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Abstract. In this paper, we study the Gevrey class regularity for solutions of the spatially homogeneous Landau equations in the hard potential case and the Maxwellian molecules case.

1. Introduction

In this paper we study the smoothness effects of solutions for the following Cauchy problem of the spatially homogeneous Landau equation

$$\begin{cases} \partial_t f = \nabla_v \cdot \{ \int_{\mathbb{R}^3} a(v-v_*)[f(v_*)\nabla_v f(v) - f(v)\nabla_v f(v_*)]dv_* \}, \\ f(0,v) = f_0(v), \end{cases} \tag{1}$$

where $f(t,v) \geq 0$ stands for the density of particles with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$, and (a_{ij}) is a nonnegative symmetric matrix given by

$$a_{ij}(v) = \left(\delta_{ij} - \frac{v_i v_j}{|v|^2}\right)|v|^{\gamma+2}. \tag{2}$$

We only consider here the condition $\gamma \in [0,1]$. It's called the hard potential case when $\gamma \in]0,1]$ and the Maxwellian molecules case when $\gamma = 0$.

Set $c = \sum_{i,j=1}^3 \partial_{v_i v_j} a_{ij} = -2(\gamma+3)|v|^\gamma$ and

$$\bar{a}_{ij}(t,v) = \Big(a_{ij} * f\Big)(t,v) = \int_{\mathbb{R}^3} a_{ij}(v-v_*) f(t,v_*) dv_*, \quad \bar{c} = c * f.$$

Then the Cauchy problem (1) can be rewritten as the following form,

$$\begin{cases} \partial_t f = \sum_{i,j=1}^3 \bar{a}_{ij} \partial_{v_i v_j} f - \bar{c} f, \\ f(0,v) = f_0(v). \end{cases} \tag{3}$$

This is a non-linear diffusion equation, and the coefficients $\bar{a}_{ij}, \bar{c}$ depend on the solution f.

Here we are mainly concerned with the Gevrey class regularity for the solution of the Landau equation. This equation is obtained as a limit of the Boltzmann equation when the collisions become grazing (see [8] and references therein). Recently, a lot of progress has been made on the study of the Sobolev regularizing property, cf. [6, 11, 13, 18, 19] and references therein, which shows that in some sense the Landau equation can be regarded as a non-linear and non-local analog of the hypo-elliptic Fokker-Planck equation. That means the weak solution, which constructed under rather weak hypothesis on the initial datum, will become smooth or, even more, rapidly decreasing in v at infinity. This behavior is quite similar to that of the spatially homogeneous Boltzmann equation without cut-off (see [2, 10] for more details).

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In the Gevrey class frame, some results have been obtained concerning the propagation property for solutions of the Boltzmann equation, e.g. the solutions having the Gevrey regularity have been constructed in [17] for initial data having the same Gevrey regularity, and the uniform propagation in all time of the Gevrey regularity has been proved in [9] in the case of Maxwellian molecules. Recently a general Gevrey regularity result have been given in [15] for spatially homogeneous and linear Boltzmann equation for any initial data. On the other hand, the local Gevrey regularity for all variables t, x, v is obtained in [4] for some semi-linear Fokker-Planck equations, which implies that, in this case, there are also the smoothness effects which is similar to the heat equation case.

Now we give some notations used throughout the paper. For a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we denote $|\alpha| = \alpha_1+\alpha_2+\alpha_3$, $\alpha! = \alpha_1!\alpha_2!\alpha_3!$ and $\partial^\alpha = \partial_{v_1}^{\alpha_1}\partial_{v_2}^{\alpha_2}\partial_{v_3}^{\alpha_3}$. We say $\beta = (\beta_1,\beta_2,\beta_3) \le (\alpha_1, \alpha_2, \alpha_3) = \alpha$ if $\beta_i \le \alpha_i$ for each i. Denote by $M(f(t))$, $E(f(t))$ and $H(f(t))$ respectively the mass, energy and entropy of the function $f(t)$, i.e.,

$$M(f(t)) = \int_{\mathbb{R}^3} f(t,v)\,dv, \quad E(f(t)) = \frac{1}{2}\int_{\mathbb{R}^3} f(t,v)\,|v|^2\,dv,$$

$$H(f(t)) = \int_{\mathbb{R}^3} f(t,v)\log f(t,v)\,dv.$$

and denote $M_0 = M(f(0))$, $E_0 = E(f(0))$ and $H_0 = H(f(0))$. We know that the solution of the Landau equation satisfies the formal conservation laws:

$$M(f(t)) = M_0, \quad E(f(t)) = E_0, \quad H(f(t)) \le H_0, \qquad \forall\, t \ge 0.$$

Also we adopt the following notations,

$$\|\partial^\alpha f(t,\cdot)\|_{L^p_s}^p = \|\partial^\alpha f(t)\|_{L^p_s}^p = \int_{\mathbb{R}^3} \left|\partial^\alpha f(t,v)\right|^p \left(1+|v|^2\right)^{s/2} dv, \text{ for } p \ge 1,$$

$$\|f(t,\cdot)\|_{H^m_s}^2 = \|f(t)\|_{H^m_s}^2 = \sum_{|\alpha|\le m} \|\partial^\alpha f(t,\cdot)\|_{L^2_s}^2.$$

Before stating our main result, let us recall the definition of the Gevrey class function space $G^\sigma(\mathbb{R}^N)$, where $\sigma \ge 1$ is the Gevrey index (cf. [5, 16]). Let u be a real function defined in $\mathbb{R}^N$. We say $u \in G^\sigma(\mathbb{R}^N)$ if $u \in C^\infty(\mathbb{R}^N)$ and there exists a positive constant C such that for all multi-indices $\alpha \in \mathbb{N}^N$, we have

$$\|\partial^\alpha u\|_{L^2(\mathbb{R}^3)} \le C^{|\alpha|+1}(|\alpha|!)^\sigma.$$

We denote by $G_0^\sigma(\mathbb{R}^3)$ the space of Gevrey functions with compact support. Note that $G^1(\mathbb{R}^N)$ is the space of real analytic functions.

In the hard potential case, the existence, uniqueness and Sobolev regularity of the weak solution had been studied by Desvillettes and Villani (see [11], Theorem 5, Theorem 6 and Theorem 7). Actually they proved that, under rather weak assumptions on the initial datum (for instance if $f_0 \in L^1_{2+\delta}$ with $\delta > 0$), there exists a weak solution f of the Cauchy problem (3) such that for all time $t_0 > 0$, all integer $m \ge 0$, and all $s > 0$,

$$\sup_{t\ge t_0} \|f(t,\cdot)\|_{H^m_s} \le C,$$

where C is a constant depending only on γ, M_0, E_0, H_0, m, s and t_0. Furthermore they proved that $f(t,v) \in C^\infty(\mathbb{R}^+_t; \mathcal{S}(\mathbb{R}^3_v))$, where $\mathbb{R}^+_t =]0,+\infty[$ and $\mathcal{S}(\mathbb{R}^3_v)$ denotes the space of smooth functions rapidly decreasing at infinity. If $f_0 \in L^2_p$ with $p > 5\gamma + 15$, then the Cauchy problem admits a unique smooth solution .

In the Maxwellian case, Villani [19] proved that the Cauchy problem admits a unique classical solution f for any initial datum and for all $t > 0$, and that f is bounded and lies in $C^\infty(\mathbb{R}^3_v)$.

Now starting from the smooth solution, we state our main result on the Gevrey regularity as follows.

Theorem 1.1. *Let f_0 be the initial datum with finite mass, energy and entropy and f be any solution of the Cauchy problem (3) such that for all t_0, t_1 with $0 < t_0 < t_1 < +\infty$, and all integer $m \geq 0$,*

$$\sup_{t\in[t_0,t_1]} \|f(t,\cdot)\|_{H^m_\gamma} < +\infty. \tag{4}$$

Then for any number $\sigma > 1$, we have $f(t,\cdot) \in G^\sigma(\mathbb{R}^3)$ for all time $t > 0$.

Remark 1.2. *Our result, which is given here for the space dimension to be equal to* 3*, will also be true for any space dimension.*

This paper includes three sections. The proof of the main result Theorem 1.1 will be given in Section 2, and in Section 3 we shall mainly prove Lemma 2.2, which is crucial in the proof of Section 2.

2. Proof of the main results

This section is devoted to the proof of the main result. In the sequel, we always use $\sum_{1\leq|\beta|\leq|\mu|}$ to denotes the summation over all the multi-indices β satisfying $\beta \leq \mu$ and $1 \leq |\beta| \leq |\mu|$. Likewise $\sum_{1\leq|\beta|\leq|\mu|-1}$ denotes the summation over all the multi-indices β satisfying $\beta \leq \mu$ and $1 \leq |\beta| \leq |\mu| - 1$. Firstly we have

Lemma 2.1. *For any $\sigma > 1$, there exists a constant C_σ, depending only on σ, such that for all multi-indices $\mu \in \mathbb{N}^3, |\mu| \geq 1$,*

$$\sum_{1\leq|\beta|\leq|\mu|} \frac{1}{|\beta|^3} \leq C_\sigma\, |\mu|^{\sigma-1}\,, \tag{5}$$

and

$$\sum_{1\leq|\beta|\leq|\mu|-1} \frac{1}{|\beta|^2\,(|\mu|-|\beta|)} \leq C_\sigma\, |\mu|^{\sigma-1}\,. \tag{6}$$

Proof For each positive integer l, we denote by $N\{|\beta| = l\}$ the number of the multi-indices β with $|\beta| = l$. In the case when the space dimension equals to 3, one has

$$N\{|\beta| = l\} = \frac{(l+2)!}{2!\,l!} = \frac{1}{2}(l+1)(l+2).$$

It's easy to see that

$$\sum_{1\leq|\beta|\leq|\mu|} \frac{1}{|\beta|^3} \leq \sum_{l=1}^{|\mu|} \sum_{|\beta|=l} \frac{1}{l^3} = \sum_{l=1}^{|\mu|} \frac{N\{|\beta| = l\}}{l^3}$$

We combine these estimates to compute

$$\sum_{1\leq|\beta|\leq|\mu|} \frac{1}{|\beta|^3} \leq \frac{1}{2} \sum_{l=1}^{|\mu|} \frac{(l+1)(l+2)}{l^3} \leq 3 \sum_{l=1}^{|\mu|} \frac{1}{l}.$$

Together with the fact that $3\sum_{l=1}^{|\mu|} l^{-1} \le C_\sigma |\mu|^{\sigma-1}$ for some constant C_σ, this gives the estimate (5). In a similar way as above we can prove the estimate (6). The proof is completed.

The following lemma, which will be proved in the next section, is of great of use to us. For simplicity, in the following discussions, we shall use the notation $\gamma - j$, with γ a multi-index and j an integer, to denotes some multi-index $\tilde{\gamma}$ satisfying $\tilde{\gamma} \le \gamma$ and $|\tilde{\gamma}| = |\mu| - j$.

Lemma 2.2. *Let $\sigma > 1$. There exist constants $B, C_1, C_2 > 0$ with B depending only on the Gevrey index σ and C_1, C_2 depending only on M_0, E_0, H_0, σ and γ, such that for all multi-indices $\mu \in \mathbb{N}^3$ with $|\mu| \ge 2$ and all $t > 0$, we have*

$$\begin{aligned}\frac{d}{dt} \quad \| \quad & \|\partial^\mu f(t)\|_{L^2}^2 + C_1 \|\nabla_v \partial^\mu f(t)\|_{L^2_\gamma}^2 \le C_2 |\mu|^2 \|\nabla_v \partial^{\mu-1} f(t)\|_{L^2_\gamma}^2 \\ + \quad & C_2 \sum_{2\le|\beta|\le|\mu|} C_\mu^\beta \|\nabla_v \partial^{\mu-\beta+1} f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\mu-1} f(t)\|_{L^2_\gamma} \cdot [G_\sigma(f(t))]_{\beta-2} \\ + \quad & C_2 \sum_{0\le|\beta|\le|\mu|} C_\mu^\beta \|\partial^\beta f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\mu-1} f(t)\|_{L^2_\gamma} \cdot [G_\sigma(f(t))]_{\mu-\beta},\end{aligned}$$

where $C_\mu^\beta = \frac{\mu!}{(\mu-\beta)!\beta!}$ is the binomial coefficients, and

$$[G_\sigma(f(t))]_{\beta-2} = \|\partial^{\beta-2} f(t)\|_{L^2} + B^{|\beta|-2} ((|\beta|-2)!)^\sigma .$$

From now on, Ω will be used to denote an arbitrary fixed interval $[T_0, T_1]$ with $0 < T_0 < T_1 < T_0 + 1$. We denote

$$\Omega_\rho = [T_0 + \rho, T_1 - \rho], \quad 0 < \rho < \frac{T_1 - T_0}{2} < \frac{1}{2}.$$

For any integer k with $k \ge 2$ and any ρ with $0 < \rho < (T_1 - T_0)/2$, take a function $\varphi_{\rho,k}(t) \in C_0^\infty(\mathbb{R})$ satisfying $0 \le \varphi_{\rho,k} \le 1$, and Supp $\varphi_{\rho,k} \subset \Omega_{\tilde{\rho}}$ with $\tilde{\rho} = \frac{k-1}{k}\rho$, and $\varphi_{\rho,k} = 1$ in Ω_ρ. it is easy to verify

$$\sup \left| \frac{d^j \varphi_{\rho,k}}{dt^j} \right| \le \tilde{C}_j (k/\rho)^j, \quad \forall\ j \in \mathbb{N}. \tag{7}$$

And for $\tilde{\rho} = (k-1)\rho/k,\ k \ge 2$, the following simple fact is clear,

$$\frac{1}{\rho^k} \le \frac{1}{\tilde{\rho}^k} = \frac{1}{\rho^k} \times \left(\frac{k}{k-1}\right)^k \le \frac{5}{\rho^k}. \tag{8}$$

Now we are prepare to prove the main results, which can be deduced easily from the following

Proposition 2.3. *Let f_0 be the initial datum with finite mass, energy and entropy and f be any solution of the Cauchy problem satisfying (4). Then for $\sigma > 1$, there exists a constant A, depending only on $T_0, T_1, M_0, E_0, H_0, \gamma$ and σ, such that for any $k \in \mathbb{N}$, $k \ge 0$,*

$$(Q)_k \quad \sup_{t\in\Omega_\rho} \|\partial^\alpha f(t)\|_{L^2} + \left\{ \int_{T_0+\rho}^{T_1-\rho} \|\nabla_v \partial^{\tilde{\alpha}} f(t)\|_{L^2_\gamma}^2 \, dt \right\}^{1/2} \le \frac{A^k}{\rho^k} [(k-1)!]^\sigma$$

holds for any multi-indices $\alpha, \tilde{\alpha}$ with $|\alpha| = |\tilde{\alpha}| = k$ and all ρ with $0 < \rho < (T_1 - T_0)/2$. Here we assume $(-1)! = 0! = 1$.

Proof We use induction on k. $(Q)_0, (Q)_1$ obviously hold if we take A large enough such that

$$A \ge \sup_{s\in[T_0,T_1]} \|f(s)\|_{H^2_\gamma} + 1 \tag{9}$$

The term on the right hand of (9) is finite by virtue of (4). Now assuming $(Q)_{k-1}$ holds, we shall show the truth of $(Q)_k, k \geq 2$. In this proof C_j, $j \geq 3$, are used to denote different constants depending only on $T_0, T_1, M_0, E_0, H_0, \gamma$ and σ.

Firstly we shall prove

$$\sup_{t\in\Omega_\rho} \|\partial^\alpha f(t)\|_{L^2} \leq \frac{1}{2}\frac{A^{|\alpha|}}{\rho^{|\alpha|}}[(|\alpha|-1)!]^\sigma, \quad \forall\ |\alpha| = k, \quad \forall\ 0 < \rho < \frac{T_1 - T_2}{2}. \tag{10}$$

In the following discussion, let α be any multi-index with $|\alpha| = k$ and ρ be any number with $0 < \rho < (T_1 - T_2)/2$. Applying Lemma 2.2 with $\mu = \alpha$, we obtain

$$\begin{aligned}
\frac{d}{dt} \quad & \|\partial^\alpha f(t)\|_{L^2}^2 + C_1\|\nabla_v \partial^\alpha f(t)\|_{L^2_\gamma}^2 \leq C_2\,|\alpha|^2\,\|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma}^2 \\
+ \quad & C_2 \sum_{2\leq|\beta|\leq|\alpha|} C_\alpha^\beta \|\nabla_v \partial^{\alpha-\beta+1} f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma} \cdot [G_\sigma(f(t))]_{\beta-2} \\
+ \quad & C_2 \sum_{0\leq|\beta|\leq|\alpha|} C_\alpha^\beta \|\partial^\beta f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma} \cdot [G_\sigma(f(t))]_{\alpha-\beta}.
\end{aligned}$$

Write the last term of the right side as

$$\begin{aligned}
& C_2\|f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma} \cdot [G_\sigma(f(t))]_\alpha \\
& + C_2 \textstyle\sum_{1\leq|\beta|\leq|\alpha|} C_\alpha^\beta \|\partial^\beta f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma} \cdot [G_\sigma(f(t))]_{\alpha-\beta}.
\end{aligned}$$

And then we get

$$\begin{aligned}
\frac{d}{dt} \quad & \|\partial^\alpha f(t)\|_{L^2}^2 + C_1\|\nabla_v \partial^\alpha f(t)\|_{L^2_\gamma}^2 \leq C_2\,|\alpha|^2\,\|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma}^2 \\
+ \quad & C_2\|f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma} \cdot [G_\sigma(f(t))]_\alpha \\
+ \quad & C_2 \sum_{2\leq|\beta|\leq|\alpha|} C_\alpha^\beta \|\nabla_v \partial^{\alpha-\beta+1} f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma} \cdot [G_\sigma(f(t))]_{\beta-2} \\
+ \quad & C_2 \sum_{1\leq|\beta|\leq|\alpha|} C_\alpha^\beta \|\partial^\beta f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma} \cdot [G_\sigma(f(t))]_{\alpha-\beta}.
\end{aligned}$$

Multiplying by $\varphi_{\rho,k}(t)$ the both sides of the above inequality, one has

$$\begin{aligned}
\frac{d}{dt} \quad & [\varphi_{\rho,k}(t)\|\partial^\alpha f(t)\|_{L^2}^2] + C_1\varphi_{\rho,k}(t)\|\nabla_v \partial^\alpha f(t)\|_{L^2_\gamma}^2 \\
\leq \quad & \frac{d\varphi_{\rho,k}}{dt} \cdot \|\partial^\alpha f(t)\|_{L^2}^2 + C_2 \cdot \varphi_{\rho,k}(t)\,|\alpha|^2\,\|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma}^2 \\
+ \quad & C_2 \cdot \varphi_{\rho,k}(t)\,\|f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma} \cdot [G_\sigma(f(t))]_\alpha \\
+ \quad & C_2 \cdot \varphi_{\rho,k}(t) \sum_{2\leq|\beta|\leq|\alpha|} C_\alpha^\beta \|\nabla_v \partial^{\alpha-\beta+1} f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma} \cdot [G_\sigma(f(t))]_{\beta-2} \\
+ \quad & C_2 \cdot \varphi_{\rho,k}(t) \sum_{1\leq|\beta|\leq|\alpha|} C_\alpha^\beta \|\partial^\beta f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma} \cdot [G_\sigma(f(t))]_{\alpha-\beta}.
\end{aligned}$$

To simplify the notation, we set

$$[G_\sigma(f)]_{\rho,\beta} = \sup_{s\in\Omega_\rho} [G_\sigma(f(s))]_\beta = \sup_{s\in\Omega_\rho} \|\partial^\beta f(s)\|_{L^2} + B^{|\beta|}(|\beta|!)^\sigma.$$

Recall Supp $\varphi_{\rho,k} \subset \Omega_{\tilde\rho}$ with $\tilde\rho = (k-1)\rho/k$ and $\varphi_{\rho,k}(t) = 1$ for all $t \in \Omega_\rho$ and $\varphi_{\rho,k}(T_0) = 0$. Then for any $t \in \Omega_\rho$, we integrate the above inequality over the interval $[T_0, t] \subset [T_0, T_1-\rho]$

and then use Cauchy inequality to get

$$
\begin{aligned}
\| \quad & \|\partial^\alpha f(t)\|_{L^2}^2 = \varphi_{\rho,k}(t)\|\partial^\alpha f(t)\|_{L^2}^2 - \varphi_{\rho,k}(T_0)\,\|\partial^\alpha f(T_0)\|_{L^2}^2 \\
\le \quad & \sup\left|\frac{d\varphi_{\rho,k}}{dt}\right| \int_{T_0+\tilde\rho}^{T_1-\tilde\rho} \|\partial^\alpha f(s)\|_{L^2}^2 ds + C_2\,|\alpha|^2 \int_{T_0+\tilde\rho}^{T_1-\tilde\rho} \|\nabla_v \partial^{\alpha-1} f(s)\|_{L^2_\gamma}^2 ds \\
+ \quad & C_2 \sup_{s\in\Omega_{\tilde\rho}} \|f(s)\|_{L^2_\gamma} \cdot \left\{\int_{T_0+\tilde\rho}^{T_1-\tilde\rho} [G_\sigma(f(s))]_\alpha^2 ds\right\}^{\frac12} \left\{\int_{T_0+\tilde\rho}^{T_1-\tilde\rho} \|\nabla_v \partial^{\alpha-1} f(s)\|_{L^2_\gamma}^2 ds\right\}^{\frac12} \\
+ \quad & C_2 \sum_{2\le|\beta|\le|\alpha|} C_\alpha^\beta [G_\sigma(f)]_{\tilde\rho,\,\beta-2} \left\{\int_{T_0+\tilde\rho}^{T_1-\tilde\rho} \|\nabla_v \partial^{\alpha-\beta+1} f(s)\|_{L^2_\gamma}^2 ds\right\}^{\frac12} \\
\times \quad & \left\{\int_{T_0+\tilde\rho}^{T_1-\tilde\rho} \|\nabla_v \partial^{\alpha-1} f(s)\|_{L^2_\gamma}^2 ds\right\}^{\frac12} \\
+ \quad & C_2 \sum_{1\le|\beta|\le|\alpha|} C_\alpha^\beta [G_\sigma(f)]_{\tilde\rho,\alpha-\beta} \left\{\int_{T_0+\tilde\rho}^{T_1-\tilde\rho} \|\partial^\beta f(s)\|_{L^2_\gamma}^2 ds\right\}^{\frac12} \left\{\int_{T_0+\tilde\rho}^{T_1-\tilde\rho} \|\nabla_v \partial^{\alpha-1} f(s)\|_{L^2_\gamma}^2 ds\right\}^{\frac12} \\
= \quad & (S_1)+(S_2)+(S_3)+(S_4)+(S_5).
\end{aligned}
$$

In order to treat the above five terms, we need the following estimates which are deduced directly from the the induction hypothesis. By the truth of $(Q)_{k-1}$, we have

$$
\left\{\int_{T_0+\tilde\rho}^{T_1-\tilde\rho} \|\nabla_v \partial^{\alpha-1} f(s)\|_{L^2_\gamma}^2 ds\right\}^{1/2} \le \frac{A^{|\alpha|-1}}{\tilde\rho^{|\alpha|-1}} [(|\alpha|-2)!]^\sigma; \tag{11}
$$

$$
\left\{\int_{T_0+\tilde\rho}^{T_1-\tilde\rho} \|\nabla_v \partial^{\alpha-\beta+1} f(s)\|_{L^2_\gamma}^2 ds\right\}^{\frac12} \le \frac{A^{|\alpha|-|\beta|+1}}{\tilde\rho^{|\alpha|-|\beta|+1}} [(|\alpha|-|\beta|)!]^\sigma,\ 2\le|\beta|\le|\alpha|; \tag{12}
$$

and

$$
\left\{\int_{T_0+\tilde\rho}^{T_1-\tilde\rho} \|\partial^\beta f(s)\|_{L^2_\gamma}^2 ds\right\}^{1/2} \le \frac{A^{|\beta|-1}}{\tilde\rho^{|\beta|-1}} [(|\beta|-2)!]^\sigma, \quad 2\le|\beta|\le|\alpha|. \tag{13}
$$

the last inequality using the fact $\|\partial^\beta f\|_{L^2_\gamma} \le \|\nabla_v \partial^{\beta-1} f\|_{L^2_\gamma}$ for any β with $2 \le |\beta| \le \alpha$. Observe that there exists a constant $\tilde B > 1$, depending only on B and σ, such that

$$
\begin{cases}
B^m (m!)^\sigma \le \tilde B^m ((m-1)!)^\sigma, & 1 \le m \le |\alpha|-1, \\
B^{|\alpha|} (|\alpha|!)^\sigma \le \tilde B^{|\alpha|-1} ((|\alpha|-2)!)^\sigma.
\end{cases}
$$

With (13), one has, by taking A such that $A \ge \tilde B$,

$$
\left\{\int_{T_0+\tilde\rho}^{T_1-\tilde\rho} [G_\sigma(f(s))]_\alpha^2 ds\right\}^{1/2} \le \frac{2A^{|\alpha|-1}}{\tilde\rho^{|\alpha|-1}} [(|\alpha|-2)!]^\sigma. \tag{14}
$$

Next we shall treat the term $[G_\sigma(f)]_{\tilde\rho,\lambda}$ which equals to $\sup_{s\in\Omega_{\tilde\rho}} \|\partial^\lambda f(s)\|_{L^2} + B^{|\lambda|}(|\lambda|!)^\sigma$ by definition. It follows from the truth of $(Q)_{k-1}$ again that

$$
\sup_{t\in\Omega_{\tilde\rho}} \|\partial^\lambda f(t)\|_{L^2} \le \frac{A^{|\lambda|}}{\tilde\rho^{|\lambda|}} ((|\lambda|-1)!)^\sigma, \quad \forall \lambda,\ 1\le|\lambda|\le k-1,
$$

from which and the fact $A \ge \tilde B$, we get the estimate on $[G_\sigma(f)]_{\tilde\rho,\lambda}$, that is,

$$
[G_\sigma(f)]_{\tilde\rho,\lambda} \le \frac{2A^{|\lambda|}}{\tilde\rho^{|\lambda|}} ((|\lambda|-1)!)^\sigma, \quad 1\le|\lambda|\le|\alpha|-1. \tag{15}
$$

Now the above estimates allow us to deal with the terms from (S_1) to (S_5). Note that $\|\partial^\alpha f(s)\|_{L^2} \le \|\partial^\alpha f(s)\|_{L^2_\gamma}$ and hence from (7) and (13), it follows immediately that

$$(S_1) \le \frac{C_3 k}{\rho}\left\{\frac{A^{|\alpha|-1}}{\tilde\rho^{|\alpha|-1}}[(|\alpha|-2)!]^\sigma\right\}^2.$$

This along with the facts $k = |\alpha|$ and $\rho^{-1} < \tilde\rho^{-1} < \tilde\rho^{-2}$ shows at once

$$(S_1) \le C_4\left\{\frac{A^{|\alpha|-1}}{\tilde\rho^{|\alpha|}}[(|\alpha|-1)!]^\sigma\right\}^2. \tag{16}$$

And by virtue of (11), we obtain

$$(S_2) \quad \le C_2\,|\alpha|^2\left\{\tfrac{A^{|\alpha|-1}}{\tilde\rho^{|\alpha|-1}}[(|\alpha|-2)!]^\sigma\right\}^2 \le C_5\left\{\tfrac{A^{|\alpha|-1}}{\tilde\rho^{|\alpha|-1}}[(|\alpha|-1)!]^\sigma\right\}^2. \tag{17}$$

For the term (S_3), Combining (9), (11) and (14), one has

$$(S_3) \le C_6 A\left\{\frac{A^{|\alpha|-1}}{\tilde\rho^{|\alpha|-1}}[(|\alpha|-2)!]^\sigma\right\}^2. \tag{18}$$

The treatment of the terms (S_4) is a little more complicated. Write $(S_4) = (S_4)' + (S_4)''$ with

$$(S_4)' \quad = C_2\sum_{|\beta|=2} C_\alpha^\beta [G_\sigma(f)]_{\tilde\rho,0}\left\{\int_{T_0+\tilde\rho}^{T_1-\tilde\rho}\|\nabla_v\partial^{\alpha-\beta+1}f(s)\|^2_{L^2_\gamma}ds\right\}^{\frac12}\left\{\int_{T_0+\tilde\rho}^{T_1-\tilde\rho}\|\nabla_v\partial^{\alpha-1}f(s)\|^2_{L^2_\gamma}ds\right\}^{\frac12}$$

and

$$\begin{aligned}(S_4)'' \quad &= \quad C_2\sum_{3\le|\beta|\le|\alpha|} C_\alpha^\beta [G_\sigma(f)]_{\tilde\rho,\beta-2}\left\{\int_{T_0+\tilde\rho}^{T_1-\tilde\rho}\|\nabla_v\partial^{\alpha-\beta+1}f(s)\|^2_{L^2_\gamma}ds\right\}^{\frac12}\\ &\quad\times\quad \left\{\int_{T_0+\tilde\rho}^{T_1-\tilde\rho}\|\nabla_v\partial^{\alpha-1}f(s)\|^2_{L^2_\gamma}ds\right\}^{\frac12}.\end{aligned}$$

It is easy to verify that, by (9), (11) and (12),

$$(S_4)' \le C_7 A\,|\alpha|^2\left\{\frac{A^{|\alpha|-1}}{\tilde\rho^{|\alpha|-1}}[(|\alpha|-2)!]^\sigma\right\}^2 \le C_8 A\left\{\frac{A^{|\alpha|-1}}{\tilde\rho^{|\alpha|-1}}[(|\alpha|-1)!]^\sigma\right\}^2.$$

And by virtue of (11) and (15), we know $(S_4)''$ is bounded from above by

$$\textstyle\sum_{3\le|\beta|\le|\alpha|}\frac{C_3|\alpha|!}{|\beta|!(|\alpha|-|\beta|)!}\frac{A^{|\beta|-2}}{\tilde\rho^{|\beta|-2}}[(|\beta|-3)!]^\sigma\frac{A^{|\alpha|-|\beta|+1}}{\tilde\rho^{|\alpha|-|\beta|+1}}[(|\alpha|-|\beta|)!]^\sigma\frac{A^{|\alpha|-1}}{\tilde\rho^{|\alpha|-1}}[(|\alpha|-2)!]^\sigma,$$

from which we get

$$(S_4)'' \le C_9\left\{\frac{A^{|\alpha|-1}}{\tilde\rho^{|\alpha|-1}}\right\}^2[(|\alpha|-2)!]^\sigma\sum_{3\le|\beta|\le|\alpha|}\frac{|\alpha|!}{|\beta|!(|\alpha|-|\beta|)!}[(|\beta|-3)!]^\sigma[(|\alpha|-|\beta|)!]^\sigma.$$

Direct verification shows that

$$\begin{aligned}\sum_{3\le|\beta|\le|\alpha|}\frac{|\alpha|!}{|\beta|!(|\alpha|-|\beta|)!}[(|\beta|-3)!]^\sigma[(|\alpha|-|\beta|)!]^\sigma &\le [(|\alpha|-1)!]^\sigma\sum_{3\le|\beta|\le|\alpha|}\frac{6\,|\alpha|}{|\beta|^3}\\ &\le C_{10}[(|\alpha|-1)!]^\sigma\,|\alpha|^\sigma \quad \text{(by(5))}.\end{aligned}$$

Combining these, we get the estimate of $(S_4)''$; that is

$$(S_4)'' \le C_{11}\left\{\frac{A^{|\alpha|-1}}{\tilde{\rho}^{|\alpha|-1}}[(|\alpha|-1)!]^{\sigma}\right\}^2.$$

This with the estimate of $(S_4)'$ gives

$$(S_4) = (S_4)' + (S_4)'' \le C_{12}A\left\{\frac{A^{|\alpha|-1}}{\tilde{\rho}^{|\alpha|-1}}[(|\alpha|-1)!]^{\sigma}\right\}^2. \tag{19}$$

The term (S_5) can be handled exactly as above, and we have, by virtue of (6), (9), (11) and (13),

$$(S_5) \le C_{13}A\left\{\frac{A^{|\alpha|-1}}{\tilde{\rho}^{|\alpha|-1}}[(|\alpha|-1)!]^{\sigma}\right\}^2. \tag{20}$$

Now combination of (16), (17), (18),(19) and (20) gives that, for all $t \in \Omega\rho$,

$$\|\partial^{\alpha} f(t)\|^2_{L^2} \le (S_1)+(S_2)+(S_3)+(S_4)+(S_5) \le C_{14}A\left\{\frac{A^{|\alpha|-1}}{\tilde{\rho}^{|\alpha|}}[(|\alpha|-1)!]^{\sigma}\right\}^2.$$

Note that $\tilde{\rho}^{-|\alpha|} = \tilde{\rho}^{-k} \le 5\rho^{-|\alpha|}$ by (8), and hence the above inequality yields

$$\|\partial^{\alpha} f(t)\|^2_{L^2} \le C_{15}A\left\{\frac{A^{|\alpha|-1}}{\rho^{|\alpha|}}[(|\alpha|-1)!]^{\sigma}\right\}^2, \quad \forall\ t \in \Omega\rho.$$

Taking A large enough such that $A \ge 16\max\{\sup_{s\in[T_0,T_1]}\|f(s)\|_{H^2_\gamma}+1,\ \tilde{B},\ C_{15}\}$, then we get finally

$$\|\partial^{\alpha} f(t)\|^2_{L^2} \le \left\{\frac{1}{2}\frac{A^{|\alpha|}}{\rho^{|\alpha|}}[(|\alpha|-1)!]^{\sigma}\right\}^2.$$

The above inequality holds for all $t \in \Omega_\rho$, and hence (10) follows.

In order to finish the proof, it remains to prove

$$\left\{\int_{T_0+\rho}^{T_1-\rho}\|\nabla_v\partial^{\tilde{\alpha}} f(t)\|^2_{L^2_\gamma}dt\right\}^{\frac{1}{2}} \le \frac{1}{2}\frac{A^{|\alpha|}}{\rho^{|\alpha|}}[(|\alpha|-1)!]^{\sigma}, \quad \forall\ |\tilde{\alpha}|=k, \quad \forall\ 0<\rho<\frac{T_1-T_0}{2}. \tag{21}$$

And it can be handled exactly as (10). The only difference is that the multi-index α and the term $\|\partial^{\alpha} f(t)\|^2_{L^2} = \varphi_{\rho,k}(t)\|\partial^{\alpha} f(t)\|^2_{L^2} - \varphi_{\rho,k}(T_0)\|\partial^{\alpha} f(T_0)\|^2_{L^2}$ appearing in the above argument will be replace respectively by $\tilde{\alpha}$ and

$$C_1\int_{T_0+\rho}^{T_1-\rho}\|\nabla_v\partial^{\tilde{\alpha}} f(s)\|^2_{L^2_\gamma}ds \le C_1\int_{T_0+\tilde{\rho}}^{T_1-\tilde{\rho}}\varphi_{\rho,k}(s)\|\nabla_v\partial^{\tilde{\alpha}} f(s)\|^2_{L^2_\gamma}ds.$$

Finally, combination of (10) and (21) gives the truth of $(Q)_k$. This completes the proof of Proposition 2.3.

3. Proof of Lemma 2.2

For simplicity , in this section, $\partial_{v_iv_j}\overline{a}_{ij}$ and $\overline{a}_{ij}$ will stand for $\sum_{1\le i,j\le 3}\partial_{v_iv_j}\overline{a}_{ij}$ and $\sum_{i,j}\overline{a}_{ij}$, respectively. In the sequel C is used to denote different constants which can be replaced by a larger one and depend only on γ, the Gevrey index σ, and the initial mass M_0, the initial energy E_0 and the initial entropy H_0.

Our starting point is the following uniformly ellipticity property of the matrix $(\overline{a}_{ij})$, cf. Proposition 4 of [11].

Lemma 3.1. *There exists a constant K, depending only on γ and M_0, E_0, H_0, such that*

$$\sum_{1\le i,j\le 3} \bar{a}_{ij}(t,v)\xi_i\xi_j \ge K(1+|v|^2)^{\gamma/2}|\xi|^2, \quad \forall\, \xi\in\mathbb{R}^3. \tag{22}$$

Remark 3.2. *Although the ellipticity of (a_{ij}) was proved in* [11] *under the hard potential case $\gamma > 0$, it's still true for $\gamma = 0$, the Maxwellian case. This can be seen in the proof of Proposition 4 of* [11].

Lemma 3.3. *There exists a constant B, depending only on the Gevrey index $\sigma > 1$, such that for all multi-indices β with $|\beta| \ge 2$ and all $g, h \in L^2_\gamma(\mathbb{R}^3)$,*

$$\int_{\mathbb{R}^3}(\partial^\beta \bar{a}_{ij}(t,v))g(v)h(v)dv \le C\|g\|_{L^2_\gamma}\|h\|_{L^2_\gamma}[G_\sigma(f(t))]_{\beta-2}, \quad \forall\, t>0,$$

where $[G_\sigma(f(t))]_{\beta-2} = \left\{\|\partial^{\beta-2}f(t)\|_{L^2} + B^{|\beta|-2}[(|\beta|-2)!]^\sigma\right\}$.

Proof For $\sigma > 1$, there exists a function $\psi \in G_0^\sigma(\mathbb{R}^3)$ compact support in $\left\{v\in\mathbb{R}^3 \mid |v|\le 2\right\}$, satisfying $\psi(v) = 1$ on the ball $\left\{v\in\mathbb{R}^3 \mid |v|\le 1\right\}$, and moreover for some constant $L > 4$ depending only on σ,

$$\sup\left|\partial^\lambda \psi\right| \le L^{|\lambda|}(|\lambda|!)^\sigma, \quad \forall\, \lambda. \tag{23}$$

For the construction of ψ, see [16] for example. Write $a_{ij} = \psi a_{ij} + (1-\psi)a_{ij}$. Then $\bar{a}_{ij} = (\psi a_{ij}) * f + [(1-\psi)a_{ij}] * f$, and hence

$$\partial^\beta \bar{a}_{ij} = [\partial^{\tilde{\beta}}(\psi a_{ij})] * (\partial^{\beta-\tilde{\beta}} f) + \left\{\partial^\beta[(1-\psi)a_{ij}]\right\} * f,$$

where $\tilde{\beta}$ is an arbitrary multi-index satisfying $\tilde{\beta} \le \beta$ and $|\tilde{\beta}| = 2$. We firstly treat the term $[\partial^{\tilde{\beta}}(\psi a_{ij})] * (\partial^{\beta-\tilde{\beta}} f)$. It is easy to verify that for all $\tilde{\beta}$ with $\left|\tilde{\beta}\right| = 2$,

$$\left|(\partial^{\tilde{\beta}} a_{ij})(v-v_*)\right| \le C|v-v_*|^\gamma,$$

from which, we can compute

$$\begin{aligned}\left|[\partial^{\tilde{\beta}}(\psi a_{ij})] * (\partial^{\beta-\tilde{\beta}} f)(v)\right| &= \left|\int_{\mathbb{R}^3}[\partial^{\tilde{\beta}}(\psi a_{ij})](v-v_*)\cdot(\partial^{\beta-\tilde{\beta}} f)(v_*)dv_*\right| \\ &\le C\int_{\{|v_*-v|\le 2\}}\left|(\partial^{\beta-\tilde{\beta}} f)(v_*)\right|dv_* \\ &\le C\|\partial^{\beta-2} f(t)\|_{L^2}.\end{aligned}$$

Next the term $\left\{\partial^\beta[(1-\psi)a_{ij}]\right\} * f$, we use Leibniz's formula to get

$$\begin{aligned}&\left|(\partial^\beta[(1-\psi)a_{ij}]) * f(v)\right| \\ &= \left|\sum_{0\le|\lambda|\le|\beta|} C_\beta^\lambda \int_{\mathbb{R}^3}[\partial^{\beta-\lambda}(1-\psi)](v-v_*)\cdot\left(\partial^\lambda a_{ij}\right)(v-v_*)\cdot f(t,v_*)dv_*\right| \\ &\le \left|\sum_{0\le|\lambda|\le|\beta|-1} C_\beta^\lambda \int_{\{1\le|v_*-v|\le 2\}}\left(\partial^{\beta-\lambda}\psi\right)(v-v_*)\cdot\left(\partial^\lambda a_{ij}\right)(v-v_*)\cdot f(t,v_*)dv_*\right| \\ &\quad + \left|\int_{\{|v_*-v|\ge 1\}}[(1-\psi)](v-v_*)\cdot\left(\partial^\beta a_{ij}\right)(v-v_*)\cdot f(t,v_*)dv_*\right| \\ &= J_1 + J_2.\end{aligned}$$

In view of (2), we can find a constant $\tilde{C}$, such that for all multi-indices λ,

$$\left|\left(\partial^\lambda a_{ij}\right)(v-v_*)\right| \le \tilde{C}^{|\lambda|}|\lambda|! \quad \text{for } 1\le|v_*-v|\le 2.$$

And for all β with $|\beta| \ge 2$,

$$\left|\left(\partial^\beta a_{ij}\right)(v-v_*)\right| \le \tilde{C}^{|\beta|}|\beta|!(1+|v_*|^\gamma+|v|^\gamma) \quad \text{for } |v_*-v|\ge 1.$$

These along with (23) give the upper bound of J_1 and J_2,

$$J_1 \le L^{|\beta|}(|\beta|!)^\sigma \cdot \|f(t)\|_{L^1} \sum_{0\le|\lambda|\le|\beta|-1} \left(\frac{\tilde{C}}{L}\right)^{|\lambda|}; \quad J_2 \le 2\tilde{C}^{|\beta|}(|\beta|!)^\sigma \cdot \|f(t)\|_{L^1_\gamma}(1+|v|^\gamma).$$

By taking L large enough such that $L \ge 2\tilde{C}$, then we get

$$J_1 + J_2 \le C\|f(t)\|_{L^1_\gamma} L^{|\beta|}(|\beta|!)^\sigma (1+|v|^2)^{\gamma/2}.$$

This along with the fact $\|f(t)\|_{L^1_\gamma} \le M_0 + 2E_0$ gives at once

$$\left|(\partial_v^\beta[(1-\psi)a_{ij}]) * f(v)\right| \le J_1 + J_2 \le CL^{|\beta|}(|\beta|!)^\sigma (1+|v|^2)^{\gamma/2}.$$

Now we choose a constant B such that $L^{|\beta|}(|\beta|!)^\sigma \le B^{|\beta|}[(|\beta|-2)!]^\sigma$. From the above inequality it follows immediately that

$$\left|(\partial_v^\beta[(1-\psi)a_{ij}]) * f(v)\right| \le CB^{|\beta|-2}[(|\beta|-2)!]^\sigma (1+|v|^2)^{\gamma/2}.$$

Combining the estimate on the term $[\partial^2(\psi a_{ij})] * (\partial^{|\beta|-2} f)$, one has

$$\begin{aligned}\left|\partial^\beta \bar{a}_{ij}(v)\right| &\le C\left\{\|\partial^{\beta-2} f(t)\|_{L^2} + B^{|\beta|-2}[(|\beta|-2)!]^\sigma \cdot (1+|v|^2)^{\gamma/2}\right\} \\ &\le C[G_\sigma(f(t))]_{\beta-2} \cdot (1+|v|^2)^{\gamma/2}.\end{aligned}$$

Together with Cauchy's inequality, we get the desired inequality.

In quite similar argument, we have the following

Lemma 3.4. *For all multi-indices β with $|\beta| \ge 0$ and all $g, h \in L^2_\gamma(\mathbb{R}^3)$, one has*

$$\int_{\mathbb{R}^3} (\partial^\beta \bar{c}(t,v)) g(v) h(v) dv \le C\|g\|_{L^2_\gamma}\|h\|_{L^2_\gamma} \cdot [G_\sigma(f(t))]_\beta, \quad \forall\, t \ge 0.$$

The rest of the paper is devoted to

Proof of Lemma 2.2. Let $b_j = \partial_{v_i} a_{ij}(v) = -2|v|^\gamma v_j$. Then we have the following relation

$$\partial_{v_i}\bar{a}_{ij}(v) = \bar{b}_j(v), \quad \partial_{v_j}\bar{b}_j = \bar{c}.$$

Since f satisfies $\partial_t f = \bar{a}_{ij}\partial_{v_i v_j} f - \bar{c} f$, then direct verification shows

$$\begin{aligned}\frac{d}{dt}\|\partial^\mu f(t)\|^2_{L^2} &= 2\int_{\mathbb{R}^3} [\partial_t \partial^\mu f(t,v)] \cdot [\partial^\mu f(t,v)] dv \\ &= 2\int_{\mathbb{R}^3} [\partial^\mu(\bar{a}_{ij}\partial_{v_i v_j} f - \bar{c} f)] \cdot [\partial^\mu f(t,v)] dv.\end{aligned}$$

Moreover, using Leibniz's formula on the term $\partial^\mu\left(\bar{a}_{ij}\partial_{v_i v_j} f - \bar{c} f\right)$, one has

$$\begin{aligned}\partial_t\|\partial^\mu f(t)\|^2_{L^2} &= 2\int_{\mathbb{R}^3} \bar{a}_{ij}\left(\partial_{v_i v_j}\partial^\mu f\right)\cdot(\partial^\mu f)\, dv \\ &\quad +2\sum_{|\beta|=1} C^\beta_\mu \int_{\mathbb{R}^3} \left(\partial^\beta \bar{a}_{ij}\right)\left(\partial_{v_i v_j}\partial^{\mu-\beta} f\right)\cdot(\partial^\mu f)\, dv \\ &\quad +2\sum_{2\le|\beta|\le|\mu|} C^\beta_\mu \int_{\mathbb{R}^3} \left(\partial^\beta \bar{a}_{ij}\right)\left(\partial_{v_i v_j}\partial^{\mu-\beta} f\right)\cdot(\partial^\mu f)\, dv \\ &\quad -2\sum_{0\le|\beta|\le|\mu|} C^\beta_\mu \int_{\mathbb{R}^3} \left(\partial^{\mu-\beta}\bar{c}\right)\left(\partial^\beta f\right)\cdot(\partial^\mu f)\, dv \\ &= (I)+(II)+(III)+(IV).\end{aligned}$$

We shall proceed to treat the above terms by the following steps.

Step 1. Upper bound for the term (I).

Integrating by parts, one has

$$\begin{aligned}(I) &= -2\int_{\mathbb{R}^3}\bar{a}_{ij}\left(\partial_{v_j}\partial^\mu f\right)\cdot\left(\partial_{v_i}\partial^\mu f\right)dv - 2\int_{\mathbb{R}^3}\bar{b}_j\left(\partial_{v_j}\partial^\mu f\right)\cdot\left(\partial^\mu f\right)dv\\ &= (I)_1 + (I)_2.\end{aligned}$$

The ellipticity property (22) of (a_{ij}) gives that

$$(I)_1 \le -2K\int_{\mathbb{R}^3}\left|\nabla_v\partial^\mu f\right|^2(1+|v|^2)^{\gamma/2}dv = -2K\|\nabla_v\partial^\mu f(t)\|^2_{L^2_\gamma}.$$

For the term $(I)_2$, integrating by parts again, we have

$$(I)_2 = -(I)_2 + 2\int_{\mathbb{R}^3}\bar{c}\left(\partial^\mu f\right)\cdot\left(\partial^\mu f\right)dv.$$

This along with the fact $|\bar{c}(v)| \le C\|f(t)\|_{L^1_\gamma}(1+|v|^2)^{\gamma/2} \le C(1+|v|^2)^{\gamma/2}$ shows immediately

$$(I)_2 \le C\|\partial^\mu f(t)\|^2_{L^2_\gamma} \le C\|\nabla_v\partial^{\mu-1}f(t)\|^2_{L^2_\gamma}.$$

Combining these, we get the estimate on the term (I), that is

$$(I) \le -2K\|\nabla_v\partial^\mu f(t)\|^2_{L^2_\gamma} + C\|\nabla_v\partial^{\mu-1}f(t)\|^2_{L^2_\gamma}. \tag{24}$$

Step 2. Upper bound for the term (II).

Recall $(II) = 2\sum_{|\beta|=1}C^\beta_\mu\int_{\mathbb{R}^3}\left(\partial^\beta\bar{a}_{ij}\right)\left(\partial_{v_iv_j}\partial^{\mu-\beta}f\right)\cdot\left(\partial^\mu f\right)dv$. Integrating by parts, we get

$$\begin{aligned}(II) \quad &= -2\sum_{|\beta|=1}C^\beta_\mu\int_{\mathbb{R}^3}\left(\partial^\beta\bar{b}_j\right)\left(\partial_{v_j}\partial^{\mu-\beta}f\right)\cdot\left(\partial^\mu f\right)dv\\ &\quad - 2\sum_{|\beta|=1}C^\beta_\mu\int_{\mathbb{R}^3}\left(\partial^\beta\bar{a}_{ij}\right)\left(\partial_{v_j}\partial^{\mu-\beta}f\right)\cdot\left(\partial_{v_i}\partial^\mu f\right)dv\\ &= (II)_1 + (II)_2.\end{aligned}$$

Note $\left|\partial^\beta\bar{b}_j(t,v)\right| \le C(1+|v|^2)^{\gamma/2}$ for any β with $|\beta|=1$ and hence

$$(II)_1 \le C|\mu|\cdot\|\nabla_v\partial^{\mu-\beta}f(t)\|_{L^2_\gamma}\|\partial^\mu f(t)\|_{L^2_\gamma} \le C|\mu|\cdot\|\nabla_v\partial^{\mu-1}f(t)\|^2_{L^2_\gamma}.$$

For the term $(II)_2$, noticing that $\mu = \beta + (\mu-\beta)$, it can be rewritten as the following form

$$(II)_2 \quad = -2\sum_{|\beta|=1}C^\beta_\mu\int_{\mathbb{R}^3}\left(\partial^\beta\bar{a}_{ij}\right)\left(\partial_{v_j}\partial^{\mu-\beta}f\right)\cdot\left(\partial^\beta\partial_{v_i}\partial^{\mu-\beta}f\right)dv.$$

Since $|\beta|=1$, we can integrate by parts to get

$$\begin{aligned}(II)_2 = \quad & 2\sum_{|\beta|=1}C^\beta_\mu\int_{\mathbb{R}^3}\left(\partial^\beta\bar{a}_{ij}\right)\left(\partial_{v_j}\partial^\mu f\right)\cdot\left(\partial_{v_i}\partial^{\mu-\beta}f\right)dv\\ &+2\sum_{|\beta|=1}C^\beta_\mu\int_{\mathbb{R}^3}\left(\partial^{\beta+\beta}\bar{a}_{ij}\right)\left(\partial_{v_j}\partial^{\mu-\beta}f\right)\cdot\left(\partial_{v_i}\partial^{\mu-\beta}f\right)dv\\ = \quad & -(II)_2 + 2\sum_{|\beta|=1}C^\beta_\mu\int_{\mathbb{R}^3}\left(\partial^{\beta+\beta}\bar{a}_{ij}\right)\left(\partial_{v_j}\partial^{\mu-\beta}f\right)\cdot\left(\partial_{v_i}\partial^{\mu-\beta}f\right)dv.\end{aligned}$$

Hence

$$(II)_2 \quad = \sum_{|\beta|=1}C^\beta_\mu\int_{\mathbb{R}^3}\left(\partial^{\beta+\beta}\bar{a}_{ij}\right)\left(\partial_{v_j}\partial^{\mu-\beta}f\right)\cdot\left(\partial_{v_i}\partial^{\mu-\beta}f\right)dv.$$

This along with the fact $\left|\partial^{\beta+\beta}\bar{a}_{ij}(v)\right| \le C(1+|v|^2)^{\gamma/2}$ for all β with $|\beta|=1$ shows at once

$$(II)_2 \le C\sum_{|\beta|=1}C^\beta_\mu\cdot\|\nabla_v\partial^{\mu-\beta}f\|^2_{L^2_\gamma} \le C|\mu|\cdot\|\nabla_v\partial^{\mu-1}f\|^2_{L^2_\gamma}.$$

Combining these, we obtain

$$(II) \le C|\mu|\cdot\|\nabla_v\partial^{|\mu|-1}f\|^2_{L^2_\gamma}. \tag{25}$$

Step 3. Upper bound for the term (III) and (IV) and the conclusion.

Recall $(III) = 2\sum_{2\le|\beta|\le|\mu|} C_\mu^\beta \int_{\mathbb{R}^3} \left(\partial^\beta \bar{a}_{ij}\right)\left(\partial_{v_iv_j}\partial^{\mu-\beta} f\right)\cdot(\partial^\mu f)\,dv$ and

$$(IV) = -2\sum_{0\le|\beta|\le|\mu|} C_\mu^\beta \int_{\mathbb{R}^3} \left(\partial^{\mu-\beta}\bar{c}\right)\left(\partial^\beta f\right)\cdot(\partial^\mu f)\,dv.$$

By virtue of Lemma 3.3 and lemma 3.4, it follows that

$$\begin{aligned}(III) &\le C\sum_{2\le|\beta|\le|\mu|} C_\mu^\beta \|\partial_{v_iv_j}\partial^{\mu-\beta} f(t)\|_{L^2_\gamma}\cdot\|\partial^\mu f(t)\|_{L^2_\gamma}[G_\sigma(f(t))]_{\beta-2}\\ &\le C\sum_{2\le|\beta|\le|\mu|} C_\mu^\beta \|\nabla_v\partial^{\mu-\beta+1} f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\mu-1} f(t)\|_{L^2_\gamma}[G_\sigma(f(t))]_{\beta-2},\end{aligned} \tag{26}$$

and

$$(IV) \le C\sum_{0\le|\beta|\le|\mu|} C_\mu^\beta \|\partial^\beta f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\mu-1} f(t)\|_{L^2_\gamma}\cdot[G_\sigma(f(t))]_{\mu-\beta}. \tag{27}$$

Combination of (24)-(27) gives the desired inequality

$$\begin{aligned}\frac{d}{dt}\|\partial^\mu f(t)\|^2_{L^2} + C_1\|\nabla_v\partial^\mu f(t)\|^2_{L^2_\gamma} &\le C_2|\mu|^2\|\nabla_v\partial^{\mu-1} f(t)\|^2_{L^2_\gamma}\\ +C_2\sum_{2\le|\beta|\le|\mu|} C_\mu^\beta\|\nabla_v\partial^{\mu-\beta+1} f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\mu-1} f(t)\|_{L^2_\gamma}\cdot[G_\sigma(f(t))]_{\beta-2}&\\ +C_2\sum_{0\le|\beta|\le|\mu|} C_\mu^\beta\|\partial^\beta f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\mu-1} f(t)\|_{L^2_\gamma}\cdot[G_\sigma(f(t))]_{\mu-\beta},&\end{aligned}$$

where C_1, C_2 are two constants depending only on M_0, E_0, H_0, σ and γ. This completes the proof of Lemma 2.2.

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School of Mathematics and Statistics, Wuhan University, Wuhan 430072, P. R. China
E-mail address: chenhua@whu.edu.cn

School of Mathematics and Statistics, Wuhan University,Wuhan 430072, P. R. China
E-mail address: weixi.li@yahoo.com

School of Mathematics and Statistics, Wuhan University, Wuhan 430072, P. R. China
and
Université de Rouen, UMR 6085-CNRS, Mathématiques, Avenue de l'Université, BR.12,
F76801 Saint Etienne du Rouvray, France
E-mail address: Chao-Jiang.Xu@univ-rouen.fr