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# Asymptotic behaviour of a family of time-inhomogeneous diffusions

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**Abstract:** Let  $X$  a solution of the time-inhomogeneous stochastic differential equation driven by a Brownian motion with drift coefficient  $b(t, x) = \rho \operatorname{sgn}(x) \frac{|x|^\alpha}{t^\beta}$ . This process can be viewed as a distorted Brownian motion in a potential possibly singular, depending on time. After obtaining results on the existence and the uniqueness of solution, we study its asymptotic behaviour and made a precise description, in terms of parameters  $\rho, \alpha$  and  $\beta$ , of the recurrence, transience and convergence. More precisely, asymptotic distributions, iterated logarithm type laws and rates of transience are proved for such processes.

**Key words:** time-inhomogeneous diffusions, singular stochastic differential equations, explosion times, scaling transformations and changes of time, recurrence and transience, iterated logarithm type laws, asymptotic distributions

**2000 Mathematics Subject Classification:** 60J60, 60H10, 60J65, 60G17, 60F15, 60F05

## 1 Introduction

Let us consider the one-dimensional process  $X$  satisfying the following time-inhomogeneous stochastic differential equation

$$X_t = x_0 + B_t + \rho \int_{t_0}^t \frac{\operatorname{sgn}(X_s) |X_s|^\alpha}{s^\beta} ds, \quad t \geq t_0. \quad (1.1)$$

Here  $B$  denote a standard Brownian motion,  $x_0, \rho, \alpha, \beta$  are some real constants and  $t_0 \in (0, \infty)$ . In this paper we shall study the asymptotic behaviour of such stochastic process. Our main goal is to give conditions which characterise recurrence, transience and convergence of  $X$ , in terms of parameters  $\rho, \alpha$  and  $\beta$ .

The equation (1.1) can be viewed as a continuous counterpart of a discrete time model considered in [16] and which is related, when  $\alpha = \beta = 1$ , to the Friedman's urn model [7] but also to the random walk with an asymptotically zero drift studied in [14]. The discrete time process studied in [16] is a random walk on the real half line such that its drift satisfies

$$\mathbb{E}(X_{t+1} - X_t \mid X_t = x) \underset{t \rightarrow \infty}{\sim} \rho \frac{|x|^\alpha}{t^\beta}.$$

In the repulsive case  $\rho \in (0, \infty)$ , the authors establish when the process is recurrent and transient for certain values of parameters  $\alpha, \beta$  and present open problems. Their approach is based on a precise study of some sub-martingales and super-martingales connected to the problem.

In the one-dimensional case, for time-homogeneous stochastic differential equations driven by a Brownian motion, there exists criteria for recurrence and transience (see for instance [12] or [8]) or explosion of solutions (see for instance [13]). Some of these criteria are extended to the time-inhomogeneous case for dimension greater or equal than two in [4]. Unfortunately, the results in [4] does not apply to the equation (1.1), even it is stated that the method can be adapted to the one-dimensional case. Recall also that there exists some general results on recurrence or transience (see for instance [10]) and explosion (see for instance [20] or [18]) based on the construction of some convenient Lyapunov functions. However, in case of equation (1.1) the construction of such Lyapunov functions seems to be more delicate.

Here are another two natural questions:

- Does there exist a well chosen normalization of  $X_t$  to ensure that the re-normalized process converges in distribution or almost surely, as  $t \rightarrow \infty$ ?
- Is it possible to obtain the pathwise largest deviations of  $X$ , for instance the iterated logarithm type law?

This kind of questions are treated for instance in [9], [2] or [1] for different equations which have some common features with (1.1). More precisely, in [9] or [2] the following one-dimensional stochastic differential equation is considered:

$$dx_t = dB_t + f(x_t)dt, \quad \text{with} \quad (1.2)$$

$$\int_{-\infty}^{\infty} f(x)dx = 0 \quad (H_1) \quad \text{or} \quad f(x) \underset{x \rightarrow \infty}{\sim} \rho x^\alpha, \quad \rho \in (0, \infty) \text{ and } \alpha \in [-1, 1) \quad (H_2)$$

Under additional assumptions (see [9],) the hypothesis  $(H_1)$  implies that  $x$  is recurrent and that the normalized process  $x_t/\sqrt{t}$  converge in distribution as  $t \rightarrow \infty$  to a standard Gaussian random variable. On the other hand, when  $\alpha \in (-1, 1)$ , the hypothesis  $(H_2)$  ensures that  $x$  is transient and asymptotically behaves as a solution of the ordinary differential equation  $h'(t) = f(h(t))$  that is  $\lim_{t \rightarrow \infty} \frac{x_t}{h(t)} = 1$  a.s. The case  $\alpha = -1$  is more difficult and its study is related to the Bessel process (see [2].) In that case  $x$  satisfies the iterated logarithm law and recurrence or transience depends on the position of  $\rho$  with respect to  $1/2$ . In [1] the damped stochastic differential equation is considered:

$$dx_t = \sigma(t)dB_t + f(x_t)dt, \quad \text{with} \quad (1.3)$$

$$\lim_{t \rightarrow \infty} \downarrow \sigma(t) = 0, \quad \sigma \in L^2 \quad \text{and} \quad f(x) \underset{x \rightarrow 0}{\sim} \rho \operatorname{sgn}(x)|x|^\alpha, \quad \rho \in (-\infty, 0) \text{ and } \alpha \in (1, \infty).$$

It is proved (see [1] and [3]) that  $x_t$  converges almost surely to 0 and it is polynomially stable, that is there exists  $p \in (0, \infty)$  such that

$$\limsup_{t \rightarrow \infty} \frac{\ln(|x_t|)}{\ln t} \leq -p \quad \text{a.s.} \quad (1.4)$$

In the present paper we firstly study the existence, uniqueness and explosion of solutions for (1.1). Different situations are distinguished following the values of  $\rho$  and  $\alpha$  and existence

and uniqueness are proved. We point out that in the case where  $\alpha \in (-\infty, 0)$  the existence of a solution is not obvious since the drift has a singularity in 0. For the time-homogenous case in [5] it is given a solution to this problem by using scale functions and speed measures. These ideas does not apply to one-dimensional time-inhomogeneous stochastic differential equations and this is the main difficulty of this part. Our idea is to use an appropriate change of time, taking full advantage of the scaling property of the Brownian motion, the Girsanov transformation and the classification of isolated singular points. Furthermore, these different tools allow us to answer the question of explosion of the solution when  $\alpha \in (1, \infty)$ . For instance we point out that if  $\beta \in (\frac{\alpha+1}{2}, \infty)$  then the solution explodes in finite time with a positive probability but not almost surely. Note that in the case when  $\alpha = -1$  the process can be viewed as a Bessel process with a dimension depending on time.

Secondly, we describe for all values of parameters  $\rho, \alpha, \beta$ , the recurrent, transient features of the solution  $X$  of (1.1) but also its convergence. We present in Figure 1 below, the diagram of phase transition that we obtain in the attractive case  $\rho \in (-\infty, 0)$ :

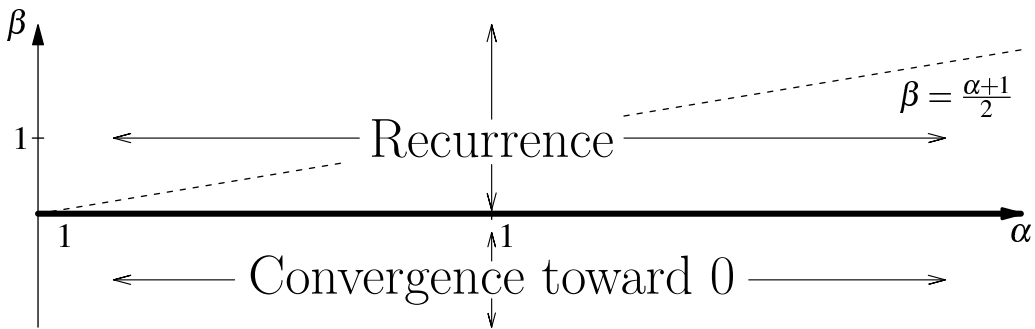


Figure 1: Phase transition in the attractive case  $\rho \in (-\infty, 0)$

Note that  $\alpha \in (-\infty, -1]$  is not allowed, since in that case any solution is only defined up to the reaching time of 0, which is finite almost surely. The critical line separating the two phases (*Recurrence* and *Convergence toward 0*) is given by  $\beta = 0$  and on this line the process is recurrent. The line  $\beta = \frac{\alpha+1}{2}$  can be called subcritical in the sense that the recurrent behaviour is different below and above (we will see that  $(1, 1)$  is a particular point of this line.) As for the proof of existence, we use a suitable change of time and an associated scaling transformation, we give the asymptotic distribution of  $X$  and its pathwise largest deviations under a convenient normalization. In fact, we show that the asymptotic behaviour of the process is strongly connected to the paths and to the stationary distribution of an ergodic diffusion  $Y$  through a scaling transformation and an appropriate change of time. For example, if  $\beta \in (-\infty, 0)$  and  $\varphi'(t) = \varphi(t)^{\frac{2\beta}{\alpha+1}}$ , then we show that, as  $t \rightarrow \infty$ ,

$$\frac{X_{\varphi(t)}}{\sqrt{\varphi'(t)}} \text{ behaves as } dY_t = dB_t + \rho \operatorname{sgn}(Y_t)|Y_t|^\alpha dt.$$

It only remains to apply the ergodic theorem and Motoo's theorem (see [17]) to  $Y$  in order to obtain the convergence in distribution of  $X_t/t^{\frac{\beta}{\alpha+1}}$  to the stationary distribution of  $Y$  and the polynomial stability of  $X$  (see (1.4)). Remark also that if we set  $\phi(t) := t^{\frac{1}{1-\beta}}$  then  $x_t := X_{\phi(t)}$  satisfies the damped stochastic differential equation (1.3) without the  $L^2$ -integrability condition for  $\sigma$  and  $\alpha \in (-1, \infty)$ . We get that  $x$  is polynomially stable under

different hypothesis. In fact we try to be more accurate as in [1] and we give the precise largest pathwise deviations.

We present in Figure 2 below, the diagram of phase transition that we obtain in the repulsive case  $\rho \in (0, \infty)$ :

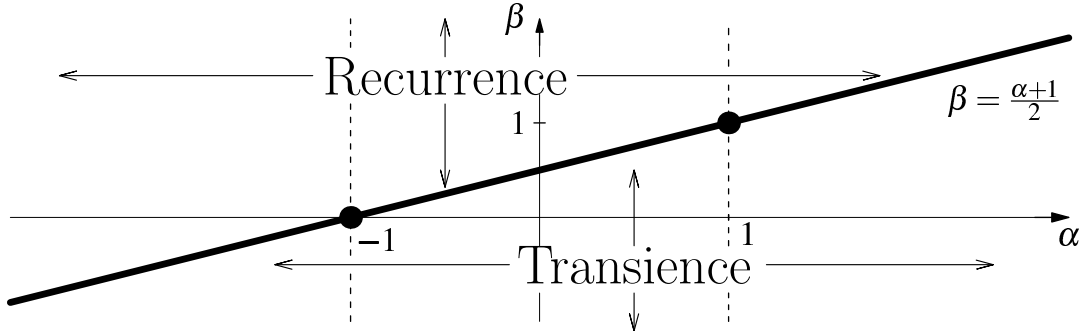


Figure 2: Phase transition in the repulsive case  $\rho \in (0, \infty)$

Remark that in the case when  $\alpha \in (1, \infty)$  and  $\beta \in (-\infty, \frac{\alpha+1}{2}]$  the process explodes in finite time almost surely and we say that it is transient. Here the non-trivial critical line is given by  $\beta = \frac{\alpha+1}{2}$  and the process is either recurrent or transient and we obtain similar results as in [16]. On this line we must distinguish the two particular points  $(-1, 0)$  and  $(1, 1)$  because these are the only cases where recurrence and transience depend on the position of  $\rho$  with respect to  $\frac{1}{2}$ . The first corresponds to the well known Bessel process and the second one is a continuous time counterpart of the Friedman's urn model. In this case we obtain similar results as in [7] concerning recurrence and transience but, again, we are more accurate regarding the asymptotic distribution and the pathwise largest deviations. For the other points of the critical line, the process is recurrent. We point out that this is an open problem in [16]. In the present paper there is no restriction on the parameters  $\rho, \alpha$  and  $\beta$ . The lines  $\alpha = -1$  and  $\alpha = 1$  can be called subcritical in the sense that the behaviour of the process is slightly different to the right or left, in particular the domain of recurrence depends on  $\alpha$ . The proof of recurrence is based on the same ideas as for the attractive case by associating an ergodic diffusion  $Y$  to an appropriate normalization of  $X$  obtained by a scaling transformation and an suitable change of time. For instance, if  $\beta \in (\frac{\alpha+1}{2}, \infty)$  and  $\alpha \in (-1, 1)$  then, as  $t \rightarrow \infty$ ,

$$\frac{X_{e^t}}{e^{\frac{t}{2}}} \text{ behaves as } dY_t = dB_t - \frac{1}{2}Y_t dt.$$

By applying the ergodic theorem and Motoo's theorem (see [17]) to the Ornstein-Uhlenbeck process  $Y$  we prove that  $X_t$  behaves as a standard Brownian motion. In particular it satisfies the iterated logarithm law and  $\frac{X_t}{\sqrt{t}}$  converge in distribution to a standard Gaussian random variable. We can say that the drift is asymptotically negligible compared to the noise in the opposite of the transient case. Concerning the proof of the transience the tools are similar to those used in [16]. When  $\alpha \in (-\infty, 1)$ , we use sub-martingale and super-martingale arguments to show that  $\frac{|X_t|}{h(t)}$  converge almost surely to 1 with  $h$  a non-negative solution of the ordinary differential equation  $h'(t) = \rho \operatorname{sgn}(h(t)) \frac{|h(t)|^\alpha}{t^\beta}$  and this result can be viewed as an extension of a result proved in [9]. If  $\alpha = 1$  the process  $\frac{X_t}{h(t)}$  converges almost surely to a standard Gaussian random variable and hence we obtain the probabilities of explosion at  $\pm\infty$  explicitly.

Some of these results could be proved with similar arguments for a general drift  $b$  under additional assumptions. In particular in the case when  $b(t, x) = f(t)g(x)$  with  $f(t) \underset{t \rightarrow \infty}{\sim} t^{-\beta}$  and  $g(x) \underset{x \rightarrow \infty}{\sim} \rho x^\alpha$ . These generalizations will be studied elsewhere. The case of a general diffusion coefficient  $\sigma$  seems more difficult, but some partial results could be obtained under the assumption  $0 < c \leq \sigma^2 \leq d < \infty$ . Other interesting cases could be when one replaces the Brownian motion by a stable Levy process or by a fractional Brownian motion and will be studied in a future work.

The paper is organised as follows: in the next section we also recall the important properties of three well-known processes which are solutions of (1.1) for particular values of parameters. In Section 3 we study the existence, uniqueness and explosions of solutions, while in Section 4 we analyse the asymptotic behaviour of the process  $X$ .

## 2 Three particular processes

We shall study the equation (1.1) in its equivalent form as a time-inhomogeneous stochastic differential equation:

$$dX_t = dB_t + \rho \operatorname{sgn}(X_t) \frac{|X_t|^\alpha}{t^\beta} dt, \quad X_{t_0} = x_0, \quad (2.1)$$

This equation is driven by a standard Brownian motion  $B$  given on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ . By symmetry of the equation and by scaling transformation, we can assume without loss of generality that  $x_0 \in [0, \infty)$  and  $t_0 = 1$ . We will keep these assumption all along the paper. To begin with, let us recall the behaviour of three important processes which are solutions of the equation (2.1) with some particular parameters: the Brownian motion, the Ornstein-Uhlenbeck process and the Bessel process.

Firstly, if  $\rho = 0$  then  $X \equiv B$  is the standard Brownian motion and it is classical that  $B$  is recurrent and it satisfies

$$\lim_{t \rightarrow \infty} \frac{B_t}{\sqrt{t}} \stackrel{\mathcal{L}}{=} \mathcal{N}(0, 1) \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{B_t}{\sqrt{2t \ln \ln t}} = 1 \quad \text{a.s.} \quad (2.2)$$

Secondly, if  $\rho \neq 0$ ,  $\alpha = 1$  and  $\beta = 0$  then  $X \equiv O$  is the Ornstein-Uhlenbeck process. By using the martingale representation theorem and a suitable harmonic function it is not difficult to prove that  $O$  is recurrent if  $\rho \in (-\infty, 0)$  and in that case it satisfies

$$\lim_{t \rightarrow \infty} O_t \stackrel{\mathcal{L}}{=} \mathcal{N}(0, (2|\rho|)^{-1}) \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{O_t}{\sqrt{\ln t}} = |\rho|^{-\frac{1}{2}} \quad \text{a.s.} \quad (2.3)$$

and transient if  $\rho \in (0, \infty)$  and in that case it satisfies

$$\lim_{t \rightarrow \infty} \frac{O_t}{e^{\rho t}} = G \quad \text{a.s.} \quad \text{with} \quad G \sim \mathcal{N}(x_0 e^{-\rho}, (2\rho)^{-1} e^{-2\rho}). \quad (2.4)$$

Finally, if  $\rho \in (0, \infty)$ ,  $\alpha = -1$  and  $\beta = 0$  then there exists a unique non-negative pathwise solution  $X \equiv R$  to the equation (2.1), it is the Bessel process of dimension  $2\rho + 1$ . Moreover, the square of any solution of the equation (2.1) is the unique solution of the equation

$$dY_t = 2\sqrt{|Y_t|} dB_t + (2\rho + 1)dt, \quad Y_1 = x_0^2. \quad (2.5)$$

It is known that  $Y$  is the square Bessel process of dimension  $2\rho + 1$ . We note that the drift of (2.1) is not locally integrable in 0 (which means that 0 is an isolated singular point.) By using the classification of isolated singular points in [5], we can prove the following properties:

- if  $\rho \in (0, \frac{1}{2})$  then we can construct different weak solutions to the equation (2.1) and the set  $\{t \geq 1 : R_t = 0\}$  is unbounded a.s.;
- if  $\rho \in [1/2, \infty)$  and  $x_0 \in (0, \infty)$  then  $R$  is the unique pathwise solution to the equation (2.1) and it is positive;
- if  $\rho \in [\frac{1}{2}, \infty)$  and  $x_0 = 0$  then, for all  $t > 0$ ,  $R_t > 0$  a.s. and the set of weak solutions to the equation (2.1) is exactly the set of all mixing distributions of  $R$  and  $-R$ ;

If  $\rho \in (-\infty, 0)$  it can be shown that every solution is defined up to the reaching time of 0, which is finite a.s. If  $\rho \in (0, \infty)$  then the asymptotic behaviour can be described as follows:

$$\lim_{t \rightarrow \infty} \frac{R_t}{\sqrt{t}} \stackrel{\mathcal{L}}{=} \sqrt{\Gamma_\rho}, \quad \text{with} \quad \Gamma_\rho \sim \gamma(\rho + 1/2, 2) \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{R_t}{\sqrt{t \ln \ln t}} = 1 \quad \text{a.s.} \quad (2.6)$$

where  $\gamma$  denote the gamma distribution. Moreover,

- if  $\rho \in (0, \frac{1}{2})$  then  $R$  is recurrent in  $[0, \infty)$
- if  $\rho = \frac{1}{2}$  then  $R$  is recurrent in  $(0, \infty)$
- if  $\rho \in (\frac{1}{2}, \infty)$  then  $R$  is transient and it satisfies

$$\liminf_{t \rightarrow \infty} \frac{\ln \left( \frac{R_t}{\sqrt{t}} \right)}{\ln \ln t} = (1 - 2\rho)^{-1} \quad \text{a.s.} \quad (2.7)$$

To obtain (2.6) we perform an appropriate time change: set  $Z_s := \frac{R_{e^s}^2}{e^s}$ . Then  $Z$  satisfies the following equation

$$dZ_s = 2\sqrt{Z_s} dW_s + (2\rho + 1 - Z_s) ds, \quad Z_0 = x_0^2.$$

Here  $W$  denote a standard Brownian motion. At this level it suffices to use Motoo's theorem (see [17] or [2], p. 915) applied to the time-homogeneous diffusion  $Z$ . Let us recall this result because it will be used several times in the following sections:

**Theorem 2.1** (Motoo). *Let  $I \subset \mathbb{R}$  be an unbounded interval and consider  $x_t \in I$  the unique solution of the one-dimensional time-homogeneous stochastic differential equation*

$$dx_t = b(x_t)dt + \sigma(x_t)dB_t.$$

*Let us consider the scale function and the speed measure of  $x$  defined for some  $c \in \bar{I}$  by*

$$s(u) := \int_c^u \exp \left( -2 \int_c^v \frac{b(z)}{\sigma(z)^2} dz \right) dv, \quad m(du) := \frac{du}{s'(u)\sigma(u)^2}, \quad u \in \bar{I},$$

*and assume that  $x$  is positive recurrent, that is  $m(I) < \infty$ . If  $h : (0, \infty) \rightarrow (0, \infty)$  is an increasing function with  $\lim_{t \rightarrow \infty} h(t) = \infty$ , then*

$$\mathbb{P} \left( \limsup_{t \rightarrow \infty} \frac{x_t}{h(t)} \geq 1 \right) = 0 \text{ or } 1$$

*according to whether  $t \mapsto s(h(t))^{-1}$  is integrable or not in a neighbourhood of infinity.*

By further using the ergodic theorem (see, for instance [12], Theorem 23.15, p. 465) applied to the time-homogeneous diffusion  $Z$ , we can deduce (2.6). The equation (2.7) is a consequence of Theorem 4.3 in [2], p. 928, obtain by applying Motoo's theorem to  $\frac{1}{Z_s}$ .

### 3 Preliminary study of solutions

In this section we study the existence and the uniqueness of solutions for the equation (2.1). Set  $\Omega := C([1, \infty); \mathbb{R})$ ,  $\Omega^* := C([0, t_1]; \mathbb{R})$ , with  $t_1 \in (0, \infty]$ . We can see that every change of time  $\varphi : [0, t_1] \rightarrow [1, \infty)$  ( $C^2$ -diffeomorphism) define a scaling transformation  $\Phi : \Omega \rightarrow \Omega^*$  given by

$$\Phi(\omega)(s) := \frac{\omega(\varphi(s))}{\sqrt{\varphi'(s)}}, \quad s \in [0, t_1), \omega \in \Omega. \quad (3.1)$$

By Ito's formula and the martingale representation theorem, it can be shown that  $\Phi$  induces a bijection between solutions of equation (2.1) and solutions of equation

$$dY_s = dW_s + \rho \frac{\varphi'(s)^{\frac{\alpha+1}{2}}}{\varphi(s)^\beta} \operatorname{sgn}(Y_s) |Y_s|^\alpha ds - \frac{1}{2} \frac{\varphi''(s)}{\varphi'(s)} Y_s ds, \quad Y_0 = \frac{x_0}{\sqrt{\varphi'(0)}}. \quad (3.2)$$

Here  $\{W_s : s \in [0, t_1]\}$  denotes a standard Brownian motion.

The existence and uniqueness of solutions for equation (2.1) are not obvious since the drift could be singular in 0 and/or not time-homogeneous. If  $\alpha \in (-\infty, 0)$  and  $\beta = 0$  the equation (2.1) is time-homogeneous and we can use the classification of isolated singular points in [5]. We will perform an appropriate scaling transformation as in (3.1) and we will use Girsanov's transformation in order to reduce to the time-homogenous case.

First we study the case where  $\alpha \in (-1, \infty)$ . Here the function  $x \mapsto |x|^\alpha$  is locally integrable in a neighbourhood of 0. As for the homogeneous case (see [5], Proposition 2.2, p. 28) we show that there exists a unique solution to the equation (2.1) up to the explosion time.

**Proposition 3.1.** *Assume that  $\rho \in \mathbb{R}^*$ ,  $\alpha \in (-1, \infty)$  and  $\beta \in \mathbb{R}$ . There exists a unique pathwise solution  $X$  to the equation (2.1) defined up to the explosion time  $\tau_e$ . Moreover, if  $\rho \in (-\infty, 0)$  or if  $\alpha \in (-1, 1]$  then  $\tau_e$  is infinite a.s.*

**Proof.** Assume that  $\alpha \in (1, \infty)$ . We construct a truncated stochastic differential equation as follows: for every integer  $n \geq 1$  set  $b_n(t, x) := \rho \operatorname{sgn}(x) \frac{x^\alpha}{t^\beta} \mathbf{1}_{\{|x| \leq n\}}$  and

$$dX_t^{(n)} = dB_t + b_n(t, X_t^{(n)}) dt, \quad X_1^{(n)} = x_0.$$

This equation has locally Lipschitz coefficients with linear growth. Let  $X^{(n)}$  be its unique pathwise solution. Choose an integer  $n_0 \geq 1$  such that  $x_0 \in [0, n)$  for all  $n \geq n_0$  and denote

$$\tau_n := \inf\{t \geq 1 : |X_t^{(n)}| \geq n\}, \quad n \geq n_0 \quad \text{and} \quad \tau_e := \sup_{n \geq n_0} \tau_n.$$

For all  $t \in [1, \tau_e)$  we set  $X_t := X_t^{(n)}$  a.s., where  $n$  is chosen such that  $t \in [1, \tau_n]$ . The process  $X$  is well defined up to the explosion time  $\tau_e$  and it is the desired solution for (2.1). The



pathwise uniqueness is a consequence of the pathwise uniqueness for the truncated stochastic differential equation.

Moreover, when  $\rho \in (-\infty, 0)$ , we can prove by using a Lyapunov function that the explosion time of the solution is infinite a.s. Indeed, let us consider the infinitesimal generator of the space-time diffusion  $t \mapsto (t, X_t)$  given by

$$\mathcal{L} := \partial_t + \rho \operatorname{sgn}(x) \frac{x^\alpha}{t^\beta} \partial_x + \frac{1}{2} \partial_{xx}. \quad (3.3)$$

The non-negative function  $V$  defined by  $V(t, x) := x^2 + 1$  satisfies for a well chosen  $\lambda > 0$ :

$$\lim_{|x| \rightarrow \infty} \inf_{t \geq 1} V(t, x) = \infty \quad \text{and} \quad \mathcal{L}V \leq \lambda V,$$

Noting that the process  $t \mapsto e^{-\lambda t} V(t, X_t)$  is a non-negative super-martingale, we can prove (see, for instance, [20], Theorem 10.2.1, p. 254) that the explosion time is infinite a.s.

In the case where  $\alpha \in [0, 1)$ , weak existence and uniqueness in law are consequence of the Girsanov theorem, the Novikov's criterion being satisfied. Moreover, pathwise uniqueness follows from Proposition 3.2 and Corollary 3.4 in [19], pp. 359-360, using local times.

Assume this time that  $\alpha \in (-1, 0)$ . By using a suitable scaling transformation as in (3.1), it suffices to study the equation (3.2). Consider the following Cauchy problem:

$$\varphi'(s) = \varphi(s)^\gamma, \quad \varphi(0) = 1, \quad \text{where} \quad \gamma := \frac{2\beta}{\alpha + 1}. \quad (3.4)$$

Its unique maximal solution  $\varphi \in C^2([0, t_1]; [1, \infty))$  is a diffeomorphism.

- If  $\gamma > 1$  then  $\varphi_\gamma(s) = (1 + (1 - \gamma)s)^{\frac{1}{1-\gamma}}$  with  $t_1 = \frac{1}{\gamma-1}$ ,
- if  $\gamma = 1$  then  $\varphi_\gamma(s) = e^s$  with  $t_1 = \infty$ ,
- if  $\gamma < 1$  then  $\varphi_\gamma(s) = (1 + (1 - \gamma)s)^{\frac{1}{1-\gamma}}$  with  $t_1 = \infty$ .

The corresponding scaling transformation  $\Phi : \Omega \rightarrow \Omega^*$  is given by

$$\Phi(\omega)(s) = \frac{\omega(\varphi_\gamma(s))}{\varphi_\gamma(s)^{\gamma/2}}, \quad s \in [0, t_1), \omega \in \Omega. \quad (3.5)$$

Then, it suffices to study the equation (3.2) which can be written

$$dY_s = dW_s + \rho \operatorname{sgn}(Y_s) |Y_s|^\alpha ds - \frac{\gamma}{2} \varphi(s)^{\gamma-1} Y_s ds, \quad Y_0 = x_0. \quad (3.6)$$

Since the function  $x \mapsto |x|^\alpha$  belongs to  $L_{\text{loc}}^1(\mathbb{R})$  there exists a unique solution, defined up to the explosion time (see [5], Proposition 2.2, p. 28), for the time-homogeneous equation

$$dH_s = dW_s + \rho \operatorname{sgn}(H_s) |H_s|^\alpha ds, \quad H_0 = x_0. \quad (3.7)$$

By using Feller's test for explosion and the properties of the scale function and the speed measure (see, for example, [13], Propositions 5.22 and 5.32, pp. 345-350,) we can see that the explosion time is infinite a.s. We deduce by Girsanov's transformation that there exists

a unique solution to the equation (3.6) defined for all  $s \in [0, t_1)$ . In order, we consider for every integer  $n \geq 1$ , the truncated equation

$$dY_s^{(n)} = dW_s + \rho \operatorname{sgn}(Y_s^{(n)}) |Y_s^{(n)}|^\alpha ds - \frac{\gamma}{2} \varphi^{\gamma-1}(s) Y_s^{(n)} \mathbf{1}_{\{|Y_s^{(n)}| \leq n\}} ds, \quad Y_0^{(n)} = x_0. \quad (3.8)$$

By applying Girsanov's theorem to the equations (3.7) and (3.8) we can deduce the existence and uniqueness for the equation (3.8). Hence we have existence and the uniqueness for the equation (3.6) up to  $t_1$ . Finally, we deduce that there exists a unique solution  $X := \Phi^{-1}(Y)$  to the equation (2.1) defined for all  $t \geq 1$ . Pathwise uniqueness follows from Proposition 3.2 and Corollary 3.4 in [19], pp. 359-360, using local time.  $\square$

Let us complete the previous proposition with a result concerning the explosion time of the solution when  $\rho \in (0, \infty)$  and  $\alpha \in (1, \infty)$ . Since the equation is not time-homogeneous, the Feller test for explosion does not apply.

**Proposition 3.2.** *Assume that  $\alpha \in (1, \infty)$ ,  $\rho \in (0, \infty)$  and  $\beta \in \mathbb{R}$  and let us consider  $\tau_e$  the explosion time of  $X$ . The following properties hold:*

1. *if  $\beta \in (-\infty, \frac{\alpha+1}{2}]$  then  $\tau_e$  is finite a.s.;*
2. *if  $\beta \in (\frac{\alpha+1}{2}, \infty)$  then  $\tau_e$  satisfies  $\mathbb{P}(\tau_e = \infty) \in (0, 1)$ .*

**Proof.** Firstly, assume that  $\gamma = \frac{2\beta}{\alpha+1} \leq 1$ . Denote by  $Y$  the solution of the equation obtained from (2.1) by applying the change of time (3.5) and show that its explosion time is finite a.s. By using the comparison theorem and Ito's formula, we can see that  $Y^2 \leq H$ , where  $H$  denotes the pathwise unique solution of the time-homogeneous equation:

$$dH_s = 2\sqrt{H_s} dW_s + \left( \rho H_s^{\frac{\alpha+1}{2}} - \frac{|\gamma|}{2} H_s + 1 \right) ds, \quad H_0 = x_0^2.$$

We use properties of the scale function and of the speed measure to prove that the explosion time of  $H$  is finite a.s. (see for instance, [13], Propositions 5.22 and 5.32, pp. 345-350), which implies that the explosion time of  $X$  is also finite a.s.

Secondly, assume that  $\gamma > 1$  and let  $t_1 = \frac{1}{\gamma-1}$ . By using the change of time (3.5) the equation (3.6) can be written:

$$dY_s = dW_s - \frac{\delta Y_s}{t_1 - s} ds + \rho \operatorname{sgn}(Y_s) |Y_s|^\alpha ds, \quad Y_0 = x_0, \quad \text{where} \quad \delta := \frac{\gamma}{2(\gamma-1)}. \quad (3.9)$$

We have  $\tau_e = \infty$  if and only if the explosion time  $\eta_e$  of (3.9) satisfies  $\eta_e \in \{t_1, \infty\}$ . Thanks to Girsanov's theorem, we see that the distribution function of  $\eta_e$  can be expressed in terms of the expectation of a functional of the  $\delta$ -Brownian bridge  $b$ , solution of the equation

$$db_s = dW_s - \frac{\delta b_s}{t_1 - s} ds, \quad b_0 = x_0. \quad (3.10)$$

Indeed, we construct a truncated stochastic differential as follows: for every integer  $n \geq 1$ , we set  $v_n(y) := \rho \operatorname{sgn}(y) |y|^\alpha \mathbf{1}_{\{|y| \leq n\}}$  and consider the equation

$$dY_s^{(n)} = dW_s - \frac{\delta Y_s^{(n)}}{t_1 - s} ds + v_n(Y_s^{(n)}) ds, \quad Y_0^{(n)} = x_0. \quad (3.11)$$

Note that  $b$  is defined for all  $s \in [0, t_1)$  and  $\lim_{s \rightarrow t_1} b_s = 0$  a.s. (see, for instance, [15]). By using Girsanov's theorem we can define on  $[0, t_1]$  the unique solution  $Y^{(n)}$  to the equation (3.11) and hence we can also construct the unique solution  $Y$  to the equation (3.9) defined up to the explosion time  $\eta_e \in [0, t_1] \cup \{+\infty\}$ . Choose  $n_0$  such that  $x_0 \in [0, n_0)$  and set, for every  $n \geq n_0$ ,

$$\sigma_n := \inf\{s \in [0, t_1) : |b_s| \geq n\} \quad \text{and} \quad \eta_n := \inf\{s \in [0, t_1) : |Y_s^{(n)}| \geq n\}.$$

For any integer  $n \geq 1$ ,  $s \in [0, t_1]$  and any bounded measurable real function  $F_s$  defined on  $C([0, s]; \mathbb{R})$ , we have

$$\mathbb{E} [F_s(Y) \mathbf{1}_{\{\eta_n > s\}}] = \mathbb{E} [F_s(b) \mathcal{E}(s) \mathbf{1}_{\{\sigma_n > s\}}],$$

where for all  $s \geq 0$ ,

$$\mathcal{E}(s) := \exp \left( \int_0^s \rho \operatorname{sgn}(b_s) b_s^\alpha db_s - \int_0^s \frac{\rho^2 b_s^{2\alpha}}{2} ds \right).$$

Letting  $n \rightarrow \infty$  and we obtain:

$$\mathbb{E} [F_s(Y) \mathbf{1}_{\{\eta_e > s\}}] = \mathbb{E} [F_s(b) \mathcal{E}(s)]. \quad (3.12)$$

In particular, the distribution function of  $\eta_e$  is given, for all  $s \in [0, t_1]$ , by  $\mathbb{P}(\eta_e > s) = \mathbb{E} [\mathcal{E}(s)]$ . Since  $\mathcal{E}$  is a positive super-martingale, the distribution function  $s \mapsto \mathbb{P}(\eta_e > s)$  is continuous on  $[0, t_1]$ . Hence,  $\mathbb{P}(\eta_e = t_1) = 0$ ,  $\mathbb{P}(\tau_e = \infty) = \mathbb{P}(\eta_e > t_1) = \mathbb{E} [\mathcal{E}(t_1)]$  and since  $\mathcal{E}(t_1) > 0$  a.s. it is clear that  $\mathbb{P}(\tau_e = \infty) > 0$ .

Besides, we note that since  $b$  converges toward 0 a.s., by using (3.12) we obtain that  $Y$  converges toward 0 a.s. on  $\{\eta_e > t_1\}$  and by the scaling transformation (3.5) we deduce that:

$$\lim_{t \rightarrow \infty} \frac{X_t}{t^{\frac{\beta}{\alpha+1}}} = 0 \quad \text{a.s. on } \{\tau_e = \infty\}. \quad (3.13)$$

To complete the proof it remains to show that  $\mathbb{P}(\tau_e = \infty) < 1$ . Let  $\lambda$  and  $c$  two positive constants and we set

$$V(t, x) := \exp \left( \int_0^x \frac{\lambda dy}{\rho |y|^\alpha + c} \right).$$

Let  $\mathcal{L}$  be the infinitesimal generator of the space-time diffusion  $t \mapsto (t, X_t)$  given by (3.3). Then, we can prove that for every  $c$ , one can choose  $\lambda$  such that for all  $x \in \mathbb{R}$  and  $t \in [1, 2]$ ,

$$\mathcal{L}V(t, x) \geq \frac{\lambda}{2^{\beta+1}} V(t, x).$$

Moreover, for every  $\lambda$ , one can choose  $c$  such that

$$\ln \left( \frac{\sup_{x \in \mathbb{R}} V(2, x)}{V(1, x_0)} \right) = \int_{x_0}^\infty \frac{1}{\rho |y|^\alpha + c} dy < 1.$$

The conditions of Theorem 10.2.1 in [20], p. 254 being satisfied, we conclude that  $X$  explodes with positive probability.  $\square$

Let us consider now the case where  $\alpha \in (-\infty, -1)$ . Here the function  $x \mapsto |x|^\alpha$  is not integrable in a neighbourhood of 0. As for the time-homogeneous situation we show that when  $\rho \in (0, \infty)$ , there exists a unique non-negative solution. If  $\rho \in (-\infty, 0)$  then it can be shown by using the comparison theorem that every solution is defined up to the reaching time of 0, which is finite a.s.

**Proposition 3.3.** *Assume that  $\rho \in (0, \infty)$ ,  $\alpha \in (-\infty, -1)$  and  $\beta \in \mathbb{R}$ . There exists a unique non-negative pathwise solution  $X$  to the equation (2.1) defined for all  $t \geq 1$ . Moreover,*

1. *if  $x_0 \in (0, \infty)$  then  $X$  is the unique pathwise solution and it is positive;*
2. *if  $x_0 = 0$  then for all  $t > 1$ ,  $X_t > 0$  a.s. and the set of all weak solutions is the set of all distributions which are mixing of the distributions of  $X$  and  $-X$ .*

**Proof.** Assume first that  $x_0 \in (0, \infty)$ . The proof is based on the same arguments as for the proof of Proposition 3.1, case  $\alpha \in (-1, 0)$ . From the classification of isolated singular points (see [5], Theorem 2.16, p. 37) we deduce that equation (3.7) admits a unique solution. Note that this solution is positive (since Girsanov's transformation between (3.6) and (3.7) gives an equivalent measure) and, as previously, it can be proved that there exists a unique solution to the equation (3.6) up to  $t_1$ , the maximal time of existence of the solution of (3.4). We can conclude by using the scaling transformation (3.5).

Secondly, assume that  $x_0 = 0$ . Similarly, by using the classification of isolated singular points (see [5], Theorems 2.16 and 3.6, p. 37 and p. 66), we can deduce that the equation (3.7) admits a unique non-negative pathwise solution  $Z$  and for all  $s > 0$ ,  $Z_s > 0$  a.s. By symmetry,  $-Z$  is the unique non-positive solution and the set of all weak solutions is the set of all distributions which are mixing of distributions of  $Z$  and  $-Z$ . By using the scaling transformation (3.5) and by noting that the Girsanov transformation induce a bijection between the solutions of equations (3.7) and (3.6) we can conclude as for the case  $x_0 \in (0, \infty)$ .  $\square$

It remains to study the case  $\rho \in \mathbb{R}^*$ ,  $\alpha = -1$  and  $\beta \in \mathbb{R}^*$ . Set  $v_t := \frac{\rho}{t^\beta}$  and consider

$$dY_t = 2\sqrt{|Y_t|} dB_t + (2v_t + 1)dt, \quad Y_0 = x_0^2. \quad (3.14)$$

As for the Bessel processes, the square of any solution of the equation (2.1) is the unique solution of the equation (3.14) and if  $\rho \in (-\infty, 0)$  it can be shown that every solution is defined up to the reaching time of 0, which is finite a.s. We show that if  $\rho \in (0, \infty)$  we have existence and uniqueness.

**Proposition 3.4.** *Assume that  $\rho \in (0, \infty)$ ,  $\alpha = -1$  and  $\beta \in \mathbb{R}^*$ . There exists a unique non-negative pathwise solution  $X$  to the equation (2.1) defined for all  $t \geq 1$ . Moreover,*

1. *if  $\rho \in [\frac{1}{2}, \infty)$ ,  $\beta \in (-\infty, 0)$  and  $x_0 \in (0, \infty)$  then  $X$  is the unique pathwise solution and it is positive;*
2. *if  $\beta \in (0, \infty)$  or if  $\rho \in (0, \frac{1}{2})$  and  $\beta \in (-\infty, 0)$  then we can construct different solutions to the equation (2.1) and in the first case, the set  $\{t \geq 1 : X_t = 0\}$  is unbounded a.s.;*
3. *if  $\rho \in [\frac{1}{2}, \infty)$ ,  $\beta \in (-\infty, 0)$  and  $x_0 = 0$  then the set of all solutions is the set of all distributions which are mixing of distributions of  $X$  and  $-X$ .*

**Proof.** By using the comparison theorem we show that there exists a unique pathwise solution  $Y$  to the equation (3.14) and by Ito's formula, for all  $\varepsilon > 0$ ,

$$\sqrt{Y_t + \varepsilon} = \sqrt{x_0^2 + \varepsilon} + \int_1^t \sqrt{\frac{Y_s}{Y_s + \varepsilon}} dB_s + \int_1^t \frac{v_s}{\sqrt{Y_s + \varepsilon}} ds + \frac{\varepsilon}{2} \int_1^t \frac{ds}{\sqrt{(Y_s + \varepsilon)^3}}.$$

Again, by using the comparison theorem, there exists a Bessel process  $R$  of dimension  $\delta \in (1, \infty)$ , such that  $Y \geq R^2$  on  $[1, t]$ . On the one hand, we can see that

$$\int_1^t \frac{ds}{R_s} < \infty \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{2} \int_1^t \frac{ds}{\sqrt{(R_s^2 + \varepsilon)^3}} = 0 \quad \text{a.s.}$$

and on the other hand,

$$\lim_{\varepsilon \rightarrow 0} \int_1^t \sqrt{\frac{Y_s}{Y_s + \varepsilon}} dB_s \stackrel{L^2}{=} B_t - B_1.$$

Hence, by letting  $\varepsilon \rightarrow 0$ , we deduce that  $X := \sqrt{Y}$  is a non-negative solution of the equation (2.1) and it is pathwise unique since uniqueness holds for (3.14).

Firstly, if  $\rho \in [1/2, \infty)$ ,  $\beta \in (-\infty, 0)$  and  $x_0 \in (0, \infty)$  then  $X$  is the unique pathwise solution. Indeed, every solution is positive by comparison with a Bessel process of dimension 2.

Secondly, if  $\beta \in (0, \infty)$  then  $\lim_{t \rightarrow \infty} v_t = 0^+$ . By using Markov's property and the comparison with a Bessel process of dimension  $\delta \in (1, 2)$  we get that the reaching time of 0 is finite a.s. and the set  $\{t \geq 1 : X_t = 0\}$  is unbounded a.s. If  $X$  is a solution starting from  $x_0 = 0$  then  $-X$  is also a solution. We deduce that different solutions could be constructed by gluing the paths of  $X$  or of  $-X$  each time when the process returns in 0.

Thirdly, if  $\rho \in (0, 1/2)$  and  $\beta \in (-\infty, 0)$  then by comparison with a Bessel process of dimension  $\delta \in (1, 2)$ , the reaching time of 0 belongs to  $[1, (2\rho)^{1/\beta})$  with a positive probability. Again, as in the preceding case, different solutions can be constructed.

Finally, if  $\rho \in [1/2, \infty)$ ,  $\beta \in (-\infty, 0)$  and  $x_0 = 0$  then by the comparison theorem with a Bessel process of dimension 2, every solution  $Z$  satisfies  $Z_t^2 \neq 0$  for all  $t > 1$  a.s. Let us introduce

$$\Omega^\pm := \{\omega \in \Omega : \forall t > 0, \pm Z_t > 0\} \in \cap_{t>0} \mathcal{F}_t \quad \text{and} \quad \mathbb{P}^\pm := \mathbb{P}(\cdot | \Omega^\pm).$$

The process  $\{B_t - B_1\}_{t \geq 1}$  is a standard Brownian motion under probabilities  $\mathbb{P}^\pm$ . By uniqueness of the non-negative solution and also, by symmetry, of the non-positive solution of (2.1), the distribution of  $Z$  under  $\mathbb{P}^\pm$  equals to the distribution of  $\pm X$ , where  $X$  is the pathwise unique non-negative solution. Hence, the set of all weak solutions is exactly the set of all distributions which are mixing of distributions of  $X$  and  $-X$ .  $\square$

**Remark.** The pathwise solution  $X$  we have constructed all along this section satisfies the strong Markov property. Indeed, for  $\alpha \in (-1, \infty)$ , this is true by uniqueness of the solution to (2.1) (see, for instance [12], Theorem 21.11, p. 421). The cited result does not apply for  $\alpha \in (-\infty, -1]$  because in this case it may exist different solutions. However, we have  $X \stackrel{\mathcal{L}}{=} Y^{\frac{1}{|\alpha|+1}}$  where  $Y$  is the unique strong Markov process solution of

$$dY_t = (|\alpha| + 1) Y_t^{\frac{|\alpha|}{|\alpha|+1}} dB_t + \left( \frac{\rho(|\alpha| + 1)}{t^\beta} + (|\alpha| + 1) |\alpha| Y_t^{\frac{|\alpha|-1}{|\alpha|+1}} \right) dt, \quad Y_{t_0} = x_0^{|\alpha|+1}.$$

Which implies our claim.  $\square$

To conclude let us sum up the informations which will be useful in the next section:

- attractive case  $\rho \in (-\infty, 0)$ :

- if  $\alpha \in (-\infty, -1]$  then every solution to the equation (2.1) is defined up to the reaching time of 0, which is finite a.s.
- if  $\alpha \in (-1, \infty)$  then there exists a unique pathwise solution  $X$  to the equation (2.1).
- repulsive case  $\rho \in (0, \infty)$ :
  - if  $\alpha \in (-\infty, -1)$  then there exists a unique pathwise solution  $X$  to the equation (2.1), taking its values in  $(0, \infty)$ , except perhaps at the initial time.
  - if  $\alpha = -1$  then there exists a unique non-negative pathwise solution  $X$  to the equation (2.1) and if  $\beta \in (0, \infty)$  then the set  $\{t \geq 1 : X_t = 0\}$  is unbounded a.s.
  - if  $\alpha \in (-1, 1]$  then there exists a unique pathwise solution  $X$  to the equation (2.1).
  - if  $\alpha \in (1, \infty)$  then there exists a unique pathwise solution  $X$  to the equation (2.1) defined up to the explosion time  $\tau_e$  and if  $\beta \in ((\alpha+1)/2, \infty)$  then  $\mathbb{P}(\tau_e < \infty) \in (0, 1)$ , otherwise  $\tau_e < \infty$  a.s.

## 4 Asymptotic behaviour of solutions

We present here the systematic study of the recurrence, transience or convergence of the solution  $X$  of the equation (2.1). We also give the asymptotic behaviour of its sample paths. For latter reference and to simplify our statements, we shall say that:

- $X$  has the Gaussian typical behaviour if, conditionally to the set of non-explosion:

$$\lim_{t \rightarrow \infty} \frac{|X_t|}{\sqrt{t}} \stackrel{\mathcal{L}}{=} |G|, \quad \text{with } G \sim \mathcal{N}(0, 1), \quad (4.1)$$

- $X$  satisfy the iterated logarithm law if, conditionally to the set of non-explosion:

$$\limsup_{t \rightarrow \infty} \frac{|X_t|}{\sqrt{2t \ln \ln t}} = 1 \quad \text{a.s.} \quad (4.2)$$

- $X$  is recurrent if, conditionally to the set of non-explosion,  $X$  is recurrent in

$$S = \begin{cases} \mathbb{R} & \text{provided } \alpha > -1; \\ [0, \infty) & \text{provided } \alpha = -1; \\ (0, \infty) & \text{provided } \alpha < -1. \end{cases}$$

**Remark.** We have already seen that the conditional probability of non-explosion is not trivial if and only if  $\alpha \in (1, \infty)$  and  $\beta \in (\frac{\alpha+1}{2}, \infty)$  (see Propositions 3.1 and 3.2). Moreover, if  $\alpha \in (-1, \infty)$  then, by symmetry, we can remove the absolute values in (4.1) and (4.2) and we can replace  $X$  by  $-X$ .  $\square$

We split our analysis in five cases. The first two cases are essentially based on the behaviour of the Brownian motion and the results are simple to prove. For the time-homogeneous case and for the general case, the proofs are more technical. From this point of view, the general

case contains a number of ideas which are simpler to be explained in the time-homogeneous situation. The Bessel type case ( $\alpha = -1$ ) can be handled by the same technics as the general case.

**a. Brownian motion case:**  $\rho = 0$ .

The solution is the standard Brownian motion which is a recurrent process and (2.2) holds.

**b. Linear case:**  $\rho \in \mathbb{R}^*$ ,  $\alpha = 1$  and  $\beta$  arbitrary

The Ornstein-Uhlenbeck process ( $\beta = 0$ ) has already been treated in Section 2. More generally, if  $\alpha = 1$  then  $X$  is again a Gaussian process and it can be expressed as a time changed Brownian motion. The result is obtained by finding a harmonic function for the space-time diffusion  $t \mapsto (t, X_t)$ . First we consider a particular case which is the analogue, in continuous time, of the Friedman's urn model [7]. The recurrent and the transient behaviour depend only on the position of  $\rho$  with respect to  $\frac{1}{2}$ .

**Proposition 4.1.** *Assume that  $\rho \in \mathbb{R}^*$  and  $\alpha = \beta = 1$ .*

1.  $X$  is recurrent if and only if  $\rho \in (-\infty, \frac{1}{2}]$ .

i) If  $\rho \in (-\infty, \frac{1}{2})$  then it has the Gaussian typical behaviour (4.1) and it satisfies

$$\limsup_{t \rightarrow \infty} \frac{X_t}{\sqrt{2t \ln \ln t}} = (1 - 2\rho)^{-\frac{1}{2}} \quad a.s. \quad (4.3)$$

ii) If  $\rho = \frac{1}{2}$  then it satisfies

$$\lim_{t \rightarrow \infty} \frac{X_t}{\sqrt{t \ln t}} \stackrel{\mathcal{L}}{=} \mathcal{N}(0, 1) \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{X_t}{\sqrt{2t \ln t \ln \ln t}} = 1 \quad a.s. \quad (4.4)$$

2.  $X$  is transient if and only if  $\rho \in (1/2, \infty)$  and it satisfies

$$\lim_{t \rightarrow \infty} \frac{X_t}{t^\rho} = G \quad a.s. \quad \text{with} \quad G \sim \mathcal{N}\left(x_0, (2\rho - 1)^{-1}\right). \quad (4.5)$$

**Proof.** Let us note that  $(t, x) \mapsto x/t^\rho$  is a harmonic function for the space-time diffusion  $t \mapsto (t, X_t)$ . Hence, by Ito's formula and the martingale representation theorem:

$$\frac{X_t}{t^\rho} = x_0 + \int_1^t \frac{dB_s}{s^\rho} = \tilde{B}_{\phi(t)}, \quad \text{with} \quad \phi(t) := \frac{t^{1-2\rho} - 1}{1 - 2\rho}.$$

Here  $\tilde{B}$  denotes a standard Brownian motion. By using (2.2) we prove the convergence in distribution but also the pathwise largest deviations in (4.3), (4.4) and (4.5). Moreover, we obtain  $\limsup_{t \rightarrow \infty} X_t = \infty$  a.s. and by symmetry,  $\liminf_{t \rightarrow \infty} X_t = -\infty$  a.s. Therefore we have the recurrent behaviour if and only if  $\rho \in (-\infty, 1/2)$ . The transient feature is obtained as a consequence of the convergence in (4.5).  $\square$

We proceed with the case where  $\beta$  is arbitrary.

**Proposition 4.2.** *Assume that  $\rho \in \mathbb{R}^*$ ,  $\alpha = 1$  and  $\beta \in \mathbb{R} \setminus \{1\}$ .*

1.  $X$  is recurrent if and only if  $\beta \in (1, \infty)$  or  $\rho \in (-\infty, 0)$  and  $\beta \in [0, 1)$ .

- i) If  $\beta \in (1, \infty)$  then it has the Gaussian typical behaviour (4.1) and it satisfies the iterated logarithm law (4.2).
- ii) If  $\rho \in (-\infty, 0)$  and  $\beta \in [0, 1)$  then it satisfies

$$\lim_{t \rightarrow \infty} \frac{X_t}{t^{\frac{\beta}{2}}} \stackrel{\mathcal{L}}{=} \mathcal{N}(0, (2|\rho|)^{-1}) \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{X_t}{t^{\frac{\beta}{2}} (\ln t)^{\frac{1}{2}}} = \left( \frac{\beta - 1}{\rho} \right)^{\frac{1}{2}} \quad \text{a.s.} \quad (4.6)$$

2.  $X$  is transient if and only if  $\rho \in (0, \infty)$  and  $\beta \in (-\infty, 1)$  and it satisfies

$$\lim_{t \rightarrow \infty} \frac{X_t}{\exp\left(\frac{\rho t^{1-\beta}}{1-\beta}\right)} = G \quad \text{a.s.} \quad \text{with} \quad G \sim \mathcal{N}(m, \sigma^2), \quad (4.7)$$

where

$$m := x_0 \exp\left(\frac{\rho}{\beta - 1}\right) \quad \text{and} \quad \sigma^2 := \int_1^\infty \exp\left(\frac{2\rho s^{1-\beta}}{\beta - 1}\right) ds.$$

3.  $X$  converges almost surely toward 0 if and only if  $\rho \in (-\infty, 0)$  and  $\beta \in (-\infty, 0)$  and, in that case, (4.6) holds.

The same ideas as for the latter proof are employed. We use the harmonic function  $(t, x) \mapsto x \exp\left(\frac{\rho t^{1-\beta}}{\beta - 1}\right)$  and Hospital's rule to obtain the precise equivalent for the change of time. The proof is left to the reader.

**c. Homogeneous case:  $\rho \in \mathbb{R}^*$ ,  $\beta = 0$  and  $\alpha$  arbitrary.**

The Bessel process ( $\alpha = -1$ ) has already been treated in the Section 2, whereas the linear case ( $\alpha = 1$ ) has been studied previously. In the time-homogeneous case,  $X$  is either positive-recurrent, null-recurrent or transient. The proof is based on properties of scale function and speed measure. To obtain the asymptotic properties of the sample paths it suffices to apply Motoo's theorem (see Theorem 2.1.) The limit in distribution is a consequence of the ergodic theorem together with, in the null-recurrent case, an appropriate change of time and the scaling transformation (3.1). By using the properties of the scale function, we can compute the probability to be transient at  $\pm\infty$ .

**Proposition 4.3.** *Assume that  $\rho \in \mathbb{R}^*$ ,  $\alpha \in \mathbb{R} \setminus \{\pm 1\}$  and  $\beta = 0$ .*

1.  $X$  is positive recurrent if and only if  $\rho \in (-\infty, 0)$  and  $\alpha \in (-1, \infty)$  and it satisfies

$$\lim_{t \rightarrow \infty} X_t \stackrel{\mathcal{L}}{=} \Pi \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{X_t}{(\ln t)^{\frac{1}{\alpha+1}}} = \left( \frac{\alpha + 1}{2|\rho|} \right)^{\frac{1}{\alpha+1}} \quad \text{a.s.} \quad (4.8)$$

where  $\Pi$  is the speed measure associated to  $X$ , normalized to be a probability.

2.  $X$  is null recurrent if and only if  $\rho \in (0, \infty)$  and  $\alpha \in (-\infty, -1)$ . It has the Gaussian typical behaviour (4.1) and it satisfies the iterated logarithm law (4.2).
3.  $X$  is transient if and only if  $\rho \in (0, \infty)$  and  $\alpha \in (-1, 1)$  and it satisfies

$$\lim_{t \rightarrow \infty} \frac{|X_t|}{t^{\frac{1}{1-\alpha}}} = (\rho(1 - \alpha))^{\frac{1}{1-\alpha}} \quad \text{a.s.} \quad (4.9)$$



**Proof.** Assume that  $\rho \in (-\infty, 0)$  and  $\alpha \in (-1, \infty)$ . The scale function and the speed measure of  $X$  are given on  $\mathbb{R}$  by:

$$s(x) := \int_0^x \exp\left(\frac{2|\rho||y|^{\alpha+1}}{\alpha+1}\right) dy \quad \text{and} \quad m(dx) := \exp\left(\frac{2\rho|x|^{\alpha+1}}{\alpha+1}\right) dx. \quad (4.10)$$

Since  $s(\mathbb{R}) = \mathbb{R}$ , we can conclude  $X$  is a recurrent process (see, for instance, [13], Proposition 5.22, p. 345) and since  $m(\mathbb{R}) < \infty$ , we obtain the first equality in (4.8), by using the ergodic theorem. Furthermore, by using Motoo's theorem (see Theorem 2.1) with the functions  $h(t) := (\lambda \ln t)^{\frac{1}{\alpha+1}}$ , we deduce the second equality in (4.8).

We go further and suppose that  $\rho \in (0, \infty)$  and  $\alpha \in (-1, 1)$ . Since  $s(+\infty) < +\infty$  and  $s(-\infty) > -\infty$ , we obtain this time that  $X$  is transient. Note that (4.9) is a direct consequence of Theorems 2 and 5 in [9], pp. 126-132.

Finally, it remains to study the null-recurrent case. Assume that  $\rho \in (0, \infty)$  and  $\alpha \in (-\infty, -1)$ . As previously it is easy to check that  $X$  is recurrent in  $(0, \infty)$ . To prove the convergence in distribution we shall use the following important:

**Lemma 4.1.** *Let  $Z$  be the unique Markov solution of the stochastic differential equation with continuous coefficients:*

$$dZ_s = a(s, Z_s)dW_s + b(s, Z_s)ds, \quad Z_0 = z_0. \quad (4.11)$$

*Assume that this equation is asymptotically time-homogeneous, in the sense that*

$$\lim_{s \rightarrow \infty} a(s, z) = a_\infty(z) \quad \text{and} \quad \lim_{s \rightarrow \infty} b(s, z) = b_\infty(z) \quad \text{uniformly on compact subsets of } \mathbb{R} \quad (4.12)$$

*and asymptotically  $\Pi$ -ergodic, in the sense that the diffusion  $\zeta$  satisfying*

$$d\zeta_s = a_\infty(\zeta_s)dW_s + b_\infty(\zeta_s)ds \quad (4.13)$$

*is an ergodic diffusion having  $\Pi$  as stationary distribution. Moreover, assume that  $Z$  is bounded in probability: for all  $\varepsilon > 0$  there exists  $r > 0$  such that*

$$\sup_{s \geq 0} \mathbb{P}(|Z_s| \geq r) < \varepsilon. \quad (4.14)$$

*Then*

$$\lim_{s \rightarrow \infty} Z_s \stackrel{\mathcal{L}}{=} \Pi. \quad (4.15)$$

We postpone the proof of the lemma and finish the proof of Proposition 4.3. We denote  $Y$  the non-negative process obtained from  $X$  by the scaling transformation (3.1) with  $\varphi(s) = e^s$  and set  $Z := Y^{|\alpha|+1}$ . It is not difficult to see that  $X$  has the Gaussian typical behaviour if and only if

$$\lim_{s \rightarrow \infty} Z_s \stackrel{\mathcal{L}}{=} |G|^{|\alpha|+1}, \quad \text{with } G \sim \mathcal{N}(0, 1). \quad (4.16)$$

By using Ito's formula, we can see that the diffusion  $Z$  satisfies

$$dZ_s = a(Z_s)dW_s + h(Z_s)ds + \ell(s)ds,$$

where, for every  $s, z \in [0, \infty)$ ,

$$a(z) := (|\alpha| + 1)z^{\frac{|\alpha|}{1+|\alpha|}}, \quad 2h(z) := (|\alpha| + 1)(\alpha z^{\frac{|\alpha|-1}{|\alpha|+1}} - z) \quad \text{and} \quad \ell(s) := \rho(|\alpha| + 1)e^{\frac{\alpha+1}{2}s}.$$

Note that the coefficients  $a$ ,  $h$  and  $\ell$  are continuous. Then (4.16) is a consequence of Lemma 4.1. Indeed,  $Z$  is asymptotically homogeneous and the time-homogeneous diffusion  $\zeta$ , associated to  $Z$ ,

$$d\zeta_s = a(\zeta_s)dW_s + h(\zeta_s)ds,$$

is the image under the application  $x \mapsto |x|^\alpha$  of the Ornstein-Uhlenbeck process, which in turn, it has the standard Gaussian probability as a stationary distribution. To conclude, it remains to prove that  $Z$  is bounded in probability. This is clear by the comparison theorem, since  $Z \leq \xi$ , with  $\xi$  the ergodic diffusion (and therefore bounded in probability), satisfying

$$d\xi_s = a(\xi_s)dW_s + h(\xi_s)ds + \ell(0)ds.$$

This reasoning allows to deduce (4.1). Moreover, by using Motoo's theorem it can be shown that the diffusions  $\xi$  and  $\zeta$  satisfy

$$\limsup_{s \rightarrow \infty} \frac{\xi_s^{\frac{1}{|\alpha|+1}}}{2 \ln s} = \limsup_{s \rightarrow \infty} \frac{\zeta_s^{\frac{1}{|\alpha|+1}}}{2 \ln s} = 1 \quad \text{a.s.}$$

Hence, recalling that  $\zeta \leq Z \leq \xi$  a.s., we get  $\limsup_{s \rightarrow \infty} \frac{Y_s}{2 \ln s} = 1$  a.s. By using the scaling transformation (3.1) we obtain (4.2). The proof of the proposition is complete except for the lemma.  $\square$

**Proof of Lemma 4.1.** Let us denote  $\mathbb{P}_{(u,z)}$  the distribution of the diffusion  $Z$  with  $Z_u = z$  and  $\{T_{u,s} : 0 \leq u \leq s\}$  the associated time-inhomogeneous semi-group. Similarly, we denote  $\mathbb{P}_z$  the distribution of the diffusion  $\zeta$  and  $\{T_s : s \geq 0\}$  the associated semi-group. Clearly, the diffusion coefficient  $(s, z) \mapsto a(u+s, z)$  and the drift  $(s, z) \mapsto b(u+s, z)$  of the diffusion  $s \mapsto Z_{u+s}$  satisfy the hypothesis of Theorem 11.1.4 in [20], p. 264. We deduce that, for every  $f \in C_b([0, \infty); \mathbb{R})$  and  $s \in [0, \infty)$ ,

$$\lim_{u \rightarrow \infty} T_{u,u+s}f(z) = T_s f(z) \quad \text{uniformly in } z \text{ on compact subsets of } \mathbb{R}. \quad (4.17)$$

Moreover,

$$\lim_{s \rightarrow \infty} T_s f(z) = \Pi(f) \quad \text{uniformly in } z \text{ on compact subsets of } \mathbb{R}. \quad (4.18)$$

Indeed, assume that  $z$  belongs to the compact  $[a, b]$  with  $-\infty < a \leq b < \infty$ . By using the strong Markov property, we can prove that for all  $s \in [0, \infty)$  and  $v \in \mathbb{R}$ ,

$$\mathbb{P}_a(\zeta_s > v) \leq \mathbb{P}_z(\zeta_s > v) \leq \mathbb{P}_b(\zeta_s > v).$$

By using the ergodic theorem and these last inequalities we get the uniform convergence on compact subsets of  $\mathbb{R}$  in (4.18). To finish the proof of (4.15) it suffices to show that

$$\lim_{u \rightarrow \infty} T_{0,u}f(z_0) = \Pi(f). \quad (4.19)$$

By using Markov's property we can write, for all  $s, u \in [0, \infty)$ ,

$$T_{0,u+s}f(z_0) - \Pi(f) = T_{0,u} [T_{u,u+s}f - T_s f](z_0) + T_{0,u} [T_s f - \Pi(f)](z_0).$$

Clearly, for arbitrary  $r, s, u$  non-negative real numbers,

$$|T_{0,u+s}f(z_0) - \Pi(f)| \leq \sup_{z \in [-r, r]} \{|T_{u,u+s}f(z) - T_s f(z)| + |T_s f(z) - \Pi(f)|\} + 4\|f\|_\infty \mathbb{P}(|Z_u| \geq r).$$

Thanks to (4.17) and (4.18) for all  $r, \varepsilon > 0$  there exists  $s, u_0 \in [0, \infty)$  such that for all  $u \geq u_0$ ,

$$|\mathbb{T}_{0, u+s} f(z_0) - \Pi(f)| \leq 2(1 + 2\|f\|_\infty)\varepsilon + 4\|f\|_\infty \sup_{s \geq 0} \mathbb{P}(|Z_s| \geq r).$$

Since  $Z$  is bounded in probability the proof of the lemma is done.  $\square$

#### d. General case

Let us begin with the attractive case  $\rho \in (-\infty, 0)$ . The main difficulty is to obtain the boundedness in probability in Lemma 4.1, but also the pathwise largest deviations. The proof essentially uses a local comparison theorem (see Theorem 4.2 below).

Consider the following two diffusions:

$$dY_s = dW_s + \rho \operatorname{sgn}(Y_s) |Y_s|^\alpha ds - \frac{1}{2} Y_s ds \quad (4.20)$$

and

$$d\Upsilon_s = dW_s + \rho \operatorname{sgn}(\Upsilon_s) |\Upsilon_s|^\alpha ds. \quad (4.21)$$

**Theorem 4.1.** *Assume that  $\rho \in (-\infty, 0)$ ,  $\alpha \in (-1, \infty)$  and  $\beta \in \mathbb{R}$ .*

1.  *$X$  is recurrent if and only if  $\beta \in [0, \infty)$ .*

i) *If  $\beta = \frac{\alpha+1}{2}$  then it satisfies*

$$\lim_{t \rightarrow \infty} \frac{X_t}{\sqrt{t}} \stackrel{\mathcal{L}}{=} \Lambda, \quad (4.22)$$

where  $\Lambda$  is the stationary distribution of (4.20). Moreover, if  $\alpha \in (1, \infty)$  then

$$\limsup_{t \rightarrow \infty} \frac{X_t}{(t \ln \ln t)^{\frac{1}{\alpha+1}}} = \left( \frac{\alpha+1}{2|\rho|} \right)^{\frac{1}{\alpha+1}} \quad (4.23)$$

and if  $\alpha \in (-1, 1)$  then  $X$  satisfies the iterated logarithm law (4.2).

ii) *If  $\beta \in (\frac{\alpha+1}{2}, \infty)$  then it has the Gaussian typical behaviour (4.1) and it satisfies the iterated logarithm law (4.2).*

iii) *If  $\beta \in [0, \frac{\alpha+1}{2})$  then it satisfies*

$$\lim_{t \rightarrow \infty} \frac{X_t}{t^{\frac{\beta}{\alpha+1}}} \stackrel{\mathcal{L}}{=} \Pi \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{X_t}{t^{\frac{\beta}{\alpha+1}} (\ln t)^{\frac{1}{\alpha+1}}} = \left( \frac{\alpha+1-2\beta}{2|\rho|} \right)^{\frac{1}{\alpha+1}} \quad a.s. \quad (4.24)$$

Here  $\Pi$  is the stationary distribution of (4.21).

2.  *$X$  converges almost surely toward 0 if and only if  $\beta \in (-\infty, 0)$  and it satisfies (4.24).*

**Proof of Theorem 4.1.** Assume first that  $\beta = \frac{\alpha+1}{2} \in (0, \infty)$  and note that  $Y$  given by (4.20) is the process obtained from  $X$  by the scaling transformation (3.5). By applying the ergodic theorem to  $Y$  we get (4.22). Moreover, by using Motoo's theorem

$$\limsup_{t \rightarrow \infty} \frac{Y_t}{\sqrt{2 \ln t}} = 1 \quad \text{a.s. provided } \alpha \in (-1, 1),$$

and

$$\limsup_{t \rightarrow \infty} \frac{Y_t}{(\ln t)^{\frac{1}{\alpha+1}}} = \left( \frac{\alpha+1}{2|\rho|} \right)^{\frac{1}{\alpha+1}} \quad \text{a.s. provided } \alpha \in (1, \infty)$$

which implies (4.2) and (4.23). Furthermore, we obtain that  $\limsup_{t \rightarrow \infty} X_t = \infty$  a.s. and by symmetry,  $\liminf_{t \rightarrow \infty} X_t = -\infty$  a.s. and  $X$  is recurrent.

The proof of the case when  $\beta \in (\frac{\alpha+1}{2}, \infty)$  is based on the same arguments as in the second point of Proposition 4.3. We present only the different points and leave the details to the reader. In order, let us consider  $\tilde{Y}$  the process obtained from  $X$  by using the scaling transformation (3.1) with  $\varphi(s) = e^s$ . Let us note that the equation satisfied by  $\tilde{Y}$  has 0 as a singular point and so we can not apply Lemma 4.1 to  $\tilde{Y}$ . To prove that  $X$  has a Gaussian typical behaviour it suffices to show that

$$\lim_{s \rightarrow \infty} \tilde{Y}_s^3 \stackrel{\mathcal{L}}{=} G^3, \quad \text{with } G \sim \mathcal{N}(0, 1).$$

We shall apply Lemma 4.1 to the process  $\tilde{Y}^3$  which verifies a stochastic differential equation having continuous coefficients. All we need to do is to prove that  $\tilde{Y}^3$  is bounded in probability. Since  $\rho \in (-\infty, 0)$  this property can be obtained by comparing the non-negative diffusion  $\gamma = \tilde{Y}^2$ , which satisfies

$$d\gamma_s = 2\sqrt{\gamma_s} dw_s + (1 - \gamma_s) ds + 2\rho e^{(\frac{\alpha+1}{2}-\beta)s} \gamma_s^{\frac{\alpha+1}{2}} ds, \quad \gamma_0 = y_0^2, \quad (4.25)$$

with the ergodic (and therefore bounded in probability) non-negative diffusion  $r$  satisfying

$$dr_s = 2\sqrt{r_s} dw_s + (1 - r_s) ds, \quad r_0 = y_0^2.$$

Here  $w$  is a standard Brownian motion. Hence the stochastic processes  $\tilde{Y}^2$ , and then  $\tilde{Y}^3$ , are bounded in probability and we can apply Lemma 4.1. Moreover, by using Motoo's theorem and the same comparison, it is not difficult to see that

$$\limsup_{s \rightarrow \infty} \frac{\gamma_s}{2 \ln s} \leq 1 \quad \text{a.s.} \quad (4.26)$$

To deduce that  $X$  satisfies the iterated logarithm law (4.2) we need to prove the opposite inequality. In fact, we show that the time-inhomogeneous part in (4.25) could be omitted. Indeed, since  $\beta \in (\frac{\alpha+1}{2}, \infty)$  and  $\alpha \in (-1, \infty)$ , by using (4.26), there exists a random time  $\tau$  a.s. finite such that for all  $s \geq \tau$ ,

$$\left| 2\rho e^{(\frac{\alpha+1}{2}-\beta)s} \gamma_s^{\frac{\alpha+1}{2}} \right| \leq \frac{1}{2} \quad \text{a.s.}$$

Let  $v \geq 0$  and denote  $\varrho$  the unique pathwise solution of the equation

$$d\varrho_s = 2\sqrt{\varrho_s} dw_s + \left( \frac{1}{2} - \varrho_s \right) ds, \quad \varrho_v = \gamma_v.$$

At this level we need to recall a useful version of the comparison theorem which proof follows from similar arguments as in Theorem 1.1 in [11], pp. 352–353 by localisation:

**Theorem 4.2** (local comparison). *For  $i = 1, 2$  consider the one-dimensional stochastic differential equation*

$$dx_t^{(i)} = \sigma(x_t^{(i)})dB_t + b_i(t, x_t^{(i)})dt,$$

*with  $B$  a standard Brownian motion. Assume that there exists a function  $h : \mathbb{R} \rightarrow \mathbb{R}$  locally integrable in a neighbourhood of 0, such that for all  $x, y \in \mathbb{R}$*

$$|\sigma(x) - \sigma(y)| \leq h(|x - y|).$$

*For  $i = 1, 2$ , let  $x^{(i)}$  be the pathwise unique solution of the equation and denote by  $\tau_i$  the exit time of the space-time process  $t \mapsto (t, x_t^{(i)})$  from a Borel set  $D \subset [0, \infty) \times \mathbb{R}$ . If  $b_1 \leq b_2$  on  $D$ , then for all  $t \geq 0$ ,*

$$x_{t \wedge \tau_1}^{(1)} \leq x_{t \wedge \tau_2}^{(2)} \quad \text{a.s. on } \{x_0^{(1)} = x_0^{(2)}\}. \quad (4.27)$$

We return to the proof of Theorem 4.1 and use this local comparison: for all  $s \geq v$ ,  $\varrho_s \leq r_s$  a.s. on  $\{\tau \leq v\}$ . We deduce that

$$\limsup_{s \rightarrow \infty} \frac{\gamma_s}{2 \ln s} \geq \limsup_{s \rightarrow \infty} \frac{\varrho_s}{2 \ln s} = 1 \quad \text{a.s. on } \{\tau \leq v\}.$$

Since the last inequality holds for all  $v \geq 0$ , and since  $\tau$  is finite a.s., we obtain the opposite inequality in (4.26). Hence  $X$  satisfies the iterated logarithm law by using the scaling transformation. We deduce that  $\limsup_{t \rightarrow \infty} X_t = \infty$  a.s. and, by symmetry,  $\liminf_{t \rightarrow \infty} X_t = -\infty$  a.s. This involves that  $X$  is recurrent.

Finally, assume that  $\beta \in [0, \frac{\alpha+1}{2})$  and denote  $\bar{Y}$  the process obtained from  $X$  by the scaling transformation (3.5). We shall apply again Lemma 4.1 to the process  $\bar{Y}^3$ . The only technical point is to show that  $\bar{Y}^3$  is bounded in probability. Set again  $\gamma := \bar{Y}^2$  and we see by Itô's formula that there exists  $k \in \mathbb{R}$  and  $c \in (0, \infty)$  such that

$$d\gamma_s = 2\sqrt{\gamma_s} dw_s + \left(1 + 2\rho \gamma_s^{\frac{\alpha+1}{2}}\right) ds + \frac{k \gamma_s}{c+s} ds, \quad \gamma_0 = y_0^2. \quad (4.28)$$

As previously, we show that the time-inhomogeneous part in (4.28) could be omitted. Set  $\delta := (\alpha+1)/(\alpha+3)$ . Since  $\rho \in (-\infty, 0)$ , by the comparison theorem and the scaling transformation it can be shown that

$$\limsup_{s \rightarrow \infty} \frac{\gamma_s}{2s \ln \ln s} \leq 1 \quad \text{a.s.}$$

Hence, there exists a random time  $\tau$ , a.s. finite, such that for all  $s \geq \tau$ ,  $|(k\gamma_s)/(c+s)| \leq \gamma_s^\delta$  a.s. Let  $v \geq 0$  and denote  $\varrho$  the unique pathwise solution of the equation

$$d\varrho_s = 2\sqrt{\varrho_s} dw_s + \left(1 + \varrho_s^\delta + 2\rho \varrho_s^{\frac{\alpha+1}{2}}\right) ds, \quad \varrho_v = \gamma_v.$$

By using Theorem 4.2 again for all  $s \geq v$ ,  $\gamma_s \leq \varrho_s$  a.s. on  $\{\tau \leq v\}$ . Let  $\eta > 0$  and  $v \geq 0$  such that  $\mathbb{P}(\tau > v) \leq \eta$ . Since  $\varrho$  is ergodic, there exists  $\kappa \geq 0$  such that for all  $s \geq 0$ ,  $\mathbb{P}(\varrho_s \geq \kappa) \leq \eta$ . Putting together the last three inequalities we obtain that  $\sup_{s \geq 0} \mathbb{P}(\gamma_s \geq \kappa) \leq 2\eta$  which implies that  $\bar{Y}^3$  is bounded in probability. Therefore, by using Motoo's theorem, the preceding comparison implies that

$$\limsup_{s \rightarrow \infty} \frac{\gamma_s}{(\ln s)^{\frac{2}{\alpha+1}}} \leq \limsup_{s \rightarrow \infty} \frac{\varrho_s}{(\ln s)^{\frac{2}{\alpha+1}}} = \left(\frac{\alpha+1}{2|\rho|}\right)^{\frac{2}{\alpha+1}} \quad \text{a.s.}$$

The opposite inequality can be obtained in the same way. Hence, since  $\beta \in [0, \infty)$ , we have  $\limsup_{t \rightarrow \infty} X_t = \infty$  a.s. and  $\liminf_{t \rightarrow \infty} X_t = -\infty$  a.s. and  $X$  is recurrent. The proof of the second part of the theorem is identical, the only difference being that  $\beta \in (-\infty, 0)$  (which implies that  $X$  converges almost surely to 0) and it will be omitted.  $\square$

We turn now to the repulsive case,  $\rho \in (0, \infty)$ . The main difficulty is the case where  $\alpha \in (1, \infty)$  and  $\beta \in (\frac{\alpha+1}{2}, \infty)$  because the process explodes with a positive probability but not almost surely. In this case we can not apply Lemma 4.1 and we need to adapt it. The transient case is proved by using some martingale arguments as in [16].

**Theorem 4.3.** *Assume that  $\rho \in (0, \infty)$ ,  $\alpha \in \mathbb{R} \setminus \{-1, 1\}$  and  $\beta \in \mathbb{R}$ .*

1.  *$X$  is recurrent if and only if  $\beta \in [\frac{\alpha+1}{2}, \infty)$ .*

i) *If  $\beta \in (\frac{\alpha+1}{2}, \infty)$  then it has the Gaussian typical behaviour (4.1) and it satisfies the iterated logarithm law (4.2).*

ii) *If  $\beta = \frac{\alpha+1}{2}$  and  $\alpha \in (-\infty, 1)$  then it satisfies the iterated logarithm law (4.2) and*

$$\lim_{t \rightarrow \infty} \frac{X_t}{\sqrt{t}} \stackrel{\mathcal{L}}{=} \Lambda, \quad (4.29)$$

where  $\Lambda$  is the stationary distribution of (4.20).

2.  *$X$  is transient if and only if  $\beta \in (-\infty, \frac{\alpha+1}{2})$  and if  $\alpha \in (-\infty, 1)$  it satisfies*

$$\lim_{t \rightarrow \infty} \frac{|X_t|}{t^{\frac{1-\beta}{1-\alpha}}} = \left( \frac{\rho(1-\alpha)}{1-\beta} \right)^{\frac{1}{1-\alpha}} \quad a.s. \quad (4.30)$$

**Proof.** 1) *Recurrence.*

Assume first that  $\beta = \frac{\alpha+1}{2}$  and note that the process  $Y$  given by (4.20) is obtained from  $X$  by the scaling transformation (3.5). We can apply Motoo's theorem and the ergodic theorem to the process  $Y$  and we deduce the statement in 1 ii). Note that  $Y$  is ergodic in  $(0, \infty)$  for  $\alpha \in (-\infty, -1)$  and ergodic in  $\mathbb{R}$  for  $\alpha \in (-1, 1)$ . Hence  $X$  is recurrent in  $(0, \infty)$  for  $\alpha \in (-\infty, -1)$  and recurrent in  $\mathbb{R}$  for  $\alpha \in (-1, 1)$ . Recall also that this is an open problem for the discrete time problem considered in [16].

Assume now that  $\beta \in (\frac{\alpha+1}{2}, \infty)$  and note by  $\tilde{Y}$  the process obtained from  $X$  by the scaling transformation (3.1), with  $\varphi(s) = e^s$ . If  $\alpha \in (-\infty, -1)$  by applying Lemma 4.1 to the process  $\tilde{Y}^{|\alpha|+1}$  we can prove the convergence in distribution and the iterated logarithm law by the same method as was already used for the time-homogeneous case in Proposition 4.3. If  $\alpha \in (-1, 1)$  then the same arguments apply by considering the process  $\tilde{Y}^3$ . Next, assume that  $\alpha \in (1, \infty)$ . Recall that, with positive probability,  $X$  does not explode (see the proof of Proposition 3.2). Employing (3.13) it is a simple calculation to see that the drift of (2.1) satisfies

$$\lim_{t \rightarrow \infty} \frac{\int_1^t \frac{|X_s|^\alpha}{s^\beta} ds}{\sqrt{t}} = 0 \quad a.s. \text{ on } \{\tau_e = \infty\}.$$

Whence  $X$  satisfies the iterated logarithm law and therefore we have  $\limsup_{t \rightarrow \infty} X_t = \infty$  and  $\liminf_{t \rightarrow \infty} X_t = -\infty$  a.s. on  $\{\tau_e = \infty\}$ , it is recurrent when it does not explode. It remains to prove the convergence in distribution, that is it suffices to verify that

$$\lim_{s \rightarrow \infty} \mathbb{P}(\tilde{Y}_s > x \mid \tau_e = \infty) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{y^2}{2}\right) dy.$$

Recall that  $\tilde{Y}$  satisfies

$$d\tilde{Y}_s = dW_s + \rho e^{(\frac{\alpha+1}{2}-\beta)s} \tilde{Y}_s^\alpha ds - \frac{1}{2} \tilde{Y}_s ds. \quad (4.31)$$

Note that Lemma 4.1 does not apply directly here, since  $\tilde{Y}$  can explode in finite time. We show that the time-inhomogeneous part in (4.31) could be omitted. Let  $\varepsilon > 0$ . There exists a random time  $\tau$  a.s. finite on the set of non-explosion, such that for all  $s \geq \tau$ ,

$$\rho e^{(\frac{\alpha+1}{2}-\beta)s} |\tilde{Y}_s|^\alpha \leq \varepsilon \quad \text{a.s.}$$

Let  $v \geq 0$  and denote  $\tilde{Y}^{(\pm\varepsilon)}$  the unique pathwise solution of the equation

$$d\tilde{Y}_s^{(\pm\varepsilon)} = dW_s \pm \varepsilon ds - \frac{1}{2} \tilde{Y}_s^{(\pm\varepsilon)} ds, \quad \tilde{Y}_v^{(\pm\varepsilon)} = \tilde{Y}_v \mathbb{1}_{\{\tau_e > v\}}.$$

Since  $\tilde{Y}^{(\pm\varepsilon)}$  is Feller and ergodic the strong mixing property hold (see [12], Theorem 20.20, p. 408) and we obtain

$$\lim_{s \rightarrow \infty} \mathbb{P}(\tilde{Y}_s^{(\pm\varepsilon)} > x \mid \tau \leq v) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{(y \pm 2\varepsilon)^2}{2}\right) dy.$$

By using Theorem 4.2 we have a.s. on  $\{\tau \leq v\}$  the comparison  $\tilde{Y}_s^{(-\varepsilon)} \leq \tilde{Y}_s \leq \tilde{Y}_s^{(+\varepsilon)}$  for all  $s \geq v$  and letting  $\varepsilon \rightarrow 0$ , we get

$$\lim_{s \rightarrow \infty} \mathbb{P}(\tilde{Y}_s > x \mid \tau \leq v) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{y^2}{2}\right) dy.$$

Noting that the set of non-explosion is  $\{\tau_e = \infty\} = \cup_{v \geq 0} \{\tau \leq v\}$  we can conclude.

2) *Transience.* To simplify our computations, let us denote the limit and the exponent of  $t$  in (4.30) respectively by

$$\ell := \left( \frac{\rho(1-\alpha)}{1-\beta} \right)^{\frac{1}{1-\alpha}} \quad \text{and} \quad \nu := \frac{1-\beta}{1-\alpha}.$$

If  $S$  denotes the process  $t \mapsto \frac{X_t^2}{t^{2\nu}}$  it suffices to verify that  $\lim_{t \rightarrow \infty} S_t = \ell^2$  a.s. In fact we shall prove that, for all  $\varepsilon > 0$ ,

$$\limsup_{t \rightarrow \infty} S_t \leq \ell^2 + 3\varepsilon \quad \text{and} \quad \liminf_{t \rightarrow \infty} S_t \geq \ell^2 - 3\varepsilon \quad \text{a.s.} \quad (4.32)$$

We prove only the first inequality in (4.32), the second one being obtained in a similar way. We split the proof in four steps.

*Step a)* We show that  $\{t \geq 1 : S_t \leq \ell^2 + \varepsilon\}$  is unbounded a.s.

Set  $\eta_u := \inf \{v \geq u : S_v \leq \ell^2 + \varepsilon\}$ . It suffices to prove that for all  $u \geq 1$  large enough,  $\eta_u < \infty$  a.s. Thanks to Ito's formula we can see that

$$S_{t \wedge \eta_u} = S_u + \int_u^{t \wedge \eta_u} \mathcal{L}V(s, X_s) ds + \int_u^{t \wedge \eta_u} \partial_x V(s, X_s) dB_s := S_u + A_t + M_t, \quad (4.33)$$

where  $V(t, x) := x^2/t^{2\nu}$  and  $\mathcal{L}$  is given by (3.3). Moreover, since  $\alpha \in (-\infty, 1)$  and  $\nu \in (1/2, \infty)$ , there exists  $t_1 \geq 1$  and  $k > 0$  such that, for all  $s \geq t_1$  and  $x \in \mathbb{R}$  for which  $V(s, x) \geq \ell^2 + 3\varepsilon$ ,

$$\mathcal{L}V(s, x) = \frac{2\rho}{s} \left( V(s, x)^{\frac{\alpha-1}{2}} - \ell^{\alpha-1} \right) V(s, x) + \frac{1}{s^{2\nu}} \leq -\frac{k}{s}. \quad (4.34)$$

This implies that the local martingale part  $M$  of the non-negative semi-martingale in (4.33) together with  $\{S_{t \wedge \eta_u}\}_{t \geq u}$  itself are non-negative super-martingales. Therefore, the bounded variation part  $A$  will be a convergent process, when  $t \rightarrow \infty$ , as the difference of two convergent super-martingales. Thanks to (4.34), this is possible only if  $\eta_u < \infty$  a.s. and this fact the claim in *Step a)*.

*Step b)* Let us introduce an increasing sequence of stopping times as follows:

$$\tau_1 := \inf\{t \geq t_1 : S_t = \ell^2 + 2\varepsilon\}, \quad \sigma_1 := \inf\{t \geq \tau_1 : S_t \notin (\ell^2 + \varepsilon, \ell^2 + 3\varepsilon)\},$$

and for every integer  $n \geq 2$ ,

$$\tau_n := \inf\{t \geq \sigma_{n-1} : S_t = \ell^2 + 2\varepsilon\}, \quad \sigma_n := \inf\{t \geq \tau_n : S_t \notin (\ell^2 + \varepsilon, \ell^2 + 3\varepsilon)\}.$$

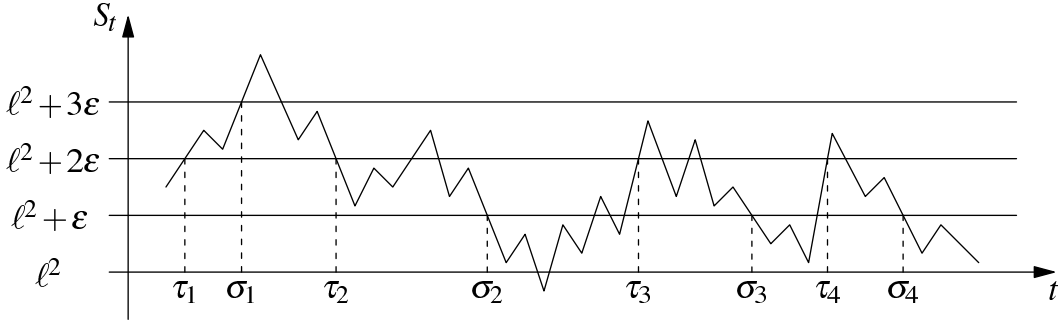


Figure 3: The increasing sequence of stopping times

Denote by  $N := \bigcap_{n \geq 1} \{\tau_n < \infty\}$  the set of all almost surely finite stopping times. Thanks to *Step a)* we can deduce that

$$\limsup_{t \rightarrow \infty} S_t \leq \ell^2 + 2\varepsilon \quad \text{a.s. on } N^c. \quad (4.35)$$

To prove the first inequality in (4.32) we need to show that

$$\limsup_{t \rightarrow \infty} S_t \leq \ell^2 + 3\varepsilon \quad \text{a.s. on } N \quad \text{or equivalently} \quad \mathbb{1}_N \sum_{n \geq 1} \mathbb{1}_{\{S_{\sigma_n} = \ell^2 + 3\varepsilon\}} < \infty \quad \text{a.s.} \quad (4.36)$$

By using a conditional version of Borel-Cantelli's lemma (see, for instance, [6] Proposition 3.2, p. 240) we see that it suffices to prove that

$$\sum_{n=1}^{\infty} \mathbb{P}(S_{\sigma_n} = \ell^2 + 3\varepsilon, \tau_n < \infty \mid \mathcal{F}_{\tau_n}) < \infty \quad \text{a.s.} \quad (4.37)$$

*Step c)* We show that there exists some positive constants  $c_1, c_2$ , such that for every  $n \geq 1$ ,

$$\mathbb{P}(S_{\sigma_n} = \ell^2 + 3\varepsilon, \tau_n < \infty \mid \mathcal{F}_{\tau_n}) \leq c_1 \tau_n^{\frac{1-2\nu}{2}} \exp(-c_2 \tau_n^{2\nu-1}) \quad \text{a.s. on } \{\tau_n < \infty\}. \quad (4.38)$$

By the strong Markov property applied to the space-time diffusion  $t \mapsto (t, X_t)$  we obtain

$$\mathbb{P}(S_{\sigma_n} = \ell^2 + 3\varepsilon, \tau_n < \infty \mid \mathcal{F}_{\tau_n}) = \mathbb{P}(S_{\sigma_n^*}^* = \ell^2 + 3\varepsilon) \quad \text{a.s. on } \{\tau_n < \infty\}.$$



Here we denote

$$S_t^* = V(t, X_t^*) \quad \text{and} \quad \sigma_n^* := \inf\{t \geq \tau_n : S_t^* \notin (\ell^2 + \varepsilon, \ell^2 + 3\varepsilon)\},$$

with  $X^*$  the solution of the equation (2.1) starting from  $X_{\tau_n}$  at time  $\tau_n$ . As in (4.33) we can write the canonical decomposition as a local martingale plus a bounded variation process:

$$S_{t \wedge \sigma_n^*}^* = S_{\tau_n}^* + A_t^* + M_t^*. \quad (4.39)$$

By the martingale representation theorem, there exists a standard Brownian motion  $W$  such that  $M^* = W_{\langle M^* \rangle}$ . Besides, it is easy to check that the quadratic variation  $\langle M^* \rangle$  is bounded a.s. and it satisfies:

$$\langle M^* \rangle_\infty \leq \frac{4(\ell^2 + 3\varepsilon)}{(2\nu - 1)\tau_n^{2\nu-1}} := v_n$$

Since the bounded variation part  $A^*$  is strictly negative (thanks to (4.34)) we can see that

$$\left\{ \sup_{0 \leq t \leq v_n} W_t < \varepsilon \right\} \subset \left\{ \sup_{t \geq \tau_n} M_t^* < \varepsilon \right\} \subset \left\{ S_{\sigma_n^*}^* = \ell^2 + \varepsilon \right\}.$$

Hence, noting that  $\sup_{0 \leq t \leq v_n} W_t \stackrel{\mathcal{L}}{=} \sqrt{v_n} |G|$ , with  $G \sim \mathcal{N}(0, 1)$ , we obtain

$$\mathbb{P}(S_{\sigma_n^*}^* = \ell^2 + 3\varepsilon) = 1 - \mathbb{P}(S_{\sigma_n^*}^* = \ell^2 + \varepsilon) \leq 2\mathbb{P}(G \geq \frac{\varepsilon}{\sqrt{v_n}})$$

and by the usual estimate for the standard Gaussian tail we get (4.38).

*Step d)* To insure the convergence of the series in (4.37) we show that the sequence  $\tau_n$  increases to infinity sufficiently fast. More precisely, we show that there exists a constant  $c > 1$  such that

$$\tau_n \geq \tau_1 c^n \quad \text{a.s. on } N. \quad (4.40)$$

The last inequality will be a consequence of a second conditional Borel-Cantelli's Lemma (see, for example, [6] Proposition 4.11, p. 256,) once we show that for every  $n \geq 1$ ,

$$\mathbb{P}(\tau_{n+1} \geq \delta \tau_n \mid \mathcal{F}_{\tau_n}) \geq p \quad \text{a.s. on } \{\tau_n < \infty\}, \quad \text{with } \delta > 1 \text{ and } p > 0. \quad (4.41)$$

It suffices to prove that

$$\mathbb{P}(\sigma_n \geq \delta \tau_n \mid \mathcal{F}_{\tau_n}) = \mathbb{P}(\sigma_n^* \geq \delta \tau_n) \geq p \quad \text{a.s. on } \{\tau_n < \infty\}. \quad (4.42)$$

In opposite to (4.34) we can see that there exists a constant  $\kappa > 0$  such that for all  $t \geq 1$  and  $x \in \mathbb{R}$  for which  $V(t, x) \leq \ell^2 + 3\varepsilon$ ,  $\mathcal{L}V(t, x) \geq -\frac{\kappa}{t}$ . If we set  $\delta := e^{\varepsilon/2\kappa}$  then

$$-\varepsilon/2 < -\kappa \ln(t/\tau_n) \leq A_t^* \leq 0, \quad t \in [\tau_n, \delta \tau_n], \quad (4.43)$$

Thanks to (4.43), (4.39) and noting that  $v_n$  is a decreasing sequence, we can write

$$\{\sigma_n^* \geq \delta \tau_n\} \supset \left\{ \left| \sup_{\tau_n \leq t \leq \delta \tau_n} M_t^* \right| < \frac{\varepsilon}{2} \right\} \supset \left\{ \left| \sup_{0 \leq t \leq v_n} W_t^* \right| < \frac{\varepsilon}{2} \right\} \supset \left\{ \left| \sup_{0 \leq t \leq v_1} W_t^* \right| < \frac{\varepsilon}{2} \right\}.$$

Here  $W^*$  is the Brownian motion in the martingale representation theorem for  $M^*$ . Therefore the inequality (4.42) is satisfied with  $p := \mathbb{P}(|\sup_{0 \leq t \leq v_1} W_t^*| < \varepsilon/2) > 0$ . This ends the proof

of the transience and hence the proof of the theorem.  $\square$

### e. Bessel type case

We have already seen that if  $\rho \in (-\infty, 0)$  and  $\alpha = -1$  then the solution of (2.1) is defined up to the reaching time of 0, which is finite a.s. Hence we need to consider only the repulsive case  $\rho \in (0, \infty)$ . Recall that if  $\beta \in (0, \infty)$  the set  $\{t \geq 1 : X_t = 0\}$  is unbounded a.s. and we will see that  $X$  is recurrent in  $[0, \infty)$ . The proof of the following result is identical to the proof of Theorem 4.3 and we leave it to the reader.

**Theorem 4.4.** *Assume that  $\rho \in (0, \infty)$ ,  $\alpha = -1$  and  $\beta \in \mathbb{R}^*$ .*

1.  *$X$  is recurrent if and only if  $\beta \in (0, \infty)$ . Moreover  $X$  has the Gaussian typical behaviour (4.1) and it satisfies the iterated logarithm law (4.2).*
2.  *$X$  is transient if and only if  $\beta \in (-\infty, 0)$  and it satisfies (4.30).*

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