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# GEVREY REGULARIZING EFFECT OF THE CAUCHY PROBLEM FOR NON-CUTOFF HOMOGENEOUS KAC'S EQUATION

NADIA LEKRINE AND CHAO-JIANG XU

ABSTRACT. In this work, we consider a spatially homogeneous Kac's equation with a non cutoff cross section. We prove that the weak solution of the Cauchy problem is in the Gevrey class for positive time. This is a Gevrey regularizing effect for non smooth initial datum. The proof relies on the Fourier analysis of Kac's operators and on an exponential type mollifier.

## 1. INTRODUCTION

In this work, we consider the following Cauchy problem for spatially homogeneous non linear Kac's equation,

$$(1.1) \quad \begin{cases} \frac{\partial f}{\partial t} = K(f, f), & v \in \mathbb{R}, t > 0, \\ f|_{t=0} = f_0. \end{cases}$$

where  $f = f(t, v)$  is the nonnegative density distribution function of particles with velocity  $v \in \mathbb{R}$  at time  $t$ . The right hand side of equation (1.1) is given by Kac's bilinear collisional operator

$$K(f, g) = \int_{\mathbb{R}} \int_{-\pi/2}^{\pi/2} \beta(\theta) \{f(v'_*)g(v') - f(v_*)g(v)\} d\theta dv_*,$$

where

$$v' = v \cos \theta - v_* \sin \theta, \quad v'_* = v \sin \theta + v_* \cos \theta.$$

We suppose that the cross-section kernel is non cut-off. To simplify the notations, we suppose (see [7, 8] for the precise description of cross-section kernel) that

$$(1.2) \quad \beta(\theta) = C_0 \frac{|\cos \theta|}{|\sin \theta|^{1+2s}}, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},$$

where  $0 < s < 1$  and  $C_0 > 0$ , then

$$\int_{-\pi/2}^{\pi/2} \beta(\theta) d\theta = +\infty,$$

and

$$(1.3) \quad \begin{cases} \int_{-\pi/2}^{\pi/2} \beta(\theta) |\theta| d\theta = C_s < +\infty, & 0 < s < 1/2, \\ \int_{-\pi/2}^{\pi/2} \beta(\theta) \theta^2 d\theta = C_s < +\infty, & 0 < s < 1. \end{cases}$$

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Hereafter, use the following function spaces: For  $1 \leq p \leq +\infty, \ell \in \mathbb{R}$ ,

$$L_\ell^p(\mathbb{R}) = \left\{ f; \|f\|_{L_\ell^p} = \left( \int_{\mathbb{R}} |\langle v \rangle^\ell f(v)|^p dv \right)^{1/p} < +\infty \right\}$$

where  $\langle v \rangle = (1 + |v|^2)^{1/2}$ .

$$L \log L(\mathbb{R}) = \left\{ f; \|f\|_{L \log L} = \int_{\mathbb{R}} |f(v)| \log(1 + |f(v)|) dv < +\infty \right\}.$$

For  $k, \ell \in \mathbb{R}$ ,

$$H_\ell^k(\mathbb{R}) = \left\{ f \in \mathcal{S}'(\mathbb{R}); \langle v \rangle^\ell f \in H^k(\mathbb{R}) \right\}.$$

We assume that the initial datum  $f_0 \not\equiv 0$  satisfies the natural boundedness on the mass, energy and entropy, that is,

$$(1.4) \quad f_0 \geq 0, \quad \int_{\mathbb{R}} f_0(v) (1 + |v|^2 + \log(1 + f_0(v))) dv < +\infty.$$

In [7], L. Desvillettes has proved the existence of a nonnegative weak solution to the Cauchy problem (1.1), (see also [11] by using a stochastic calculus),

$$(1.5) \quad f \in L^\infty([0, +\infty[; L_k^1(\mathbb{R})),$$

if  $f_0 \in L_k^1(\mathbb{R})$  for some  $k \geq 2$ . The weak solution satisfies the conservation of mass

$$(1.6) \quad \int_{\mathbb{R}} f(t, v) dv = \int_{\mathbb{R}} f_0(v) dv, \quad \forall t > 0,$$

the conservation of energy

$$(1.7) \quad \int_{\mathbb{R}} f(t, v) |v|^2 dv = \int_{\mathbb{R}} f_0(v) |v|^2 dv, \quad \forall t > 0,$$

and also the entropy inequality

$$(1.8) \quad \int_{\mathbb{R}} f(t, v) \log f(t, v) dv \leq \int_{\mathbb{R}} f_0(v) \log f_0(v) dv, \quad \forall t > 0,$$

but does not conserve the momentum.

L. Desvillettes proved also in [7] (see also [10]), the  $C^\infty$ -regularity of weak solutions if  $f_0 \in L_\ell^1(\mathbb{R})$  for any  $\ell \in \mathbb{N}$ . This regularizing effect properties is now well-known for non cut-off homogeneous Boltzmann equations (see also [3, 4, 9, 13]).

In this work, we consider the higher order regularity, the Gevrey regularity of solutions of the Cauchy problem (1.1). We start by recalling the definition of the Gevrey class functions.  $u \in G^\alpha(\mathbb{R}^n)$  (the Gevrey class function space with index  $\alpha$ ), if for  $\alpha \geq 1$ , there exists  $C > 0$  such that for any  $k \in \mathbb{N}$ ,

$$\|D^k u\|_{L^2(\mathbb{R}^n)} \leq C^{k+1} (k!)^\alpha,$$

or equivalently, there exists  $c_0 > 0$  such that  $e^{c_0 \langle D \rangle^{1/\alpha}} u \in L^2(\mathbb{R}^n)$ , where

$$\langle D \rangle = (1 + |D_v|^2)^{1/2}, \quad \|D^k u\|_{L^2(\mathbb{R}^n)}^2 = \sum_{|\beta|=k} \|D^\beta u\|_{L^2(\mathbb{R}^n)}^2.$$

Note that  $G^1(\mathbb{R}^n)$  is the usual analytic function space. If  $0 < \alpha < 1$ , the above definition gives the ultra-analytical function class. Recall that we give here the Gevrey class functions on  $\mathbb{R}^n$ , and so we can use the Fourier transformation and give an equivalent definition by using a Fourier multiplier  $e^{c_0 \langle D \rangle^{1/\alpha}}$ , we can also replace  $L^2$ -norm by  $L^\infty$ -norm.

Our result on the Gevrey regularity can be stated as follows.

**Theorem 1.1.** *Assume that the initial datum  $f_0 \in L^1_{2+2s} \cap L \log L(\mathbb{R})$ , and the cross-section  $\beta$  satisfy (1.2) with  $0 < s < \frac{1}{2}$ . For  $T_0 > 0$ , if  $f \in L^\infty([0, T_0]; L^1_{2+2s} \cap L \log L(\mathbb{R}))$  is a nonnegative weak solution of the Cauchy problem (1.1), then for any  $0 < s' < s$ , there exists  $0 < T_* \leq T_0$  such that*

$$f(t, \cdot) \in G^{\frac{1}{2s'}}(\mathbb{R})$$

for any  $0 < t \leq T_*$ .

**Remark 1.2.** *The above results is a smoothing effect property in the Gevrey class for the Cauchy problem. We suppose nothing about regularity and high order moment controls for the initial datum.*

Recall that Kac's equation is obtained when one considers radially symmetric solutions of the spatially homogeneous Boltzmann equation for Maxwellian molecules (see [7]). The Cauchy problem for the spatially homogeneous Boltzmann equation is defined by :

$$(1.9) \quad \frac{\partial g}{\partial t} = Q(g, g), \quad v \in \mathbb{R}^3, \quad t > 0; \quad g|_{t=0} = g_0,$$

where the Boltzmann collision operator  $Q(g, f)$  is a bi-linear functional given by

$$(1.10) \quad Q(g, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma) \{g(v'_*)f(v') - g(v_*)f(v)\} d\sigma dv_*,$$

for  $\sigma \in \mathbb{S}^2$  and where

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma.$$

The non-negative function  $B(z, \sigma)$  called the Boltzmann collision kernel depends only on  $|z|$  and the scalar product  $\langle \frac{z}{|z|}, \sigma \rangle$ . In most of the cases, the collision kernel  $B$  can not be expressed explicitly. However, to capture its main property, it can be assumed to be in the form

$$B(|v - v_*|, \cos \theta) = \Phi(|v - v_*|)b(\cos \theta), \quad \cos \theta = \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

The Maxwellian case corresponds to  $\Phi \equiv 1$ . Except for hard sphere model, the function  $b(\cos \theta)$  has a singularity at  $\theta = 0$ . We assume that

$$(1.11) \quad \sin \theta b(\cos \theta) \approx K\theta^{-1-2s} \quad \text{when } \theta \rightarrow 0,$$

where  $K > 0, 0 < s < 1$ . Remark that the solution of Boltzmann equation satisfies also the conservation of mass, energy and the entropy inequality.

A function  $g$  is radially symmetric with respect to  $v \in \mathbb{R}^3$ , if it satisfy the property

$$g(t, v) = g(t, Av), \quad v \in \mathbb{R}^3$$

for any rotation  $A$  in  $\mathbb{R}^3$ . We proved the following results.

**Theorem 1.3.** *Assume that the initial datum  $g_0 \in L^1_{2+2s} \cap L \log L(\mathbb{R}^3)$ ,  $g_0 \geq 0$  is radially symmetric. Let  $\Phi \equiv 1$  and let  $b$  satisfy (1.11) with  $0 < s < \frac{1}{2}$ . If  $g$  is a nonnegative radially symmetric weak solution of the Cauchy problem (1.9) such that  $g \in L^\infty([0, +\infty[; L^1_{2+2s} \cap L \log L(\mathbb{R}^3))$ , then*

$$g(t, \cdot) \in G^{\frac{1}{2s'}}(\mathbb{R}^3)$$

for any  $t > 0$  and any  $0 < s' < s$ .

Remark that for the non cut-off spatially homogeneous Boltzmann equation, we have the  $H^\infty$ -regularizing effect of weak solutions (see also [9, 12, 13, 4]). Namely if  $f$  is a weak solution of the Cauchy problem (1.9) and the cross section  $b$  satisfy (1.11), then we have  $f(t, \cdot) \in H^{+\infty}(\mathbb{R})$  for any  $0 < t$ .

Notice that, for the Boltzmann equation, the local solutions having the Gevrey regularity have been constructed in [16] for initial data having higher Gevrey regularity, and the propagation of Gevrey regularity for solutions of Boltzmann equation is studied in [8]. The result given here is concerned with the production of the Gevrey regularity for weak solutions whose initial data have no assumption on the regularity. This regularizing effect property of the Cauchy problem is analogous to the results of [13] where linearized Boltzmann equation is considered. In [14], we have the ultra-analytical regularizing effect of the Cauchy problem in  $G^{\frac{1}{2}}(\mathbb{R}^3)$  for the homogeneous Landau equations, which is optimal as seen from the Cauchy problem of heat equation.

## 2. FOURIER ANALYSIS OF KAC'S OPERATORS

We will now be interested in studying the Fourier analysis of the Kac's collision operator. This is a key step in the regularity analysis of weak solutions. For simplification of notations, we use  $(\cdot, \cdot)$  instead of  $(\cdot, \cdot)_{L^2(\mathbb{R}_v)}$ . We have firstly the following coercivity estimate deduced from the non cut-off of collision kernel.

**Proposition 2.1.** *Assume that the cross-section is non cut-off, satisfies the assumption (1.2). Let  $f \geq 0, f \neq 0, f \in L^1_1(\mathbb{R}) \cap L \log L(\mathbb{R})$ , then there exists a constant  $c_f > 0$ , depending only on  $\beta, \|f\|_{L^1_1}$ , and  $\|f\|_{L \log L}$ , such that*

$$(2.1) \quad -\left(K(f, g), g\right) \geq c_f \|g\|_{H^s(\mathbb{R}_v)}^2 - C \|f\|_{L^1_1} \|g\|_{L^2}^2$$

for any smooth function  $g \in H^1(\mathbb{R})$ .

**Remark 2.2.** *In the proof of Proposition 2.1, the following properties are essential (see (44) in [1])*

(H-1) *there exists a  $r > 0$  such that  $\int_{\{v \in \mathbb{R}; |v| \leq r\}} f(v) dv \geq \frac{3}{4} \|f\|_{L^1_1}$*

(H-2) *there exists a  $\delta > 0$  such that  $\int_A f(v) dv < \frac{1}{4} \|f\|_{L^1_1}$  for any measurable set  $A \subset \mathbb{R}$  satisfying  $|A| < \delta$ .*

As stated in Lemma 2.2 of [8](p.1738), Lebesgue's dominated convergence theorem shows that both properties follow only from the assumption  $f \in L^1$ . However, the proof of Theorem 1.1 and Theorem 1.3 require that  $r$  and  $\delta$  can be chosen uniformly with respect to  $t$  if Proposition 2.1 is applied to solution  $f(t, v)$ . Under the conservation of mass (1.6), (H-1) and (H-2), respectively, follow from (1.7) and (1.8), respectively. In the proof of Theorem 1.6, the property (H-2) will be checked directly without the entropy inequality (see Lemma 5.1 below).

Recall the following weak formulation for collision operators

$$(K(f, g), h) = \iint_{\mathbb{R}^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) f(v_*) g(v) (h(v') - h(v)) d\theta dv_* dv,$$

for suitable functions  $f, g, h$  with reals values. Then

$$\begin{aligned} (-K(f, g), g) &= \frac{1}{2} \iint_{\mathbb{R}^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) f(v_*) (g(v') - g(v))^2 d\theta dv_* dv \\ &\quad - \frac{1}{2} \iint_{\mathbb{R}^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) f(v_*) (g(v')^2 - g(v)^2) d\theta dv_* dv. \end{aligned}$$

The second term of right hand side can be estimated by using the Cancellation lemma of [1]. But in the Maxwellien case, by an appropriate change of variable, we then have,

$$\begin{aligned} &\left| \frac{1}{2} \iint_{\mathbb{R}^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) f(v_*) (g(v')^2 - g(v)^2) d\theta dv_* dv \right| \\ &= \left| \frac{1}{2} \iint_{\mathbb{R}^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) f(v_*) g(v)^2 \left( \frac{1}{\cos \theta} - 1 \right) d\theta dv_* dv \right| \\ &\leq C \iint_{\mathbb{R}^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin(\theta)|^{-1-2s} \left| \sin\left(\frac{\theta}{2}\right) \right|^2 |f(v_*)| g(v)^2 d\theta dv_* dv \\ &\leq C \|f\|_{L^1} \|g\|_{L^2}^2. \end{aligned}$$

The coercivity term in  $H^s$  is deduced from the following positive term,

$$\frac{1}{2} \iint_{\mathbb{R}^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) f(v_*) (g(v') - g(v))^2 d\theta dv_* dv.$$

Here we need the Bobylev formula, i. e. the Fourier transform of collision operators :

$$(2.2) \quad \mathcal{F}(K(f, g))(\xi) = \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \{ \hat{f}(\xi \sin \theta) \hat{g}(\xi \cos \theta) - \hat{f}(0) \hat{g}(\xi) \} d\theta,$$

for suitable functions  $f$  and  $g$  and by using both properties (1) and (2) and the unifom integrability of  $f_t$ . (see [1, 4, 13]). From the above formula, we can get also the following upper bound estimates (see [12, 13]). For  $m, \ell \in \mathbb{R}$ , and for suitable functions  $f, g$ , we have

$$(2.3) \quad \|K(f, g)\|_{H_t^m(\mathbb{R}_v)} \leq C \|f\|_{L_{t^+ + 2s}^1(\mathbb{R}_v)} \|g\|_{H_{(t+2s)^+}^{m+2s}(\mathbb{R}_v)},$$

where  $\alpha^+ = \max\{\alpha, 0\}$ .

To study the Gevrey regularity of the weak solution, as in [13, 14], we consider the exponential type mollifier. For  $0 < \delta < 1$ ,  $c_0 > 0$  and  $0 < s' < s$ , we set

$$G_\delta(t, \xi) = \frac{e^{c_0 t \langle \xi \rangle^{2s'}}}{1 + \delta e^{c_0 t \langle \xi \rangle^{2s'}}$$

where

$$\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}, \quad \xi \in \mathbb{R}.$$

Then, for any  $0 < \delta < 1$ ,

$$(2.4) \quad G_\delta(t, \xi) \in L^\infty(]0, T[ \times \mathbb{R}),$$

and

$$(2.5) \quad \lim_{\delta \rightarrow 0} G_\delta(t, \xi) = e^{c_0 t \langle \xi \rangle^{2s'}}.$$

Denote by  $G_\delta(t, D_v)$ , the Fourier multiplier of symbol  $G_\delta(t, \xi)$ ,

$$G_\delta g(t, v) = G_\delta(t, D_v)g(t, v) = \mathcal{F}_{\xi \rightarrow v}^{-1}(G_\delta(t, \xi) \hat{g}(t, \xi)).$$

Then our aim is to prove the uniform boundedness (with respect to  $0 < \delta < 1$ ) of the term  $\|G_\delta(t, D_\nu)f(t, \cdot)\|_{L^2(\mathbb{R})}$  for the weak solution of the Cauchy problem (1.1). In what follows, we will use the same notation  $G_\delta$  for the pseudo-differential operators  $G_\delta(t, D_\nu)$  and also its symbol  $G_\delta(t, \xi)$ .

**Lemma 2.3.** *Let  $T > 0, c_0 > 0$ , We have that for any  $0 < \delta < 1$  and  $0 \leq t \leq T, \xi \in \mathbb{R}$ ,*

$$\begin{aligned} |\partial_t G_\delta(t, \xi)| &\leq c_0 \langle \xi \rangle^{2s'} G_\delta(t, \xi), \\ |\partial_\xi G_\delta(t, \xi)| &\leq 2s' c_0 t \langle \xi \rangle^{2s'-1} G_\delta(t, \xi) \end{aligned}$$

and

$$|\partial_\xi^2 G_\delta(t, \xi)| \leq C \langle \xi \rangle^{2(2s'-1)} G_\delta(t, \xi)$$

with  $C > 0$  independent of  $\delta$ .

In fact, we have the following formulas

$$(2.6) \quad \partial_t G_\delta(t, \xi) = c_0 \langle \xi \rangle^{2s'} G_\delta(t, \xi) \frac{1}{1 + \delta e^{c_0 t \langle \xi \rangle^{2s'}}},$$

$$(2.7) \quad \partial_\xi G_\delta(t, \xi) = 2s' c_0 t (1 + |\xi|^2)^{s'-1} \xi G_\delta(t, \xi) \frac{1}{1 + \delta e^{c_0 t \langle \xi \rangle^{2s'}}},$$

and

$$(2.8) \quad \begin{aligned} \partial_\xi^2 G_\delta(t, \xi) &= (2s' c_0 t (1 + |\xi|^2)^{s'-1} \xi)^2 G_\delta(t, \xi) \frac{1 - \delta e^{c_0 t \langle \xi \rangle^{2s'}}}{(1 + \delta e^{c_0 t \langle \xi \rangle^{2s'}})^2} \\ &\quad + 2s' c_0 t \left( (1 + |\xi|^2)^{s'-1} + 2(s'-1) \xi^2 (1 + |\xi|^2)^{s'-2} \right) G_\delta(t, \xi) \frac{1}{1 + \delta e^{c_0 t \langle \xi \rangle^{2s'}}}. \end{aligned}$$

**Lemma 2.4.** *There exists  $C > 0$  such that for all  $0 < \delta < 1$  and  $\xi \in \mathbb{R}$ , we have,*

$$(2.9) \quad |G_\delta(\xi) - G_\delta(\xi \cos \theta)| \leq C \sin^2(\theta/2) \langle \xi \rangle^{2s'} G_\delta(\xi \cos \theta) G_\delta(\xi \sin \theta),$$

and

$$(2.10) \quad |(\partial_\xi G_\delta)(\xi) - (\partial_\xi G_\delta)(\xi \cos \theta)| \leq C \sin^2(\theta/2) \langle \xi \rangle^{(4s'-1)^+} G_\delta(\xi \cos \theta) G_\delta(\xi \sin \theta),$$

where  $(4s'-1)^+ = \max\{4s'-1, 0\}$ .

*Proof.* For the estimate (2.9), we have, by using the Taylor formula

$$G_\delta(\xi) - G_\delta(\xi \cos \theta) = (\xi - \xi \cos \theta) \int_0^1 (\partial_\xi G_\delta)(\xi \cos \theta + \tau(\xi - \xi \cos \theta)) d\tau$$

where  $\xi_\tau = \xi \cos \theta + \tau(\xi - \xi \cos \theta)$ . Then (2.7) implies

$$|G_{t,\delta}(\xi) - G_\delta(t, \xi \cos \theta)| \leq 4s' c_0 t |\xi| \sin^2(\theta/2) \int_0^1 G_\delta(t, \xi_\tau) \langle \xi_\tau \rangle^{2s'-1} d\tau.$$

For  $0 \leq \tau \leq 1$  and  $-\pi/4 \leq \theta \leq \pi/4$ ,

$$\frac{\sqrt{2}}{2} |\xi| \leq |\xi_\tau| = |\xi \cos \theta + \tau(\xi - \xi \cos \theta)| \leq |\xi|,$$

which implies, for  $0 < 2s' < 1$ , that there exists  $C_{s'} > 0$  such that

$$\langle \xi_\tau \rangle^{2s'} \leq \langle \xi \rangle^{2s'}, \quad \langle \xi_\tau \rangle^{2s'-1} \leq C_{s'} \langle \xi \rangle^{2s'-1}.$$

On the other hand,  $G_\delta(t, \xi) = G_\delta(t, |\xi|)$  is increasing with respect to  $|\xi|$ , since for  $\xi > 0$ ,  $\partial_\xi G_\delta(t, \xi) > 0$ , then

$$G_\delta(t, \xi_\tau) \leq G_\delta(t, \xi).$$

By using

$$|\xi|^2 = |\xi \cos \theta|^2 + |\xi \sin \theta|^2,$$

and

$$(1 + a + b)^{2s'} \leq (1 + a)^{2s'} + (1 + b)^{2s'}, \quad (1 + \delta e^\alpha)(1 + \delta e^\beta) \leq 3(1 + \delta e^{\alpha+\beta}),$$

we get

$$(2.11) \quad G_\delta(\xi) \leq 3G_\delta(\xi \cos \theta)G_\delta(\xi \sin \theta).$$

Thus

$$|G_\delta(\xi) - G_\delta(\xi \cos \theta)| \leq C \sin^2(\theta/2) \langle \xi \rangle^{2s'} G_\delta(\xi \cos \theta) G_\delta(\xi \sin \theta).$$

We have proved the estimate (2.9) when  $|\theta| \leq \pi/4$ . If  $\pi/4 \leq |\theta| \leq \pi/2$ , we have

$$\begin{aligned} |G_\delta(\xi) - G_\delta(\xi \cos \theta)| &\leq |G_\delta(\xi)| + |G_\delta(\xi \cos \theta)| \leq 2|G_\delta(\xi)| \\ &\leq 6 G_\delta(\xi \cos \theta) G_\delta(\xi \sin \theta) \leq C \sin^2(\theta/2) G_\delta(\xi \cos \theta) G_\delta(\xi \sin \theta). \end{aligned}$$

For the estimate (2.10), by using (2.8), we have that if  $|\theta| \leq \pi/4$ ,

$$\begin{aligned} |(\partial_\xi G_\delta)(\xi) - (\partial_\xi G_\delta)(\xi \cos \theta)| &= \left| (\xi - \xi \cos \theta) \int_0^1 (\partial_\xi^2 G_\delta)(\xi_\tau) d\tau \right| \\ &\leq C |\xi| \sin^2(\theta/2) \langle \xi \rangle^{2(2s'-1)} \int_0^1 G_\delta(\xi_\tau) d\tau \\ &\leq C \sin^2(\theta/2) \langle \xi \rangle^{4s'-1} G_\delta(\xi \sin \theta) G_\delta(t, \xi \cos \theta). \end{aligned}$$

The case  $\pi/4 \leq |\theta| \leq \pi/2$  is similar to (2.9). Thus, we have proved Lemma 2.4.  $\square$

We now study the commutators of Kac's collision operators with the above mollifier operators.

**Proposition 2.5.** *Assume that  $0 < s' < 1/2$ , Let  $f, g \in L^2_1(\mathbb{R}_v)$  and  $h \in H^{s'}(\mathbb{R}_v)$ , then we have that*

$$(2.12) \quad \begin{aligned} &\left| (G_\delta K(f, g), h) - (K(f, G_\delta g), h) \right| \\ &\leq C \|G_\delta f\|_{L^2_1(\mathbb{R})} \|G_\delta g\|_{H^{s'}(\mathbb{R})} \|h\|_{H^{s'}(\mathbb{R})}, \end{aligned}$$

and

$$(2.13) \quad \begin{aligned} &\left| ((v G_\delta) K(f, g), h) - (K(f, (v G_\delta) g), h) \right| \\ &\leq C (\|f\|_{L^1_1(\mathbb{R})} + \|G_\delta f\|_{L^2_1(\mathbb{R})}) \|G_\delta g\|_{H^{s'}(\mathbb{R})} \|h\|_{H^{s'}(\mathbb{R})}. \end{aligned}$$

*Proof.* By definition, we have, for a suitable function  $F$ ,

$$(2.14) \quad \mathcal{F}(G_\delta F)(\xi) = G_\delta(t, \xi) \hat{F}(\xi),$$

and

$$(2.15) \quad \mathcal{F}((v G_\delta) F)(\xi) = i \partial_\xi (G_\delta \hat{F})(\xi) = i (\partial_\xi G_\delta)(\xi) \hat{F}(\xi) + i G_\delta(\xi) (\partial_\xi \hat{F})(\xi).$$



By using the Bobylev formula (2.2) and the Plancherel formula,

$$\begin{aligned}
& (2\pi)^{1/2} \left\{ (G_\delta K(f, g), h) - (K(f, G_\delta g), h) \right\} \\
&= \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) G_\delta(\xi) \left\{ \hat{f}(\xi \sin \theta) \hat{g}(\xi \cos \theta) - \hat{f}(0) \hat{g}(\xi) \right\} d\theta \overline{\hat{h}(\xi)} d\xi \\
&\quad - \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \left\{ \hat{f}(\xi \sin \theta) (\mathcal{F}(G_\delta g))(\xi \cos \theta) - \hat{f}(0) (\mathcal{F}(G_\delta g))(\xi) \right\} d\theta \overline{\hat{h}(\xi)} d\xi \\
&= \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \hat{f}(\xi \sin \theta) \left\{ G_\delta(\xi) - G_\delta(\xi \cos \theta) \right\} \hat{g}(\xi \cos \theta) \overline{\hat{h}(\xi)} d\theta d\xi.
\end{aligned}$$

The above formula can be justified by the cutoff approximation of collision kernel  $\beta(\theta)$ , then (2.9) and (1.3) imply

$$\begin{aligned}
& \left| (G_\delta K(f, g), h) - (K(f, G_\delta g), h) \right| \\
&\leq C \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin^2(\theta/2) |G_\delta(\xi \sin \theta) \hat{f}(\xi \sin \theta)| \\
&\quad \times |G_\delta(\xi \cos \theta) \hat{g}(\xi \cos \theta)| \langle \xi \rangle^{2s'} |\hat{h}(\xi)| d\theta d\xi \\
&\leq C \|G_\delta \hat{f}\|_{L^\infty(\mathbb{R}_\xi)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin^2(\theta/2) \\
&\quad \times \left( \int_{\mathbb{R}_\xi} \langle \xi \rangle^{2s'} |G_\delta(\xi \cos \theta) \hat{g}(\xi \cos \theta)|^2 d\xi \right)^{1/2} \|h\|_{H^{s'}(\mathbb{R})} d\theta \\
&\leq C \|G_\delta f\|_{L^1(\mathbb{R}_v)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \frac{\sin^2(\theta/2)}{|\cos \theta|^{1/2+s'}} d\theta \|\langle \cdot \rangle^{s'} G_\delta \hat{g}\|_{L^2(\mathbb{R}_\xi)} \|h\|_{H^{s'}(\mathbb{R})} \\
&\leq C \|G_\delta f\|_{L^1_1(\mathbb{R}_v)} \|G_\delta g\|_{H^{s'}(\mathbb{R})} \|h\|_{H^{s'}(\mathbb{R})},
\end{aligned}$$

where we have used the following continuous embedding

$$L^2_\alpha(\mathbb{R}) \subset L^1(\mathbb{R}), \quad \alpha > 1/2.$$

We have proved (2.12).

To treat (2.13), by using (2.15), we similarly have,

$$\begin{aligned}
& (2\pi)^{1/2} \left\{ ((v G_\delta) K(f, g), h) - (K(f, (v G_\delta) g), h) \right\} \\
&= \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \left\{ i \partial_\xi (G_\delta(\xi) \hat{f}(\xi \sin \theta) \hat{g}(\xi \cos \theta)) \right. \\
&\quad \left. - \hat{f}(\xi \sin \theta) \mathcal{F}((v G_\delta) g)(\xi \cos \theta) \right\} \overline{\hat{h}(\xi)} d\xi d\theta \\
&= i \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin \theta (\partial_\xi \hat{f})(\xi \sin \theta) G_\delta(\xi) \hat{g}(\xi \cos \theta) \overline{\hat{h}(\xi)} d\xi d\theta \\
&+ i \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \hat{f}(\xi \sin \theta) \left\{ \partial_\xi (G_\delta(\xi) \hat{g}(\xi \cos \theta)) - (\partial_\xi (G_\delta \hat{g}))(\xi \cos \theta) \right\} \overline{\hat{h}(\xi)} d\xi d\theta \\
&= (I) + (II).
\end{aligned}$$

For the term  $(I)$ , we have

$$\begin{aligned} |(I)| &\leq \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) |\sin \theta| |(\partial_\xi \hat{f})(\xi \sin \theta)| |G_\delta(\xi \cos \theta) \hat{g}(\xi \cos \theta)| |\overline{\hat{h}(\xi)}| d\xi d\theta \\ &+ \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) |\sin \theta| |(\partial_\xi \hat{f})(\xi \sin \theta)| |G_\delta(\xi) - G_\delta(\xi \cos \theta)| |\hat{g}(\xi \cos \theta)| |\overline{\hat{h}(\xi)}| d\xi d\theta \\ &\leq I_1 + I_2. \end{aligned}$$

Firstly, (1.3) with the hypothesis  $0 < s < 1/2$  implies that

$$\begin{aligned} I_1 &\leq \|\partial_\xi \hat{f}\|_{L^\infty(\mathbb{R}_\xi)} \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) |\sin \theta| |G_\delta(\xi \cos \theta) \hat{g}(\xi \cos \theta)| |\hat{h}(\xi)| d\xi d\theta \\ &\leq C \|f\|_{L^1_1(\mathbb{R}_v)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \frac{|\sin \theta|}{|\cos \theta|^{1/2}} d\theta \|\hat{h}\|_{L^2(\mathbb{R}_\xi)} \\ &\quad \times \left( \int_{\mathbb{R}_\xi} |G_\delta(\xi \cos \theta) \hat{g}(\xi \cos \theta)|^2 d(\xi \cos \theta) \right)^{1/2} \\ &\leq C \|f\|_{L^1_1(\mathbb{R}_v)} \|G_\delta g\|_{L^2(\mathbb{R}_v)} \|h\|_{L^2(\mathbb{R}_v)}. \end{aligned}$$

For the term  $I_2$ , by using (2.9) we have the following estimates which are also true for  $0 < s < 1$ ,

$$\begin{aligned} I_2 &\leq \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) |\sin \theta| \sin^2(\theta/2) |G_\delta(\xi \sin \theta) (\partial_\xi \hat{f})(\xi \sin \theta)| \\ &\quad \times |G_\delta(\xi \cos \theta) \hat{g}(\xi \cos \theta)| \langle \xi \rangle^{2s'} |\hat{h}(\xi)| d\xi d\theta \\ &\leq C \|\langle \cdot \rangle^{s'} G_\delta \hat{g}\|_{L^\infty(\mathbb{R}_\xi)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \frac{|\sin \theta| \sin^2(\theta/2)}{|\sin \theta|^{1/2}} d\theta \|h\|_{H^{s'}(\mathbb{R}_v)} \\ &\quad \times \left( \int_{\mathbb{R}_\xi} |G_\delta(\xi \sin \theta) (\partial_\xi \hat{f})(\xi \sin \theta)|^2 d(\xi \sin \theta) \right)^{1/2} \\ &\leq C \|\langle D_v \rangle^{s'} G_\delta g\|_{L^1(\mathbb{R}_v)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \frac{|\sin \theta| \sin^2(\theta/2)}{|\sin \theta|^{1/2}} d\theta \|h\|_{H^{s'}(\mathbb{R}_v)} \\ &\quad \times \left( \int_{\mathbb{R}_\xi} |G_\delta(\xi \sin \theta) (\partial_\xi \hat{f})(\xi \sin \theta)|^2 d(\xi \sin \theta) \right)^{1/2} \\ &\leq C \|G_\delta g\|_{H^1_1(\mathbb{R}_v)} \|G_\delta(vf)\|_{L^2(\mathbb{R}_v)} \|h\|_{H^{s'}(\mathbb{R}_v)}. \end{aligned}$$

Moreover, for a suitable function  $F$ , we have

$$G_\delta(vF) = vG_\delta F + [G_\delta, v]F,$$

and

$$\mathcal{F}([G_\delta, v]F)(\xi) = i(\partial_\xi G_\delta)(\xi) \hat{F}(\xi).$$

Then the symbolic calculus (2.7) implies that, for  $0 < 2s' < 1$ , we have

$$(2.16) \quad \|G_\delta(vF)\|_{H^\alpha(\mathbb{R}_v)} \leq C \|G_\delta F\|_{H^1_1(\mathbb{R}_v)}$$

for any  $\alpha \geq 0$ , then

$$(2.17) \quad |(I)| \leq C \left\{ \|f\|_{L^1_1(\mathbb{R}_v)} + \|G_\delta f\|_{L^2_1(\mathbb{R}_v)} \right\} \|G_\delta g\|_{H^1_1(\mathbb{R}_v)} \|h\|_{H^{s'}(\mathbb{R}_v)}$$

On the other hand, for the term  $(II)$ , we have

$$\begin{aligned} & \partial_\xi(G_\delta(\xi)\hat{g}(\xi \cos \theta)) - (\partial_\xi(G_\delta \hat{g}))(\xi \cos \theta) = \{G_\delta(\xi) - G_\delta(\xi \cos \theta)\}(\partial_\xi \hat{g})(\xi \cos \theta) \\ & + G_\delta(\xi)(\cos \theta - 1)(\partial_\xi \hat{g})(\xi \cos \theta) + \{(\partial_\xi G_\delta)(\xi) - (\partial_\xi G_\delta)(\xi \cos \theta)\} \hat{g}(\xi \cos \theta) \\ & = A_1 + A_2 + A_3. \end{aligned}$$

Thus

$$|(II)| \leq C \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) |\hat{f}(\xi \sin \theta)| |A_1 + A_2 + A_3| |\hat{h}(\xi)| d\xi d\theta.$$

We study now the above 3 terms on the right-hand side. By using (2.9),

$$\begin{aligned} & \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) |\hat{f}(\xi \sin \theta)| |A_1| |\hat{h}(\xi)| d\xi d\theta \\ & \leq \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin^2(\theta/2) |G_\delta(\xi \sin \theta) \hat{f}(\xi \sin \theta)| \\ & \quad \times |G_\delta(\xi \cos \theta) (\partial_\xi \hat{g})(\xi \cos \theta)| \langle \xi \rangle^{2s'} |\hat{h}(\xi)| d\xi d\theta \\ & \leq C \|G_\delta f\|_{L^1(\mathbb{R}_v)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \frac{\sin^2(\theta/2)}{|\cos \theta|^{1/2}} d\theta \|G_\delta(vg)\|_{H^{s'}(\mathbb{R}_v)} \|h\|_{H^{s'}(\mathbb{R}_v)} \\ & \leq C \|G_\delta f\|_{L^2_1(\mathbb{R}_v)} \|G_\delta(vg)\|_{H^{s'}(\mathbb{R}_v)} \|h\|_{H^{s'}(\mathbb{R}_v)}. \end{aligned}$$

The estimate (2.11) and  $\cos \theta - 1 = -2 \sin^2(\theta/2)$  imply

$$\begin{aligned} & \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) |\hat{f}(\xi \sin \theta)| |A_2| |\hat{h}(\xi)| d\xi d\theta \\ & \leq C \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin^2(\theta/2) |G_\delta(\xi \sin \theta) \hat{f}(\xi \sin \theta)| \\ & \quad \times |G_\delta(\xi \cos \theta) (\partial_\xi \hat{g})(\xi \cos \theta)| |\hat{h}(\xi)| d\xi d\theta \\ & \leq C \|G_\delta f\|_{L^2_1(\mathbb{R}_v)} \|G_\delta(vg)\|_{L^2(\mathbb{R}_v)} \|h\|_{L^2(\mathbb{R}_v)}. \end{aligned}$$

Finally, the hypothesis  $0 < s < 1/2$  implies  $(4s' - 1)^+ < 2s'$ , then (2.10) yields,

$$\begin{aligned} & \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) |\hat{f}(\xi \sin \theta)| |A_3| |\hat{h}(\xi)| d\xi d\theta \\ & \leq C \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin^2(\theta/2) |G_\delta(\xi \sin \theta) \hat{f}(\xi \sin \theta)| \\ & \quad \times \langle \xi \rangle^{2s'} |G_\delta(\xi \cos \theta) \hat{g}(\xi \cos \theta)| |\hat{h}(\xi)| d\xi d\theta \\ & \leq C \|G_\delta f\|_{L^2_1(\mathbb{R}_v)} \|G_\delta g\|_{H^{s'}(\mathbb{R}_v)} \|h\|_{H^{s'}(\mathbb{R}_v)}. \end{aligned}$$

By summing the above 3 estimates, (2.16) implies that

$$(2.18) \quad |(II)| \leq C \|G_\delta f\|_{L^2_1(\mathbb{R}_v)} \|G_\delta g\|_{H^{s'}(\mathbb{R}_v)} \|h\|_{H^{s'}(\mathbb{R}_v)}.$$

Proof of Proposition 2.5 is established.  $\square$

**Remark 2.6.** In the proof of estimate for the term  $I_1$  and the last term of  $(II)$ , we have used crucially the restrict assumption  $0 < s < 1/2$ .

## 3. SOBOLEV REGULARIZING EFFECT OF WEAK SOLUTIONS

We will first give an  $H^{+\infty}$ -regularizing effect results for Kac's equation. The following Theorem is more precise than Theorem 1.1 of [13] where the homogeneous Boltzmann equation with Maxwellian molecules has been studied.

**Theorem 3.1.** *Assume that the initial datum  $f_0 \in L^1_{2+2s} \cap L \log L(\mathbb{R})$ , and the cross-section  $\beta$  satisfy (1.2) with  $0 < s < \frac{1}{2}$ . If  $f \in L^\infty(]0, +\infty[; L^1_{2+2s} \cap L \log L(\mathbb{R}))$  is a nonnegative weak solution of the Cauchy problem (1.1), then  $f(t, \cdot) \in H^{+\infty}_2(\mathbb{R})$  for any  $t > 0$ .*

**Remark 3.2.** 1) *This is a  $H^{+\infty}$ -smoothing effect results for the Cauchy problem, it is different from that of [7, 11] where their assumption is that all moments of the initial datum are bounded.*

2) *The results of theorem 3.1 is also true if we assume the following Debye-Yukawa type collision kernel :*

$$\beta(\theta) = C_0 \frac{|\cos \theta|}{|\sin \theta|} \left( \log |\theta|^{-1} \right)^m, \quad 0 < m.$$

To prove the Theorem 3.1, we use, as in [13], the mollifier of polynomial type

$$M_\delta(t, \xi) = \langle \xi \rangle^{tN-1} (1 + \delta |\xi|^2)^{-N_0},$$

for  $0 < \delta < 1$ ,  $t \in [0, T_0]$  and  $2N_0 = T_0N + 4$ .

The idea is the same as the section 3 of [13], but now we need to estimate the commutators with weighted  $\langle v \rangle^2$ . It is analogous to the computation of preceding section. We give here only the main points of the proof,

**Lemma 3.3.** *We have that for any  $0 < \delta < 1$  and  $0 \leq t \leq T_0$ ,  $\xi \in \mathbb{R}$ ,*

$$|\partial_t M_\delta(t, \xi)| \leq N \log \langle \xi \rangle M_\delta(t, \xi).$$

For  $-\pi/4 \leq \theta \leq \pi/4$ ,

$$|M_\delta(\xi) - M_\delta(\xi \cos \theta)| \leq C \sin^2(\theta/2) M_\delta(\xi \cos \theta),$$

$$|(\partial_\xi M_\delta)(\xi) - (\partial_\xi M_\delta)(\xi \cos \theta)| \leq C \sin^2(\theta/2) \langle \xi \rangle^{-1} M_\delta(\xi \cos \theta),$$

and

$$|(\partial_\xi^2 M_\delta)(\xi) - (\partial_\xi^2 M_\delta)(\xi \cos \theta)| \leq C \sin^2(\theta/2) \langle \xi \rangle^{-2} M_\delta(\xi \cos \theta),$$

where the constant  $C$  depends on  $T_0, N$ , but is independents of  $0 < \delta < 1$ .

We prove also this Lemma by using the Taylor formula, and for any  $k \in \mathbb{N}$ ,

$$|\partial_\xi^k M_\delta(\xi)| \leq C_k \langle \xi \rangle^{-k} M_\delta(\xi), \quad \xi \in \mathbb{R}$$

with  $C_k$  depends on  $T_0, N$ , but is independents of  $0 < \delta < 1$ . Moreover, for the polynomial mollifier, we can substitute the inequality (2.11) by the following inequality,

$$(3.1) \quad M_\delta(\xi) \leq C M_\delta(\xi \cos \theta), \quad -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4},$$

here again  $C$  depending on  $N_0, T$ , and independents of  $\delta > 0$ . We have therefore

**Proposition 3.4.** *Assume that  $0 < s < 1/2$ , we have that*

$$(3.2) \quad \left| \left( (v M_\delta) K(f, g), h \right) - \left( K(f, (v M_\delta) g), h \right) \right| \leq C \|f\|_{L^1_1(\mathbb{R})} \|M_\delta g\|_{L^2_1(\mathbb{R})} \|h\|_{L^2(\mathbb{R})},$$

and

$$(3.3) \quad \left| \left( \langle v \rangle^2 M_\delta K(f, g), h \right) - \left( K(f, \langle v \rangle^2 M_\delta g), h \right) \right| \leq C \|f\|_{L^1_2(\mathbb{R})} \|M_\delta g\|_{L^2_2(\mathbb{R})} \|h\|_{L^2(\mathbb{R})},$$

The proof of (3.2) is similar to (2.13) where we substitute Lemma 2.4 by Lemma 3.3, and replace (2.11) by (3.1). Consider now the estimate (3.3), we have, as in the proof of the proposition 2.5,

$$\begin{aligned} & (2\pi)^{1/2} \left\{ \left( \langle v \rangle^2 M_\delta K(f, g), h \right) - \left( K(f, \langle v \rangle^2 M_\delta g), h \right) \right\} \\ &= - \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin^2 \theta (\partial_\xi^2 \hat{f})(\xi \sin \theta) M_\delta(\xi) \hat{g}(\xi \cos \theta) \overline{\hat{h}(\xi)} d\xi d\theta \\ & - 2 \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin \theta (\partial_\xi \hat{f})(\xi \sin \theta) \left( \partial_\xi (M_\delta(\xi) \hat{g}(\xi \cos \theta)) \right) \overline{\hat{h}(\xi)} d\xi d\theta \\ & - \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \hat{f}(\xi \sin \theta) \left\{ \partial_\xi^2 (M_\delta(\xi) \hat{g}(\xi \cos \theta)) - (\partial_\xi^2 (M_\delta \hat{g}))(\xi \cos \theta) \right\} \overline{\hat{h}(\xi)} d\xi d\theta \\ &= B_1 + B_2 + B_3. \end{aligned}$$

Then

$$|B_1| \leq C \|\partial_\xi^2 \hat{f}\|_{L^\infty(\mathbb{R})} \|M_\delta g\|_{L^2(\mathbb{R})} \|h\|_{L^2(\mathbb{R})} \leq C \|f\|_{L^1_2(\mathbb{R})} \|M_\delta g\|_{L^2(\mathbb{R})} \|h\|_{L^2(\mathbb{R})},$$

and for  $0 < 2s < 1$ ,

$$|B_2| \leq C \|f\|_{L^1_1(\mathbb{R})} \left( \|M_\delta g\|_{L^2(\mathbb{R})} + \|M_\delta \langle v \rangle g\|_{L^2(\mathbb{R})} \right) \|h\|_{L^2(\mathbb{R})}.$$

The term  $B_3$  is evidently more complicate, but the idea is the same, we omit here their computations.

Using the continuous embedding

$$L^1_\ell(\mathbb{R}) \subset H^{-1}_\ell(\mathbb{R}),$$

the upper bounded (2.3) with  $m = -2$ ,  $\ell = 2$  and  $0 < 2s < 1$  imply ,

$$\|K(g, h)\|_{H^{-2}_{2+2s}(\mathbb{R}_v)} \leq C \|g\|_{L^1_{2+2s}(\mathbb{R}_v)} \|h\|_{H^{-2+2s}(\mathbb{R}_v)} \leq C \|g\|_{L^1_{2+2s}(\mathbb{R}_v)} \|h\|_{L^1_{2+2s}(\mathbb{R}_v)}.$$

Let  $f \in L^\infty([0, +\infty[; L^1_{2+2s}(\mathbb{R}))$  be a weak solution of the Cauchy problem (1.1), then we can take

$$f_1 = M_\delta(t, D_v) \langle v \rangle^4 M_\delta(t, D_v) f \in L^\infty([0, T_0]; H^{-2+2s}(\mathbb{R})),$$

as test functions of the Cauchy problem (1.1). By using similar manipulations as in [13], we can obtain the regularity with respect to  $t$  variable, to simplify the notations we suppose that  $f_1 \in C^1([0, T_0]; H^{-2+2s}(\mathbb{R}))$ . We have

$$\left( \partial_t f(t, \cdot), f_1(t, \cdot) \right)_{L^2(\mathbb{R}_v)} = \left( K(f, f), f_1 \right)_{L^2(\mathbb{R}_v)}.$$

Then Lemma 3.3, Proposition 3.4, the coercivity estimate (2.1) and the conservations (1.6), (1.7), (1.8) imply that

$$\begin{aligned} & \frac{d}{dt} \|M_\delta f(t)\|_{L^2_2(\mathbb{R}_v)}^2 + c_{f_0} \|M_\delta f(t)\|_{H^2_2(\mathbb{R}_v)}^2 \\ & \leq C_{f_0} \|\log^{1/2}(|D_v|) M_\delta f(t)\|_{L^2_2(\mathbb{R}_v)}^2 + C \|f_0\|_{L^1_2(\mathbb{R})} \|M_\delta f(t)\|_{L^2_2(\mathbb{R}_v)}^2. \end{aligned}$$

We now use the following interpolation inequality, for any small  $\varepsilon > 0$

$$(3.4) \quad \|\log^{1/2}(|D_v|) M_\delta f(t)\|_{L^2_2(\mathbb{R}_v)}^2 \leq \varepsilon \|M_\delta f(t)\|_{H^2_2(\mathbb{R}_v)}^2 + C_\varepsilon \|M_\delta f(t)\|_{L^2_2(\mathbb{R}_v)}^2.$$

Then for  $t \in [0, T_0]$ ,

$$\frac{d}{dt} \|M_\delta f(t)\|_{L^2(\mathbb{R}_v)}^2 \leq C_1 \|M_\delta f(t)\|_{L^2(\mathbb{R})}^2$$

where  $C_1$  depends on  $T_0, N$ , but independent of  $0 < \delta < 1$ . So that for  $t \in [0, T_0]$ ,

$$\|M_\delta f(t)\|_{L^2(\mathbb{R}_v)} \leq e^{C_1 t} \|M_\delta f(0)\|_{L^2(\mathbb{R})} \leq e^{C_1 t} \|f_0\|_{H^1_2(\mathbb{R})} \leq e^{C_1 t} \|f_0\|_{L^1_2(\mathbb{R})}.$$

We have therefore proved for  $t \in [0, T_0]$ ,

$$(1 + |D_v|^2)^{tN-1} f(t, \cdot) \in L^2_2(\mathbb{R}).$$

Since we can choose arbitrary  $N > 0$  and  $T_0 > 0$ , we have proved Theorem 3.1.

#### 4. GEVREY REGULARIZING EFFECT OF SOLUTIONS

Theorem 3.1 implies that the weak solution of the Cauchy problem (1.1) satisfies  $f \in L^\infty([t_0, T_0]; H^1_2(\mathbb{R}))$  for any  $t_0 > 0$ . Then  $f$  is a solution of the following Cauchy problem :

$$\begin{cases} \frac{\partial f}{\partial t} = K(f, f), & v \in \mathbb{R}, t > t_0, \\ f|_{t=t_0} = f(t_0, \cdot) \in H^1_2(\mathbb{R}). \end{cases}$$

We now study the local Gevrey regularizing effect of the Cauchy problem, and suppose that the initial datum is  $f_0 \in H^1_2 \cap L^1_2(\mathbb{R})$ . We state this result as the:

**Theorem 4.1.** *Assume that the initial datum  $f_0 \in H^1_2 \cap L^1_2(\mathbb{R})$ , and the cross-section  $\beta$  satisfy (1.2) with  $0 < s < \frac{1}{2}$ . For  $T_0 > 0$ , if  $f \in L^\infty([0, T_0]; H^1_2 \cap L^1_2(\mathbb{R}))$  is a nonnegative weak solution of the Cauchy problem (1.1), then for any  $0 < s' < s$ , there exists  $0 < T_* \leq T_0$  such that  $f(t, \cdot) \in G^{\frac{1}{2s'}}(\mathbb{R})$  for any  $0 < t \leq T_*$ . More precisely, there exists  $c_0 > 0$ ,*

$$e^{c_0 t (D_v)^{2s'}} f \in L^\infty([0, T_*]; L^1_1(\mathbb{R})).$$

**Remark 4.2.** *The above Gevrey smoothing effect property of Cauchy problem is for any weak solution  $f \in L^\infty([0, T_0]; H^1_2 \cap L^1_2(\mathbb{R}))$ , so that we don't need to use the uniqueness of solution for Kac's equation.*

We prove the above theorem by construction of *a priori* estimates for the mollified weak solution. Take  $f \in L^\infty([0, T_0]; H^1_2 \cap L^1_2(\mathbb{R}))$  to be a weak solution of the Cauchy problem (1.1), then (2.3) with  $m = \ell = 0$  implies that, (recall the assumption  $0 < s < 1/2$ )

$$K(f, f) \in L^\infty([0, T_0]; L^2(\mathbb{R}_v)).$$

So that we need to choose a test function  $\varphi \in C^1([0, T_0]; L^2(\mathbb{R}_v))$  to make sense

$$(K(f, f), \varphi)_{L^2(\mathbb{R}_v)}.$$

The right way is to choose a mollified weak solution  $f$ , we first have

$$\tilde{f}(t, \cdot) = \left( G_\delta(t, D_v) \langle v \rangle^2 G_\delta(t, D_v) f \right)(t, \cdot) \in L^\infty([0, T_0]; H^1(\mathbb{R})).$$

Here again we suppose that  $\tilde{f} \in C^1([0, T_0]; H^1(\mathbb{R}_v))$ , and study the equation of (1.1) in the following weak formulation

$$(4.1) \quad \left( \partial_t f(t, \cdot), \tilde{f}(t, \cdot) \right)_{L^2(\mathbb{R}_v)} = \left( K(f, f), \tilde{f} \right)_{L^2(\mathbb{R}_v)}.$$

First, the left hand side term is

$$\begin{aligned} \left( \partial_t f(t, \cdot), \tilde{f}(t, \cdot) \right)_{L^2(\mathbb{R}_v)} &= \frac{1}{2} \frac{d}{dt} \|G_\delta f(t)\|_{L^2_1(\mathbb{R}_v)}^2 \\ &\quad - \left( (\partial_t G_\delta)(t, D_v) f(t, \cdot), G_\delta(t, D_v) f(t, \cdot) \right)_{L^2(\mathbb{R}_v)} \\ &\quad - \left( v (\partial_t G_\delta)(t, D_v) f(t, \cdot), v G_\delta(t, D_v) f(t, \cdot) \right)_{L^2(\mathbb{R}_v)}. \end{aligned}$$

Then we estimate the two terms on right hand side by using the following lemma.

**Lemma 4.3.** *There exists  $C > 0$  such that*

$$(4.2) \quad \left| \left( (\partial_t G_\delta)(t, D_v) f(t, \cdot), G_\delta(t, D_v) f(t, \cdot) \right)_{L^2(\mathbb{R}_v)} \right| \leq C \|G_\delta f\|_{H^{s'}(\mathbb{R}_v)}^2,$$

and

$$(4.3) \quad \left| \left( v (\partial_t G_\delta)(t, D_v) f(t, \cdot), v G_\delta(t, D_v) f(t, \cdot) \right)_{L^2(\mathbb{R}_v)} \right| \leq C \|G_\delta f\|_{H^{s'}(\mathbb{R}_v)}^2.$$

*Proof.* (4.2) can be deduced directly from (2.6) by using the Plancherel formula.

For (4.3), we have

$$\begin{aligned} &\left| \left( v (\partial_t G_\delta)(t, D_v) f(t, \cdot), v G_\delta(t, D_v) f(t, \cdot) \right)_{L^2(\mathbb{R}_v)} \right| \\ &= C \left| \int_{\mathbb{R}} \left( \partial_\xi \left( c_0 \langle \xi \rangle^{2s'} G_\delta(t, \xi) \frac{1}{1 + \delta e^{c_0 t \langle \xi \rangle^{2s'}}} \hat{f}(t, \xi) \right) \overline{\mathcal{F}(v G_\delta f)(t, \xi)} d\xi \right| \\ &\leq C \int_{\mathbb{R}} \langle \xi \rangle^{2s'} \left| \partial_\xi \left( G_\delta(t, \xi) \hat{f}(t, \xi) \right) \right| \left| \mathcal{F}(v G_\delta f)(t, \xi) \right| d\xi \\ &\quad + C \int_{\mathbb{R}} \left| \partial_\xi \left( \langle \xi \rangle^{2s'} \frac{1}{1 + \delta e^{c_0 t \langle \xi \rangle^{2s'}}} \right) \right| \left| G_\delta(t, \xi) \hat{f}(t, \xi) \right| \left| \mathcal{F}(v G_\delta f)(t, \xi) \right| d\xi \\ &\leq C \|G_\delta f\|_{H^{s'}(\mathbb{R}_v)}^2, \end{aligned}$$

where we use the fact that

$$\left| \partial_\xi \left( \langle \xi \rangle^{2s'} \frac{1}{1 + \delta e^{c_0 t \langle \xi \rangle^{2s'}}} \right) \right| \leq C \langle \xi \rangle^{2s'}.$$

Hence Lemma 4.3 is proved □

Then (4.1) and Lemma 4.3 give

$$(4.4) \quad \frac{1}{2} \frac{d}{dt} \|G_\delta f(t)\|_{L^2_1(\mathbb{R}_v)}^2 - \left( K(f, f), \tilde{f} \right)_{L^2(\mathbb{R}_v)} \leq C \|G_\delta f\|_{H^{s'}(\mathbb{R}_v)}^2.$$

On the other hand, we have

$$\begin{aligned} \left( K(f, f), \tilde{f} \right)_{L^2(\mathbb{R}_v)} &= \left( G_\delta K(f, f), (1 + v^2) G_\delta f \right)_{L^2(\mathbb{R}_v)} \\ &= \left( K(f, G_\delta f), G_\delta f \right)_{L^2(\mathbb{R}_v)} + \left( K(f, v G_\delta f), v G_\delta f \right)_{L^2(\mathbb{R}_v)} \\ &\quad + \left( G_\delta K(f, f) - K(f, G_\delta f), G_\delta f \right)_{L^2(\mathbb{R}_v)} \\ &\quad + \left( v G_\delta K(f, f) - K(f, v G_\delta f), v G_\delta f \right)_{L^2(\mathbb{R}_v)}. \end{aligned}$$

Then Proposition 2.5 implies

$$\left| \left( G_\delta K(f, f) - K(f, G_\delta f), G_\delta f \right)_{L^2(\mathbb{R}_v)} \right| \leq C \|G_\delta f\|_{L^2_1(\mathbb{R}_v)} \|G_\delta f\|_{H^{s'}(\mathbb{R}_v)}^2$$

and

$$\begin{aligned} & \left| (v G_\delta K(f, f) - K(f, v G_\delta f), v G_\delta f) \right|_{L^2(\mathbb{R}_v)} \\ & \leq C \left( \|f\|_{L^1_1(\mathbb{R})} + \|G_\delta f\|_{L^2_1(\mathbb{R})} \right) \|G_\delta f\|_{H^s_1(\mathbb{R})}^2. \end{aligned}$$

The Proposition 2.1 implies

$$\begin{aligned} -\left(K(f, G_\delta f), G_\delta f\right) & \geq c_f \|G_\delta f\|_{H^s(\mathbb{R}_v)}^2 - C \|f\|_{L^1(\mathbb{R}_v)} \|G_\delta f\|_{L^2(\mathbb{R}_v)}^2, \\ -\left(K(f, v G_\delta f), v G_\delta f\right) & \geq c_f \|v G_\delta f\|_{H^s(\mathbb{R}_v)}^2 - C \|f\|_{L^1(\mathbb{R}_v)} \|v G_\delta f\|_{L^2(\mathbb{R}_v)}^2. \end{aligned}$$

Since

$$\|G_\delta f\|_{H^s_1(\mathbb{R}_v)}^2 \leq \|G_\delta f\|_{H^s(\mathbb{R}_v)}^2 + \|v G_\delta f\|_{H^s(\mathbb{R}_v)}^2 + C \|G_\delta f\|_{L^2_1(\mathbb{R}_v)}^2$$

By summing all the above estimates and (4.4), we obtain

$$\begin{aligned} (4.5) \quad & \frac{d}{dt} \|G_\delta f(t)\|_{L^2_1(\mathbb{R}_v)}^2 + c_{f(t)} \|G_\delta f(t)\|_{H^s_1(\mathbb{R}_v)}^2 \\ & \leq C \|G_\delta f(t)\|_{H^s_1(\mathbb{R}_v)}^2 + C \|f(t)\|_{L^1(\mathbb{R}_v)} \|G_\delta f(t)\|_{L^2_1(\mathbb{R}_v)}^2 \\ & \quad + C \left( \|f(t)\|_{L^1_1(\mathbb{R})} + \|G_\delta f(t)\|_{L^2_1(\mathbb{R})} \right) \|G_\delta f(t)\|_{H^s_1(\mathbb{R})}^2. \end{aligned}$$

#### End of proof of Theorem 4.1

By using (1.6) and (1.7), we have

$$\|f(t)\|_{L^1(\mathbb{R}_v)} + \|f(t)\|_{L^2_2(\mathbb{R}_v)} \leq C \|f_0\|_{L^2_2(\mathbb{R}_v)}, \quad c_{f(t)} \geq c_{f_0} > 0.$$

Then (4.5) yields

$$\begin{aligned} (4.6) \quad & \frac{d}{dt} \|G_\delta f(t)\|_{L^2_1(\mathbb{R}_v)}^2 + c_{f_0} \|G_\delta f(t)\|_{H^s_1(\mathbb{R}_v)}^2 \\ & \leq C_{f_0} \|G_\delta f(t)\|_{H^s_1(\mathbb{R}_v)}^2 + C \|G_\delta f(t)\|_{L^2_1(\mathbb{R})} \|G_\delta f(t)\|_{H^s_1(\mathbb{R})}^2. \end{aligned}$$

We now need the following interpolation inequality, for  $0 < s' < s$  and any  $\lambda > 0$ ,

$$(4.7) \quad \|u\|_{H^{s'}}^2 \leq \lambda \|u\|_{H^s}^2 + \lambda^{-\frac{s}{s-s'}} \|u\|_{L^2}^2.$$

Then for any small  $\varepsilon > 0$ ,

$$C_{f_0} \|G_\delta f(t)\|_{H^s_1(\mathbb{R}_v)}^2 \leq \varepsilon \|G_\delta f(t)\|_{H^s_1(\mathbb{R}_v)}^2 + C_{\varepsilon, f_0} \|G_\delta f(t)\|_{L^2_1(\mathbb{R}_v)}^2$$

and

$$C \|G_\delta f(t)\|_{L^2_1(\mathbb{R})} \|G_\delta f(t)\|_{H^s_1(\mathbb{R}_v)}^2 \leq \varepsilon \|G_\delta f(t)\|_{H^s_1(\mathbb{R}_v)}^2 + C_\varepsilon \|G_\delta f(t)\|_{L^2_1(\mathbb{R})}^{\frac{s}{s-s'}+2}.$$

We finally get from (4.6), that for any  $0 < \varepsilon$  and  $0 < s' < s$ , there exists  $C_\varepsilon > 0$  such that

$$\begin{aligned} & \frac{d}{dt} \|G_\delta f(t)\|_{L^2_1(\mathbb{R}_v)}^2 + (c_{f_0} - 2\varepsilon) \|G_\delta f(t)\|_{H^s_1(\mathbb{R}_v)}^2 \\ & \leq C_{\varepsilon, f_0} \|G_\delta f(t)\|_{L^2_1(\mathbb{R}_v)}^2 + C_\varepsilon \|G_\delta f(t)\|_{L^2_1(\mathbb{R})}^{\frac{s}{s-s'}+2}. \end{aligned}$$

We choose  $0 < 2\varepsilon \leq c_{f_0}$ , we get

$$(4.8) \quad \frac{d}{dt} \|G_\delta f(t)\|_{L^2_1(\mathbb{R}_v)} \leq C_1 \|G_\delta f(t)\|_{L^2_1(\mathbb{R}_v)} + C_2 \|G_\delta f(t)\|_{L^2_1(\mathbb{R})}^{\frac{s}{s-s'}+1}, \quad t \in [0, T_0],$$

with  $C_1, C_2 > 0$  and independent of  $\delta > 0$ . Then

$$\frac{d}{dt} \left( e^{-C_1 t} \|G_\delta f(t)\|_{L^2_1(\mathbb{R}_v)} \right) \leq C_2 e^{\tilde{C}_1 t} \left( e^{-C_1 t} \|G_\delta f(t)\|_{L^2_1(\mathbb{R}_v)} \right)^{\frac{s}{s-s'}+1}$$



where  $\widetilde{C}_1 = \frac{s' C_1}{s-s'}$ , thus for  $t \in ]0, T_0]$

$$\int_0^t \frac{d}{d\tau} \left( e^{-C_1 \tau} \|G_\delta f(\tau)\|_{L_1^2(\mathbb{R}_v)} \right)^{-\frac{s'}{s-s'}} d\tau \geq \frac{C_2}{C_1} (1 - e^{-\widetilde{C}_1 t}).$$

So that, for  $0 < \delta < 1$ ,

$$\|G_\delta f(t)\|_{L_1^2(\mathbb{R}_v)} \leq \frac{\widetilde{C}_1 e^{C_1 t} \|f_0\|_{L_1^2(\mathbb{R}_v)}}{\left( C_1 + C_2 (1 - e^{-\widetilde{C}_1 t}) \|f_0\|_{L_1^2(\mathbb{R}_v)}^{\frac{s'}{s-s'}} \right)^{\frac{s-s'}{s'}}}.$$

We now choose  $0 < T_* \leq T_0$  small enough so that

$$\left( C_1 + C_2 (1 - e^{-\widetilde{C}_1 t}) \|f_0\|_{L_1^2(\mathbb{R}_v)}^{\frac{s'}{s-s'}} \right)^{\frac{s-s'}{s'}} \geq C_3 > 0, \quad t \in [0, T_*],$$

then by compactness and by taking limit  $\delta \rightarrow 0$ , we have for  $t \in [0, T_*]$ ,

$$(4.9) \quad \|e^{c_0 t (D_v)^{2s'}} f\|_{L^\infty([0, T_*]; L_1^2(\mathbb{R}_v))}^2 \leq e^{C_1 T_*} \frac{\widetilde{C}_1}{C_3} \|f_0\|_{L_1^2(\mathbb{R}_v)}^2.$$

We therefore have proved Theorem 4.1.

## 5. RADIALLY SYMMETRIC BOLTZMANN EQUATIONS

We consider now the Boltzmann collision operators (1.10). In the Maxwellian case, the Bobylev's formula takes the form

$$(5.1) \quad \mathcal{F}(Q(g, f))(\xi) = \int_{\mathbb{S}^2} b \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \left\{ \hat{g}(\xi^-) \hat{f}(\xi^+) - \hat{g}(0) \hat{f}(\xi) \right\} d\sigma$$

where  $\xi \in \mathbb{R}^3$ ,

$$\xi^+ = \frac{\xi + |\xi| \sigma}{2}, \quad \xi^- = \frac{\xi - |\xi| \sigma}{2}.$$

On the other hand

$$|\xi^+|^2 = |\xi|^2 \frac{1 + \frac{\xi}{|\xi|} \cdot \sigma}{2}, \quad |\xi^-|^2 = |\xi|^2 \frac{1 - \frac{\xi}{|\xi|} \cdot \sigma}{2},$$

so that if we define  $\theta$  by

$$\cos \theta = \frac{\xi}{|\xi|} \cdot \sigma,$$

we obtain

$$|\xi^+|^2 = |\xi|^2 \cos^2 \left( \frac{\theta}{2} \right), \quad |\xi^-|^2 = |\xi|^2 \sin^2 \left( \frac{\theta}{2} \right).$$

We now consider the radially symmetric function with respect to  $v \in \mathbb{R}^3$ , namely the function satisfy the property

$$h(v) = h(Av), \quad v \in \mathbb{R}^3$$

for any proper orthogonal  $3 \times 3$  matrix  $A$ , then  $h(v) = h(0, 0, |v|)$ . Denote by  $\mathcal{F}_{\mathbb{R}^3}$  the Fourier transformation in  $\mathbb{R}^3$  and  $\mathcal{F}_{\mathbb{R}^1}$  the Fourier transformation in  $\mathbb{R}^1$ . Then  $\mathcal{F}_{\mathbb{R}^3}(h)(\xi)$  is also radially symmetric with respect to  $\xi \in \mathbb{R}^3$ , and it is in the form

$$\mathcal{F}_{\mathbb{R}^3}(h)(\xi) = \mathcal{F}_{\mathbb{R}^3}(h)(0, 0, |\xi|) = \int_{\mathbb{R}} e^{-i|\xi|v_3} \left( \int_{\mathbb{R}^2} h(v_1, v_2, v_3) dv_1 dv_2 \right) dv_3.$$

So that

$$(5.2) \quad \mathcal{F}_{\mathbb{R}^1}^{-1}(\mathcal{F}_{\mathbb{R}^3}(h)(0, 0, \cdot))(u) = \int_{\mathbb{R}^2} h(v_1, v_2, u) dv_1 dv_2$$

is an even function in  $\mathbb{R}$ , and we have

**Lemma 5.1.** *Assume that  $h \in L^1_k(\mathbb{R}^3)$ ,  $h \geq 0$  is a radially symmetric function for certain  $k \geq 0$ , and uniformly integrable in  $\mathbb{R}^3$ , then*

$$\mathcal{F}_{\mathbb{R}^1}^{-1}(\mathcal{F}_{\mathbb{R}^3}(h)(0, 0, \cdot)) \in L^1_k(\mathbb{R})$$

is a nonnegative even function, and uniformly integrable in  $\mathbb{R}$ .

*Proof.* By using (5.2), it is evident that  $h \in L^1_k(\mathbb{R}^3)$  implies  $\mathcal{F}_{\mathbb{R}^1}^{-1}(\mathcal{F}_{\mathbb{R}^3}(h)(0, 0, \cdot)) \in L^1_k(\mathbb{R})$ , and  $h \geq 0$  implies  $\mathcal{F}_{\mathbb{R}^1}^{-1}(\mathcal{F}_{\mathbb{R}^3}(h)(0, 0, \cdot)) \geq 0$ . Hence we need only to check the uniform integrability of  $\mathcal{F}_{\mathbb{R}^1}^{-1}(\mathcal{F}_{\mathbb{R}^3}(h)(0, 0, \cdot))$  in  $\mathbb{R}$ . Since  $h \in L^1(\mathbb{R}^3)$ , for any  $\varepsilon > 0$ , there exists  $R_0 > 0$  such that

$$\int_{\{v \in \mathbb{R}^3; |v| \geq R_0\}} |h(v_1, v_2, v_3)| dv_1 dv_2 dv_3 < \frac{\varepsilon}{2}.$$

The uniform integrability of  $h$  in  $\mathbb{R}^3$  imply that, there exists  $\delta_1 > 0$  such that

$$\int_B |h(v_1, v_2, v_3)| dv_1 dv_2 dv_3 < \frac{\varepsilon}{2},$$

for any  $B \subset \mathbb{R}^3$  with  $|B| \leq \delta_1$ . Choose new  $\delta_0 = \delta_1(R_0^2)^{-1}$ , then for any  $A \subset \mathbb{R}$ , if  $|A| \leq \delta_0$ , we have

$$\begin{aligned} \int_A |\mathcal{F}_{\mathbb{R}^1}^{-1}(\mathcal{F}_{\mathbb{R}^3}(h)(0, 0, \cdot))(u)| du &\leq \int_{\mathbb{R}^2 \times A} |h(v_1, v_2, v_3)| dv_1 dv_2 dv_3 \\ &\leq \int_{\{(v_1, v_2, v_3) \in \mathbb{R}^3; |v_1| \leq R_0, |v_2| \leq R_0, v_3 \in A\}} |h(v_1, v_2, v_3)| dv_1 dv_2 dv_3 + \frac{\varepsilon}{2} < \varepsilon, \end{aligned}$$

because of

$$|\{(v_1, v_2, v_3) \in \mathbb{R}^3; |v_1| \leq R_0, |v_2| \leq R_0, v_3 \in A\}| \leq R_0^2 |A| \leq \delta_1.$$

□

**Remark 5.2.** *In the proof of above Lemma, if  $h \in L \log L(\mathbb{R}^3)$  then  $h$  is uniformly integrable in  $\mathbb{R}^3$  with  $\delta_1$  depends only on  $\varepsilon, \|h\|_{L \log L(\mathbb{R}^3)}$  and  $\|h\|_{L^1(\mathbb{R}^3)}$ . Therefore,  $\mathcal{F}_{\mathbb{R}^1}^{-1}(\mathcal{F}_{\mathbb{R}^3}(h)(0, 0, \cdot))$  is uniformly integrable in  $\mathbb{R}^1$  with  $\delta_0$  also depends only on  $\varepsilon, \|h\|_{L \log L(\mathbb{R}^3)}$  and  $\|h\|_{L^1(\mathbb{R}^3)}$ .*

### End of proof of Theorem 1.3

Suppose now  $g \in L^\infty([0, +\infty[; L^1_{2+2s} \cap L \log L(\mathbb{R}^3))$  is a non negative radially symmetric weak solution of the Cauchy problem (1.9). Setting, for  $t \geq 0, u \in \mathbb{R}$ ,

$$(5.3) \quad f(t, u) = \mathcal{F}_{\mathbb{R}^1}^{-1}(\mathcal{F}_{\mathbb{R}^3}(g)(t, 0, \cdot))(u) = \int_{\mathbb{R}^2} g(t, v_1, v_2, u) dv_1 dv_2,$$

hereafter, the time variable  $t$  is always considered as parameters for the Fourier transformation, then  $f(t, u)$  is an even function with respect to  $u \in \mathbb{R}$ , and

$$\hat{f}(t, \tau) = \mathcal{F}_{\mathbb{R}^1}(f(t, \cdot))(\tau) = \mathcal{F}_{\mathbb{R}^3}(g)(t, 0, \tau).$$

So that the Bobylev's formula (5.1) give, for  $\xi \in \mathbb{R}^3$ ,

$$(5.4) \quad \mathcal{F}_{\mathbb{R}^3}(Q(g, g))(\xi) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(|\theta|) \{ \hat{f}(t, |\xi| \sin(\theta/2)) \hat{f}(t, |\xi| \cos(\theta/2)) - \hat{f}(t, 0) \hat{f}(t, |\xi|) \} d\theta$$

where

$$\beta(|\theta|) = \frac{1}{2} |\sin \theta| b(\cos \theta).$$

Then the right hand side of (5.4) is Fourier transformation of Kac's operator  $K(f, f)$ . We have proved that if  $g(t, v)$  is a non negative radially symmetric weak solution of the Cauchy problem (1.9), then  $f(t, u)$  is a weak solution of the Cauchy problem of Kac's equation :

$$(5.5) \quad \begin{cases} \frac{\partial f}{\partial t}(t, u) = K(f, f)(t, u), \\ f(0, u) = f_0(u) = \int_{\mathbb{R}^2} g_0(v_1, v_2, u) dv_1 dv_2, \end{cases}$$

or equivalently in the Fourier variable:

$$\begin{cases} \frac{\partial \hat{f}}{\partial t}(t, \tau) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(|\theta|) \{ \hat{f}(t, \tau \sin(\theta/2)) \hat{f}(t, \tau \cos(\theta/2)) - \hat{f}(t, 0) \hat{f}(t, \tau) \} d\theta, \\ \hat{f}(0, \tau) = \hat{f}_0(\tau) = \hat{g}_0(0, 0, \tau). \end{cases}$$

Under the assumption of Theorem 1.3 for  $g(t, v)$ , Lemma 5.1 and Remark 5.2 implies that  $f(t, u)$  satisfy the hypothesis of Theorem 1.1 except  $f$  belong to  $L \log L$  which substituted by the uniform integrability of  $f = f_i(\cdot)$  in  $\mathbb{R}$ . As it is point out in the Remark 2.2, this property is enough to assure the coercivity (2.1). Then we apply Theorem 1.1 to the Cauchy problem (5.5), thus there exists  $T_* > 0$  such that for  $0 < t \leq T_*$ ,

$$e^{c_0 t \langle |\tau| \rangle^{2s'}} \hat{f}(t, \tau) = e^{c_0 t \langle |\tau| \rangle^{2s'}} \mathcal{F}_{\mathbb{R}^3}(g)(t, 0, 0, \tau) \in H^1(\mathbb{R}_\tau).$$

It remain to prove the Gevrey smoothing effect in the global time interval. Kac's equation shares with the homogeneous Boltzmann equation for Maxwellian molecules the existence and uniqueness theory for the Cauchy problem, see [15] for the uniqueness of weak solution for the non-cut-off Boltzmann equation. We take  $0 < t_0 < t_1 \leq T_*$ , and consider the Cauchy problem (5.5) with even initial datum  $\hat{f}(t_1, \tau)$ . The Sobolev embedding  $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$  imply that

$$\begin{aligned} \|e^{c_0 t_1 \langle \cdot \rangle^{2s'}} \hat{f}(t_1, \cdot)\|_{L^\infty(\mathbb{R})} &\leq C \|e^{c_0 t_1 \langle \cdot \rangle^{2s'}} \hat{f}(t_1, \cdot)\|_{H^1(\mathbb{R})} \\ &\leq C \|e^{c_0 t_1 \langle |D_u| \rangle^{2s'}} f(t_1, \cdot)\|_{L^2_1(\mathbb{R})} < +\infty \end{aligned}$$

Now the following propagation of Gevrey regularity results deduces the Gevrey smoothing effect in the global time interval.

**Theorem 5.3. (Theorem 2.3 of [8])**

Let  $f_0$  be a non negative, even function, satisfying

$$\sup_{\xi \in \mathbb{R}} (|\hat{f}_0(\xi)| e^{c_1 \langle \xi \rangle^{2s}}) < +\infty,$$

for some  $c_1 > 0$  and the cross-section  $\beta$  satisfying (1.2) with  $0 < s < 1$ . Then the solution of the Cauchy problem (1.1) satisfies  $f(t, \cdot) \in G^{\frac{1}{2s}}(\mathbb{R})$  for any  $t \geq 0$ .

In conclusion, if  $g \in L^\infty(]0, +\infty[; L^1_{2+2s} \cap L \log L(\mathbb{R}^3))$  is a non negative radially symmetric weak solution of the Cauchy problem (1.9), then under the assumption of Theorem 1.3, we have proved that for any fixed  $0 < t < +\infty$ , there exists  $c_0 > 0$  such that

$$e^{c_0 \langle |D_u| \rangle^{2s'}} f(t, \cdot) \in L^2(\mathbb{R})$$

where  $f$  is the function defined by (5.3). We can finish now the proof by the following estimations, for fixed  $t > 0$ ,

$$\begin{aligned}
\|e^{\frac{c_0}{2}\langle D_v \rangle^{2s'}} g(t, \cdot)\|_{L^2(\mathbb{R}^3)}^2 &= \int_{\mathbb{R}^3} \left| e^{\frac{c_0}{2}\langle |\xi| \rangle^{2s'}} \mathcal{F}_{\mathbb{R}^3}(g)(t, \xi_1, \xi_2, \xi_3) \right|^2 d\xi \\
&= \int_{\mathbb{R}^3} \left| e^{\frac{c_0}{2}\langle |\xi| \rangle^{2s'}} \mathcal{F}_{\mathbb{R}^3}(g)(t, 0, 0, |\xi|) \right|^2 d\xi \\
&= C \int_0^\infty \left| e^{\frac{c_0}{2}\langle \tau \rangle^{2s'}} \hat{f}(t, \tau) \right|^2 \tau^2 d\tau \\
&\leq C \int_0^\infty \left| e^{c_0\langle \tau \rangle^{2s'}} \hat{f}(t, \tau) \right|^2 d\tau \\
&\leq C \|e^{c_0\langle D_u \rangle^{2s'}} f(t, \cdot)\|_{L^2(\mathbb{R})}^2 < +\infty.
\end{aligned}$$

We finished the proof of Theorem 1.3.

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