi<0} \mathcal{M}_{i+1}^n(\xi)}^{\text{negative transmission}} + \overbrace{\mathbb{1}_{\xi>0, \xi^2 + 2g\Delta\phi_{i+1/2} < 0} \mathcal{M}_{i+1}^n(-\xi)}^{\text{reflection}} \\ &+ \underbrace{\mathbb{1}_{\xi>0, \xi^2 + 2g\Delta\phi_{i+1/2} > 0} \mathcal{M}_i^n\left(\sqrt{\xi^2 + 2g\Delta\phi_{i+1/2}}\right)}_{\text{positive transmission}} \end{aligned} \quad (15)$$

The term $\Delta\phi_{i\pm 1/2}$ in (15) is the upwinded source term (10) and $\xi^2 \pm 2g\Delta\phi_{i+1/2}$ is the jump condition for a particle with the kinetic speed ξ which is necessary to

- be reflected: this means that the particle has not enough kinetic energy $\xi^2/2$ to overpass the potential barrier (reflection in (15)),
- overpass the potential barrier with a positive speed (positive transmission in (15)),
- overpass the potential barrier with a negative speed (negative transmission in (15)).

Using the definition of the non-conservative product ϕ (11), the potential barrier $\Delta\phi_{i\pm 1/2}$ has the following expression:

$$\Delta\phi_{i+1/2} = \begin{cases} \left[\left[Z + \int_x K(x, S)u|u| dx \right] \right]_{i+1/2} \\ - [[S]]_{i+1/2} \int_0^1 \frac{c^2}{g} \left(\frac{\psi_A(s) - \psi_S(s)}{\psi_A(s) \psi_S(s)} \right) ds \\ - [[S]]_{i+1/2} \int_0^1 \frac{\gamma(s, \psi_S(s)) \psi_{\cos \theta}(s)}{\psi_A(s)} ds \\ + [[\cos \theta]]_{i+1/2} \int_0^1 \bar{Z}(s, \psi_S(s)) ds & \text{if } E_i^n = 1 \\ \\ \left[\left[Z + \int_x K(x, A)u|u| dx \right] \right]_{i+1/2} \\ - [[A]]_{i+1/2} \int_0^1 \frac{\gamma(s, \psi_A(s))}{\psi_A(s)} (\psi_{\cos \theta}(s)) ds \\ + [[\cos \theta]]_{i+1/2} \int_0^1 \bar{Z}(s, \psi_A(s)) ds & \text{if } E_i^n = 0 \end{cases} \quad (16)$$

where ψ_K is the straight lines path connecting the left state K_i to the right one K_{i+1} .

The final step of the construction of the kinetic scheme is the choice of the density function χ . The choice of the density χ function which defines the equilibrium function \mathcal{M} can be chosen as the solution of the stationary equation (9): consequently, the scheme obtained belongs to the category of exactly well-balanced scheme. Here, we focus only on the simplest choice of the χ -function [1] which allows to compute easily numerical fluxes, namely:

$$\chi(\omega) = \frac{1}{2\sqrt{3}} \mathbb{1}_{[-\sqrt{3}, \sqrt{3}]}(\omega).$$

We have then:

Theorem 3.2

1. Assuming the CFL condition $\frac{\Delta t^n}{\Delta x} \max_{i \in \mathbb{Z}} (|u_i^n| + \sqrt{3}c) < 1$, the numerical scheme (14) keeps A positive.

2. The kinetic scheme allows to compute the drying and flooding area.

4 Numerical results

The numerical validation in the case of a pressurised flow has been previously studied by the authors in [3] (and thus is not presented in this paper). Since experimental data for mixed flows in non uniform pipe are not available, we compare the result obtained by the presented kinetic scheme with an upwinded VFRoe method [4]. The first numerical test is a validation test. The numerical experiments are performed in the case of an expanding 5 m long closed circular pipe at altitude $Z_0 = 1$ m with 0 slope (slope of the main pipe axis). The upstream diameter is 2 and the downstream diameter is 2.2. The friction is not considered for the first test and is set to 0. The simulation starts from a steady free surface with a discharge $Q = 0$ m³/s. The upstream boundary condition is a prescribed hydrograph (increasing linearly from 1 m to 3.2 m in 5 s) while the downstream discharge is kept constant to 0 m³/s.

Let us recall that the zero water level corresponds to the main pipe axis. The piezometric head is defined by:

$$piezo = z + p \quad \text{with} \quad \begin{cases} p = 2R + \frac{c^2(A - S)}{gS} & \text{if the flow is pressurised} \\ p = h & \text{if the flow is free surface,} \end{cases}$$

where h is the water height.

The results are compared with the VFRoe scheme [4]. In spite of missing experimental data, the results on FIG. 1 are in a good agreement.

A second test case is performed in order to compare the effect of the presented kinetic scheme with upwinded friction and the explicit kinetic scheme with cell-centered friction (treated as a source term) in a symmetric flow for different values of K_s . The numerical experiment is performed on a 100 m long closed pipe with constant section of diameter 2. The simulation starts from a double dam break, that is, one from the upstream and the other from the downstream end (see the curve at time $t = 0$ FIG. 3-2). The upstream and downstream condition are identical: the piezometric line increases linearly from 1 to 2.1 meters. From a physical point of view, we expect a symmetrical flow. The results in FIG. 3-2 show that the explicit cell-centered friction scheme does not preserve the symmetry of the flow; in particular at time $t = 8.047$ (see FIG. 3 on top) we observe a small disymmetry for $K_s = 10$ which evolves drastically at time $t = 3.492$ with $K_s = 100$ (see FIG. 2 on top). We observe that decreasing the value of the Strickler coefficient K_s has catastrophic consequences on the results (see

FIG. 2 on top) of the explicit cell-centered friction whereas the scheme with upwinded friction perfectly preserves this property (see FIG. 2 on bottom).

5 Conclusion

We have presented a method to upwind the dynamic source term depending on unknowns following the principle of interfacial upwind using a FVK framework. We have also proposed a way to upwind the friction given by the Manning-Strickler law by the notion of *dynamic* slope. The numerical results show that the approximation of the source term provide an efficient first order scheme for industrial application.

As a well-known result, the upwinding of source terms for approximate Godunov schemes, introduce a stationnary wave with a vanishing denominator when critical flows occurs. We also cannot deal with drying and flooding with this kind of scheme. All these drawbacks are avoided with the presented scheme. Actually, the friction term is manually set to 0 when drying or flooding cells appears.

Although the numerical scheme is not based on an exactly well-balanced scheme, it seems to be numerically valid.

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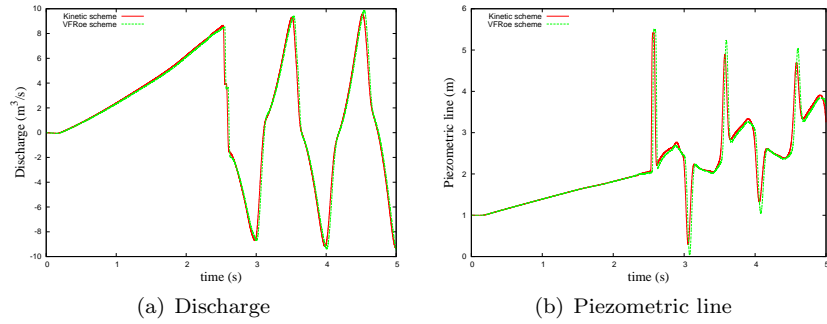


Figure 1: Comparison between the kinetic scheme and the upwinded VFRoe scheme at $x = 0.5$ m.

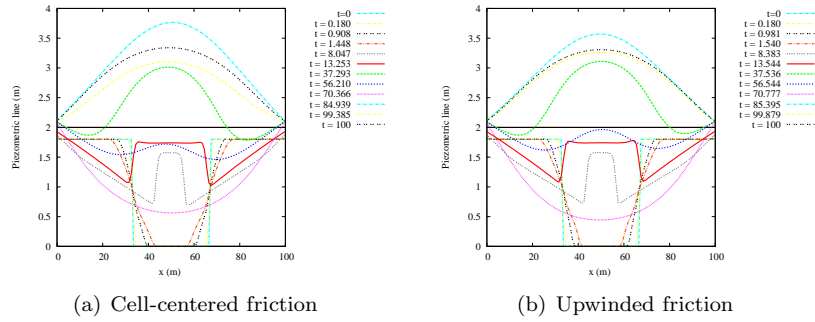


Figure 2: Comparison of the cell-centered friction and upwinded friction for $K_s = 100$.

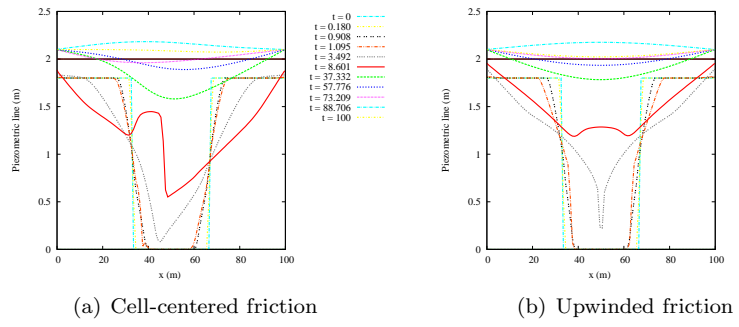


Figure 3: Comparison of the cell-centered friction and upwinded friction for $K_s = 10$.