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Existence and asymptotic behaviour of some time-inhomogeneous diffusions

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Abstract: Let us consider a solution of a one-dimensional stochastic differential equation driven by a standard Brownian motion with time-inhomogeneous drift coefficient $d(t, x) = \rho \operatorname{sgn}(x)|x|^\alpha/t^\beta$. This process can be viewed as a Brownian motion evolving in a potential, possibly singular, depending on time. We prove results on the existence and uniqueness of solution, study its asymptotic behaviour and made a precise description, in terms of parameters ρ, α and β , of the recurrence, transience and convergence. More precisely, asymptotic distributions, iterated logarithm type laws and rates of transience and explosion are proved for such processes.

Key words: time-inhomogeneous diffusions, time dependent potential, singular stochastic differential equations, explosion times, scaling transformations and change of time, recurrence and transience, iterated logarithm type laws, asymptotic distributions.

2000 Mathematics Subject Classification: 60J60, 60H10, 60J65, 60G17, 60F15, 60F05

1 Introduction

Let X be a one-dimensional process describing a Brownian motion dynamics in a moving, possibly singular, potential V :

$$dX_t = dB_t - \frac{1}{2} \partial_x V(t, X_t) dt, \quad X_{t_0} = x_0, \quad \text{with } V(t, x) := \begin{cases} \frac{-2\rho \log|x|}{t^\beta}, & \text{when } \alpha = 1, \\ \frac{-2\rho |x|^{\alpha+1}}{\alpha+1 t^\beta}, & \text{elsewhere.} \end{cases} \quad (1.1)$$

Here B denotes a standard linear Brownian motion, $t_0 > 0$ and x_0, ρ, α, β are some real constants. In this paper we shall study the asymptotic behaviour of such process. More precisely, our main goal is to give conditions which characterise the recurrence, transience and convergence in terms of parameters ρ, α and β . Here are the natural questions one can ask: does there exist pathwise unique strong solution X for equation (1.1)? is this solution X recurrent or transient? does there exist a well chosen normalisation of X to ensure that the normalised process converges in distribution or almost surely? is it possible to obtain pathwise largest deviations of X , for instance iterated logarithm type law?

Questions as the last two ones are treated in [1, 2, 7] for different equations having some common features with (1.1). For instance, Gihman and Skorohod in [7], Chap. 4, §17, consider the following equation

$$dY_t = dB_t + d(Y_t)dt, \quad \text{with } d(y) \underset{|y| \rightarrow \infty}{\sim} \rho |y|^\alpha, \quad \rho > 0 \text{ and } -1 < \alpha < 1. \quad (1.2)$$

Under additional assumptions, one proves that Y_t is transient and asymptotically behaves as a solution of the deterministic underlying dynamical system, that is

$$Y_t \underset{t \rightarrow \infty}{\sim} h_t \quad \text{a.s., with } h'_t = d(h_t).$$

Equation (1.2) is also considered by Appleby and Wu [2] with particular $\alpha = -1$. Its study is related to the Bessel process and the situation is more difficult. One proves that Y_t satisfies the iterated logarithm law and recurrence or transience depends on the position of ρ with respect to $1/2$. Appleby and Mackey [1] study the following damped stochastic differential equation

$$dY_t = \sigma(t)dB_t + d(Y_t)dt, \quad \text{with } d(y) \underset{y \rightarrow 0}{\sim} \rho \operatorname{sgn}(y)|y|^\alpha, \quad \rho < 0 \quad \text{and} \quad \alpha > 1. \quad (1.3)$$

Here the diffusion coefficient $\sigma \in L^2$ is such that $\sigma(t) \downarrow 0$, as $t \rightarrow \infty$. It is proved that Y_t converges almost surely to 0 with polynomial rate. We will see that equation (1.3) is connected to equation (1.1) by performing a suitable change of time.

For time-homogeneous stochastic differential equations driven by a one-dimensional Brownian motion, there exist precise criteria for recurrence or transience (see, for instance, Kallenberg [10], Chap. 23), or explosion (see, for instance, Ikeda and Watanabe [9], Chap. VI, §3), using the scale function. Some of these criteria are extended to the time-inhomogeneous situation for dimension greater or equal than two in Bhattacharya and Ramasubramanian [3]. Unfortunately, the results in [3] do not apply to equation (1.1), even it is stated that the method can be adapted to the one-dimensional case. Recall also that there exist some general results on recurrence or transience (see, for instance, Has'minskii [8], Chap. III), and explosion (see for instance Narita [14] or Stroock and Varadhan [16], Chap. 10), based on the construction of some convenient Lyapunov functions. However, for equation (1.1), the construction of such functions seems to be more delicate.

Equation (1.1) can be also viewed as a continuous counterpart of a discrete time model considered recently by Menshikov and Volkov [12]. Indeed, the discrete time process studied in [12] is a random walk on the real positive half line such that

$$\mathbb{E}(X_{t+1} - X_t \mid X_t = x) \underset{t \rightarrow \infty}{\sim} \rho \frac{x^\alpha}{t^\beta}.$$

The authors establish when the process is recurrent or transient for certain values of parameters ρ, α, β , give the answer to a open question concerning the Friedman's urn model (see Freedman [6]), and present some open problems. Their approach is based on a precise study of some submartingales and supermartingales.

In the present paper we firstly study existence, uniqueness and explosion of solutions for (1.1). Different situations are distinguished, following the values of ρ and α , and existence and uniqueness are proved. We point out that, when $\alpha < 0$, the existence of a solution is not obvious, since the drift has a singularity. For the time-homogeneous case, a solution to this problem is given by Cherny and Engelbert in [4], by using the scale function. These ideas do not apply to one-dimensional time-inhomogeneous stochastic differential equations, and this is the main difficulty of this part of our paper. The idea is to use an appropriate change of time, taking full advantage of the scaling property of the Brownian motion, of the Girsanov transformation, but also of the classification of isolated singular points in [4]. These different tools also allow to answer the question of explosion of the solution when $\alpha > 1$. As

an example, we point out that, when $2\beta > \alpha + 1$, the solution explodes in finite time with a positive probability, but not almost surely.

Secondly, we describe, for all values of parameters ρ, α, β , the recurrent or the transient feature of the solution, but also its convergence. We present in Figure 1 below, the diagram of phase transition that we obtain in the attractive case $\rho < 0$:

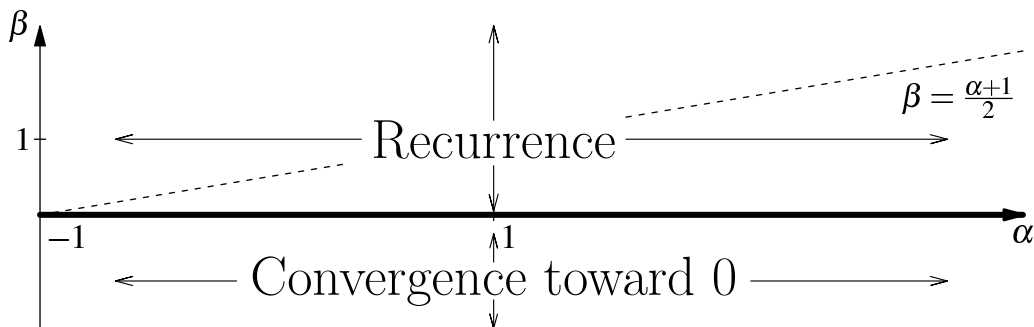


Figure 1: Phase transition in the attractive case $\rho < 0$

Note that $\alpha \leq -1$ is not allowed, since in that case, any solution is only defined up to the time of first reaching 0 (which is finite almost surely), and cannot be continued after it has reached this point. The critical line separating the two phases (*recurrence* and *convergence toward 0*) is $\beta = 0$ and on this line the process is recurrent. The line $2\beta = \alpha + 1$ could be called subcritical, in the sense that, the rate of the asymptotic behaviour is different on both sides. As for the proof of the existence, we use a suitable scaling transformation to obtain the asymptotic distribution of X and its pathwise largest deviations, under a convenient normalisation. In fact, we show that the asymptotic behaviour of the process is strongly connected to the paths, and to the stationary distribution, of an ergodic diffusion. For example, when $2\beta < \alpha + 1$, if φ is the positive solution of $\varphi'(t) = \varphi(t)^{\frac{2\beta}{\alpha+1}}$, then

$$\frac{X_{\varphi(t)}}{\sqrt{\varphi'(t)}} \quad \text{"behaves as"} \quad H_t = B_t + \int_0^t \rho \operatorname{sgn}(H_s) |H_s|^\alpha ds.$$

We obtain the convergence in distribution of $X_t/t^{\frac{\beta}{\alpha+1}}$ to the stationary distribution of H , and also its pathwise largest deviation. In particular, when $\beta < 0$, we get the so-called polynomial stability of X . Furthermore, note that, if we set $Y_t := X(\phi_t)$, with $\phi_t := t^{\frac{1}{1-\beta}}$, then Y_t satisfies the damped stochastic differential equation (1.3). We prove similar results as in [1] under slightly different hypothesis, and we obtain sharp rates of convergence.

We present in Figure 2 below the diagram of phase transition that we obtain in the repulsive case $\rho > 0$. When $\alpha > 1$ and $2\beta \leq \alpha + 1$ the explosion time is almost surely finite. The critical curve is composed from two half-lines, $\beta = 0$, when $\alpha \leq -1$, and $2\beta = \alpha + 1$, when $\alpha \geq -1$, and the process is either recurrent or transient. We prove similar results as in [12], and again we obtain sharp rates of convergence. On the critical curve one needs to distinguish two particular points $(-1, 0)$ and $(1, 1)$, because these are the only cases where recurrence and transience depend on the position of ρ with respect to $1/2$. $(\alpha, \beta) = (-1, 0)$ corresponds to the well known Bessel process, whereas $(\alpha, \beta) = (1, 1)$ is a continuous time counterpart of the Friedman's urn model. In the latter case, we obtain similar results as in [6] and [12], concerning recurrence and transience, but also regarding the asymptotic distribution

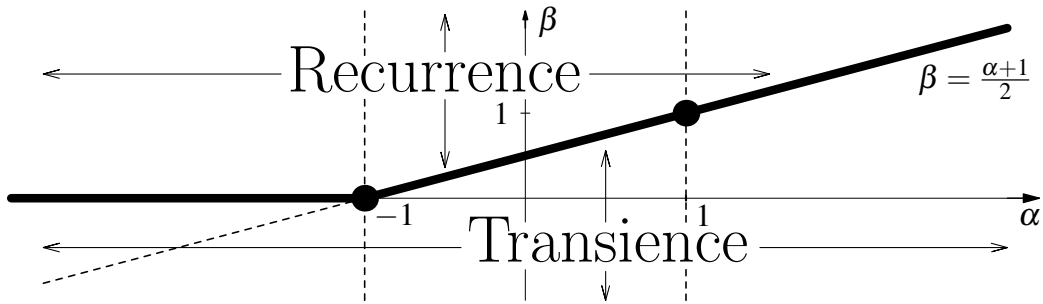


Figure 2: Phase transition in the repulsive case $\rho > 0$

and the pathwise largest deviations. For the other points of the critical curve, the process is recurrent. We point out that this is an open problem in [12]. The lines $\alpha = -1$ and $\alpha = 1$ could be called subcritical, in the sense that, the behaviour of the process is slightly different to the right or left. In particular, the domain of recurrence depends on α . The proof of recurrence is based on the same ideas as for the attractive case: by an appropriate scaling transformation of X we associate an ergodic diffusion, whose asymptotic behaviour is easier to obtain. For instance, when $2\beta > \alpha + 1$ and $-1 < \alpha < 1$, we show that

$$\frac{X_{e^t}}{e^{\frac{t}{2}}} \quad \text{''behaves as''} \quad U_t = B_t - \int_0^t \frac{U_s}{2} ds.$$

We get that X behaves as a standard Brownian motion: it satisfies the iterated logarithm law and X_t/\sqrt{t} converges in distribution to a standard Gaussian random variable. Roughly speaking, this means that the drift is asymptotically negligible compared to the noise. Concerning the proof of the transient case, when $\alpha < 1$, the tools are similar to those used in [12]. We obtain similar results as in [7], for equation (1.2), and we show that X behaves as a solution of the deterministic underlying dynamical system, that is

$$|X_t| \underset{t \rightarrow \infty}{\sim} |h_t| \quad \text{a.s.,} \quad \text{with} \quad h'_t = \rho \operatorname{sgn}(h_t) \frac{|h_t|^\alpha}{t^\beta}.$$

Some results in the present paper could be obtained, with similar arguments, for a general potential V , under convenient assumptions, for instance, when $\partial_x V(t, x) = -2f(t)g(x)$ with $f(t) \sim_{t \rightarrow \infty} t^{-\beta}$ and $|g(x)| \sim_{|x| \rightarrow \infty} \rho|x|^\alpha$. These results will be presented elsewhere. The case of a multiplicative noise seems more difficult. Another interesting situation is obtained when one replaces the Brownian motion by a (stable) Lévy process, and it is object of some works in progress. Some methods in the present paper can be used in the study of a time-inhomogeneous diffusion in random environment of the form $V(t, x) = t^{-\beta} W(x)$, with W a self-similar process (for instance, a Brownian motion). This situation it is also object of some works in progress.

The paper is organised as follows: in the next section we introduce the scaling transformations and list the associated equations associated to some particular transformations. In Section 3 we perform the complete study of the existence, uniqueness and explosion of the solutions for equation (1.1). Section 4 is devoted to a systematic study of the asymptotic behaviour of the solutions. Three cases are considered: on the critical line $2\beta = \alpha + 1$, above and under this line. Proofs of some technical results are given in the Appendix.

2 Scaling transformation and associated equations

We shall study equation (1.1) in its equivalent form:

$$dX_t = dB_t + \rho \operatorname{sgn}(X_t) \frac{|X_t|^\alpha}{t^\beta} dt, \quad X_{t_0} = x_0, \quad (2.1)$$

B being a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. By symmetry of the equation and by the usual scaling transformation, we can assume without loss of generality that $x_0 \geq 0$ and $t_0 = 1$. We will keep these assumptions all along the paper.

We begin by defining a transformation of equation (2.1) which takes full advantage of the scaling property of the Brownian motion B and the homogeneous properties of the drift $d(t, x) := \rho \operatorname{sgn}(x)|x|^\alpha/t^\beta$, that is for any $\lambda, \mu > 0$,

$$(t \mapsto B_{\lambda t}) \stackrel{\mathcal{L}}{=} (t \mapsto \lambda^{\frac{1}{2}} B_t) \quad \text{and} \quad d(\mu t, \lambda x) = \lambda^\alpha \mu^{-\beta} d(t, x).$$

This transformation will provide some important equations related to our problem and it will be useful later to study the existence, the uniqueness and the asymptotic behaviour of solutions of equation (2.1).

2.1 Scaling transformation

For any $T \in (0, \infty]$ let $\overline{\mathcal{C}}([0, T])$ be the set of functions $\omega : [0, T] \rightarrow \mathbb{R} \cup \{\Delta\}$ such that there exists a time $\tau_e(\omega) \in (0, T]$ (called the killing time of ω) such that ω is continuous on $[0, \tau_e(\omega))$ and $\omega = \infty$ on $[\tau_e(\omega), T]$. We set $\Omega := \overline{\mathcal{C}}([1, \infty))$ and $\Omega^* := \overline{\mathcal{C}}([0, t_1))$, with $t_1 \in (0, \infty]$. For every C^2 -diffeomorphism (change of time) $\varphi : [0, t_1) \rightarrow [1, \infty)$ we introduce the scaling transformation $\Phi_\varphi : \Omega \rightarrow \Omega^*$ given by

$$\Phi_\varphi(\omega)(s) := \frac{\omega(\varphi(s))}{\sqrt{\varphi'(s)}}, \quad \text{with } s \in [0, t_1), \omega \in \Omega. \quad (2.2)$$

Proposition 2.1. *The scaling transformation Φ_φ induces a bijection between weak solutions (possibly explosive) of equation (2.1) and weak solutions (possibly explosive) of equation*

$$dX_s^{(\varphi)} = dW_s + \rho \frac{\varphi'(s)^{\frac{\alpha+1}{2}}}{\varphi(s)^\beta} \operatorname{sgn}(X_s^{(\varphi)}) |X_s^{(\varphi)}|^\alpha ds - \frac{\varphi''(s)}{\varphi'(s)} \frac{X_s^{(\varphi)}}{2} ds, \quad X_0^{(\varphi)} = \frac{x_0}{\sqrt{\varphi'(0)}}. \quad (2.3)$$

Here $\{W_s : s \in [0, t_1)\}$ denotes a standard Brownian motion. More precisely,

i) if (X, B) is a solution of equation (2.1) then $(X^{(\varphi)}, W)$ is a solution of equation (2.3) where

$$X^{(\varphi)} = \Phi_\varphi(X) \quad \text{and} \quad W_t := \int_0^t \frac{dB(\varphi(s))}{\sqrt{\varphi'(s)}}; \quad (2.4)$$

ii) if $(X^{(\varphi)}, W)$ is a solution of equation (2.3) then (X, B) is a solution of equation (2.1) where

$$X = \Phi_\varphi^{-1}(X^{(\varphi)}) \quad \text{and} \quad B_t - B_1 := \int_1^t \sqrt{(\varphi' \circ \varphi^{-1})(s)} dW(\varphi^{-1}(s)). \quad (2.5)$$

It follows that uniqueness in law, pathwise uniqueness or strong existence holds for equation (2.1) if and only if it holds for equation (2.3).

Proof. Let (X, B) be a solution of equation (2.1). By using P. Lévy's characterisation theorem of Brownian motion, we can see that W defined in (2.4) is a standard Brownian motion. Moreover, by performing the change of variable $t := \varphi(s)$ in (2.1), we get

$$X_{\varphi(s)} - X_{\varphi(0)} = \int_0^s \sqrt{\varphi'(u)} dW_u + \rho \int_0^s \operatorname{sgn}(X_{\varphi(u)}) \frac{|X_{\varphi(u)}|^\alpha}{\varphi(u)^\beta} \varphi'(u) du.$$

By the integration by parts formula written in its differential form we obtain

$$d\left(\frac{X_{\varphi(s)}}{\sqrt{\varphi'(s)}}\right) = dW_s + \rho \frac{\varphi'(s)^{\frac{\alpha+1}{2}}}{\varphi(s)^\beta} \operatorname{sgn}\left(\frac{X_{\varphi(s)}}{\sqrt{\varphi'(s)}}\right) \left|\frac{X_{\varphi(s)}}{\sqrt{\varphi'(s)}}\right|^\alpha ds - \frac{X_{\varphi(s)}}{\sqrt{\varphi'(s)}} \frac{\varphi''(s)}{2\varphi'(s)} ds.$$

We conclude that equation (2.3) is satisfied by $(X^{(\varphi)}, W)$. The proof of (2.5) is similar by noting that Φ_φ is a bijection and its inverse function is given by

$$\Phi_\varphi^{-1}(\omega)(s) = \sqrt{\varphi' \circ \varphi^{-1}(s)} \omega(\varphi^{-1}(s)), \quad \text{with } s \in [1, \infty), \omega \in \Omega^*.$$

The final remark is a simple application of parts i) and ii). □

2.2 Two particular transformations

We give here two scaling transformations which produce at least one time-homogeneous coefficient among the two terms of the drift in (2.3). We also introduce some equations related to (2.1) which will be useful later in our study.

2.2.1 Exponential scaling transformation

The transformation (2.2) associated to the exponential change of time $\varphi_e(t) := e^t$, and denoted by Φ_e , is given by

$$\Phi_e(\omega)(s) = \frac{\omega(e^s)}{e^{s/2}}, \quad \text{with } s \in [0, \infty), \omega \in \Omega.$$

The process $X^{(e)} := \Phi_e(X)$ satisfies equation (2.3) which can be written

$$dX_s^{(e)} = dW_s - \frac{X_s^{(e)}}{2} ds + \rho e^{(\frac{\alpha+1}{2}-\beta)s} \operatorname{sgn}(X_s^{(e)}) |X_s^{(e)}|^\alpha ds, \quad X_0^{(e)} = x_0. \quad (2.6)$$

On the one hand, if we leave out the third term in the right hand side of equation (2.6), then we obtain the classical equation of the Ornstein-Uhlenbeck process:

$$dU_s = dW_s - \frac{U_s}{2} ds, \quad U_0 = x_0. \quad (2.7)$$

Note that equation (2.7) is a particular case of equation (2.1) with particular parameters $\rho = -1/2$, $\alpha = 1$ and $\beta = 0$. On the other hand, when $\alpha = -1$, by using Ito's formula we can see that $Y := X^2$ satisfies

$$dY_t = 2\sqrt{Y_t} dW_t + \left(\frac{2\rho}{t^\beta} + 1\right) dt, \quad Y_0 = x_0^2, \quad Y \geq 0. \quad (2.8)$$

This process can be viewed as a square Bessel process whose dimension depends on time. Clearly, when $\beta = 0$, this process is the classical square Bessel process R of dimension $2\rho + 1$ and it satisfies

$$dR_t = 2\sqrt{R_t} dW_t + (2\rho + 1) dt, \quad R_0 = x_0^2, \quad R \geq 0. \quad (2.9)$$

Furthermore, the process $R^{(e)} := \Phi_e(R)$ satisfies

$$dR_t^{(e)} = 2\sqrt{R_t^{(e)}} dW_t + \left(2\rho + 1 - \frac{R_t^{(e)}}{2}\right) dt, \quad R_0^{(e)} = x_0^2, \quad R^{(e)} \geq 0. \quad (2.10)$$

2.2.2 Power scaling transformation

Assume that $\alpha \neq -1$ and consider the Cauchy problem:

$$\varphi'_\gamma(s) = \varphi_\gamma(s)^\gamma, \quad \varphi_\gamma(0) = 1, \quad \text{with} \quad \gamma := \frac{2\beta}{\alpha + 1}. \quad (2.11)$$

There exists a unique maximal solution $\varphi_\gamma \in C^2([0, t_1]; [1, \infty))$ and we can see that

$$\varphi_\gamma(s) = (1 + (1 - \gamma)s)^{\frac{1}{1-\gamma}}, \quad \text{when } \gamma \neq 1 \quad \text{and} \quad \varphi_\gamma(s) = e^s, \quad \text{when } \gamma = 1,$$

with $t_1 = \infty$, when $\gamma \in (-\infty, 1]$, and $t_1 = \frac{1}{\gamma-1}$, when $\gamma \in (1, \infty)$. The transformation (2.2) associated to this change of time will be denoted Φ_γ , and is given by

$$\Phi_\gamma(\omega)(s) = \frac{\omega(\varphi_\gamma(s))}{\varphi_\gamma(s)^{\frac{\alpha}{2}}}, \quad \text{with } s \in [0, t_1), \omega \in \Omega.$$

The process $X^{(\gamma)} := \Phi_\gamma(X)$ satisfies equation (2.3) which can be written

$$dX_s^{(\gamma)} = dW_s + \rho \operatorname{sgn}(X_s^{(\gamma)}) |X_s^{(\gamma)}|^\alpha ds - \gamma \varphi_\gamma^{\gamma-1}(s) \frac{X_s^{(\gamma)}}{2} ds, \quad X_0^{(\gamma)} = x_0, \quad s \in [0, t_1). \quad (2.12)$$

i) If $\gamma \in (-\infty, 1)$, then (2.12) takes the form:

$$dX_s^{(\gamma)} = dW_s + \rho \operatorname{sgn}(X_s^{(\gamma)}) |X_s^{(\gamma)}|^\alpha ds - \frac{\gamma X_s^{(\gamma)}}{2(1 + (1 - \gamma)s)} ds, \quad X_0^{(\gamma)} = x_0, \quad s \in [0, \infty). \quad (2.13)$$

ii) If $\gamma \in (1, \infty)$, then (2.12) takes the form:

$$dX_s^{(\gamma)} = dW_s + \rho \operatorname{sgn}(X_s^{(\gamma)}) |X_s^{(\gamma)}|^\alpha ds - \delta \frac{X_s^{(\gamma)}}{t_1 - s} ds, \quad X_0^{(\gamma)} = x_0, \quad s \in [0, t_1), \quad (2.14)$$

with

$$t_1 = \frac{1}{\gamma - 1} \quad \text{and} \quad \delta := \frac{\gamma}{2(\gamma - 1)}. \quad (2.15)$$

iii) If $\gamma = 1$, then (2.12) takes the form:

$$dZ_s = dW_s + \left(\rho \operatorname{sgn}(Z_s) |Z_s|^\alpha - \frac{Z_s}{2} \right) ds, \quad Z_0 = x_0, \quad s \in [0, \infty). \quad (2.16)$$

Note that the transformations Φ_e and Φ_γ coincide when $\gamma = 1$. Finally, let us introduce another two stochastic differential equations related to (2.12). First, we leave out the third term on the right hand side of (2.12) and we get

$$dH_s = dW_s + \rho \operatorname{sgn}(H_s)|H_s|^\alpha ds, \quad H_0 = x_0, \quad s \in [0, t_1]. \quad (2.17)$$

Note that the latter equation is nothing but (2.1) with $\beta = 0$. Second, we leave out the second term in the right hand side of (2.14) and we obtain

$$db_s = dW_s - \delta \frac{b_s}{t_1 - s} ds, \quad b_0 = x_0, \quad s \in [0, t_1]. \quad (2.18)$$

The process b is the so-called δ -Brownian bridge (see also [11]) and it is the classical Brownian bridge when $\delta = 1$.

3 Preliminary study of solutions

3.1 Existence and uniqueness

Existence and uniqueness for equation (2.1) are not obvious since the drift could be singular in 0 and/or not time-homogeneous. However, with the help of transformation (2.2) and the Girsanov transformation, we shall reduce to the study of equation (2.17).

Lemma 3.1. *Assume that $\alpha \in \mathbb{R} \setminus \{-1\}$. The Girsanov transformation induces a linear bijection between weak solutions of equation (2.12) and weak solutions of equation (2.17).*

Proof. This result is classical, but we give the proof for the sake of completeness. We can take place on the canonical space $\Omega^* = \overline{\mathbb{C}}([0, t_1])$ (defined in the begining of section 2.1) endowed with the canonical σ -field \mathcal{F} and the canonical filtration $\{\mathcal{F}_t : t \in [0, t_1]\}$. Set, for every $n \geq 1$ and $\omega \in \Omega^*$, $\tau_n(\omega) := \inf\{s \geq 0 : |\omega(s)| \geq n\}$, and denote $\tau_e := \sup\{\tau_n : n \geq 1\}$, the explosion time. Let \mathbb{P} be a weak solution of equation (2.17), defined on \mathcal{F}_{τ_e-} . Denote by $\mathbb{P}^{(n)}$ the restriction of \mathbb{P} on \mathcal{F}_{τ_n} . Then, under $\mathbb{P}^{(n)}$, the canonical process satisfies the equation

$$\omega(t \wedge \tau_n) = x_0 + W_{t \wedge \tau_n}(\omega) + \int_0^{t \wedge \tau_n} \rho \operatorname{sgn}(\omega(s))|\omega(s)|^\alpha ds,$$

where W denotes a standard Brownian motion. The Girsanov density is given by

$$D_n(\omega) := \exp \left(\int_0^{\tau_n} \gamma \varphi_\gamma^{\gamma-1}(s) \frac{\omega(s)}{2} dW_s(\omega) - \frac{1}{2} \int_0^{\tau_n} \gamma^2 \varphi_\gamma^{2(\gamma-1)}(s) \frac{\omega^2(s)}{4} ds \right).$$

For any integer $n \geq 1$, define the probability $\mathbb{Q}^{(n)}(d\omega) := D_n(\omega)\mathbb{P}^{(n)}(d\omega)$. It is clear that, under $\mathbb{Q}^{(n)}$, the canonical process satisfies the equation

$$\omega(t \wedge \tau_n) = x_0 + \tilde{W}_{t \wedge \tau_n}(\omega) + \int_0^{t \wedge \tau_n} \rho \operatorname{sgn}(\omega(s))|\omega(s)|^\alpha ds - \int_0^{t \wedge \tau_n} \gamma \varphi_\gamma^{\gamma-1}(s) \frac{\omega(s)}{2} ds,$$

\tilde{W} denoting also a standard Brownian motion. Therefore, we can construct a unique weak solution \mathbb{Q} to equation (2.12), defined on \mathcal{F}_{τ_e-} , such that $\mathbb{Q}|_{\mathcal{F}_{\tau_n}} = \mathbb{Q}^{(n)}$. Since the Girsanov density is positive, we can reverse the construction above and deduce that the Girsanov transformation induces a linear bijection between weak solutions of equation (2.12) and weak solutions of equation (2.17). \square

Remark 3.2. Let \mathbb{P} be a weak solution of equation (2.17) and \mathbb{Q} the corresponding solution of equation (2.12) obtained by the Girsanov transformation. We can see that

$$\begin{aligned} \mathbb{P}(\{\omega : \omega \geq 0\}) = 1 \quad (\text{respectively, } \mathbb{P}(\{\omega : \omega(0) = 0, \forall t > 0 \omega(t) > 0\}) = 1), \\ \iff \mathbb{Q}(\{\omega : \omega \geq 0\}) = 1 \quad (\text{respectively, } \mathbb{Q}(\{\omega : \omega(0) = 0, \forall t > 0 \omega(t) > 0\}) = 1). \end{aligned}$$

3.1.1 Nonsingular case: $\alpha \geq 0$

Proposition 3.3. If $\alpha \in [0, \infty)$ and $\beta \in \mathbb{R}$, then there exists a pathwise unique strong solution X to equation (2.1) defined up to the explosion time τ_e .

Proof. Assume first that $\alpha \in (1, \infty)$. We construct a truncated stochastic differential equation as follows: for every integer $n \geq 1$ set $d_n(t, x) := \rho \operatorname{sgn}(x)(|x| \wedge n)^\alpha / t^\beta$ and

$$d^{(n)}X_t = dB_t + d_n(t, {}^{(n)}X_t)dt, \quad {}^{(n)}X_1 = x_0.$$

This equation has locally Lipschitz coefficients with linear growth. Let ${}^{(n)}X$ be its pathwise unique strong solution. Let us introduce

$$\tau_n := \inf\{t \geq 1 : |{}^{(n)}X_t| \geq n\}, \quad n \geq 1 \quad \text{and} \quad \tau_e := \sup\{\tau_n : n \geq 1\}.$$

For all $t \in [1, \tau_e)$ we set $X_t := {}^{(n)}X_t$ a.s. where n is chosen such that $t \in [1, \tau_n]$. The process X is well defined up to the explosion time τ_e and it is the desired solution for (2.1). Secondly assume that $\alpha \in [0, 1)$. weak existence and uniqueness in law are consequence of the Girsanov theorem, the Novikov criterion being satisfied. To verify this criterion, let W be a standard Brownian motion starting from x_0 , set $d(t, x) := \rho \operatorname{sgn}(x)|x|^\alpha / t^\beta$ and let $T \in (1, \infty)$ be. By using the Jensen inequality applied to the exponential function, we get

$$\mathbb{E} \left[\exp \left(\frac{1}{2} \int_1^T d^2(t, W_t) dt \right) \right] \leq \int_1^T \mathbb{E} \left[\frac{e^{\lambda |W_t|^{2\alpha}}}{T-1} \right] dt < \infty, \quad \text{with } \lambda := \sup_{1 \leq t \leq T} \frac{(T-1)|d(T, 1)|}{2}.$$

Hence the Novikov criterion holds. Pathwise uniqueness is a consequence of Proposition 3.2 and Corollary 3.4, Chap. IX in [15], pp. 389-390, using local times. Finally, when $\alpha = 1$, the result is clear, since the coefficients of equation (2.1) are locally Lipschitz functions with linear growth. Remark also that, when $\alpha \in [0, 1]$, $\tau_e = \infty$ a.s. \square

3.1.2 Singular case: $-1 < \alpha < 0$

In this case the function $x \mapsto |x|^\alpha$ is locally integrable in a neighbourhood of 0. As for the time-homogeneous equation (2.17), we show that there exists a pathwise unique strong solution to equation (2.1).

Proposition 3.4. If $\alpha \in (-1, 0)$ and $\beta \in \mathbb{R}$, then there exists a pathwise unique strong solution X to equation (2.1).

Proof. By using Proposition 2.1 and Lemma 3.1 we see that it suffices to study equation (2.17). More precisely, since pathwise uniqueness holds for equation (2.1) (by Proposition 3.2 and Corollary 3.4, Chap. IX in [15], pp. 389-390) it suffices to prove that there exists a unique weak solution H to equation (2.17). Indeed, in this case, Lemma 3.1 implies that

there exists a unique weak solution $X^{(\gamma)}$ to equation (2.12) and Proposition 2.1 implies that equation (2.1) has a unique weak solution X . By pathwise uniqueness, we deduce that there exists a pathwise unique strong solution. To study equation (2.17), we note that the function $x \mapsto |x|^\alpha$ is locally integrable on the real line. By using Proposition 2.2 in [4], p. 28, there exists a unique weak solution for (2.17) defined up to the explosion time, and we are done. \square

3.1.3 Singular case: $\alpha < -1$

In this case the function $x \mapsto |x|^\alpha$ is not locally integrable in a neighbourhood of 0. Nevertheless, as in the proof of Proposition 3.4, it suffices to study equation (2.17).

Proposition 3.5. *If $\rho \in (0, \infty)$, $\alpha \in (-\infty, -1)$ and $\beta \in \mathbb{R}$, then there exists a pathwise unique nonnegative strong solution X to equation (2.1):*

- i) if $x_0 \in (0, \infty)$, then X is the pathwise unique strong solution and it is positive;*
- ii) if $x_0 = 0$, then for all $t > 1$, $X_t > 0$ a.s. and the set of all weak solutions is the set of all distributions which are mixture of the distributions of X and $-X$.*

Proof. Recall that we are assuming that $x_0 \in [0, \infty)$. By using the results on power equations in Chap. 5 in [4], there exists a unique nonnegative weak solution H defined up to the explosion time for equation (2.17). Therefore, by Lemma 3.1, there exists a unique nonnegative weak solution $X^{(\gamma)}$ to equation (2.12), and by Proposition 2.1 there exists a unique nonnegative weak solution X to equation (2.1). To conclude it remains to show that pathwise uniqueness holds for nonnegative solutions of equation (2.1). We point out that there is no uniqueness in law for equation (2.1) and we cannot apply directly Proposition 3.2, Chap. IX in [15], p. 389. However, we can state the following:

Lemma 3.6. *If uniqueness in law holds for nonnegative solutions of a one dimensional stochastic differential equation driven by a standard Brownian motion and if the local time at the level 0, $L^0(X^1 - X^2) \equiv 0$, for any pair of nonnegative solutions such that $X_0^1 \equiv X_0^2$ a.s., then pathwise uniqueness holds for nonnegative solutions.*

By using this lemma and Corollary 3.4, Chap. IX in [15], p. 390, pathwise uniqueness holds for nonnegative solutions of equation (2.1) and we conclude that there exists a pathwise unique nonnegative strong solution X for this equation. Moreover, if $x_0 \in (0, \infty)$, then by using the results on power equations in Chap. 5 in [4] (see, for instance, Theorem 3.5 in [4], p. 66) any solution H of equation (2.17) is positive. We deduce from Proposition 2.1 and Lemma 3.1 that there exists a pathwise unique strong solution to equation (2.1) and this solution is positive. If $x_0 = 0$, by using again Theorem 3.5 in [4], p. 66, the set of all solutions of equation (2.17) is (by symmetry of the equation) the set of all distributions which are mixture of the distributions of $\pm H$ (here H denotes the unique nonnegative weak solution for (2.17)). Once again, we deduce the point ii) from Proposition 2.1 and Lemma 3.1. \square

Proof of Lemma 3.6. The proof is modeled on the proof of Proposition 3.2, Chap. IX in [15], p. 389. By Proposition 3.1, Chap. IX in [15], p. 389, $X^1 \vee X^2$ is a nonnegative solution. We deduce that $X^1 \stackrel{\mathcal{L}}{=} X^1 \vee X^2$ and $X^1 = X^2$ a.s. \square

3.1.4 Bessel type case: $\alpha = -1$

Proposition 3.7. *If $\rho \in (0, \infty)$, $\alpha = -1$ and $\beta \in \mathbb{R}$, then there exists a pathwise unique nonnegative strong solution X to equation (2.1):*

- i) if $\rho \in [1/2, \infty)$, $\beta \in (-\infty, 0]$ and $x_0 \in (0, \infty)$, then X is the pathwise unique strong solution and it is positive;*
- ii) if $\rho \in [1/2, \infty)$, $\beta \in (-\infty, 0]$ and $x_0 = 0$, the set of all weak solutions is the set of all distributions which are mixture of distributions of X and $-X$ and $\forall t > 0$, $X_t > 0$ a.s.;*
- iii) if $\beta \in (0, \infty)$ or if $(\rho, \beta) \in (0, 1/2) \times (-\infty, 0]$, we can construct different weak solutions to equation (2.1) and in the first case the set $\{t \geq 1 : X_t = 0\}$ is unbounded a.s.*

Proof. The proof of the existence of a pathwise unique nonnegative strong solution Y of equation (2.8) is not difficult. Indeed, weak existence of a nonnegative solution holds since the drift in equation (2.8) is deterministic and nonnegative, whereas pathwise uniqueness can be obtained by using Lemma 3.6 and Corollary 3.4, Chap. IX in [15], p. 389. We shall prove that $X := \sqrt{Y}$ is a nonnegative weak solution of equation (2.1). By applying Ito's formula, for all $t \geq 1$ and $\varepsilon > 0$,

$$(X_t^2 + \varepsilon)^{\frac{1}{2}} = (x_0^2 + \varepsilon)^{\frac{1}{2}} + \int_1^t \left(\frac{X_s^2}{X_s^2 + \varepsilon} \right)^{\frac{1}{2}} dW_s + \int_1^t \frac{\rho ds}{s^\beta (X_s^2 + \varepsilon)^{\frac{1}{2}}} + \int_1^t \frac{\varepsilon ds}{2(X_s^2 + \varepsilon)^{\frac{3}{2}}}. \quad (3.1)$$

We need to let $\varepsilon \rightarrow 0$ in the last equation. Firstly, it is clear that

$$\lim_{\varepsilon \rightarrow 0} \int_1^t \left(\frac{X_s^2}{X_s^2 + \varepsilon} \right)^{\frac{1}{2}} dW_s = W_t - W_1, \quad \text{in probability.}$$

Secondly, by monotone convergence theorem, the third term in the right hand side of (3.1) converges a.s. We show that the limit is finite a.s. and that the fourth term converges toward 0 in probability. To this end, consider the pathwise unique nonnegative strong solution of

$$Q_s = x_0 + W_s - W_1 + \int_1^s \frac{\rho_1}{Q_u} du, \quad \text{with } \rho_1 := \inf \left\{ \frac{\rho}{s^\beta} : s \in [1, t] \right\} > 0. \quad (3.2)$$

Q_s is a classical Bessel process of dimension $2\rho_1 + 1$. By using a comparison theorem (see Theorem 1.1, Chap. VI in [11], p. 437) and Ito's formula, $X_s^2 \geq Q_s^2$ for all $s \in [1, t]$ and

$$(Q_t^2 + \varepsilon)^{\frac{1}{2}} = (x_0^2 + \varepsilon)^{\frac{1}{2}} + \int_1^t \left(\frac{Q_s^2}{Q_s^2 + \varepsilon} \right)^{\frac{1}{2}} dW_s + \int_1^t \frac{\rho_1 ds}{(Q_s^2 + \varepsilon)^{\frac{1}{2}}} + \int_1^t \frac{\varepsilon ds}{2(Q_s^2 + \varepsilon)^{\frac{3}{2}}}.$$

Since Q_s is a solution of (3.2) we obtain, by letting $\varepsilon \rightarrow 0$ in the last equality,

$$\int_1^t \frac{\rho ds}{s^\beta X_s} \leq \int_1^t \frac{\rho_2}{Q_s} ds < \infty \quad \text{a.s. with } \rho_2 := \sup \left\{ \frac{\rho}{s^\beta} : s \in [1, t] \right\} < \infty,$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_1^t \frac{\varepsilon ds}{2(X_s^2 + \varepsilon)^{\frac{3}{2}}} \leq \lim_{\varepsilon \rightarrow 0} \int_1^t \frac{\varepsilon ds}{2(Q_s^2 + \varepsilon)^{\frac{3}{2}}} = 0 \quad \text{in probability.}$$

We deduce that X is a nonnegative weak solution of (2.1). Pathwise uniqueness is obtained by using Lemma 3.6 and Corollary 3.4, Chap. IX in [15], p. 389. We proceed with the proof of statements i)-iii) in the proposition. Firstly, if $\rho \in [1/2, \infty)$, $\beta \in (-\infty, 0]$ and $x_0 \in (0, \infty)$, then X is the pathwise unique strong solution, since every solution is positive by comparison with a Bessel process of dimension 2. Secondly, if $\beta \in (0, \infty)$, then $\lim_{t \rightarrow \infty} \rho/t^\beta = 0^+$. By using the Markov property and the comparison with a Bessel process of dimension $\delta \in (1, 2)$, we get that the reaching time of 0 is finite a.s. and that the set $\{t > 1 : X_t = 0\}$ is unbounded a.s. Besides, if X is a solution starting from $x_0 = 0$ then $-X$ is also a solution. We deduce that different solutions could be constructed by gluing the paths of X and $-X$ each time when the process returns in 0. Thirdly, if $\rho \in (0, 1/2)$ and $\beta \in (-\infty, 0]$, then by comparison with a Bessel process of dimension $\delta \in (1, 2)$, the reaching time of 0 belongs to $[1, (2\rho)^{1/\beta})$, with a positive probability. Again, as in the preceding case, different solutions can be constructed. Finally, if $\rho \in [1/2, \infty)$, $\beta \in (-\infty, 0]$ and $x_0 = 0$, then by comparison with a Bessel process of dimension 2 every solution \tilde{X} of (2.1) satisfies $\tilde{X}_t^2 \neq 0$, for all $t > 1$ a.s. Let us introduce

$$\Omega^\pm := \{\omega \in \Omega : \forall t > 1, \pm \tilde{X}_t > 0\} \quad \text{and} \quad \mathbb{P}^\pm := \mathbb{P}(\cdot | \Omega^\pm).$$

Note that for all $\varepsilon > 0$, $\Omega^\pm = \{\omega \in \Omega : \forall 1 < t < 1 + \varepsilon, \pm \tilde{X}_t > 0\} \in \mathcal{F}_{t+\varepsilon}$ and then $\Omega^\pm \in \mathcal{F}_{1+}$. Therefore the process $\{B_t - B_1\}_{t \geq 1}$ is again a standard Brownian motion under probabilities \mathbb{P}^\pm . By uniqueness of the nonnegative weak solution and also, by symmetry, of the nonpositive solution of (2.1), the distribution of \tilde{X} under \mathbb{P}^\pm equals to the distribution of $\pm X$, where X is the pathwise unique nonnegative strong solution. We deduce that

$$\mathcal{L}(\tilde{X}) = \lambda \mathcal{L}(X) + (1 - \lambda) \mathcal{L}(-X), \quad \text{with} \quad \lambda := \mathbb{P}(\Omega^+).$$

Then it is clear that the set of all weak solutions is exactly the set of all distributions which are mixture of distributions of X and $-X$. This completes the proof of the proposition. \square

3.2 Explosion of solutions

Proposition 3.8. *The explosion time τ_e of any solution X of (2.1) is infinite a.s. when $\rho \in (-\infty, 0)$ (attractive case) or $\alpha \in (-\infty, 1]$ (linear growth case). It is finite a.s. when $\rho \in (0, \infty)$, $\alpha \in (1, \infty)$ and $2\beta \in (-\infty, \alpha + 1]$.*

Proof. Assume first that $\rho \in (-\infty, 0)$ or $\alpha \in (-\infty, 1]$. Let V be a twice continuous differentiable nonnegative function such that $V(x) := 1 + x^2$ for all $|x| \geq 1$, $V(x) = 0$ for all $x \in [1/2, 1/2]$. Let \mathcal{L} be the infinitesimal generator of X given by

$$\mathcal{L} := \partial_t + \rho \operatorname{sgn}(x) \frac{|x|^\alpha}{t^\beta} \partial_x + \frac{1}{2} \partial_{xx}. \quad (3.3)$$

For all $T \geq 1$, we denote c_T the supremum of $\mathcal{L}V$ on $[1, T] \times [-1, 1]$. Then it is a simple calculation to see that for all $t \in [1, T]$ and $x \in \mathbb{R}$,

$$\mathcal{L}V(t, x) \leq c_T + \lambda_T V(t, x), \quad \text{with} \quad \lambda_T := \begin{cases} 1, & \text{when } \rho \in (-\infty, 0), \\ 1 + \sup_{1 \leq t \leq T} \rho t^{-\beta}, & \text{when } \rho \in (0, \infty). \end{cases}$$

By using Theorem 10.2.1 in [16], p. 254, we deduce that τ_e is a.s. finite. Assume that $\rho \in (0, \infty)$, $\alpha \in (1, \infty)$ and $2\beta \in (-\infty, \alpha + 1]$. By using Proposition 2.1 it suffices to show

that the solution $X^{(\gamma)}$ of equation (2.12) explodes in finite time a.s. Let us introduce Q_s and C_s , the pathwise unique strong solutions of

$$dQ_s = 2\sqrt{Q_s} dW_s + \left(2\rho Q_s^{\frac{\alpha+1}{2}} - |\gamma|Q_s + 1 \right) ds, \quad Q_0 = x_0^2$$

and, respectively,

$$dC_s = 2\sqrt{C_s} dW_s + \left(2\rho C_s^{\frac{\alpha+1}{2}} - \gamma\varphi_\gamma^{\gamma-1}(s)C_s + 1 \right) ds, \quad C_0 = x_0^2.$$

By using Ito's formula, we can see that the square of $X^{(\gamma)}$ satisfies the latter equation and then $C \stackrel{\mathcal{L}}{=} (X^{(\gamma)})^2$. Moreover, since $0 \leq \varphi_\gamma^{\gamma-1} \leq 1$ (remark that $\gamma \leq 1$) we can show, by using comparison theorem (Theorem 1.1, Chap. VI in [9], p. 437), that $0 \leq Q_s \leq C_s$ a.s. Besides, by using Theorem 5.7 in [4], p. 97, the explosion time of the time-homogeneous diffusion Q_s is finite a.s. This implies that the explosion time of $X^{(\gamma)}$ is finite a.s. and ends the proof of the proposition. \square

Proposition 3.9. *If $\rho \in (0, \infty)$, $\alpha \in (1, \infty)$ and $2\beta \in (\alpha + 1, \infty)$, then,*

$$\mathbb{P}(\tau_e = \infty) = \mathbb{E} \left[\exp \left(\int_0^{t_1} \rho \operatorname{sgn}(b_u) |b_u|^\alpha dW_u - \int_0^{t_1} \frac{\rho^2 |b_u|^{2\alpha}}{2} du \right) \right] \in (0, 1). \quad (3.4)$$

Here b denotes the solution of equation (2.18) and τ_e the explosion time of X .

Proof. Let $X^{(\gamma)}$ be the pathwise unique strong solution of equation (2.14) and b be the pathwise unique strong solution of equation (2.18). Denote by η_e the explosion time of $X^{(\gamma)}$ and note that a.s. $\eta_e \in [0, t_1] \cup \{\infty\}$ and $\{\eta_e \geq t_1\} = \{\tau_e = \infty\}$. We need to show that $\mathbb{P}(\eta_e \geq t_1)$ is equal to the right hand side of (3.4) and belongs to $(0, 1)$. First of all, b is a continuous process on $[0, t_1]$, with $b_{t_1} = 0$ a.s., and it is the so-called δ -Brownian bridge (see Definition 1 in [11], p. 1022). By using the Girsanov transformation between b and $X^{(\gamma)}$, we can write for every integer $n \geq 1$, $s \in [0, t_1]$ and \mathcal{F}_s -measurable bounded function F_s ,

$$\mathbb{E} \left[F_s(X_{\bullet \wedge \eta_n}^{(\gamma)}) \mathbf{1}_{\{\eta_n > s\}} \right] = \mathbb{E} \left[F_s(b_{\bullet \wedge \sigma_n}) \mathcal{E}(s \wedge \sigma_n) \mathbf{1}_{\{\sigma_n > s\}} \right],$$

where

$$\eta_n := \inf\{s \in [0, t_1] : |X_s^{(\gamma)}| \geq n\}, \quad \sigma_n := \inf\{s \in [0, t_1] : |b_s| \geq n\},$$

and

$$\mathcal{E}(s) := \exp \left(\int_0^s \rho \operatorname{sgn}(b_u) |b_u|^\alpha dW_u - \int_0^s \frac{\rho^2 |b_u|^{2\alpha}}{2} du \right).$$

Letting $n \rightarrow \infty$, we obtain

$$\mathbb{E} \left[F_s(X^{(\gamma)}) \mathbf{1}_{\{\eta_e > s\}} \right] = \mathbb{E} \left[F_s(b) \mathcal{E}(s) \right]. \quad (3.5)$$

In particular, we have proved that for all $s \in [0, t_1]$, $\mathbb{P}(\eta_e > s) = \mathbb{E}[\mathcal{E}(s)]$. Furthermore it is clear that $\mathbb{P}(\tau_e = \infty) = \mathbb{P}(\eta_e \geq t_1) \geq \mathbb{E}[\mathcal{E}(t_1)] > 0$. At this level we state a technical result which proof is postponed to the Appendix.

Lemma 3.10. *Assume that $\rho \in (0, \infty)$, $\alpha \in (1, \infty)$ and $2\beta \in (\alpha + 1, \infty)$, and denote by $\eta_e \in [0, t_1] \cup \{\infty\}$ the explosion time of the pathwise unique strong solution $X^{(\gamma)}$ of equation (2.14). Then $\mathbb{P}(\eta_e = t_1) = 0$.*

We deduce from this lemma that $\mathbb{P}(\tau_e = \infty) = \mathbb{P}(\eta_e \geq t_1) = \mathbb{P}(\eta_e > t_1) = \mathbb{E}(\mathcal{E}(t_1))$ and the equality in (3.4) is proved. It remains to show that $\mathbb{P}(\tau_e = \infty) < 1$. Recall that $\alpha \in (1, \infty)$ and let $a \in (1, \alpha)$. Set $\psi(x) := 1 \wedge |x|^{-a}$ and note that, for any $T > 1$, there exists $k \geq 1$, such that $\int_0^\infty \psi(y)dy < k(T-1)$. Moreover, we can see that there exists a continuous differentiable odd function ϕ , defined on \mathbb{R} , vanishing only at $x = 0$, such that $|\phi| \leq \psi$, and

$$\phi(x) := kx, \quad x \in [-1/2k, 1/2k], \quad \lim_{|x| \rightarrow \infty} |x|^\alpha |\phi(x)| = \infty \quad \text{and} \quad \lim_{|x| \rightarrow \infty} \phi'(x) = 0.$$

For $\mu > 0$ we introduce the bounded twice continuous differentiable function

$$V_\mu(x) := \exp\left(\mu \int_0^x \phi(y)dy\right), \quad x \in \mathbb{R}.$$

We shall apply Theorem 10.2.1 in [16], p. 254, to the diffusion X , solution of (2.1), with the function V_μ for some $\mu > 0$: it will implies that $\mathbb{P}(\tau_e \leq T) > 0$ for any $T > 1$. We need to verify that there exists $\lambda > 0$ and $\mu > 0$ such that for all $t \in [1, T]$ and $x \in \mathbb{R}$,

$$\mathcal{L}V_\mu(t, x) \geq \lambda V_\mu(x) \quad \text{and} \quad \ln\left(\frac{\sup_{x \in \mathbb{R}} V_\mu(x)}{V_\mu(x_0)}\right) < \lambda(T-1). \quad (3.6)$$

Here \mathcal{L} is given in (3.3). In order, to prove (3.6), note that for all $t \in [1, T]$ and $x \in \mathbb{R}$,

$$\mathcal{L}V_\mu(t, x) = \mu V_\mu(x)(\rho t^{-\beta} |x|^\alpha |\phi(x)| + \frac{\mu}{2} \phi^2(x) + \frac{1}{2} \phi'(x)).$$

The assumptions on ϕ imply that there exists $r \geq 1$ such that, for all $\mu > 0$,

$$\mathcal{L}V_\mu \geq k\mu V_\mu \quad \text{on} \quad [1, T] \times ([-1/2k, 1/2k] \cup [-r, r]^c).$$

Besides, since ϕ^2 is bounded away from zero, while $|\phi'|$ is bounded on $[-1/2k, -r] \cup [1/2k, r]$, we deduce that there exists $\mu_0 > 0$ such that

$$\mathcal{L}V_{\mu_0} \geq k\mu_0 V_{\mu_0} \quad \text{on} \quad [1, T] \times ([-1/2k, -r] \cup [1/2k, r]).$$

Hence, for all $t \in [1, T]$ and $x \in \mathbb{R}$, $\mathcal{L}V_{\mu_0}(t, x) \geq k\mu_0 V_{\mu_0}(x)$ and we can see that

$$\ln\left(\frac{\sup_{x \in \mathbb{R}} V_{\mu_0}(x)}{V_{\mu_0}(x_0)}\right) = \mu_0 \int_{|x_0|}^\infty \phi(y)dy \leq \mu_0 \int_0^\infty \psi(y)dy < k\mu_0(T-1).$$

Therefore Theorem 10.2.1 in [16], p. 254, applies with $\lambda := \mu_0 k$ and V_{μ_0} . We obtain that X explodes in finite time, with positive probability. We have proved that $\mathbb{P}(\tau_e = \infty) \in (0, 1)$. This ends the proof of proposition, excepted for Lemma 3.10. \square

Remark 3.11. *We study equation (2.1), only for parameters $(\rho, \alpha, \beta) \in \mathcal{P} := \mathcal{P}_- \cup \mathcal{P}_+$, where*

$$\mathcal{P}_- := (-\infty, 0] \times (-1, \infty) \times \mathbb{R} \quad (\text{attractive case}), \quad \mathcal{P}_+ := [0, \infty) \times \mathbb{R} \times \mathbb{R} \quad (\text{repulsive case}).$$

By using similar methods as in Propositions 3.3, 3.4 and 3.5, when $(\rho, \alpha, \beta) \in \mathbb{R}^3 \setminus \mathcal{P}$, we can prove that the solutions of equation (2.1) are only defined up to the reaching time of 0, which is finite a.s. and cannot be continued after this time. Therefore, this case is out of range for the study of the asymptotic behaviour.

Remark 3.12. For parameters $(\rho, \alpha, \beta) \in \mathcal{P}$ and $x_0 \in [0, \infty)$, the solution X of equation (2.1) is a time-inhomogeneous diffusion, in the sense that, it is a regular strong Markov process (see, for instance, Chap. 23 in [10], p. 455) in \mathbb{R} , when $\alpha \in (-1, \infty)$, in $(0, \infty)$, when $\alpha \in (-\infty, -1)$ ($x_0 = 0$ being an entrance boundary) and in $[0, \infty)$, when $\alpha = -1$.

4 Asymptotic behaviour of solutions

We present here the systematic study of the recurrence, transience or convergence of the solution X of (2.1) for parameters $(\rho, \alpha, \beta) \in \mathcal{P}$ (see Remark 3.11 above).

Definition 4.1. We shall say that the process X is recurrent in $E \subset \mathbb{R}$ (under a given probability \mathbb{P}), when for all $x \in E$, the set $\{t \geq 1 : X_t = x\}$ is \mathbb{P} -a.s. unbounded, and we shall say that the process X is transient (under a given probability \mathbb{P}), when $\lim_{t \rightarrow \tau_e} |X_t| = \infty$ \mathbb{P} -a.s., where τ_e denotes the explosion time of X .

For later reference, we introduce the following three functions:

$$L(t) := (2t \ln \ln t)^{\frac{1}{2}}, \quad L_{\rho, \alpha}(t) := t^{\frac{1}{2}} (c_{\rho, \alpha} \ln \ln t)^{\frac{1}{\alpha+1}} \quad \text{and} \quad L_{\rho, \alpha, \beta}(t) := (c_{\rho, \alpha, \beta} t^\beta \ln t)^{\frac{1}{\alpha+1}}, \quad (4.1)$$

where

$$c_{\rho, \alpha} := \frac{|\alpha + 1|}{2|\rho|} \quad \text{and} \quad c_{\rho, \alpha, \beta} := \frac{|\alpha + 1 - 2\beta|}{2|\rho|}. \quad (4.2)$$

Set $E_\alpha := \mathbb{R}$, when $\alpha \in (-1, \infty)$, and $E_\alpha := (0, \infty)$, when $\alpha \in (-\infty, -1]$, and introduce, when they are well defined, the distributions $\Lambda_{\rho, \alpha}$ and $\Pi_{\rho, \alpha}$ on E_α , given by

$$\Lambda_{\rho, \alpha}(dx) := (Z_{\rho, \alpha}^\Lambda)^{-1} e^{\frac{2\rho|x|^{\alpha+1}}{\alpha+1}} e^{-\frac{x^2}{2}} \mathbf{1}_{E_\alpha}(x) dx, \quad \text{with} \quad Z_{\rho, \alpha}^\Lambda := \int_{E_\alpha} e^{\frac{2\rho|x|^{\alpha+1}}{\alpha+1}} e^{-\frac{x^2}{2}} dx, \quad (4.3)$$

and

$$\Pi_{\rho, \alpha}(dx) := (Z_{\rho, \alpha}^\Pi)^{-1} e^{\frac{2\rho|x|^{\alpha+1}}{\alpha+1}} \mathbf{1}_{E_\alpha}(x) dx, \quad \text{with} \quad Z_{\rho, \alpha}^\Pi := \int_{E_\alpha} e^{\frac{2\rho|x|^{\alpha+1}}{\alpha+1}} dx. \quad (4.4)$$

4.1 Behaviour on the critical line: $2\beta = \alpha + 1$

The scaling transformation (2.2) associated with the exponential change of time provides a time-homogeneous equation (2.16). With the help of Motoo's theorem (see Theorem 4.3 below) and of the ergodic theorem we obtain the asymptotic behaviour of solutions to (2.1).

Theorem 4.2 (Attractive case). *If $(\rho, \alpha, \beta) \in \mathcal{P}_-$ and $2\beta = \alpha + 1$, then X is recurrent in \mathbb{R} and it satisfies*

$$\lim_{t \rightarrow \infty} \frac{X_t}{\sqrt{t}} \stackrel{\mathcal{L}}{=} \Lambda_{\rho, \alpha}. \quad (4.5)$$

Note that $\Lambda_{\rho, \alpha} \sim \mathcal{N}(0, (1 - 2\rho)^{-1})$ when $\alpha = 1$. Moreover,

$$\limsup_{t \rightarrow \infty} \frac{X_t}{L(t)} = 1 \text{ a.s., when } \alpha \in (-1, 1), \quad \limsup_{t \rightarrow \infty} \frac{X_t}{L_{\rho, \alpha}(t)} = 1 \text{ a.s., when } \alpha \in (1, \infty), \quad (4.6)$$

and

$$\limsup_{t \rightarrow \infty} \frac{X_t}{(1 - 2\rho)^{-1/2} L(t)} = 1 \text{ a.s., when } \alpha = 1. \quad (4.7)$$

Proof. Let $Z = \Phi_e(X) \equiv \Phi_\gamma(X)$ be the weak solution of equation (2.16). The scale function and the speed measure of Z (see Chap. VI in [9], pp. 446,449) are respectively given by

$$s(x) := \int_0^x e^{-\frac{2\rho|y|^{\alpha+1}}{\alpha+1}} e^{\frac{y^2}{2}} dy \quad \text{and} \quad m(dx) := e^{\frac{2\rho|x|^{\alpha+1}}{\alpha+1}} e^{-\frac{x^2}{2}} dx.$$

We remark that m is a finite measure on \mathbb{R} and $m(dx)/m(\mathbb{R}) = \Lambda_{\rho,\alpha}(dx)$. By using the ergodic theorem (see, for instance, Theorem 23.15 in [10], p. 465), we obtain

$$\lim_{t \rightarrow \infty} \frac{X_t}{\sqrt{t}} = \lim_{t \rightarrow \infty} Z_{\ln t} \stackrel{\mathcal{L}}{=} \Lambda_{\rho,\alpha}.$$

It remains to prove equalities (4.6) and (4.7) and the recurrent feature in \mathbb{R} . We shall apply Motoo's theorem (see [13]). We recall this result since it will be used several times:

Theorem 4.3 (Motoo). *Let X be a regular continuous strong Markov process in (a, ∞) , $a \in [-\infty, \infty)$, which is homogeneous in time, with scale function s and finite speed measure m . For every real positive increasing function h ,*

$$\mathbb{P} \left(\limsup_{t \rightarrow \infty} \frac{X_t}{h(t)} \geq 1 \right) = 0 \quad \text{or} \quad 1 \quad \text{according to whether} \quad \int^\infty \frac{dt}{s(h(t))} < \infty \quad \text{or} \quad = \infty.$$

Firstly, assume that $\alpha \in (-1, 1)$. Note that, if $d(x)$ is the drift of Z , then $d(x) \sim_{x \rightarrow \infty} -\frac{x}{2}$. Roughly speaking, the pathwise largest deviations of Z and of the Ornstein-Uhlenbeck process will be the same. Indeed, by using L'Hôpital's rule, $s(x) \sim_{x \rightarrow \infty} x^{-1} \exp(-\frac{2\rho|x|^{\alpha+1}}{\alpha+1} + \frac{x^2}{2})$, and a simple computation gives

$$\int^\infty \frac{dt}{s(\sqrt{c \ln t})} < \infty \quad \text{or} \quad = \infty \quad \text{according to whether} \quad c > 2 \quad \text{or} \quad c < 2.$$

Motoo's theorem applies and we deduce that for all $\varepsilon > 0$,

$$\mathbb{P} \left(\limsup_{t \rightarrow \infty} \frac{Z_t}{\sqrt{2 \ln t}} \geq 1 + \varepsilon \right) = 0, \quad \mathbb{P} \left(\limsup_{t \rightarrow \infty} \frac{Z_t}{\sqrt{2 \ln t}} \geq 1 - \varepsilon \right) = 1.$$

Therefore,

$$\limsup_{t \rightarrow \infty} \frac{X_t}{L(t)} = \limsup_{t \rightarrow \infty} \frac{Z_{\ln t}}{\sqrt{2 \ln t}} = 1 \quad \text{a.s.}$$

Secondly, assume that $\alpha \in (1, \infty)$. The drift of Z satisfies $d(x) \underset{x \rightarrow \infty}{\sim} \rho \operatorname{sgn}(x)|x|^\alpha$, hence, roughly speaking, the pathwise largest deviation of Z and of H given by (2.17) will be the same. By using again L'Hôpital's rule, $s(x) \sim_{x \rightarrow \infty} (2|\rho|)^{-1} x^{-\alpha} \exp(-\frac{2\rho x^{\alpha+1}}{\alpha+1} + \frac{x^2}{2})$. Let us set $h_c(t) := (c \ln t)^{\frac{1}{\alpha+1}}$. It is not difficult to see that

$$\int^\infty \frac{dt}{s(h_c(t))} < \infty \quad \text{or} \quad = \infty \quad \text{according to whether} \quad c > c_{\rho,\alpha} \quad \text{or} \quad c < c_{\rho,\alpha}.$$

Here $c_{\rho,\alpha}$ is given in (4.2). Then Motoo's theorem implies

$$\limsup_{t \rightarrow \infty} \frac{X_t}{L_{\rho,\alpha}(t)} = \limsup_{t \rightarrow \infty} \frac{Z_{\ln t}}{(c_{\rho,\alpha} \ln \ln t)^{\frac{1}{\alpha+1}}} = 1 \quad \text{a.s.}$$

Finally, when $\alpha = 1$, we use Motoo's theorem as previously and we obtain (4.7). Furthermore, by symmetry of equation (2.1), we can replace X by $-X$ in equations (4.6) and (4.7) and we deduce that $\limsup_{t \rightarrow \infty} X_t = \infty$ and $\liminf_{t \rightarrow \infty} X_t = -\infty$ a.s., and we can conclude that X is recurrent in \mathbb{R} . \square

In the repulsive case, the same ideas as for the proof of Theorem 4.2 apply. However, we need to distinguish two particular cases when $\alpha = -1$ (Bessel case) or $\alpha = 1$ (the continuous time analogue of the Friedman's urn model in [6] and [12]). We note that in these cases the recurrent or transient features depend on the position of ρ with respect to $1/2$.

Theorem 4.4 (Repulsive case). *Assume that $(\rho, \alpha, \beta) \in \mathcal{P}_+$ and $2\beta = \alpha + 1$.*

i) If $\alpha \in (-1, 1)$, then X is recurrent in \mathbb{R} and it satisfies

$$\lim_{t \rightarrow \infty} \frac{X_t}{\sqrt{t}} \stackrel{\mathcal{L}}{=} \Lambda_{\rho, \alpha} \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{X_t}{L(t)} = 1 \quad \text{a.s.} \quad (4.8)$$

ii) If $\alpha \in (-\infty, -1)$, then X is transient and it satisfies

$$\lim_{t \rightarrow \infty} \frac{X_t}{\sqrt{t}} \stackrel{\mathcal{L}}{=} \Lambda_{\rho, \alpha}, \quad \limsup_{t \rightarrow \infty} \frac{X_t}{L(t)} = 1 \quad \text{a.s.} \quad \text{and} \quad \liminf_{t \rightarrow \infty} \frac{X_t}{L_{\rho, \alpha}(t)} = 1 \quad \text{a.s.} \quad (4.9)$$

iii) If $\alpha \in (1, \infty)$, then the explosion time τ_e is finite a.s. and X satisfies

$$|X_t| \underset{t \rightarrow \tau_e}{\sim} \frac{\tau_e^{\frac{\alpha+1}{2(\alpha-1)}}}{(\rho(\alpha-1)(\tau_e - t))^{\frac{1}{\alpha-1}}} \quad \text{a.s.} \quad (4.10)$$

iv) If $\alpha = -1$, then X is the classical Bessel process of dimension $2\rho + 1$. It is recurrent in $[0, \infty)$, when $\rho \in (0, 1/2)$, recurrent in $(0, \infty)$, when $\rho = 1/2$ and transient, when $\rho \in (1/2, \infty)$. In these three cases,

$$\lim_{t \rightarrow \infty} \frac{X_t}{\sqrt{t}} \stackrel{\mathcal{L}}{=} \sqrt{\Gamma_\rho} \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{X_t}{L(t)} = 1 \quad \text{a.s.}, \quad \text{with } \Gamma_\rho \sim \text{Gamma}\left(\rho + \frac{1}{2}, 2\right). \quad (4.11)$$

Moreover,

$$\liminf_{t \rightarrow \infty} \frac{\ln\left(\frac{X_t}{\sqrt{t}}\right)}{\ln \ln t} = -\frac{1}{2\rho - 1} \quad \text{a.s.}, \quad \text{when } \rho \in \left(\frac{1}{2}, \infty\right). \quad (4.12)$$

v) If $\alpha = 1$, then X is a Gaussian process, recurrent in \mathbb{R} , when $\rho \in (0, 1/2]$, and transient, when $\rho \in (1/2, \infty)$. The process X satisfies

$$\lim_{t \rightarrow \infty} \frac{X_t}{\sqrt{t}} \stackrel{\mathcal{L}}{=} \mathcal{N}\left(0, \frac{1}{1-2\rho}\right) \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{X_t}{\sqrt{\frac{2}{1-2\rho}L(t)}} = 1 \quad \text{a.s.}, \quad \text{when } \rho \in \left(0, \frac{1}{2}\right), \quad (4.13)$$

$$\lim_{t \rightarrow \infty} \frac{X_t}{\sqrt{t \ln t}} \stackrel{\mathcal{L}}{=} \mathcal{N}(0, 1) \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{X_t}{\sqrt{2t \ln t \ln \ln t}} = 1 \quad \text{a.s.}, \quad \text{when } \rho = \frac{1}{2}, \quad (4.14)$$

and

$$\lim_{t \rightarrow \infty} \frac{X_t}{t^\rho} = G_{\rho, x_0} \quad \text{a.s.}, \quad \text{with } G_{\rho, x_0} \sim \mathcal{N}\left(x_0, \frac{1}{2\rho - 1}\right), \quad \text{when } \rho \in \left(\frac{1}{2}, \infty\right). \quad (4.15)$$

Proof. To begin with, let us point out that the proof of (4.8), when $\alpha \in (-\infty, 1) \setminus \{-1\}$, and the recurrent feature in \mathbb{R} , when $\alpha \in (-1, 1)$, is the same as the proof of Theorem 4.2. If $\alpha \in (-\infty, -1)$ we need to prove the equality in (4.9). To this end, consider $Z = \Phi_e(X) \equiv \Phi_\gamma(X)$, the pathwise unique strong solution of (2.16). By Ito's formula,

$$d\tilde{Z}_t = \tilde{Z}_t^2 dW_t + (\tilde{Z}_t^3 - \rho\tilde{Z}_t^{2-\alpha} + \frac{\tilde{Z}_t}{2}) dt, \quad \tilde{Z}_0 = \frac{1}{x_0}, \quad \text{with } \tilde{Z} := \frac{1}{Z}. \quad (4.16)$$

By performing a change of time and by using Motoo's theorem, we deduce that

$$\liminf_{t \rightarrow \infty} \frac{X_t}{L_{\rho, \alpha}(t)} = \left(\limsup_{t \rightarrow \infty} \frac{\tilde{Z}_t}{(c_{\rho, \alpha} \ln t)^{\frac{1}{|\alpha+1|}}} \right)^{-1} = 1 \quad \text{a.s.}$$

Note that this relation insure the transient feature, since $\lim_{t \rightarrow \infty} L_{\rho, \alpha}(t) = \infty$, when $\alpha < -1$. Assume that $\alpha \in (1, \infty)$: by Proposition 3.8 the explosion time η_e of Z is finite a.s. Moreover, we can see that the process $z_t := Z_t - W_t$ satisfies the random ordinary differential equation

$$z'_t = \rho \operatorname{sgn}(z_t + W_t) |z_t + W_t|^\alpha - \frac{z_t + W_t}{2}.$$

We deduce that

$$\frac{|z_t|^{1-\alpha}}{\alpha-1} = \int_t^{\eta_e} \frac{z'_s ds}{\operatorname{sgn}(z_s) |z_s|^\alpha} \underset{t \rightarrow \eta_e}{\sim} \rho(\eta_e - t) \quad \text{and} \quad |Z_t| \underset{t \rightarrow \eta_e}{\sim} \frac{1}{(\rho(\alpha-1)(\eta_e - t))^{\frac{1}{\alpha-1}}} \quad \text{a.s.}$$

Remark also that the explosion time τ_e of X satisfies $\tau_e = e^{\eta_e}$ a.s. Therefore

$$|X_t| = \sqrt{t} |Z_{\ln t}| \underset{t \rightarrow \tau_e}{\sim} \frac{\sqrt{\tau_e}}{(\rho(\alpha-1)(\ln \tau_e - \ln t))^{\frac{1}{\alpha-1}}} \underset{t \rightarrow \tau_e}{\sim} \frac{\tau_e^{\frac{\alpha+1}{2(\alpha-1)}}}{(\rho(\alpha-1)(\tau_e - t))^{\frac{1}{\alpha-1}}} \quad \text{a.s.}$$

Assume that $\alpha = -1$ and let $R^{(e)}$ be the pathwise unique strong solution of equation (2.10). By applying Lemma 2.2 in [2], p. 916 and the ergodic theorem to $R^{(e)}$ (see, for instance, Theorem 23.15 in [10], p. 465), we obtain, by change of time, (4.11). Equality (4.12) is a consequence of Lemma 4.1 in [2], p. 926. The recurrent or the transient features are proved in Chap. IX in [15]. Finally, assume that $\alpha = 1$. Let us note that $(t, x) \mapsto x/t^\rho$ is a harmonic function for the space-time diffusion $t \mapsto (t, X_t)$. By Ito's formula we get

$$\frac{X_t}{t^\rho} = x_0 + M_t, \quad \text{with} \quad M_t := \int_1^t \frac{dB_s}{s^\rho}.$$

Moreover, by using the Dambis-Dubins-Schwartz theorem (see Theorem 1.7, Chap. V in [15], p. 182), there exists a standard Brownian motion W such that

$$M_t = W_{\langle M \rangle_t}, \quad \text{with} \quad \langle M \rangle_t = \int_1^t \frac{ds}{s^{2\rho}} = \begin{cases} \frac{t^{1-2\rho}-1}{1-2\rho} & \text{if } \rho \neq 1/2 \\ \ln t & \text{if } \rho = 1/2. \end{cases}$$

By using the well known properties of the Brownian motion, we deduce the convergence in distribution and the pathwise largest deviations of X when $\alpha = 1$. Furthermore, the recurrent or transient features are simple consequences and the theorem is proved. \square

4.2 Behaviour above the critical line: $2\beta > \alpha + 1$

The scaling transformation (2.2) associated with the exponential change of time does not provides a time-homogeneous equation. However, we shall prove that the asymptotic behaviour of equation (2.6) is related to the asymptotic behaviour of the Ornstein-Uhlenbeck process (2.7) with the help of the following result, which proof is postponed to the Appendix.

Lemma 4.5. *Let Z and H be regular strong Markov processes which are, respectively, weak solutions of the stochastic differential equations with continuous coefficients:*

$$dZ_s = \sigma(s, Z_s) dB_t + d(s, Z_s) ds \quad \text{and} \quad dH_s = \sigma_\infty(H_s) dB_s + d_\infty(H_s) ds.$$

Assume (Z, H) is asymptotically time-homogeneous and Π -ergodic, in the sense that

$$\lim_{s \rightarrow \infty} \sigma(s, z) = a_\infty(z) \quad \text{and} \quad \lim_{s \rightarrow \infty} d(s, z) = d_\infty(z), \quad \text{uniformly on compact subsets of } \mathbb{R}$$

and

$$\lim_{s \rightarrow \infty} H_s \stackrel{\mathcal{L}}{=} \Pi.$$

If Z is bounded in probability, then it converges in distribution to Π , that is

$$(\forall \varepsilon > 0, \exists r > 0, \sup_{s \geq 0} \mathbb{P}(|Z_s| \geq r) < \varepsilon) \implies \lim_{s \rightarrow \infty} Z_s \stackrel{\mathcal{L}}{=} \Pi.$$

Theorem 4.6 (Attractive case). *If $(\rho, \alpha, \beta) \in \mathcal{P}_-$ and $2\beta \in (\alpha + 1, \infty)$, then X is recurrent in \mathbb{R} and it satisfies*

$$\lim_{t \rightarrow \infty} \frac{X_t}{\sqrt{t}} \stackrel{\mathcal{L}}{=} \mathcal{N}(0, 1) \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{X_t}{L(t)} = 1 \quad \text{a.s.} \quad (4.17)$$

Proof. Let us consider $X^{(e)} = \Phi_e(X)$ the pathwise unique strong solution of equation (2.6). Note that the equalities in (4.17) are equivalent to

$$\lim_{t \rightarrow \infty} \left(X_t^{(e)} \right)^3 \stackrel{\mathcal{L}}{=} G^3 \quad \text{and} \quad \limsup_{t \rightarrow \infty} \left(\frac{X_t^{(e)}}{\sqrt{2 \ln t}} \right)^3 = 1 \quad \text{a.s.,} \quad \text{with } G \sim \mathcal{N}(0, 1). \quad (4.18)$$

We can see that these convergences are satisfied by the process $Q_t = U_t^3$, where U is the Ornstein-Uhlenbeck process solution of equation (2.7). This process is the pathwise unique strong solution of the time-homogeneous equation

$$dQ_t = 3Q_t^{\frac{2}{3}} dW_t + 3\left(Q_t^{\frac{1}{3}} - \frac{Q_t}{2}\right) dt, \quad Q_0 = x_0^3.$$

Let us consider the pathwise unique strong solution of equation, with continuous coefficients,

$$dC_t = 3C_t^{\frac{2}{3}} dW_t + 3\left(C_t^{\frac{1}{3}} - \frac{C_t}{2} + \rho e^{(\frac{\alpha+1}{2}-\beta)t} \operatorname{sgn}(C_t) |C_t|^{\frac{\alpha+2}{3}}\right) dt, \quad C_0 = x_0^3. \quad (4.19)$$

By Ito's formula, we can see that $C \stackrel{\mathcal{L}}{=} (X^{(e)})^3$ and it is a simple calculation to see that (C_t, Q_t) is asymptotically homogeneous and $\mathcal{L}(G^3)$ -ergodic (see Lemma 4.5 for a definition). In order to apply Lemma 4.5 and to prove the first equality in (4.18) it suffices to show that

C_t is bounded in probability. To this end, denote by C_t^\pm the pathwise unique strong solutions of the time-homogeneous equations, with continuous coefficients,

$$dC_t^\pm = 3 (C_t^\pm)^{\frac{2}{3}} dW_t + 3 \left((C_t^\pm)^{\frac{1}{3}} - \frac{C_t^\pm}{2} + \rho \operatorname{sgn}(C_t^\pm) |C_t^\pm|^{\frac{\alpha+2}{3}} \mathbf{1}_{\{\mp C_t^\pm \geq 0\}} \right) dt, \quad C_0^\pm = x_0^3.$$

By using comparison theorem (see Theorem 1.1, Chap. VI in [9], p. 437) we get, for all $t \geq 0$, $C_t^- \leq C_t \leq C_t^+$, a.s. Moreover, by computation of the speed measure as in the proof of Theorem 4.2, we can see that C_t^\pm are ergodic diffusions and therefore there are bounded in probability. By comparison, it is the same for C_t , and this fact implies the first equality in (4.18), hence the first equality in (4.17). Besides, by applying Motoo's theorem to C_t^+ we get

$$\limsup_{t \rightarrow \infty} \frac{C_t}{(2 \ln t)^{\frac{3}{2}}} \leq \limsup_{t \rightarrow \infty} \frac{C_t^+}{(2 \ln t)^{\frac{3}{2}}} = 1 \quad \text{a.s.} \quad (4.20)$$

To deduce the second equality in (4.18), we need to prove the opposite inequality in (4.20). We can see that equality (4.20) holds for $-C_t$ by symmetry of (4.19) and then it implies that

$$\lim_{t \rightarrow \infty} \rho e^{(\frac{\alpha+1}{2}-\beta)t} \operatorname{sgn}(C_t) |C_t|^{\frac{\alpha+2}{3}} = 0 \quad \text{a.s.} \quad (4.21)$$

Let $u \geq 0$ be and let us introduce the pathwise unique strong solution of equation

$$dC_t(u) = 3 C_t(u)^{\frac{2}{3}} dW_t + 3(C_t(u)^{\frac{1}{3}} - \frac{C_t(u)}{2} - 1) dt, \quad C_u(u) = C_u.$$

We shall prove that for all $t \geq u$,

$$C_t(u) \leq C_t \quad \text{a.s. on} \quad \Omega_u := \left\{ \sup_{t \geq u} \rho e^{(\frac{\alpha+1}{2}-\beta)t} |C_t|^{\frac{\alpha+2}{3}} \leq 1 \right\}. \quad (4.22)$$

Indeed, we introduce the stopping time τ_u defined by

$$\tau_u := \inf \left\{ t \geq u : \rho e^{(\frac{\alpha+1}{2}-\beta)t} |C_t|^{\frac{\alpha+2}{3}} > 1 \right\}.$$

Then by comparison theorem (see Theorem 1.1, Chap. VI in [9], p. 437) and a classical argument of localisation, we obtain $C_{\bullet \wedge \tau_u}(u) \leq C_{\bullet \wedge \tau_u}$ a.s. and since $\{\tau_u = \infty\} = \Omega_u$ we deduce comparison (4.22). By applying Motoo's theorem to $C_t(u)$, as in the proof of Theorem 4.2, and by using (4.21), we deduce

$$1 = \limsup_{t \rightarrow \infty} \frac{C_t(u)}{(2 \ln t)^{\frac{3}{2}}} \leq \limsup_{t \rightarrow \infty} \frac{C_t}{(2 \ln t)^{\frac{3}{2}}} \quad \text{a.s. on} \quad \Omega_u, \quad \text{and} \quad \mathbb{P}(\cup_{u \geq 0} \Omega_u) = 1.$$

We obtain the opposite inequality in (4.20) and finish the proof of (4.17). By symmetry of equation (2.1), we can replace X by $-X$ in the second equality in (4.17), and we conclude that X is recurrent in \mathbb{R} . \square

In the repulsive case the proof follows the same lines as for the proof of Theorem 4.6. Nevertheless, a distinctive feature appears when $\alpha \in (-\infty, -1]$. We need to modify the approach of the proof of Theorem 4.6, since equation (4.19) will have a singularity in 0. Moreover, when $\alpha \in (1, \infty)$ the process does not explode with positive probability (see Proposition 3.9) and we need to adapt Lemma 4.28 to prove that, under the conditional probability of nonexplosion, the solution of equation (2.6) behaves as the Ornstein-Uhlenbeck (2.7).

Theorem 4.7 (Repulsive case). *Assume that $(\rho, \alpha, \beta) \in \mathcal{P}_+$ and $2\beta \in (\alpha + 1, \infty)$.*

i) If $\alpha \in (-1, 1]$, then X is recurrent in \mathbb{R} and it satisfies (4.17).

ii) If $\alpha \in (-\infty, -1]$, then X satisfies

$$\lim_{t \rightarrow \infty} \frac{X_t}{\sqrt{t}} \stackrel{\mathcal{L}}{=} |G| \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{X_t}{L(t)} = 1 \quad \text{a.s.}, \quad \text{with } G \sim \mathcal{N}(0, 1). \quad (4.23)$$

Moreover, X is recurrent in $[0, \infty)$, when $\alpha = -1$, X is recurrent in $(0, \infty)$, when $\alpha \in (-\infty, -1)$ and $\beta \in [0, \infty)$ and X is transient, when $\alpha \in (-\infty, -1)$ and $\beta \in (-\infty, 0)$. In the two last situations

$$\liminf_{t \rightarrow \infty} \frac{X_t}{L_{\rho, \alpha, \beta}(t)} \geq 1, \quad \text{a.s.} \quad (4.24)$$

iii) If $\alpha \in (1, \infty)$ then, under the conditional probability of nonexplosion $\mathbb{P}(\bullet | \tau_e = \infty)$, X is recurrent in \mathbb{R} and it satisfies (4.17). Moreover, under the probability of explosion $\mathbb{P}(\bullet | \tau_e < \infty)$, it satisfies (4.10).

Proof. First, let us note that, when $\alpha \in (-1, 1]$, the proof is exactly the same as the proof of Theorem 4.6. Assume that $\alpha \in (-\infty, -1]$ and let $X^{(e)} = \Phi_e(X)$ be the pathwise unique nonnegative strong solution of equation (2.6). We proceed as for the proof of Theorem 4.6. We remark that relations in (4.23) are equivalent to

$$\lim_{t \rightarrow \infty} \left(X_t^{(e)} \right)^{|\alpha|+1} \stackrel{\mathcal{L}}{=} |G|^{|\alpha|+1} \quad \text{and} \quad \limsup_{t \rightarrow \infty} \left(\frac{X_t^{(e)}}{\sqrt{2 \ln t}} \right)^{|\alpha|+1} = 1 \quad \text{a.s.}, \quad \text{with } G \sim \mathcal{N}(0, 1). \quad (4.25)$$

We can see that (4.25) is satisfied by the process $Q_t = |U_t|^{|\alpha|+1}$, where U is given by (2.7) and

$$dQ_t = (|\alpha| + 1) Q_t^{\frac{|\alpha|}{|\alpha|+1}} dW_t + \frac{|\alpha| + 1}{2} \left(|\alpha| Q_t^{\frac{|\alpha|-1}{|\alpha|+1}} - Q_t \right) dt, \quad Q_0 = x_0^{|\alpha|+1}.$$

Moreover, let us consider the pathwise unique nonnegative strong solution of

$$dC_t = (|\alpha| + 1) C_t^{\frac{|\alpha|}{|\alpha|+1}} dW_t + \frac{|\alpha| + 1}{2} \left(|\alpha| C_t^{\frac{|\alpha|-1}{|\alpha|+1}} - C_t + 2\rho e^{\left(\frac{\alpha+1}{2}-\beta\right)t} C_t^{\frac{|\alpha|+\alpha}{|\alpha|+1}} \right) dt, \quad C_0 = x_0^{|\alpha|+1}.$$

By Ito's formula, we see that $C \stackrel{\mathcal{L}}{=} |X^{(e)}|^{|\alpha|+1}$ and it is clear that (C_t, Q_t) is asymptotically time-homogeneous and $\mathcal{L}(|G|^{|\alpha|+1})$ -ergodic, with $G \sim \mathcal{N}(0, 1)$. We show that C_t is bounded in probability, by comparing C_t with the ergodic nonnegative diffusion satisfying

$$dC_t^+ = (|\alpha| + 1) (C_t^+)^{\frac{|\alpha|}{|\alpha|+1}} dW_t + \frac{|\alpha| + 1}{2} \left(|\alpha| (C_t^+)^{\frac{|\alpha|-1}{|\alpha|+1}} - C_t^+ + 2\rho (C_t^+)^{\frac{|\alpha|+\alpha}{|\alpha|+1}} \right) dt, \quad C_0^+ = x_0^{|\alpha|+1}.$$

Lemma 4.5 applies and we deduce

$$\lim_{t \rightarrow \infty} \frac{X_t}{\sqrt{t}} \stackrel{\mathcal{L}}{=} \lim_{t \rightarrow \infty} C_t^{\frac{1}{|\alpha|+1}} \stackrel{\mathcal{L}}{=} |G|.$$

By applying Motoo's theorem to Q_t and C_t^+ and by comparison theorem we obtain

$$1 = \limsup_{t \rightarrow \infty} \frac{Q_t}{(2 \ln t)^{\frac{|\alpha|+1}{2}}} \leq \limsup_{t \rightarrow \infty} \frac{C_t}{(2 \ln t)^{\frac{|\alpha|+1}{2}}} \leq \limsup_{t \rightarrow \infty} \frac{C_t^+}{(2 \ln t)^{\frac{|\alpha|+1}{2}}} = 1 \quad \text{a.s.}$$

We deduce that

$$\limsup_{t \rightarrow \infty} \frac{X_t}{L(t)} \stackrel{\mathcal{L}}{=} \limsup_{t \rightarrow \infty} \left(\frac{C_t}{(2 \ln t)^{\frac{|\alpha|+1}{2}}} \right)^{\frac{1}{|\alpha|+1}} = 1 \quad \text{a.s.} \quad (4.26)$$

This implies (4.23). Moreover, when $\alpha = -1$, we have seen that the point 0 is recurrent for X , by using the second equality in (4.23), we deduce the recurrent feature in $[0, \infty)$. Assume that $\alpha \in (-\infty, -1)$ and consider the process $\tilde{X}^{(\gamma)} := 1/X^{(\gamma)}$, where $X^{(\gamma)} = \Phi_\gamma(X)$ is the pathwise unique nonnegative strong solution of equation (2.13). By Ito's formula, $\tilde{X}^{(\gamma)}$ is the pathwise unique nonnegative strong solution of equation

$$d\tilde{X}_t^{(\gamma)} = (\tilde{X}_t^{(\gamma)})^2 dW_t + \left((\tilde{X}_t^{(\gamma)})^3 - \rho (\tilde{X}_t^{(\gamma)})^{2-\alpha} + \frac{\gamma \tilde{X}_t^{(\gamma)}}{2(1 - (1 - \gamma)t)} \right) dt, \quad \tilde{X}_0^{(\gamma)} := \frac{1}{x_0}.$$

Let us introduce the pathwise unique nonnegative strong solution of equation

$$d\tilde{V}_t = \tilde{V}_t^2 dW_t + (\tilde{V}_t^3 - \rho \tilde{V}_t^{2-\alpha} + \frac{|\gamma| \tilde{V}_t}{2}) dt, \quad \tilde{V}_0 := \frac{1}{x_0}.$$

By using the comparison theorem for $\tilde{X}^{(\gamma)}$ and \tilde{V} , and by applying Motoo's theorem to \tilde{V} ,

$$\liminf_{t \rightarrow \infty} \frac{X_t}{L_{\rho, \alpha, \beta}(t)} = \left(\limsup_{s \rightarrow \infty} \frac{X_s^{(\gamma)}}{(c_{\rho, \alpha} \ln s)^{\frac{1}{|\alpha|+1}}} \right)^{-1} \geq \left(\limsup_{s \rightarrow \infty} \frac{\tilde{V}_s}{(c_{\rho, \alpha} \ln s)^{\frac{1}{|\alpha|+1}}} \right)^{-1} = 1 \quad \text{a.s.}$$

In the latter relation, the first equality is obtained by the change of time $s = \varphi_\gamma^{-1}(t)$ (where φ_γ is defined in (2.11)) Noting that $\lim_{t \rightarrow \infty} L_{\rho, \alpha, \beta}(t) = \infty$, when $\beta \in (-\infty, 0)$, we deduce that X is transient in this case. Moreover, if $\beta = 0$, then X is an homogeneous diffusion and by usual criteria, using the scale function, we can see that X is recurrent in $(0, \infty)$. If $\beta \in [0, \infty)$, by comparison theorem with the process obtained for $\beta = 0$, we deduce that X is recurrent in $(0, \infty)$. This ends the proof of ii) in Theorem 4.7.

Assume that $\alpha \in (1, \infty)$. Consider $X^{(\gamma)} = \Phi_\gamma(X)$ and b respectively, the pathwise unique strong solutions of equations (2.14) and (2.18), and denote by η_e the explosion time of $X^{(\gamma)}$. Recall that $\{\eta_e \geq t_1\} = \{\tau_e = \infty\}$, that $\mathbb{P}(\eta_e = t_1) = 0$ (Lemma 3.10), and $\lim_{t \rightarrow t_1} b_t = 0$ a.s. By the Girsanov transformation (3.5), we obtain

$$\mathbb{P} \left(\lim_{t \rightarrow t_1} X_t^{(\gamma)} = 0, \eta_e \geq t_1 \right) = \mathbb{E} \left(\mathbb{1}_{\left\{ \lim_{t \rightarrow t_1} b_t = 0 \right\}} \mathcal{E}(t_1) \right) = \mathbb{E}(\mathcal{E}(t_1)) = \mathbb{P}(\eta_e \geq t_1).$$

By change of time, we deduce

$$\lim_{t \rightarrow \infty} \frac{X_t}{t^{\frac{\beta}{\alpha+1}}} = \lim_{t \rightarrow t_1} X_t^{(\gamma)} = 0 \quad \text{a.s. on } \{\tau_e = \infty\}. \quad (4.27)$$

Then

$$\lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} \int_1^t \rho \frac{|X_s|^\alpha}{s^\beta} ds = \lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} \int_1^t \rho \left| \frac{X_s}{s^{\frac{\beta}{\alpha+1}}} \right|^\alpha s^{-\frac{\beta}{\alpha+1}} ds = 0 \quad \text{a.s. on } \{\tau_e = \infty\}.$$

We deduce that X satisfies, under the conditional probability of nonexplosion, the iterated logarithm law. As a consequence it is recurrent in \mathbb{R} . We prove the convergence in distribution (4.17) under the conditional probability of nonexplosion. For this end, it suffices to show that

$$\lim_{s \rightarrow \infty} \mathbb{P}(X_s^{(e)} > x \mid \sigma_e = \infty) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{y^2}{2}\right) dy. \quad (4.28)$$

Here σ_e denotes the explosion time of the pathwise unique strong solution of equation (2.6) $X^{(e)} = \Phi_e(X)$. Note that Lemma 4.5 does not apply directly to $(X^{(e)}, U)$, since σ_e could be finite with positive probability. By using (4.27), we remark that

$$\lim_{s \rightarrow \infty} \rho e^{(\frac{\alpha+1}{2}-\beta)s} |X_s^{(e)}|^\alpha = \lim_{t \rightarrow \infty} \rho t^{(\frac{1}{2}-\frac{\beta}{\alpha+1})} \left| \frac{X_t}{t^{\frac{\beta}{\alpha+1}}} \right|^\alpha = 0 \quad \text{a.s. on } \{\sigma_e = \infty\}. \quad (4.29)$$

Let $\varepsilon > 0, v \geq 0$ be and denote $U^{(\pm\varepsilon)}$ the pathwise unique strong solutions of equations

$$dU_s^{(\pm\varepsilon)} = dW_s - \frac{U_s^{(\pm\varepsilon)}}{2} ds \pm \varepsilon ds, \quad U_v^{(\pm\varepsilon)} = X_v^{(e)} \mathbf{1}_{\{\sigma_e > v\}}.$$

It is classical that $U^{(\pm\varepsilon)}$ is Feller and ergodic. Furthermore, the strong mixing property holds (see [10], Theorem 20.20, p. 408), hence we obtain

$$\lim_{s \rightarrow \infty} \mathbb{P}(U_s^{(\pm\varepsilon)} > x \mid \Omega_v^\varepsilon) = F_{\mp\varepsilon}(x) := \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{(y \mp \varepsilon)^2}{2}\right) dy, \quad (4.30)$$

with

$$\Omega_v^\varepsilon := \left\{ \sup_{s \geq v} \rho e^{(\frac{\alpha+1}{2}-\beta)s} |X_s^{(e)}|^\alpha \leq \varepsilon \right\}.$$

Similarly as for (4.22), we can show, by using comparison theorem and a classical argument of localisation, that, for all $s \geq v$, $U_s^{(-\varepsilon)} \leq X_s^{(e)} \leq U_s^{(+\varepsilon)}$ a.s. on Ω_v^ε . We deduce, from (4.30),

$$F_{+\varepsilon}(x) \leq \liminf_{s \rightarrow \infty} \mathbb{P}\left(X_s^{(e)} > x \mid \Omega_v^\varepsilon\right) \leq \limsup_{s \rightarrow \infty} \mathbb{P}\left(X_s^{(e)} > x \mid \Omega_v^\varepsilon\right) \leq F_{-\varepsilon}(x). \quad (4.31)$$

Thanks to (4.29) the set of nonexplosion is $\{\sigma_e = \infty\} = \cup_{v \geq 0} \Omega_v^\varepsilon$. Letting $v \rightarrow \infty$, and then $\varepsilon \rightarrow 0$ in (4.31), we deduce (4.28). To finish the proof we need to study the process X under the probability of explosion $\mathbb{P}(\bullet \mid \tau_e < \infty)$ and prove that it also satisfies (4.10). The method is the same as in the proof of Theorem 4.4: we show that

$$|X_t^{(e)}| \underset{t \rightarrow \infty}{\sim} \frac{1}{(\rho(\alpha-1)(\eta_e - t))^{\frac{1}{\alpha-1}}} \quad \text{a.s. on } \{\eta_e < \infty\},$$

and we conclude by change of time. This ends the proof of Theorem 4.7. \square

4.3 Behaviour under the critical line: $2\beta < \alpha + 1$

With the help of Lemma 4.5 and of the comparison theorem, we shall prove that the asymptotic behaviour of equation (2.13) is related to the asymptotic behaviour of the time-homogeneous equation (2.16). By change of time, we shall obtain the asymptotic behaviour for (2.1).

Theorem 4.8 (Attractive case). *If $(\rho, \alpha, \beta) \in \mathcal{P}_-$ and $2\beta \in (-\infty, \alpha + 1)$, then X is recurrent in \mathbb{R} , when $\beta \in [0, \infty)$, and X converges a.s. towards 0, when $\beta \in (-\infty, 0)$. In these cases the process satisfies*

$$\lim_{t \rightarrow \infty} \frac{X_t}{t^{\frac{\beta}{\alpha+1}}} \stackrel{\mathcal{L}}{=} \Pi_{\rho, \alpha} \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{X_t}{L_{\rho, \alpha, \beta}(t)} = 1 \quad \text{a.s.} \quad (4.32)$$

Proof. The proof will follow the same lines as the proof of Theorem 4.6. Consider $X^{(\gamma)} = \Phi_\gamma(X)$ the pathwise unique strong solution to equation (2.13). By change of time, the equalities in (4.32) are equivalent to

$$\lim_{t \rightarrow \infty} (X_t^{(\gamma)})^3 = S^3 \quad \text{and} \quad \limsup_{t \rightarrow \infty} \left(\frac{X_t^{(\gamma)}}{(c_{\rho, \alpha} \ln t)^{\frac{1}{\alpha+1}}} \right)^3 = 1 \quad \text{a.s.,} \quad \text{with} \quad S \sim \Pi_{\rho, \alpha}. \quad (4.33)$$

Note that these convergences are satisfied for $Q_t := H_t^3$, where H_t is the solution of equation (2.17). This process is the pathwise unique strong solution of equation

$$dQ_t = 3Q_t^{\frac{2}{3}} dW_t + 3\left(\rho \operatorname{sgn}(Q_t)|Q_t|^{\frac{\alpha+2}{3}} + Q_t^{\frac{1}{3}}\right)dt, \quad Q_0 = x_0^3.$$

Let us consider the pathwise unique strong solution of equation with continuous coefficient

$$dC_t = 3C_t^{\frac{2}{3}} dW_t + 3\left(\rho \operatorname{sgn}(C_t)|C_t|^{\frac{\alpha+2}{3}} + C_t^{\frac{1}{3}} - \frac{\gamma C_t}{2(1 - (1 - \gamma)t)}\right)dt, \quad C_0 = x_0^3.$$

By Ito's formula we can see that $C \stackrel{\mathcal{L}}{=} (X^{(\gamma)})^3$ and it is a simple calculation to see that (C_t, Q_t) is asymptotically homogeneous and $\mathcal{L}(S^3)$ -ergodic. In order, to apply Lemma 4.5 we have to prove that C_t is bounded in probability. Since ρ is negative, it is not difficult to prove, by using the comparison theorem (see Theorem 1.1 in [9], pp. 352-353) and Ito's formula, that $X_t^2 \leq \widetilde{W}_t^2$, where \widetilde{W}_t is a Brownian motion with $\widetilde{W}_1 = x_0$. By the change of time $s = \varphi_\gamma^{-1}(t)$,

$$\limsup_{t \rightarrow \infty} \frac{C_t}{(2t \ln \ln t)^{\frac{3}{2}}} \stackrel{\mathcal{L}}{=} \limsup_{s \rightarrow \infty} \frac{X_s^3}{\left(\frac{2}{1-\gamma} s \ln \ln s\right)^{\frac{3}{2}}} \leq (1 - \gamma)^3 \quad \text{a.s.} \quad (4.34)$$

Fix $\theta \in (1/3, (\alpha + 2)/3)$. Thanks to (4.34), we deduce

$$\lim_{t \rightarrow \infty} \frac{1}{|C_t|^\theta} \cdot \frac{\gamma C_t}{1 + (1 - \gamma)t} = 0 \quad \text{a.s.} \quad (4.35)$$

Let $v \geq 0$ be and introduce C_t^\pm the pathwise unique strong solutions of equations

$$dC_t^\pm = 3C_t^{\frac{2}{3}} dW_t + 3\left(\rho \operatorname{sgn}(C_t)|C_t|^{\frac{\alpha+2}{3}} + C_t^{\frac{1}{3}} \pm |C_t|^\theta\right)dt, \quad C_v^\pm = C_v.$$

As for the proof of comparison (4.22) we can prove (by using Theorem 1.1 Chap. VI in [9], p. 437-438, and a classical argument of localisation) that for all $s \geq v$,

$$C_t^- \leq C_t \leq C_t^+ \quad \text{a.s. on} \quad \Omega_v := \left\{ \sup_{t \geq v} \frac{1}{C_t^\theta} \cdot \frac{\gamma C_t}{1 + (1 - \gamma)t} \leq 1 \right\}. \quad (4.36)$$

By (4.35), for any $\varepsilon > 0$, we can choose $v \geq 0$ such that $\mathbb{P}(\Omega_v) \geq 1 - \varepsilon$. Moreover, there exists $r \geq 0$ such that for all $t \geq v$, $\mathbb{P}(|C_t^\pm| \geq r) \leq \varepsilon$ since C_t^\pm are ergodic diffusions (by computation of the speed measure). Combining the latter inequality with (4.36) which holds on Ω_v , we obtain that $\mathbb{P}(|C_t| \geq r) \leq 2\varepsilon$, for all $t \geq v$ and therefore we conclude that C_t is bounded in probability. Moreover, Motoo's theorem (applied to C^\pm) and the preceding comparison imply

$$1 = \limsup_{t \rightarrow \infty} \frac{C_t^-}{(c_{\rho, \alpha} \ln t)^{\frac{3}{\alpha+1}}} \leq \limsup_{t \rightarrow \infty} \frac{C_t}{(c_{\rho, \alpha} \ln t)^{\frac{3}{\alpha+1}}} \leq \limsup_{t \rightarrow \infty} \frac{C_t^+}{(c_{\rho, \alpha} \ln t)^{\frac{3}{\alpha+1}}} = 1 \quad \text{a.s.}$$

This ends the proof of (4.33) and of the theorem. \square

Theorem 4.9 (Repulsive case). *Assume that $(\rho, \alpha, \beta) \in \mathcal{P}_+$ and $2\beta \in (-\infty, \alpha + 1)$. Then X is transient and it satisfies*

$$\lim_{t \rightarrow \infty} \frac{|X_t|}{t^{\frac{1-\beta}{1-\alpha}}} = \left(\frac{\rho(1-\alpha)}{1-\beta} \right)^{\frac{1}{1-\alpha}} \quad \text{a.s., when } \alpha \in (-\infty, 1), \quad (4.37)$$

and

$$|X_t| \underset{t \rightarrow \infty}{\sim} \frac{\varphi_Y^{\frac{\gamma}{\alpha-1}} \circ \varphi_Y^{-1}(\tau_e) \cdot \tau_e^{\frac{\gamma}{2}}}{(\rho(\alpha-1)(\tau_e-t))^{\frac{1}{\alpha-1}}} \quad \text{a.s., when } \alpha \in (1, \infty), \quad (4.38)$$

and

$$\lim_{t \rightarrow \infty} \frac{X_t}{\exp\left(\frac{\rho t^{1-\beta}}{1-\beta}\right)} = G \quad \text{a.s., when } \alpha = 1, \quad (4.39)$$

where $G \sim \mathcal{N}(m, \sigma^2)$, with $m := x_0 \exp(\frac{\rho}{\beta-1})$ and $\sigma^2 := \int_1^\infty \exp(\frac{2\rho s^{1-\beta}}{\beta-1}) ds$.

Proof. Assume first that $\alpha \in (-\infty, 1)$. To simplify the computations, let us denote the limit and the exponent of t in (4.37), respectively by

$$\ell := \left(\frac{\rho(1-\alpha)}{1-\beta} \right)^{\frac{1}{1-\alpha}} \quad \text{and} \quad \nu := \frac{1-\beta}{1-\alpha}.$$

If we set $S_t := X_t^2/t^{2\nu}$, it suffices to verify that $\lim_{t \rightarrow \infty} S_t = \ell^2$ a.s., that is, for all $\varepsilon > 0$,

$$\limsup_{t \rightarrow \infty} S_t \leq \ell^2 + 3\varepsilon \quad \text{and} \quad \liminf_{t \rightarrow \infty} S_t \geq \ell^2 - 3\varepsilon \quad \text{a.s.} \quad (4.40)$$

We shall prove only the first inequality in (4.40), the second one being obtained in a similar way. We split the proof of this inequality in four steps.

Step a). We begin by proving that, for all $\varepsilon > 0$,

$$\{t \geq 1 : S_t \leq \ell^2 + \varepsilon\} \quad \text{is unbounded a.s.} \quad (4.41)$$

For this end, set $\eta_u := \inf \{v \geq u : S_v \leq \ell^2 + \varepsilon\}$, $u \geq 1$. Then, it suffices to prove that for all $u \geq 1$ large enough, $\eta_u < \infty$ a.s. By using Ito's formula, we can see that

$$S_{t \wedge \eta_u} = S_u + \int_u^{t \wedge \eta_u} \mathcal{L}V(s, X_s) ds + \int_u^{t \wedge \eta_u} \partial_x V(s, X_s) dB_s := S_u + M_t + A_t, \quad (4.42)$$

where $V(t, x) := x^2/t^{2\nu}$ and where \mathcal{L} is given by (3.3). Moreover, we can see that there exist $s_0 \geq 1$ and $c > 0$ such that, for all $s \geq s_0$ and $x \in \mathbb{R}$, for which $V(s, x) \geq \ell^2 + \varepsilon$ and

$$\mathcal{L}V(s, x) = \frac{2\rho}{s} \left(V(s, x)^{\frac{\alpha-1}{2}} - \ell^{\alpha-1} \right) V(s, x) + \frac{1}{s^{2\nu}} \leq -\frac{c}{s} \leq 0. \quad (4.43)$$

This implies that the local martingale part $S_u + M$ of the non-negative semimartingale in (4.42), together with $S_{\bullet \wedge \eta_u}$ itself, are nonnegative supermartingales for all $u \geq s_0$. Therefore, the bounded variation part A will be a convergent process as the difference of two convergent supermartingales. Thanks to (4.43), this is possible if and only if $\eta_u < \infty$ a.s.

Step b). We introduce an increasing sequence of stopping times as follows:

$$\tau_1 := \inf\{t \geq s_0 : S_t = \ell^2 + 2\varepsilon\}, \quad \sigma_1 := \inf\{t \geq \tau_1 : S_t \in \{\ell^2 + \varepsilon, \ell^2 + 3\varepsilon\}\}$$

and for every integer $n \geq 2$,

$$\tau_n := \inf\{t \geq \sigma_{n-1} : S_t = \ell^2 + 2\varepsilon\}, \quad \sigma_n := \inf\{t \geq \tau_n : S_t \in \{\ell^2 + \varepsilon, \ell^2 + 3\varepsilon\}\}.$$

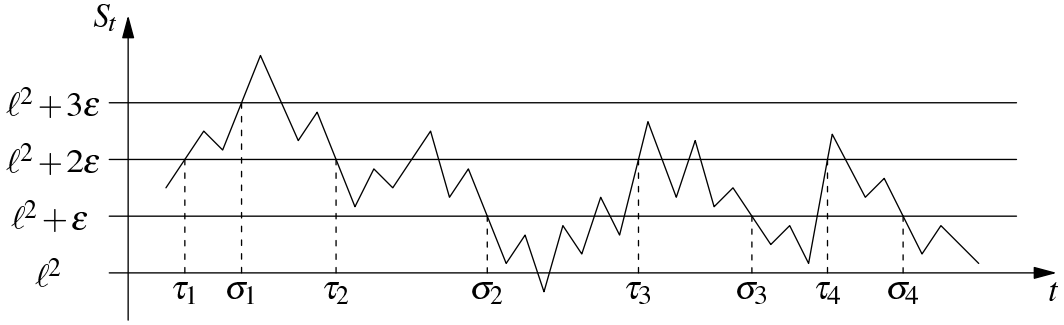


Figure 3: The increasing sequence of stopping times

Set $F := \bigcap_{n \geq 1} \{\tau_n < \infty\}$. Thanks to (4.41), we obtain that $\limsup_{t \rightarrow \infty} S_t \leq \ell^2 + 2\varepsilon$ a.s. on F^c . To prove the first inequality in (4.40), we need to show that

$$\limsup_{t \rightarrow \infty} S_t \leq \ell^2 + 3\varepsilon \quad \text{a.s. on } F, \quad \text{or, equivalently,} \quad \mathbb{1}_F \sum_{n \geq 1} \mathbb{1}_{\{S_{\sigma_n} = \ell^2 + 3\varepsilon\}} < \infty \quad \text{a.s.}$$

By using a conditional version of the Borel-Cantelli lemma (see, for instance, Corollary 7.20 in [10], p. 131), it is equivalent to prove that

$$\sum_{n=1}^{\infty} \mathbb{P}(S_{\sigma_n} = \ell^2 + 3\varepsilon, \tau_n < \infty \mid \mathcal{F}_{\tau_n}) < \infty \quad \text{a.s.} \quad (4.44)$$

Step c). We show that there exist positive constants λ_1 and λ_2 such that for all integer $n \geq 1$,

$$\mathbb{P}(S_{\sigma_n} = \ell^2 + 3\varepsilon, \tau_n < \infty \mid \mathcal{F}_{\tau_n}) \leq \lambda_1 \tau_n^{\left(\frac{1}{2} - \nu\right)} \exp(-\lambda_2 \tau_n^{2\nu-1}) \quad \text{a.s. on } \{\tau_n < \infty\}. \quad (4.45)$$

To this end, let us denote by $\mathbb{P}_{s,x}$ the distribution of the weak solution of (2.1) such that $X_s = x$. The strong Markov property applies and this yields

$$\mathbb{P}(S_{\sigma_n} = \ell^2 + 3\varepsilon, \tau_n < \infty \mid \mathcal{F}_{\tau_n}) = \mathbb{P}_{\tau_n, X_{\tau_n}}(S_{\sigma_n} = \ell^2 + 3\varepsilon) \quad \text{a.s. on } \{\tau_n < \infty\}.$$

As in (4.42), we can write under the conditional probability $\mathbb{P}_{\tau_n, X_{\tau_n}}$, the canonical decomposition $S_{t \wedge \sigma_n} = S_{\tau_n} + M_t^n + A_t^n$, sum of a local martingale and a bounded variation process. Besides, we can show that $v_n := \langle M^n \rangle_\infty$ satisfies

$$v_n = \int_{\tau_n}^{\sigma_n} \frac{4S_u}{u^{2\nu}} du \leq \int_{\tau_n}^{\infty} \frac{4(\ell^2 + 3\varepsilon)}{u^{2\nu}} du = \frac{4(\ell^2 + 3\varepsilon)}{(2\nu - 1)\tau_n^{2\nu-1}} \leq \frac{4(\ell^2 + 3\varepsilon)}{(2\nu - 1)s_0^{2\nu-1}} =: v_0.$$

Then, by the Dambis-Dubins-Schwarz theorem, there exists a standard Brownian motion W^n (under the conditional probability $\mathbb{P}_{\tau_n, X_{\tau_n}}$) such that $M^n = W_{\langle M^n \rangle}^n$ and since A^n is strictly negative, we can see that

$$\left\{ \sup_{0 \leq t \leq v_n} W_t^n < \varepsilon \right\} \subset \left\{ \sup_{t \geq \tau_n} M_t^n < \varepsilon \right\} \subset \{S_{\sigma_n} = \ell^2 + \varepsilon\}.$$

It is classical that $\sup\{W_t^n : 0 \leq t \leq v_n\} \stackrel{\mathcal{L}}{=} |G_n|$, with $G_n \stackrel{\mathcal{L}}{=} \mathcal{N}(0, v_n)$, under the conditional probability $\mathbb{P}_{\tau_n, X_{\tau_n}}$ and therefore we obtain

$$\mathbb{P}_{\tau_n, X_{\tau_n}}(S_{\sigma_n} = \ell^2 + 3\varepsilon) = 1 - \mathbb{P}_{\tau_n, X_{\tau_n}}(S_{\sigma_n} = \ell^2 + \varepsilon) \leq \mathbb{P}_{\tau_n, X_{\tau_n}}(|G_n| \geq \varepsilon/\sqrt{v_n}).$$

By the usual estimate of tails for the standard Gaussian random variables, we get (4.45).

Step d). To insure the convergence of the series in (4.44) we show that the sequence (τ_n) increases to infinity sufficiently fast. More precisely, we show that there exists $\lambda > 1$ such that $\tau_n \geq \lambda^n \tau_1$ a.s. on F . This inequality will be a consequence of a sharper form of the Borel-Cantelli lemma (see, for example, Theorem 1 in [5], p. 800) once we show that there exist some constants $q > 1$ and $p > 0$ such that for all $n \geq 1$,

$$\mathbb{P}(\tau_{n+1} \geq q\tau_n \mid \mathcal{F}_{\tau_n}) \geq \mathbb{P}_{\tau_n, X_{\tau_n}}(\sigma_n \geq q\tau_n) \geq p \quad \text{a.s. on } \{\tau_n < \infty\}. \quad (4.46)$$

In opposite to (4.43), we can see that there exists a constant $k > 0$ such that for all $t \geq 1$ and $x \in \mathbb{R}$, for which $V(t, x) \leq \ell^2 + 3\varepsilon$, $\mathcal{L}V(t, x) \geq -k/t$. We deduce that for all $t \in [\tau_n, q\tau_n]$,

$$-\frac{\varepsilon}{2} \leq k \ln\left(\frac{\tau_n}{t}\right) \leq A_t^n \leq 0, \quad \text{with } q := e^{\frac{k\varepsilon}{2}} > 1.$$

By using this inequality, we can write

$$\{\sigma_n \geq q\tau_n\} \supset \left\{ \sup_{\tau_n \leq t \leq q\tau_n} |M_t^n| < \frac{\varepsilon}{2} \right\} \supset \left\{ \sup_{0 \leq t \leq v_0} |W_t^n| < \frac{\varepsilon}{2} \right\}.$$

Therefore, inequality (4.46) is satisfied with the deterministic positive constant

$$p := \mathbb{P}_{\tau_n, X_{\tau_n}}\left(\sup_{0 \leq t \leq v_0} |W_t^n| < \frac{\varepsilon}{2}\right) = \mathbb{P}\left(\sup_{0 \leq t \leq v_0} |B_t| < \frac{\varepsilon}{2}\right).$$

Here B denotes a standard Brownian motion. The sharper form of the Borel-Cantelli lemma applies and we obtain that $\tau_n \geq \lambda^n \tau_1$ a.s. on F . We deduce (4.44) and (4.37) holds.

Assume that $\alpha \in (1, \infty)$. The proof of (4.38) follows the same lines as the proof of (4.10). We show that $X^{(\gamma)} = \Phi_\gamma(X)$ satisfies

$$|X_s^{(\gamma)}| \underset{s \rightarrow \infty}{\sim} \frac{1}{(\rho(\alpha - 1)(\tau_e - s))^{\frac{1}{\alpha-1}}} \quad \text{a.s., when } \alpha \in (1, \infty),$$

and we conclude by applying the change of time $t = \varphi_\gamma^{-1}(s)$.

Finally, assume that $\alpha = 1$. The same ideas as for the proof of (4.14) are employed. By Ito's formula and the Dambis-Dubins-Schwartz theorem there exists a standard Brownian motion W such that

$$\frac{X_t}{v(t)} = \frac{x_0}{v(1)} + W_{\phi(t)}, \quad \text{with} \quad \phi(t) := \int_1^t \frac{ds}{v^2(s)} \quad \text{and} \quad v(t) := \exp\left(\rho \frac{t^{1-\beta}}{1-\beta}\right).$$

There are three situations: i) if $\beta \in (1, \infty)$, then $\lim_{t \rightarrow \infty} v(t) = 1$ and $\phi(t) \sim_{t \rightarrow \infty} t$; ii) if $\rho \in (0, \infty)$ and $\beta \in (-\infty, 1)$, then $\lim_{t \rightarrow \infty} v(t) = \infty$ $\lim_{t \rightarrow \infty} \phi(t) = \sigma^2 < \infty$; iii) if $\rho \in (-\infty, 0)$ and $\beta \in (-\infty, 1)$, then $\lim_{t \rightarrow \infty} v(t) = 0$ and $\phi(t) \sim_{t \rightarrow \infty} 2|\rho|t^\beta v^{-2}(t)$. The first two points are obvious, whereas the last one is obtained by using L'Hôpital's rule.

By using the usual properties of Brownian motion we get the convergence in distribution and the pathwise largest deviations. The recurrent or transient features are then deduced. This ends the proof of the theorem. \square

5 Appendix

Proof of Lemma 3.10. To begin with, let us recall that $\rho \in (0, \infty)$, $\alpha \in (1, \infty)$, $2\beta \in (\alpha + 1, \infty)$, that γ, t_1 and δ are given in (2.11) and (2.15), and that $X_s^{(\gamma)}$ is the pathwise unique strong solution of equation (2.14), which explosion time is $\eta_e \in [0, t_1] \cup \{\infty\}$. The goal is to prove that $\eta_e \neq t_1$ a.s. By Ito's formula, we can see that

$$dX_s^{(\gamma, t_1)} = (t_1 - s)^{\frac{1}{\alpha-1}} dW_s + d(s, X_s^{(\gamma, t_1)}) ds, \quad \text{with} \quad X_s^{(\gamma, t_1)} := (t_1 - s)^{\frac{1}{\alpha-1}} X_s^{(\gamma)},$$

and

$$d(s, x) := \rho \frac{x(|x|^{\alpha-1} - \ell^{\alpha-1})}{t_1 - s} \quad \text{and} \quad \ell := \left(\frac{1 + \delta(\alpha - 1)}{\rho(\alpha - 1)}\right)^{\frac{1}{\alpha-1}} \in (0, \infty).$$

Roughly speaking, since $x \cdot d(s, x) \geq 0$ (respectively ≤ 0), according as $|x| \geq \ell$ (respectively $\leq \ell$), $0, -\infty$ and ∞ are ‘‘attractive’’ levels, whereas $-\ell$ and ℓ are ‘‘repulsive’’ levels for the process $X^{(\gamma, t_1)}$. The strategy of the proof is as follows: firstly, we show that

$$\lim_{s \rightarrow t_1} |X_s^{(\gamma, t_1)}| \in \{0, \ell, \infty\} \quad \text{a.s. on} \quad F := \{\eta_e = t_1\}. \quad (5.1)$$

Secondly, we shall prove that the following three events are of probability zero:

$$F_0 := F \cap \left\{ \lim_{s \rightarrow t_1} |X_s^{(\gamma, t_1)}| = 0 \right\}, \quad F_\ell := F \cap \left\{ \lim_{s \rightarrow t_1} |X_s^{(\gamma, t_1)}| = \ell \right\} \quad \text{and}, \\ F_\infty := F \cap \left\{ \lim_{s \rightarrow t_1} |X_s^{(\gamma, t_1)}| = \infty \right\}. \quad (5.2)$$

We stress that the following reasoning will be performed by taking place on the event F . For simplicity, this condition will be understood and will be dropped along the following steps.

Step a). We verify (5.1). Introduce $E := \{\omega \in F : \liminf_{s \rightarrow t_1} X_s^{(\gamma, t_1)} < \limsup_{s \rightarrow t_1} X_s^{(\gamma, t_1)}\}$. Fix $\omega \in E$ and suppose that $\limsup_{s \rightarrow t_1} X_s^{(\gamma, t_1)}(\omega) > \ell$. We can pick two sequences of real

numbers (which depends on ω), (s_n) and (u_n) , such that $0 \leq u_n \leq s_n < t_1$ for all integers n , $\lim_{n \rightarrow \infty} u_n = t_1$, and

$$X_{u_n}^{(\gamma, t_1)}(\omega) - X_{s_n}^{(\gamma, t_1)}(\omega) = \frac{1}{2} \left(\limsup_{s \rightarrow t_1} X_s^{(\gamma, t_1)}(\omega) - \ell \right) > 0.$$

Moreover, this choice could be done such that for any $s \in [u_n, s_n]$, $X_s^{(\gamma, t_1)}(\omega) \geq \ell$. Denote by M the martingale part of $X^{(\gamma, t_1)}$. Since the drift $d(s, x)$ is nonnegative for all $x \geq \ell$, we deduce that

$$\begin{aligned} |M_{s_n}(\omega) - M_{u_n}(\omega)| &= \left| X_{u_n}^{(\gamma, t_1)}(\omega) - X_{s_n}^{(\gamma, t_1)}(\omega) + \int_{u_n}^{s_n} d(s, X_s^{(\gamma, t_1)}(\omega)) ds \right| \\ &\geq \frac{1}{2} \left(\limsup_{s \rightarrow t_1} X_s^{(\gamma, t_1)}(\omega) - \ell \right) > 0. \end{aligned}$$

Clearly a similar argument works when $0 < \limsup_{s \rightarrow t_1} X_s^{(\gamma, t_1)}(\omega) \leq \ell$, but also for the two symmetric situations $\liminf_{s \rightarrow t_1} X_s^{(\gamma, t_1)}(\omega) < -\ell$ and $-\ell \leq \limsup_{s \rightarrow t_1} X_s^{(\gamma, t_1)}(\omega) < 0$. This means that

$$\liminf_{n \rightarrow \infty} |M_{s_n}(\omega) - M_{u_n}(\omega)| > 0, \quad \text{a.s. on } E.$$

Since M is a.s. uniformly continuous on $[0, t_1]$, necessarily $\mathbb{P}(E) = 0$. We obtain equality (5.1) by noting that

$$\lim_{s \rightarrow t_1} |d(s, X_s^{(\gamma, t_1)})| = \infty \quad \text{a.s. on } F \cap \left\{ \lim_{s \rightarrow t_1} |X_s^{(\gamma, t_1)}| \notin \{0, \ell\} \right\}.$$

Step b). Note that $h_s := X_s^{(\gamma)} - W_s$ is the solution of the ordinary differential equation

$$h'_s = \rho \operatorname{sgn}(h_s + W_s) |h_s + W_s|^\alpha - \delta \frac{h_s + W_s}{t_1 - s}.$$

We re-write the latter equation

$$h'_s = -\varepsilon_1(X_s^{(\gamma, t_1)}) \cdot \frac{\delta(h_s + W_s)}{t_1 - s}, \quad \text{with } \varepsilon_1(x) := 1 - \frac{\rho}{\delta} |x|^{\alpha-1}, \quad (5.3)$$

and

$$h'_s = \varepsilon_2(X_s^{(\gamma, t_1)}) \cdot \rho \operatorname{sgn}(h_s + W_s) |h_s + W_s|^\alpha, \quad \text{with } \varepsilon_2(x) := 1 - \frac{\delta}{\rho} |x|^{1-\alpha}. \quad (5.4)$$

Step c). Recall that η_e is the explosion time of $X^{(\gamma)}$. If we prove that $X^{(\gamma)}$ is bounded on $[0, t_1]$, a.s. on F_0 , necessarily $\mathbb{P}(F_0) = 0$. Since W is a.s. continuous on the compact $[0, t_1]$, it suffices to prove that h is bounded on $[0, t_1]$, a.s. on F_0 . Set $\kappa := \sup_{s \in [0, t_1]} |W_s|$. We note that $\lim_{s \rightarrow t_1} \varepsilon_1(X_s^{(\gamma, t_1)}) = 1$ a.s. on F_0 . Therefore, for any $\omega \in F_0$, there exists $u \in [0, t_1]$ such that, for all $s \in [u, t_1)$, $h_s(\omega) h'_s(\omega) \mathbb{1}_{\{|h_s(\omega)| \geq \kappa\}} \leq 0$, bu using (5.3). This implies that $h_s^2(\omega)$ is bounded on $[0, t_1]$ and we are done.

Step d). If we prove that $\lim_{s \rightarrow t_1} |X_s^{(\gamma, t_1)}| = (\rho(\alpha - 1))^{\frac{1}{1-\alpha}}$ a.s. on F_∞ , then, necessarily $\mathbb{P}(F_\infty) = 0$. Clearly, $\lim_{s \rightarrow t_1} \varepsilon_2(X_s^{(\gamma, t_1)}) = 1$ a.s. on F_∞ . Then, by using (5.4) and the fact that W is bounded on $[0, t_1]$,

$$\frac{|h_s|^{1-\alpha}}{\alpha - 1} = \int_s^{t_1} \frac{h'_u}{\text{sgn}(h_u)|h_u|^\alpha} du \underset{s \rightarrow t_1}{\sim} \rho(t_1 - s), \quad \text{a.s. on } F_\infty.$$

To conclude, it suffices to recall that $X_s^{(\gamma, t_1)} = (t_1 - s)^{\frac{1}{\alpha-1}}(h_s + W_s)$.

Step e). Similarly, if we prove that $\lim_{s \rightarrow t_1} |X_s^{(\gamma, t_1)}| = \infty$ a.s. on F_ℓ , then, necessarily $\mathbb{P}(F_\ell) = 0$. First, we show that

$$\lim_{s \rightarrow t_1} (X_s^{(\gamma, t_1)} - \ell)^2 = \infty, \quad \text{a.s. on } F_\ell^+ := F_\ell \cap \left\{ \lim_{s \rightarrow t_1} X_s^{(\gamma, t_1)} = \ell \right\}. \quad (5.5)$$

Denote $K_s := (X_s^{(\gamma, t_1)} - \ell)^2$. By Ito's formula, we can write

$$dK_s = 2(t_1 - s)^{\frac{1}{\alpha-1}} \sqrt{K_s} dB_s + \left(2q(X_s^{(\gamma, t_1)}) \frac{K_s}{t_1 - s} + \frac{1}{2}(t_1 - s)^{\frac{2}{\alpha-1}} \right) ds,$$

where

$$B_s := \int_0^s \text{sgn}(K_u) dW_u \quad \text{and} \quad q(x) := \rho x \frac{|x|^{\alpha-1} - \ell^{\alpha-1}}{x - \ell}.$$

Introduce, for $v \in [0, t_1]$, $C_s(v)$ the pathwise unique strong solution of

$$dC_s(v) = 2(t_1 - s)^{\frac{1}{\alpha-1}} \sqrt{C_s(v)} dB_s + \left(q_\infty \frac{C_s(v)}{t_1 - s} + \frac{1}{2}(t_1 - s)^{\frac{2}{\alpha-1}} \right) ds, \quad C_v(v) = K_v \mathbf{1}_{\{\eta_e > v\}},$$

where $q_\infty := \lim_{x \rightarrow \ell} q(x) = \rho(\alpha - 1)\ell^{\alpha-1}$. By comparison and localisation (see also the proof of (4.22)), we can show that for all $s \in [v, t_1]$,

$$K_s \geq C_s(v) \quad \text{a.s. on} \quad \Omega_v := F_\ell^+ \cap \left\{ \inf_{v \in [0, t_1]} |2q(X_s^{(\gamma, t_1)})| \geq q_\infty \right\}.$$

By Ito's formula, the law of the process $C(v)$ equals to the law of the square of the unique weak solution $Q(v)$ of the equation

$$dQ_s(v) = (t_1 - s)^{\frac{1}{\alpha-1}} dB_s + \frac{q_\infty Q_s(v)}{2(t_1 - s)} ds, \quad Q_v(v) = \sqrt{C_v(v)}.$$

Since $Q(v)$ is the solution of a linear equation, it is not difficult to see that $\lim_{s \rightarrow t_1} |Q_s(v)| = \infty$ a.s. and then we deduce that $\lim_{s \rightarrow t_1} C_s(v) = \infty$ a.s. Hence, for any $v \in [0, t_1]$, $\lim_{s \rightarrow t_1} K_s = \infty$ a.s. on Ω_v . Since $\lim_{s \rightarrow t_1} 2q(X_s^{(\gamma, t_1)}) > q_\infty$ a.s. on F_ℓ^+ we obtain that $\lim_{s \rightarrow t_1} K_s = \infty$ a.s. on $\cup_{v \in [0, t_1]} \Omega_v = F_\ell^+$, which is (5.5). We conclude that $\mathbb{P}(F_\ell \cap \{\lim_{s \rightarrow t_1} X_s^{(\gamma, t_1)} = \ell\}) = 0$. Clearly by similar arguments, we can prove that $\mathbb{P}(F_\ell \cap \{\lim_{s \rightarrow t_1} X_s^{(\gamma, t_1)} = -\ell\}) = 0$. Hence $\mathbb{P}(F_\ell) = 0$. The proof of the lemma is now complete. \square

Proof of Lemma 4.5. Denote by $\mathbb{P}_{u,z}$ the distribution of the diffusion Z with $Z_u = z$ and $\{\mathbb{T}_{u,s} : 0 \leq u \leq s\}$ the associated time-inhomogeneous semi-group. Similarly, denote by \mathbb{P}_z the distribution of the diffusion H starting from z at initial time and $\{\mathbb{T}_s : s \geq 0\}$

the associated semi-group. Clearly, the diffusion coefficient $(s, z) \mapsto a(u + s, z)$ and the drift $(s, z) \mapsto d(u + s, z)$ of the diffusion $s \mapsto Z_{u+s}$ satisfy the hypothesis of Theorem 11.1.4 in [16], p. 264. We deduce that, for every $f \in C_b([0, \infty); \mathbb{R})$ and $s \in [0, \infty)$,

$$\lim_{u \rightarrow \infty} T_{u, u+s} f(z) = T_s f(z) \quad \text{uniformly in } z \text{ on compact subsets of } \mathbb{R}. \quad (5.6)$$

Moreover,

$$\lim_{s \rightarrow \infty} T_s f(z) = \Pi(f) \quad \text{uniformly in } z \text{ on compact subsets of } \mathbb{R}. \quad (5.7)$$

Indeed, assume that z belongs to the compact set $[b, c]$. By using the strong Markov property, we can prove that for all $s \in [0, \infty)$ and $v \in \mathbb{R}$,

$$\mathbb{P}_b(H_s > v) \leq \mathbb{P}_z(H_s > v) \leq \mathbb{P}_c(H_s > v).$$

By using the ergodic theorem and these last inequalities we get the uniform convergence on compact subsets of \mathbb{R} in (5.7). Besides, by the Markov property, for all $s, u \in [0, \infty)$,

$$T_{0, u+s} f(z_0) - \Pi(f) = T_{0, u} [T_{u, u+s} f - T_s f](z_0) + T_{0, u} [T_s f - \Pi(f)](z_0)$$

and clearly, for arbitrary r, s, u nonnegative real numbers,

$$|T_{0, u+s} f(z_0) - \Pi(f)| \leq \sup_{z \in [-r, r]} \{|T_{u, u+s} f(z) - T_s f(z)| + |T_s f(z) - \Pi(f)|\} + 4\|f\|_\infty \mathbb{P}_{z_0}(|Z_u| \geq r).$$

Thanks to (5.6) and (5.7), for all $r, \varepsilon > 0$ there exists $s_0, u_0 \in [0, \infty)$ such that for all $u \geq u_0$,

$$|T_{0, u+s_0} f(z_0) - \Pi(f)| \leq \varepsilon + 4\|f\|_\infty \sup_{s \geq 0} \mathbb{P}(|Z_s| \geq r).$$

Since Z is bounded in probability we deduce that $\lim_{u \rightarrow \infty} T_{0, u} f(z_0) = \Pi(f)$. \square

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