Simultaneous Identification of the Diffusion Coefficient and the Potential for the Schrödinger Operator with only one Observation

Laure Cardoulis, Patricia Gaitan

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Laure Cardoulis *and Patricia Gaitan †

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Abstract

This article is devoted to prove a stability result for two independent coefficients for a Schrödinger operator in an unbounded strip. The result is obtained with only one observation on an unbounded subset of the boundary and the data of the solution at a fixed time on the whole domain.

1 Introduction

Let $\Omega = \mathbb{R} \times (d, 2d)$ be an unbounded strip of $\mathbb{R}^2$ with a fixed width $d > 0$. Let ν be the outward unit normal to Ω on $\Gamma = \partial\Omega$. We denote $x = (x_1, x_2)$ and $\Gamma = \Gamma^+ \cup \Gamma^-$, where $\Gamma^+ = \{x \in \Gamma;\ x_2 = 2d\}$ and $\Gamma^- = \{x \in \Gamma;\ x_2 = d\}$. We consider the following Schrödinger equation

$$\begin{cases} Hq := i\partial_t q + a\Delta q + bq = 0 \ \text{ in } \ \Omega \times (0,T), \\ q(x,t) = F(x,t) \ \text{ on } \ \partial\Omega \times (0,T), \\ q(x,0) = q_0(x) \ \text{ in } \ \Omega, \end{cases} \tag{1.1}$$

*Université de Toulouse, UT1 CEREMATH, CNRS, Institut de Mathématiques de Toulouse, UMR 5219, 21 Alles de Brienne 31042 Toulouse, France

†Université d'Aix-Marseille, IUT Aix-en-Provence Avenue Gaston Berger 413 av Gaston Berger, 13625 Aix-en-Provence et LATP, UMR CNRS 6632, 39, rue Joliot Curie, 13453 Marseille Cedex 13, France

where a and b are real-valued functions such that $a \in \mathcal{C}^3(\overline{\Omega})$, $b \in \mathcal{C}^2(\overline{\Omega})$ and $a(x) \geq a_{min} > 0$. Moreover, we assume that a is bounded and b and all its derivatives up to order two are bounded. If we assume that q_0 belongs to $H^4(\Omega)$ and $F \in H^2(0,T,H^2(\partial\Omega)) \cap H^1(0,T,H^4(\partial\Omega)) \cap H^3(0,T,L^2(\partial\Omega))$, then (1.1) admits a solution in $H^1(0,T,H^2(\Omega)) \cap H^2(0,T,L^2(\Omega))$.
Our problem can be stated as follows:
Is it possible to determine the coefficients a and b from the measurement of $\partial_\nu(\partial_t^2 q)$ on Γ^+?

Let q (resp. $\widetilde{q}$) be a solution of (1.1) associated with (a, b, F, q_0) (resp. ($\widetilde{a}$, $\widetilde{b}$, F, q_0)). We assume that q_0 is a real valued function.
Our main result is

$$\begin{aligned} \|a-\widetilde{a}\|^2_{L^2(\Omega)} + \|b-\widetilde{b}\|^2_{L^2(\Omega)} &\leq C\|\partial_\nu(\partial_t^2 q) - \partial_\nu(\partial_t^2 \widetilde{q})\|^2_{L^2((-T,T)\times\Gamma^+)} \\ &+ C\sum_{i=0}^{2} \|\partial_t^i(q-\widetilde{q})(\cdot,0)\|^2_{H^2(\Omega)}, \end{aligned}$$

where C is a positive constant which depends on (Ω,Γ,T) and where the above norms are weighted Sobolev norms.
This paper is an improvement of the work [10] in the sense that we simultaneously determine with only one observation, two independent coefficients, the diffusion coefficient and the potential. We use for that two important tools: Carleman estimate (2.5) and Lemma 2.4.
Carleman inequalities constitute a very efficient tool to derive observability estimates. The method of Carleman estimates has been introduced in the field of inverse problems by Bukhgeim and Klibanov (see [5], [6], [13], [14]). Carleman estimates techniques are presented in [15] for standard coefficients inverse problems for both linear and non-linear partial differential equations. These methods give a local Lipschitz stability around a single known solution.
A lot of works using the same strategy concern the wave equation (see [16], [3], [2]) and the heat equation (see [18], [12], [4]). For the determination of a time-independent potential in Schrödinger evolution equation, we can refer to [1] for bounded domains and [10] for unbounded domains. We can also cite [17] where the authors use weight functions satisfying a relaxed pseudo-convexity condition which allows to prove Carleman inequalities with less restrictive boundary observations.

Up to our knowledge, there are few results concerning the simultaneous identification of two coefficients with only one observation. In [11] a stability result is given for the particular case where each coefficient only depends on one variable ($a = a(x_2)$ and $b = b(x_1)$) for the operator $i\partial_t q + \nabla \cdot (a\nabla q) + bq$ in an unbounded strip of $\mathbb{R}^2$. The authors give a stability result for the diffusion coefficient a and the potential b with only one observation in an unbounded part of the boundary.
A physical background could be the reconstruction of the diffusion coefficient and the potential in a strip in geophysics. There are also applications in quantum mechanics: inverse problems associated with curved quantum guides (see [7], [8], [9]).
This paper is organized as follows. Section 2 is devoted to some usefull estimates. We first give an adapted global Carleman estimate for the operator H. We then recall the crucial Lemma given in [15]. In Section 3 we state and prove our main result.

2 Some Usefull Estimates

2.1 Global Carleman Inequality

Let a be a real-valued function in $\mathcal{C}^3(\overline{\Omega})$ and b be a real-valued function in $\mathcal{C}^2(\overline{\Omega})$ such that

Assumption 2.1.
- *$a \geq a_{min} > 0$, a and all its derivatives up to order three are bounded,*
- *b and its derivatives up to order two are bounded.*

Let $q(x,t)$ be a function equals to zero on $\partial\Omega \times (-T,T)$ and solution of the Schrödinger equation

$$i\partial_t q + a\Delta q + bq = f.$$

We prove here a global Carleman-type estimate for q with a single observation acting on a part Γ^+ of the boundary Γ in the right-hand side of the estimate.
Note that this estimate is quite similar to the one obtained in [10], but the computations are different. Indeed, the weigth function β does not satisfy

the same pseudo-convexity assumptions (see Assumption 2.2) and the decomposition of the operator H is different (see (2.3)).
Let $\widetilde{\beta}$ be a $\mathcal{C}^4(\overline{\Omega})$ positive function such that there exists positive constants C_0, C_{pc} which satisfy

Assumption 2.2.
- $|\nabla\widetilde{\beta}| \geq C_0 > 0 \quad in \quad \Omega, \quad \partial_\nu\widetilde{\beta} \leq 0 \quad on \quad \Gamma^-$,
- $\widetilde{\beta}$ *and all its derivatives up to order four are bounded in* $\overline{\Omega}$,
- $2\Re(D^2\widetilde{\beta}(\zeta,\bar{\zeta})) - \nabla a \cdot \nabla\widetilde{\beta}|\zeta|^2 + 2a^2|\nabla\widetilde{\beta}\cdot\zeta|^2 \geq C_{pc}|\zeta|^2$, *for all* $\zeta \in \mathbb{C}$

where

$$D^2\widetilde{\beta} = \begin{pmatrix} \partial_{x_1}(a^2\partial_{x_1}\widetilde{\beta}) & \partial_{x_1}(a^2\partial_{x_2}\widetilde{\beta}) \\ \partial_{x_2}(a^2\partial_{x_1}\widetilde{\beta}) & \partial_{x_2}(a^2\partial_{x_2}\widetilde{\beta}) \end{pmatrix}.$$

Note that the last assertion of Assumption 2.2 expresses the pseudo-convexity condition for the function β. This Assumption imposes restrictive conditions for the choice of the diffusion coefficient a in connection with the function $\widetilde{\beta}$ as in [10].
Note that there exist functions satisfying such assumptions. Indeed if we assume that $\tilde{\beta}(x) := \tilde{\beta}(x_2)$, these conditions can be written in the following form:

$$A = 2\partial_{x_2}(a^2\partial_{x_2}\tilde{\beta}) - \partial_{x_2}a\ \partial_{x_2}\tilde{\beta} + 2a^2(\partial_{x_2}\tilde{\beta})^2 \geq cst > 0$$

and

$$-\frac{(\partial_{x_1}(a^2\partial_{x_2}\tilde{\beta}))^2}{A} - \partial_{x_2}a\ \partial_{x_2}\tilde{\beta} \geq cst > 0.$$

For example $\tilde{\beta}(x) = e^{-x_2}$ with $a(x) = \frac{1}{2}(x_2^2+5)$ satisfy the previous conditions (with $x_2 \in (d, 2d)$).
Then, we define $\beta = \widetilde{\beta} + K$ with $K = m\|\widetilde{\beta}\|_\infty$ and $m > 1$. For $\lambda > 0$ and $t \in (-T, T)$, we define the following weight functions

$$\varphi(x,t) = \frac{e^{\lambda\beta(x)}}{(T+t)(T-t)}, \qquad \eta(x,t) = \frac{e^{2\lambda K} - e^{\lambda\beta(x)}}{(T+t)(T-t)}.$$

We set $\psi = e^{-s\eta}q$, $M\psi = e^{-s\eta}H(e^{s\eta}\psi)$ for $s > 0$. Let H be the operator defined by

$$Hq := i\partial_t q + a\Delta q + bq \ \text{ in } \ \Omega \times (-T, T). \tag{2.2}$$

Following [1], we introduce the operators :

$$\begin{aligned} M_1\psi &:= i\partial_t\psi + a\Delta\psi + s^2a|\nabla\eta|^2\psi + (b - s\nabla\eta\cdot\nabla a)\psi, \\ M_2\psi &:= is\partial_t\eta\psi + 2as\nabla\eta\cdot\nabla\psi + s\nabla\cdot(a\nabla\eta)\psi. \end{aligned} \tag{2.3}$$

Then

$$\begin{aligned} \int_{-T}^{T}\int_{\Omega}|M\psi|^2dx\ dt &= \int_{-T}^{T}\int_{\Omega}|M_1\psi|^2dx\ dt + \int_{-T}^{T}\int_{\Omega}|M_2\psi|^2dx\ dt \\ &+ 2\Re\Big(\int_{-T}^{T}\int_{\Omega}M_1\psi\ \overline{M_2\psi}\ dx\ dt\Big), \end{aligned}$$

where $\overline{z}$ is the conjugate of z, $\Re\ (z)$ its real part and $\Im\ (z)$ its imaginary part. Then the following result holds.

Theorem 2.3. *Let H, M_1, M_2 be the operators defined respectively by (2.2), (2.3). We assume that Assumptions 2.1 and 2.2 are satisfied. Then there exist $\lambda_0 > 0$, $s_0 > 0$ and a positive constant $C = C(\Omega, \Gamma, T)$ such that, for any $\lambda \geq \lambda_0$ and any $s \geq s_0$, the next inequality holds:*

$$\begin{aligned} &s^3\lambda^4\int_{-T}^{T}\int_{\Omega}e^{-2s\eta}|q|^2\ dx\ dt + s\lambda\int_{-T}^{T}\int_{\Omega}e^{-2s\eta}|\nabla q|^2\ dx\ dt + \|M_1(e^{-s\eta}q)\|^2_{L^2(\Omega\times(-T,T))} \\ &+\|M_2(e^{-s\eta}q)\|^2_{L^2(\Omega\times(-T,T))} \leq Cs\lambda\int_{-T}^{T}\int_{\Gamma^+}e^{-2s\eta}|\partial_\nu q|^2\ \partial_\nu\beta\ d\sigma\ dt \\ &\qquad + \int_{-T}^{T}\int_{\Omega}e^{-2s\eta}\ |Hq|^2\ dx\ dt, \end{aligned} \tag{2.4}$$

for all q satisfying $q \in L^2(-T,T;H_0^1(\Omega)\cap H^2(\Omega))\cap H^1(-T,T;L^2(\Omega))$, $\partial_\nu q \in L^2(-T,T;L^2(\Gamma))$. Moreover we have

$$\begin{aligned} &s^3\lambda^4\int_{-T}^{T}\int_{\Omega}e^{-2s\eta}|q|^2\ dx\ dt + s\lambda\int_{-T}^{T}\int_{\Omega}e^{-2s\eta}|\nabla q|^2\ dx\ dt + \|M_1(e^{-s\eta}q)\|^2_{L^2(\Omega\times(-T,T))} \\ &+\|M_2(e^{-s\eta}q)\|^2_{L^2(\Omega\times(-T,T))} + s^{-1}\lambda^{-1}\int_{-T}^{T}\int_{\Omega}e^{-2s\eta}|i\partial_t q + a\Delta q|^2\ dx\ dt \\ &\leq C\left[s\lambda\int_{-T}^{T}\int_{\Gamma^+}e^{-2s\eta}|\partial_\nu q|^2\ \partial_\nu\beta\ d\sigma\ dt + \int_{-T}^{T}\int_{\Omega}e^{-2s\eta}\ |Hq|^2\ dx\ dt\right]. \end{aligned} \tag{2.5}$$

Proof:
We have to estimate the scalar product

$$\Re\left(\int_{-T}^{T}\int_{\Omega} M_1\psi\ \overline{M_2\psi}\ dx\ dt\right) = \sum_{i=1}^{4}\sum_{j=1}^{3} I_{ij}$$

with

$$I_{11} = \Re\left(\int_{-T}^{T}\int_{\Omega}(i\partial_t\psi)(-is\partial_t\eta\ \overline{\psi})\ dx\ dt\right),\quad I_{12} = \Re\left(\int_{-T}^{T}\int_{\Omega}(i\partial_t\psi)(2as\nabla\eta\cdot\nabla\overline{\psi})\ dx\ dt\right),$$

$$I_{13} = \Re\left(\int_{-T}^{T}\int_{\Omega}(i\partial_t\psi)(s\nabla\cdot(a\nabla\eta)\overline{\psi})\ dx\ dt\right),\quad I_{21} = \Re\left(\int_{-T}^{T}\int_{\Omega}(a\Delta\psi)(-is\partial_t\eta\ \overline{\psi})\ dx\ dt\right),$$

$$I_{22} = \Re\left(\int_{-T}^{T}\int_{\Omega}(a\Delta\psi)(2as\nabla\eta\cdot\nabla\overline{\psi})\ dx\ dt\right),\quad I_{23} = \Re\left(\int_{-T}^{T}\int_{\Omega}(a\Delta\psi)(s\nabla\cdot(a\nabla\eta)\ \overline{\psi})\ dx\ dt\right),$$

$$I_{31} = \Re\left(\int_{-T}^{T}\int_{\Omega}(s^2a|\nabla\eta|^2\psi)(-is\partial_t\eta\ \overline{\psi})\ dx\ dt\right),\quad I_{32} = \Re\left(\int_{-T}^{T}\int_{\Omega}(s^2a|\nabla\eta|^2\psi)(2as\nabla\eta\cdot\nabla\overline{\psi})\ dx\ dt\right),$$

$$I_{33} = \Re\left(\int_{-T}^{T}\int_{\Omega}(s^2a|\nabla\eta|^2\psi)(s\nabla\cdot(a\nabla\eta)\ \overline{\psi})\ dx\ dt\right),\quad I_{41} = \Re\left(\int_{-T}^{T}\int_{\Omega}((b-s\nabla\eta\cdot\nabla a)\psi)(-is\partial_t\eta\ \overline{\psi})\ dx\ dt\right),$$

$$I_{42} = \Re\left(\int_{-T}^{T}\int_{\Omega}((b-s\nabla\eta\cdot\nabla a)\psi)(2as\nabla\eta\cdot\nabla\overline{\psi})\ dx\ dt\right),\quad I_{43} = \Re\left(\int_{-T}^{T}\int_{\Omega}((b-s\nabla\eta\cdot\nabla a)\psi)(s\nabla\cdot(a\nabla\eta)\ \overline{\psi})\ dx\ dt\right).$$

Following [1], using integrations by part and Young estimates, we get (2.4). Moreover from (2.3) we have:

$$i\partial_t q + a\Delta q = M_1 q - s^2a|\nabla\eta|^2 q + (b - s\nabla\eta\cdot\nabla a)q.$$

So

$$\begin{aligned} i\partial_t q + a\Delta q &= e^{s\eta}M_1(e^{-s\eta}q) + is\partial_t\eta q - ae^{s\eta}\Delta(e^{-s\eta})q - 2ae^{s\eta}\nabla(e^{-s\eta})\cdot\nabla q \\ &- s^2a|\nabla\eta|^2 q + (b - s\nabla\eta\cdot\nabla a)q. \end{aligned}$$

And we deduce (2.5) from (2.4).

■

2.2 The Crucial Lemma

We recall in this section the proof of a very important lemma proved by Klibanov and Timonov (see for example [14], [15]).

Lemma 2.4. *There exists a positive constant κ such that*

$$\int_{-T}^{T}\int_{\Omega}\left|\int_{0}^{t} q(x,\xi)d\xi\right|^2 e^{-2s\eta}dxdt \leq \frac{\kappa}{s}\int_{-T}^{T}\int_{\Omega}|q(x,t)|^2 e^{-2s\eta}dxdt,$$

for all $s > 0$.

Proof :
By the Cauchy-Schwartz inequality, we have

$$\int_{-T}^{T}\int_{\Omega}\left|\int_{0}^{t} q(x,\xi)d\xi\right|^2 e^{-2s\eta}dxdt \leq \int_{\Omega}\int_{-T}^{T}|t|\left|\int_{0}^{t}|q(x,\xi)|^2 d\xi\right| e^{-2s\eta}dxdt$$
$$\leq \int_{\Omega}\int_{0}^{T} t\left(\int_{0}^{t}|q(x,\xi)|^2 d\xi\right)e^{-2s\eta}dxdt + \int_{\Omega}\int_{-T}^{0}(-t)\left(\int_{t}^{0}|q(x,\xi)|^2 d\xi\right)e^{-2s\eta}dxdt. \tag{2.6}$$

Note that

$$\partial_t(e^{-2s\eta(x,t)}) = -2s(e^{2\lambda K} - e^{\lambda\beta(x)})\frac{2t}{(T^2-t^2)^2}e^{-2s\eta(x,t)}.$$

So, if we denote by $\alpha(x) = e^{2\lambda K} - e^{\lambda\beta(x)}$, we have

$$te^{-2s\eta(x,t)} = -\frac{(T^2-t^2)^2}{4s\alpha(x)}\partial_t(e^{-2s\eta(x,t)}).$$

For the first integral of the right hand side of (2.6), by integration by parts we have

$$\int_{\Omega}\int_{0}^{T} t\left(\int_{0}^{t}|q(x,\xi)|^2 d\xi\right)e^{-2s\eta}dxdt = \int_{\Omega}\int_{0}^{T}\left(\int_{0}^{t}|q(x,\xi)|^2 d\xi\right)\frac{(T^2-t^2)^2}{-4s\alpha(x)}\partial_t(e^{-2s\eta})dt\,dx$$

$$= \int_{\Omega}\left[\left(\int_{0}^{t}|q(x,\xi)|^2 d\xi\right)\frac{(T^2-t^2)^2}{-4s\alpha(x)}e^{-2s\eta}\right]_{t=0}^{t=T}dx + \int_{\Omega}\int_{0}^{T}|q(x,t)|^2\frac{(T^2-t^2)^2}{4s\alpha(x)}e^{-2s\eta}dt\,dx$$
$$+\int_{\Omega}\int_{0}^{T}\left(\int_{0}^{t}|q(x,\xi)|^2 d\xi\right)\frac{t(t^2-T^2)}{s\alpha(x)}e^{-2s\eta}dt\,dx.$$

Here we used $\alpha(x) > 0$ for all $x \in \overline{\Omega}$ and we obtain

$$\int_\Omega \int_0^T t \left(\int_t^0 |q(x,\xi)|^2 d\xi \right) e^{-2s\eta} dxdt \leq \frac{1}{4s} \sup_{x\in\overline{\Omega}} \left(\frac{1}{\alpha(x)} \right) \int_\Omega \int_0^T |q(x,t)|^2 e^{-2s\eta} (T^2 - t^2)^2 dxdt.$$

Similarly for the second integral of the right hand side of (2.6)

$$\int_\Omega \int_{-T}^0 (-t) \left(\int_t^0 |q(x,\xi)|^2 d\xi \right) e^{-2s\eta} dxdt \leq \frac{1}{4s} \sup_{x\in\overline{\Omega}} \left(\frac{1}{\alpha(x)} \right) \int_\Omega \int_{-T}^0 |q(x,t)|^2 e^{-2s\eta} (T^2 - t^2)^2 dxdt.$$

Thus the proof of Lemma 2.4 is completed.

∎

3 Stability result

In this section, we establish a stability inequality for the diffusion coefficient a and the potential b.
Let $q \in \mathcal{C}^2(\Omega \times (0,T))$ be solution of

$$\begin{cases} i\partial_t q + a\Delta q + bq = 0 \ \text{ in } \ \Omega \times (0,T), \\ q(x,t) = F(x,t) \ \text{ on } \ \partial\Omega \times (0,T), \\ q(x,0) = q_0(x) \ \text{ in } \ \Omega, \end{cases}$$

and $\widetilde{q} \in \mathcal{C}^2(\Omega \times (0,T))$ be solution of

$$\begin{cases} i\partial_t \widetilde{q} + \widetilde{a}\Delta \widetilde{q} + \widetilde{b}\widetilde{q} = 0 \ \text{ in } \ \Omega \times (0,T), \\ \widetilde{q}(x,t) = F(x,t) \ \text{ on } \ \partial\Omega \times (0,T), \\ \widetilde{q}(x,0) = q_0(x) \ \text{ in } \ \Omega, \end{cases}$$

where (a,b) and $(\widetilde{a},\widetilde{b})$ both satisfy Assumption 2.1.

Assumption 3.1.

- *All the time-derivatives up to order three and the space-derivatives up to order four for $\tilde{q}$ exist and are bounded.*
- *There exists a positive constant $C > 0$ such that $|\tilde{q}| \geq C$, $|\partial_t(\frac{\Delta\tilde{q}}{\tilde{q}})| \geq C$, $|\Delta\tilde{q}| \geq C$, $|\partial_t(\frac{\tilde{q}}{\Delta\tilde{q}})| \geq C$.*
- *q_0 is a real-valued function.*

Since q_0 is a real-valued function, we can extend the function q (resp. $\widetilde{q}$) on $\Omega\times(-T,T)$ by the formula $q(x,t)=\overline{q}(x,-t)$ for every $(x,t)\in\Omega\times(-T,0)$. Note that this extension satisfies the previous Carleman estimate. Our main stability result is

Theorem 3.2. *Let q and $\widetilde{q}$ be solutions of (1.1) in $\mathcal{C}^2(\Omega\times(0,T))$ such that $q-\widetilde{q}\in H^2((-T,T);H^2(\Omega))$. We assume that Assumptions 2.1, 2.2, 3.1 are satisfied. Then there exists a positive constant $C=C(\Omega,\Gamma,T)$ such that for s and λ large enough,*

$$\int_{-T}^{T}\int_{\Omega} e^{-2s\eta}(|\widetilde{a}-a|^2+|\widetilde{b}-b|^2)\,dx\,dt \leq Cs\lambda^2\int_{-T}^{T}\int_{\Gamma^+}\varphi\, e^{-2s\eta}\partial_\nu\beta\,|\partial_\nu(\partial_t^2 q-\partial_t^2\widetilde{q})|^2\,d\sigma\,dt$$

$$+C\lambda\int_{-T}^{T}\int_{\Omega} e^{-2s\eta}\Big(\sum_{i=0}^{2}|\partial_t^i(q-\tilde{q})(.,0)|^2+\nabla(q-\tilde{q})(.,0)|^2$$

$$+|\partial_t\nabla(q-\tilde{q})(.,0)|^2+|\partial_t\Delta(q-\tilde{q})(.,0)|^2\Big)\,dx\,dt.$$

Therefore

$$\begin{aligned}\|a-\widetilde{a}\|^2_{L^2(\Omega)}+\|b-\widetilde{b}\|^2_{L^2(\Omega)} &\leq C\|\partial_\nu(\partial_t^2 q)-\partial_\nu(\partial_t^2\widetilde{q})\|^2_{L^2((-T,T)\times\Gamma^+)}\\ &+ C\sum_{i=0}^{2}\|\partial_t^i(q-\widetilde{q})(\cdot,0)\|^2_{H^2(\Omega)},\end{aligned}$$

where the previous norms are weighted Sobolev norms.

Proof:
We denote by $u=q-\tilde{q}$, $\alpha=\tilde{a}-a$ and $\gamma=\tilde{b}-b$, so we get:

$$\begin{cases} i\partial_t u+a\Delta u+bu=\alpha\Delta\tilde{q}+\gamma\tilde{q} \ \text{ in } \ \Omega\times(-T,T),\\ u(x,t)=0 \ \text{ on } \ \partial\Omega\times(-T,T),\\ u(x,0)=0 \ \text{ in } \ \Omega.\end{cases}\tag{3.7}$$

The proof will be done in two steps: in a first step we prove an estimation for α and in a second step for γ.
First step: We set $u_1=\dfrac{u}{\tilde{q}}$. Then from (3.7) u_1 is solution of

$$\begin{cases} i\partial_t u_1+a\Delta u_1+bu_1+A_{11}u_1+B_{11}\cdot\nabla u_1=\alpha\dfrac{\Delta\tilde{q}}{\tilde{q}}+\gamma \ \text{ in } \ \Omega\times(-T,T),\\ u_1(x,t)=0 \ \text{ on } \ \partial\Omega\times(-T,T)\end{cases}$$

where $A_{11} = i\dfrac{\partial_t \tilde{q}}{\tilde{q}} + a\dfrac{\Delta \tilde{q}}{\tilde{q}}$ and $B_{11} = \dfrac{2a}{\tilde{q}}\nabla \tilde{q}$.
Then defining $u_2 = \partial_t u_1$ we get that u_2 satisfies

$$\begin{cases} i\partial_t u_2 + a\Delta u_2 + bu_2 + \sum_{i=1}^2 A_{i2}u_i + \sum_{i=1}^2 B_{i2}\cdot\nabla u_i = \alpha\partial_t(\dfrac{\Delta\tilde{q}}{\tilde{q}}) \ \text{ in } \ \Omega\times(-T,T), \\ u_2(x,t) = 0 \ \text{ on } \ \partial\Omega\times(-T,T) \end{cases}$$

where $A_{12} = \partial_t A_{11},\ A_{22} = A_{11},\ B_{12} = \partial_t B_{11},\ B_{12} = B_{11}$.
Now let $u_3 = \dfrac{u_2}{\partial_t(\frac{\Delta\tilde{q}}{\tilde{q}})}$, then u_3 is solution of

$$\begin{cases} i\partial_t u_3 + a\Delta u_3 + bu_3 + \sum_{i=1}^3 A_{i3}u_i + \sum_{i=1}^3 B_{i3}\cdot\nabla u_i = \alpha \ \text{ in } \ \Omega\times(-T,T), \\ u_3(x,t) = 0 \ \text{ on } \ \partial\Omega\times(-T,T) \end{cases} \tag{3.8}$$

where A_{i3} and B_{i3} are bounded functions.
If we denote by $g = \partial_t(\dfrac{\Delta\tilde{q}}{\tilde{q}})$, then

$$A_{13} = \frac{1}{g}A_{12},\ A_{23} = \frac{1}{g}A_{22},\ A_{33} = \frac{1}{g}(i\partial_t g + \Delta g),\ B_{13} = \frac{1}{g}B_{12},\ B_{23} = \frac{1}{g}B_{22},\ B_{33} = \frac{2a}{g}\nabla g.$$

At last we define $u_4 = \partial_t u_3$ and u_4 satisfies

$$\begin{cases} i\partial_t u_4 + a\Delta u_4 + bu_4 + \sum_{i=1}^4 A_{i4}u_i + \sum_{i=1}^4 B_{i4}\cdot\nabla u_i = 0 \ \text{ in } \ \Omega\times(-T,T), \\ u_4(x,t) = 0 \ \text{ on } \ \partial\Omega\times(-T,T) \end{cases}$$

where A_{i4} and B_{i4} are still bounded functions. Note that $A_{14} = \partial_t A_{13},\ A_{24} = \partial_t A_{23} + A_{13},\ A_{34} = \partial_t A_{33} + A_{23}\partial_t g + B_{23}\cdot\nabla(\partial_t g),\ A_{44} = A_{23}g + A_{33} + B_{23}\cdot\nabla g,\ B_{14} = \partial_t B_{13},\ B_{24} = \partial_t B_{23} + B_{13},\ B_{34} = \partial_t B_{33} + \partial_t g B_{23},\ B_{44} = B_{33} + gB_{23}$.
Applying the Carleman inequality (2.5) for u_4 we obtain (for s and λ sufficiently large):

$$s^3\lambda^4\int_{-T}^T\int_\Omega e^{-2s\eta}|u_4|^2\ dx\ dt + s\lambda\int_{-T}^T\int_\Omega e^{-2s\eta}|\nabla u_4|^2\ dx\ dt \tag{3.9}$$

$$+s^{-1}\lambda^{-1}\int_{-T}^T\int_\Omega e^{-2s\eta}|i\partial_t u_4 + a\Delta u_4|^2\ dx\ dt$$

$$\leq C\left[s\lambda\int_{-T}^T\int_{\Gamma^+} e^{-2s\eta}|\partial_\nu u_4|^2\ \partial_\nu\beta\ d\sigma\ dt + \sum_{i=1}^3\int_{-T}^T\int_\Omega e^{-2s\eta}\ (|u_i|^2 + |\nabla u_i|^2)\ dx\ dt\right].$$

Note that $\int_{-T}^{T}\int_\Omega e^{-2s\eta}|u_1|^2\ dx\ dt = \int_{-T}^{T}\int_\Omega e^{-2s\eta}|\int_0^t \partial_t u_1|^2\ dx\ dt$, so from Lemma 2.4 we get

$$\int_{-T}^{T}\int_\Omega e^{-2s\eta}|u_1|^2\ dx\ dt \leq \frac{C}{s}\int_{-T}^{T}\int_\Omega e^{-2s\eta}|u_3|^2\ dx\ dt$$

$$\leq \frac{C}{s^2}\int_{-T}^{T}\int_\Omega e^{-2s\eta}|u_4|^2\ dx\ dt + \frac{C}{s}\int_{-T}^{T}\int_\Omega e^{-2s\eta}|u_3(.,0)|^2\ dx\ dt.$$

By the same way, we have

$$\int_{-T}^{T}\int_\Omega e^{-2s\eta}|\nabla u_1|^2\ dx\ dt \leq \frac{C}{s^2}\int_{-T}^{T}\int_\Omega e^{-2s\eta}|\nabla u_4|^2\ dx\ dt$$

$$+\frac{C}{s}\int_{-T}^{T}\int_\Omega e^{-2s\eta}|\nabla u_3(.,0)|^2\ dx\ dt + C\int_{-T}^{T}\int_\Omega e^{-2s\eta}|\nabla u_1(.,0)|^2\ dx\ dt.$$

So (3.9) becomes

$$s^3\lambda^4\int_{-T}^{T}\int_\Omega e^{-2s\eta}|u_4|^2\ dx\ dt + s\lambda\int_{-T}^{T}\int_\Omega e^{-2s\eta}|\nabla u_4|^2\ dx\ dt \tag{3.10}$$

$$+s^{-1}\lambda^{-1}\int_{-T}^{T}\int_\Omega e^{-2s\eta}|i\partial_t u_4 + a\Delta u_4|^2\ dx\ dt \leq Cs\lambda\int_{-T}^{T}\int_{\Gamma^+} e^{-2s\eta}|\partial_\nu u_4|^2\ \partial_\nu\beta\ d\sigma\ dt$$

$$+C\int_{-T}^{T}\int_\Omega e^{-2s\eta}(|u_3(.,0)|^2 + |\nabla u_3(.,0)|^2 + \nabla u_1(.,0)|^2)\ dx\ dt.$$

Furthermore from (3.8) we have (with C a positive constant)

$$|\alpha|^2 \leq C\left(|i\partial_t u_3 + a\Delta u_3|^2 + \sum_{i=1}^{3}(|u_i|^2 + |\nabla u_i|^2)\right).$$

Therefore for s sufficiently large, from Lemma 2.4

$$\int_{-T}^{T}\int_\Omega e^{-2s\eta}|\alpha|^2\ dx\ dt \leq \frac{C}{s}\int_{-T}^{T}\int_\Omega e^{-2s\eta}\left(|i\partial_t u_4 + a\Delta u_4|^2 + |u_4|^2 + |\nabla u_4|^2\right)\ dx\ dt$$

$$+C\int_{-T}^{T}\int_\Omega e^{-2s\eta}|(i\partial_t u_3 + a\Delta u_3)(0)|^2\ dx\ dt + C\int_{-T}^{T}\int_\Omega e^{-2s\eta}|\nabla u_1(.,0)|^2\ dx\ dt$$

$$+C\int_{-T}^{T}\int_{\Omega} e^{-2s\eta}(|u_3(.,0)|^2+|\nabla u_3(.,0)|^2)\ dx\ dt.$$

Using (3.10) we get

$$\begin{aligned}\frac{1}{\lambda}\int_{-T}^{T}\int_{\Omega} e^{-2s\eta}|\alpha|^2\ dx\ dt &\leq Cs\lambda\int_{-T}^{T}\int_{\Gamma^+} e^{-2s\eta}|\partial_\nu u_4|^2\ \partial_\nu\beta\ d\sigma\ dt\\ &+ \frac{C}{\lambda}\int_{-T}^{T}\int_{\Omega} e^{-2s\eta}|(i\partial_t u_3+a\Delta u_3)(.,0)|^2\ dx\ dt\\ &+ C\int_{-T}^{T}\int_{\Omega} e^{-2s\eta}|\nabla u_1(.,0)|^2\ dx\ dt\\ &+ C\int_{-T}^{T}\int_{\Omega} e^{-2s\eta}(|u_3(.,0)|^2+|\nabla u_3(.,0)|^2)\ dx\ dt\end{aligned}$$

and then

$$\frac{1}{\lambda}\int_{-T}^{T}\int_{\Omega} e^{-2s\eta}|\alpha|^2\ dx\ dt \leq Cs\lambda\int_{-T}^{T}\int_{\Gamma^+} e^{-2s\eta}|\partial_\nu u_4|^2\ \partial_\nu\beta\ d\sigma\ dt \qquad (3.11)$$

$$+C\int_{-T}^{T}\int_{\Omega} e^{-2s\eta}\left(\sum_{i=0}^{2}|\partial_t^i u(.,0)|^2+|\nabla u(.,0)|^2+|\partial_t\nabla u(.,0)|^2+|\partial_t\Delta u(.,0)|^2\right)\ dx\ dt.$$

Second step: By the same way we obtain an estimation of γ. We set

$$v_1=\frac{u}{\Delta\tilde{q}},\quad v_2=\partial_t v_1,\quad v_3=\frac{v_2}{\partial_t(\frac{\tilde{q}}{\Delta\tilde{q}})}.$$

Following the same methodology as in the first step, we obtain:

$$\frac{1}{\lambda}\int_{-T}^{T}\int_{\Omega} e^{-2s\eta}|\gamma|^2\ dx\ dt \leq Cs\lambda\int_{-T}^{T}\int_{\Gamma^+} e^{-2s\eta}|\partial_\nu u_4|^2\ \partial_\nu\beta\ d\sigma\ dt \qquad (3.12)$$

$$+C\int_{-T}^{T}\int_{\Omega} e^{-2s\eta}\left(\sum_{i=0}^{2}|\partial_t^i u(.,0)|^2+|\nabla u(.,0)|^2+|\partial_t\nabla u(.,0)|^2+|\partial_t\Delta u(.,0)|^2\right)\ dx\ dt.$$

From (3.11) and (3.12) we can conclude.

∎

Remark 3.3. *1. Note that the following function* $\tilde{q}(x,t)=e^{-it}+x_2^2+5$ *with* $\tilde{a}(x)=\dfrac{x_2^2+5}{2}$, $\tilde{b}(x)=-1$ *satisfies Assumption 3.1.*

2. *This method works for the Schrödinger operator in the divergential form:*

$$i\partial_t q + \nabla \cdot (a \nabla q) + bq.$$

We still obtain a similar stability result but with more restrictive hypotheses on the regularity of the function $\widetilde{q}$.

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