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# SURVIVAL PROBABILITY OF THE BRANCHING RANDOM WALK KILLED BELOW A LINEAR BOUNDARY

JEAN BÉRARD, JEAN-BAPTISTE GOUÉRÉ

ABSTRACT. We give an alternative proof of a result by N. Gantert, Y. Hu and Z. Shi on the asymptotic behavior of the survival probability of the branching random walk killed below a linear boundary.

## 1. INTRODUCTION

Consider a random variable  $\zeta$  taking values in a bounded interval  $[\zeta_-, \zeta_+]$ , and a branching random walk on  $\mathbb{R}$  with deterministic binary branching, and i.i.d. random walk steps whose common distribution is that of  $\zeta$ . We denote by  $X = (X_n)_{n \geq 0}$  the (random) sequence of subsets formed by the population of the branching random walk at successive time steps, and we use the notation  $\mathbb{P}_x$  to refer to the case where  $X_0$  consists in a single particle located at  $x$ . We then denote by  $Y(v) = (Y_n(v))_{n \geq 0}$  the branching random walk obtained from  $X$  by killing, for all  $n \geq 0$ , every particle in  $X_n$  whose location is  $< vn$ . (Here, killing means that not only the particle, but also all its descendants, are removed from the process). We say that  $Y(v)$  survives for ever when  $Y_n(v) \neq \emptyset$  for every  $n \geq 0$ .

Introduce the log-Laplace transform of  $\zeta$ , defined for  $t \in \mathbb{R}$  by

$$\Lambda(t) := \log E(\exp(t\zeta)).$$

Due to the fact that  $\zeta$  has bounded support, the log-Laplace transform is well-defined and finite for all  $t \in \mathbb{R}$ , and is  $C^\infty$  as a function of the parameter  $t$ . Now assume that there exists  $t^* \in ]0, \infty[$  such that

$$(1) \quad \Lambda(t^*) - t^* \Lambda'(t^*) = -\log(2).$$

Then set

$$(2) \quad v^* = \Lambda'(t^*).$$

It can then be shown that, for all  $x \geq 0$ ,  $\mathbb{P}_x(Y(v) \text{ survives for ever}) > 0$  for all  $v < v^*$ , while  $\mathbb{P}_x(Y(v) \text{ survives for ever}) = 0$  for all  $v \geq v^*$ . Moreover, N. Gantert, Y. Hu and Z. Shi recently proved in [6] the following result:

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The authors would like to thank J. Quastel for suggesting that the approach developed in [7] might be used to re-derive the asymptotic behavior of the survival probability of the branching random walk with killing studied in the present paper.

**Theorem 1.** *As  $v - v^* \rightarrow 0^+$ ,*

$$\log \mathbb{P}_0(Y(v) \text{ survives for ever}) \sim -\pi \sqrt{\frac{\Lambda''(t^*)t^*}{2(v^* - v)}}.$$

In fact, the theorem quoted above is less general than the one proved in [6], since we have assumed deterministic binary branching and bounded random walk steps, whereas [6] allows for a more general branching mechanism, and possibly unbounded steps provided that the Laplace transform is finite in a neighborhood of zero. In the present paper, we give a proof of Theorem 1 which is completely different from the one in [6]. While the proof in [6] is based, among other things, on a change-of-measure argument combined with refined large deviations estimates for random walk paths, the present proof relies on the characterization of the survival probability of the branching random walk with killing, by a non-linear convolution equation. This idea has been used by B. Derrida and D. Simon in [5, 8] to derive a heuristic justification of Theorem 1, treating the corresponding equation by the methods developed by Brunet and Derrida to study perturbations of equations of F-KPP type (see [2, 3, 4]). The rigorous treatment given in the present paper was inspired by the paper by C. Mueller, L. Mytnik and J. Quastel [7], where a continuous version of the non-linear convolution equation is studied in order to give a mathematical proof of a result by Brunet and Derrida on the speed of travelling-wave solutions of the F-KPP equation with small noise.

The rest of the paper is organized as follows. In Section 2, we give the non-linear convolution equation characterizing the survival probability, and show how, given super- and sub-solutions of this equation, one obtains upper and lower bounds on the survival probability. In Section 3, we find explicit solutions to a linear convolution equation, and study their asymptotic behavior. Then, in Section 4 we transform these solutions in order to obtain super- and sub-solutions to the original non-linear convolution equation. Section 5 puts together the arguments needed to prove Theorem 1, and Section 6 contains some concluding remarks.

## 2. EQUATIONS CHARACTERIZING THE SURVIVAL PROBABILITY

Let  $\psi : [0, 1] \rightarrow [0, 1]$  be defined by

$$\psi(x) = 2s - s^2,$$

and let  $v$  denote a real number such that

$$v < v^*.$$

Let  $\mathcal{H}$  denote the following set of maps:

$$\mathcal{H} := \{h : \mathbb{R} \rightarrow [0, 1], h \text{ non-decreasing, } h \equiv 0 \text{ on } ]-\infty, 0[ \},$$

and, for all  $h \in \mathcal{H}$ , define  $T(h) \in \mathcal{H}$  by<sup>1</sup>

$$\begin{cases} T(h)(x) := \psi(E(h(x + \zeta - v))), & x \geq 0 \\ T(h)(x) := 0, & x < 0 \end{cases}$$

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<sup>1</sup>The fact that  $T(h) \in \mathcal{H}$  is a straightforward consequence of the fact that  $h \in \mathcal{H}$ .

On the other hand, for all  $x \in \mathbb{R}$ , let

$$q_n(x) := \mathbb{P}_x(Y_n(v) \neq \emptyset), \quad q(x) := \mathbb{P}_x(Y(v) \text{ survives forever}).$$

Note that, for obvious reasons<sup>2</sup>,  $q_n$  and  $q$  belong to  $\mathcal{H}$ , and  $q_0 = \mathbf{1}_{[0,+\infty)}$ . Given  $h_1, h_2 \in \mathcal{H}$ , we say that  $h_1 \leq h_2$  when  $h_1(x) \leq h_2(x)$  for all  $x \in \mathbb{R}$ .

The following proposition gives the non-linear convolution equation satisfied by  $q_n$  and  $q$ , on which our analysis is based.

**Proposition 1.** *For all  $n \geq 0$ ,  $q_{n+1} = T(q_n)$ , and  $T(q) = q$ .*

*Proof.* Analysis of the first step performed by the walk. □

An important fact is that being a non-trivial fixed point of  $T$  uniquely characterizes  $q$  among the elements of  $\mathcal{H}$ , as stated in the following proposition.

**Proposition 2.** *Any  $r \in \mathcal{H}$  such that  $r \not\equiv 0$  and  $T(r) = r$  is such that  $r = q$ .*

Our strategy for estimating  $q(x)$  is based on the following two comparison results.

**Proposition 3.** *Assume that  $h \in \mathcal{H}$  is such that  $h(0) > 0$  and  $T(h) \leq h$ . Then  $q \leq h$ .*

**Proposition 4.** *Assume that  $h \in \mathcal{H}$  is such that  $T(h) \geq h$ . Then  $q \geq h$ .*

Let us now record the following simple but crucial property of  $T$ .

**Proposition 5.** *If  $h_1, h_2 \in \mathcal{H}$  satisfy  $h_1 \leq h_2$ , then  $T(h_1) \leq T(h_2)$ .*

*Proof.* Immediate. □

As a first useful consequence of Proposition 5, we can prove Proposition 4.

*Proof of Proposition 4.* Since  $T(h) \geq h$ , we can iteratively apply Proposition 5 to prove that for all  $n \geq 0$ ,  $T^{n+1}(h) \geq T^n(h)$ , whence the inequality  $T^n(h) \geq h$ . On the other hand, since  $h \in \mathcal{H}$ , we have that  $h \leq \mathbf{1}_{[0,+\infty[}$ , whence the inequality  $T^n(h) \leq T^n(\mathbf{1}_{[0,+\infty[}) = q_n$ . By dominated convergence,  $\lim_{n \rightarrow +\infty} q_n(x) = q(x)$ , so we deduce that  $h \leq q$ . □

We now collect some elementary lemmas that are used in the subsequent proofs.

**Lemma 1.** *One has that  $P(\zeta \leq v^*) < 1$ .*

*Proof.* By assumption, there exists  $t^* > 0$  such that  $\Lambda'(t^*) = v^*$ . But  $\Lambda'(t^*) = \frac{E(\zeta e^{t^* \zeta})}{E(e^{t^* \zeta})}$ , so that  $E(\zeta e^{t^* \zeta}) = E(v^* e^{t^* \zeta})$ . If moreover  $P(\zeta \leq v^*) = 1$ , we deduce from the previous identity that  $P(\zeta = v^*) = 1$ , so that  $\Lambda(t^*) = t^* v^*$ , which contradicts the assumption that  $\Lambda(t^*) - t^* v^* = -\log(2)$ . □

**Lemma 2.** *One has that  $q(0) > 0$ .*

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<sup>2</sup>For instance by coupling.

*Proof.* In Section 4, for  $v^* - v$  small enough, we exhibit  $c_- \in \mathcal{H}$  such that  $T(c_-) \geq c_-$  and  $c_-(x) > 0$  for  $x > 0$ . We deduce (with Lemma 1) that  $T(c_-)(0) > 0$ . Now, letting  $h := T(c_-)$ , Proposition 5 shows that  $T(h) \geq h$ . Proposition 4 then yields that  $q(0) \geq h(0) > 0$ . This conclusion is valid for small enough  $v^* - v$ . Since clearly<sup>3</sup>  $q(0)$  is non-increasing with respect to  $v$ , the conclusion of the lemma is in fact valid for all  $v < v^*$ .  $\square$

**Lemma 3.** *There exists a constant  $\kappa$  depending only on the distribution of  $\zeta$ , such that, for all  $v < v^*$ ,  $q(0) \geq \kappa q(1)$ .*

*Proof.* Using the fact that  $\psi(s) \geq s$  for all  $x \in [0, 1]$ , we have that, for all  $x \in [0, +\infty[$ ,  $q(x) \geq E(q(x + \zeta - v))$ . From Lemma 1, we can find  $\eta > 0$  such that  $P(\zeta \geq v^* + \eta) > 0$ . Using the fact that  $q$  is non-decreasing, we obtain that  $q(x) \geq P(\zeta \geq v^* + \eta)q(x + \eta)$ . Iterating, we see that, for all  $n \geq 0$ ,  $q(0) \geq P(\zeta \geq v^* + \eta)^n q(n\eta)$ . Choosing  $n$  large enough so that  $n\eta \geq 1$ , we get the desired result.  $\square$

We now prove Proposition 2.

*Proof of Proposition 2.* Let  $r \in \mathcal{H}$  satisfy  $T(r) = r$  and  $r \not\equiv 0$ . According to Proposition 4, we already have that  $r \leq q$ . Our next step is to show that  $r(0) > 0$ . To this end, let

$$D := \{x \in [0, +\infty[; r(x) > 0\}.$$

Since we assume that  $r \not\equiv 0$ ,  $D$  is non-empty, and since, moreover,  $r$  is non-decreasing,  $D$  has to be an interval, unbounded to the right. Since  $v < v^*$ , we know from Lemma 1 that  $P(\zeta \leq v) < 1$ , whence the existence of  $\eta > 0$  such that  $P(\zeta - v > \eta) > 0$ . Let  $x$  be such that  $x + \eta$  belongs to  $D$ . Then,  $x + \zeta - v$  belongs to  $D$  with positive probability, so that  $E(r(x + \zeta - v)) > 0$ . If, moreover,  $x \geq 0$ , we have that:

$$r(x) = T(r)(x) = \psi(E(r(x + \zeta - v))),$$

so that  $r(x) > 0$  since  $\psi(s) > 0$  for all  $0 < s \leq 1$ . We have therefore proved that

$$(D - \eta) \cap [0, +\infty[ \subset D.$$

Since  $D$  is a subinterval of  $[0, +\infty[$ , unbounded to the right, this implies that  $D = [0, +\infty[$ , whence  $r(0) > 0$ . Now let

$$F := \{\lambda \geq 0; \forall x \in [0, +\infty[, r(x) \geq \lambda q(x)\}.$$

Since  $q(0) > 0$  by Lemma 2 and  $r \leq q$ ,  $F$  must have the form  $[0, \lambda_0]$  for some  $\lambda_0 \in [0, 1]$ , and, we need to prove that indeed  $\lambda_0 = 1$  to finish our argument. We first show that  $\lambda_0 > 0$ . Since  $r$  is non-decreasing, we have that, for all  $x \in \mathbb{R}$ ,  $r \geq r(0)\mathbf{1}_{[0, +\infty[}$ , whence  $r \geq r(0)q$  since  $\mathbf{1}_{[0, +\infty[} \geq q$ . As a consequence,  $\lambda_0 \geq r(0)$ , and we have seen that  $r(0) > 0$ . Now, using the fact that  $T(r) = r$ , Proposition 5, and the definition of  $\lambda_0$ , we see that

$$r = T(r) \geq T(\lambda_0 q).$$

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<sup>3</sup>For instance by coupling. If one wants a non-probabilistic proof based on the non-linear convolution equation, this is still possible by using the monotonicity of  $T$  with respect to  $v$ , applying  $T^n$  to  $q_0$  to recover  $q$  in the limit  $n \rightarrow +\infty$ .

For  $x \geq 0$ , we also have that  $q(x) \geq q(0) > 0$ , and  $E(q(x + \zeta - v)) > 0$ . Since  $\lambda_0 > 0$ , we can write:

$$r(x) \geq \lambda_0 q(x) \frac{T(\lambda_0 q)(x)}{\lambda_0 q(x)} = \lambda_0 q(x) \frac{T(\lambda_0 q)(x)}{\lambda_0 T(q)(x)},$$

whence the inequality

$$r(x) \geq \lambda_0 q(x) \chi(E(q(x + \zeta - v))),$$

where  $\chi$  is the map defined, for  $s \in ]0, 1]$ , by

$$\chi(s) := \frac{\psi(\lambda_0 s)}{\lambda_0 \psi(s)} = \frac{2 - \lambda_0 s}{2 - s},$$

with the extension  $\chi(0) := 1$ . Since  $q$  is non-decreasing, this is also the case of the map  $x \mapsto E(q(x + \zeta - v))$ . Moreover,  $\chi$  too is non-decreasing on  $[0, 1]$ , so we get that:

$$r(x) \geq \lambda_0 \chi(E(q(\zeta - v))) q(x).$$

Therefore,  $\lambda_0 \chi(E(q(\zeta - v)))$  is an element of the set  $F$ . If  $\lambda_0 < 1$ , the fact that  $E(q(\zeta - v)) > 0$  and strict monotonicity of  $\chi$  show that  $\chi(E(q(\zeta - v))) > \chi(0) = 1$ . We would thus have the existence of an element in  $F$  strictly greater than  $\lambda_0$ , a contradiction. We thus conclude that  $\lambda_0$  equals 1. Therefore one must have  $r \geq q$ .  $\square$

*Proof of Proposition 3.* Since  $T(h) \leq h$ , we deduce from Proposition 5 that, for all  $x \in \mathbb{R}$ , the sequence  $(T^n(h)(x))_{n \geq 0}$  is non-increasing. We deduce the existence of a map  $T^\infty(h)$  in  $\mathcal{H}$  such that, for all  $x \in \mathbb{R}$ ,  $T^\infty(h)(x) = \lim_{n \rightarrow +\infty} T^n(h)(x)$ , and it is easily checked by dominated convergence that  $T(T^\infty(h))(x) = T^\infty(h)(x)$  for all  $x \in \mathbb{R}$ , while  $T^\infty(h)(x) \leq h(x)$ . It remains to check that  $T^\infty(h) \not\equiv 0$  to obtain the result. Since  $h$  is non-decreasing, we have that, for all  $x \geq 0$ ,  $h(x) \geq h(0) \mathbf{1}_{[0, +\infty[}(x)$ . Moreover, it is easily checked that, for all  $\lambda, s \in [0, 1]$ ,  $\psi(\lambda s) \geq \lambda \psi(s)$ . Using Proposition 5, we thus obtain that  $T^n(h)(x) \geq h(0) T^n(\mathbf{1}_{[0, +\infty[})(x)$ , whence  $T^\infty(h)(x) \geq h(0) q(x)$  by letting  $n \rightarrow +\infty$ . We deduce that  $T^\infty(h)(x) \not\equiv 0$  since we have assumed that  $h(0) > 0$ .  $\square$

### 3. EXPONENTIAL SOLUTIONS OF LINEAR EQUATIONS

In this section, we consider linear convolution equations of the form

$$(3) \quad c(x) = e^{-a} E(c(x + \zeta - v)),$$

where  $c : \mathbb{R} \rightarrow \mathbb{C}$  is a measurable map,  $v$  is close to  $v^*$  and  $e^{-a}$  is close to 2. Looking for solutions of the form

$$(4) \quad c(x) = e^{\phi x},$$

where  $\phi \in \mathbb{C}$ , we see that a necessary and sufficient condition for (4) to yield a solution is that

$$(5) \quad E\left(e^{\phi(\zeta - v)}\right) = e^a.$$

Due to our initial assumption on  $\zeta$ , we have that

$$E\left(e^{t^*(\zeta - v^*)}\right) = 1/2,$$

so that one can hope to find solutions to (5) by performing a perturbative analysis. This is precisely what is done in Section 3.1. Then, in Section 3.2, some of the properties of the corresponding solutions of (3) are studied.

**3.1. Existence of the exponent.** Consider the extension of the Laplace transform of  $\zeta$  when the parameter  $t \in \mathbb{C}$ . Since  $\zeta$  has bounded support,  $t \mapsto E(\exp(t\zeta))$  defines a holomorphic map from  $\mathbb{C}$  to  $\mathbb{C}$ , and, since  $E(\exp(t^*\zeta)) \notin \mathbb{R}_-$ , we may extend the definition of  $\Lambda$  to an open neighborhood  $U$  of  $t^*$  in  $\mathbb{C}$ , by setting  $\Lambda(t) := \log E(\exp(t\zeta))$  for  $t \in U$ , using the principal determination of the logarithm. We thus obtain a holomorphic map on  $U$ .

**Proposition 6.** *Let  $a$  be a holomorphic function defined on a neighborhood of zero, such that  $a(0) = -\log(2)$  and  $a'(0) = 0$ . There exists  $\epsilon_0 > 0$  and a map  $\phi : [0, \epsilon_0] \rightarrow U$  such that, for all  $v \in [v^* - \epsilon_0, v^*]$ , the following identity holds:*

$$(6) \quad E\left(e^{\phi(v^*-v)(\zeta-v)}\right) = e^{a(v^*-v)},$$

and such that, as  $\epsilon \rightarrow 0$ ,

$$(7) \quad \phi(\epsilon) = t^* + i\sqrt{\frac{2t^*\epsilon}{\Lambda''(t^*)}} + o(\sqrt{\epsilon}).$$

*Proof.* Given  $v$  in the vicinity of  $v^*$ , we are looking for a  $t \in U$  such that

$$(8) \quad E\left(e^{t(\zeta-v)}\right) = e^{a(v^*-v)}.$$

For  $t$  close enough to  $t^*$ , and  $v$  close enough to  $v^*$ , we can use the logarithm and observe that the equation

$$(9) \quad \Lambda(t) - tv = a(v^* - v).$$

is sufficient for (8) to hold. Expanding  $\Lambda$  for  $t \in U$ , we have that

$$\Lambda(t) = \Lambda(t^*) + (t - t^*)\Lambda'(t^*) + (t - t^*)^2g(t),$$

where  $g$  is holomorphic on  $U$  and satisfies  $g(t^*) = \Lambda''(t^*)/2$ . Plugging (1) and (2), we can rewrite the above expansion as

$$\Lambda(t) = -\log(2) + tv^* + (t - t^*)^2g(t).$$

On the other hand, using our assumptions on  $a$ , we can write

$$a(v^* - v) = -\log(2) + (v^* - v)b(v^* - v),$$

where  $b$  is an holomorphic function in a neighborhood of zero such that  $b(0) = 0$ . Finally, (9) reads

$$(10) \quad (t - t^*)^2g(t) = (v - v^*)(t - b(v^* - v)).$$

Observe that, since  $g(t^*) = \Lambda''(t^*)/2 \notin \mathbb{R}_-$ , we can define  $\sqrt{g(t)}$  for  $t$  close to  $t^*$ , using the principal determination of the logarithm and the definition  $\sqrt{z} = \exp(\log z/2)$ . We can similarly define  $\sqrt{t - b(z)}$  for  $(t, z)$  close to  $(t^*, 0)$ . Now, for  $(t, u) \in \mathbb{C}$  in the vicinity of  $(t^*, 0)$ , consider the equation .

$$(11) \quad (t - t^*)\sqrt{g(t)} = u\sqrt{t - b(-u^2)}.$$

Clearly, if (11) holds with  $u = i\sqrt{v^* - v}$  when  $v^*$  and  $v$  are real numbers such that  $v < v^*$ , then (10) holds. Now consider the map  $\Xi$  defined in the neighborhood of  $(t^*, 0)$  in  $\mathbb{C} \times \mathbb{C}$  by

$$\Xi(t, u) := (t - t^*)\sqrt{g(t)} - u\sqrt{t - b(-u^2)}.$$

Observe that  $\Xi(t^*, 0) = 0$ , and that the (holomorphic) derivative of  $\Xi$  with respect to  $t$  at  $(t^*, 0)$  is equal to  $\sqrt{g(t^*)} = \sqrt{\Lambda''(t^*)/2} \neq 0$ . Identifying  $\mathbb{C}$  with  $\mathbb{R} \times \mathbb{R}$ , we can thus view  $\Xi$  as a smooth map defined on an open set of  $\mathbb{R}^4$ , and apply the implicit function theorem to deduce the existence of a smooth map  $f$  defined on the neighborhood of 0 in  $\mathbb{C}$  such that  $f(0) = t^*$  and such that, for all  $u$  near zero,

$$(12) \quad \Xi(f(u), u) = 0.$$

We now set  $\phi(\epsilon) = f(i\sqrt{\epsilon})$ , which yields (6). Then, one obtains (7) by computing the derivative of  $f$  at zero from (12) in the usual way.  $\square$

**3.2. Properties of the exponential solutions.** Now let  $a$ ,  $\epsilon_0$  and  $\phi$  be given as in Proposition 6. We assume throughout the sequel that

$$\epsilon = v^* - v.$$

An immediate consequence of the proposition is that, for all  $v \in [v^* - \epsilon_0, v^*]$ , the map defined on  $\mathbb{R}$  by  $x \mapsto e^{\phi(\epsilon)x}$ , solves the equation

$$(13) \quad c(x) = e^{-a(\epsilon)}E(c(x + \zeta - v)).$$

If  $a(\epsilon) \in \mathbb{R}$  when  $\epsilon \in \mathbb{R}$  (this will be the case in all the examples we consider below), then the map  $x \mapsto e^{\overline{\phi(\epsilon)}x}$  is also a solution of (13), where  $\bar{z}$  denotes the conjugate complex number of  $z$ . Let us set  $\alpha(\epsilon) := \Re(\phi(\epsilon))$  and  $\beta(\epsilon) := \Im(\phi(\epsilon))$ . Thus, for all  $A, \theta \in \mathbb{R}$ , we obtain a solution of (13) if we set

$$(14) \quad c(x) := Ae^{\alpha(\epsilon)x} \sin(\beta(\epsilon)x + \theta).$$

Assume for the moment that  $A > 0$ , and consider  $\epsilon$  small enough so that  $\alpha(\epsilon) > 0$  and  $\beta(\epsilon) > 0$ . For notational convenience, introduce

$$d(x) := c(x - \theta/\beta(\epsilon)) = Ae^{\alpha(\epsilon)(x - \theta/\beta(\epsilon))} \sin(\beta(\epsilon)x).$$

Thus,

$$d(0) = d(\pi/\beta(\epsilon)) = 0,$$

and one has that

$$\begin{cases} d \leq 0 \text{ on } [-\pi/\beta(\epsilon), 0], \\ d \geq 0 \text{ on } [0, \pi/\beta(\epsilon)], \\ d \leq 0 \text{ on } [\pi/\beta(\epsilon), 2\pi/\beta(\epsilon)]. \end{cases}$$

The derivative of  $d$  is given by

$$d'(x) = A\alpha(\epsilon)e^{\alpha(\epsilon)(x - \theta/\beta(\epsilon))} \sin(\beta(\epsilon)x) + A\beta(\epsilon)e^{\alpha(\epsilon)(x - \theta/\beta(\epsilon))} \cos(\beta(\epsilon)x).$$

One thus checks that  $d$  attains a unique maximum on the interval  $[0, \pi/\beta(\epsilon)]$ , at a value  $x = L(\epsilon)$  satisfying

$$\tan(\beta(\epsilon)L(\epsilon)) = -\beta(\epsilon)/\alpha(\epsilon),$$

and that  $d$  is increasing on the interval  $[0, L(\epsilon)]$ .



As  $\epsilon$  goes to 0, we know from (7) that

$$\alpha(\epsilon) = t^* + o(\sqrt{\epsilon}), \quad \beta(\epsilon) = \sqrt{\frac{2t^*\epsilon}{\Lambda''(t^*)}} + o(\sqrt{\epsilon}),$$

and we deduce that, as  $\epsilon$  goes to 0,

$$(15) \quad L(\epsilon) = \frac{\pi}{\beta(\epsilon)} - 1/t^* + o(1),$$

whence

$$(16) \quad L(\epsilon) \sim \pi \sqrt{\frac{\Lambda''(t^*)}{2t^*\epsilon}}.$$

#### 4. BUILDING A SUPER-SOLUTION AND A SUB-SOLUTION TO THE ORIGINAL EQUATION

In this section, we explain how to transform the exponential solutions of the linear equation obtained in the previous section, into super- and sub- solutions of the non-linear convolution equation of Section 2. In the sequel,  $\Delta$  denotes a real number (whose existence is guaranteed by the assumption that  $\zeta$  has bounded support) such that, for all  $v$  in the vicinity of  $v^*$ ,

$$P(|\zeta - v| \leq \Delta) = 1.$$

**4.1. The super-solution.** Let us choose the function  $a$  in Proposition 6 as the constant function  $a(\epsilon) := -\log(2)$ . Consider the function  $c(\cdot)$  defined in (14), that is,

$$c(x) := A(\epsilon)e^{\alpha(\epsilon)x} \sin(\beta(\epsilon)x + \theta(\epsilon)),$$

whose parameters are specified as follows. First,  $\theta(\epsilon) := \beta(\epsilon)\Delta$ , so that, with the notations of the previous section,  $c(x) = d(x + \Delta)$ . Then,  $A(\epsilon)$  is implicitly defined by the requirement that

$$c(L(\epsilon) - 2\Delta) = 1.$$

This last condition rewrites more explicitly as

$$(17) \quad A(\epsilon)e^{\alpha(\epsilon)(L(\epsilon) - 2\Delta)} \sin(\beta(\epsilon)(L(\epsilon) - \Delta)) = 1.$$

From (15), we have that, as  $\epsilon \rightarrow 0$ ,

$$\sin(\beta(\epsilon)(L(\epsilon) - \Delta)) \sim \beta(\epsilon)(1/t^* + \Delta).$$

Combining with (17), (16) and (7), we deduce that, as  $\epsilon \rightarrow 0$ ,  $A(\epsilon) > 0$  and

$$(18) \quad \log A(\epsilon) \sim -\pi \sqrt{\frac{\Lambda''(t^*)t^*}{2\epsilon}}.$$

For notational convenience, we introduce

$$C(\epsilon) := L(\epsilon) - 2\Delta.$$

The next proposition summarizes the properties of  $c$  that we shall use in the sequel.

**Proposition 7.** *For small enough  $\epsilon$ , the following properties hold.*

- (i)  $c(x) \geq 0$  for all  $x \in [-\Delta, 0]$ ;

- (ii)  $0 \leq c(x) \leq 1$  for all  $x \in [0, C(\epsilon)]$ ;
- (iii)  $c(x) \geq 1$  for all  $x \in [C(\epsilon), C(\epsilon) + \Delta]$ .
- (iv)  $c$  is non-decreasing on  $[0, C(\epsilon)]$

Moreover, as  $\epsilon$  goes to zero,

$$(19) \quad \log c(1) \sim -\pi \sqrt{\frac{\Lambda''(t^*)t^*}{2\epsilon}}.$$

*Proof.* (i), (ii), (iii) and (iv) are rather direct consequences of the definition and of the analysis of Section 3.2. As for (19), it is readily derived from (18).  $\square$

We now define the map  $c_+ : \mathbb{R} \rightarrow [0, 1]$  by

- $c_+(x) = 0$  for all  $x < 0$ ;
- $c_+(x) = c(x)$  for all  $x \in [0, L(\epsilon) - 2\Delta]$ ;
- $c_+(x) = 1$  for all  $x > L(\epsilon) - 2\Delta$ .

Now let  $\psi_+$  be defined on  $[0, 1]$  by

$$\psi_+(s) := \min(2s, 1).$$

**Proposition 8.** For all  $s \in [0, 1]$ ,  $\psi_+(s) \geq \psi(s)$ .

*Proof.* Immediate.  $\square$

**Proposition 9.** For small enough  $\epsilon$ , one has that  $c_+ \in \mathcal{H}$  and, for all  $x \geq 0$ ,

$$c_+(x) \geq \psi_+(E(c_+(x + \zeta - v))).$$

*Proof.* The fact that  $c_+ \in \mathcal{H}$  is guaranteed by the definition and properties (i) to (iv) of Proposition 7. Consider first the case  $x \in [0, C(\epsilon)]$ . By definition, one has that

$$c(x) = 2E(c(x + \zeta - v)).$$

Since, by construction,  $c_+ \leq c$  on  $[-\Delta, C(\epsilon) + \Delta]$ , we deduce that

$$c(x) \geq 2E(c_+(x + \zeta - v)).$$

Since  $\psi_+(s) \leq 2s$  for all  $s \in [0, 1]$ , we deduce that

$$c(x) \geq \psi_+(E(c_+(x + \zeta - v))),$$

whence, remembering that  $c_+(x) = c(x)$ ,

$$c_+(x) \geq \psi_+(E(c_+(x + \zeta - v))).$$

Now, for  $x > C(\epsilon)$ , we have that  $c_+(x) = 1$ , so that  $c_+(x) \geq \psi_+(s)$  for all  $s \in [0, 1]$ . In particular,

$$c_+(x) \geq \psi_+(E(c_+(x + \zeta - v))).$$

$\square$

**Corollary 1.** For small enough  $\epsilon$ , one has that  $c_+ \geq T(c_+)$ .

*Proof.* Direct consequence of Propositions 8 and 9.  $\square$

**4.2. The sub-solution.** In this section, we choose the function  $a$  in Proposition 6 as the function  $a(\epsilon) := -\log(2) + \epsilon^2$ . For notational convenience, we also introduce

$$\gamma(\epsilon) := e^{-a(\epsilon)}, \quad h(\epsilon) := \gamma(\epsilon)(2 - \gamma(\epsilon)).$$

As above, we consider the function  $c(\cdot)$  defined in (14), which is of the form

$$c(x) := A(\epsilon)e^{\alpha(\epsilon)x} \sin(\beta(\epsilon)x + \theta(\epsilon)),$$

the parameters being specified as follows. First,  $\theta(\epsilon) := 0$ . Then,  $A(\epsilon)$  is implicitly defined by the requirement that

$$c(L(\epsilon)) = h(\epsilon).$$

The same argument as in the previous section then shows that, as  $\epsilon \rightarrow 0$ ,  $A(\epsilon) > 0$  and

$$(20) \quad \log A(\epsilon) \sim -\pi \sqrt{\frac{\Lambda''(t^*)t^*}{2\epsilon}}.$$

Here are the properties of  $c$  that we shall use in the sequel.

**Proposition 10.** *For small enough  $\epsilon$ , the following properties hold.*

- (i)  $c(x) \leq 0$  for all  $x \in [-\Delta, 0]$ ;
- (ii)  $0 \leq c(x) \leq h(\epsilon)$  for all  $x \in [0, L(\epsilon)]$ ;
- (iii)  $c(x) \leq h(\epsilon)$  for all  $x \in [L(\epsilon), L(\epsilon) + \Delta]$ ;
- (iv)  $c$  is non-decreasing on  $[0, L(\epsilon)]$ .

Moreover, as  $\epsilon$  goes to zero,

$$(21) \quad \log c(1) \sim -\pi \sqrt{\frac{\Lambda''(t^*)t^*}{2\epsilon}}.$$

*Proof.* (i), (ii), (iii) and (iv) are rather direct consequences of the definition and of the analysis of Section 3.2. As for (21), it is readily derived from (20).  $\square$

We define the map  $c_- : \mathbb{R} \rightarrow [0, 1]$  by

- $c_-(x) = 0$  for all  $x < 0$ ;
- $c_-(x) = c(x)$  for all  $x \in [0, L(\epsilon)]$ ;
- $c_-(x) = h(\epsilon)$  for all  $x > L(\epsilon)$ .

Now define  $\psi_-$  on  $[0, 1]$  by

$$\psi_-(s) := \min(\gamma(\epsilon)s, h(\epsilon)).$$

**Proposition 11.** *For small enough  $\epsilon$ , for all  $s \in [0, 1]$ ,  $\psi_-(s) \leq \psi(s)$ .*

*Proof.* Immediate.  $\square$

**Proposition 12.** *For small enough  $\epsilon$ , one has  $c_- \in \mathcal{H}$  and, for all  $x \geq 0$ ,*

$$c_-(x) \leq \psi_-(E(c_-(x + \zeta - v))).$$

*Proof.* The fact that  $c_- \in \mathcal{H}$  is a direct consequence of the definition and of properties (i) to (iv) of Proposition 10. Consider first the case  $x \in [0, L(\epsilon)]$ . By construction, one has that

$$c(x) = \gamma(\epsilon)E(c(x + \zeta - v)).$$

Since, by construction,  $c_- \geq c$  on  $[-\Delta, L(\epsilon) + \Delta]$ , we deduce that

$$(22) \quad c(x) \leq \gamma(\epsilon)E(c_-(x + \zeta - v)).$$

Since we have  $c(x) \leq h(\epsilon)$  and  $c_-(x) = c(x)$ , we deduce that

$$c_-(x) \leq \psi_-(E(c_-(x + \zeta - v))).$$

Consider now  $x > L(\epsilon)$ . Since  $c_-$  is non-decreasing, so is  $y \mapsto \gamma(\epsilon)E(c_-(y + \zeta - v))$ . We deduce that

$$\gamma(\epsilon)E(c_-(x + \zeta - v)) \geq \gamma(\epsilon)E(c_-(L(\epsilon) + \zeta - v)).$$

Using again the fact that  $c_- \geq c$  on  $[-\Delta, L(\epsilon) + \Delta]$ , we have that

$$\gamma(\epsilon)E(c_-(L(\epsilon) + \zeta - v)) \geq \gamma(\epsilon)E(c(L(\epsilon) + \zeta - v)) = c(L(\epsilon)) = h(\epsilon).$$

Since  $c_- \leq h(\epsilon)$ , we finally deduce that

$$c_-(x) \leq \psi_-(E(c_-(x + \zeta - v))).$$

□

**Corollary 2.** *For small enough  $\epsilon$ , one has that  $c_- \leq T(c_-)$ .*

*Proof.* Direct consequence of Propositions 11 and 12. □

## 5. PROOF OF THEOREM 1

For small enough  $\epsilon$ , Corollary 1 yields a function  $c_+ \in \mathcal{H}$  such that  $c_+ \geq T(c_+)$ . Proposition 3 then implies that  $q \leq c_+$ . As a consequence,  $q(0) \leq q(1) \leq c_+(1)$ , so that (19) yields that

$$\limsup_{\epsilon \rightarrow 0} \sqrt{\epsilon} \log q(0) \leq -\pi \sqrt{\frac{\Lambda''(t^*)t^*}{2}}.$$

Similarly, Corollary 2 yields a function  $c_- \in \mathcal{H}$  such that  $c_- \leq T(c_-)$ . Proposition 4 then implies that  $q \geq c_-$ . Using Lemma 3, we obtain that  $q(0) \geq \kappa q(1) \geq \kappa c_-(1)$ , so that (21) yields that

$$\liminf_{\epsilon \rightarrow 0} \sqrt{\epsilon} \log q(0) \geq -\pi \sqrt{\frac{\Lambda''(t^*)t^*}{2}}.$$

## 6. CONCLUDING REMARKS

Note that the analogue of Theorem 1 in the case of Branching Brownian Motion is a direct consequence of the results in [7]. Indeed, the equation corresponding to our non-linear convolution equation  $T(q) = q$  is the differential equation  $q'' = vq' - q(1 - q)$ , which is treated in [7] by a linear stability analysis. Unfortunately, we could not directly translate this analysis to the discrete equation  $T(q) = q$ .

A nice feature of the present proof is that it provides a more Brunet-Derrida-like argument for Theorem 1 than [6]. Indeed, Brunet and Derrida's analysis is based on comparing (non-rigorously) the solutions of non-linear equations such as  $T(q) = q$  to linear equations such as (3). Since Theorem 1 is a key result in our rigorous proof of the Brunet-Derrida behavior of branching-selection particle systems, given in [1], the present paper somehow bridges the gap between Brunet and Derrida's approach of the problem and [1], and also between [1] and [7].

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