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# Low-frequency scattering from perfectly conducting spheroidal bodies in a conductive medium with magnetic dipole excitation

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Inductive electromagnetic means that are currently employed in the exploration of the Earth's subsurface and embedded voluminous bodies often call for an intensive use, primary at the modeling stage and later on at the inversion stage, of analytically demanding tools of field calculation. Under the aim of modeling implementation, this contribution is concerned with some interesting aspects of the low-frequency interaction of arbitrarily oriented (i.e. three-dimensional) time-harmonic magnetic dipoles, with 3-D perfectly conducting spheroidal bodies embedded in an otherwise homogeneous conductive medium. For many practical applications involving buried obstacles such as Earth's subsurface electromagnetic probing at low-frequency or any other physical cases (e.g. geoelectromagnetics), non-axisymmetric spheroidal geometry approximates sufficiently such kind of metallic shapes. On the other hand, our analytical approach deals with prolate spheroids, since the corresponding results for the oblate spheroidal geometry can be readily obtained through a simple transformation. The particular physical model concerns a solid impenetrable (metallic) body under a magnetic dipole excitation, where the scattering boundary value problem is attacked via rigorous low-frequency expansions for the incident, scattered and total electric and magnetic fields in terms of positive integral powers of  $(ik)$ , that is  $(ik)^n$  for  $n \geq 0$ , where  $k$  stands for the complex wavenumber of the exterior medium. The purpose of the modeling is to contribute to a simple yet versatile tool to infer information on an unknown body from measurements of the three-component electric and magnetic fields nearby. Our goal is to obtain the most important terms of the low-frequency expansions of the electromagnetic fields, that is the static (for  $n = 0$ ) and the dynamic ( $n = 1, 2, 3$ ) terms. In particular, for  $n = 1$  there are no incident fields and thus no scattered ones, while for  $n = 0$  the Rayleigh electromagnetic expression is easily obtained in terms of infinite series. Emphasis is given on the calculation of the next two non-trivial terms (at  $n = 2$  and at  $n = 3$ ) of the aforementioned fields. Consequently, those are found in closed form from exact solutions of coupled (at  $n = 2$ , to the one at  $n = 0$ ) or uncoupled (at  $n = 3$ ) Laplace equations and they are given in compact fashion, as infinite series expansions for  $n = 2$  or finite forms for  $n = 3$ . Nevertheless, the difficulty of the Poisson's equation that has to be solved for  $n = 2$  is presented, whereas our analytical approach demands the use of the well-known cut-off method in order to obtain an analytical closed solution. Finally, this research adds useful reference results to the already ample library of scattering by simple shapes using analytical methods.

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## 1. Introduction

The main field of application of the present research, especially in down-hole mineral exploration using inductive electromagnetic means, is Earth's subsurface electromagnetic probing, where we are often faced with the problem of retrieving an anomaly of certain kind, specified as metallic ores, from three-dimensional magnetic fields, which are measured along a borehole when a low-frequency [1] time-harmonic source is placed nearby, usually fixed at the surface of the Earth. By deciphering such fields, interesting information concerning main parameters such as orientations, sizes, shapes, magnetic and electric properties, as well as distribution of conductivity of the metallic bodies, are identified in order to bring insight to the field behavior. However, this is not an easy task, since the inverse problem cannot be tackled in robust fashion unless proper models of the field interaction are available.

Nowadays, there are many computational codes and corresponding inversion algorithms available to that matter as illustrated early on in [2], which approach simply enough the three-dimensional field interaction and the inversion problem, while several complementary investigations are presented in [3]. Nevertheless, simple methodologies and much material on exploration by electromagnetic means and models thereof, which are described in the classical book [4], remain useful, while a recent and detailed contribution by Ref. [5] illustrates this type of approach. In addition, under the low-frequency hypothesis [1], a close link with the present investigation is offered by three important Refs. [6–8], where low-contrast cases (it is assumed in this case that the body is a non-metallic obstacle of finite conductivity) based on the well-known Green's function [9,10], were approached by hybrid means from integral formulations of the fields conveniently approximated [7] and from expansions of the Green's formula or of the resulting fields [6,8]. More specific, Ref. [7] provides us with interesting results, which are concerned with a penetrable sphere illuminated by a dipolar magnetic field within a conductive host medium via a Mie series expansion, while in [6] and [8] the problem deals with the general case of ellipsoidal [11] penetrable obstacles and it is faced via expansions of the Green's function or of the electromagnetic fields inside and outside the body, within the framework of the localized non-linear approximation. On the other hand, for high-contrast cases the body is considered to be metallic, hence impenetrable, whereas in view of this aspect, a recent work [12] for perfectly conducting ellipsoids is successfully dealt with the first term (the static one case) of the low-frequency expansion. In [4] a low-frequency expansion is found by the manipulation of Debye potentials, but the approach does not accommodate for a perfectly conducting sphere via an asymptotic analysis. Moreover, in addition to the interest for practical applications, like mining exploration as in our case, or so-called unexploded ordinance (UXO) investigations [13] or like the exploration of natural structures such as water-filled cavities and other possibly conductive materials in subsoil at shallow depths, one can also find useful results to the already ample library of scattering by simple shapes (e.g. spheroids) using analytical methods in Refs. [14,15]. Here we must point out a recent numerical work of electromagnetic imaging of a three-dimensional perfectly conducting object using a boundary integral formulation [16]. Within this frame of numerical approaches we also refer to two purely numerical papers, one dealing with the characterization of spheroidal metallic objects using electromagnetic induction [17] and the other one dealing with a spheroidal-mode approach for unexploded ordinance (UXO) inversion under time-harmonic excitations in the magneto-quasistatic regime [18], wherein both of them important inversion algorithms are developed. However, these methods, as well as those mentioned in the beginning of our reference sequence, involve the use of elaborate computer codes for each case under consideration. There is always room and need for analytical methods, which capture (albeit only roughly) the essential features of the electromagnetic process under consideration in an analytical formula, which contains the appropriate geometrical and physical dimensionless groups. Even the low-frequency analytical expansions in the spherical case were not available until three years ago when the low-frequency solution for a perfectly conducting sphere in a conductive medium with dipolar excitation was published [19] or when a mathematical formulation concerning the electromagnetic induction response to spheroidal anomalies embedded in a weakly conducting host medium was introduced [20]. However, this work [20] covers only the case of constant incident magnetic and electric fields, whereas even though the scattered fields are given in a compact closed fashion, the corresponding part of this work, which concerns the perfectly conducting spheroidal bodies case, provides us only with a small part of the scattered field. Nevertheless, for their application this part is adequate and their results are recovered from our generalized analytical method (for three-dimensional and any – not necessary constant – incident electromagnetic fields) as special cases in a separate application paragraph.

Here, the goal is to obtain a versatile set of mathematical and numerical tools in order to infer information on an unknown subsurface body, which it scatters off when illuminated by a known primary source, let us say a time-harmonic magnetic dipole, operated at low-frequency on the surface of the Earth in an arbitrary direction and producing three-dimensional incident waves. In order to put those tools together, we work within the framework of the well-known low-frequency diffusive scattering theory [1], i.e., by expanding the electromagnetic fields (incident, as well as scattered) in an Rayleigh-like manner of positive integral powers of  $(ik)$ ,  $k$  being the complex wavenumber of the exterior medium at the operation frequency, and by appropriately calculating the 3-D vector fields at each  $n$  of the expansion  $(ik)^n$  for  $n \geq 0$ . Our aim is to construct a simple yet robust model of the fields to identify the anomaly when magnetic fields are collected nearby. High-contrast cases, for which the ratio between the conductivity of the body and the one of the embedding homogeneous medium is high, are approximated via the introduction of bounded perfectly impenetrable (high conductive) bodies, and the scattering problem is transformed into a succession, one at each  $n$ , of possibly coupled boundary value problems formulated according to second-order Laplace's partial differential equations with proper perfectly reflecting boundary conditions, resulting (after several cumbersome but rigorous analytical calculations) in closed-form solutions and providing with the proper analytical

tools for solving the forward and the inverse problems. Consequently, here one pursues the mathematical analysis in the infinitely conducting case, which can be illustrated in realistic situations of mineral exploration by considering simple shapes for the buried objects.

Only simplified shapes can effectively be retrieved for analytical work at low frequencies. For obvious reasons most of the literature concern spherical obstacles, a fact that does not hold true for the case of general buried bodies. But even if the obstacles are spherical, deposition of impurities or other kind of substances will establish a preferable direction, which affects the geometry of the spherical obstacles. Despite the fact that the most general consideration and the real challenge concern the ellipsoidal approximation, the simplest and important case is the spheroid [11]. Thus, we shall limit ourselves to the non-axisymmetric spheroidal case in order to obtain the most important terms of the low-frequency expansions of the electromagnetic fields, that is, the static (for  $n = 0$ ) and the dynamic ( $n = 1, 2, 3$ ) terms in the three-dimensional space of the conductive medium using the appropriate boundary conditions on the surface of the body, i.e. cancellation of the normal component of the total magnetic field and cancellation of the two tangential components of the total electric field. In particular, for  $n = 1$  there are no incident fields as shown from the corresponding low-frequency electric and magnetic expansions. Thus, we limit ourselves to the spheroid case for  $n = 0, 2, 3$ , where for  $n = 0$  the Rayleigh electromagnetic term is obtained with straightforward calculations, while most emphasis is given on the calculation of the next two non-trivial terms (at  $n = 2$  and at  $n = 3$ ) of the electromagnetic fields. Those are found in closed form from exact solutions of coupled at  $n = 2$ , to the one at  $n = 0$  (leading to a kind of Poisson's equation) or uncoupled at  $n = 3$  Laplace equations. Hence, our results are given in a three-dimensional compact spheroidal fashion, as infinite series expansions ( $n = 0, 2$ ) and finite forms ( $n = 3$ ).

The prolate spheroidal shape seems to occur more frequently than the oblate one in practical applications. Therefore, we restrict our attention to prolate spheroids, since the results for the oblate spheroid can be obtained through a well-known transformation [11], where the geometrical as well as the analytical correspondence is established.

The rest of the paper itself is organized as follows. In Section 2, the theoretical basis via an analytic mathematical formulation is sketched, while in Section 3 some interesting information concerning the spheroidal (prolate and oblate) harmonic eigenfunctions [10] is provided. In addition, this section provides us with the necessary analytical tools for the geometrical reduction of the prolate spheroidal system to the spherical one. In Section 4, the main results of the forward problem of the calculation of the electromagnetic fields are displayed and discussed, while the mathematical degeneracy of our results (in view of Section 3) to the corresponding and already known expressions for the sphere-problem [19] is demonstrated. Moreover, in order to supplement this paper with an analytical application of our generalized method, in addition to the aforementioned reduction of our results to the corresponding spherical ones [19], we invoke Section 5, which is concerned with a simple application of a part of our general spheroidal results handled in such a manner so as to recover the already known, but simplified expressions derived in the recent Ref. [20]. An outline of our work and future steps follows in Section 6. Finally, an appendix is devoted to some useful material in order to make this work complete and independent.

## 2. Mathematical development

Proceeding to the analysis, the particular physics concerns a solid impenetrable body of surface  $S$  under a magnetic dipole excitation. More specifically, we are focused onto the case of a 3-D bounded, highly conductive body considered in the sequel (Section 3) as a metallic prolate spheroidal obstacle, which is buried in a less conductive, homogeneous, linear and isotropic, nonmagnetic, medium with conductivity  $\sigma^+$  and permeability  $\mu^+$  (which is in fact  $\mu_0$ ), where in terms of the imaginary unit  $i$

$$k^+ \equiv k = \sqrt{i\omega\mu^+\sigma^+} = \sqrt{\frac{\omega\mu^+\sigma^+}{2}}(1 + i) \quad (1)$$

stands for the wavenumber, at low circular frequency  $\omega$  and for permittivity  $\varepsilon^+ \ll \sigma^+/\omega$ , the time-dependence  $\exp(-i\omega t)$  of all field quantities being implied from now on. In what follows we shall refer to smooth, bounded or not bounded, three-dimensional domains  $V^+(\mathbb{R}^3)$  of electromagnetic action in the external space, where every field or property will be generally written in terms of the position vector  $\mathbf{r} = x_1\hat{\mathbf{x}}_1 + x_2\hat{\mathbf{x}}_2 + x_3\hat{\mathbf{x}}_3$  expressed via the Cartesian basis  $\hat{\mathbf{x}}_\kappa$ ,  $\kappa = 1, 2, 3$  in Cartesian coordinates  $(x_1, x_2, x_3)$ . This dependence in every symbol will be omitted for convenience in writing. Here, one is dealing with a single voluminous anomaly, described as an impenetrable three-dimensional spheroid of arbitrary position and orientation with respect to the surface of the Earth. This body is illuminated by a low-frequency localized vector source, which is modeled with little restriction as a magnetic dipole  $\mathbf{m}$  with arbitrary location at  $\mathbf{r}_0$  and arbitrary orientation, meaning  $\mathbf{m} = \sum_{\kappa=1}^3 m_\kappa \hat{\mathbf{x}}_\kappa$ . The electromagnetic incident fields  $\mathbf{H}^i, \mathbf{E}^i$  are scattered by the solid body, creating the corresponding scattered fields  $\mathbf{H}^s, \mathbf{E}^s$  whose magnetic components are sought, while the total magnetic and electric fields  $\mathbf{H}^t, \mathbf{E}^t$ , which are collected along a single line passing outside the body, are given by the summation of both of the incident and scattered fields, i.e.

$$\mathbf{H}^t = \mathbf{H}^i + \mathbf{H}^s \quad \text{and} \quad \mathbf{E}^t = \mathbf{E}^i + \mathbf{E}^s, \quad \mathbf{r} \in V^+(\mathbb{R}^3) - \{\mathbf{r}_0\}. \quad (2)$$

The boundary value problem is attacked via low-frequency expansions [1] in terms of integral powers of  $(ik)$ , that is

$$\mathbf{H}^x = \sum_{n=0}^{\infty} \mathbf{H}_n^x (ik)^n \quad \text{and} \quad \mathbf{E}^x = \sum_{n=0}^{\infty} \mathbf{E}_n^x (ik)^n, \quad x = i, s, t \quad (3)$$

for all the electromagnetic fields, incident ( $i$ ), scattered ( $s$ ) and total ( $t$ ). As a consequence of the low-frequency assumption and in terms of the gradient operator  $\nabla$  operated at  $\mathbf{r}$  or  $\mathbf{r}_0$ , whenever the case may be (here for reasons of writing convenience we shall define as  $\nabla \equiv \nabla_{\mathbf{r}}$  and similarly for the Laplace operator  $\Delta \equiv \Delta_{\mathbf{r}}$ , otherwise it will be designated), the rigorous and well-known Maxwell's equations [1], i.e.  $\nabla \times \mathbf{E}^x = i\omega\mu^+ \mathbf{H}^x$  and  $\nabla \times \mathbf{H}^x = (-i\omega\varepsilon^+ + \sigma^+) \mathbf{E}^x$   $\stackrel{k^+ \ll \sigma^+/\omega}{\approx} \sigma^+ \mathbf{E}^x$ , which connect the magnetic and the electric fields, are reduced as follows (with the aid of Eq. (1))

$$\sigma^+ \nabla \times \mathbf{E}_n^x = -\mathbf{H}_{n-2}^x, \quad n \geq 2 \quad \text{and} \quad \nabla \times \mathbf{H}_n^x = \sigma^+ \mathbf{E}_n^x, \quad n \geq 0, \quad (4)$$

whereas  $\nabla \cdot \mathbf{H}_n^x = \nabla \cdot \mathbf{E}_n^x = 0$ ,  $n \geq 0$  in  $\mathbf{r} \in V^+(\mathbb{R}^3) - \{\mathbf{r}_0\}$  for  $x = i, s, t$ . Thus, we come up with several first-order differential relationships for the magnetic and electric fields, which are divergence free.

In view of the definitions  $\mathbf{R} = \mathbf{r} - \mathbf{r}_0$  and  $R = |\mathbf{r} - \mathbf{r}_0|$ , the electromagnetic incident fields generated by the magnetic dipole  $\mathbf{m}$  assume the forms [1]

$$\mathbf{H}^i = \frac{1}{4\pi} \left[ \left( k^2 + \frac{ik}{R} - \frac{1}{R^2} \right) \mathbf{m} - \left( k^2 + \frac{3ik}{R} - \frac{3}{R^2} \right) \frac{\mathbf{R} \otimes \mathbf{R} \cdot \mathbf{m}}{R^2} \right] \frac{e^{ikR}}{R} \quad (5)$$

and

$$\mathbf{E}^i = \left[ \frac{\omega\mu^+ k}{4\pi} \left( 1 + \frac{ik}{R} \right) \frac{\mathbf{m} \times \mathbf{R}}{R} \right] \frac{e^{ikR}}{R}, \quad (6)$$

whereas the symbol “ $\otimes$ ” denotes juxtaposition. Some extended algebraic calculations on the incident fields (5) and (6) produced by the magnetic dipole, which are based on the Taylor's expansion of the exponential functions and by virtue of Eq. (1), yield low-frequency relations as powers of  $(ik)$  for the incident fields, where it is proved that the first four powers of the fields, that is the static term for  $n = 0$  and the dynamic terms for  $n = 1, 2, 3$ , are sufficient enough to describe the problem. In detail, the primary (incident) fields enjoy the expressions

$$\mathbf{H}^i = \left[ \mathbf{H}_0^i + \mathbf{H}_2^i (ik)^2 + \mathbf{H}_3^i (ik)^3 \right] + O((ik)^4) \quad (7)$$

and

$$\mathbf{E}^i = \left[ \mathbf{E}_2^i (ik)^2 \right] + O((ik)^4), \quad (8)$$

where in terms of the gradient differential operator  $\nabla \equiv \nabla_{\mathbf{r}}$  (operated at  $\mathbf{r}$ ), we obtain the non-trivial terms

$$\mathbf{H}_0^i = \frac{\mathbf{m}}{4\pi} \cdot \left( \frac{3\mathbf{R} \otimes \mathbf{R}}{R^2} - \tilde{\mathbf{I}} \right) \frac{1}{R^3} = \frac{\mathbf{m}}{4\pi} \cdot \left( \nabla \otimes \nabla \frac{1}{R} \right), \quad (9)$$

$$\mathbf{H}_2^i = -\frac{\mathbf{m}}{4\pi} \cdot \left( \frac{\mathbf{R} \otimes \mathbf{R}}{R^2} + \tilde{\mathbf{I}} \right) \frac{1}{2R} = \frac{\mathbf{m}}{4\pi} \cdot \frac{1}{2} \left( \nabla \frac{1}{R} \otimes \mathbf{R} - \tilde{\mathbf{I}} \right), \quad (10)$$

$$\mathbf{H}_3^i = \frac{\mathbf{m}}{4\pi} \cdot \left( -\frac{2}{3} \tilde{\mathbf{I}} \right), \quad (11)$$

for the magnetic incident fields, while

$$\mathbf{E}_2^i = -\frac{\mathbf{m}}{4\pi\sigma^+} \times \frac{\mathbf{R}}{R^3} = \frac{\mathbf{m}}{4\pi\sigma^+} \times \nabla \frac{1}{R} \quad (12)$$

for the electric incident field, where  $\tilde{\mathbf{I}}$  stands for the unit dyadic. In view of the result  $\nabla \otimes \mathbf{r} = \tilde{\mathbf{I}}$  and identity (A.1), we utilized two basic relations in order to derive the second forms of expressions (9), (10) and (12), those being

$$\nabla_{\mathbf{r}} \frac{1}{R} \equiv \nabla \frac{1}{R} = -\frac{\mathbf{R}}{R^3} \quad \text{and} \quad \nabla_{\mathbf{r}} \frac{1}{R} \equiv \nabla \frac{1}{R} = -\nabla_{\mathbf{r}_0} \frac{1}{R}. \quad (13)$$

Let us notice that magnetic terms of order  $n$  vary like  $1/R^{3-n}$  and the electric ones vary like  $1/R^{4-n}$  when range  $R$  increases to infinity. We also observe that for the incident magnetic field the dynamic term for  $n = 1$  is not present, while for the incident electric field the only term that survives is the dynamic term for  $n = 2$ , reflecting exactly the same physical and mathematical treatment to the scattered fields. Hence, we are forced to calculate the corresponding non-trivial scattered fields by solving four mixed Maxwell's type problems for each  $n = 0, 2, 3$ .

Straightforward calculations on Maxwell's Eq. (4) for  $x = s$  and elaborate use of the identity (A.8), result in a set of boundary value problems possibly coupled to one another, from the static one at  $n = 0$  to dynamic ones at higher values of  $n$  up to  $n = 3$ , which are given in terms of the harmonic potentials  $\Phi_0^s$ ,  $\Phi_2^s$  and  $\Phi_3^s$  via

$$\Delta \mathbf{H}_0^s = \mathbf{0} \Rightarrow \mathbf{H}_0^s = \nabla \Phi_0^s \quad (\nabla \cdot \mathbf{H}_0^s = 0, \nabla \times \mathbf{H}_0^s = \mathbf{0}), \quad (14)$$

$$\Delta \mathbf{H}_2^s = \mathbf{H}_0^s \Rightarrow \mathbf{H}_2^s = \Phi_2^s + \frac{1}{2}(\mathbf{r}\Phi_0^s) \quad \text{and} \quad \sigma^+ \mathbf{E}_2^s = \nabla \times \mathbf{H}_2^s \quad (\nabla \cdot \mathbf{H}_2^s = \nabla \cdot \mathbf{E}_2^s = 0), \quad (15)$$

$$\Delta \mathbf{H}_3^s = \mathbf{0} \Rightarrow \mathbf{H}_3^s = \nabla \Phi_3^s \quad (\nabla \cdot \mathbf{H}_3^s = 0, \nabla \times \mathbf{H}_3^s = \mathbf{0}), \quad (16)$$

where the fields  $\mathbf{H}_0^s$ ,  $\mathbf{H}_2^s$ ,  $\mathbf{E}_2^s$  and  $\mathbf{H}_3^s$  are to be calculated, while  $\mathbf{H}_1^s = \mathbf{E}_0^s = \mathbf{E}_1^s = \mathbf{E}_3^s = \mathbf{0}$  as a direct consequence of the type of the incident fields. Note that  $\Delta \Phi_0^s = \Delta \Phi_3^s = 0$ , while  $\Delta \Phi_2^s = \mathbf{0}$ . Here, we must point out that the inhomogeneous vector Laplace Eq. (15), coupled with the solution of (14), is solved by writing down the second-order scattered magnetic field as the summation of a general vector harmonic function  $\Phi_2^s$  and of a particular solution  $\frac{1}{2}(\mathbf{r}\Phi_0^s)$ , whose form is an immediate result of the use of identity (A.9). This set of problems must be solved with the proper boundary conditions being applied on the body surface  $S$  for the total fields at each order  $n$ , that is, in view of the outward unit normal vector  $\hat{\mathbf{n}}$ , cancellation of the normal component of the total magnetic field ( $\hat{\mathbf{n}} \cdot \mathbf{H}^t = 0$ ) and of the tangential component of the electric field ( $\hat{\mathbf{n}} \times \mathbf{E}^t = \mathbf{0}$ ), which in combination with Eqs. (2) and (3), yields

$$\hat{\mathbf{n}} \cdot (\mathbf{H}_n^i + \mathbf{H}_n^s) = 0, \quad n = 0, 2, 3 \quad \text{and} \quad \hat{\mathbf{n}} \times (\mathbf{E}_2^i + \mathbf{E}_2^s) = \mathbf{0} \quad \text{for} \quad \mathbf{r} \in S. \quad (17)$$

Hence, the boundary value problems that have to be solved for the magnetic field are the following ones: the static one ( $n = 0$ ), reduced to a potential problem with Neumann boundary condition, at  $n = 2$ , where the problem is far more complicated due to the coupling of the static term and the solution is provided by the summation of a general and a particular one, and at  $n = 3$ , where we arrive again at a potential problem with Neumann boundary condition. Here, we mention that the scattered electric field for  $n = 2$  is given by the *curl* of the corresponding magnetic field as shown from Eq. (15). At this stage we are ready to apply the particular prolate and oblate spheroidal geometry of our boundary value problem.

### 3. Spheroidal harmonic eigenfunctions and eigenexpansions

Let us now specialize our previous analysis to the spheroidal case, by considering in the beginning a three-dimensional prolate spheroidal obstacle, which is buried and must be identified. Given a fixed positive number  $c > 0$ , which we consider to be the semifocal distance of our system, we define the transformed prolate spheroidal coordinates  $(\tau, \zeta, \varphi)$ ,  $1 \leq \tau < +\infty$ ,  $-1 \leq \zeta \leq 1$  and  $0 \leq \varphi < 2\pi$  (see Fig. 1), which are given by the following relations (notice that  $\mathbf{r} = (x_1, x_2, x_3)$  and  $\mathbf{r}_0 = (x_{10}, x_{20}, x_{30})$ )

$$x_1 = c\tau\zeta, \quad x_2 = c\sqrt{\tau^2 - 1}\sqrt{1 - \zeta^2} \cos \varphi, \quad x_3 = c\sqrt{\tau^2 - 1}\sqrt{1 - \zeta^2} \sin \varphi, \quad (18)$$

while the outward unit normal vector on the surface  $S$  of the particular spheroid  $\tau = \tau_s = a_1/c$  (with main axis  $a_1 = c\tau_s$ ,  $a_2 = a_3 = c\sqrt{\tau_s^2 - 1}$  and eccentricity  $e = 1/\tau_s$ ) is furnished by the Cartesian-basis formula

$$\hat{\mathbf{n}} \equiv \hat{\boldsymbol{\tau}} = \frac{1}{\sqrt{\tau^2 - \zeta^2}} \left( \zeta\sqrt{\tau^2 - 1}\hat{\mathbf{x}}_1 + \tau\sqrt{1 - \zeta^2} \cos \varphi \hat{\mathbf{x}}_2 + \tau\sqrt{1 - \zeta^2} \sin \varphi \hat{\mathbf{x}}_3 \right) \quad \text{for} \quad \tau = \tau_s \quad (19)$$

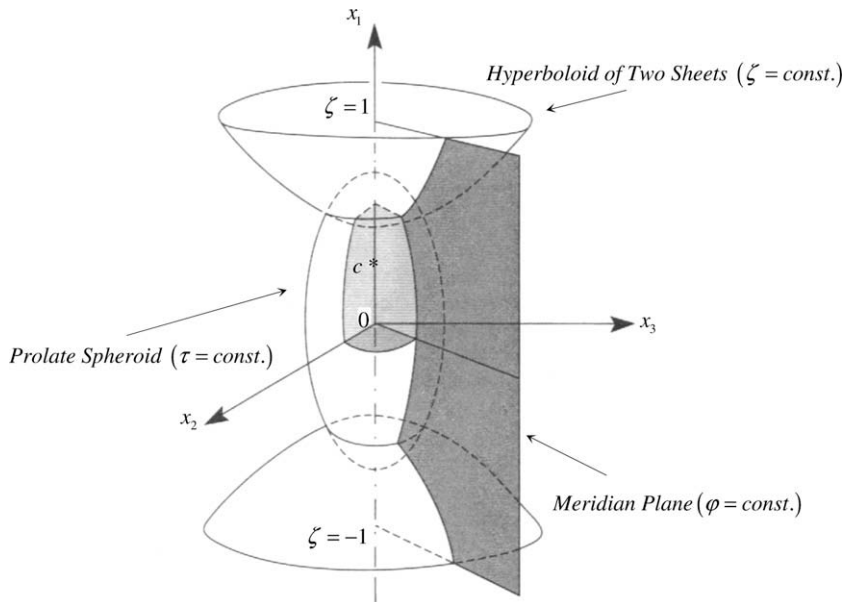


Fig. 1. The system of prolate spheroidal coordinates.

and the differential operator of gradient  $\nabla$  and Laplace  $\Delta$ , in this geometry, assumes the form

$$\nabla_{\mathbf{r}} \equiv \nabla = \frac{1}{c\sqrt{\tau^2 - \zeta^2}} \left[ \sqrt{\tau^2 - 1} \hat{\tau} \frac{\partial}{\partial \tau} - \sqrt{1 - \zeta^2} \hat{\zeta} \frac{\partial}{\partial \zeta} \right] + \frac{1}{c\sqrt{\tau^2 - 1}\sqrt{1 - \zeta^2}} \hat{\phi} \frac{\partial}{\partial \varphi} \quad (20)$$

and

$$\Delta_{\mathbf{r}} \equiv \Delta = \frac{1}{c^2(\tau^2 - \zeta^2)} \left\{ \frac{\partial}{\partial \tau} \left[ (\tau^2 - 1) \frac{\partial}{\partial \tau} \right] + \frac{\partial}{\partial \zeta} \left[ (1 - \zeta^2) \frac{\partial}{\partial \zeta} \right] \right\} + \frac{1}{c^2(\tau^2 - 1)(1 - \zeta^2)} \frac{\partial^2}{\partial \varphi^2}, \quad (21)$$

respectively. Especially, as  $\tau \rightarrow 1^+$  the prolate spheroid degenerates to the particular focal segment  $S_0 = \{(t, 0, 0) : t \in [-c, c], c > 0\}$ , while as  $\tau \rightarrow +\infty$  it approaches a sphere located at infinity. The orthonormal vectors  $\hat{\tau}, \hat{\zeta}, \hat{\phi}$  denote the coordinate vectors of the system,  $\hat{\tau}$  given by (19) and

$$\hat{\zeta} = \frac{1}{\sqrt{\tau^2 - \zeta^2}} (-\tau \sqrt{1 - \zeta^2} \hat{\mathbf{x}}_1 + \zeta \sqrt{\tau^2 - 1} \cos \varphi \hat{\mathbf{x}}_2 + \zeta \sqrt{\tau^2 - 1} \sin \varphi \hat{\mathbf{x}}_3), \quad (22)$$

$$\hat{\phi} = -\sin \varphi \hat{\mathbf{x}}_2 + \cos \varphi \hat{\mathbf{x}}_3 \quad (23)$$

for every  $1 \leq \tau < +\infty$ ,  $-1 \leq \zeta \leq 1$  and  $0 \leq \varphi < 2\pi$ . The unit dyadic is provided by the expression

$$\tilde{\mathbf{I}} = \hat{\mathbf{x}}_1 \otimes \hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_2 \otimes \hat{\mathbf{x}}_2 + \hat{\mathbf{x}}_3 \otimes \hat{\mathbf{x}}_3 = \hat{\tau} \otimes \hat{\tau} + \hat{\zeta} \otimes \hat{\zeta} + \hat{\phi} \otimes \hat{\phi} \quad (24)$$

in Cartesian and prolate spheroidal coordinates.

On the other hand, we introduce the interior  $u_{\ell}^{(i)ms}$  and the exterior  $u_{\ell}^{(e)ms}$  harmonic eigenfunctions of degree  $\ell$  ( $\ell = 0, 1, 2, \dots$ ) and of order  $m$  ( $m = 0, 1, 2, \dots, \ell$ ) in terms of the associated Legendre functions [10]  $P_{\ell}^m$  of the first and  $Q_{\ell}^m$  of the second kind (see also appendix) via the formulae

$$u_{\ell}^{(i)ms} = P_{\ell}^m(\tau) P_{\ell}^m(\zeta) f_{\ell}^{ms}(\varphi) \quad \text{and} \quad u_{\ell}^{(e)ms} = Q_{\ell}^m(\tau) P_{\ell}^m(\zeta) f_{\ell}^{ms}(\varphi), \quad (25)$$

respectively, which both are regular on the axis of symmetry, while the angular dependence is given by

$$f_{\ell}^{ms}(\varphi) = \begin{cases} \cos m\varphi, & s = e \\ \sin m\varphi, & s = o \end{cases} \Rightarrow f_{\ell}^{ms'}(\varphi) = \begin{cases} -m \sin m\varphi, & s = e \\ m \cos m\varphi, & s = o \end{cases} \quad \text{for every } \varphi \in [0, 2\pi), \quad (26)$$

with  $s$  denoting the even ( $e$ ) or the odd ( $o$ ) part of the eigenfunctions. Then, every harmonic function  $u$  in prolate spheroidal geometry is written as follows:

$$\Delta u = 0 \quad \Rightarrow \quad u = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \sum_{s=e,o} \left[ A_{\ell}^{(i)ms} u_{\ell}^{(i)ms} + A_{\ell}^{(e)ms} u_{\ell}^{(e)ms} \right], \quad (27)$$

where  $A_{\ell}^{(i)ms}$  and  $A_{\ell}^{(e)ms}$  are constant coefficients. In addition, for the region of observation and of interest here for the boundary conditions, which is for  $\mathbf{r} < \mathbf{r}_0$ , we utilize the eigenexpansion

$$\frac{1}{R} \equiv \frac{1}{|\mathbf{r} - \mathbf{r}_0|} = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \sum_{s=e,o} \left[ \rho_{\ell}^{ms} P_{\ell}^m(\tau) P_{\ell}^m(\zeta) f_{\ell}^{ms}(\varphi) \right], \quad (28)$$

where at  $\mathbf{r}_0 = (\tau_0, \zeta_0, \varphi_0)$  we get

$$\rho_{\ell}^{ms} \equiv \rho_{\ell}^{ms}(\mathbf{r}_0) = \frac{(2\ell + 1)}{c} \frac{[(\ell - m)!]^2}{[(\ell + m)!]} (-1)^m \varepsilon_m Q_{\ell}^m(\tau_0) P_{\ell}^m(\zeta_0) f_{\ell}^{ms}(\varphi_0), \quad (29)$$

while  $\varepsilon_m = 1$  for  $m = 0$  and  $\varepsilon_m = 2$  for  $m \geq 1$ . In what follows, in order to solve the boundary value problems (14)–(17), we use many useful recurrence relations for the Legendre and the trigonometric functions, as well as certain identities concerning the differential operators, which can be found in the appendix.

Concluding the needed information, the corresponding results for the oblate spheroidal geometry are obtained through the simple transformation [11]

$$\tau \rightarrow i\lambda \quad \text{and} \quad c \rightarrow -i\bar{c}, \quad (30)$$

where  $0 \leq \lambda < +\infty$  and  $\bar{c} > 0$  are the new characteristic variables. The asymptotic case of the needle can be reached by a prolate spheroid where  $0 < a_3 = a_2 \ll a_1 < +\infty$ , while in the case where  $0 < a_3 \ll a_2 = a_1 < +\infty$  the oblate spheroid takes the shape of a circular disk.

The spheroidal geometry degenerates to the spherical one [11] in the limit, as the semifocal distance  $c$  tends to zero,  $c \rightarrow 0^+$ . For the corresponding analytical reduction, the limiting process involves an appropriate combination of  $c$  with the coordinate variables such as  $r \equiv \|\mathbf{r}\| = c\sqrt{\tau^2 + \zeta^2 - 1}$  for  $\tau > 1$  and  $|\zeta| \leq 1$  as well as the following limits:

$$\lim_{c \rightarrow 0^+} c\tau = r \quad \text{and} \quad \lim_{c \rightarrow 0^+} \frac{1}{2c} \ln \frac{\tau + 1}{\tau - 1} = \frac{1}{r}. \quad (31)$$

That way we recover the radial component  $r$  (as well as  $1/r$ ) of the spherical coordinate system

$$\mathbf{r} = \sum_{i=1}^3 x_i \hat{\mathbf{x}}_i = r \zeta \hat{\mathbf{x}}_1 + r \sqrt{1 - \zeta^2} \cos \varphi \hat{\mathbf{x}}_2 + r \sqrt{1 - \zeta^2} \sin \varphi \hat{\mathbf{x}}_3, \quad 0 < r < +\infty, \quad -1 \leq \zeta \leq 1, \quad \varphi \in [0, 2\pi) \quad (32)$$

and it is obvious that the spherical normal unit vector on the surface of the corresponding spherical buried obstacle is

$$\lim_{c \rightarrow 0^+} \hat{\boldsymbol{\tau}} = \hat{\mathbf{r}} = \frac{\mathbf{r}}{r}. \quad (33)$$

In order to obtain the corresponding mathematical form for the sphere-case, we need the definitions (A.11)–(A.17) of the associated Legendre functions of the first and of the second kind, which leads to

$$\lim_{c \rightarrow 0^+} c^\ell P_\ell^m(\tau) = p_\ell \frac{\ell!}{(\ell - m)!} r^\ell \quad \text{and} \quad \lim_{c \rightarrow 0^+} c^{-(\ell+1)} Q_\ell^m(\tau) = q_\ell (-1)^m \frac{(\ell + m)!}{\ell!} r^{-(\ell+1)} \quad (34)$$

with the aim of the reduction formulas (31), whereas

$$p_\ell = \frac{(2\ell)!}{2^\ell (\ell!)^2} \quad \text{and} \quad q_\ell = \frac{1}{2^\ell} \sum_{\kappa=0}^{\lfloor \ell/2 \rfloor} \frac{(-1)^\kappa (2\ell - 2\kappa)!}{\kappa! (\ell - \kappa)! (\ell - 2\kappa)! (2\ell - 2\kappa + 1)}, \quad \text{where} \quad (2\ell + 1) p_\ell q_\ell = 1. \quad (35)$$

Finally, introducing a certain scaling for the constant coefficients appearing into the expansion (27), according to which  $c_\ell^{(i)ms}$  are multiplied by  $c^\ell$  and  $c_\ell^{(e)ms}$  are multiplied by  $c^{-(\ell+1)}$  for every  $\ell \geq 0$ ,  $m \leq \ell$  and  $s = e, o$ , we regain the corresponding expansion in spherical harmonic eigenfunctions and we are able to recover the corresponding spherical electromagnetic fields.

#### 4. Spheroidal low-frequency electromagnetic fields

Our purpose is to derive closed analytical expressions for the non-vanishing scattered magnetic fields  $\mathbf{H}_0^s$ ,  $\mathbf{H}_2^s$  and  $\mathbf{H}_3^s$ , as well as for the scattered electric field  $\mathbf{E}_2^s$ , since  $\mathbf{H}_1^s = \mathbf{E}_0^s = \mathbf{E}_1^s = \mathbf{E}_3^s = \mathbf{0}$ . In order to achieve that, we must solve independently the problems (14) and (16) to obtain  $\mathbf{H}_0^s$  and  $\mathbf{H}_3^s$ , respectively and then proceed to the problem (15) to obtain  $\mathbf{H}_2^s$  and, thus,  $\mathbf{E}_2^s$ , which is much more complicated due to its coupling with (14). The proper boundary conditions for the total electromagnetic fields (2) on the surface  $\tau = \tau_s$  of the metallic object given by (17) fit the aforementioned boundary value problems and the expressions (9)–(12) for the incident fields, as well as the unit dyadic representation (24), are properly used, in view of the eigenexpansion (28), (29). The position of the magnetic dipole  $\mathbf{m}$  at  $\mathbf{r} = \mathbf{r}_0$  is also taken into consideration utilizing Eq. (18), while the harmonic potentials  $\Phi_0^s$ ,  $\Phi_2^s$  and  $\Phi_3^s$  follow the previous analysis based on relations (25)–(27). Under this aspect and with respect to relation (20) (with Eqs. (19), (22), and (23)), we perform a number of long and tedious calculations in order to obtain the electromagnetic fields, by making extensive use of Ref. [12] and of the appendix in the end of the article.

The particular steps for the extraction of the final forms of the magnetic and the electric fields described above at each  $n = 0, 2, 3$  (for  $n = 1$  there does not exist any kind of fields, while for  $n \geq 4$  any additional terms do not offer an approximation of great importance) demand the action of the gradient differential operator  $\nabla \equiv \nabla_\tau$ , on the solid harmonic eigenfunctions (interior and exterior) as it is easily demonstrated by the boundary value problems (14)–(16). Hence, we apply relationship (20) to the harmonic eigenfunctions (25), where by virtue of the recurrence expressions (A.20) and (A.21) for the corresponding derivations at  $\zeta$  and  $\tau$  respectively, we conclude to

$$\nabla u_0^{(i)0e} = \mathbf{0} \quad \text{for} \quad \ell = m = 0 \quad \text{and} \quad s = e, \quad (36)$$

$$\begin{aligned} \nabla u_\ell^{(i)ms} &= \frac{c f^{ms}(\varphi)}{(2\ell + 1)} \left\{ \frac{P_\ell^m(\zeta) \hat{\boldsymbol{\tau}}}{c^2 \sqrt{\tau^2 - \zeta^2} \sqrt{\tau^2 - 1}} [\ell(\ell - m + 1) P_{\ell+1}^m(\tau) - (\ell + 1)(\ell + m) P_{\ell-1}^m(\tau)] \right. \\ &\quad \left. + \frac{P_\ell^m(\tau) \hat{\boldsymbol{\zeta}}}{c^2 \sqrt{\tau^2 - \zeta^2} \sqrt{1 - \zeta^2}} [\ell(\ell - m + 1) P_{\ell+1}^m(\zeta) - (\ell + 1)(\ell + m) P_{\ell-1}^m(\zeta)] \right\} \\ &\quad + \frac{f^{ms'}(\varphi) \hat{\boldsymbol{\phi}}}{c \sqrt{\tau^2 - 1} \sqrt{1 - \zeta^2}} P_\ell^m(\tau) P_\ell^m(\zeta) \quad \text{for} \quad \ell \geq 1, \quad m = 0, 1, 2, \dots, \ell \quad \text{and} \quad s = e, o, \end{aligned} \quad (37)$$

as far as the gradient on the interior solid harmonics is concerned, while

$$\nabla u_0^{(e)0e} = - \frac{\hat{\boldsymbol{\tau}}}{c \sqrt{\tau^2 - \zeta^2} \sqrt{\tau^2 - 1}} \quad \text{for} \quad \ell = m = 0 \quad \text{and} \quad s = e, \quad (38)$$

$$\begin{aligned} \nabla u_\ell^{(e)ms} &= \frac{c f^{ms}(\varphi)}{(2\ell + 1)} \left\{ \frac{P_\ell^m(\zeta) \hat{\boldsymbol{\tau}}}{c^2 \sqrt{\tau^2 - \zeta^2} \sqrt{\tau^2 - 1}} [\ell(\ell - m + 1) Q_{\ell+1}^m(\tau) - (\ell + 1)(\ell + m) Q_{\ell-1}^m(\tau)] \right. \\ &\quad \left. + \frac{Q_\ell^m(\tau) \hat{\boldsymbol{\zeta}}}{c^2 \sqrt{\tau^2 - \zeta^2} \sqrt{1 - \zeta^2}} [\ell(\ell - m + 1) P_{\ell+1}^m(\zeta) - (\ell + 1)(\ell + m) P_{\ell-1}^m(\zeta)] \right\} \\ &\quad + \frac{f^{ms'}(\varphi) \hat{\boldsymbol{\phi}}}{c \sqrt{\tau^2 - 1} \sqrt{1 - \zeta^2}} Q_\ell^m(\tau) P_\ell^m(\zeta) \quad \text{for} \quad \ell \geq 1, \quad m = 0, 1, 2, \dots, \ell \quad \text{and} \quad s = e, o, \end{aligned} \quad (39)$$



as far as the gradient on the exterior solid harmonics is concerned, where  $\mathbf{r} \in V^+(\mathbb{R}^3) - \{\mathbf{r}_0\}$ . It is obvious that expressions (36)–(39) are also valid for  $\mathbf{r} = \mathbf{r}_0$ , so as to be provided with the corresponding relationships of the  $\nabla_{\mathbf{r}_0}$  acting on harmonics defined at the location of the magnetic dipole. Nevertheless, since the region of observation and scattering is located outside the metallic body, we are obliged to utilize the exterior harmonic eigenfunctions (25) for the potential problems, however the gradient on interior harmonics will be used in our analysis via the incident fields (9)–(12) with the eigenexpansion (28) and (29). Thus, we are now ready to proceed to the calculation of the scattered fields  $\mathbf{H}_0^s$ ,  $\mathbf{H}_2^s$ ,  $\mathbf{E}_2^s$  and  $\mathbf{H}_3^s$  (we remind that  $\mathbf{H}_1^s = \mathbf{E}_0^s = \mathbf{E}_1^s = \mathbf{E}_3^s = \mathbf{0}$ ) and to recover the corresponding scattered fields for the spherical case (one assumes the model of a spherical buried body), which can be found separately in [19]. We begin from the easiest case for  $n = 3$ , we continue to the fields for  $n = 0$  and we conclude with the most cumbersome case for  $n = 2$ .

#### 4.1. The potential field problem at $n = 3$ and calculation of $\mathbf{H}_3^s$ ( $\mathbf{E}_3^s = \mathbf{0}$ )

The most trivial calculations concern the scattered magnetic field  $\mathbf{H}_3^s$ . This is due to the fact that the incident field (11) for  $n = 3$  is a constant vector, while here we have to solve the potential boundary value problem (16) with the Neumann boundary condition (17) on  $S$  for  $n = 3$ , which for  $\hat{\mathbf{n}} \equiv \hat{\tau}$  is

$$\mathbf{H}_3^s = \nabla \Phi_3^s \quad \text{with} \quad \Delta \Phi_3^s = 0, \quad \text{where} \quad \hat{\tau} \cdot (\mathbf{H}_3^i + \mathbf{H}_3^s) = 0 \quad \text{on} \quad \tau = \tau_s. \quad (40)$$

Then, the exterior harmonic structure of the potential  $\Phi_3^s$ , with the aim of definitions (25)–(27), yields

$$\mathbf{H}_3^s = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \sum_{s=e,o} a_{\ell}^{(e)ms} \nabla [Q_{\ell}^m(\tau) P_{\ell}^m(\zeta) f^{ms}(\varphi)], \quad \mathbf{r} \in V^+(\mathbb{R}^3) - \{\mathbf{r}_0\}, \quad (41)$$

where  $a_{\ell}^{(e)ms}$  for  $\ell = 0, 1, 2, \dots, m = 0, 1, 2, \dots, \ell$  and  $s = e, o$  stand for the constant coefficients that have to be determined. Thus, in terms of the primary field (11), in view of Eq. (24) and taking the three projections of the magnetic dipole on the Cartesian coordinate system, i.e.  $\mathbf{m} = \sum_{\kappa=1}^3 m_{\kappa} \hat{\mathbf{x}}_{\kappa}$ , then boundary condition (40), Eqs. (38), (39) and relations (19), (40) leave us with the boundary expression on  $\tau = \tau_s$

$$\left\{ -\frac{2}{3} \frac{1}{4\pi} \frac{1}{\sqrt{\tau_s^2 - \zeta^2}} \left( m_1 \zeta \sqrt{\tau_s^2 - 1} + m_2 \tau_s \sqrt{1 - \zeta^2} \cos \varphi + m_3 \tau_s \sqrt{1 - \zeta^2} \sin \varphi \right) - \frac{a_0^{(e)0e}}{c \sqrt{\tau_s^2 - \zeta^2} \sqrt{\tau_s^2 - 1}} \right. \\ \left. + \sum_{\ell=1}^{\infty} \sum_{m=0}^{\ell} \sum_{s=e,o} \frac{a_{\ell}^{(e)ms} f^{ms}(\varphi)}{c(2\ell+1) \sqrt{\tau_s^2 - \zeta^2} \sqrt{\tau_s^2 - 1}} \left[ \ell(\ell-m+1) Q_{\ell+1}^m(\tau_s) - (\ell+1)(\ell+m) Q_{\ell-1}^m(\tau_s) \right] P_{\ell}^m(\zeta) \right\} = 0. \quad (42)$$

At this stage we are able to evaluate the unknown constant coefficients for  $\mathbf{H}_3^s$  after we proceed with the proper orthogonality arguments for the trigonometric functions (26) and those provided by (A.18) for the associated Legendre functions of the first kind. By definition of the last ones (see Eqs. (A.11) and (A.13)), we can write  $P_0(\zeta) = 1$ ,  $P_1(\zeta) = \zeta$ ,  $P_1^1(\zeta) = \sqrt{1 - \zeta^2}$  and directly arrive from (42) at

$$a_{\ell}^{(e)ms} = 0, \quad \text{for} \quad \ell = 0, 2, 3, \dots, m = 0, 1, 2, \dots, \ell \quad \text{and} \quad s = e, o, \quad (43)$$

while for  $\ell = 1, m = 0, 1$  and  $s = e, o$  as we conclude with the three coefficients that do not vanish

$$a_1^{(e)0e} = \frac{cm_1}{4\pi} \frac{2}{3Q_1'(\tau_s)} \quad \text{and} \quad a_1^{(e)1e} = \frac{cm_2}{4\pi} \frac{2\tau_s}{3\sqrt{\tau_s^2 - 1} Q_1^1(\tau_s)}, \quad a_1^{(e)1o} = \frac{cm_3}{4\pi} \frac{2\tau_s}{3\sqrt{\tau_s^2 - 1} Q_1^1(\tau_s)}, \quad (44)$$

where the primes denote  $\tau$ -derivation of the associated Legendre functions  $Q_1$  and  $Q_1^1$  of the second kind (see Ref. [10] and appendix) at the surface  $\tau = \tau_s$ . Consequently, the problem for  $n = 3$  is solved and it remains to substitute the constant coefficients (43) and (44) into the harmonic expression (41), which due to the fact that only three constants survive and after some trivial manipulation based on relation (39) for  $\ell = 1$ , assumes the following finite prolate spheroidal dyadic form

$$\mathbf{H}_3^s = \frac{2}{3} \frac{\mathbf{m}}{4\pi} \cdot \left\{ \frac{\sqrt{\tau^2 - 1}}{\sqrt{\tau^2 - \zeta^2}} \left[ \frac{\partial \mathbf{f}(\tau)}{\partial \tau} \zeta + \frac{\partial \mathbf{g}(\tau, \varphi)}{\partial \tau} \sqrt{1 - \zeta^2} \right] \otimes \hat{\tau} + \frac{1}{\sqrt{\tau^2 - \zeta^2}} \left[ -\mathbf{f}(\tau) \sqrt{1 - \zeta^2} + \mathbf{g}(\tau, \varphi) \zeta \right] \otimes \hat{\zeta} + \frac{1}{\sqrt{\tau^2 - 1}} \frac{\partial \mathbf{g}(\tau, \varphi)}{\partial \varphi} \otimes \hat{\phi} \right\} \quad (45)$$

for every  $\mathbf{r} \in V^+(\mathbb{R}^3) - \{\mathbf{r}_0\}$ , where  $\tau_s = a_1/c$  and the new functions  $\mathbf{f}$  and  $\mathbf{g}$  are expressed via the Cartesian basis  $\hat{\mathbf{x}}_{\kappa}$ ,  $\kappa = 1, 2, 3$

$$\mathbf{f}(\tau) = \hat{\mathbf{x}}_1 \frac{Q_1(\tau)}{Q_1'(\tau_s)} \quad \text{and} \quad \mathbf{g}(\tau, \varphi) = (\hat{\mathbf{x}}_2 \cos \varphi + \hat{\mathbf{x}}_3 \sin \varphi) \frac{\tau_s}{\sqrt{\tau_s^2 - 1}} \frac{Q_1^1(\tau)}{Q_1^1(\tau_s)}. \quad (46)$$

As we stated previously, the primes mean derivation with respect to the argument  $\tau$  of the associated Legendre functions of the second kind at the surface of the spheroid  $\tau = \tau_s$ , while let us remind that the scattered electric field vanishes, i.e.  $\mathbf{E}_3^s = \mathbf{0}$ .

We cross-check our results for the scattered field  $\mathbf{H}_3^s$  by following the limiting procedure as  $c \rightarrow 0^+$  described earlier within Eqs. (31)–(35), so as to recover the already known scattered field  $\mathbf{H}_{3,sphere}^s$  for the corresponding sphere-case problem from [19]. In order to avoid complicated calculations through the reduction process, we leave aside the expressions (45), (46) and we use the initial form (41) with the coefficients (43) and (44). It is not the purpose of this paper to show analytically how we proceed with the reduction (we provided the proper tools (31)–(35)), since it contains straightforward calculations. Hence, some trivial algebra and the analogous  $\nabla_{sphere}$  of the gradient operator in spherical coordinates [11,19] lead us to

$$\lim_{c \rightarrow 0^+} \mathbf{H}_3^s = -\frac{(\mathbf{m} \cdot \hat{\mathbf{x}}_1)}{4\pi} \frac{a^2}{3} \nabla_{sphere} (r^{-2} P_1(\zeta)) = \frac{(\mathbf{m} \cdot \hat{\mathbf{x}}_1)}{4\pi} \frac{1}{3} \left(\frac{a}{r}\right)^3 \left(2\zeta \hat{\mathbf{r}} + \sqrt{1 - \zeta^2} \hat{\boldsymbol{\zeta}}\right) \equiv \mathbf{H}_{3,sphere}^s, \quad (47)$$

where  $\hat{\mathbf{x}}_1$  is the main spherical projection-axis of the operated magnetic dipole  $\mathbf{m}$ , ( $r, \zeta$ ) for  $r \geq 0$  and  $|\zeta| \leq 1$  are the spherical coordinates (the azimuthal angle  $\varphi \in [0, 2\pi)$  is absent here) and  $a$  is the radius of the specified buried spherical obstacle. The scattered field (47) for  $n = 3$  is identical to the already known one from Ref. [19], thus the prolate spheroidal relations (45) and (46) are reduced in a suitable analytical manner.

At this point we must mention that the corresponding results for the oblate spheroidal geometry, as well as for the needle and the disk, can be obtained with the aid of the transformation (30) and the use of the definitions described below (30), respectively.

#### 4.2. The potential field problem at $n = 0$ and calculation of $\mathbf{H}_0^s = (\mathbf{E}_0^s = \mathbf{0})$

A much more complicated analysis based on the previous steps has to be followed in order to obtain the scattered field in the case where  $n = 0$ , that is the static term  $\mathbf{H}_0^s$ , which is an immediate consequence of the complexity of the corresponding incident field  $\mathbf{H}_0^i$ , given by (9). This primary field admits the double action of the gradient operator (at the position  $\mathbf{r} \neq \mathbf{r}_0$ ) on the quantity  $1/R$  for  $R = |\mathbf{r} - \mathbf{r}_0|$ . If we act in the same way as previously, we confront once more a potential boundary value problem of the form (14) and we also apply the Neumann boundary condition (17) on  $S$  for  $n = 0$ , whereas for  $\hat{\mathbf{n}} \equiv \hat{\boldsymbol{\tau}}$  it is stated by

$$\mathbf{H}_0^s = \nabla \Phi_0^s \quad \text{with} \quad \Delta \Phi_0^s = 0, \quad \text{where} \quad \hat{\boldsymbol{\tau}} \cdot (\mathbf{H}_0^i + \mathbf{H}_0^s) = 0 \quad \text{on} \quad \tau = \tau_s. \quad (48)$$

Similarly, the exterior harmonic potential  $\Phi_0^s$ , with the aim of relations (25)–(27), enters into Eq. (48) to provide us with

$$\mathbf{H}_0^s = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \sum_{s=e,o} b_{\ell}^{(e)ms} \nabla [Q_{\ell}^m(\tau) P_{\ell}^m(\zeta) f^{ms}(\varphi)], \quad \mathbf{r} \in V^+(\mathbb{R}^3) - \{\mathbf{r}_0\}, \quad (49)$$

where  $b_{\ell}^{(e)ms}$  for  $\ell = 0, 1, 2, \dots, m = 0, 1, 2, \dots, \ell$  and  $s = e, o$  denote the constant coefficients that have to be evaluated by the appropriate boundary condition (48), which is the matching of the magnetic boundary condition at the surface  $\tau = \tau_s$  of the spheroid, i.e. cancellation of the normal derivative of the total potential ( $\hat{\boldsymbol{\tau}} \cdot (\mathbf{H}_0^i + \mathbf{H}_0^s) = 0$  on  $\tau = \tau_s$ ), where initially we shall calculate the two summands separately. Then, in view of expressions (38) and (39), as well as of identity (A.21), we come up with

$$\hat{\boldsymbol{\tau}} \cdot \mathbf{H}_0^s \Big|_{\tau=\tau_s} = \frac{1}{c \sqrt{\tau_s^2 - \zeta^2} \sqrt{\tau_s^2 - 1}} \left\{ -b_0^{(e)0e} + \sum_{\ell=1}^{\infty} \sum_{m=0}^{\ell} \sum_{s=e,o} b_{\ell}^{(e)ms} (\tau_s^2 - 1) Q_{\ell}^{m'}(\tau_s) P_{\ell}^m(\zeta) f^{ms}(\varphi) \right\}, \quad (50)$$

where the prime indicates derivation with respect to the  $\tau$ -variable. Yet, the expression of the incident field does not appear easily amenable to further processing and an alternative approach has been followed. We do not apply twice the operator  $\nabla \equiv \nabla_{\mathbf{r}}$  on  $1/R$ , as indicated by Eq. (9), but we firstly evaluate the inner product  $\hat{\boldsymbol{\tau}} \cdot \nabla$  with respect to relations (19) and (20) to obtain

$$\hat{\boldsymbol{\tau}} \cdot \mathbf{H}_0^i \Big|_{\tau=\tau_s} = \frac{1}{c \sqrt{\tau_s^2 - \zeta^2} \sqrt{\tau_s^2 - 1}} \left\{ (\tau_s^2 - 1) \frac{\partial}{\partial \tau} \left[ \nabla_{\mathbf{r}} \frac{1}{|\mathbf{r} - \mathbf{r}_0|} \right] \cdot \frac{\mathbf{m}}{4\pi} \right\}_{\tau=\tau_s}, \quad (51)$$

since the dyadic  $\nabla \otimes \nabla_{\mathbf{r}}$  is symmetric, while the second relation of (13) helps us to avoid the double-derivation by changing the argument of derivation of  $1/R$ . Thus, Eq. (51) can be rewritten as

$$\hat{\boldsymbol{\tau}} \cdot \mathbf{H}_0^i \Big|_{\tau=\tau_s} = -\frac{1}{c \sqrt{\tau_s^2 - \zeta^2} \sqrt{\tau_s^2 - 1}} \left\{ (\tau_s^2 - 1) \frac{\partial}{\partial \tau} \left[ \nabla_{\mathbf{r}_0} \frac{1}{|\mathbf{r} - \mathbf{r}_0|} \right] \cdot \frac{\mathbf{m}}{4\pi} \right\}_{\tau=\tau_s}, \quad (52)$$

which, upon introduction of eigenexpansion (28), becomes

$$\hat{\boldsymbol{\tau}} \cdot \mathbf{H}_0^i \Big|_{\tau=\tau_s} = -\frac{1}{c \sqrt{\tau_s^2 - \zeta^2} \sqrt{\tau_s^2 - 1}} \left\{ (\tau_s^2 - 1) \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \sum_{s=e,o} \left[ (\nabla_{\mathbf{r}_0} \rho_{\ell}^{ms}(\mathbf{r}_0)) P_{\ell}^{m'}(\tau_s) P_{\ell}^m(\zeta) f^{ms}(\varphi) \right] \cdot \frac{\mathbf{m}}{4\pi} \right\}, \quad (53)$$

where  $\rho_\ell^{ms} \equiv \rho_\ell^{ms}(\mathbf{r}_0)$  is provided by relation (29), whose gradient  $\nabla_{\mathbf{r}_0} \rho_\ell^{ms}(\mathbf{r}_0)$  is a known quantity which can be determined by Eqs. (38) and (39) with the aid of (25) at  $\mathbf{r} = \mathbf{r}_0$ , while the prime coincides once more with the derivative of  $P_\ell^m(\tau)$  for  $\tau = \tau_s$ . Hence, we achieved to reduce the difficulty of this boundary condition (48) using this technique, where by combination of (50) and (53), it suggests

$$-b_0^{(e)0e} + \sum_{\ell=1}^{\infty} \sum_{m=0}^{\ell} \sum_{s=e,o} \left\{ (\tau_s^2 - 1) \left[ b_\ell^{(e)ms} Q_\ell^m(\tau_s) - \frac{\mathbf{m}}{4\pi} \cdot (\nabla_{\mathbf{r}_0} \rho_\ell^{ms}(\mathbf{r}_0)) P_\ell^m(\tau_s) \right] P_\ell^m(\zeta) f^{ms}(\varphi) \right\} = 0. \quad (54)$$

Orthogonality arguments of the eigenproducts  $P_\ell^m(\zeta) f^{ms}(\varphi)$  for  $\ell \geq 0$ ,  $0 \leq m \leq \ell$  and  $s = e, o$  results in

$$b_\ell^{(e)ms} = \frac{P_\ell^m(\tau_s)}{Q_\ell^m(\tau_s)} \left( \frac{\mathbf{m}}{4\pi} \cdot \nabla_{\mathbf{r}_0} \rho_\ell^{ms}(\mathbf{r}_0) \right), \quad \text{for } \ell = 0, 1, 2, \dots, m = 0, 1, 2, \dots, \ell \text{ and } s = e, o, \quad (55)$$

noticing that  $b_0^{(e)0e} = 0$ . Eventually, we substitute the constant coefficients from Eq. (55) to the potential field (49), we introduce Eq. (20) and we expand the quantity  $\rho_\ell^{ms}(\mathbf{r}_0)$  by virtue of the relation (29), in order to obtain the prolate spheroidal expression for  $\mathbf{H}_0^s$  in terms of the unit normal vectors  $\hat{\tau}$ ,  $\hat{\zeta}$  and  $\hat{\phi}$  given by (19), (22) and (23), i.e. the infinite series scattered field

$$\begin{aligned} \mathbf{H}_0^s &= \sum_{\ell=1}^{\infty} \sum_{m=0}^{\ell} \sum_{s=e,o} \frac{(2\ell+1)}{c^2} (-1)^m \varepsilon_m \frac{[(\ell-m)!]^2}{[(\ell+m)!]} \frac{P_\ell^m(\tau_s)}{Q_\ell^m(\tau_s)} \left[ \frac{\mathbf{m}}{4\pi} \cdot \nabla_{\mathbf{r}_0} (Q_\ell^m(\tau_0) P_\ell^m(\zeta_0) f^{ms}(\varphi_0)) \right] \\ &\quad \times \left\{ \frac{1}{\sqrt{\tau^2 - \zeta^2}} \left[ \sqrt{\tau^2 - 1} Q_\ell^m(\tau) P_\ell^m(\zeta) \hat{\tau} - \sqrt{1 - \zeta^2} Q_\ell^m(\tau) P_\ell^m(\zeta) \hat{\zeta} \right] f^{ms}(\varphi) + \frac{1}{\sqrt{\tau^2 - 1} \sqrt{1 - \zeta^2}} Q_\ell^m(\tau) P_\ell^m(\zeta) f^{ms}(\varphi) \hat{\phi} \right\}, \\ &\quad \mathbf{r} \in V^+(\mathbb{R}^3) - \{\mathbf{r}_0\}, \end{aligned} \quad (56)$$

where the primes denote derivation with respect to the argument,  $\varepsilon_0 = 1$  and  $\varepsilon_m = 2$  for  $m \neq 0$ , while  $\tau_s = a_1/c$  and the magnetic dipole is assumed to be  $\mathbf{m} = \sum_{\kappa=1}^3 m_\kappa \hat{\mathbf{x}}_\kappa$ . Obviously, the operator  $\nabla_{\mathbf{r}_0}$  is formulated by (20) in the case of  $\mathbf{r}(\tau, \zeta, \varphi) = \mathbf{r}_0(\tau_0, \zeta_0, \varphi_0)$ , that is

$$\begin{aligned} \nabla_{\mathbf{r}_0} (Q_\ell^m(\tau_0) P_\ell^m(\zeta_0) f^{ms}(\varphi_0)) &= \left\{ \frac{1}{c \sqrt{\tau_0^2 - 1} \sqrt{1 - \zeta_0^2}} Q_\ell^m(\tau_0) P_\ell^m(\zeta_0) f^{ms}(\varphi_0) \hat{\phi}_0 + \frac{1}{c \sqrt{\tau_0^2 - \zeta_0^2}} \left[ \sqrt{\tau_0^2 - 1} Q_\ell^m(\tau_0) P_\ell^m(\zeta_0) \hat{\tau}_0 \right. \right. \\ &\quad \left. \left. - \sqrt{1 - \zeta_0^2} Q_\ell^m(\tau_0) P_\ell^m(\zeta_0) \hat{\zeta}_0 \right] f^{ms}(\varphi_0) \right\}, \end{aligned} \quad (57)$$

all calculated at the position of the magnetic dipole  $\mathbf{r} = \mathbf{r}_0$ .

The limiting process as  $c \rightarrow 0^+$  is described via Eqs. (31)–(35) and if we wish to recover the already known scattered field  $\mathbf{H}_{0,sphere}^s$  for the corresponding sphere-case potential problem from [19], we are forced to follow the very same procedure described in Section 4.1, since expressions (41) and (49) are identical. However, we can in effect bypass the analytical reduction steps based on complicate, though straightforward application of formulae (31)–(35). Hence, some trivial algebra and the analogous  $\nabla_{sphere}$  of the gradient operator in spherical coordinates [11,19], applied both at  $\mathbf{r}$  and  $\mathbf{r}_0$ , provide us with

$$\begin{aligned} \lim_{c \rightarrow 0^+} \mathbf{H}_0^s &= -\frac{(\mathbf{m} \cdot \hat{\mathbf{x}}_1)}{4\pi} \sum_{\ell=1}^{\infty} \sum_{m=0}^{\ell} \frac{\ell \varepsilon_m a^{2\ell+1}}{(\ell+1)} \frac{(\ell-m+1)!}{(\ell+m)!} \frac{P_{\ell+1}^m(\zeta_0)}{r_0^{\ell+2}} \\ &\quad \times \frac{1}{r^{\ell+2}} \left\{ \hat{\mathbf{r}}(\ell+1) P_\ell^m(\zeta) \cos[m(\varphi - \varphi_0)] + \hat{\zeta} \sqrt{1 - \zeta^2} P_\ell^m(\zeta) \cos[m(\varphi - \varphi_0)] + \frac{m \hat{\phi}}{\sqrt{1 - \zeta^2}} P_\ell^m(\zeta) \sin[m(\varphi - \varphi_0)] \right\} \\ &\equiv \mathbf{H}_{0,sphere}^s, \end{aligned} \quad (58)$$

where relations (A.26)–(A.29) from the appendix have been used, while the Cartesian vector  $\hat{\mathbf{x}}_1$  stands for the main direction of a sphere where the magnetic dipole  $\mathbf{m}$  is located,  $(r, \zeta, \varphi)$  for  $r \geq 0$ ,  $|\zeta| \leq 1$  and  $\varphi \in [0, 2\pi)$  are the spherical coordinates. On the other hand,  $r_0$ ,  $\zeta_0$  and  $\varphi_0$  correspond to  $\mathbf{r} = \mathbf{r}_0$  and  $a$  is the radius of the buried spherical body under identification. The scattered field (58) for  $n = 0$  matches exactly with the one obtained in Ref. [19], thus the prolate spheroidal relations (56) and (57) are reduced properly.

Similarly, we point out that the corresponding results for an oblate spheroidal obstacle, as well as for the needle and the disk case, can be recovered via the transformation (30) and the use of the definitions described there.

#### 4.3. The potential field problem at $n = 2$ and calculation of $\mathbf{H}_2^s$ and $\mathbf{E}_2^s$

Let us now concentrate upon the potential problem at  $n = 2$ , where a very difficult and cumbersome manipulation of the boundary value problem (15) with (17) results into the dynamic scattered fields  $\mathbf{H}_2^s$  and  $\mathbf{E}_2^s$ . There exist two reasons that are

responsible for this difficulty. The first one is the coupling of the particular model with the zero-order field  $\mathbf{H}_0^s$  (static term), while the second one refers to the extra electric field  $\mathbf{E}_2^s$ , which enters our problem with the corresponding additional boundary conditions. However, the importance of the  $\mathbf{H}_2^s$  (as well as  $\mathbf{E}_2^s$ ) term is of major significance, since those fields provide purely imaginary-valued field components in the conductive medium (see Section 4.4 in combination with Eq. (1)) and therefore those are needed as they contribute to at least most of the imaginary part (quadrature) of the scattered field  $\mathbf{H}^s$ . On the other hand, the real part (in-phase) is being essentially made of the static  $\mathbf{H}_0^s$  contribution, as it is directly obvious in Section 4.4. The mathematical problem that has to be solved is presented by the modeling (15) and (17), which in terms of the normal unit vector  $\hat{\tau}$  provided by (19) is presented as

$$\mathbf{H}_2^s = \Phi_2^s + \frac{1}{2}(\mathbf{r}\Phi_0^s) \quad \text{with} \quad \Delta\Phi_2^s = \mathbf{0}, \Delta\Phi_0^s = 0, \quad \text{where} \quad \hat{\tau} \cdot (\mathbf{H}_2^i + \mathbf{H}_2^s) = 0 \quad \text{on} \quad \tau = \tau_s, \quad (59)$$

and

$$\mathbf{E}_2^s = \frac{1}{\sigma^+} \nabla \times \mathbf{H}_2^s = \frac{1}{\sigma^+} \left( \nabla \times \Phi_2^s + \frac{1}{2} \nabla \Phi_0^s \times \mathbf{r} \right), \quad \text{where} \quad \hat{\tau} \times (\mathbf{E}_2^i + \mathbf{E}_2^s) = \mathbf{0} \quad \text{on} \quad \tau = \tau_s, \quad (60)$$

where the second equality for the scattered electric field in (60) comes from an immediate application of the identity (A.3), while even though the divergence-free character of  $\mathbf{E}_2^s$  is obvious, this is not the case for the scattered magnetic field  $\mathbf{H}_2^s$ , whose property

$$\nabla \cdot \mathbf{H}_2^s = 0 \Rightarrow \nabla \cdot \Phi_2^s + \frac{3}{2} \Phi_0^s + \frac{1}{2} (\mathbf{r} \cdot \nabla \Phi_0^s) = 0, \quad (61)$$

which is a consequence of (A.2) acting onto (59) and of  $\nabla \cdot \mathbf{r} = 3$ , stands for the extra condition that must be satisfied in addition with the three (one scalar and two components of a vector) boundary conditions (59) and (60).

As shown in relation (59), the scattered magnetic field at order  $n = 2$  is made of two sets of functions. As far as the first one is concerned, which coincides with the particular solution of the form  $\frac{1}{2}(\mathbf{r}\Phi_0^s)$ , it is readily ensured from identity (A.9), as well as from the harmonic character of the position vector  $\mathbf{r}$  and from the harmonic potential  $\Phi_0^s$ , that

$$\Delta \left[ \frac{1}{2}(\mathbf{r}\Phi_0^s) \right] = \nabla \Phi_0^s \equiv \mathbf{H}_0^s, \quad (62)$$

since  $\nabla \otimes \mathbf{r} = \tilde{\mathbf{I}}$ , where the coupling with the  $n = 0$  problem is demonstrated through Eq. (62) for the non-harmonic part of the field  $\mathbf{H}_2^s$ . Hence, in view of formulae (48) and (49) and in terms of the constant coefficients  $b_\ell^{e/ms}$  for  $\ell = 0, 1, 2, \dots$ ,  $m = 0, 1, 2, \dots, \ell$  and  $s = e, o$ , provided by (55), we identify the potential  $\Phi_0^s$  as

$$\Phi_0^s = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \sum_{s=e,o} \left\{ \frac{P_\ell^m(\tau_s)}{Q_\ell^m(\tau_s)} \left( \frac{\mathbf{m}}{4\pi} \cdot \nabla_{\mathbf{r}_0} \rho_\ell^{ms}(\mathbf{r}_0) \right) Q_\ell^m(\tau) P_\ell^m(\zeta) f^{ms}(\varphi) \right\}, \quad \mathbf{r} \in V^+(\mathbb{R}^3) - \{\mathbf{r}_0\}, \quad (63)$$

whereas

$$\nabla_{\mathbf{r}_0} \rho_\ell^{ms}(\mathbf{r}_0) = \frac{(2\ell+1)}{c} \left[ \frac{(\ell-m)!}{(\ell+m)!} \right]^2 (-1)^m \varepsilon_m \nabla_{\mathbf{r}_0} (Q_\ell^m(\tau_0) P_\ell^m(\zeta_0) f^{ms}(\varphi_0)), \quad \ell \geq 0, m \leq \ell, s = e, o \quad (64)$$

and the quantity  $\nabla_{\mathbf{r}_0} (Q_\ell^m(\tau_0) P_\ell^m(\zeta_0) f^{ms}(\varphi_0))$  is given by (57) evaluated at the position of the magnetic dipole  $\mathbf{r}(\tau, \zeta, \varphi) = \mathbf{r}_0(\tau_0, \zeta_0, \varphi_0)$ , while we recall that  $\varepsilon_0 = 1$  and  $\varepsilon_{m \neq 0} = 2$ . Note that for  $\ell = m = 0$  there exists no field, however we retain this term without affecting the nature of the  $\Phi_0^s$  field for reasons of convenience and coherence in our forthcoming calculations. The function  $f^{ms}$  defined both at  $\varphi$  and  $\varphi_0$  secures the azimuthal dependence via Eq. (26), while let us remind that all the information on the associated Legendre functions of the first  $P_\ell^m$  and of the second  $Q_\ell^m$  kind can be retrieved from the appendix. The second set of functions of the dynamic field  $\mathbf{H}_2^s$  is constructed via the harmonic character of  $\Phi_2^s$  for external domains, which yields

$$\Phi_2^s = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \sum_{s=e,o} \mathbf{h}_\ell^{(e)ms} [Q_\ell^m(\tau) P_\ell^m(\zeta) f^{ms}(\varphi)], \quad \mathbf{r} \in V^+(\mathbb{R}^3) - \{\mathbf{r}_0\}, \quad (65)$$

where

$$\mathbf{h}_\ell^{(e)ms} = c_\ell^{(e)ms} \hat{\mathbf{x}}_1 + d_\ell^{(e)ms} \hat{\mathbf{x}}_2 + e_\ell^{(e)ms} \hat{\mathbf{x}}_3 \quad \text{for} \quad \ell = 0, 1, 2, \dots, m = 0, 1, 2, \dots, \ell \text{ and } s = e, o, \quad (66)$$

denote the vector character of the unknown constant coefficients  $\mathbf{h}_\ell^{(e)ms}$ . Thus, according to the expansions (63) and (65) and in terms of the spheroidal representation of the position vector

$$\mathbf{r} = \sum_{\kappa=1}^3 x_\kappa \hat{\mathbf{x}}_\kappa = \frac{c}{\sqrt{\tau^2 - \zeta^2}} \left( \tau \sqrt{\tau^2 - 1} \hat{\tau} - \zeta \sqrt{1 - \zeta^2} \hat{\zeta} \right), \quad 1 \leq \tau < +\infty, -1 \leq \zeta \leq 1, \quad (67)$$

the full solution (59) for the scattered magnetic field is expressed as

$$\mathbf{H}_2^s = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \sum_{s=e,o} \left\{ \left[ \mathbf{h}_{\ell}^{(e)ms} + \mathbf{r} \frac{P_{\ell}^{m'}(\tau_s)}{2Q_{\ell}^{m'}(\tau_s)} \left( \frac{\mathbf{m}}{4\pi} \cdot \nabla_{\mathbf{r}_0} \rho_{\ell}^{ms}(\mathbf{r}_0) \right) \right] Q_{\ell}^m(\tau) P_{\ell}^m(\zeta) f^{ms}(\varphi) \right\}, \quad (68)$$

while the generated scattered electric field (60) assumes the form

$$\mathbf{E}_2^s = \frac{1}{\sigma^+} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \sum_{s=e,o} \nabla \left( Q_{\ell}^m(\tau) P_{\ell}^m(\zeta) f^{ms}(\varphi) \right) \times \left[ \mathbf{h}_{\ell}^{(e)ms} + \mathbf{r} \frac{P_{\ell}^{m'}(\tau_s)}{2Q_{\ell}^{m'}(\tau_s)} \left( \frac{\mathbf{m}}{4\pi} \cdot \nabla_{\mathbf{r}_0} \rho_{\ell}^{ms}(\mathbf{r}_0) \right) \right] \quad (69)$$

by straightforward application of the identities (A.3) and (A.5), where both the electromagnetic fields for  $n = 2$  are defined at the exterior domain  $V^+(\mathbb{R}^3) - \{\mathbf{r}_0\}$ . In order to represent the fields (68) and (69) in spheroidal coordinates, we introduce the spheroidal relation of  $\mathbf{h}_{\ell}^{(e)ms}$ , i.e.

$$\begin{aligned} \mathbf{h}_{\ell}^{(e)ms} = & \left( c_{\ell}^{(e)ms} \zeta \sqrt{\tau^2 - 1} + d_{\ell}^{(e)ms} \tau \sqrt{1 - \zeta^2} \cos \varphi + e_{\ell}^{(e)ms} \tau \sqrt{1 - \zeta^2} \sin \varphi \right) \frac{\hat{\tau}}{\sqrt{\tau^2 - \zeta^2}} \\ & + \left( -c_{\ell}^{(e)ms} \tau \sqrt{1 - \zeta^2} + d_{\ell}^{(e)ms} \zeta \sqrt{\tau^2 - 1} \cos \varphi + e_{\ell}^{(e)ms} \zeta \sqrt{\tau^2 - 1} \sin \varphi \right) \frac{\hat{\zeta}}{\sqrt{\tau^2 - \zeta^2}} \\ & + \left( -d_{\ell}^{(e)ms} \sin \varphi + e_{\ell}^{(e)ms} \cos \varphi \right) \hat{\phi} \end{aligned} \quad (70)$$

for  $\ell = 0, 1, 2, \dots$ ,  $m = 0, 1, 2, \dots, \ell$  and  $s = e, o$

for  $1 \leq \tau < +\infty$ ,  $-1 \leq \zeta \leq 1$  and  $0 \leq \varphi < 2\pi$ , which is a consequence of relationships (19), (22) and (23) that are inserted in Eq. (66). The three sets  $c_{\ell}^{(e)ms}$ ,  $d_{\ell}^{(e)ms}$  and  $e_{\ell}^{(e)ms}$  ( $\ell = 0, 1, 2, \dots$ ,  $m = 0, 1, 2, \dots, \ell$ ,  $s = e, o$ ) of scalar coefficients introduced above have to be constructed in accord with the primary (incident) field data (10) and (12), with the value of the particular solution  $\frac{1}{2}(\mathbf{r}\Phi_0^s)$  and with the boundary conditions (59) and (60). We recall from the analytical point of view the additional imposition (61), which is adequate, as a physical property of  $\mathbf{H}_2^s$ , for the confirmation of our results beyond the difficult task of the reduction to the spherical case for  $n = 2$ .

The procedure that is followed for the evaluation of those unknown sets of constant coefficients is considered below, where only the main steps are being emphasized in view of the large amount of calculations that have to be performed. Consequently, we have to enforce the three aforementioned scalar boundary conditions (59) and (60) to the total magnetic and electric fields  $\mathbf{H}_2^s$  (10) with (68) and  $\mathbf{E}_2^s$  (12) with (69) on the particular surface  $\tau = \tau_s = a_1/c$ , i.e. the  $\tau$ -component of  $\hat{\tau} \cdot (\mathbf{H}_2^i + \mathbf{H}_2^s) = 0$  defined as B.C.1 and the  $\zeta$ ,  $\varphi$ -components of  $\hat{\tau} \times (\mathbf{E}_2^i + \mathbf{E}_2^s) = \mathbf{0}$  specified as B.C.2 and B.C.3, respectively. These boundary conditions are sufficient enough to provide us with the unknown constant coefficients  $c_{\ell}^{(e)ms}$ ,  $d_{\ell}^{(e)ms}$  and  $e_{\ell}^{(e)ms}$  for  $\ell = 0, 1, 2, \dots$ ,  $m = 0, 1, 2, \dots, \ell$  and  $s = e, o$  as far as the imposed condition of divergence-free scattered magnetic field (61) is ensured. As one can suppose, the field problem for  $n = 2$  has an enormous additional difficulty and mathematical complexity, thus we are forced to define certain analytical quantities frequently used into our forthcoming calculations. Hence, one step before we apply the boundary conditions and in terms of the position of the magnetic dipole  $\mathbf{r}_0 = (x_{10}, x_{20}, x_{30}) = (\tau_0, \zeta_0, \varphi_0)$ , we introduce the  $C_{\ell}^{ms}$ ,  $D_{\ell}^{ms}$ ,  $E_{\ell}^{ms}$  coefficients as

$$\left\{ \begin{array}{l} C_{\ell}^{ms} \\ D_{\ell}^{ms} \\ E_{\ell}^{ms} \end{array} \right\} = \frac{1}{(2\ell + 1)} \left\{ \begin{array}{l} \sqrt{\tau_s^2 - 1} \\ 2^{-1} \tau_s \\ 2^{-1} \tau_s \end{array} \right\} \left[ \left\{ \frac{\mathbf{m}}{4\pi} \cdot \begin{pmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \\ \hat{\mathbf{x}}_3 \end{pmatrix} \right\} \rho_{\ell}^{ms}(\mathbf{r}_0) P_{\ell}^m(\tau_s) - Q_{\ell}^m(\tau_s) \left\{ \begin{array}{l} c_{\ell}^{(e)ms} \\ d_{\ell}^{(e)ms} \\ e_{\ell}^{(e)ms} \end{array} \right\} \right] \quad (71)$$

and the  $C_{\ell}^{ms'}$ ,  $D_{\ell}^{ms'}$ ,  $E_{\ell}^{ms'}$  coefficients as

$$\left\{ \begin{array}{l} C_{\ell}^{ms'} \\ D_{\ell}^{ms'} \\ E_{\ell}^{ms'} \end{array} \right\} = \frac{1}{(2\ell + 1)} \left\{ \begin{array}{l} \tau_s \sqrt{\tau_s^2 - 1} \\ 2^{-1} (\tau_s^2 - 1) \\ 2^{-1} (\tau_s^2 - 1) \end{array} \right\} \left[ \left\{ \frac{\mathbf{m}}{4\pi} \cdot \begin{pmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \\ \hat{\mathbf{x}}_3 \end{pmatrix} \right\} \rho_{\ell}^{ms'}(\mathbf{r}_0) P_{\ell}^{m'}(\tau_s) - Q_{\ell}^{m'}(\tau_s) \left\{ \begin{array}{l} c_{\ell}^{(e)ms} \\ d_{\ell}^{(e)ms} \\ e_{\ell}^{(e)ms} \end{array} \right\} \right], \quad (72)$$

which contain the coefficients under calculation. Moreover, we define the  $B_{\ell}^{ms}$ ,  $B_{\ell}^{ms'}$  constants

$$\left\{ \begin{array}{l} B_{\ell}^{ms} \\ B_{\ell}^{ms'} \end{array} \right\} = \frac{c \sqrt{\tau_s^2 - 1}}{2(2\ell + 1)} \left\{ \begin{array}{l} \tau_s Q_{\ell}^m(\tau_s) P_{\ell}^{m'}(\tau_s) \\ Q_{\ell}^{m'}(\tau_s) P_{\ell}^m(\tau_s) \end{array} \right\} \left( \frac{\mathbf{m}}{4\pi} \cdot \nabla_{\mathbf{r}_0} \rho_{\ell}^{ms}(\mathbf{r}_0) \right), \quad (73)$$

as well as the  $M_{\ell}^{ms}$  and  $M_{\ell, \kappa}^{ms}$ ,  $\kappa = 1, 2, 3$  constants as

$$M_{\ell}^{ms} = \left( \frac{\mathbf{m}}{4\pi} \cdot \mathbf{r}_0 \right) \frac{\sqrt{\tau_s^2 - 1}}{2c} \rho_{\ell}^{ms}(\mathbf{r}_0) P_{\ell}^{m'}(\tau_s) \quad (74)$$

and

$$\begin{Bmatrix} M_{\ell,1}^{ms} \\ M_{\ell,2}^{ms} \\ M_{\ell,3}^{ms} \end{Bmatrix} = \frac{\rho_\ell^{ms}(\mathbf{r}_0)}{2(2\ell+1)} \begin{Bmatrix} \mathbf{m} \\ \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \\ \hat{\mathbf{x}}_3 \end{Bmatrix} \left[ \begin{Bmatrix} \tau_s \sqrt{\tau_s^2 - 1} \\ 2^{-1}(\tau_s^2 - 1) \\ 2^{-1}(\tau_s^2 - 1) \end{Bmatrix} P_\ell^{m'}(\tau_s) + \begin{Bmatrix} \sqrt{\tau_s^2 - 1} \\ 2^{-1}\tau_s \\ 2^{-1}\tau_s \end{Bmatrix} P_\ell^m(\tau_s) \right], \quad (75)$$

respectively. The definitions (71)–(75) are valid for every  $\ell = 0, 1, 2, \dots$ ,  $m = 0, 1, 2, \dots, \ell$  and  $s = e, o$  and necessary in order to simplify the appearing formulae during our analytical work in progress. In addition

$$\mathbf{m} = \sum_{\kappa=1}^3 m_\kappa \hat{\mathbf{x}}_\kappa, \quad \text{while} \quad P_\ell^{m'}(\tau_s) = \frac{dP_\ell^m(\tau)}{d\tau} \Big|_{\tau=\tau_s} \quad \text{and} \quad Q_\ell^{m'}(\tau_s) = \frac{dQ_\ell^m(\tau)}{d\tau} \Big|_{\tau=\tau_s}, \quad (76)$$

while recalling that  $\varepsilon_m = 1$  for  $m = 0$  and  $\varepsilon_m = 2$  for  $m \geq 1$  we bring back Eq. (29)

$$\rho_\ell^{ms}(\mathbf{r}_0) = \frac{(2\ell+1)}{c} \left[ \frac{(\ell-m)!}{(\ell+m)!} \right]^2 (-1)^m \varepsilon_m Q_\ell^m(\tau_0) P_\ell^m(\zeta_0) f^{ms}(\varphi_0). \quad (77)$$

Continuing, at every three-dimensional position  $\mathbf{r} = (\tau, \zeta, \varphi) \in V^+(\mathbb{R}^3) - \{\mathbf{r}_0\}$ , we apply the boundary conditions (59) and (60) (suitably using expansion (28) with (77)) on the prolate spheroidal surface  $\tau = \tau_s$  as we discussed earlier to obtain

$$\begin{aligned} & \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \sum_{s=e,o} \left\{ ((2\ell+1)B_\ell^{ms} - M_\ell^{ms}) P_\ell^m(\zeta) f^{ms}(\varphi) + (M_{\ell,1}^{ms} - C_\ell^{ms}) ((2\ell+1)\zeta P_\ell^m(\zeta)) f^{ms}(\varphi) \right. \\ & \left. + \left( (2\ell+1)\sqrt{1-\zeta^2} P_\ell^m(\zeta) \right) \left[ (M_{\ell,2}^{ms} - D_\ell^{ms}) (2 \cos \varphi f^{ms}(\varphi)) + (M_{\ell,3}^{ms} - E_\ell^{ms}) (2 \sin \varphi f^{ms}(\varphi)) \right] \right\} = 0 \end{aligned} \quad (78)$$

as far as the first boundary condition B.C.1 is concerned, while the application of relation (60) provides us with two by far more complicated explicit expressions for the rest of the boundary conditions that have to be enforced. In terms of the derivatives  $P_\ell^{m'}(\zeta)$  for  $\zeta \in [-1, 1]$  those are

$$\begin{aligned} & \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \sum_{s=e,o} \left\{ \left[ B_\ell^{ms'} \sqrt{1-\zeta^2} \sqrt{1-\zeta^2} ((2\ell+1)\zeta P_\ell^m(\zeta)) - B_\ell^{ms} ((2\ell+1)(1-\zeta^2) P_\ell^{m'}(\zeta)) \right] f^{ms}(\varphi) \right. \\ & + \left[ -C_\ell^{ms'} \sqrt{1-\zeta^2} ((2\ell+1)\sqrt{1-\zeta^2} P_\ell^m(\zeta)) + C_\ell^{ms} \zeta ((2\ell+1)(1-\zeta^2) P_\ell^{m'}(\zeta)) \right] f^{ms}(\varphi) \\ & + \left[ D_\ell^{ms'} \sqrt{1-\zeta^2} ((2\ell+1)\zeta P_\ell^m(\zeta)) + D_\ell^{ms} \sqrt{1-\zeta^2} ((2\ell+1)(1-\zeta^2) P_\ell^{m'}(\zeta)) \right] 2 \cos \varphi f^{ms}(\varphi) \\ & \left. + \left[ E_\ell^{ms'} \sqrt{1-\zeta^2} ((2\ell+1)\zeta P_\ell^m(\zeta)) + E_\ell^{ms} \sqrt{1-\zeta^2} ((2\ell+1)(1-\zeta^2) P_\ell^{m'}(\zeta)) \right] 2 \sin \varphi f^{ms}(\varphi) \right\} = 0 \end{aligned} \quad (79)$$

for the  $\zeta$ -component, identified as (B.C.2) and

$$\begin{aligned} & \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \sum_{s=e,o} \left\{ [-(2\ell+1)B_\ell^{ms} P_\ell^m(\zeta) + C_\ell^{ms} ((2\ell+1)\zeta P_\ell^m(\zeta))] f^{ms'}(\varphi) + \left( (2\ell+1)\sqrt{1-\zeta^2} P_\ell^m(\zeta) \right) [D_\ell^{ms'} 2 \sin \varphi f^{ms}(\varphi) \right. \\ & \left. + D_\ell^{ms} 2 \cos \varphi f^{ms'}(\varphi) - E_\ell^{ms'} 2 \cos \varphi f^{ms}(\varphi) + E_\ell^{ms} 2 \sin \varphi f^{ms'}(\varphi) \right\} = 0 \end{aligned} \quad (80)$$

for the  $\varphi$ -component, specified as (B.C.3), where the derivatives  $f^{ms'}(\varphi)$  follow the definition (26). The boundary conditions (78)–(80) carry the unknown constant coefficients, which must be determined. However, their form is not appropriate to obey orthogonality rules neither in the  $\zeta$ -dependence nor in the  $\varphi$ -dependence. Nevertheless, by extensive and careful use of the recurrence relations (A.19), (A.20), (A.22), (A.24) and (A.25) for the associated Legendre functions  $P_\ell^m(\zeta)$  for  $\zeta \in [-1, 1]$ , as well as by application of the expressions (A.26)–(A.29) to the angular functions  $f^{ms}(\varphi)$  for  $\varphi \in [0, 2\pi)$ , we manage to transform Eqs. (78)–(80) to a set of boundary relations of the form  $\sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \sum_{s=e,o} (\text{const.}) P_\ell^m(\zeta) f^{ms}(\varphi)$ , which are ready to admit orthogonality manipulation of the set of functions  $P_\ell^m(\zeta) f^{ms}(\varphi)$  for every  $\ell = 0, 1, 2, \dots$ ,  $m = 0, 1, 2, \dots, \ell$  and  $s = e, o$ . Hence, in view of this statement, an appropriate rearrangement of the indexes of the constants and some very cumbersome and difficult elaboration lead us to the following three relations, which interconnect the coefficients to each other,

i.e.

$$\begin{aligned} & (\ell+m+1)C_{\ell+1}^{ms} + (\ell-m)C_{\ell-1}^{ms} + (D_{\ell-1}^{m-1s} - (-1)^j E_{\ell-1}^{m-1s}) - (D_{\ell+1}^{m-1s} - (-1)^j E_{\ell+1}^{m-1s}) + (\ell+m+1)(\ell+m+2)(D_{\ell+1}^{m+1s} \\ & + (-1)^j E_{\ell+1}^{m+1s}) - (\ell-m-1)(\ell-m)(D_{\ell-1}^{m+1s} + (-1)^j E_{\ell-1}^{m+1s}) = ((2\ell+1)B_\ell^{ms} - M_\ell^{ms}) + (\ell+m+1)M_{\ell+1,1}^{ms} \\ & + (\ell-m)M_{\ell-1,1}^{ms} + (M_{\ell-1,2}^{m-1s} - (-1)^j M_{\ell-1,3}^{m-1s}) - (M_{\ell+1,2}^{m-1s} - (-1)^j M_{\ell+1,3}^{m-1s}) + (\ell+m+1)(\ell+m+2)(M_{\ell+1,2}^{m+1s} \\ & + (-1)^j M_{\ell+1,3}^{m+1s}) - (\ell-m-1)(\ell-m)(M_{\ell-1,2}^{m+1s} + (-1)^j M_{\ell-1,3}^{m+1s}) \end{aligned} \quad (81)$$

as a consequence of (B.C.1) (relationship (78)), while

$$\begin{aligned}
& \frac{(\ell+m+1)(\ell+m+2)}{(2\ell+3)} (C_{\ell+2}^{ms'} + (\ell+3)C_{\ell+2}^{ms}) + \frac{(\ell-m-1)(\ell-m)}{(2\ell-3)} (C_{\ell-2}^{ms'} - (\ell-2)C_{\ell-2}^{ms}) \\
& + \frac{(2\ell+1)}{(2\ell-1)(2\ell+3)} ((\ell^2-3m^2+\ell)C_{\ell}^{ms} - 2(\ell^2+m^2+\ell-1)C_{\ell}^{ms'}) \\
& + \frac{(2\ell+1)(\ell-m)(\ell+m+1)}{(2\ell-1)(2\ell+3)} \left[ (2m+1)(D_{\ell}^{m+1s'} + (-1)^j E_{\ell}^{m+1s'}) - (2\ell^2+2\ell-3m-3)(D_{\ell}^{m+1s} + (-1)^j E_{\ell}^{m+1s}) \right] \\
& + \frac{(2\ell+1)}{(2\ell-1)(2\ell+3)} \left[ (2m-1)(D_{\ell}^{m-1s'} - (-1)^j E_{\ell}^{m-1s'}) + (2\ell^2+2\ell+3m-3)(D_{\ell}^{m-1s} - (-1)^j E_{\ell}^{m-1s}) \right] \\
& - \frac{(\ell-m-2)(\ell-m-1)(\ell-m)}{(2\ell-1)} \left[ (D_{\ell-2}^{m+1s'} + (-1)^j E_{\ell-2}^{m+1s'}) - (\ell-2)(D_{\ell-2}^{m+1s} + (-1)^j E_{\ell-2}^{m+1s}) \right] \\
& + \frac{(\ell-m)}{(2\ell-1)} \left[ (D_{\ell-2}^{m-1s'} - (-1)^j E_{\ell-2}^{m-1s'}) - (\ell-2)(D_{\ell-2}^{m-1s} - (-1)^j E_{\ell-2}^{m-1s}) \right] \\
& + \frac{(\ell+m+1)(\ell+m+2)(\ell+m+3)}{(2\ell+3)} \left[ (D_{\ell+2}^{m+1s'} + (-1)^j E_{\ell+2}^{m+1s'}) + (\ell+3)(D_{\ell+2}^{m+1s} + (-1)^j E_{\ell+2}^{m+1s}) \right] \\
& - \frac{(\ell+m+1)}{(2\ell+3)} \left[ (D_{\ell+2}^{m-1s'} - (-1)^j E_{\ell+2}^{m-1s'}) + (\ell+3)(D_{\ell+2}^{m-1s} - (-1)^j E_{\ell+2}^{m-1s}) \right] \\
& = \left[ \frac{(\ell+m+1)(\ell+m+2)(\ell+m+3)}{(2\ell+3)(2\ell+5)} B_{\ell+3}^{ms'} + \frac{(\ell-m-2)(\ell-m-1)(\ell-m)}{(2\ell-3)(2\ell-1)} B_{\ell-3}^{ms'} \right] \\
& - (\ell+m+1) \left[ \frac{(\ell+m-1)(\ell-m+3) + (2m-1)^2}{(2\ell-1)(2\ell+5)} B_{\ell+1}^{ms'} - (\ell+2)B_{\ell+1}^{ms} \right] \\
& - (\ell-m) \left[ \frac{(\ell+m-2)(\ell-m+2) + (2m-1)^2}{(2\ell-3)(2\ell+3)} B_{\ell-1}^{ms'} + (\ell-1)B_{\ell-1}^{ms} \right] \tag{82}
\end{aligned}$$

for (B.C.2) (relation (79)) and

$$\begin{aligned}
& m(-1)^j ((\ell+m+1)C_{\ell+1}^{ms} + (\ell-m)C_{\ell-1}^{ms}) - (\ell+m+1)(\ell+m+2) \left[ (E_{\ell+1}^{m+1s'} - (-1)^j D_{\ell+1}^{m+1s'}) + (m+1)(E_{\ell+1}^{m+1s} - (-1)^j D_{\ell+1}^{m+1s}) \right] \\
& + (\ell-m-1)(\ell-m) \left[ (E_{\ell-1}^{m+1s'} - (-1)^j D_{\ell-1}^{m+1s'}) + (m+1)(E_{\ell-1}^{m+1s} - (-1)^j D_{\ell-1}^{m+1s}) \right] \\
& - \left[ (E_{\ell-1}^{m-1s'} + (-1)^j D_{\ell-1}^{m-1s'}) - (m-1)(E_{\ell-1}^{m-1s} + (-1)^j D_{\ell-1}^{m-1s}) \right] + \left[ (E_{\ell+1}^{m-1s'} + (-1)^j D_{\ell+1}^{m-1s'}) - (m-1)(E_{\ell+1}^{m-1s} + (-1)^j D_{\ell+1}^{m-1s}) \right] \\
& = m(-1)^j (2\ell+1)B_{\ell}^{ms} \tag{83}
\end{aligned}$$

as far as boundary condition (B.C.3) (Eq. (80)) is concerned. Boundary relations (81)–(83) hold true for  $\ell = 0, 1, 2, \dots, m = 0, 1, 2, \dots, \ell$  and  $s = e, o$  (whenever a negative index appears, the relative constant is set to nil), while we operate on system of Eqs. (81)–(83) by defining  $\bar{s}$  and  $j$  as follows:

$$s = e \Rightarrow \bar{s} = o \Rightarrow j = 2 \quad \text{and} \quad s = o \Rightarrow \bar{s} = e \Rightarrow j = 1 \tag{84}$$

for reasons of notational convenience. The complicated recurrence relations (81)–(83) contain the constant coefficients  $c_{\ell}^{(e)ms}$ ,  $d_{\ell}^{(e)ms}$  and  $e_{\ell}^{(e)ms}$  with different upper and lower indexes. Thus, writing separately (for  $s = e$  and for  $s = o$ ) six even–odd equations of the form (81)–(83), substituting Eqs. (71)–(75) and gathering in pertinent fashion the constants, we are forced to solve the non-homogeneous linear system of equations

$$\tilde{\mathbf{A}}_{\ell}^m \mathbf{x}_{\ell}^m = \mathbf{b}_{\ell}^m \quad \text{for} \quad \ell = 0, 1, 2, \dots \quad \text{and} \quad m = 0, 1, 2, \dots, \ell, \tag{85}$$

where the  $6 \times 50$  matrix  $\tilde{\mathbf{A}}_{\ell}^m$  of the coefficients of the unknowns is stated by

$$\tilde{\mathbf{A}}_{\ell}^m = \begin{bmatrix} A_{BC1,1}^e & A_{BC1,2}^e & \cdots & A_{BC1,49}^e & A_{BC1,50}^e \\ A_{BC2,1}^e & A_{BC2,2}^e & \cdots & A_{BC2,49}^e & A_{BC2,50}^e \\ A_{BC3,1}^e & A_{BC3,2}^e & \cdots & A_{BC3,49}^e & A_{BC3,50}^e \\ A_{BC1,1}^o & A_{BC1,2}^o & \cdots & A_{BC1,49}^o & A_{BC1,50}^o \\ A_{BC2,1}^o & A_{BC2,2}^o & \cdots & A_{BC2,49}^o & A_{BC2,50}^o \\ A_{BC3,1}^o & A_{BC3,2}^o & \cdots & A_{BC3,49}^o & A_{BC3,50}^o \end{bmatrix}, \tag{86}$$

while the  $50 \times 1$  vector  $\mathbf{x}_\ell^m$  of the unknown coefficients that interests us, is assumed to be

$$\mathbf{x}_\ell^m = \begin{bmatrix} c_{\ell-2}^{(e)me} & \dots & c_{\ell+2}^{(e)me} & c_{\ell-2}^{(e)mo} & \dots & c_{\ell+2}^{(e)mo} & d_{\ell-2}^{(e)m-1e} & \dots & d_{\ell+2}^{(e)m-1e} & d_{\ell-2}^{(e)m+1e} & \dots & d_{\ell+2}^{(e)m+1e} & d_{\ell-2}^{(e)m-1o} & \dots & d_{\ell+2}^{(e)m-1o} \\ d_{\ell-2}^{(e)m+1o} & \dots & d_{\ell+2}^{(e)m+1o} & e_{\ell-2}^{(e)m-1e} & \dots & e_{\ell+2}^{(e)m-1e} & e_{\ell-2}^{(e)m+1e} & \dots & e_{\ell+2}^{(e)m+1e} & e_{\ell-2}^{(e)m-1o} & \dots & e_{\ell+2}^{(e)m-1o} & e_{\ell-2}^{(e)m+1o} & \dots & e_{\ell+2}^{(e)m+1o} \end{bmatrix}^\top \quad (87)$$

and the  $6 \times 1$  vector  $\mathbf{b}_\ell^m$  of the known constants yields

$$\mathbf{b}_\ell^m = [b_{BC1}^e \quad b_{BC2}^e \quad b_{BC3}^e \quad b_{BC1}^o \quad b_{BC2}^o \quad b_{BC3}^o]^\top. \quad (88)$$

The elements of the matrices  $\tilde{\mathbf{A}}_\ell^m$  and  $\mathbf{b}_\ell^m$  are known complicated expressions resulting from (81)–(83). It is obvious that in order to solve analytically the system (85)–(88), we are obliged to limit ourselves to some necessary simplifications so as to obtain the solution of  $\mathbf{H}_2^s$  and  $\mathbf{E}_2^s$  in a closed form. Consequently, in view of the assumptions that we have to make, we apply the well-known “cut-off” method, usual in obtaining closed solutions. This method is based on the selective choice of certain constant coefficients of different indexes so as to have to solve as many equations as the unknowns. Thus, if we choose to keep six sets of coefficients, e.g.  $c_{\ell-1}^{(e)ms}$ ,  $d_{\ell-1}^{(e)m-1s}$  and  $e_{\ell-1}^{(e)m-1s}$  for  $\ell = 0, 1, 2, \dots$ ,  $m = 0, 1, 2, \dots, \ell$  and  $s = e, o$ , then we solve a system of six equations with six unknowns to obtain the solution in closed form. On the other hand, if we introduce the inverted matrix of  $\tilde{\mathbf{A}}_\ell^m$  as  $\tilde{\mathbf{A}}_\ell^{m\top}$ , then there exist the following theorem, which provides us with the full unique solution of the system (85)–(88). In details, if the aforementioned system has not a solution then the linear and non-homogeneous system

$$(\tilde{\mathbf{A}}_\ell^{m\top} \tilde{\mathbf{A}}_\ell^m) \mathbf{x}_\ell^m = \tilde{\mathbf{A}}_\ell^{m\top} \mathbf{b}_\ell^m \quad \text{for } \ell = 0, 1, 2, \dots \text{ and } m = 0, 1, 2, \dots, \ell \quad (89)$$

has the unique solution

$$\mathbf{x}_\ell^m = (\tilde{\mathbf{A}}_\ell^{m\top} \tilde{\mathbf{A}}_\ell^m)^{-1} \tilde{\mathbf{A}}_\ell^{m\top} \mathbf{b}_\ell^m \quad \text{for } \ell = 0, 1, 2, \dots \text{ and } m = 0, 1, 2, \dots, \ell, \quad (90)$$

whenever the homogeneous system

$$(\tilde{\mathbf{A}}_\ell^{m\top} \tilde{\mathbf{A}}_\ell^m) \mathbf{x}_\ell^m = \mathbf{0} \quad \text{for } \ell = 0, 1, 2, \dots \text{ and } m = 0, 1, 2, \dots, \ell \quad (91)$$

assumes the zero-solution  $\mathbf{x}_\ell^m = \mathbf{0}$  for every  $\ell = 0, 1, 2, \dots$  and  $m = 0, 1, 2, \dots, \ell$ . Finally, the scattered electromagnetic fields  $\mathbf{H}_2^s$  and  $\mathbf{E}_2^s$  in prolate spheroidal geometry are given by collecting all previous results and substituting them into Eqs. (68) and (69), with the aim of (67) and (70) to obtain the corresponding spheroidal expressions.

Now, we are left to satisfy the divergence-free relation (61), where straightforward calculations on the potentials (63), (65) and combined use of the relations of the appendix, as well as utilization of Eqs. (20), (66) and (67), provides us with the desirable result. In addition, the task of reducing our results to the spherical ones is not easy and it is not with the purpose of the present work to enter into so many calculations, since the confirmation of the divergence-free scattered magnetic field secures sufficiently enough our analytical results. However, one can follow the very same limiting procedure, as the semifocal distance tends to zero, described via relationships (31)–(35) taking

$$\lim_{c \rightarrow 0^+} \mathbf{H}_2^s \equiv \mathbf{H}_{2,sphere}^s \quad \text{and} \quad \lim_{c \rightarrow 0^+} \mathbf{E}_2^s \equiv \mathbf{E}_{2,sphere}^s, \quad (92)$$

whereas the spherical scattered fields  $\mathbf{H}_{2,sphere}^s$  and  $\mathbf{E}_{2,sphere}^s$  are given in [19] and it is not worth to write down these relations, which assume very complicated closed forms.

#### 4.4. The scattered electromagnetic fields $\mathbf{H}^s$ and $\mathbf{E}^s$ for $n = 0, 2, 3$

Recapitulating, for the degree of interest here  $n = 0, 2, 3$ , where we obtain the most interesting parts of the electromagnetic fields, the scattered magnetic field is expressed as

$$\mathbf{H}^s = [\mathbf{H}_0^s + \mathbf{H}_2^s(ik)^2 + \mathbf{H}_3^s(ik)^3] + O((ik)^4), \quad \mathbf{r} \in V^+(\mathbb{R}^3) - \{\mathbf{r}_0\}, \quad (93)$$

while the scattered electric field assumes the form

$$\mathbf{E}^s = [\mathbf{E}_2^s(ik)^2] + O((ik)^4), \quad \mathbf{r} \in V^+(\mathbb{R}^3) - \{\mathbf{r}_0\}, \quad (94)$$

where the fields  $\mathbf{H}_0^s$ ,  $\mathbf{H}_2^s$ ,  $\mathbf{H}_3^s$  and  $\mathbf{E}_2^s$  have being evaluated within Sections 4.1–4.3 in closed form. Here, we insert the wave-number  $k$  from (1) into the relations (93) and (94), whereas some trivial analysis leads to

$$\mathbf{H}^s = \left[ \mathbf{H}_0^s + (\omega\mu^+\sigma^+) \sqrt{\frac{\omega\mu^+\sigma^+}{2}} \mathbf{H}_3^s \right] + (\omega\mu^+\sigma^+) \left[ \sqrt{\frac{\omega\mu^+\sigma^+}{2}} \mathbf{H}_3^s - \mathbf{H}_2^s \right] i + O((ik)^4) \quad (95)$$

and

$$\mathbf{E}^s = [-(\omega\mu^+\sigma^+) \mathbf{E}_2^s] i + O((ik)^4), \quad (96)$$

respectively for every  $\mathbf{r} \in V^+(\mathbb{R}^3) - \{\mathbf{r}_0\}$ . We observe that the electric field is purely imaginary, while the magnetic field is a complex number, indicating that the electromagnetic fields at  $n = 2$  are adequate for the full solutions, since the contribution of the  $\mathbf{H}_3^s$  is of minor significance.



## 5. Analytical application of the method

In order to demonstrate the usefulness of our generalized analytical results we supplement the paper by an example of application of our methodology. That way, one can realize that this work does belong to the applied mathematics area, but also covers the corresponding engineering science field. In this sense, the start point is Ref. [20], which refers to electromagnetic induction response to spheroidal anomalies. A significant part of this work is devoted to the response to a perfectly conducting spheroidal body by considering separately the two cases of an axial and a transverse constant magnetic incident field, produced by the corresponding magnetic dipole. Under those physical circumstances, in this paper [20], simple analytical solutions are obtained. However, these results are concerned with some special reduced cases of our general method. Hence, in order to recover the results given in [20] from our general three-dimensional expressions, we shall follow the straightforward analytical procedure of extracting the already simplified results from ours.

Under this aim, we consider the full incident magnetic field (7), where in order to be in accordance with the application Ref. [20] and in view of the non-trivial terms (9)–(11), the constant part of (7), called as  $\mathbf{H}_{\text{cst}}^i$ , takes the form

$$\mathbf{H}_{\text{cst}}^i = \mathbf{H}_3^i(ik)^3 = (\omega\mu^+\sigma^+)\sqrt{\frac{\omega\mu^+\sigma^+}{2}}(1+i)\mathbf{H}_3^i = -\frac{2}{3}(\omega\mu^+\sigma^+)\sqrt{\frac{\omega\mu^+\sigma^+}{2}}(1+i)\frac{\mathbf{m}}{4\pi}, \quad (97)$$

with the aim of Eqs. (1) and (11) (note that  $\mathbf{m} \cdot \tilde{\mathbf{I}} = \mathbf{m}$ ). Expression (97) stands for the constant analogous of the incident magnetic field (7). Here, we must recall the three-dimensional representation of the magnetic dipole  $\mathbf{m} = \sum_{\kappa=1}^3 m_\kappa \hat{\mathbf{x}}_\kappa$ . Consequently, we observe that the constant magnetic incident field comes from our general magnetic incident field using the  $n=3$  term from Eq. (11), which is also responsible for the corresponding magnetic scattered field. This field, indicated as  $\mathbf{H}_{(i)\text{-cst}}^s$ , is derived from the general form (93) or (95), that is

$$\mathbf{H}_{(i)\text{-cst}}^s = \mathbf{H}_3^s(ik)^3 = (\omega\mu^+\sigma^+)\sqrt{\frac{\omega\mu^+\sigma^+}{2}}(1+i)\mathbf{H}_3^s, \quad \mathbf{r} \in V^+(\mathbb{R}^3) - \{\mathbf{r}_0\}, \quad (98)$$

whereas  $\mathbf{H}_3^s$  has been found in the general form provided by relationships (45) and (46). Thus, we are ready to take the two explicit cases under consideration as in paper [20].

### 5.1. Calculation of the axial response to a perfectly conducting prolate or oblate spheroid

We firstly assume the case of the prolate spheroid as described previously in Section 4.1. Here, the constant incident magnetic field is directed along the  $\hat{\mathbf{x}}_1$  axis and aligned along the long axis of the prolate spheroid, i.e.,  $\mathbf{m} = m_1 \hat{\mathbf{x}}_1$  and relation (97) is written as follows:

$$\mathbf{H}_{\text{cst}}^{i.ax.} = -\frac{2}{3}(\omega\mu^+\sigma^+)\sqrt{\frac{\omega\mu^+\sigma^+}{2}}(1+i)\frac{m_1}{4\pi}\hat{\mathbf{x}}_1. \quad (99)$$

The particular direction of the magnetic dipole, in view of expressions (46), results in

$$\frac{\mathbf{m}}{4\pi} \cdot \mathbf{f}(\tau) = \frac{m_1}{4\pi} \frac{Q_1(\tau)}{Q_1'(\tau_s)} \quad \text{and} \quad \frac{\mathbf{m}}{4\pi} \cdot \mathbf{g}(\tau, \varphi) = 0, \quad \mathbf{r} \in V^+(\mathbb{R}^3) - \{\mathbf{r}_0\}, \quad (100)$$

where substituting Eq. (100) to our general magnetic scattered field (98) and in terms of the  $\mathbf{H}_3^s$  field from (45), we obtain

$$\mathbf{H}_{(i)\text{-cst}}^{s.ax.} = \frac{2}{3}(\omega\mu^+\sigma^+)\sqrt{\frac{\omega\mu^+\sigma^+}{2}}(1+i)\frac{m_1}{4\pi} \frac{1}{\sqrt{\tau^2 - \zeta^2}} \left[ \sqrt{\tau^2 - 1} \frac{Q_1'(\tau)}{Q_1'(\tau_s)} \zeta \hat{\boldsymbol{\tau}} - \frac{Q_1(\tau)}{Q_1'(\tau_s)} \sqrt{1 - \zeta^2} \hat{\boldsymbol{\zeta}} \right], \quad (101)$$

for every  $\mathbf{r} \in V^+(\mathbb{R}^3) - \{\mathbf{r}_0\}$ , where  $\tau_s = a_1/c$ . Expression (101) for the axial scattered field produced by an axial constant magnetic field, which assumes an axisymmetric form, is *identical* to the corresponding expression derived in Ref. [20, relation B11, p. 2206], of course with the introduction of a different notation. Hence, their results can be taken from ours much more general for the particular case of a constant axial incident magnetic field.

Finally, the corresponding results for the oblate spheroidal geometry are obtained by substitution of the transformation (30) into Eq. (101), while the spherical limit is analysed extensively within our previous paragraphs.

### 5.2. Calculation of the transverse response to a perfectly conducting prolate or oblate spheroid

For once more we take the case of a prolate spheroidal obstacle described in Section 4.1. Here, the constant incident magnetic field is directed along the  $\hat{\mathbf{x}}_2$  axis in the coordinate system of the spheroid, i.e.,  $\mathbf{m} = m_2 \hat{\mathbf{x}}_2$  and relation (97) reduces to

$$\mathbf{H}_{\text{cst}}^{i.tr.} = -\frac{2}{3}(\omega\mu^+\sigma^+)\sqrt{\frac{\omega\mu^+\sigma^+}{2}}(1+i)\frac{m_2}{4\pi}\hat{\mathbf{x}}_2. \quad (102)$$

Now, this particular direction of the magnetic dipole and in view again of expressions (46), leave the corresponding components of the functions  $\mathbf{f}(\tau)$  and  $\mathbf{g}(\tau, \varphi)$ , that is

$$\frac{\mathbf{m}}{4\pi} \cdot \mathbf{f}(\tau) = 0 \quad \text{and} \quad \frac{\mathbf{m}}{4\pi} \cdot \mathbf{g}(\tau, \varphi) = \frac{m_2}{4\pi} \frac{\tau_s}{\sqrt{\tau_s^2 - 1}} \frac{Q_1^1(\tau)}{Q_1^1(\tau_s)} \cos \varphi, \quad \mathbf{r} \in V^+(\mathbb{R}^3) - \{\mathbf{r}_0\}. \quad (103)$$

We now compute the corresponding scattered field by inserting Eq. (103) into the general magnetic scattered field (98), whereas in terms of the  $\mathbf{H}_3^s$  field from (45), we conclude that

$$\mathbf{H}_{(i) \rightarrow \text{cst}}^{\text{s.tr.}} = \frac{2}{3} (\omega \mu^+ \sigma^+) \sqrt{\frac{\omega \mu^+ \sigma^+}{2}} (1 + i) \frac{m_2}{4\pi} \times \frac{\tau_s}{\sqrt{\tau_s^2 - 1}} \left[ \frac{\sqrt{\tau^2 - 1}}{\sqrt{\tau^2 - \zeta^2}} \frac{Q_1^1(\tau)}{Q_1^1(\tau_s)} \sqrt{1 - \zeta^2} \cos \varphi \hat{\tau} + \frac{1}{\sqrt{\tau^2 - \zeta^2}} \frac{Q_1^1(\tau)}{Q_1^1(\tau_s)} \zeta \cos \varphi \hat{\zeta} - \frac{1}{\sqrt{\tau^2 - 1}} \frac{Q_1^1(\tau)}{Q_1^1(\tau_s)} \sin \varphi \hat{\phi} \right], \quad (104)$$

where  $\mathbf{r} \in V^+(\mathbb{R}^3) - \{\mathbf{r}_0\}$  and  $\tau_s = a_1/c$ . Relationship (104) for the transverse scattered field produced by a transverse constant magnetic field, which assumes a non-axisymmetric form, is for once again *identical* to the corresponding expression derived in Ref. [20, relation B20, p. 2207], with the introduction of the proper notation. Hence, their results can be taken from ours much more general for the particular case of a constant transverse incident magnetic field.

Obviously, the corresponding results for an oblate spheroidal buried obstacle are obtained by substitution of the transformation (30) into Eq. (101), while the spherical limit is analysed extensively within our previous paragraphs.

The application of our results to more simple forms, already evaluated by other authors [20], shows the generality of our method. More specific, the kind of incident field that we use is the most general three-dimensional, thus non-axisymmetric, type of fields, where all kind of incident fields are taken into account, constant or not constant, for an arbitrary direction and location of the magnetic dipole. Our generalized method is unique and permits general manipulation of the low-frequency equations.

## 6. Conclusions

The large amount of vector data, the electromagnetic and geometrical complexity of the Earth, the many configurations of sources and receivers, the uncertainty resulting from datasets containing both the contribution of the primary field (observed if the ore bodies were, hypothetically, taken out) and the contribution of the secondary field (resulting from the interaction of that primary field with the ore bodies), explain the continuous interest of elaborating within the frame of analytical and numerical methods of solving forward and inverse electromagnetic scattering problems. One is confronted with a near-field problem, where planar skin depths are significantly larger than source-body or body-sensor distances and only diffusion phenomena occur (conduction currents are predominant). As far as the spheroidal shape is concerned, it is highly versatile and easily matches single obstacles of smooth surface and arbitrary proportions, while simplified shapes provide a proper first model when probing ore bodies in the Earth. On the other hand, the assumption of perfectly conducting spheroidal bodies is primary realistic in view of the high conductivity of most mineral ores, their huge conductivity ratio with their surrounding medium, and the low operation frequencies. Present investigations confirm that simple models as ours appear reliable when used to model the response of a general three-dimensional spheroid to a localized vector source in a homogeneous conductive medium both for low-contrast cases and high-contrast cases. Our devised modeling tools are based on a rigorous low-frequency analysis of the electromagnetic fields, where both their real and imaginary part are of equivalent significance in the development of a reliable model, while the first three non-zero terms of the analytical expansion of the magnetic fields and the corresponding first non-zero term of the one of the electric field are sufficient enough for further numerical elaboration or implementation in view of a future inversion scheme. In view of this aspect, mathematical and computational work is currently in progress in several directions, such as different and more complicated geometries.

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## Appendix

In the interest of making this work complete and independent we provide some useful material, which was used during our calculations and can be found in [10].

We begin with the introduction of certain identities. Let  $u$ ,  $v$  and  $\mathbf{f}$ ,  $\mathbf{g}$  denote two scalar and two vector fields, respectively. Then, if we define by  $\mathbb{S}$  a dyadic, the basic identities used in this project concern the action of the gradient or the Laplace operator on the following expressions:

$$\nabla \otimes (\mathbf{u}\mathbf{f}) = u(\nabla \otimes \mathbf{f}) + \nabla u \otimes \mathbf{f}, \quad (\text{A.1})$$

$$\nabla \cdot (\mathbf{u}\mathbf{f}) = u\nabla \cdot \mathbf{f} + \nabla u \cdot \mathbf{f}, \quad (\text{A.2})$$

$$\nabla \times (\mathbf{u}\mathbf{f}) = u\nabla \times \mathbf{f} + \nabla u \times \mathbf{f}, \quad (\text{A.3})$$

$$\nabla(\mathbf{f} \cdot \mathbf{g}) = (\nabla \otimes \mathbf{f}) \cdot \mathbf{g} + (\nabla \otimes \mathbf{g}) \cdot \mathbf{f}, \quad (\text{A.4})$$

$$\nabla(uv) = u\nabla v + v\nabla u, \quad (\text{A.5})$$

$$\nabla \otimes (\tilde{\mathbb{S}} \cdot \mathbf{f}) = (\nabla \otimes \tilde{\mathbb{S}}) \cdot \mathbf{f} + (\nabla \otimes \mathbf{f}) \cdot \tilde{\mathbb{S}}^\top, \quad (\text{A.6})$$

$$\nabla \otimes (\mathbf{f} \otimes \mathbf{g}) = (\nabla \otimes \mathbf{f}) \otimes \mathbf{g} + [\mathbf{f} \otimes (\nabla \otimes \mathbf{g})]^{213}, \quad (\text{A.7})$$

$$\nabla \times \nabla \times \mathbf{f} = \nabla \nabla \cdot \mathbf{f} - \Delta \mathbf{f}, \quad (\text{A.8})$$

$$\Delta(\mathbf{u}\mathbf{f}) = \mathbf{f}\Delta u + u\Delta \mathbf{f} + 2\nabla u \cdot (\nabla \otimes \mathbf{f}), \quad (\text{A.9})$$

whereas the symbol “ $\otimes$ ” denotes juxtaposition,  $\tilde{\mathbb{S}}^\top$  is the inverted dyadic and the symbol  $(\ )^{213}$  denotes left transposition for a triadic.

The associated Legendre functions of the first  $P_\ell^m(x)$  and of the second  $Q_\ell^m(x)$  kind [10] are linear independent solutions of the associated Legendre differential equation

$$(1-x^2)y''(x) - 2xy'(x) + \left[ \ell(\ell+1) - \frac{m^2}{1-x^2} \right]y(x) = 0 \quad (\text{A.10})$$

for every  $\ell = 0, 1, 2, \dots$  and  $m = 0, 1, 2, \dots, \ell$ , which is valid for  $|x| < 1$  and  $x > 1$ . These functions are defined as follows:

$$P_\ell^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x), \quad |x| < 1 \quad (\text{A.11})$$

and

$$P_\ell^m(x) = (x^2-1)^{m/2} \frac{d^m}{dx^m} P_\ell(x), \quad x > 1, \quad (\text{A.12})$$

where the Legendre polynomials  $P_\ell(x)$  are furnished by the Rodrigues formula

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2-1)^\ell, \quad x \in \mathbb{R} \quad (\text{A.13})$$

for every  $\ell = 0, 1, 2, \dots$  and  $m = 0, 1, 2, \dots, \ell$ . Equivalent, for the same values of  $\ell$  and  $m$

$$Q_\ell^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} Q_\ell(x), \quad |x| < 1 \quad (\text{A.14})$$

and

$$Q_\ell^m(x) = (x^2-1)^{m/2} \frac{d^m}{dx^m} Q_\ell(x), \quad x > 1, \quad (\text{A.15})$$

and here the Legendre functions of the second kind appear in the form

$$Q_\ell(x) = P_\ell(x)Q_0(x) - \sum_{\kappa=1}^{\lfloor \ell/2 \rfloor} \frac{(2\ell-4\kappa+3)}{(2\kappa-1)(\ell-\kappa+1)} P_{\ell-2\kappa+1}(x), \quad x \in \mathbb{R}, \quad (\text{A.16})$$

with

$$Q_0(x) = \frac{1}{2} \ln \frac{1+x}{1-x}, \quad |x| < 1 \quad \text{and} \quad Q_0(x) = \frac{1}{2} \ln \frac{x+1}{x-1}, \quad x > 1. \quad (\text{A.17})$$

The functions  $P_\ell^m(x)$ ,  $|x| < 1$ ,  $\ell \geq 0$ ,  $m \leq \ell$  satisfy the orthogonality relation

$$(P_\ell^m(x), P_{\ell'}^m(x)) = \int_{-1}^{+1} P_\ell^m(x) P_{\ell'}^m(x) dx = \frac{2}{2n+1} \frac{(\ell+m)!}{(\ell-m)!} \delta_{\ell\ell'}, \quad \ell \geq 0, \quad \ell' \geq 0 \quad (\text{A.18})$$

with  $\delta_{\ell\ell'}$  being the Kronecker delta and  $m = 0, 1, 2, \dots, \ell$ . This integral becomes singular for the set of functions  $Q_\ell^m(x)$ ,  $|x| < 1$ ,  $\ell \geq 0$ ,  $m \leq \ell$ .

The associated Legendre functions of the first kind satisfy

$$(2\ell+1)xP_\ell^m(x) = (\ell+m)P_{\ell-1}^m(x) + (\ell-m+1)P_{\ell+1}^m(x), \quad x \in \mathbb{R}, \quad (\text{A.19})$$

while

$$(2\ell+1)(1-x^2) \frac{d}{dx} P_\ell^m(x) = (\ell+1)(\ell+m)P_{\ell-1}^m(x) - \ell(\ell-m+1)P_{\ell+1}^m(x), \quad |x| < 1 \quad (\text{A.20})$$

or

$$(2\ell + 1)(x^2 - 1) \frac{d}{dx} P_\ell^m(x) = \ell(\ell - m + 1)P_{\ell+1}^m(x) - (\ell + 1)(\ell + m)P_{\ell-1}^m(x), \quad x > 1 \quad (\text{A.21})$$

and

$$(2\ell + 1)\sqrt{1 - x^2}P_\ell^m(x) = (\ell + m)(\ell + m - 1)P_{\ell-1}^{m-1}(x) - (\ell - m + 1)(\ell - m + 2)P_{\ell+1}^{m-1}(x) = P_{\ell+1}^{m+1}(x) - P_{\ell-1}^{m+1}(x), \quad |x| < 1. \quad (\text{A.22})$$

or

$$(2\ell + 1)\sqrt{x^2 - 1}P_\ell^m(x) = (\ell - m + 1)(\ell - m + 2)P_{\ell+1}^{m-1}(x) - (\ell + m)(\ell + m - 1)P_{\ell-1}^{m-1}(x) = P_{\ell+1}^{m+1}(x) - P_{\ell-1}^{m+1}(x), \quad x > 1. \quad (\text{A.23})$$

These recurrence relations hold true also for the associated Legendre functions of the second kind  $Q_\ell^m(x)$ ,  $x \in \mathbb{R}$  for the values of  $\ell = 0, 1, 2, \dots$  and  $m = 0, 1, 2, \dots, \ell$ . By definition

$$P_\kappa^l(x) \equiv 0, \quad l > \kappa \quad \text{and} \quad P_\kappa^l(x) = Q_\kappa^l(x) \equiv 0, \quad l < 0, \quad \kappa < 0, \quad x \in \mathbb{R}, \quad (\text{A.24})$$

$$P_\ell^m(\pm 1) = 0, \quad m \neq 0 \quad \text{and} \quad P_\ell(1) = 1, \quad P_\ell(-1) = (-1)^\ell, \quad \ell \geq 0. \quad (\text{A.25})$$

Finally, as far as the trigonometric functions  $\sin m\varphi$  and  $\cos m\varphi$  are concerned for  $0 \leq m \leq \ell$  and  $\ell = 0, 1, 2, \dots$ , the following expressions hold true

$$\sin \varphi \sin m\varphi = \frac{1}{2} [\cos(m-1)\varphi - \cos(m+1)\varphi], \quad (\text{A.26})$$

$$\cos \varphi \cos m\varphi = \frac{1}{2} [\cos(m-1)\varphi + \cos(m+1)\varphi], \quad (\text{A.27})$$

$$\cos \varphi \sin m\varphi = \frac{1}{2} [\sin(m+1)\varphi + \sin(m-1)\varphi], \quad (\text{A.28})$$

$$\sin \varphi \cos m\varphi = \frac{1}{2} [\sin(m+1)\varphi - \sin(m-1)\varphi], \quad (\text{A.29})$$

where  $\varphi \in [0, 2\pi)$  stands for the azimuthal angle, taken for the first-period of the trigonometric circle, while the orthogonal-ity here is obvious for the functions  $\sin m\varphi$  and  $\cos m\varphi$ .

## References

- [1] G. Dassios, R.E. Kleinman, *Low Frequency Scattering*, Oxford University Press, Oxford, 2000.
- [2] Z. Xiong, A.C. Tripp, Electromagnetic scattering of large structures in layered Earths using integral equations, *Radio Sci.* 30 (1995) 921–929.
- [3] M.L. Oristaglio, B.R. Spies (Eds.), *Three Dimensional Electromagnetics*, SEG, Tulsa, 1999.
- [4] A.A. Kaufman, G.V. Keller, *Inductive Mining Prospecting*, Elsevier Science, New York, 1985.
- [5] B. Bourgeois, K. Suignard, G. Perrusson, Electric and magnetic dipoles for geometric interpretation of three-component electromagnetic data in geophysics, *Inverse Prob.* 16 (2000) 1225–1262.
- [6] G. Perrusson, D. Lesselier, M. Lambert, B. Bourgeois, A. Charalambopoulos, G. Dassios, Conductive masses in a half-space Earth in the diffusive regime: fast hybrid modeling of a low-contrast ellipsoid, *IEEE Trans. Geosci. Remote Sens.* 38 (2000) 1585–1599.
- [7] T.M. Habashy, R.W. Groom, B.R. Spies, Beyond the Born and Rytov approximations: a nonlinear approach to electromagnetic scattering, *J. Geophys. Res.* 98 (1993) 1759–1775.
- [8] A. Charalambopoulos, G. Dassios, G. Perrusson, D. Lesselier, The localized nonlinear approximation in ellipsoidal geometry: a novel approach to the low frequency problem, *Int. J. Eng. Sci.* 40 (2002) 67–91.
- [9] P.M. Morse, H. Feshbach, *Methods of Theoretical Physics*, vol. I, II, McGraw-Hill Inc., New York, 1953.
- [10] E.W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics*, Chelsea Publishing Company, New York, 1965.
- [11] P. Moon, E. Spencer, *Field Theory Handbook*, second ed., Springer-Verlag, Berlin, 1971.
- [12] G. Perrusson, D. Lesselier, P. Vafeas, G. Kamvyssas, G. Dassios, Low-frequency electromagnetic modelling and retrieval of simple bodies in a conductive Earth, *World Scientific (Progress in Analysis)* 2 (2003) 1413–1422.
- [13] O.C. Ao, H. Braunisch, K. O'Neill, J.A. Kong, Quasi-magnetostatic solution for a conductive and permeable spheroid with arbitrary excitation, *IEEE Trans. Geosci. Remote Sens.* 39 (2001) 2689–2701.
- [14] J.J. Bowman, P.L. Uslenghi, T.B. Senior (Eds.), *Electromagnetic and Acoustic Scattering by Simple Shapes*, North Holland, Amsterdam, 1969.
- [15] V.K. Varadan, V.V. Varadan, *Acoustic, Electromagnetic and Elastic Wave Scattering, Low and High Frequency Asymptotics*, Amsterdam, North Holland, 1987.
- [16] H. Tortel, Electromagnetic imaging of a three-dimensional perfectly conducting object using a boundary integral formulation, *Inverse Prob.* 20 (2004) 385–398.
- [17] X. Chen, K. O'Neill, T.M. Grzegorzczuk, J.A. Kong, Spheroidal mode approach for the characterization of metallic objects using electromagnetic induction, *IEEE Trans. Geosci. Remote Sens.* 45 (2007) 697–706.
- [18] X. Chen, K. O'Neill, B.E. Barrowes, T.M. Grzegorzczuk, J.A. Kong, Application of a spheroidal-mode approach and a differential evolution algorithm for inversion of magneto-quasistatic data in UXO discrimination, *Inverse Prob.* 20 (2004) S27–S40.
- [19] P. Vafeas, G. Perrusson, D. Lesselier, Low-frequency solution for a perfectly conducting sphere in a conductive medium with dipolar excitation, *Progr. Electromagnetics Res.* 49 (2004) 87–111.
- [20] S.J. Norton, W.A. SanFilipo, I.J. Won, Eddy-current and current-channeling response to spheroidal anomalies, *IEEE Trans. Geosci. Remote Sens.* 43 (2005) 2200–2209.