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# Asymptotic behaviour of a family of one-dimensional time-inhomogeneous diffusions

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**Abstract:** Let us consider a solution of the time-inhomogeneous stochastic differential equation driven by a Brownian motion with drift coefficient  $b(t, x) = \rho \operatorname{sgn}(x)|x|^\alpha/t^\beta$ . This process can be viewed as a distorted Brownian motion in a potential, possibly singular, depending on time. After obtaining results on existence and uniqueness of solution, we study its asymptotic behaviour and made a precise description, in terms of parameters  $\rho, \alpha$  and  $\beta$ , of the recurrence, transience and convergence. More precisely, asymptotic distributions, iterated logarithm type laws and rates of transience and explosion are proved for such processes.

**Key words:** time-inhomogeneous diffusions, singular stochastic differential equations, explosion times, scaling transformations and changes of time, recurrence and transience, iterated logarithm type laws, asymptotic distributions.

**2000 Mathematics Subject Classification:** 60J60, 60H10, 60J65, 60G17, 60F15, 60F05

## 1 Introduction

Let  $X$  be a one-dimensional process satisfying the following equation

$$X_t = x_0 + B_t + \rho \int_{t_0}^t \frac{\operatorname{sgn}(X_s)|X_s|^\alpha}{s^\beta} ds, \quad t \geq t_0. \quad (1.1)$$

Here  $B$  denotes a standard Brownian motion,  $x_0, \rho, \alpha, \beta$  are some real constants and  $t_0 > 0$ . In this paper we shall study the asymptotic behaviour of such process. More precisely, our main goal is to give conditions which characterise the recurrence, transience and convergence in terms of parameters  $\rho, \alpha$  and  $\beta$ .

The equation (1.1) can be viewed as a continuous counterpart of a discrete time model considered recently in [16] and which is related, when  $\alpha = \beta = 1$ , to the Friedman's urn model [7] but also to the random walk with an asymptotically zero drift studied in [14]. Indeed, the discrete time process studied in [16] is a random walk on the real half line such that

$$\mathbb{E}(X_{t+1} - X_t \mid X_t = x) \underset{t \rightarrow \infty}{\sim} \rho \frac{x^\alpha}{t^\beta}.$$

The authors establish when the process is recurrent or transient for certain values of parameters  $\rho, \alpha, \beta$  and present open problems. Their approach is based on a precise study of some submartingales and supermartingales.

In the one-dimensional case, for time-homogeneous stochastic differential equations driven by a Brownian motion, there exists precise criteria, using the scale function, for recurrence and transience (see for instance [12] or [8]) or explosion (see for instance [13]). Some of these criteria are extended to the time-inhomogeneous situation for dimension greater or equal than two in [3]. Unfortunately, the results in [3] do not apply to the equation (1.1), even it is stated that the method can be adapted to the one-dimensional case. Recall also that there exist some general results on recurrence or transience (see for instance [10]) and explosion (see for instance [20] or [18]) based on the construction of some convenient Lyapunov functions. However, for the equation (1.1), the construction of such functions seems to be more delicate.

Here are another two natural questions:

- i) Does there exist a well chosen normalisation of  $X$  to ensure that the normalised process converges in distribution or almost surely?
- ii) Is it possible to obtain the pathwise largest deviations of  $X$ , for instance the iterated logarithm law?

This kind of questions are treated in [1],[2] or [9] for different equations which have some common features with (1.1). The authors consider in [2] or [9] the following equation

$$dx_t = dB_t + F(x_t)dt, \quad \text{with}$$

$$\int_{-\infty}^{\infty} F(x)dx = 0 \quad (H_1) \quad \text{or} \quad F(x) \underset{x \rightarrow \infty}{\sim} \rho |x|^\alpha, \quad \rho \in (0, \infty) \quad \text{and} \quad \alpha \in [-1, 1) \quad (H_2).$$

Under additional assumptions (see [9]) the hypothesis  $(H_1)$  implies that  $x$  is recurrent and that the process  $x_t/\sqrt{t}$  converges in distribution to a standard Gaussian random variable. On the other hand, when  $\alpha \in (-1, 1)$ , hypothesis  $(H_2)$  ensures that  $x$  is transient and asymptotically behaves as a solution of the deterministic underlying dynamical system, that is

$$x_t \underset{t \rightarrow \infty}{\sim} h_t \quad \text{a.s.} \quad \text{with} \quad h'_t = F(h_t)$$

The case  $\alpha = -1$  (see [2]) is more difficult and its study is related to the Bessel process. In that case  $x$  satisfies the iterated logarithm law and recurrence or transience depends on the position of  $\rho$  with respect to  $1/2$ . Besides, the authors in [1] study the following damped stochastic differential equation

$$dx_t = \sigma(t)dB_t + F(x_t)dt, \quad \text{with} \tag{1.2}$$

$$\lim_{t \rightarrow \infty} \downarrow \sigma(t) = 0, \quad \sigma \in L^2 \quad \text{and} \quad F(x) \underset{x \rightarrow 0}{\sim} \rho \operatorname{sgn}(x)|x|^\alpha, \quad \rho \in (-\infty, 0) \quad \text{and} \quad \alpha \in (1, \infty).$$

It is proved that  $x_t$  converges almost surely to 0 with polynomial speed. We will see that equation (1.1) is connected to the last one by using a suitable change of time.

In this paper we firstly study existence, uniqueness and explosion of solutions for (1.1). Different situations are distinguished following the values of  $\rho$  and  $\alpha$  and existence and uniqueness are proved. We point out that when  $\alpha \in (-\infty, 0)$  the existence of a solution is not obvious since the drift has a singularity in 0. For the time-homogeneous case, a solution to this problem is given in [4] by using the scale function. These ideas does not apply to one-dimensional time-inhomogeneous stochastic differential equations and this is the main difficulty of this part. Our idea is to use an appropriate change of time, taking full advantage of the scaling

property of the Brownian motion, the Girsanov's transformation and the classification of isolated singular points in [4]. These different tools allow us to answer the question of explosion of the solution when  $\alpha \in (1, \infty)$ . We note that when  $2\beta \in (\alpha + 1, \infty)$  we shall prove that the solution explodes in finite time with a positive probability, but not almost surely.

Secondly, we describe for all values of parameters  $\rho, \alpha, \beta$ , the recurrent and the transient features of the solution but also its convergence. We present in Figure 1 below, the diagram of phase transition that we obtain in the attractive case  $\rho \in (-\infty, 0)$ :

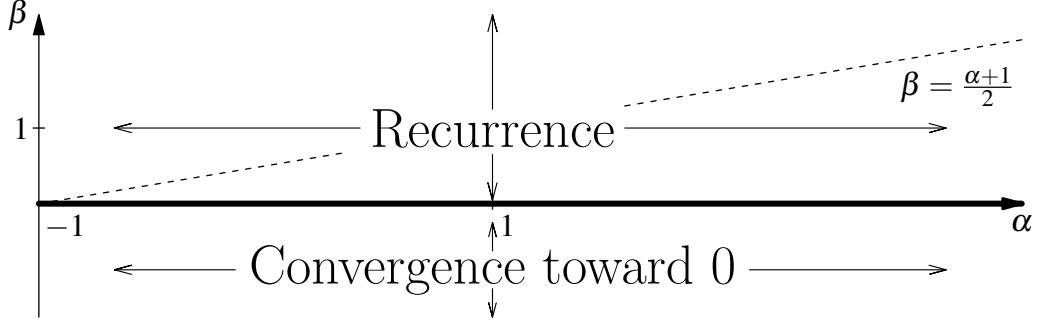


Figure 1: Phase transition in the attractive case  $\rho \in (-\infty, 0)$

Note that  $\alpha \in (-\infty, -1]$  is not allowed since in that case we will show that any solution is only defined up to the reaching time of 0, which is finite almost surely. The critical line separating the two phases (*Recurrence* and *Convergence toward 0*) is given by  $\beta = 0$  and on this line the process is recurrent. The line  $2\beta = \alpha + 1$  can be called subcritical in the sense that the recurrent behaviour is different below and above. As for the proof of the existence, we use a suitable change of time and an associated scaling transformation, we give the asymptotic distribution of  $X$  and its pathwise largest deviations under a convenient normalisation. In fact, we show that the asymptotic behaviour of the process is strongly connected to the paths and to the stationary distribution of an ergodic diffusion.

For instance, when  $2\beta \in (-\infty, \alpha + 1)$ , we show that

$$\frac{X_{\varphi(t)}}{\sqrt{\varphi'(t)}} \quad \text{"behaves as"} \quad H_t = B_t + \int_0^t \rho \operatorname{sgn}(H_s) |H_s|^\alpha ds, \quad \text{with} \quad \varphi'(t) = \varphi(t)^{\frac{2\beta}{\alpha+1}}.$$

In that case, we obtain the convergence in distribution of  $X_t/t^{\frac{\beta}{\alpha+1}}$  to the stationary distribution of  $H$  and its pathwise largest deviation. In particular, when  $\beta$  is negative, we get the polynomial stability of  $X$  and if we denote  $x_t := X(\phi_t)$ , with  $\phi_t := t^{\frac{1}{1-\beta}}$ , we see that  $x$  satisfies the damped stochastic differential equation (1.2) and we prove similar results as in [1] under slightly different hypothesis, but we try to be more accurate.

Then we present in Figure 2 below, the diagram of phase transition that we obtain in the repulsive case  $\rho \in (0, \infty)$ . We will remark that when  $\alpha \in (1, \infty)$  and  $2\beta \in (-\infty, \alpha + 1)$  the process explodes in finite time almost surely. Here the nontrivial critical line is given by  $2\beta = \alpha + 1$  and the process is either recurrent or transient and we obtain similar results as in [16], but again we try to be more accurate. On this line we must distinguish the two particular points  $(-1, 0)$  and  $(1, 1)$  because these are the only cases where recurrence and transience depend on the position of  $\rho$  with respect to  $1/2$ . The first corresponds to the well known Bessel process and the second one is a continuous time counterpart of the Friedman's urn

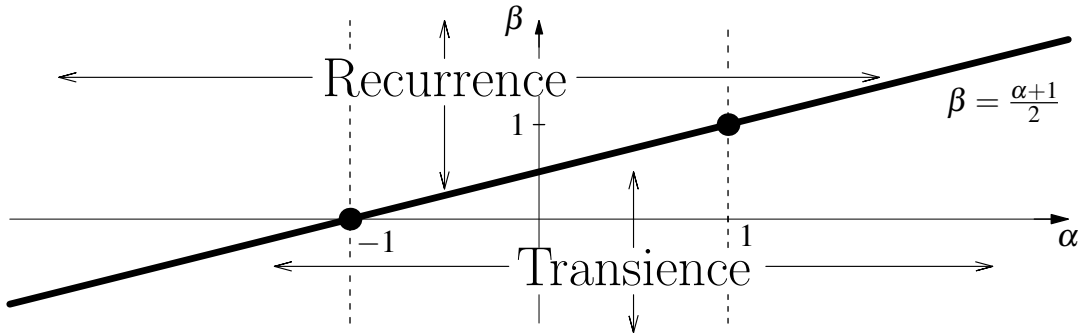


Figure 2: Phase transition in the repulsive case  $\rho \in (0, \infty)$

model. In this case, we obtain similar results as in [7] concerning recurrence and transience, but also regarding the asymptotic distribution and the pathwise largest deviations. For the other points of the critical line, the process is recurrent. We point out that this is an open problem in [16]. In the present paper there is no restriction on the parameters  $\rho, \alpha$  and  $\beta$ . The lines  $\alpha = -1$  and  $\alpha = 1$  can be called subcritical in the sense that the behaviour of the process is slightly different to the right or left, in particular, the domain of recurrence depends on  $\alpha$ . The proof of recurrence is based on the same ideas as for the attractive case by associating an ergodic diffusion to an appropriate transformation of  $X$ .

For instance, when  $2\beta \in (\alpha + 1, \infty)$  and  $\alpha \in (-1, 1)$ , we show that

$$\frac{X_{e^t}}{e^{\frac{t}{2}}} \quad \text{"behaves as"} \quad U_t = B_t - \int_0^t \frac{U_s}{2} ds.$$

In that case, we get that  $X$  behaves as a standard Brownian motion, that is it satisfies the iterated logarithm law and  $X_t/\sqrt{t}$  converge in distribution to a standard Gaussian random variable. We can say that the drift is asymptotically negligible compared to the noise in the opposite of the transient case. Concerning the proof of this one when  $\alpha \in (-\infty, 1)$ , the tools are similar to those used in [16]. We show that  $X$  behaves as a solution of the deterministic underlying dynamical system, that is

$$|X_t| \underset{t \rightarrow \infty}{\sim} |h_t| \quad \text{a.s.} \quad \text{with} \quad h'_t = \rho \operatorname{sgn}(h_t) |h_t|^\alpha / t^\beta.$$

Some of these results could be proved with similar arguments for a general drift  $b$  under additional assumptions, for instance when  $b(t, x) = f(t)g(x)$  with  $f(t) \sim_{t \rightarrow \infty} t^{-\beta}$  and  $g(x) \sim_{x \rightarrow \infty} \rho|x|^\alpha$ . The case of a general diffusion coefficient  $\sigma$  seems more difficult. Other interesting cases would be when one replaces the Brownian motion by a stable Lévy process or by a fractional Brownian motion. These generalisations will be studied elsewhere.

The paper is organised as follows: in the next section we also recall the important properties of three well-known processes which are solutions of (1.1) for particular values of parameters. In Section 3 we introduce the scaling transformation and we study the existence, uniqueness and explosions of solutions, while in Section 4 we analyse the asymptotic behaviour of the process.

## 2 Three particular processes

We shall study the equation (1.1) in its equivalent differential form:

$$dX_t = dB_t + \rho \operatorname{sgn}(X_t) \frac{|X_t|^\alpha}{t^\beta} dt, \quad X_{t_0} = x_0. \quad (2.1)$$

Here  $B$  is a standard Brownian motion defined on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ . By symmetry of the equation and by scaling transformation, we can assume without loss of generality that  $x_0 \geq 0$  and  $t_0 = 1$ . We will keep these assumptions all along the paper.

To begin with, let us recall the behaviour of three important processes which are solutions of (2.1) with some particular parameters.

### 2.1 The Brownian Motion and the Ornstein-Uhlenbeck Process

Firstly, if  $\rho = 0$  then  $X \equiv B$  is the standard Brownian motion and it is classical that  $B$  is recurrent and it satisfies

$$\lim_{t \rightarrow \infty} \frac{B_t}{\sqrt{t}} \stackrel{\mathcal{L}}{=} \mathcal{N}(0, 1) \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{B_t}{\sqrt{2t \ln \ln t}} = 1 \quad \text{a.s.} \quad (2.2)$$

Secondly, if  $\rho \neq 0$ ,  $\alpha = 1$  and  $\beta = 0$  then  $X \equiv U$  is the Ornstein-Uhlenbeck process. By using the harmonic function  $V(t, x) := e^{-\rho t} x$  it is not difficult to prove that  $U$  is recurrent if and only if  $\rho \in (-\infty, 0)$  and in that case it satisfies

$$\lim_{t \rightarrow \infty} U_t \stackrel{\mathcal{L}}{=} \mathcal{N}\left(0, \frac{1}{2|\rho|}\right) \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{U_t}{\sqrt{\ln t}} = \frac{1}{\sqrt{\rho}} \quad \text{a.s.}$$

Moreover, it is transient if and only if  $\rho \in (0, \infty)$  and in that case it satisfies

$$\lim_{t \rightarrow \infty} \frac{U_t}{e^{\rho t}} = G \quad \text{a.s.} \quad \text{with} \quad G \sim \mathcal{N}\left(\frac{x_0}{e^\rho}, \frac{e^{2\rho}}{2\rho}\right).$$

Let us note that the probabilities to be transient in  $\pm\infty$  can be computed.

### 2.2 The Bessel process

Finally, if  $\rho \in (0, \infty)$ ,  $\alpha = -1$  and  $\beta = 0$ , then there exists a pathwise unique nonnegative strong solution  $X \equiv R$  to the equation (2.1), it is the Bessel process of dimension  $2\rho + 1$  (see, for instance, Proposition A.21 in [4], p. 109). The square of any weak solution of the equation (2.1) is the unique weak solution of the equation

$$dr_t = 2\sqrt{|r_t|} dw_t + (2\rho + 1)dt, \quad r_1 = x_0^2.$$

Here  $w$  is a standard Brownian motion. This is the square Bessel process of dimension  $2\rho + 1$ . By using Theorems 3.4 and 3.5 in [4], p. 66, and Proposition 3.2 and corollary 3.4 in [19], pp. 359-360, using local times, we can prove the following properties:

- i) if  $\rho \in (0, 1/2)$  then we can construct different weak solutions to the equation (2.1) and the set  $\{t \geq 1 : R_t = 0\}$  is unbounded a.s.;

- ii) if  $\rho \in [1/2, \infty)$  and  $x_0 \in (0, \infty)$  then  $R$  is the pathwise unique strong solution to the equation (2.1) and it is positive;
- iii) if  $\rho \in [1/2, \infty)$  and  $x_0 = 0$  then, for all  $t > 0$ ,  $R_t > 0$  a.s. and the set of weak solutions to the equation (2.1) is the set of distributions which are mixture of the laws of  $\pm R$ .

Furthermore, the asymptotic behaviour can be described as follows:

$$\limsup_{t \rightarrow \infty} \frac{R_t}{\sqrt{2t \ln \ln t}} = 1 \quad \text{a.s.} \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{R_t}{\sqrt{t}} \stackrel{\mathcal{L}}{=} \sqrt{\Gamma}, \quad \text{with} \quad \Gamma \sim \gamma\left(\rho + \frac{1}{2}, 2\right). \quad (2.3)$$

Here  $\gamma$  denotes the gamma distribution. Moreover,

- i) if  $\rho \in (0, 1/2)$  then  $R$  is recurrent in  $[0, \infty)$ ;
- ii) if  $\rho = 1/2$  then  $R$  is recurrent in  $(0, \infty)$ ;
- iii) if  $\rho \in (1/2, \infty)$  then  $R$  is transient and it satisfies

$$\liminf_{t \rightarrow \infty} \frac{\ln\left(\frac{R_t}{\sqrt{t}}\right)}{\ln \ln t} = -\frac{1}{2\rho - 1} \quad \text{a.s.} \quad (2.4)$$

To obtain (2.3) it suffices to perform an appropriate change of time as in [2], p. 914 and to use Ito's formula in order to be reduced to the study of the time-homogeneous equation:

$$dZ_s = 2\sqrt{Z_s} dW_s + (2\rho + 1 - Z_s)ds, \quad Z_0 = x_0^2, \quad \text{with} \quad Z_s := R_{e^s}^2 / e^s.$$

Here  $W$  is a standard Brownian motion. By using Lemma 2.2 in [2], p. 916 and the ergodic theorem (see, for instance, Theorem 23.15 in [12], p. 465) both applied to  $Z$ , we get

$$\limsup_{t \rightarrow \infty} \frac{Z_t}{2 \ln t} = 1 \quad \text{a.s.} \quad \text{and} \quad \lim_{t \rightarrow \infty} Z_t \stackrel{\mathcal{L}}{=} \Gamma.$$

Then (2.3) is deduced by change of time whereas the recurrent or the transient features are explained in [19], p. 442. Finally, equation (2.4) is a consequence of Lemma 4.1 in [2], p. 926.

Let us note that when  $\rho \in (-\infty, 0)$ ,  $\alpha = -1$  and  $\beta = 0$ , the reaching time of 0 by  $X$  is finite a.s. (see for instance [19], pp. 442-453). Moreover, by using Theorem 3.2 in [4], p. 66, we can see that every solution can not be continued after it has reached 0.

### 3 Preliminary study of solutions

In this section we study existence and uniqueness of solutions for (2.1). To this end, we first introduce our main tool, the scaling transformation, which will be useful later to study the asymptotic behaviour.

### 3.1 Scaling transformation

Let us set  $\Omega := C([1, \infty); \mathbb{R})$ ,  $\Omega^* := C([0, t_1]; \mathbb{R})$ , with  $t_1 \in (0, \infty]$ . For every change of time ( $C^2$ -diffeomorphism)  $\varphi : [0, t_1] \rightarrow [1, \infty)$  we introduce the map  $\Phi_\varphi : \Omega \rightarrow \Omega^*$  given by

$$\Phi_\varphi(\omega)(s) := \frac{\omega(\varphi(s))}{\sqrt{\varphi'(s)}}, \quad s \in [0, t_1], \omega \in \Omega.$$

**Proposition 3.1.** *The scaling transformation  $\Phi_\varphi$  induces a bijection between weak solutions of equation (2.1) and weak solutions of equation*

$$d\mathfrak{X}_s = d\mathfrak{W}_s + \rho \frac{\varphi'(s)^{\frac{\alpha+1}{2}}}{\varphi(s)^\beta} \operatorname{sgn}(\mathfrak{X}_s) |\mathfrak{X}_s|^\alpha ds - \frac{\varphi''(s)}{\varphi'(s)} \frac{\mathfrak{X}_s}{2} ds, \quad \mathfrak{X}_0 = \frac{x_0}{\sqrt{\varphi'(0)}}. \quad (3.1)$$

Here  $\{\mathfrak{W}_s : s \in [0, t_1]\}$  denotes a standard Brownian motion. More precisely,

i) if  $(X, B)$  is a solution of equation (2.1) then  $(\mathfrak{X}, \mathfrak{W})$  is a solution of equation (3.1) where

$$\mathfrak{X} = \Phi_\varphi(X) \quad \text{and} \quad \mathfrak{W}_s := \int_0^s \frac{dB(\varphi(u))}{\sqrt{\varphi'(u)}}; \quad (3.2)$$

ii) if  $(\mathfrak{X}, \mathfrak{W})$  is a solution of equation (3.1) then  $(X, B)$  is a solution of equation (2.1) where

$$X = \Phi_\varphi^{-1}(\mathfrak{X}) \quad \text{and} \quad B_t - B_1 := \int_1^t \sqrt{\varphi' \circ \varphi^{-1}(u)} d\mathfrak{W}(\varphi^{-1}(u)). \quad (3.3)$$

It follows that uniqueness in law, pathwise uniqueness or strong existence holds for equation (2.1) if and only if it holds for equation (3.1).

**Proof.** Let  $X$  a solution (possibly explosive) of the equation (2.1) with a standard Brownian motion  $B$ . By using P. Lévy's characterisation theorem we can see that  $\mathfrak{W}$  defined in (3.2) is a standard Brownian motion. Moreover,

$$dX_{\varphi(s)} = \sqrt{\varphi'(s)} d\mathfrak{W}_s + \rho \operatorname{sgn}(X_{\varphi(s)}) \sqrt{\varphi'(s)} |X_{\varphi(s)}|^\alpha / \varphi(s)^\beta ds.$$

We conclude that equation (3.1) is satisfied by  $\mathfrak{X} := \Phi_\varphi(X)$  by the integration by parts' formula. The proof of (3.3) is similar and the final remark is a simple application.  $\square$

### 3.2 Two particular changes of time

In the following results we provide some changes of time which produce time-homogeneous coefficients for each term of the drift in (3.1).

**Corollary 3.1.** *Assume that  $\alpha \neq -1$  and consider the Cauchy problem:*

$$\varphi'_\gamma(s) = (\varphi_\gamma(s))^\gamma, \quad \varphi_\gamma(0) = 1, \quad \text{with} \quad \gamma := \frac{2\beta}{\alpha+1}.$$

*There exists a unique maximal solution  $\varphi_\gamma \in C^2([0, t_1]; [1, \infty))$ . Moreover,*

i) if  $\gamma > 1$  then  $\varphi_\gamma(s) = (1 + (1 - \gamma)s)^{\frac{1}{1-\gamma}}$ , with  $t_1 = \frac{1}{\gamma-1}$ ;



ii) if  $\gamma = 1$  then  $\varphi_\gamma(s) = e^s$ , with  $t_1 = \infty$ ;

iii) if  $\gamma < 1$  then  $\varphi_\gamma(s) = (1 + (1 - \gamma)s)^{\frac{1}{1-\gamma}}$ , with  $t_1 = \infty$ .

The scaling transformation  $\Phi_\gamma : \Omega \rightarrow \Omega^*$  associated to this change of time is given by

$$\Phi_\gamma(\omega)(s) = \frac{\omega(\varphi_\gamma(s))}{\varphi_\gamma(s)^{\frac{\gamma}{2}}}, \quad s \in [0, t_1), \omega \in \Omega.$$

Moreover, the equation (3.1) can be written

$$d\mathfrak{h}_s = dW_s + \rho \operatorname{sgn}(\mathfrak{h}_s)|\mathfrak{h}_s|^\alpha ds - \gamma \varphi_\gamma(s)^{\gamma-1} \frac{\mathfrak{h}_s}{2} ds, \quad \mathfrak{h}_0 = x_0, \quad \text{with } \mathfrak{h} := \Phi_\gamma(X). \quad (3.4)$$

**Corollary 3.2.** *The scaling transformation  $\Phi_e : \Omega \rightarrow \Omega^*$  associated to the exponential change of time  $\varphi_e(t) := e^t$  is given by*

$$\Phi_e(\omega)(s) = \frac{\omega(e^s)}{e^{s/2}}, \quad s \in [0, \infty), \omega \in \Omega.$$

Moreover, the equation (3.1) can be written

$$d\mathfrak{u}_s = dW_s - \frac{\mathfrak{u}_s}{2} ds + \rho e^{(\frac{\alpha+1}{2}-\beta)s} \operatorname{sgn}(\mathfrak{u}_s)|\mathfrak{u}_s|^\alpha ds, \quad \mathfrak{u}_0 = x_0, \quad \text{with } \mathfrak{u} := \Phi_e(X) \quad (3.5)$$

### 3.3 Existence, uniqueness and explosion

The existence and uniqueness for equation (2.1) are not obvious since the drift could be singular in 0 and/or not time-homogeneous. However, with the help of the scaling transformation defined in Corollary 3.1 and the Girsanov's transformation, we will reduce to study the time-homogeneous part in equation (3.4) and we will use the classification of isolated singular points in [4].

We first study the case where  $\alpha \in (-1, \infty)$ . Here the function  $x \mapsto |x|^\alpha$  is locally integrable in a neighbourhood of 0. As for the time-homogeneous case (see Proposition 2.2 in [4], p. 28) we show that there exists a unique pathwise strong solution to the equation (2.1) up to the explosion time.

**Proposition 3.2.** *Assume that  $\alpha \in (-1, \infty)$  and  $\beta \in \mathbb{R}$ . There exists a pathwise unique strong solution  $X$  to the equation (2.1) defined up to the explosion time  $\tau_e$ . Moreover, if  $\rho \in (-\infty, 0)$  (the attractive case) or  $\alpha \in (-1, 1]$  (the linear growth case) then  $\tau_e$  is infinite a.s.*

**Proof.** Assume that  $\alpha \in (1, \infty)$ . We construct a truncated stochastic differential equation as follows: for every integer  $n \geq 1$  set  $b_n(t, x) := \rho \operatorname{sgn}(x)(|x| \wedge n)^\alpha / t^\beta$  and

$$dX_t^{(n)} = dB_t + b_n(t, X_t^{(n)})dt, \quad X_1^{(n)} = x_0.$$

This equation has locally Lipschitz coefficients with linear growth. Let  $X^{(n)}$  be its pathwise unique strong solution. Let us introduce

$$\tau_n := \inf\{t \geq 1 : |X_t^{(n)}| \geq n\}, \quad n \geq n_0 \quad \text{and} \quad \tau_e := \sup\{\tau_n : n \geq 1\}.$$

For all  $t \in [1, \tau_e)$  we set  $X_t := X_t^{(n)}$  a.s. where  $n$  is chosen such that  $t \in [1, \tau_n]$ . The process  $X$  is well defined up to the explosion time  $\tau_e$  and it is the desired solution for (2.1).

Moreover, when  $\rho \in (-\infty, 0)$ , we can prove by using a Lyapunov function that the explosion time of the solution is infinite a.s. Indeed, let us consider the infinitesimal generator of the space-time diffusion  $t \mapsto (t, X_t)$  given by

$$\mathcal{L} := \partial_t + \rho \operatorname{sgn}(x) \frac{|x|^\alpha}{t^\beta} \partial_x + \frac{1}{2} \partial_{xx}. \quad (3.6)$$

The non-negative function defined by  $V(x) := x^2 + 1$  satisfies

$$\lim_{|x| \rightarrow \infty} V(x) = \infty \quad \text{and} \quad \mathcal{L}V \leq V.$$

By using Theorem 10.2.1 in [20], p. 254, we deduce that the explosion time is infinite a.s.

Assume that  $\alpha \in [0, 1)$ . Then, weak existence and uniqueness in law are consequences of the Girsanov's theorem, the Novikov's criterion being satisfied. To verify this criterion, let  $w$  be a standard Brownian motion starting from  $x_0$ , set  $b(s, x) := \rho \operatorname{sgn}(x)|x|^\alpha/s^\beta$  and fix  $t \in (1, \infty)$ . By the Jensen's inequality, applied with the convex function  $\exp$ :

$$\mathbb{E} \left[ \exp \left( \frac{1}{2} \int_1^t b^2(s, w_s) ds \right) \right] \leq \int_1^t \mathbb{E} \left[ \frac{e^{\lambda |w_s|^{2\alpha}}}{t-1} \right] dt < \infty, \quad \text{with} \quad \lambda := \sup_{1 \leq s \leq t} \frac{(t-1)|b(s, 1)|}{2}.$$

The Novikov's criterion holds. Pathwise uniqueness is a consequence of Proposition 3.2 and Corollary 3.4 in [19], pp. 359-360, using local times.

Assume now that  $\alpha \in (-1, 0)$ . By using Proposition 3.1 and Corollary 3.1 it suffices to study equation (3.4) and prove that there exists a pathwise unique strong solution  $\mathfrak{h} = \Phi_\gamma(X)$  which is nonexploding. First, let us introduce the time-homogeneous part in (3.4):

$$dH_s = dW_s + \rho \operatorname{sgn}(H_s)|H_s|^\alpha ds. \quad (3.7)$$

Since  $x \mapsto |x|^\alpha$  belongs to  $L_{\text{loc}}^1(\mathbb{R})$  there exists a unique weak solution for (3.7) with  $H_0 = x_0$  defined up to the explosion time (see Proposition 2.2 in [4], p. 28). By using Theorem 5.5 in [4], p. 97, concerning power equations, this explosion time is infinite a.s. Second, consider for every integer  $n \geq 1$ , the truncated equation

$$d\mathfrak{h}_s^{(n)} = dW_s + \rho \operatorname{sgn}(\mathfrak{h}_s^{(n)})|\mathfrak{h}_s^{(n)}|^\alpha ds - \gamma \varphi^{\gamma-1}(s) \frac{\mathfrak{h}_s^{(n)}}{2} \mathbb{1}_{\{|\mathfrak{h}_s^{(n)}| \leq n\}} ds, \quad \mathfrak{h}_0^{(n)} = x_0. \quad (3.8)$$

By applying the Girsanov's theorem between (3.7) and (3.8), we can deduce existence and uniqueness in law of (nonexploding) solutions  $\mathfrak{h}^{(n)}$  of (3.8) and pathwise uniqueness follows again from Proposition 3.2 and Corollary 3.4 in [19], pp. 359-360. Hence, given a Brownian motion  $\{W_s : s \in [0, t_1)\}$ , there exists a pathwise unique strong solutions  $(\mathfrak{h}^{(n)}, W)$ ,  $n \geq 1$ . By extension, we get a pathwise unique strong solution  $(\mathfrak{h}, W)$  of (3.4) defined up to the explosion time. We conclude that there exists a pathwise unique strong solution  $X = \Phi_\gamma^{-1}(\mathfrak{h})$  to equation (2.1) defined up to the explosion time, which is infinite a.s., by using again Theorem 10.2.1 in [20], p. 254 and the Lyapunov function  $V(x) := x^2 + 1$ .

Finally, when  $\alpha = 1$ , the result is clear since the coefficients of equation (2.1) are locally Lipschitz functions with linear growth. This ends the proof of the proposition.  $\square$

Let us complete the previous proposition with a result concerning the explosion time of the solution when  $\rho \in (0, \infty)$  and  $\alpha \in (1, \infty)$ .

**Proposition 3.3.** Assume that  $\rho \in (0, \infty)$ ,  $\alpha \in (1, \infty)$  and  $\beta \in \mathbb{R}$  and let us consider  $\tau_e$  the explosion time of  $X$ . The following properties hold:

- i) if  $2\beta \in (-\infty, \alpha + 1]$  then  $\tau_e$  is finite a.s.;
- ii) if  $2\beta \in (\alpha + 1, \infty)$  then  $\tau_e$  satisfies  $\mathbb{P}(\tau_e = \infty) \in (0, 1)$ .

**Proof.** We use Proposition 3.1 and Corollary 3.1. Assume that  $2\beta \in (-\infty, \alpha + 1]$ . It suffices to show that  $\mathfrak{h} = \Phi_\gamma(X)$  explodes in finite times a.s. To this end, by Ito's formula

$$d\mathfrak{h}_s^2 = 2\sqrt{\mathfrak{h}_s^2} dw_s + (2\rho(\mathfrak{h}_s^2)^{\frac{\alpha+1}{2}} - \gamma\varphi(s)^{\gamma-1}\mathfrak{h}_s^2 + 1) ds, \quad \mathfrak{h}_0^2 = x_0^2.$$

Here  $w$  is a Brownian motion. Let us introduce  $Z$  the pathwise unique strong solution of

$$dZ_s = 2\sqrt{Z_s} dw_s + (2\rho Z_s^{\frac{\alpha+1}{2}} - |\gamma|Z_s + 1) ds, \quad Z_0 = x_0^2.$$

Since  $0 \leq \varphi^{\gamma-1} \leq 1$  (remark that  $\gamma \leq 1$ ) we can show by using the comparison theorem (Theorem 1.1 in [11], p. 437) that  $Z \leq \mathfrak{h}^2$  a.s. Besides, by using Theorem 5.7 in [4], p. 97, the explosion time of the time-homogeneous diffusion  $Z$  is finite a.s. This implies that the explosion time of  $\mathfrak{h}$ , and furthermore of  $X$ , is also finite a.s.

Assume now that  $2\beta \in (\alpha + 1, \infty)$  (remark that  $\gamma > 1$ ). Equation (3.4) can be written

$$d\mathfrak{h}_s = dW_s - \frac{\delta \mathfrak{h}_s}{t_1 - s} ds + \rho \operatorname{sgn}(\mathfrak{h}_s) |\mathfrak{h}_s|^\alpha ds, \quad \mathfrak{h}_0 = x_0, \quad \text{with} \quad \delta := \frac{\gamma}{2(\gamma - 1)}. \quad (3.9)$$

Denote by  $\eta_e$  its explosion time and note that a.s.  $\eta_e \in [0, t_1] \cup \{\infty\}$  and  $\{\eta_e \geq t_1\} = \{\tau_e = \infty\}$ . We show that  $\mathbb{P}(\eta_e \geq t_1) \in (0, 1)$ . Let us introduce the time-inhomogeneous part in (3.9):

$$db_s = dW_s - \frac{\delta b_s}{t_1 - s} ds, \quad b_0 = x_0. \quad (3.10)$$

There exists a unique pathwise strong solution  $b$  to the equation (3.10) which is a continuous stochastic process on  $[0, t_1]$ , with  $b_1 = 0$  a.s. (see for instance [15]). By using the Girsanov's transformation between (3.9) and (3.10) (their truncation), we can write for every integer  $n \geq 1$ ,  $s \in [0, t_1]$  and  $\mathcal{F}_s$ -measurable bounded function  $F_s$ ,

$$\mathbb{E} [F_s(\mathfrak{h}_{\cdot \wedge \eta_n}) \mathbf{1}_{\{\eta_n > s\}}] = \mathbb{E} [F_s(b_{\cdot \wedge \sigma_n}) \mathcal{E}(s \wedge \sigma_n) \mathbf{1}_{\{\sigma_n > s\}}]$$

where

$$\eta_n := \inf\{s \in [0, t_1] : |\mathfrak{h}_s| \geq n\}, \quad \sigma_n := \inf\{s \in [0, t_1] : |b_s| \geq n\}$$

and

$$\mathcal{E}(s) := \exp \left( \int_0^s \rho \operatorname{sgn}(b_u) |b_u|^\alpha dW_u - \int_0^s \frac{\rho^2 |b_u|^{2\alpha}}{2} du \right).$$

Letting  $n \rightarrow \infty$ , we obtain

$$\mathbb{E} [F_s(\mathfrak{h}) \mathbf{1}_{\{\eta_e > s\}}] = \mathbb{E} [F_s(b) \mathcal{E}(s)]. \quad (3.11)$$

In particular, we have proved that for all  $s \in [0, t_1]$ ,  $\mathbb{P}(\eta_e > s) = \mathbb{E}[\mathcal{E}(s)]$  and furthermore it is clear that  $\mathbb{P}(\tau_e = \infty) = \mathbb{P}(\eta_e \geq t_1) \geq \mathbb{E}[\mathcal{E}(t_1)] > 0$ .

To complete the proof, it remains to show that  $\mathbb{P}(\tau_e = \infty) < 1$ . We shall apply Theorem 10.2.1 in [20], p. 254. Set

$$V_{\mu,c}(x) := \exp \left( \mu \int_0^x \frac{y}{\rho|y|^{\alpha+1} + c} dy \right), \quad \text{with } \mu, c > 0.$$

Note that there exists  $c_0 \geq 2^{|\beta|}$  such that for all  $\mu > 0$ ,  $\sup\{V_{\mu,c_0}(x) : x \in \mathbb{R}\} < V(x_0)e^1$ . Moreover, we can see that there exists  $n_0 \geq 0$  such that for all  $\mu > 0$ ,  $t \in [1, 2]$  and  $|x| \geq n_0$ ,

$$\mathcal{L}V_{\mu,c_0}(t, x) = \mu \left[ \frac{\rho|x|^{\alpha+1}}{t^\beta(\rho|x|^{\alpha+1} + c_0)} + \frac{\mu x^2 + c_0 - \alpha\rho|x|^{\alpha+1}}{2(\rho|x|^{\alpha+1} + c_0)^2} \right] V_{\mu,c_0}(x) \geq \frac{\mu}{2c_0} V_{\mu,c_0}(x). \quad (3.12)$$

Here  $\mathcal{L}$  is given by (3.6). Then, there exists  $\mu_0 > 0$  such that for all  $t \in [1, 2]$  and  $|x| \leq n_0$ ,

$$\mathcal{L}V_{\mu_0,c_0}(t, x) \geq \frac{\mu_0 c_0}{2(\rho n_0^{\alpha+1} + c_0)^2} V_{\mu_0,c_0}(x) \geq V_{\mu_0,c_0}(x). \quad (3.13)$$

By using (3.12) and (3.13) we deduce that  $\mathcal{L}V_{\mu_0,c_0} \geq V_{\mu_0,c_0}$  on  $[1, 2] \times \mathbb{R}$  and Theorem 10.2.1 in [20], p. 254 applies. Hence,  $X$  explodes in finite time with positive probability.  $\square$

Let us consider now the case where  $\alpha \in (-\infty, -1)$ . Here the function  $x \mapsto |x|^\alpha$  is not integrable in a neighbourhood of 0. As for the time-homogeneous situation we show that when  $\rho \in (0, \infty)$ , there exists a unique pathwise nonnegative strong solution.

**Proposition 3.4.** *Assume that  $\rho \in (0, \infty)$ ,  $\alpha \in (-\infty, -1)$  and  $\beta \in \mathbb{R}$ . There exists a pathwise unique nonnegative strong solution  $X$  to the equation (2.1) defined for all  $t \geq 1$ . Moreover,*

- i) *if  $x_0 \in (0, \infty)$  then  $X$  is the pathwise unique strong solution and it is positive;*
- ii) *if  $x_0 = 0$  then for all  $t > 1$ ,  $X_t > 0$  a.s. and the set of all weak solutions is the set of all distributions which are mixture of the distributions of  $X$  and  $-X$ .*

**Proof.** Assume first that  $x_0 \in (0, \infty)$ . The proof is based on the same arguments as for the proof of Proposition 3.2, case  $\alpha \in (-1, 0)$ , and we present here only the main lines. We prove that equation (3.4) admits a pathwise unique strong solution  $\mathfrak{h}$  defined up to its explosion time by using the Girsanov's transformation between equations (3.4) and (3.7) (their truncation) and by noting that there exists a pathwise unique strong solution  $H$ , with  $H_0 = x_0$ , to the equation (3.7) (see Theorem 2.16 in [4], p. 37). Then, we show by using the Lyapunov function  $V(x) := 1 + |x|^{\alpha+1}$  and Theorem 10.2.1 in [20], p. 254, that the explosion time  $\eta_e$  of  $\mathfrak{h}$  satisfies  $\eta_e < t_1$  a.s. Note that  $H$  is positive and since Girsanov's transformation gives a locally equivalent measure, it is the same for  $\mathfrak{h}$ . We deduce that there exists a pathwise unique solution  $X = \Phi_\gamma^{-1}(\mathfrak{h})$  to the equation (2.1) which is nonexploding.

Secondly, assume that  $x_0 = 0$ . Similarly, we get that equation (3.4) admits a pathwise unique nonnegative strong solution  $\mathfrak{h}$  with for all  $s > 0$ ,  $\mathfrak{h}_s > 0$  a.s. and we obtain that the set of all weak solutions is given by the set of all distributions which are mixture of distributions of  $\mathfrak{h}$  and  $-\mathfrak{h}$  since it is the same for the time-homogeneous equation (3.7) by using Theorem 3.6 in [4], p. 66. We conclude again by using the scaling transformation  $\Phi_\gamma$ .  $\square$

It remains to study the case  $\rho \neq 0$ ,  $\alpha = -1$  and  $\beta \neq 0$ . Let us introduce the equation

$$dr_s = 2\sqrt{|r_s|}dw_s + \left(2\rho/s^\beta + 1\right)ds, \quad r_1 = x_0^2. \quad (3.14)$$

Here  $w$  denotes a standard Brownian motion. As for the Bessel process, the square of any weak solution of the equation (2.1) is a weak solution of equation (3.14). We show that if  $\rho \in (0, \infty)$  we have existence and uniqueness of a nonnegative solution for (2.1). This process can be viewed as a Bessel process whose dimension depends on time.

**Proposition 3.5.** *Assume that  $\rho \in (0, \infty)$ ,  $\alpha = -1$  and  $\beta \neq 0$ . There exists a pathwise unique nonnegative strong solution  $X$  to the equation (2.1) defined for all  $t \geq 1$ . Moreover,*

- i) if  $\rho \in [1/2, \infty)$ ,  $\beta \in (-\infty, 0)$  and  $x_0 \in (0, \infty)$  then  $X$  is the pathwise unique strong solution and it is positive;*
- ii) if  $\beta \in (0, \infty)$  or if  $\rho \in (0, 1/2)$  and  $\beta \in (-\infty, 0)$  then we can construct different solutions to the equation (2.1) and in the first case, the set  $\{t \geq 1 : X_t = 0\}$  is unbounded a.s.;*
- iii) if  $\rho \in [1/2, \infty)$ ,  $\beta \in (-\infty, 0)$  and  $x_0 = 0$  then the set of all solutions is the set of all distributions which are mixture of distributions of  $X$  and  $-X$ .*

**Proof.** The existence of a pathwise unique strong solution  $r$  of the equation (3.14) can be obtained by using Theorem 3.5 in [19], p. 390. Moreover, since the drift is nonnegative, by using the comparison theorem (see Theorem 1.1 in [11], p. 437) we get that  $r$  is nonnegative. We shall prove that  $X := r^{\frac{1}{2}}$  is a nonnegative weak solution of equation (2.1). To this end, by applying Ito's formula, we obtain for all  $t \geq 1$  and  $\varepsilon > 0$ ,

$$(X_t^2 + \varepsilon)^{\frac{1}{2}} = (x_0^2 + \varepsilon)^{\frac{1}{2}} + \int_1^t \left( \frac{X_s^2}{X_s^2 + \varepsilon} \right)^{\frac{1}{2}} dw_s + \int_1^t \frac{\rho ds}{s^\beta (X_s^2 + \varepsilon)^{\frac{1}{2}}} + \int_1^t \frac{\varepsilon ds}{2(X_s^2 + \varepsilon)^{\frac{3}{2}}}. \quad (3.15)$$

We need to let  $\varepsilon \rightarrow 0$  in the last equation. Firstly, it is clear that

$$\lim_{\varepsilon \rightarrow 0} \int_1^t \left( \frac{X_s^2}{X_s^2 + \varepsilon} \right)^{\frac{1}{2}} dw_s = w_t - w_1 \quad \text{in probability.}$$

Secondly, by monotone convergence theorem, the third term in the right hand side of (3.15) converges a.s. We show that the limit is finite a.s. and that the forth term converges toward 0 in probability. To this end, we consider (see section 2 on Bessel processes) the pathwise unique nonnegative strong solution of

$$R_s = x_0 + w_s - w_1 + \int_1^s \frac{\rho_m}{R_u} du, \quad \text{with} \quad \rho_m := \inf\{\rho/s^\beta : s \in [1, t]\} > 0.$$

By using the comparison theorem (see Theorem 1.1 in [11], p. 437) and Ito's formula, we get that for all  $s \in [1, t]$ ,  $X_s^2 \geq R_s^2$  and

$$(R_t^2 + \varepsilon)^{\frac{1}{2}} = (x_0^2 + \varepsilon)^{\frac{1}{2}} + \int_1^t \left( \frac{R_s^2}{R_s^2 + \varepsilon} \right)^{\frac{1}{2}} dw_s + \int_1^t \frac{\rho_m ds}{(R_s^2 + \varepsilon)^{\frac{1}{2}}} + \int_1^t \frac{\varepsilon ds}{2(R_s^2 + \varepsilon)^{\frac{3}{2}}}.$$

By letting  $\varepsilon \rightarrow 0$  in the last equality and the previous comparison, we obtain

$$\int_1^t \frac{\rho ds}{s^\beta X_s} \leq \int_1^t \frac{\rho_M}{R_s} ds < \infty \quad \text{a.s.} \quad \text{with} \quad \rho_M := \sup\{\rho/s^\beta : s \in [1, t]\} > 0.$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_1^t \frac{\varepsilon ds}{2(X_s^2 + \varepsilon)^{\frac{3}{2}}} \leq \lim_{\varepsilon \rightarrow 0} \int_1^t \frac{\varepsilon ds}{2(R_s^2 + \varepsilon)^{\frac{3}{2}}} = 0 \quad \text{in probability.}$$

We deduce that  $X$  is a nonnegative weak solution of (2.1), which is unique in law, since uniqueness in law holds for (3.14) and as previously it is possible to obtain its pathwise uniqueness by using local times. We proceed with the proof of statements i)-iii) in the proposition.

Firstly, if  $\rho \in [1/2, \infty)$ ,  $\beta \in (-\infty, 0)$  and  $x_0 \in (0, \infty)$  then  $X$  is the pathwise unique strong solution since every solution is positive by comparison with a Bessel process of dimension 2.

Secondly, if  $\beta \in (0, \infty)$  then  $\lim_{t \rightarrow \infty} \rho/t^\beta = 0^+$  and by using the Markov's property and the comparison with a Bessel process of dimension  $\delta \in (1, 2)$  we get that the reaching time of 0 is finite a.s. and that the set  $\{t > 1 : X_t = 0\}$  is unbounded a.s. Besides, if  $X$  is a solution starting from  $x_0 = 0$  then  $-X$  is also a solution. We deduce that different solutions could be constructed by gluing the paths of  $X$  or of  $-X$  each time when the process returns in 0.

Thirdly, if  $\rho \in (0, 1/2)$  and  $\beta \in (-\infty, 0)$  then by comparison with a Bessel process of dimension  $\delta \in (1, 2)$  the reaching time of 0 belongs to  $[1, (2\rho)^{1/\beta})$  with a positive probability. Again, as in the preceding case, different solutions can be constructed.

Finally, if  $\rho \in [1/2, \infty)$ ,  $\beta \in (-\infty, 0)$  and  $x_0 = 0$  then by comparison with a Bessel process of dimension 2 every solution  $Z$  of (2.1) satisfies  $Z_t^2 \neq 0$  for all  $t > 1$  a.s. Let us introduce

$$\Omega^\pm := \{\omega \in \Omega : \forall t > 1, \pm Z_t > 0\} \quad \text{and} \quad \mathbb{P}^\pm := \mathbb{P}(\cdot | \Omega^\pm).$$

Note that for all  $\varepsilon > 0$ ,  $\Omega^\pm = \{\omega \in \Omega : \forall 1 < t < 1 + \varepsilon, \pm Z_t > 0\} \in \mathcal{F}_{t+\varepsilon}$  and then  $\Omega^\pm \in \mathcal{F}_{1+}$ . Therefore the process  $\{B_t - B_1\}_{t \geq 1}$  is again a standard Brownian motion under probabilities  $\mathbb{P}^\pm$  and by uniqueness of the nonnegative weak solution and also, by symmetry, of the non-positive solution of (2.1), the distribution of  $Z$  under  $\mathbb{P}^\pm$  equals to the distribution of  $\pm X$ , where  $X$  is the pathwise unique nonnegative strong solution. We deduce that the set of all weak solutions is exactly the set of all distributions which are mixture of distributions of  $X$  and  $-X$ . This completes the proof of the proposition.  $\square$

**Remark 1.** The pathwise strong solution  $X$  that we have constructed all along this section satisfies the strong Markov property. Indeed, when  $\alpha \in (-1, \infty)$ , uniqueness in law hold for (2.1) and therefore we can apply Theorem 21.11 in [12], p. 421. If  $\alpha \in (-\infty, -1]$  the cited result does not apply directly because it may exist different solutions. Nevertheless, our claim is true since the result applies to the stochastic differential equation verified by  $|X|^{\alpha+1}$ .  $\square$

**Remark 2.** Recall that in the attractive case  $\rho \in (-\infty, 0)$ , when  $\alpha = -1$  and  $\beta = 0$ , every solution of (2.1) can not be continued after it has reached 0. The same result is true when  $\alpha \in (-\infty, -1]$  and  $\beta$  arbitrary. First of all, the reaching time of 0 is finite a.s. when  $\rho \in (-\infty, 0)$  since by comparison theorem (see Theorem 1.1 in [11], p. 437), every solution  $X$  satisfies  $X^2 \leq w^2$ , with  $w$  a standard Brownian motion starting from  $x_0$  at time  $t_0$ .

Assume that  $\alpha = -1$  and let us consider a solution  $X$  of (2.1) starting from 0 at initial time 1. Then  $X^2$  is the pathwise unique solution of (3.14) with  $X_1^2 = 0$  a.s. Besides, for every

$t > 1$ , it is not difficult to see that there exists a Bessel  $R$  process of dimension  $2\rho_M + 1 < 1$ , with  $\rho_M := \sup\{\rho/s^\beta : 1 \leq s \leq t\}$ , such that  $X_s^2 \leq R_s^2$  for all  $s \in [1, t]$ . By using Corollary 2.2, p. 199 in [5], we obtain

$$\int_0^t \frac{1}{|X_s|} ds \geq \int_0^t \frac{1}{R_s} ds = \infty \quad \text{a.s.}$$

We deduce, when  $\rho \in (-\infty, 0)$  and  $\alpha = -1$ , that any solution of (2.1) can not be continued after the reaching time of 0 by using the strong Markov property.

In the case where  $\rho \in (-\infty, 0)$ ,  $\alpha \in (-\infty, -1)$  and  $\beta \in \mathbb{R}$ , by using Proposition 3.1 and the Girsanov's transformation, we reduce to the study of equation (3.7). Since for this equation every solution can not be continued after it has reached 0 (see Theorem 3.2 in [4], p. 66), we deduce that it is the same for  $\mathfrak{h} = \Phi_\gamma(X)$  and furthermore for  $X$ .  $\square$

## 4 Asymptotic behaviour of solutions

We present here the systematic study of the recurrence, transience or convergence of the solution  $X$  of (2.1) defined up to its explosion time  $\tau_e$ . We also give the asymptotic behaviour of its sample paths. For latter reference and to simplify our statements let us introduce:

**Definition 4.1.** *We shall say that:*

1.  *$X$  has the Gaussian typical behaviour or  $X$  satisfies the iterated logarithm law (under a given probability  $\mathbb{P}$ ) when it respectively satisfies*

$$\lim_{t \rightarrow \infty} \frac{|X_t|}{\sqrt{t}} \stackrel{\mathcal{L}}{=} |G| \quad \text{under } \mathbb{P}, \quad \text{with } G \sim \mathcal{N}(0, 1) \quad (4.1)$$

or

$$\limsup_{t \rightarrow \infty} \frac{|X_t|}{\sqrt{2t \ln \ln t}} = 1 \quad \mathbb{P}\text{-a.s.} \quad (4.2)$$

2.  *$X$  is recurrent (under a given probability  $\mathbb{P}$ ) when, for every  $x \in E_\alpha$ , the following set  $\{t \geq 1 : X_t = x\}$  is  $\mathbb{P}$ -a.s. unbounded. Here we denote  $E_\alpha = \mathbb{R}, (0, \infty)$  or  $[0, \infty)$  according to whether  $\alpha \in (-1, \infty), \alpha \in (-\infty, -1)$  or  $\alpha = -1$ .*
3.  *$X$  is transient (under a given probability  $\mathbb{P}$ ) when  $\lim_{t \rightarrow \tau_e} |X_t| = \infty$   $\mathbb{P}$ -a.s.*

**Remark 3.** When  $\alpha \in (-1, \infty)$ , by symmetry of equation (2.1), we can remove the absolute values in (4.1) and replace  $X$  by  $-X$ . Moreover, when there are no confusion, we leave out the probability  $\mathbb{P}$ .  $\square$

We split our analysis in four cases. The first case is essentially based on the behaviour of the Brownian motion and the results are simple to prove. For the time-homogeneous case and for the general case, the proofs are more technical. From this point of view, the general case contains a number of ideas which are simpler to be explained in the time-homogeneous situation. The Bessel type case ( $\alpha = -1$ ) can be handled by the same techniques as for the general case.

#### 4.1 Linear case: $\rho \neq 0$ , $\alpha = 1$ and $\beta$ arbitrary

The Ornstein-Uhlenbeck process ( $\beta = 0$ ) has already been treated in Section 2. More generally, when  $\alpha = 1$ ,  $X$  is again a Gaussian process and it can be expressed as a time changed Brownian motion. The result is obtained by finding a harmonic function for the space-time diffusion  $t \mapsto (t, X_t)$ . First, we consider a particular case which is the analogue, in continuous time, of the Friedman's urn model [7]. Note that we obtain similar results.

**Proposition 4.1.** *Assume that  $\rho \neq 0$  and  $\alpha = \beta = 1$ .*

1.  *$X$  is recurrent if and only if  $\rho \in (-\infty, 1/2]$ . Moreover,*

*i) if  $\rho \in (-\infty, 1/2)$  then it has the Gaussian typical behaviour (4.1) and it satisfies*

$$\limsup_{t \rightarrow \infty} \frac{X_t}{\sqrt{2t \ln \ln t}} = \frac{1}{\sqrt{1-2\rho}} \quad a.s.;$$

*ii) if  $\rho = 1/2$  then it satisfies*

$$\lim_{t \rightarrow \infty} \frac{X_t}{\sqrt{t \ln t}} \stackrel{\mathcal{L}}{=} \mathcal{N}(0, 1) \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{X_t}{\sqrt{2t \ln t \ln \ln t}} = 1 \quad a.s.$$

2.  *$X$  is transient if and only if  $\rho \in (1/2, \infty)$  and it satisfies*

$$\lim_{t \rightarrow \infty} \frac{X_t}{t^\rho} = G \quad a.s. \quad \text{with} \quad G \sim \mathcal{N}\left(x_0, \frac{1}{2\rho-1}\right).$$

**Proof.** Let us note that  $(t, x) \mapsto x/t^\rho$  is a harmonic function for the space-time diffusion  $t \mapsto (t, X_t)$ . By Ito's formula and P. Lévy's characterisation, we can write

$$\frac{X_t}{t^\rho} = x_0 + \int_1^t \frac{dB_s}{s^\rho} = x_0 + \tilde{B}_{\phi(t)} \quad \text{with} \quad \phi(t) := \begin{cases} \frac{t^{1-2\rho}-1}{1-2\rho} & \text{if } \rho \neq 1/2; \\ \ln t & \text{if } \rho = 1/2. \end{cases}$$

Here  $\tilde{B}$  denotes a standard Brownian motion. By using (2.2) we prove the convergence in distribution but also the pathwise largest deviations. The recurrent or the transient features are simple consequences.  $\square$

We proceed with the case where  $\beta$  is arbitrary.

**Proposition 4.2.** *Assume that  $\rho \neq 0$ ,  $\alpha = 1$  and  $\beta \neq 1$ .*

1.  *$X$  is recurrent if and only if  $\beta \in (1, \infty)$  or  $\rho \in (-\infty, 0)$  and  $\beta \in [0, 1)$ . Moreover,*

*i) if  $\beta \in (1, \infty)$  then it has the Gaussian typical behaviour (4.1) and it satisfies the iterated logarithm law (4.2);*

*ii) if  $\rho \in (-\infty, 0)$  and  $\beta \in [0, 1)$  then it satisfies*

$$\lim_{t \rightarrow \infty} \frac{X_t}{t^{\frac{\beta}{2}}} \stackrel{\mathcal{L}}{=} \mathcal{N}\left(0, \frac{1}{2|\rho|}\right) \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{X_t}{t^{\frac{\beta}{2}} (\ln t)^{\frac{1}{2}}} = \left(\frac{\beta-1}{\rho}\right)^{\frac{1}{2}} \quad a.s. \quad (4.3)$$



2.  $X$  is transient if and only if  $\rho \in (0, \infty)$  and  $\beta \in (-\infty, 1)$  and it satisfies

$$\lim_{t \rightarrow \infty} \frac{X_t}{\exp\left(\frac{\rho t^{1-\beta}}{1-\beta}\right)} = G \quad a.s. \quad \text{with} \quad G \sim \mathcal{N}(m, \sigma^2),$$

where

$$m := x_0 \exp\left(\frac{\rho}{\beta-1}\right) \quad \text{and} \quad \sigma^2 := \int_1^\infty \exp\left(\frac{2\rho s^{1-\beta}}{\beta-1}\right) ds.$$

3.  $X$  converges almost surely toward 0 if and only if  $\rho \in (-\infty, 0)$  and  $\beta \in (-\infty, 0)$  and, in that case, (4.3) holds.

**Proof.** The same ideas as for the latter proof are employed. We use the harmonic function  $(t, x) \mapsto x \exp(\rho t^{1-\beta}/(\beta-1))$  and L'Hôpital's rule to obtain the precise equivalent for the change of time given by the Dambis-Dubins-Schwarz theorem (see Theorem 1.7 in [19], p. 182). The proof is left to the reader.  $\square$

**Remark 4.** In the linear case, we can compute the probability to be transient at  $\pm\infty$ .  $\square$

## 4.2 Homogeneous case: $\rho \neq 0$ , $\beta = 0$ and $\alpha$ arbitrary

The Bessel process ( $\alpha = -1$ ) has already been treated in the Section 2, whereas the linear case ( $\alpha = 1$ ) has been studied previously. In the time-homogeneous situation,  $X$  is either positive-recurrent, null-recurrent or transient, the proof of these features being based on the properties of scale function. Limits in distribution can be obtained by the ergodic theorem together with, in the null-recurrent case, an appropriate change of time and the scaling transformation defined in Corollaries 3.1 and 3.2. Furthermore, to get the asymptotic properties of the sample paths, it suffices to apply Motoo's theorem (see, for instance, [17]). Let us recall this result because it will be used several times in the following.

**Theorem 4.1** (Motoo). *Let  $X$  be a positive-recurrent continuous strong Markov process whose state space is  $[a, \infty)$  (or  $(a, \infty)$ ),  $a \in [-\infty, \infty)$  and with scale function  $s$ . Then, for every increasing real function  $h$  converging toward  $\infty$ ,*

$$\mathbb{P}\left(\limsup_{t \rightarrow \infty} \frac{X_t}{h(t)} \geq 1\right) = 0 \text{ or } 1 \quad \text{according to whether} \quad \int^\infty \frac{dt}{s(h(t))} < \infty \text{ or } = \infty.$$

**Proposition 4.3.** *Assume that  $\rho \neq 0$ ,  $\alpha \neq \pm 1$  and  $\beta = 0$ .*

1.  $X$  is positive-recurrent if and only if  $\rho \in (-\infty, 0)$  and  $\alpha \in (-1, \infty)$  and it satisfies

$$\lim_{t \rightarrow \infty} X_t \stackrel{\mathcal{L}}{=} \Pi \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{X_t}{(\ln t)^{\frac{1}{\alpha+1}}} = \left(\frac{\alpha+1}{2|\rho|}\right)^{\frac{1}{\alpha+1}} \quad a.s. \quad (4.4)$$

where  $\Pi$  is the speed measure associated to  $X$ , normalised to be a probability.

2.  $X$  is null-recurrent if and only if  $\rho \in (0, \infty)$  and  $\alpha \in (-\infty, -1)$ . It has the Gaussian typical behaviour (4.1) and it satisfies the iterated logarithm law (4.2).

3.  $X$  is transient if and only if  $\rho \in (0, \infty)$  and  $\alpha \in (-1, \infty) \setminus \{1\}$  and it satisfies

$$\lim_{t \rightarrow \infty} \frac{|X_t|}{t^{\frac{1}{1-\alpha}}} = (\rho(1-\alpha))^{\frac{1}{1-\alpha}} \quad \text{or} \quad \lim_{t \rightarrow \tau_e} \frac{|X_t|}{(\tau_e - t)^{\frac{1}{\alpha-1}}} = (\rho(\alpha-1))^{\frac{1}{\alpha-1}} \quad a.s. \quad (4.5)$$

according to whether  $\alpha \in (-1, 1)$  or  $\alpha \in (1, \infty)$ .

**Remark 5.** By using the properties of the scale function, it is possible to compute the probability to be transient at  $\pm\infty$ .  $\square$

**Proof.** When  $\rho \in (0, \infty)$  and  $\alpha \in (-1, 1)$  the first equality (4.5) is a direct consequence of Theorems 2 and 5 in [9], pp. 126-132, while the second equality is a consequence of (4.25) which is proved in Theorem 4.3 below.

Assume now that  $\rho \in (-\infty, 0)$  and  $\alpha \in (-1, \infty)$ . The scale function and the speed measure of  $X$  (see, for instance [13], p. 343) are given by

$$s(x) := \int_0^x \exp\left(-\frac{2\rho|y|^{\alpha+1}}{\alpha+1}\right) dy \quad \text{and} \quad m(dx) := \exp\left(\frac{2\rho|x|^{\alpha+1}}{\alpha+1}\right) dx.$$

Since  $s$  is a bijection of  $\mathbb{R}$ , we can conclude (see, for instance, Proposition 5.22 in [13], p. 345) that  $X$  is a recurrent process. Furthermore,  $m$  being a finite measure on  $\mathbb{R}$ ,  $X$  is positive-recurrent and we obtain the first equality in (4.4) by using the ergodic theorem (see for instance, Theorem 23.15 in [12], p. 465). It remains to prove the second equality in (4.4). To this end, we shall apply Motoo's theorem. Firstly, by using L'Hôpital's rule, observe that

$$s(x) \underset{x \rightarrow +\infty}{\sim} \frac{\exp(-2\rho x^{\alpha+1}/(\alpha+1))}{2|\rho|x^\alpha} \quad \text{and set} \quad \lambda_0 := \left(\frac{\alpha+1}{2|\rho|}\right)^{\frac{1}{\alpha+1}}.$$

Let us introduce  $h_\lambda(t) := \lambda(\ln t)^{\frac{1}{\alpha+1}}$ , with  $\lambda > 0$ . A simple computation gives

$$\int^\infty \frac{dt}{s(h_\lambda(t))} < \infty \text{ or } = \infty \quad \text{according to whether} \quad \lambda > \lambda_0 \quad \text{or} \quad \lambda < \lambda_0.$$

Then, Motoo's theorem applies, and we deduce that for all  $\varepsilon > 0$ ,

$$\mathbb{P}\left(\limsup_{t \rightarrow \infty} \frac{X_t}{h_\lambda(t)} \geq 1 + \varepsilon\right) = 0 \quad \text{and} \quad \mathbb{P}\left(\limsup_{t \rightarrow \infty} \frac{X_t}{h_\lambda(t)} \geq 1 - \varepsilon\right) = 1. \quad (4.6)$$

That is the second equality in (4.4).

Finally, assume that  $\rho \in (0, \infty)$  and  $\alpha \in (-\infty, -1)$ . As previously we obtain that  $X$  is recurrent in  $(0, \infty)$  since  $s$  is a bijection between  $(0, \infty)$  and  $\mathbb{R}$ . Furthermore, it is null-recurrent since its speed measure is not finite on  $(0, \infty)$ . To prove the convergence in distribution we shall use the following important

**Lemma 4.1.** *Suppose that  $Z$  and  $\zeta$  are the unique strong Markovian solution of the stochastic differential equations with continuous coefficients:*

$$dZ_s = a(s, Z_s)dW_s + b(s, Z_s)ds, \quad d\zeta_s = a_\infty(\zeta_s)dW_s + b_\infty(\zeta_s)ds.$$

*Assume that the equation satisfied by  $Z$  is asymptotically time-homogeneous and  $\Pi$ -ergodic:*

$$\lim_{s \rightarrow \infty} a(s, z) = a_\infty(z) \quad \text{and} \quad \lim_{s \rightarrow \infty} b(s, z) = b_\infty(z) \quad \text{uniformly on compact subsets of } \mathbb{R}$$

and

$$\lim_{s \rightarrow \infty} \zeta_s \stackrel{\mathcal{L}}{=} \Pi.$$

Then, if for all  $\varepsilon > 0$  there exists  $r > 0$  such that  $\sup\{\mathbb{P}(|Z_s| \geq r) : s \geq 0\} < \varepsilon$ , that is  $Z$  is bounded in probability,

$$\lim_{s \rightarrow \infty} Z_s \stackrel{\mathcal{L}}{=} \Pi.$$

We postpone the proof of this lemma and finish the proof of Proposition 4.3. Let us consider  $\mathbf{u} = \Phi_e(X)$  defined in Proposition (3.2). It is clear that  $X$  has the Gaussian typical behaviour if and only if

$$\lim_{s \rightarrow \infty} \mathbf{u}_s^{|\alpha|+1} \stackrel{\mathcal{L}}{=} \Pi \stackrel{\mathcal{L}}{=} |G|^{|\alpha|+1}, \quad \text{with } G \sim \mathcal{N}(0, 1). \quad (4.7)$$

By using Ito's formula, we can see that  $\mathbf{u}^{|\alpha|+1}$  satisfies

$$d\mathbf{u}_s^{|\alpha|+1} = a(\mathbf{u}_s^{|\alpha|+1})dW_s + h(\mathbf{u}_s^{|\alpha|+1})ds + \ell(s)ds, \quad \text{with}$$

$$a(z) := (|\alpha| + 1)z^{\frac{|\alpha|}{1+|\alpha|}}, \quad 2h(z) := (|\alpha| + 1)(|\alpha|z^{\frac{|\alpha|-1}{|\alpha|+1}} - z) \quad \text{and} \quad \ell(s) := \rho(|\alpha| + 1)e^{\left(\frac{\alpha+1}{2}\right)s}.$$

Note that the coefficients  $a, h$  and  $\ell$  are continuous and that  $\lim_{u \rightarrow \infty} \ell(u) = 0 \leq \ell \leq \ell(0)$ . We can see that  $\mathbf{u}^{|\alpha|+1}$  is asymptotically time-homogeneous and  $\Pi$ -ergodic. Indeed, the first part of the claim is clear and the time-homogeneous diffusion associated to  $\mathbf{u}^{|\alpha|+1}$  is the pathwise unique strong solution of the equation

$$d\zeta_s = a(\zeta_s)dW_s + h(\zeta_s)ds, \quad \zeta_0 = \mathbf{u}_0^{|\alpha|+1}.$$

Moreover, by using Ito's formula, we can see that  $\zeta$  is equal in law to  $U^{|\alpha|+1}$ , where  $U$  is an Ornstein-Uhlenbeck process, weak solution of

$$dU_t = dW_t - \frac{U_t}{2}dt. \quad (4.8)$$

We deduce that  $\mathbf{u}^{|\alpha|+1}$  is asymptotically  $\Pi$ -ergodic, where  $\Pi$  is given in (4.7). To apply Lemma 4.1 it remains to prove that  $\mathbf{u}^{|\alpha|+1}$  is bounded in probability. Let us introduce the pathwise unique strong solution of

$$d\xi_s = a(\xi_s)dW_s + h(\xi_s)ds + \ell(0)ds, \quad \xi_0 = \mathbf{u}_0^{|\alpha|+1}.$$

By applying the comparison theorem (see Theorem 1.1 in [11], p. 437) it can be shown that  $0 \leq \zeta \leq \mathbf{u}^{|\alpha|+1} \leq \xi$  a.s. Moreover, by using the ergodic theorem (see, for instance, Theorem 23.15 in [12], p. 465) the diffusions  $\zeta$  and  $\xi$  are positive-recurrent. Since ergodic diffusions are bounded in probability, we deduce the boundedness in probability of  $\mathbf{u}^{|\alpha|+1}$ . Furthermore, we obtain (4.7) from Lemma 4.1 and  $X = \Phi_e^{-1}(\mathbf{u})$  satisfies the Gaussian typical behaviour (4.1).

Finally, to obtain the iterated logarithm law for  $X$ , it suffices to show the equalities in

$$1 = \limsup_{s \rightarrow \infty} \frac{\xi_s^{\frac{1}{|\alpha|+1}}}{\sqrt{2 \ln s}} \leq \limsup_{s \rightarrow \infty} \frac{\mathbf{u}_s}{\sqrt{2 \ln s}} \leq \limsup_{s \rightarrow \infty} \frac{\zeta_s^{\frac{1}{|\alpha|+1}}}{\sqrt{2 \ln s}} = 1 \quad \text{a.s.}$$

We repeat the reasoning developed in the proof of (4.6), by applying Motoo's theorem to the diffusions  $\zeta$  and  $\xi$ , with the functions  $h_\lambda(t) := \lambda(\ln t)^{\frac{|\alpha|+1}{2}}$ ,  $\lambda > 0$ . This ends the proof of the

proposition, excepted for the Lemma (4.1).  $\square$

**Proof of Lemma 4.1.** Let us denote  $\mathbb{P}_{u,z}$  the distribution of the diffusion  $Z$  with  $Z_u = z$  and  $\{T_{u,s} : 0 \leq u \leq s\}$  the associated time-inhomogeneous semi-group. Similarly, we denote  $\mathbb{P}_z$  the distribution of the diffusion  $\zeta$  starting from  $z$  at initial time and  $\{T_s : s \geq 0\}$  the associated semi-group. Clearly, the diffusion coefficient  $(s, z) \mapsto a(u + s, z)$  and the drift  $(s, z) \mapsto b(u + s, z)$  of the diffusion  $s \mapsto Z_{u+s}$  satisfy the hypothesis of Theorem 11.1.4 in [20], p. 264. We deduce that, for every  $f \in C_b([0, \infty); \mathbb{R})$  and  $s \in [0, \infty)$ ,

$$\lim_{u \rightarrow \infty} T_{u,u+s}f(z) = T_sf(z) \quad \text{uniformly in } z \text{ on compact subsets of } \mathbb{R}. \quad (4.9)$$

Moreover,

$$\lim_{s \rightarrow \infty} T_sf(z) = \Pi(f) \quad \text{uniformly in } z \text{ on compact subsets of } \mathbb{R}. \quad (4.10)$$

Indeed, assume that  $z$  belongs to the compact set  $[a, b]$ . By using the strong Markov property, we can prove that for all  $s \in [0, \infty)$  and  $v \in \mathbb{R}$ ,

$$\mathbb{P}_a(\zeta_s > v) \leq \mathbb{P}_z(\zeta_s > v) \leq \mathbb{P}_b(\zeta_s > v).$$

By using the ergodic theorem and these last inequalities we get the uniform convergence on compact subsets of  $\mathbb{R}$  in (4.10). Besides, by the Markov property, for all  $s, u \in [0, \infty)$ ,

$$T_{0,u+s}f(z_0) - \Pi(f) = T_{0,u}[T_{u,u+s}f - T_sf](z_0) + T_{0,u}[T_sf - \Pi(f)](z_0)$$

and clearly, for arbitrary  $r, s, u$  non-negative real numbers,

$$|T_{0,u+s}f(z_0) - \Pi(f)| \leq \sup_{z \in [-r, r]} \{|T_{u,u+s}f(z) - T_sf(z)| + |T_sf(z) - \Pi(f)|\} + 4\|f\|_\infty \mathbb{P}(|Z_u| \geq r).$$

Thanks to (4.9) and (4.10), for all  $r, \varepsilon > 0$  there exists  $s_0, u_0 \in [0, \infty)$  such that for all  $u \geq u_0$ ,

$$|T_{0,u+s_0}f(z_0) - \Pi(f)| \leq 2(1 + 2\|f\|_\infty)\varepsilon + 4\|f\|_\infty \sup_{s \geq 0} \mathbb{P}(|Z_s| \geq r).$$

Since  $Z$  is bounded in probability we deduce that  $\lim_{u \rightarrow \infty} T_{0,u}f(z_0) = \Pi(f)$ . The proof of the lemma is done.  $\square$

### 4.3 General case

Let us introduce the following time-homogeneous diffusion:

$$dQ_s = dW_s + \rho \operatorname{sgn}(Q_s)|Q_s|^\alpha ds - \frac{Q_s}{2} ds. \quad (4.11)$$

Recall that  $H$  and  $U$  are the diffusions respectively defined in (3.7) and (4.8) by

$$dH_s = dW_s + \rho \operatorname{sgn}(H_s)|h_s|^\alpha ds \quad \text{and} \quad dU_s = dW_s - \frac{U_s}{2} ds.$$

We set, when it exists,

$$\lim_{s \rightarrow \infty} Q_s \stackrel{\mathcal{L}}{=} \Lambda, \quad \lim_{s \rightarrow \infty} H_s \stackrel{\mathcal{L}}{=} \Pi, \quad \text{and recall that} \quad \lim_{t \rightarrow \infty} U_s = \mathcal{N}(0, 1). \quad (4.12)$$

**Theorem 4.2.** Assume that  $\rho \in (-\infty, 0)$ ,  $\alpha \in (-1, \infty)$  and  $\beta \in \mathbb{R}$ .

1.  $X$  is recurrent if and only if  $\beta \in [0, \infty)$ . Moreover,

i) if  $2\beta = \alpha + 1$  then it satisfies

$$\lim_{t \rightarrow \infty} \frac{X_t}{\sqrt{t}} \stackrel{\mathcal{L}}{=} \Lambda. \quad (4.13)$$

It satisfies the iterated logarithm law (4.2) when  $\alpha \in (-1, 1)$  and

$$\limsup_{t \rightarrow \infty} \frac{X_t}{(t \ln \ln t)^{\frac{1}{\alpha+1}}} = \left( \frac{\alpha+1}{2|\rho|} \right)^{\frac{1}{\alpha+1}} \quad \text{a.s.} \quad \text{when } \alpha \in (1, \infty); \quad (4.14)$$

ii) if  $2\beta \in (\alpha+1, \infty)$  then it has the Gaussian typical behaviour (4.1) and it satisfies the iterated logarithm law (4.2);

iii) if  $2\beta \in [0, \alpha+1)$  then it satisfies

$$\lim_{t \rightarrow \infty} \frac{X_t}{t^{\frac{\beta}{\alpha+1}}} \stackrel{\mathcal{L}}{=} \Pi \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{X_t}{t^{\frac{\beta}{\alpha+1}} (\ln t)^{\frac{1}{\alpha+1}}} = \left( \frac{\alpha+1-2\beta}{2|\rho|} \right)^{\frac{1}{\alpha+1}} \quad \text{a.s.} \quad (4.15)$$

2.  $X$  converges almost surely toward 0 if and only if  $\beta \in (-\infty, 0)$  and it satisfies (4.15).

**Proof of Theorem 4.2.** We use again Proposition (3.1) and Corollaries (3.1) and (3.2) and we keep their notation. Assume first that  $2\beta = \alpha + 1$  and note that  $\Phi_\gamma(X) = \Phi_e(X) = Q$ , where  $Q$  satisfies (4.11). Therefore, by applying the ergodic theorem to  $Q$  we get immediately the convergence in distribution (4.13). Besides, by using Motoo's theorem as in (4.6) with

$$h_\lambda(t) := \lambda(\ln t)^{\frac{1}{\alpha+1}}, \quad \lambda > 0,$$

we obtain the pathwise largest deviation of  $Q$  and by change of time we deduce (4.2) and (4.14). The recurrent feature is a consequence of these equalities.

The proof of the case when  $2\beta \in (\alpha+1, \infty)$  is based on the same arguments as in the second point of Proposition 4.3. We present only the different points and leave the details to the reader. This time,  $X$  has the Gaussian typical behaviour if and only if

$$\lim_{s \rightarrow \infty} \mathbf{u}_s^3 \stackrel{\mathcal{L}}{=} G^3, \quad \text{with } G \sim \mathcal{N}(0, 1) \quad \text{and} \quad \mathbf{u} = \Phi_e(X). \quad (4.16)$$

We apply Lemma 4.1 to  $\mathbf{u}^3$ , which verifies a stochastic differential equation having continuous coefficients. By using Ito's formula and the comparison theorem (see Theorem 1.1 in [11], pp. 352–353) it is not difficult to prove that there exists an Ornstein-Uhlenbeck process  $U$ , weak solution of (4.8) with  $U_0 = \mathbf{u}_0$ , such that  $\mathbf{u}^2 \leq U^2$  a.s. (here we use  $\rho \in (-\infty, 0)$ ). Then we deduce that  $\mathbf{u}^3$  is bounded in probability since  $U$  is ergodic and furthermore bounded in probability. Moreover, it is a simple calculation to see that  $\mathbf{u}$  is asymptotically homogeneous (here we use  $2\beta \in (-\infty, \alpha+1)$ ) and that its associated time-homogeneous diffusion is equal in law to  $U^3$ . Hence, we deduce (4.16) and then, the Gaussian typical behaviour of  $X = \Phi_e^{-1}(\mathbf{u})$ . Besides, by using the previous comparison and Lemma 2.2 in [2], we obtain

$$\limsup_{s \rightarrow \infty} \frac{\mathbf{u}_s}{\sqrt{2 \ln s}} \leq \limsup_{s \rightarrow \infty} \frac{U_s}{\sqrt{2 \ln s}} = 1 \quad \text{a.s.} \quad (4.17)$$

We need to prove the opposite inequality to obtain (4.2) for  $X$ . In fact, we show that the time-inhomogeneous part in the equation satisfied by  $\mathbf{u}^2$  can be omitted. This implies that  $\mathbf{u}$  behaves as the Ornstein-Uhlenbeck  $U$ . First, by Ito's formula,

$$d\mathbf{u}_s^2 = 2\sqrt{\mathbf{u}_s^2} dw_s + (1 - \mathbf{u}_s^2) ds + 2\rho e^{(\frac{\alpha+1}{2}-\beta)s} (\mathbf{u}_s^2)^{\frac{\alpha+1}{2}} ds, \quad \mathbf{u}_0 = x_0.$$

Here  $w$  denotes a standard Brownian motion. Besides, by using (4.17), we get

$$\lim_{s \rightarrow \infty} q(s, \mathbf{u}_s) := \lim_{s \rightarrow \infty} 2\rho e^{(\frac{\alpha+1}{2}-\beta)s} (\mathbf{u}_s^2)^{\frac{\alpha+1}{2}} = 0 \quad \text{a.s.} \quad (4.18)$$

Second, let  $v \geq 0$  and denote by  $u^2(v)$  the pathwise unique nonnegative strong solution of

$$du_s^2(v) = 2\sqrt{u_s^2(v)} dw_s + (1/2 - u_s^2(v)) ds, \quad u_v^2(v) = \mathbf{u}_v^2.$$

We shall prove that,

$$\forall s \geq v, \quad u_s^2(v) \leq \mathbf{u}_s^2 \quad \text{a.s. on } \Omega_v := \left\{ \sup_{s \geq v} |q(s, \mathbf{u}_s)| \leq \frac{1}{2} \right\}. \quad (4.19)$$

To this end, let us introduce  $\tau := \inf \{s \geq v : |q(s, \mathbf{u}_s)| > 1/2\}$ . By using Theorem 1.1 in [11], pp. 352–353 and a classical argument of localisation, we can see that  $u_{\cdot \wedge \tau}^2(v) \leq \mathbf{u}_{\cdot \wedge \tau}^2$  a.s. and this exactly the comparison (4.19). Moreover, by using Lemma 2.2 in [2] and (4.18), we obtain respectively

$$\limsup_{s \rightarrow \infty} \frac{\mathbf{u}_s}{\sqrt{2 \ln s}} \geq \limsup_{s \rightarrow \infty} \frac{u_s(v)}{\sqrt{2 \ln s}} = 1 \quad \text{a.s. on } \Omega_v \quad \text{and} \quad \mathbb{P}(\cup_{v \geq 0} \Omega_v) = 1. \quad (4.20)$$

This implies the opposite inequality in (4.17). We conclude that  $X = \Phi_e^{-1}(\mathbf{u})$  satisfies the iterated logarithm law and we get the recurrent feature.

Finally, assume that  $2\beta \in (-\infty, \alpha + 1)$ . This proof is based on the same ideas as for the proof of case  $2\beta \in (\alpha + 1, \infty)$ . Here, to prove the convergence in distribution, it suffices to apply Lemma 4.1 to the process  $\mathfrak{h}^3 = \Phi_\gamma^3(X)$ . The only technical points is to show the boundedness in probability and to obtain its pathwise largest deviations. We only prove these two points. First, we can see that

$$d\mathfrak{h}_s^2 = 2\sqrt{\mathfrak{h}_s^2} dw_s + (1 + 2\rho (\mathfrak{h}_s^2)^{\frac{\alpha+1}{2}}) ds + \frac{k \mathfrak{h}_s^2}{c + s} ds, \quad \text{with } k \in \mathbb{R} \text{ and } c > 0.$$

Here  $w$  denotes a standard Brownian motion. Again, we show that the time-inhomogeneous part of  $\mathfrak{h}^2$  could be omitted. This implies that  $\mathfrak{h}$  behaves as  $H$  defined in (3.7). Indeed, since  $\rho$  is negative, it is not difficult to prove by comparison theorem (see Theorem 1.1 in [11], pp. 352–353) and Ito's formula that  $X^2$  is lower than a standard Brownian motion. Then, we get

$$\limsup_{s \rightarrow \infty} \frac{\mathfrak{h}_s^2}{2s \ln \ln s} = \limsup_{t \rightarrow \infty} \frac{X_t^2/t^\gamma}{2\varphi_\gamma^{-1}(t) \ln \ln \varphi_\gamma^{-1}(t)} = (1 - \gamma) \left( \limsup_{t \rightarrow \infty} \frac{X_t^2}{2t \ln \ln t} \right) \leq 1 - \gamma \quad \text{a.s.}$$

We deduce that

$$\lim_{s \rightarrow \infty} \left[ \frac{q(s, \mathfrak{h}_s^2)}{(\mathfrak{h}_s^2)^\delta} := \left( \frac{k \mathfrak{h}_s^2}{c + s} \right) \frac{1}{(\mathfrak{h}_s^2)^\delta} \right] = 0 \quad \text{a.s. with } \delta := \frac{\alpha + 1}{\alpha + 3}. \quad (4.21)$$

Let  $v \geq 0$  and denote by  $h^2(v)$  the pathwise unique nonnegative strong solution of

$$dh_s^2(v) = 2\sqrt{h_s^2} dw_s + (1 + (h_s^2(v))^\delta + 2\rho(h_s^2(v))^{\frac{\alpha+1}{2}}) ds, \quad h_v^2(v) = \mathfrak{h}_v^2.$$

As for the proof of inequality (4.19), by using Theorem 1.1 in [11], pp. 352–353, the comparison  $\mathfrak{h}_s^2 \leq h_s^2(v)$ ,  $s \geq v$ , holds a.s. on  $\Omega_v := \{\forall s \geq v, |q(s, \mathfrak{h}_s^2)| \leq (\mathfrak{h}_s^2)^\delta\}$ . Let  $\varepsilon > 0$ , then by (4.21) we can choose  $v \geq 0$  such that  $\mathbb{P}(\Omega_v) \leq \varepsilon$ . Moreover, there exists  $r \geq 0$  such that for all  $s \geq v$ ,  $\mathbb{P}(h_s^2(v) \geq r) \leq \varepsilon$  since  $h^2(v)$  is an ergodic diffusion. Putting together the last three inequalities, we obtain that  $\mathbb{P}(\mathfrak{h}_s^2 \geq r) \leq 2\varepsilon$ , for all  $s \geq v$  and therefore we conclude that  $\mathfrak{h}^3$  is bounded in probability.

Second, as for the proof of inequality (4.20), Motoo's theorem (apply to  $h(v)$ ,  $v \geq 0$ ) and the preceding comparison, imply

$$\limsup_{s \rightarrow \infty} \frac{\mathfrak{h}_s}{(\ln s)^{\frac{1}{\alpha+1}}} \leq \left( \frac{\alpha+1}{2|\rho|} \right)^{\frac{2}{\alpha+1}} \quad \text{a.s.} \quad (4.22)$$

The opposite inequality can be obtained in the same way as previously for (4.22) by using the cited comparison theorem with this time the pathwise unique nonnegative strong solution of

$$d\mathfrak{h}_s^2(v) = 2\sqrt{\mathfrak{h}_s^2(v)} dw_s + (1 - (\mathfrak{h}_s^2(v))^\delta + 2\rho(\mathfrak{h}_s^2(v))^{\frac{\alpha+1}{2}}) ds, \quad \mathfrak{h}_v^2(v) = \mathfrak{h}_v^2, \quad v \geq 0.$$

The proof is left to the reader. This implies the pathwise largest deviations in (4.15) and we deduce immediately that when  $2\beta \in [0, \alpha+1)$ ,  $X$  is a recurrent process, whereas when  $\beta \in (-\infty, 0]$ ,  $X$  converge almost surely toward 0. This ends the proof.  $\square$

We turn now to the repulsive case  $\rho \in (0, \infty)$ . The main difficulty is the case where  $\alpha \in (1, \infty)$  and  $2\beta \in (\alpha+1, \infty)$  because the process explodes with a positive probability but not almost surely. The transient case is proved by using some martingale arguments as in [16].

**Theorem 4.3.** *Assume that  $\rho \in (0, \infty)$ ,  $\alpha \in \mathbb{R} \setminus \{-1, 1\}$  and  $\beta \in \mathbb{R}$ .*

1.  *$X$  is recurrent, under the conditional probability of nonexplosion, if and only if we have  $2\beta \in (\alpha+1, \infty)$  or  $2\beta = \alpha+1$  and  $\alpha \in (-\infty, 1)$ . Moreover,*

*i) if  $2\beta \in (\alpha+1, \infty)$  then it has the Gaussian typical behaviour (4.1) and it satisfies the iterated logarithm law (4.2);*

*ii) if  $2\beta = \alpha+1$  and  $\alpha \in (-\infty, 1)$  then it satisfies the iterated logarithm law (4.2) and*

$$\lim_{t \rightarrow \infty} \frac{X_t}{\sqrt{t}} \stackrel{\mathcal{L}}{=} \Lambda. \quad (4.23)$$

2.  *$X$  is transient, under the conditional probability of explosion, if and only if we have  $2\beta \in (-\infty, \alpha+1)$  or  $2\beta = \alpha+1$  and  $\alpha \in (1, \infty)$ . Moreover,*

*i) if  $\alpha \in (-\infty, 1)$  then it satisfies*

$$\lim_{t \rightarrow \infty} \frac{|X_t|}{t^{\frac{1-\beta}{1-\alpha}}} = \left( \frac{\rho(1-\alpha)}{1-\beta} \right)^{\frac{1}{1-\alpha}} \quad \text{a.s.} \quad (4.24)$$

ii) if  $\alpha \in (1, \infty)$  then it satisfies

$$\lim_{t \rightarrow \tau_e} \frac{X_t}{(\tau_e - t)^{\frac{1}{\alpha-1}}} = \varphi_\gamma(\tau_e)^{\frac{\gamma}{2}} (\rho(\alpha - 1))^{\frac{1}{\alpha-1}} \quad \mathbb{P}(\cdot \mid \tau_e < \infty)\text{-a.s.} \quad (4.25)$$

**Proof.** 1) *Recurrence.* In this part we will use the results in Propositions 3.1 and Corollaries 3.1 and 3.2. On the one hand, if  $2\beta = \alpha + 1$  and  $\alpha \in (-\infty, 1)$  (the problem for the discrete time situation considered in [16]) or  $2\beta \in (\alpha + 1, \infty)$  and  $\alpha \in (-1, \infty)$ , the sentences in 1 i) and 1 ii) could be proved in a similar way as Theorem 4.2 points 1 i) and 1 ii). On the other hand, if  $2\beta \in (\alpha + 1, \infty)$  and  $\alpha \in (-\infty, -1)$ , it suffices to follow the reasoning already performed for the proof of the second point in Proposition 4.3.

It remains to study the case where  $\alpha \in (1, \infty)$ . Assume that  $2\beta \in (\alpha + 1, \infty)$ . We will need the following result whose proof is postponed to the end of this subsection.

**Lemma 4.2.** *Assume that  $\alpha \in (1, \infty)$  and  $2\beta \in (\alpha + 1, \infty)$ . Then,*

$$\lim_{t \rightarrow \infty} \frac{X_t}{t^{\frac{\beta}{\alpha+1}}} = 0 \quad \mathbb{P}(\cdot \mid \tau_e = \infty)\text{-a.s.} \quad (4.26)$$

By using Lemma 4.2, we can see that

$$\lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} \int_1^t \rho \frac{|X_s|^\alpha}{s^\beta} ds = \lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} \int_1^t \rho \left| \frac{X_s}{s^{\frac{\beta}{\alpha+1}}} \right|^\alpha t^{-\frac{\beta}{\alpha+1}} ds = 0 \quad \text{a.s. on } \{\tau_e = \infty\}.$$

We deduce that  $X$  satisfies, on the set of nonexplosion, the iterated logarithm law (4.2) and as a consequence it is recurrent on this set. We prove the Gaussian typical behaviour under the conditional probability of nonexplosion. For this end, it suffices to show that

$$\lim_{s \rightarrow \infty} \mathbb{P}(\mathbf{u}_s > x \mid \sigma_e = \infty) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{y^2}{2}\right) dy. \quad (4.27)$$

Here  $\sigma_e$  denotes the explosion time of  $\mathbf{u} = \Phi_e(X)$ . Note that Lemma 4.1 does not apply directly here since the explosion time of  $\mathbf{u}$  can be finite with a positive probability. We first show that the time-inhomogeneous part in (3.5) could be omitted. By using (4.26), we get

$$\lim_{s \rightarrow \infty} \left[ q(s, \mathbf{u}_s) := \rho e^{(\frac{\alpha+1}{2}-\beta)s} |\mathbf{u}_s|^\alpha \right] = \lim_{t \rightarrow \infty} \rho t^{(\frac{1}{2}-\frac{\beta}{\alpha+1})} \left| \frac{X_t}{t^{\frac{\beta}{\alpha+1}}} \right|^\alpha = 0 \quad \text{a.s. on } \{\sigma_e = \infty\}. \quad (4.28)$$

Let  $\varepsilon > 0, v \geq 0$  be and denote  $U^{(\pm\varepsilon)}$  the pathwise unique strong solution of the equation

$$dU_s^{(\pm\varepsilon)} = dW_s - \frac{U_s^{(\pm\varepsilon)}}{2} ds \pm \varepsilon ds, \quad U_v^{(\pm\varepsilon)} = \mathbf{u}_v \mathbf{1}_{\{\sigma_e > v\}}.$$

It is classical that  $U^{(\pm\varepsilon)}$  is Feller and ergodic. Furthermore, the strong mixing property holds (see [12], Theorem 20.20, p. 408) and we obtain

$$\lim_{s \rightarrow \infty} \mathbb{P}(U_s^{(\pm\varepsilon)} > x \mid \Omega_v^\varepsilon) =: F_{\mp\varepsilon}(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{(y \mp \varepsilon)^2}{2}\right) dy, \\ \text{with } \Omega_v^\varepsilon := \left\{ \sup_{s \geq v} |q(s, \mathbf{u}_s)| \leq \varepsilon \right\}. \quad (4.29)$$



As for the proof of (4.19), we can show that the comparison  $U_s^{(-\varepsilon)} \leq \mathbf{u}_s \leq U_s^{(+\varepsilon)}$  holds a.s. on  $\Omega_v^\varepsilon$ , for all  $s \geq v$ . We get from (4.29) and the previous comparison,

$$F_{+\varepsilon}(x) \leq \liminf_{s \rightarrow \infty} \mathbb{P} \left( \mathbf{u}_s > x \mid \Omega_v^{(\varepsilon)} \right) \leq \limsup_{s \rightarrow \infty} \mathbb{P} \left( \mathbf{u}_s > x \mid \Omega_v^{(\varepsilon)} \right) \leq F_{-\varepsilon}(x). \quad (4.30)$$

Due to (4.28), the set of non-explosion is  $\{\sigma_e = \infty\} = \cup_{v \geq 0} \Omega_v^\varepsilon$ . By passing to the limit in (4.30), first as  $v \rightarrow \infty$ , and then as  $\varepsilon \rightarrow 0$ , we deduce (4.27) and furthermore  $X$  has the Gaussian typical behaviour under the probability of nonexplosion.

2) *Transience.* To simplify our computations, let us denote the limit and the exponent of  $t$  in (4.24) respectively by

$$\ell := \left( \frac{\rho(1-\alpha)}{1-\beta} \right)^{\frac{1}{1-\alpha}} \quad \text{and} \quad \nu := \frac{1-\beta}{1-\alpha}.$$

If we set  $S_t := X_t^2/t^{2\nu}$ , it suffices to verify that  $\lim_{t \rightarrow \infty} S_t = \ell^2$  a.s. That is, for all  $\varepsilon > 0$ ,

$$\limsup_{t \rightarrow \infty} S_t \leq \ell^2 + 3\varepsilon \quad \text{and} \quad \liminf_{t \rightarrow \infty} S_t \geq \ell^2 - 3\varepsilon \quad \text{a.s.} \quad (4.31)$$

We shall prove only the first inequality in (4.31), the second one being obtained in a similar way. We split the proof in four steps.

*Step a).* We begin to show that, for all  $\varepsilon > 0$ ,

$$\{t \geq 1 : S_t \leq \ell^2 + \varepsilon\} \quad \text{is unbounded a.s.} \quad (4.32)$$

For this end, set  $\eta_u := \inf \{v \geq u : S_v \leq \ell^2 + \varepsilon\}$ ,  $u \geq 1$ . Then, it suffices to prove that for all  $u \geq 1$  large enough,  $\eta_u < \infty$  a.s. By using Ito's formula, we can see that

$$S_{t \wedge \eta_u} = S_u + \int_u^{t \wedge \eta_u} \mathcal{L}V(s, X_s) ds + \int_u^{t \wedge \eta_u} \partial_x V(s, X_s) dB_s := S_u + M_t + A_t, \quad (4.33)$$

where  $V(t, x) := x^2/t^{2\nu}$  and  $\mathcal{L}$  is given by (3.6). Moreover, we can see that there exist  $s_0 \geq 1$  and  $c > 0$  such that for all  $s \geq s_0$  and  $x \in \mathbb{R}$ , for which  $V(s, x) \geq \ell^2 + \varepsilon$ ,

$$\mathcal{L}V(s, x) = \frac{2\rho}{s} \left( V(s, x)^{\frac{\alpha-1}{2}} - \ell^{\alpha-1} \right) V(s, x) + \frac{1}{s^{2\nu}} \leq -\frac{c}{s} \leq 0. \quad (4.34)$$

This implies that the local martingale part  $S_u + M$  of the non-negative semimartingale in (4.33), together with  $S_{\cdot \wedge \eta_u}$  itself, are nonnegative supermartingales for all  $u \geq s_0$ . Therefore, the bounded variation part  $A$  will be a convergent process as the difference of two convergent supermartingales. Thanks to (4.34), this is possible if and only if  $\eta_u < \infty$  a.s. and this fact is the claim of *Step a*).

*Step b).* We introduce an increasing sequence of stopping times as follows:

$$\tau_1 := \inf \{t \geq s_0 : S_t = \ell^2 + 2\varepsilon\}, \quad \sigma_1 := \inf \{t \geq \tau_1 : S_t \in \{\ell^2 + \varepsilon, \ell^2 + 3\varepsilon\}\}$$

and for every integer  $n \geq 2$ ,

$$\tau_n := \inf \{t \geq \sigma_{n-1} : S_t = \ell^2 + 2\varepsilon\}, \quad \sigma_n := \inf \{t \geq \tau_n : S_t \in \{\ell^2 + \varepsilon, \ell^2 + 3\varepsilon\}\}.$$

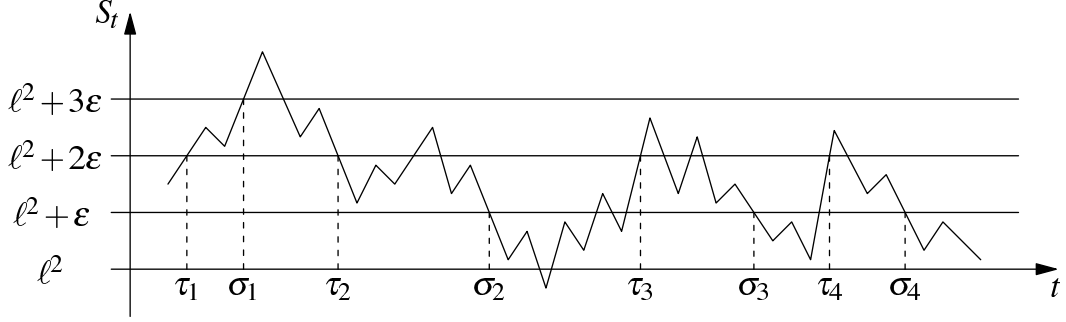


Figure 3: The increasing sequence of stopping times

Set  $F := \cap_{n \geq 1} \{\tau_n < \infty\}$ . Thanks to (4.32), we obtain that

$$\limsup_{t \rightarrow \infty} S_t \leq \ell^2 + 2\varepsilon \quad \text{a.s. on } F^c.$$

Hence, to prove the first inequality in (4.31), we need to show that

$$\limsup_{t \rightarrow \infty} S_t \leq \ell^2 + 3\varepsilon \quad \text{a.s. on } F \quad \text{or equivalently} \quad \mathbb{1}_F \sum_{n \geq 1} \mathbb{1}_{\{S_{\sigma_n} = \ell^2 + 3\varepsilon\}} < \infty \quad \text{a.s.}$$

By using a conditional version of Borel-Cantelli's lemma (see, for instance, Proposition 3.2 in [6], p. 240) it is equivalent to prove that

$$\sum_{n=1}^{\infty} \mathbb{P}(S_{\sigma_n} = \ell^2 + 3\varepsilon, \tau_n < \infty \mid \mathcal{F}_{\tau_n}) < \infty \quad \text{a.s.} \quad (4.35)$$

Step c). We show that there exist positive constants  $\lambda_1$  and  $\lambda_2$  such that for all integer  $n \geq 1$ ,

$$\mathbb{P}(S_{\sigma_n} = \ell^2 + 3\varepsilon, \tau_n < \infty \mid \mathcal{F}_{\tau_n}) \leq \lambda_1 \tau_n^{\left(\frac{1}{2} - \nu\right)} \exp(-\lambda_2 \tau_n^{2\nu-1}) \quad \text{a.s. on } \{\tau_n < \infty\}. \quad (4.36)$$

To this end, let us denote by  $\mathbb{P}_{s,x}$  the distribution of the weak solution of (2.1) such that  $X_s = x$ . The strong Markov property applies and this yields

$$\mathbb{P}(S_{\sigma_n} = \ell^2 + 3\varepsilon, \tau_n < \infty \mid \mathcal{F}_{\tau_n}) = \mathbb{P}_{\tau_n, X_{\tau_n}}(S_{\sigma_n} = \ell^2 + 3\varepsilon) \quad \text{a.s. on } \{\tau_n < \infty\}.$$

As in (4.33), we can write under the conditional probability  $\mathbb{P}_{\tau_n, X_{\tau_n}}$ , the canonical decomposition  $S_{t \wedge \sigma_n} = S_{\tau_n} + M_t^n + A_t^n$ , sum of a local martingale and a bounded variation process. Besides, we can show that  $v_n := \langle M^n \rangle_{\infty}$  satisfies

$$v_n = \int_{\tau_n}^{\sigma_n} \frac{4S_u}{u^{2\nu}} du \leq \int_{\tau_n}^{\infty} \frac{4(\ell^2 + 3\varepsilon)}{u^{2\nu}} du = \frac{4(\ell^2 + 3\varepsilon)}{(2\nu - 1)\tau_n^{2\nu-1}} \leq \frac{4(\ell^2 + 3\varepsilon)}{(2\nu - 1)s_0^{2\nu-1}} =: v_0.$$

Then, by the Dambis-Dubins-Schwarz theorem, there exists a standard Brownian motion  $W^n$  (under the conditional probability  $\mathbb{P}_{\tau_n, X_{\tau_n}}$ ) such that  $M^n = W_{\langle M^n \rangle}^n$  and since  $A^n$  is strictly negative, we can see that

$$\left\{ \sup_{0 \leq t \leq v_n} W_t^n < \varepsilon \right\} \subset \left\{ \sup_{t \geq \tau_n} M_t^n < \varepsilon \right\} \subset \{S_{\sigma_n} = \ell^2 + \varepsilon\}.$$

It is classical that the random variable  $\sup\{W_t^n : 0 \leq t \leq v_n\} \stackrel{\mathcal{L}}{=} G_n \stackrel{\mathcal{L}}{=} \mathcal{N}(0, v_n)$  (under the conditional probability  $\mathbb{P}_{\tau_n, X_{\tau_n}}$ ) and therefore we obtain

$$\mathbb{P}_{\tau_n, X_{\tau_n}}(S_{\sigma_n} = \ell^2 + 3\varepsilon) = 1 - \mathbb{P}_{\tau_n, X_{\tau_n}}(S_{\sigma_n} = \ell^2 + \varepsilon) \leq 2\mathbb{P}_{\tau_n, X_{\tau_n}}\left(G_n \geq \frac{\varepsilon}{\sqrt{v_n}}\right).$$

By the usual estimate for the standard Gaussian tail, we get (4.36).

*Step d).* To insure the convergence of the series in (4.35) we show that the sequence  $\tau_n$  increases to infinity sufficiently fast. More precisely, we show that there exists  $\lambda > 1$  such that  $\tau_n \geq \tau_1 \lambda^n$  a.s. on  $F$ . This inequality will be a consequence of a second conditional Borel-Cantelli's Lemma (see, for example, Proposition 4.11 in [6], p. 256) once we show that there exist some constants  $q > 1$  and  $p > 0$  such that for all  $n \geq 1$ ,

$$\mathbb{P}(\tau_{n+1} \geq q\tau_n \mid \mathcal{F}_{\tau_n}) \geq \mathbb{P}_{\tau_n, X_{\tau_n}}(\sigma_n \geq q\tau_n) \geq p \quad \text{a.s. on } \{\tau_n < \infty\}. \quad (4.37)$$

In opposite to (4.34), we can see that there exists a constant  $k > 0$  such that for all  $t \geq 1$  and  $x \in \mathbb{R}$ , for which  $V(t, x) \leq \ell^2 + 3\varepsilon$ ,  $\mathcal{L}V(t, x) \geq -k/t$ . We deduce that for all  $t \in [\tau_n, q\tau_n]$ ,

$$-\frac{\varepsilon}{2} \leq k \ln\left(\frac{\tau_n}{t}\right) \leq A_t^n \leq 0, \quad \text{with } q := e^{\frac{k\varepsilon}{2}} > 1.$$

By using this inequality, we can write

$$\{\sigma_n \geq q\tau_n\} \supset \left\{ \sup_{\tau_n \leq t \leq q\tau_n} |M_t^n| < \frac{\varepsilon}{2} \right\} \supset \left\{ \sup_{0 \leq t \leq v_0} |W_t^n| < \frac{\varepsilon}{2} \right\}.$$

Therefore, inequality (4.37) is satisfied with the deterministic positive constant

$$p := \mathbb{P}_{\tau_n, X_{\tau_n}}\left(\sup_{0 \leq t \leq v_0} |W_t^n| < \frac{\varepsilon}{2}\right) = \mathbb{P}\left(\sup_{0 \leq t \leq v_0} |B_t| < \frac{\varepsilon}{2}\right).$$

Here  $B$  denotes a standard Brownian motion. The Borel-Cantelli's Lemma applies and we obtain that  $\tau_n \geq \tau_1 \lambda^n$  a.s. on  $F$ . We deduce (4.35) and then we conclude that (4.24) holds.

Finally, we need to prove the statement (4.25). Assume that  $\alpha \in (1, \infty)$ , consider  $\mathfrak{h} = \Phi_\gamma(X)$  and denote by  $\eta_e$  its explosion time. We can see that  $h_s := \mathfrak{h}_s - W_s$  satisfies the (random) ordinary differential equation

$$\dot{h}_s = \rho \operatorname{sgn}(h_s + W_s) |h_s + W_s|^\alpha - \gamma \varphi_\gamma(s)^{\gamma-1} \frac{h_s + W_s}{2}. \quad (4.38)$$

Moreover, it is not difficult to see that

$$\frac{|h_s|^{1-\alpha}}{\alpha-1} = \int_s^{\eta_e} \frac{\dot{h}_u}{h_u^\alpha} du \underset{s \rightarrow \eta_e}{\sim} \rho(\eta_e - s) \quad \mathbb{P}(\cdot \mid \eta_e < \infty)\text{-a.s.}$$

This implies (4.25) by change of time. The proof of the theorem is done, excepted for the Lemma 4.2.  $\square$

**Proof of Lemma 4.2.** Let us recall that  $\mathfrak{h} = \Phi_\gamma(X)$  is the pathwise unique strong solution of (3.9) and that  $\mathbb{P}(\tau_e = \infty) = \mathbb{P}(\eta_e \in \{t_1, \infty\})$ , with  $\eta_e$  its explosion time. We show that

$$\lim_{t \rightarrow t_1} \mathfrak{h}_t = 0 \quad \text{a.s. on } \{\eta_e = \infty\} \quad \text{and} \quad \mathbb{P}(\eta_e = t_1) = 0. \quad (4.39)$$

The first equality in (4.39) is a simple consequence of the Girsanov's transformation which gives (3.11) and the fact that the  $\delta$ -Brownian bridge  $b$  defined in (3.10) satisfies  $b_{t_1} = 0$  a.s. The main difficulty is to prove the second equality in (4.39). Let us introduce

$$\ell := \left( \frac{1 + \delta(\alpha - 1)}{\rho(\alpha - 1)} \right)^{\frac{1}{\alpha-1}} \quad \text{and} \quad Z_s := (t_1 - s)^{\frac{1}{\alpha-1}} \mathfrak{h}_s$$

and note that  $Z$  satisfies the following stochastic differential equation

$$dZ_s = (t_1 - s)^{\frac{1}{\alpha-1}} dw_s + \frac{\rho Z_s (|Z_s|^{\alpha-1} - \ell^{\alpha-1})}{(t_1 - s)^{\frac{\alpha}{\alpha-1}}} ds. \quad (4.40)$$

We begin by showing that

$$\mathbb{P}(\eta_e = t_1) = \mathbb{P}\left(\eta_e = t_1, \lim_{s \rightarrow t_1} |Z_s| \in \{0, \ell, \infty\}\right). \quad (4.41)$$

First, let us prove that

$$\mathbb{P}\left(\eta_e = t_1, \liminf_{s \rightarrow t_1} Z_s < \limsup_{s \rightarrow t_1} Z_s\right) = 0. \quad (4.42)$$

Denote by  $M$  the local martingale part of  $Z$  and by  $d(s, z)$  the drift of the equation (4.40). Since  $s \mapsto M_s$  is a.s. uniformly continuous on  $[0, t_1]$ ,

$$\mathbb{P}\left(\liminf_{s, u \rightarrow t_1} |M_s - M_u| > 0\right) = 0. \quad (4.43)$$

Moreover, the points  $\{0, \pm\infty\}$  are attractive whereas the points  $\{\pm\ell\}$  are repulsive since

$$d(s, z)z \geq 0 \quad \text{or} \quad d(s, z)z \leq 0 \quad \text{according whether} \quad |z| \geq \ell \quad \text{or} \quad |z| \leq \ell. \quad (4.44)$$

Assume that  $\eta_e = t_1$  and  $\liminf_{s \rightarrow t_1} Z_s < \limsup_{s \rightarrow t_1} Z_s$ . By using (4.44), it is not difficult to construct two sequences  $(s_n)$  and  $(u_n)$  of real numbers, belonging to  $[0, t_1]$  such that

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} u_n = t_1 \quad \text{and} \quad \liminf_{n \rightarrow \infty} |M_{s_n} - M_{u_n}| > 0.$$

We deduce from (4.43) the equality (4.42). Then, we obtain equality (4.41) by noting that

$$\lim_{s \rightarrow t_1} |d(s, Z_s)| = \infty \quad \text{a.s. on} \quad \left\{ \eta_e = t_1, \lim_{s \rightarrow t_1} |Z_s| \notin \{0, \ell\} \right\}.$$

Second, to prove  $\mathbb{P}(\eta_e = t_1) = 0$ , it suffices to show that for any  $\lambda \in \{0, \ell, \infty\}$ ,

$$\mathbb{P}\left(\eta_e = t_1, \lim_{s \rightarrow t_1} |Z_s| = \lambda\right) = 0. \quad (4.45)$$

Let us recall that  $h$  is a solution of equation (4.38), which can be written

$$\dot{h}_s = \rho \operatorname{sgn}(h_s + w_s) |h_s + W_s|^\alpha - \delta \frac{h_s + W_s}{t_1 - s}.$$

Besides, it is a simple calculation to see that there exist two functions  $\epsilon_i$ ,  $i = 1, 2$ , such that

$$\dot{h}_s = -\epsilon_1(Z_s) \left[ \frac{\delta(h_s + W_s)}{t_1 - s} \right], \quad \text{with} \quad \lim_{z \rightarrow 0} \epsilon_1(z) = 1 \quad (4.46)$$

and

$$\dot{h}_s = \epsilon_2(Z_s) [\rho \operatorname{sgn}(h_s + W_s) |h_s + W_s|^\alpha], \quad \text{with} \quad \lim_{z \rightarrow \infty} \epsilon_2(z) = 1. \quad (4.47)$$

Consider  $h_s := \mathfrak{h}_s - W_s$  and we analyse three cases.

Case  $\lambda = 0$ . By using (4.46) we can show that there exists  $u \in [0, t_1)$  such that

$$\sup \left\{ h_s \dot{h}_s \mathbf{1}_{\{|y_s| \geq c\}} : u \leq s < t_1 \right\} \leq 0, \quad \text{with} \quad c := \sup\{|W_s| : 0 \leq s \leq t_1\}.$$

It is not difficult to see that  $h$ , and furthermore  $\mathfrak{h}$ , are bounded on  $[0, t_1)$ . Then, we deduce

$$\mathbb{P} \left( \eta_e = t_1, \lim_{s \rightarrow t_1} |Z_s| = 0 \right) \leq \mathbb{P} \left( \eta_e = t_1, \sup_{0 \leq s < t_1} |\mathfrak{h}_s| < \infty \right) = 0.$$

Case  $\lambda = \infty$ . By using (4.47) we obtain

$$\frac{|h_s|^{1-\alpha}}{\alpha-1} = \int_s^{t_1} \frac{\dot{h}_u}{\operatorname{sgn}(h_u) |h_u|^\alpha} du \underset{s \rightarrow t_1}{\sim} \rho(t_1 - s), \quad \text{which implies} \quad \lim_{s \rightarrow t_1} Z_s = (\rho(\alpha-1))^{\frac{1}{1-\alpha}}.$$

Therefore,

$$\mathbb{P} \left( \eta_e = t_1, \lim_{s \rightarrow t_1} |Z_s| = \infty \right) \leq \mathbb{P} \left( \eta_e = t_1, \lim_{s \rightarrow t_1} |Z_s| = (\rho(\alpha-1))^{\frac{1}{1-\alpha}} = \infty \right) = 0.$$

Case  $\lambda = \ell$ . We show that this event is of probability 0 by using the fact that  $\pm \ell$  are repulsive points. Remark that the drift of the equation (4.40) can be written

$$d(s, z) =: q(z) \frac{z \pm \ell}{(t_1 - s)^{\frac{\alpha}{\alpha-1}}}, \quad \text{with} \quad \lim_{z \rightarrow \pm \ell} q(z) =: q_\infty = \rho(\alpha-1)\ell^{\alpha-1}.$$

By Ito's formula, we can see that  $z_s := (Z_s \pm \ell)^2$  is the pathwise strong solution of

$$dz_s = 2\sqrt{z_s} (t_1 - s)^{\frac{1}{\alpha-1}} dw_s + \left( 2q(Z_s) \frac{z_s}{(t_1 - s)^{\frac{\alpha}{\alpha-1}}} + (t_1 - s)^{\frac{2}{\alpha-1}} \right) ds.$$

Here  $w$  denotes a standard Brownian motion. Let  $v \in [0, t_1)$  and  $\zeta(v)$  be the pathwise unique strong solution of

$$d\zeta_s(v) = 2\sqrt{\zeta_s(v)} (t_1 - s)^{\frac{1}{\alpha-1}} dw_s + \left( q_\infty \frac{\zeta_s(v)}{(t_1 - s)^{\frac{\alpha}{\alpha-1}}} + (t_1 - s)^{\frac{2}{\alpha-1}} \right) ds, \quad \zeta_v(v) = z_v \mathbf{1}_{\{\eta_e > v\}}.$$

By Ito's formula, the law of  $\zeta(v)$  equals to the law of  $\xi^2(v)$ , with

$$d\xi_s(v) = (t_1 - s)^{\frac{1}{\alpha-1}} dW_s + \frac{\xi_s(v)}{(t_1 - s)^{\frac{\alpha}{\alpha-1}}} ds, \quad \xi_v(v) = \sqrt{\zeta_v(v)}.$$

Since  $\xi(v)$  is the solution of a linear equation, we can see that  $\lim_{s \rightarrow t_1} \xi_s(v) = \infty$  a.s. and then we deduce that  $\lim_{s \rightarrow t_1} \zeta_s(v) = \infty$  a.s. Besides, by using the same arguments as for the proof of the comparison (4.19), we can show that for all  $s \in [v, t_1)$ ,

$$z_s \geq \zeta_s(v) \quad \text{a.s. on} \quad \Omega_v := \left\{ \eta_e = t_1, \sup_{v \leq s < t_1} |q(Z_s)| \geq \frac{q_\infty}{2} \right\}.$$

Hence, we obtain

$$\lim_{s \rightarrow t_1} |Z_s| = \infty \quad \text{a.s. on} \quad \left\{ \eta_e = t_1, \lim_{s \rightarrow t_1} |Z_s| = \ell \right\} \subset \bigcup_{0 \leq v < t_1} \Omega_v.$$

and we deduce (4.45) for  $\lambda = \ell$ . We conclude that  $\mathbb{P}(\eta_e = t_1) = 0$ .  $\square$

#### 4.4 Bessel type case

We have already seen that if  $\rho \in (-\infty, 0)$  and  $\alpha = -1$  then the solution of (2.1) is defined up to the reaching time of 0, which is finite a.s. Hence we need to consider only the repulsive case  $\rho \in (0, \infty)$ . Recall that if  $\beta \in (0, \infty)$  the set  $\{t \geq 1 : X_t = 0\}$  is unbounded a.s. and then we will see that  $X$  is recurrent in  $[0, \infty)$ . The proof of the following result is similar to the proof of Theorem 4.3 and we leave it to the reader.

**Theorem 4.4.** *Assume that  $\rho \in (0, \infty)$ ,  $\alpha = -1$  and  $\beta \neq 0$ .*

1.  *$X$  is recurrent if and only if  $\beta \in (0, \infty)$ . Moreover  $X$  has the Gaussian typical behaviour (4.1) and it satisfies the iterated logarithm law (4.2).*
2.  *$X$  is transient if and only if  $\beta \in (-\infty, 0)$  and it satisfies (4.24).*

**Proof.** We give only the main arguments. To get the recurrent feature, it suffices to consider  $\mathbf{u}^2 = \Phi_e^2(X)$  and to apply Lemma 4.1 to obtain the convergence in distribution. The pathwise largest deviation is obtained by comparing  $\mathbf{u}^2$  with a time-homogeneous diffusion and applying Motoo's theorem. The proof of the transience is the same as for the general case.  $\square$

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