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Chapter 16

Introduction to inverse scattering in acoustics and elasticity

Marc Lambert, Marc Bonnet, Dominique Lesselier



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This introduction to inverse problems for acoustic or elastic waves does not aim to be an exhaustive presentation of their theoretical and numerical difficulties, nor of their solutions implemented in many fields of applications of interest —including those of evaluation and non-destructive testing (NDT) of artificial structures, or the characterization of the human body, two capital topics among those treated in detail in other chapters of this book. We only wish to highlight general elements that can be found in a relevant way in many inverse problems which we can meet in practice. Perhaps more importantly, we also wish to give to the interested reader some guidelines for further research, based on a list of publications from the recent international literature, in addition to some examples from different works to which the authors have been contributing in acoustic and elastic topological identification, an investigation which we believe to be useful for our purpose here.

16.1 General framework

Referring to the thorough analysis conducted in [Sab00], the problem addressed can be summarized as follows: Let us consider a set of characteristic parameters of a system, artificial or natural, \mathcal{P} (density, Lamé parameters, compression or shear wave celerities, or any other relevant factor) linked to a set of observables \mathcal{O} (displacement, pressure, ...) through “laws of nature” \mathcal{L} . Let \mathcal{M} be a mathematical model, more or less sophisticated, which would be the best description of these laws with equations or systems of equations (differential, integrals, partial derivatives and combinations), and with appropriate boundary and/or initial conditions. Then, the solution of the “direct problem” consists in calculating \mathcal{O} from the known \mathcal{M} and \mathcal{P} sets, while the “inverse problem” consists in finding the parameters \mathcal{P} from the measurements \mathcal{O} and from the laws \mathcal{M} .

$$\mathcal{O} = \mathcal{L}(\mathcal{P}) \simeq \mathcal{M}(\mathcal{P}) \quad [16.1.1]$$

As stated, many fields of application exist, like for example:

- geophysics, and civil or environmental engineering [FB93; Hoo95], where the issue mainly is the identification of natural or artificial buried structures from elastic (seismic) measurements, acquired on the surface or in a borehole (there is also obvious interest in similar situations in underwater acoustics);
- non destructive testing and evaluation [LMH02; SSS02; Kun03], where the objective is to characterize defects (cracks, inclusions or corrosion, delamination, burns, ...) affecting a part or a set of pieces of metal, or composite, which may modify the nominal behavior or affect the safety during the use of this material;

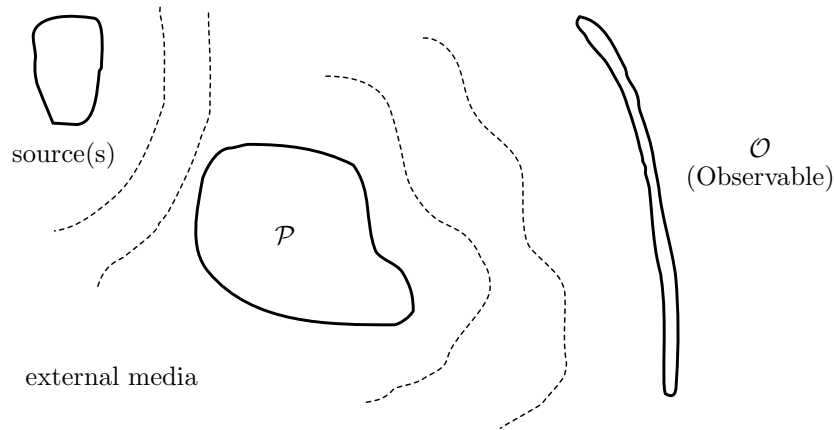


Figure 16.1.1: The usual wave diffraction problem

- biological media imaging [Sch02], particularly within the human body to detect various abnormalities (such as inclusions) affecting it, or to detect via maps of elastic parameters an abnormal behaviour of the tissues or other significant elements;
- the characterization of materials [Kun03], which is focused on the evaluation of their mechanical properties from the response provided to a known sollicitation.

Yet, reading the above shows that we already facing a huge subject, which is testified by [Sab00], as well as [Bon99] (one of a few references in French). Thus we will now only consider the specific issue of identification, characterization and/or imaging of complex passive objects at rest, active sources or vibrating bodies (in terms of acoustics and vibro-acoustics) being henceforth excluded.

16.2 Inverse problem of wave scattering

The generic problem of wave scattering, as envisaged in the following, can be presented as follows (figure 16.1.1). We consider one or more objects or obstacles (we could also use terms like inhomogeneities, inclusions, defects and so on) of limited spatial extension, and of characteristic parameters \mathcal{P} , found in a well-known host environment, whose parameters can be estimated beforehand. This host environment can be finite, semi-infinite, or infinite, with a simple or complex configuration. The whole system is excited by one or more sources of known characteristics. The so-called secondary (or scattered) fields or displacements (the observable \mathcal{O} referring to equation [16.1.1]), resulting from the interaction between the wave and the host medium, including all obstacles, are then collected by appropriate receivers. We consider here the framework of acoustics or elasticity and assume that the variations are small enough in such a way that the perturbations induced by the objects are linear with respect to the excitation.

As a matter of fact, the information concerning the investigated properties of these objects (locations, sizes, shapes, parameters of constitutive materials, etc.) appears to be present in coded form in the signal(s) acquired by receivers. Their “decoding”, i.e., the solution of the *inverse problem*, can extract from this(these) signal(s) the parameters of interest, with sometimes the higher objective that consists in their quantitative characterization, or, often in a more realistic way (with lesser degree of complexity), in order to produce images that are properly representative of these objects.

In an optimal situation, the knowledge of the “inverse laws of nature”, which we note here \mathcal{L}^{-1} (and its approached mathematical model \mathcal{M}^{-1}), should enable the reconstruction of the parameters of interest \mathcal{P} by applying a relationship similar to [16.1.1] and given by

$$\mathcal{P} = \mathcal{L}^{-1}(\mathcal{O}) \simeq \mathcal{M}^{-1}(\mathcal{O}). \quad [16.2.2]$$

But in practice, the mere existence of \mathcal{M}^{-1} is not guaranteed (e.g., the same observable \mathcal{O} can be obtained for several values of \mathcal{P}), and obtaining an explicit formulation of \mathcal{M}^{-1} , if any, is possible only in very limited cases, which reduces its use.

So, some other schemes have to be implemented to overcome this difficulty. The most oftenly chosen path is to solve equation [16.2.2] via an iterative optimization scheme (figure 16.2.2) which consists in finding the

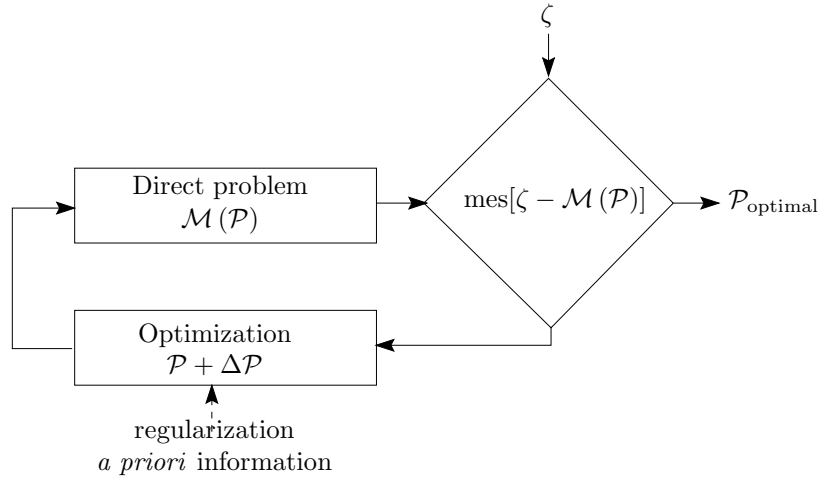


Figure 16.2.2: Solution scheme for the inverse problem, optimizing the fit of the measured data to the theoretical solution

“best” possible solution, without ever explicitly calculating \mathcal{M}^{-1} . This scheme enables the determination of an optimal set of parameters, $\mathcal{P}_{\text{optimal}}$, so that the difference between the measurements ζ and the solution $\mathcal{O} = \mathcal{M}(\mathcal{P})$ is minimum:

$$\mathcal{P}_{\text{optimal}} = \underset{\mathcal{P}}{\text{Argmin}} (\text{mes}[\zeta - \mathcal{M}(\mathcal{P})]), \quad [16.2.3]$$

where $\text{mes}[\cdot]$ is a judiciously chosen measurement of this discrepancy (this term describing the fit of the measured data to the theoretical solution is often called cost function, objective function, goal function...). Of course, the choice of this cost function depends on the solution sought, the application proposed, etc.

The successive steps of the inversion process can be sketched as follows: (i) choice of an initial set of parameters \mathcal{P} ; (ii) solution of the direct problem; (iii) computation of the discrepancy between the data ζ and the solution of the direct problem, knowing that if it is deemed sufficiently small compared to a chosen criterion, the process stops and the corresponding solution $\mathcal{P}_{\text{optimal}}$ is obtained; otherwise, the process goes on; (iv) optimization process, in which a correction is carried out on all parameters so as the discrepancy should be narrowing. At this stage of the process, constraints on corrections can be incorporated in order to favor certain solutions on the basis of *a priori* information. Then the process returns to stage (ii) and repeats.

These inverse problems are in any case not linear since the observed fields do not linearly depend upon the parameters of the objects. They are also ill-conditioned problems: without discussing the concept with the detail that would be required [TA74], existence, uniqueness and continuity of the inverse solution with respect to the data cannot be ensured simultaneously. From a practical point of view, this means in particular that some small variations in the data set can yield large changes in the calculated parameters. In addition to these fundamental obstacles, we must not forget also some more classical difficulties, related to the imperfections of the data itself (random or systematic errors), and of the models.

Thus, the chosen model yields an approximated description of a real situation, and the inversion can for example be carried out by neglecting a significant part of the interaction phenomena occurring between the host medium and the objects, such as multiple scattering. Under such approximations, linear models can be successfully applied. For example, it is often true when the wavelengths are either large or small, compared to a relevant size of the objects. We will get back to this matter later.

Taking into account the above, the key idea is the construction and implementation of resolution methods for these inverse problems that lead to solutions:

- which are precise enough to be useful and stable with respect to the errors, be they from the models or from the data, which is ensured through adjustments (typically operating on the norms of the solutions, their derivatives, or some special functions resulting from the application of particular operators on these solutions), and the incorporation before and/or during the inversion process of some *a priori* information on objects (compactness, positivity, connectivity, limited amplitudes, etc.);
- which can be obtained fairly quickly, especially in the case of complex structures or of important size (which depends, of course, on the computers used, and on the chosen numerical methods). Their construction

method must also take into account the form and the limitations of data as precisely as possible, but be simultaneously broad enough to remain useful in the real situations encountered, those being not necessarily nominal.

16.3 Direct problem

The resolution of the corresponding direct problems is a required first step, necessary in any study of inverse problems. It allows:

- a validation of the formulation of the acoustic or elastic quantities (fields, displacements, or any associated function, including potentials), chosen by comparison with synthetic results (described in the literature for example), or experimental measurements. The latter may then have been obtained either under controlled laboratory conditions (a key step in practice), or by on-site measurements (with all the constraints of real world);
- the acquisition of synthetic data that will provide a wide variety of feasibility tests, robustness, or accuracy of the inversions (we often use the term of “numerical experiment”);
- a better understanding of the physical interaction phenomena that are involved, including, in ideal situations, the identification of key parameters of the objects (and of their hosts); parameters that can be expected to be those aimed by the inversion process, or those only that can be retrieved effectively.

In addition, such a work requires the development of fast and efficient calculation modules that can be used during the inversion process. Indeed, a significant difficulty of most solutions of inverse problems lies in the need of repetitive (and usually expensive in terms of computation time) calculations of the quantities modeled for some given environments.

It is important to note that strong formulations are the most frequently used, as a common starting point for the direct problem and for the inverse problem, including intermediate cases resulting from approximations, like the Born approximation, or the Neumann series or extensions (sometimes used in the low-frequency range [DK00]), or the Kirchhoff formulation (a high-frequency approach) [LMM02]. These formulations are based on the use of Green functions, which can be either scalar (acoustic fluid propagation medium) or tensor (solid elastic propagation medium) ones, and which are obtained as solutions to properly set differential systems, including specific boundary conditions (on the boundaries of the objects themselves, of the host medium, and naturally at infinity). These Green functions generally lead to the formulation of the scattering phenomenon through integral equations, e.g., Lippmann-Schwinger integral equations.

Of course, direct calculations can also be based on weak formulations that lead to finite-element methods and their multiple variants and extensions, emphasizing that the classical works of [Hoo95] highlighted the equivalence of these two types of representations, and the similarities with traditional numerical techniques based on them, i.e. moment and finite-element methods.

It is important to emphasize here that nothing forbids the use of alternate techniques such as finite-difference methods and variants and/or extensions, especially in the case of complex environments (in terms of geometry, of distribution of the electromagnetic or elastic parameters) for which either the Green solutions are unknown or too complicated to obtain, or the finite-element method proves to be inappropriate or too costly to implement.

16.4 Optimizations

We do not aim to be exhaustive on optimization methods, so we limit ourselves to the description of the main approaches, including advantages and disadvantages. We then provide the reader with some references that we consider as pertinent in this domain.

First, let us remind that the purpose of optimization in an inversion process consists in finding, inside a restricted search space described by the set of possible solutions, the particular solution that optimizes the adequacy of the data with the theoretical solution (thus, the particular solution that minimizes the cost function). Of course, the more complex the problem, the more difficult the search of the optimal solution.

One possible approach is to browse the entire search space, to calculate for every possible solution the corresponding cost function (which requires the solution of the direct problem every time), and finally to retain the solution leading to the lowest cost function. It is obvious that this method is expensive in terms

of computation time and becomes unrealistic in the case of problems of large dimensions, or in the case of parameter spaces \mathcal{P} having large dimensions (large number of degrees of freedom).

This is the reason why some optimization methods have been developed that can be divided (at least grossly speaking) into two classes called $\ddot{\text{global}}$ and $\ddot{\text{local}}$ optimizations. We give here a short description of these two classes.

- **Local optimization:** These methods are based on the idea that, starting from an initial, well-chosen solution, the direction of the displacement in the space of solutions to estimate a better solution is obtained from the estimation of the gradient of the cost function with respect to the parameters. A fairly rapid convergence process often results; however, these methods have two major limitations: (i) the choice of the cost function, which must be differentiable with respect to the parameters, (ii) the known risk that the process converges onto a local minimum of the cost function: in this case, the algorithm is stuck at this local minimum; it fails and cannot converge to the desired global minimum. Among many books on optimization, let us mention [Bon+97].
- **Global optimization:** One of the oldest methods is the Monte Carlo method. It consists in a random walk inside the space of possible solutions, where at each explored solution, the corresponding cost function is evaluated and compared to the best solution already obtained, keeping the new solution if it provides a lower cost. The process is repeated until a stopping criterion is met. Such a method has the disadvantage that each random possible solution is independent of all previously explored ones. Thus, other methods have been developed in order to take into account the previously obtained information when a new solution is proposed. Among them, one should mention simulated annealing (based on the theory of thermodynamics) and genetic algorithms (using the theory of evolution of a population in a given area). These two well-known methods are open to many variations and adaptations [Mic97].

As might be expected, approaches combining global optimization and local optimization have been proposed, then combining the ability to browse the space of possible solutions and the rapid convergence of these two optimization classes.

In addition to this classification into two main families of minimization algorithms, some new distinction can be proposed depending on the information expected by the user.

- **Cartography:** the search area is divided into 2-D pixels or 3-D voxels in which the physical parameters are assumed constant; in all the following, we use the term $\ddot{\text{pixel}}$ for simplification. The gradient algorithms therefore estimate the value of each parameter within each pixel, and then produce a map that describes the variations of the physical properties of the proposed setting. A still relevant illustration of the gradient methods used in acoustic and electromagnetic inversion is proposed in [vK97]. While most of these algorithms solve a non-linear problem using the solution of successive linear problems (the Newton-Kantorovich method, and its equivalents or variants, such as the distorted Born method, and the iterative Born method), some are directly developed to take into account the non-linearity of the problem (the conjugate-gradient method, the modified gradient method, and the contrast-source inversion method, just to limit us to the best-known methods in the specialized literature; each of these methods can be of course extended with specific additions; each method is also characterized by specific pros and cons).
- **Contour reconstruction :** Unlike cartography, the contour reconstruction methods (a line in 2-D, a surface in 3-D) aim at determining the boundaries of obstacles, those being considered as homogeneous or at least made of a set of homogeneous parts. These boundaries evolve in space with the calculations of shape gradients, obtained in most cases from the solution to a wisely chosen problem, called the “adjoint problem”. Nowadays, these methods are rather classical ones. They are based on electromagnetic or elastic contour integral formulations, for which the unknown parameters are coefficients that describe the boundaries of the unknown obstacles. One of the major limitations, however, is that the number of obstacles to build (or at least an upper limit of that number) must be known in advance. Therefore, new methods have been recently developed to enable, if necessary and in a natural fashion, the appearance or disappearance of obstacles. Such typical methods are the controlled evolution methods, based on level sets. In practice, these methods combine two approaches. The first one is based on the work described for example by [Set99], leading to a representation of the boundary of an obstacle of unknown topology using the zero level of a level-sets; [San96] is a pioneer investigation of this approach in the field of inverse scattering problems. The second one is based on works on velocity methods, widely considered in the design of optimal shapes [DZ01; SZ92]. Finally, one can change the contour, using the global evolution of the level set, involving topological derivatives throughout the domain of search so that one optimizes its

shape via an adequacy criterion to the training data (e.g., including judicious constraints on the perimeter or surface, and/or the area or volume of the obstacle(s)).

The choice of parameters to search for is a major difficulty in solving inverse problems. This choice is based on a balance between the expected and/or hoped speed of the algorithms, the complexity of the model used, and the desired accuracy of the solution. While most of the work done here aims at the complete characterization (with a few approximations, or even better with no approximations), several paths leading to approximate solutions can be followed:

- Search of an equivalent obstacle: The obstacle being investigated is replaced by a canonical equivalent obstacle (circular cylinder, ellipsoid, and its degenerated forms –the sphere is frequently the only one considered– or other shapes adapted to the application), canonical obstacle whose position, physical parameters and relevant geometric descriptors are to be determined on a more or less complex way, as illustrated for example by [Wir02] and cited references.
- Search of equivalent sources: The obstacle being investigated is replaced by an equivalent set of discrete sources or by a distribution of equivalent sources whose positions and amplitudes contain, in more or less complex forms, electromagnetic, elastic, or geometric information that characterize the obstacle; an abundant literature refers to this approach, like, among other pioneer contributions, [AJK97]. The reader may also refer to [LL00] for an application in shallow underwater acoustics.

16.5 Other approaches

In addition to these techniques focused onto the solution of non-linear inverse problems, we can mention some other techniques, like diffraction tomography or filtered backpropagation and variants for which there are many recent references; here let us cite [PGM96; Lan02]. These techniques are based on linearized versions of the equations to be considered. This linearization of the equations is mostly done within the Born approximation framework in the case of scatterers with a low contrast of physical parameters with respect to the environment (weakly diffractive). We also use the Kirchhoff approximation (also used in physical optics) in the case of scatterers that can be considered as non penetrable (very high contrast of the parameters with respect to the environment). These formulations lead to fast solution algorithms, mostly involving Fast Fourier Transforms (case of measurements along straight lines or in planar surfaces).

Recently in [CK96], an original method for the reconstruction of the boundaries of obstacles was proposed, entitled *linear sampling*. Originally proposed for the reconstruction of the boundaries of impenetrable obstacles from the scattered far field, it is based on a linear relationship between the far field and a gauge function, the norm of which tends towards infinity on the contours of the obstacles. Then, the aim of the method is to find all points for which this norm is infinite. Many extensions have followed this initial work (including factorization methods), particularly in the case of near field inverse problems [FG04]. In connection with this, and highlighted in an original manner by [Che01], we also note that we can address the problem of the localisation of quite small scatterers (small compared to the wavelength used to explore them) with numerous established or developing works, particularly on time reversal (including a recent tutorial [FP01]) and/or about approaches involving the MUSIC (*MUltiple SIgnal Classification*) algorithm, as illustrated by [Kir02] and [AIL05]. Finally, the case of limited environments, for which complete data on the boundaries (Dirichlet-Neumann boundary conditions) are available, can be treated using the method based on the *reciprocity violation*. It is based on the fact that two acoustical states are linked together by a reciprocity relationship involving integrals on all boundary contours of the environment considered. In the absence of obstacle, that identity holds true for external measurements and for any state (known as *adjoint*) satisfying the wave equations. A violation of this reciprocity principle thus indicates the existence of an object (simple or multiple) to be identified. A judicious choice of the adjoint state allows in some cases to extract some information on the obstacle by explicit calculations. For example, a recent development [BCM04] has shown that the convex hull of the obstacle can be calculated from assessments of the instantaneous reciprocity violation, for adjoint fields like properly chosen planar propagative waves.

16.6 General configuration

It is worth mentioning that many of inversion studies are limited to a generic configuration, which is often chosen as a local idealization, more or less significant, of the structure to be tested. The generic configuration is also often used to set some assumptions or concepts, application coming later.

Indeed, the host environment is frequently assumed to be uniform by parts, like homogeneous layers separated by planar parallel interfaces; circular cylindrical tubes, or even spherical cavities or spherical blocks, can also be relevant. The sources that create the sound illumination (remember that we consider an active situation) can be, as already indicated, either plane and generated at infinity, or localized. Line sources, point-like sources and dipoles considerably simplify the analysis, but more sophisticated radiating devices, such as linear or surface arrays, must also be considered. They are located either on the first interface, or at some finite distance of it (including the possibility of a direct contact with it), or inside the host medium (inside a particular layer medium for example). The receivers (limited number of receivers) provide an access to the scattered field, which is therefore linked to the objects buried in the host medium. They are also located either on the surface or within the host environment.

The data obtained are described in the literature as being *aspect-limited*: in summary, it is not possible to turn around the whole obstacle. To partially compensate the loss of information resulting from this limitation, measurements are performed (as far as possible) for several harmonic frequencies (within a given bandwidth in the impulse regime), and for sources and receivers having several geometries, leading to mono-, bi- or multi-static experimental setups, with a more or less specific coverage in space and frequency.

As a consequence of this limited aspect, let us consider theorems for the uniqueness of inversions. Usually, these theorems, applied to acoustics, are focused on uniqueness from scattering patterns associated to the obstacles [CK98], with some subtle (and largely discussed) extensions to elasticity, also including the two fundamental cases of i) the crack and ii) the inclusion of planar section or finite volume. They appear to be only known in the unrestricted configurations of scattering objects in a (free) homogeneous propagation medium. This increases the complexity of interpretation of the inversion results. Meanwhile, the case of a stratified medium instead of a mainly homogeneous half-space increases the complexity of the mathematical formulation, and of the numerical computation of the Green functions or Green tensors, on which strong formulations, mostly used as already indicated, are based.

16.7 Illustrations

Controlled evolution of sets of levels

The following description is based on the one made in [Ram+02b]. For a more sophisticated analysis, the reader could refer to fundamental publications such as [Ram+02a] and abundant references therein.

We consider an homogeneous cylindrical object Ω , with a planar section of arbitrary (not necessarily connected) yet sufficiently smooth boundary contour. This object is assumed to be fully buried inside a domain \mathcal{D} of space (free, half-space, slice of material) and illuminated by sources operating in the harmonic regime. The field is collected by suitable receivers, providing a certain set \mathcal{S} of field values ζ . The analysis is applicable in the case of TM or TE polarization in electromagnetics, and also in fluid acoustics. In this latter situation, the variations of the velocity of compressional waves (or the variation of the compressibility) can be described through the introduction of a contrast function χ of the appropriate quantity (this contrast function can be a complex number) that is zero outside the object. The boundary conditions are in this case written as transmission conditions (an impenetrable object will be modeled through the introduction of heavy losses inside it). Consequently, some domain integral equations (rigorous and based on the application of the Green theorems) yield expressions of the scattered fields, denoted as U_d , in the measurement domain \mathcal{S} , those being functions of the fields U existing in the domain \mathcal{D} (observation equation), the latter themselves solutions of state equations (as for example the Lippmann-Schwinger ones).

The solution method itself is implicit, and is based on a fixed Cartesian grid space (allowing the pre-calculation of Green discrete functions, the numerical computations being then carried out by applying a method of moments to the integral wave equations) and time (with some possible adaptive choices of this parameter). It calculates the temporal evolution at time t of a function of level sets ϕ , the level 0 of which now defines the boundary contour Γ_t . The negative values of this function are associated with points located inside the domain Ω_t , whereas positive values correspond to points located outside Ω_t . This applies at any time t .

The existence of a velocity field V for the deformation of Ω_t , normal to that contour Γ_t (the object or the parts of the object are closed, only normal velocities are then taken into account), therefore means that ϕ satisfies an Hamilton-Jacobi evolution equation [Set99], in space \mathbf{x} and in time t :

$$\frac{\partial}{\partial t} \phi(t, \mathbf{x}) + V(t, \mathbf{r}) |\nabla \phi(t, \mathbf{x})| = 0, \mathbf{r} \in \mathcal{D}. \quad [16.7.4]$$

It remains to build one (the) velocity V that transforms an initial domain Ω_0 in a more satisfying Ω_t . In other words, the cost function $J(\Omega_t)$, defined as the norm $\|U_d - \zeta\|_{\mathcal{S}}^2$ of the difference between the measured and calculated data on \mathcal{S} , has a decreasing amplitude, or is such that dJ/dt is negative, up to numerical uncertainties and discretization errors. Some penalty terms for the area and/or perimeter can be incorporated into the cost function, and this induces suitable changes of the velocity V . This construction yields the derivative of J at time t , through the construction of a minimum-maximum of a Lagrangian \mathcal{L} , avoiding a heavy derivation of the field with respect to the deformation of the domain. Indeed, the control theory, e.g. [SZ92; DZ01], shows that the derivative which we search for is equal to the partial derivative of the Lagrangian evaluated at its saddle point (proved unique in our particular case of interest), this derivative calling for to a shape gradient g , originally defined on the contour and then extended to the whole search domain. This extension was originally completed in a heuristic manner, but later validated through the concept of *topological derivation*. This concept is still up to date in electromagnetics as well as in elasticity, as is illustrated by the example presented in the next section.

Thus, in the case of a contrast of velocity of compressional waves between the fluid environment and the object (transmission conditions), g simply boils down on the contour to the product $\chi \Re(Up)$, where \Re designates the real part, U the direct state (i.e. the solution of the direct scattering problem to be solved), and p the adjoint state that satisfies the same propagation and transmission equations as the direct state, but whose source is made of the complex conjugate distribution of the difference between data and the fields associated with the object effectively built at the considered time, $U_d - \zeta$, on \mathcal{S} .

It could be shown that the quantity $(J(\Omega + \Omega_\varepsilon) - J(\Omega)) / \text{mes}(\Omega_\varepsilon)$ also has the value $\chi \Re(Up)$ for every point \mathbf{x} of \mathcal{D} . For this purpose, we must define Ω_ε as being a domain of contrast χ of \mathcal{D} , simply connected and with fairly regular contour, for which the measure $\text{mes}(\Omega_\varepsilon)$ tends towards zero with the real parameter ε (in this case, the domain Ω_ε collapses and converges to \mathbf{x}). We also note $J(\Omega + \Omega_\varepsilon)$ and $J(\Omega)$ the values of the function J with and without this perturbation Ω_ε , respectively.

We do not address here the difficulty related to the nature of the shape gradients g during the appearance or disappearance of domains. In a few words, any change of the topology (for example by fusion or fission of domains, the normals being in particular undefined at the junction or break points) represents a problem for the definition of function spaces and for an appropriate formulation of the evolution equation. Nowadays, such problem cannot be considered as solved. Yet, the numerical approach is valid, as illustrated by many practical examples.

The application of the above is illustrated by the reconstruction of two fluid obstacles of different size (one is about $\lambda \times \lambda$, the other one $\lambda/3 \times \lambda$) buried into an homogeneous fluid environment. The wavelength in this environment λ is given by c/f , where c is the velocity of compression waves (1470 m/s) and f the operation frequency of the sources (1 MHz). Both obstacles have the same characteristic velocity (1800 m/s). The whole system is illuminated by 4 line sources, regularly distributed over a circle of radius 2λ . The scattered field is measured by 32 receivers, also regularly distributed over the same circle. The synthetic data are obtained by numerical solution of the direct problem. Consequently, they are not affected by noise measurement. Note that the domain where the obstacles are sought is located in the center of the system and has a size of $5\lambda/3 \times 5\lambda/3$.

Figure 16.7.3 displays the two best reconstructions (in the sense of solutions that minimize the cost function) obtained with two different variants of the method of level sets. Both methods have been initialized by a single square obstacle (figure 16.3(a)). In the first approach (figure 16.3(b)), the contrast is known and fixed, the boundary being in this case the only unknown. In the second variant (figure 16.3(c)), boundary and contrast of the obstacle are both unknown. This latter situation is handled by alternating the use of i) the evolution process, and ii) an evaluation of the contrast, the domain being fixed (the initial velocity for the obstacle has been chosen here as 2000 m/s). This evaluation of the contrast is performed by a non-linear Levenberg-Marquardt algorithm involving a Lagrangian formalism in order to rigorously express the derivative with respect to the contrast [Ram+03].

Topological derivative

In this section, we consider the identification of a rigid obstacle (or a set of such obstacles) in a three-dimensional acoustic environment (limited or not), excited by unitary point-like sources, successively applied at the positions \mathbf{x}_i^S ($1 \leq i \leq N$). The field u scattered by the objects is assumed to be measured by receivers located at \mathbf{x}_j^M ($1 \leq j \leq M$). The problem is equivalent to the reconstruction of the contour Γ of an obstacle delimited by Ω ,

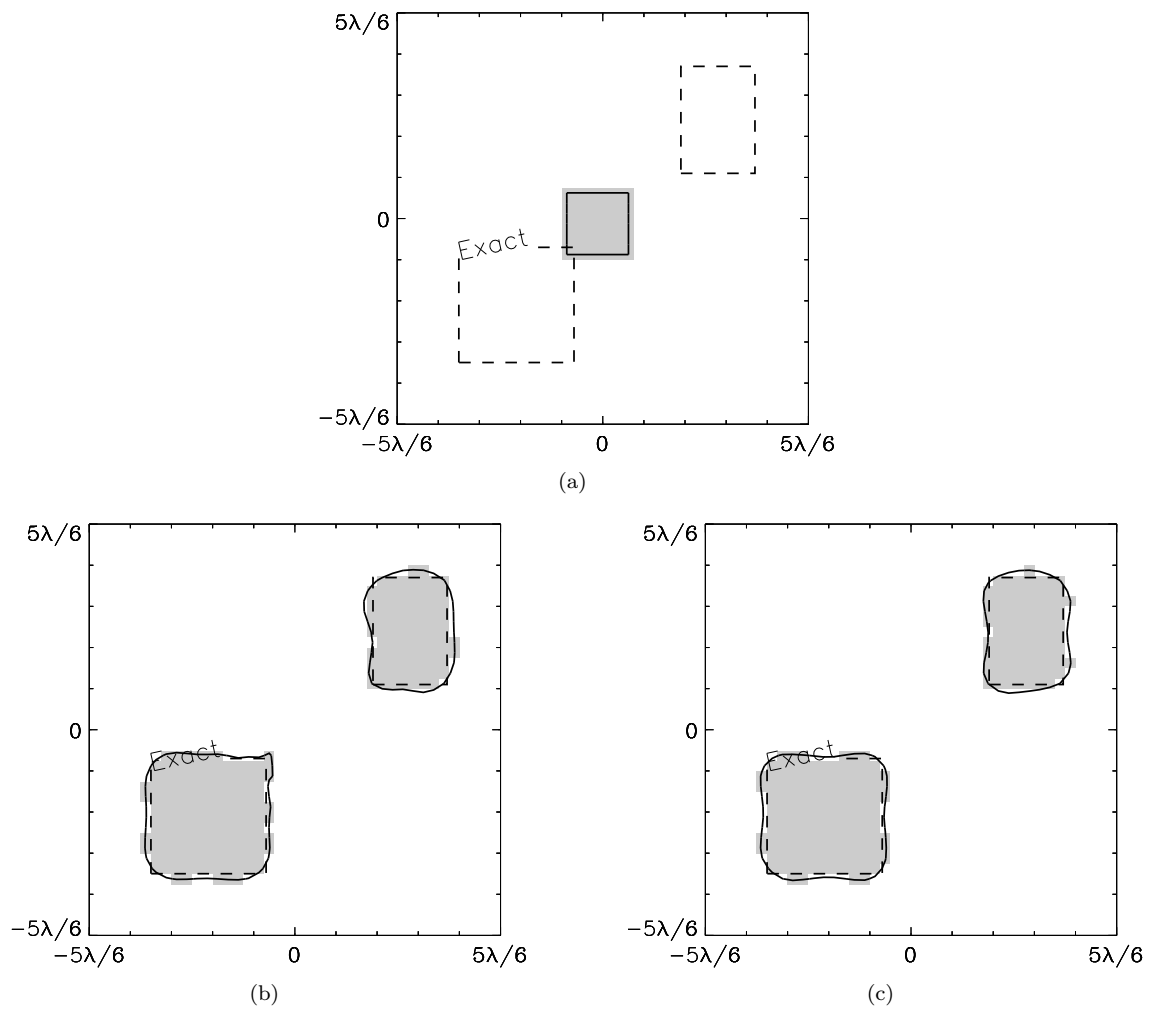


Figure 16.7.3: Reconstruction of an object consisting in two separate parts with the same contrast. Initialization process (a), best solution obtained (b) using the method of evolution of level sets, (c) same as (b), combined with a method of contrast reconstruction. The dashed and solid lines correspond to the exact and reconstructed boundaries (the level zero of all levels), respectively. The shaded area corresponds to the interior of the object

this obstacle being single or multiple. It is natural to introduce the following cost function

$$\mathcal{J}(\Omega) = \frac{1}{2} \sum_{i=1}^N \left\{ \sum_{j=1}^M \left\| u_{\Omega}^{(i)}(\mathbf{x}_j^m) - \xi^{(i)}(\mathbf{x}_j^m) \right\|^2 \right\}$$

where

- $u_{\Omega}^{(i)}$ represents the acoustic field scattered for the source \mathbf{x}_i^S and a given configuration Ω of the obstacle,
- $\xi^{(i)}$ represents the corresponding measured scattered field (the domain Ω , or in equivalent manner, the surface Γ , thus behaves as the parameter \mathcal{P} in (16.2.3)).

Minimizing $\mathcal{J}(\Omega)$ using traditional methods requires a choice of initial configuration $\Omega^{(0)}$. It is desirable to make this choice in a pertinent manner. If \mathcal{D} represents the environment without obstacle, it is interesting to try to assess the impact on the cost function of the presence of a small obstacle $\Omega_{\varepsilon}(\mathbf{x}^o)$ of well-known shape (for example a spherical obstacle), with a characteristic size ε and volume $\varepsilon^3 V$, centered at a point \mathbf{x}^o in \mathcal{D} . In this particular situation, we can establish that the cost function can be developed in the following form

$$\mathcal{J}(\Omega_{\varepsilon}(\mathbf{x}^o)) - \mathcal{J}(\emptyset) = \varepsilon^3 V \mathcal{T}(\mathbf{x}^o) + o(\varepsilon^3)$$

where $\mathcal{T}(\mathbf{x}^o)$, the *topological derivative* of \mathcal{J} at \mathbf{x}^o , is a function of

1. the incident fields $u^{(i)}$ associated with each source,
2. the adjoint fields $v^{(i)}$ generated by the superposition of the sources, whose intensities are complex conjugates of the differences $u_{\Omega}^{(i)}(\mathbf{x}_j^m) - \xi^{(i)}(\mathbf{x}_j^m)$ applied to receivers.

This function can be written as follows:

$$\mathcal{T}(\mathbf{x}^o) = \sum_{i=1}^N \left[\nabla u^{(i)} \cdot \mathbf{A} \cdot \nabla v^{(i)} - \frac{4\pi^2}{\lambda^2} u^{(i)} v^{(i)} \right] (\mathbf{x}^o) \quad [16.7.5]$$

where the second-order tensor \mathbf{A} depends only upon the chosen form for $\Gamma_{\varepsilon}(\mathbf{x}^o)$ (for a spherical shape, we show that $A_{ab} = (3/2)\delta_{ab}$). In order to minimize $\mathcal{J}(\Omega)$, the “interesting” regions of space are those for which the topological derivative $\mathcal{T}(\mathbf{x}^o)$ is negative. For example, we can heuristically consider the latter as a first approximation of the obstacle sought. This approach is currently being explored for the similar inverse problem in elastodynamics [BG04].

As an illustration, let us consider the problem of identifying a set of three rigid spherical objects (centered at $(-3, 2, -3)$, $(2, 3, -3)$ and $(0, -4, -4)$, radii 1, 1 and 2, in arbitrary units) in an acoustic half space $x_3 \leq 0$, such that the wavelength $\lambda = (2\pi/1, 1)$. We consider $N = 4$ point-like sources located at $\mathbf{x}^S = (\pm 5, \pm 5, 0)$ from the surface $\{x_3 = 0\}$, on which a condition of zero velocity is also assumed. We consider $M = 9$ receivers $\mathbf{x}^m = ((0, \pm 5), (0, \pm 5), 0)$. Figure 16.7.4 shows the topological derivative field $\mathcal{T}(\mathbf{x}^o)$ calculated on a mesh of points \mathbf{x}^o in the plane $x_3 = -3$. The lowest negative values of $\mathcal{T}(\mathbf{x}^o)$ correspond to the horizontal location of the objects, and the absolute minimum of $\mathcal{T}(\mathbf{x}^o)$ indicates the largest object.

It is important to note that the numerical computation of the topological derivative field $\mathcal{T}(\mathbf{x}^o)$ is easy and cheap, compared to the cost of a numerical optimization algorithm for the same inverse problem. But the field u is necessary to assess the initial value of the cost function. The adjoint fields are evaluated either by explicit formulations (integral representations), or by the solution of an adjoint boundary problem, for which the operator has already been built and factorized for the calculation of u . Equation (16.7.5), or its variants on other contexts, can be quickly evaluated. Many extensions of the concept of topological derivative, originally introduced for topological optimization of structures [EKS94], whose application to inverse problems is just beginning, are under consideration.

16.8 Bibliography

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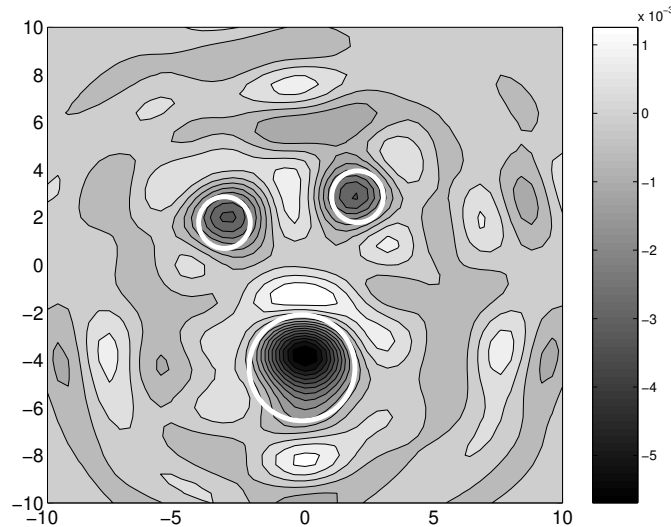


Figure 16.7.4: Identification of a set of three rigid spherical objects (centered at $(-3, 2, -3)$, $(2, 3, -3)$ and $(0, -4, -4)$, radii 1, 1 and 2, represented by white circles) in an acoustic half space $x_3 \leq 0$: distribution of \mathcal{T} in the plane $x_3 = -3$

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