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# Uniting two Control Lyapunov Functions for affine systems

Vincent Andrieu and Christophe Prieur

## Abstract

The problem of piecing together two Control Lyapunov Functions (CLFs) is addressed. The first CLF characterizes a local asymptotic controllability property toward the origin, whereas the second CLF is related to a global asymptotic controllability property with respect to a compact set. A sufficient condition is expressed to obtain an explicit solution. This sufficient condition is shown to be always satisfied for a linear second order controllable system. In a second part, it is shown how this uniting CLF problem can be used to solve the problem of piecing together two stabilizing control laws. Finally, this framework is applied on a numerical example to improve local performance of a globally stabilizing state feedback.

## I. INTRODUCTION

Smooth Control Lyapunov Functions (CLFs) are instrumental in many feedback control designs and can be traced back to Artstein who introduced this Lyapunov characterization of asymptotic controllability in [4]. For instance, one of the useful characteristic of smooth CLFs is the existence of *universal formulas* for stabilization of nonlinear affine (in the control) systems (see [6], [8]). Numerous tools for the design of global CLF are now available (for instance by backstepping [7], or by forwarding [10], [15]). On another hand, via linearization (or other local approaches), one may design local CLF yielding locally stabilizing controllers. This leads to the idea of uniting a local CLF with a global CLF. In Section II a sufficient condition to piece together a pair of CLFs is given.

This issue is closely related to the ability to piece together a local controller and a global one. This problem of unification of control laws was introduced in [18]. It has been subsequently

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developed in [12] where this problem has been solved by considering controllers with continuous and discrete dynamics (namely hybrid controller). As shown in Section IV below, solving the uniting CLF problem provides a simple solution to the uniting control problem without employing discrete dynamics. Some related results concerning the unification of different controllers can be found in [14], [19] where hybrid controllers are used, or in [1] where the patchy feedbacks design has been studied.

The problem of piecing together two CLFs seems to be challenging. Indeed, in [12], it is shown that there is a topological obstruction that it may be impossible to piece two arbitrary controllers when restricting to continuous stabilizing feedbacks. Thus, with the converse Lyapunov theory, this implies that the uniting CLF problem may have no solution either. This obstruction is a motivation to look for a sufficient condition guaranteeing the existence of a solution to the uniting CLF problem.

For linear systems, the sufficient condition guaranteeing a solution to the uniting CLF problem can be formalized (in a stronger version) as a Linear Matrix Inequality (LMI). This provides a simple and efficient test to show that the proposed algorithm provides a solution to the uniting control problem when dealing with linear systems.

This result on linear systems is interesting also for nonlinear systems since it may be helpful to change the local behavior of the trajectories based on the first order approximation. A numerical example is given in Section V showing how this framework can be used to modify the local behavior of the trajectories of a nonlinear system in order to minimize a cost function. In contrast to the solution by means of hybrid controllers (see e.g. [13]), this approach allows the design of a continuous global control and locally optimal.

The paper is organized as follows. In Section II, the uniting CLF problem is precisely stated and a sufficient condition guaranteeing its solvability is given. In Section III, the linear case is investigated through a simple example and a sufficient condition in terms of LMI is provided. Section IV is devoted to the uniting control problem. In this section it is shown how a solution can be obtained once the uniting CLF problem is solved. An illustration of the proposed result on a nonlinear example, in which a prescribed local optimality is obtained is given in Section V. Finally Section VI contains some concluding remarks.

**Notation:**  $L_f V$  denotes the Lie derivative of a differentiable function  $V$  with respect to the vector field  $f$ . Given a symmetric matrix  $Q$ , the notation  $Q < 0$  means that it is negative definite.

## II. PROBLEM STATEMENT AND MAIN RESULT

### A. Problem formulation

The class of nonlinear system considered are those which can be written in the following form:

$$\dot{x} = f(x) + g(x)u, \quad (1)$$

where  $x$  in  $\mathbb{R}^n$  is the state,  $u$  in  $\mathbb{R}^p$  is the control input, and  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $g : \mathbb{R}^n \rightarrow \mathbb{R}^p$  are locally Lipschitz functions such that  $f(0) = 0$ .

For system (1), two CLFs  $V_0$  and  $V_\infty$  satisfying the Artstein condition (see [4]) on specific sets are given. More precisely, the following assumption holds.

**Assumption 1:** *There exist a positive definite and continuously differentiable function  $V_0 : \mathbb{R}^n \rightarrow \mathbb{R}_+$ , a positive semi-definite, proper and continuously differentiable function  $V_\infty : \mathbb{R}^n \rightarrow \mathbb{R}_+$ , and positive values  $R_0$  and  $r_\infty$  such that :*

- Local CLF:  $\{x : 0 < V_0(x) \leq R_0, L_g V_0(x) = 0\} \subseteq \{x : L_f V_0(x) < 0\}$  ; (2)

- Global set-CLF:  $\{x : V_\infty(x) \geq r_\infty, L_g V_\infty(x) = 0\} \subseteq \{x : L_f V_\infty(x) < 0\}$  ; (3)

- Covering assumption:  $\{x : V_\infty(x) > r_\infty\} \cup \{x : V_0(x) < R_0\} = \mathbb{R}^n$  .

The function  $V_\infty$  characterizes the global asymptotic controllability toward the set  $\{x : V_\infty \leq R_0\}$  for system (1). Hence, this function is proper but not necessarily positive definite.

Roughly speaking the Covering assumption means that the two sets in which the asymptotic controllability property holds (the two sets in which each CLF satisfies Artstein condition) overlap and cover the entire domain.

The problem addressed in this paper can be formalized as follows:

**Uniting CLF problem:** The uniting CLF problem is to find a proper, positive definite and continuously differentiable function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  such that:

- Global CLF:  $\{x : x \neq 0, L_g V(x) = 0\} \subseteq \{x : L_f V(x) < 0\}$  ; (4)

- Local property:  $\{x : V_\infty(x) \leq r_\infty\} \subseteq \{x : V(x) = r_\infty V_0(x)\}$  ; (5)

- Global property:  $\{x : V_0(x) \geq R_0\} \subseteq \{x : V(x) = R_0 V_\infty(x)\}$  . (6)

If the local CLF  $V_0$  satisfies the small control property (see [16]), then, in view of property (5), the same holds for the function  $V$ . In this case, the so-called *universal formulas* (see [16], [8], [6]) can be used to compute a controller which renders the origin a globally asymptotically stable equilibrium.

As shown in Section IV, one of the main interest of solving the uniting CLF problem is that it provides a way to piece together (continuously) some specific stabilizing controllers.

### B. A sufficient condition and a constructive theorem

The first result establishes that, with the following additional assumption, the existence of a solution to the uniting CLF problem is obtained.

**Assumption 2:** *Given two positive values  $r_\infty$  and  $R_0$  and two functions  $V_0 : \mathbb{R}^n \rightarrow \mathbb{R}_+$  and  $V_\infty : \mathbb{R}^n \rightarrow \mathbb{R}_+$ , for all  $x$  in  $\{x : V_\infty(x) > r_\infty, V_0(x) < R_0\}$ , the following implication holds:*

$$\exists \lambda_x > 0 : L_g V_0(x) = -\lambda_x L_g V_\infty(x) \Rightarrow L_f V_0(x)|L_g V_\infty(x)| + L_f V_\infty(x)|L_g V_0(x)| < 0 . \quad (7)$$

The first result can now be stated.

**Theorem 2.1:** *Under Assumptions 1 and 2, there exists a solution to the uniting CLF problem. More precisely, the function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  defined, for all  $x$  in  $\mathbb{R}^n$ , by*

$$V(x) = R_0 \left[ \varphi_0(V_0(x)) + \varphi_\infty(V_\infty(x)) \right] V_\infty(x) + r_\infty \left[ 1 - \varphi_0(V_0(x)) - \varphi_\infty(V_\infty(x)) \right] V_0(x) , \quad (8)$$

where  $\varphi_0 : \mathbb{R}_+ \rightarrow [0, 1]$  and  $\varphi_\infty : \mathbb{R}_+ \rightarrow [0, 1]$  are two continuously differentiable non decreasing functions satisfying<sup>1</sup>:

$$\varphi_0(s) \begin{cases} = 0 & \forall s \leq r_0 \\ > 0 & \forall r_0 < s < R_0 \\ = \frac{1}{2} & \forall s \geq R_0 \end{cases} , \quad \varphi_\infty(s) \begin{cases} = 0 & \forall s \leq r_\infty \\ > 0 & \forall r_\infty < s < R_\infty \\ = \frac{1}{2} & \forall s \geq R_\infty \end{cases} , \quad (11)$$

<sup>1</sup>For instance,  $\varphi_0$  and  $\varphi_\infty$  can be defined as:

$$\varphi_0(s) = \frac{3}{2} \left( \frac{s - r_0}{R_0 - r_0} \right)^2 - \left( \frac{s - r_0}{R_0 - r_0} \right)^3 , \quad s \in [r_0, R_0] , \quad (9)$$

$$\varphi_\infty(s) = \frac{3}{2} \left( \frac{s - r_\infty}{R_\infty - r_\infty} \right)^2 - \left( \frac{s - r_\infty}{R_\infty - r_\infty} \right)^3 , \quad s \in [r_\infty, R_\infty] . \quad (10)$$

and where  $r_0 = \max_{\{x: V_\infty(x) \leq r_\infty\}} V_0(x)$  and<sup>2</sup>  $R_\infty = \min_{\{x: V_0(x) \geq R_0\}} V_\infty(x)$ , is a proper, positive definite continuously differentiable function satisfying (4), (5), and (6).

The structure of the function  $V$  is inspired by the construction given in [2].

*Proof:* The first part of the proof is devoted to show that the positive real numbers  $r_0$  and  $R_\infty$  are properly defined. Indeed, the function  $V_\infty$  being positive semi-definite and proper, the set  $\{x : V_\infty(x) \leq r_\infty\}$  is a non empty compact subset and  $r_0$  can be properly defined. For  $R_\infty$ , two cases need to be considered:

- If  $\{x : V_0(x) \geq R_0\} \neq \emptyset$ , pick any element  $x^*$  in  $\{x : V_0(x) \geq R_0\}$ . Since the function  $V_\infty$  is proper, it yields that  $\{x : V_\infty(x) \leq V(x^*)\}$  is a compact set and  $\min_{\{x: V_0(x) \geq R_0\}} V_\infty(x) = \min_{\{x: V_0(x) \geq R_0, V_\infty(x) \leq V(x^*)\}} V_\infty(x)$ . Therefore in this case,  $R_\infty$  can be defined.
- In the case where  $\{x : V_0(x) \geq R_0\} = \emptyset$  let  $R_\infty$  be any positive real number such that  $R_\infty > r_\infty$ .

Note that with the Covering assumption, it yields that:

$$r_0 < R_0 \quad , \quad r_\infty < R_\infty . \quad (12)$$

Indeed if one of the two inequalities in (12) is not satisfied then this implies the existence of  $x^*$  in  $\mathbb{R}^n$  such that  $V_\infty(x^*) \leq r_\infty$  and  $V_0(x^*) \geq R_0$  and consequently  $x^*$  is not in the set  $\{x : V_\infty(x) > r_\infty\} \cup \{x : V_0(x) < R_0\}$  which contradicts the Covering assumption.

The function  $V_0$  being positive definite and the function  $V_\infty$  being proper, it can be checked that  $V$  is positive definite and proper. Moreover it satisfies the local and asymptotic properties given in Equations (5) and (6).

It remains to show that  $V$  satisfies Artstein condition for all  $x$  in  $\mathbb{R}^n \setminus \{0\}$ . Note that the functions  $V_0$  and  $V_\infty$  satisfying the implications (2) and (3), it yields that the function  $V$  satisfies the Artstein condition on the set  $\{x : V_0(x) \geq R_0\} \cup \{x \neq 0 : V_\infty(x) \leq r_\infty\}$ .

Note that in the set  $\{x : V_0(x) < R_0, V_\infty(x) > r_\infty\}$ , the following inequality holds:

$$R_0 V_\infty(x) - r_\infty V_0(x) > 0 . \quad (13)$$

Furthermore,

$$L_f V(x) = A(x) L_f V_0(x) + B(x) L_f V_\infty(x) ,$$

$$L_g V(x) = A(x) L_g V_0(x) + B(x) L_g V_\infty(x) ,$$

<sup>2</sup>In the case where  $\{x : V_0(x) \geq R_0\} = \emptyset$  let  $R_\infty$  be such that  $R_\infty > r_\infty$ .

where the continuous functions  $A : \mathbb{R}^n \rightarrow \mathbb{R}_+$  and  $B : \mathbb{R}^n \rightarrow \mathbb{R}_+$  are defined as, for all  $x$  in  $\mathbb{R}^n$ ,

$$A(x) = [R_0 V_\infty(x) - r_\infty V_0(x)] \varphi'_0(V_0(x)) + r_\infty [1 - \varphi_0(V_0(x)) - \varphi_\infty(V_\infty(x))] ,$$

$$B(x) = [R_0 V_\infty(x) - r_\infty V_0(x)] \varphi'_\infty(V_\infty(x)) + R_0 [\varphi_0(V_0(x)) + \varphi_\infty(V_\infty(x))] .$$

In the set  $\{x : V_0(x) < R_0, V_\infty(x) > r_\infty\}$  it holds that  $A(x) > 0$  and  $B(x) > 0$ . Suppose there exists  $x^*$  in this set such that  $L_g V(x^*) = 0$ . Two cases have to be considered:

- If  $L_g V_0(x^*) = 0$ , then  $L_g V_\infty(x^*) = 0$ , and since  $V_0$  and  $V_\infty$  satisfy the Artstein condition, this implies that  $L_f V(x^*) < 0$ ;
- If  $L_g V_0(x^*) \neq 0$ , this implies:

$$L_g V_0(x^*) = -\frac{B(x^*)}{A(x^*)} L_g V_\infty(x^*) , \quad (14)$$

and

$$A(x^*) = \frac{B(x^*) |L_g V_\infty(x^*)|}{|L_g V_0(x^*)|} .$$

Consequently,

$$L_f V(x^*) = \frac{B(x^*)}{|L_g V_0(x^*)|} \left[ L_f V_0(x^*) |L_g V_\infty(x^*)| + L_f V_\infty(x^*) |L_g V_0(x^*)| \right] ,$$

and with (14) and Assumption 2, it yields  $L_f V(x^*) < 0$ .

Hence, the function  $V$  satisfies Artstein condition for all  $x$  in  $\mathbb{R}^n \setminus \{0\}$ . This concludes the proof of Theorem 2.1. ■

Note that this result is applied for linear systems in Section III below and on a nonlinear example to give prescribed optimal behavior of the trajectories around the equilibrium in Section V.

### C. About Assumption 2

Note that a way to relax this assumption is to restrict the sufficient condition in Theorem 2.1 to  $\lambda_x = \frac{B(x)}{A(x)}$ , where  $A$  and  $B$  are the continuous functions defined in the proof of Theorem 2.1.

Another way to relax Assumption 2 is to suppose that the implication (7) is valid only for all  $x$  in  $\{x : V_\infty(x) > \tilde{r}_\infty, V_0(x) < \tilde{R}_0\}$  where  $\tilde{R}_0$  and  $\tilde{r}_\infty$  are two positive real numbers satisfying:

$$\tilde{r}_\infty \geq r_\infty , \quad \tilde{R}_0 \leq r_0 ,$$

and

$$\{x : V_\infty(x) > \tilde{r}_\infty\} \cup \{x : V_0(x) < \tilde{R}_0\} = \mathbb{R}^n .$$

In this case the positive real numbers  $r_0$  and  $R_\infty$  and the functions  $\varphi_0$  and  $\varphi_\infty$  involved in the construction of the global CLF  $V$  have to be redefined accordingly (i.e. with  $\tilde{R}_0$  instead of  $R_0$  and  $\tilde{r}_\infty$  instead of  $r_\infty$ ).

Another formulation of Assumption 2 can be given as stated in the following proposition which proof can be found in [3].

**Proposition 2.2:** *Given two continuously differentiable functions  $V_0 : \mathbb{R}^n \rightarrow \mathbb{R}_+$  and  $V_\infty : \mathbb{R}^n \rightarrow \mathbb{R}_+$ , and a state  $x$  in  $\mathbb{R}^n \setminus \{0\}$  such that Artstein condition is satisfied for both functions, the implication (7) is equivalent to the existence of a control  $u_x$  in  $\mathbb{R}^p$  such that:*

$$L_f V_0(x) + L_g V_0(x) u_x < 0 , \quad L_f V_\infty(x) + L_g V_\infty(x) u_x < 0 . \quad (15)$$

*Proof:* Proof of (15)  $\Rightarrow$  (7): Let  $x^*$  in  $\mathbb{R}^n \setminus \{0\}$  and  $\lambda_{x^*}$  in  $\mathbb{R}_+$  be such that  $L_g V_0(x^*) = -\lambda_{x^*} L_g V_\infty(x^*)$ , and suppose there exists  $u_{x^*}$  in  $\mathbb{R}^p$  such that (15) is satisfied with  $x = x^*$  and  $u = u_{x^*}$ . This implies:

$$\begin{aligned} L_f V_0(x^*) < -L_g V_0(x^*) u_{x^*} &= \lambda_{x^*} L_g V_\infty(x^*) u_{x^*} , \\ &< -\lambda_{x^*} L_f V_\infty(x^*) . \end{aligned}$$

Since  $\lambda_{x^*} = \frac{|L_g V_0(x^*)|}{|L_g V_\infty(x^*)|}$  it yields (7).

Proof of (7)  $\Rightarrow$  (15): For the converse, suppose (7) is satisfied. Several cases need to be distinguished. If  $L_g V_0(x^*) = 0$ , since  $x \neq 0$  and the function  $V_0$  satisfies the Artstein condition, it yields  $L_f V_0(x^*) < 0$ . Consequently each control input  $u_{x^*}$  such that  $L_f V_\infty(x^*) + L_g V_\infty(x^*) u_{x^*} < 0$  ensures that (15) is satisfied. The case  $L_g V_\infty(x^*) = 0$  can be dealt with in a similar way. Hence, suppose that  $L_g V_0(x^*) \neq 0$  and  $L_g V_\infty(x^*) \neq 0$  and let  $u_{x^*}$  be defined by:

$$u_{x^*} = -k \left( \frac{L_g V_0(x^*)^T}{|L_g V_0(x^*)|} + \frac{L_g V_\infty(x^*)^T}{|L_g V_\infty(x^*)|} \right) ,$$

where  $k$  is a positive real number. Using the fact that

$$L_g V_0(x^*)^T L_g V_\infty(x^*) = |L_g V_0(x^*)| |L_g V_\infty(x^*)| \cos(L_g V_0(x^*), L_g V_\infty(x^*)) ,$$

it yields:

$$L_f V_0(x^*) + L_g V_0(x^*) u_{x^*} = L_f V_0(x^*) - k(1 + \cos(L_g V_0(x^*), L_g V_\infty(x^*))) |L_g V_0(x^*)| ,$$



$$L_f V_\infty(x^*) + L_g V_\infty(x^*) u_{x^*} = L_f V_\infty(x^*) - k(1 + \cos(L_g V_0(x^*), L_g V_\infty(x^*)) |L_g V_\infty(x^*)| .$$

Suppose  $\cos(L_g V_0(x^*), L_g V_\infty(x^*)) > -1$ . In this case, the result is obtained taking  $k$  sufficiently large. When  $\cos(L_g V_0(x^*), L_g V_\infty(x^*)) = -1$  (i.e the upper condition in (7) is satisfied), by Assumption 2 a real number  $\mu_{x^*}$  can be selected such that:

$$\frac{L_f V_\infty(x^*)}{|L_g V_\infty(x^*)|^2} < \mu_{x^*} < -\frac{L_f V_0(x^*)}{|L_g V_0(x^*)| |L_g V_\infty(x^*)|} . \quad (16)$$

If the control input  $u_{x^*}$  is defined as:

$$u_{x^*} = -\mu_{x^*} L_g V_\infty(x^*)^T ,$$

the second inequality of (16) yields

$$\begin{aligned} L_f V_\infty(x^*) + L_g V_\infty(x^*) u_{x^*} &= L_f V_\infty(x^*) - \mu_{x^*} |L_g V_\infty(x^*)|^2 , \\ &< 0 . \end{aligned}$$

Employing the first inequality of (16), it yields:

$$\begin{aligned} L_f V_0(x^*) + L_g V_0(x^*) u_{x^*} &= L_f V_0(x^*) - \mu_{x^*} L_g V_0(x^*) L_g V_\infty(x^*)^T , \\ &= L_f V_0(x^*) + \mu_{x^*} |L_g V_0(x^*)| |L_g V_\infty(x^*)| , \\ &< 0 . \end{aligned}$$

This concludes the proof of Proposition 2.2. ■

**Remark 2.3:** Note that if Assumption 2 is satisfied for system (1) then it is also satisfied for any system which can be written as :

$$\dot{x} = \tilde{f}(x) + g(x) u , \quad (17)$$

with  $\tilde{f}(x) = f(x) + g(x)k(x)$  where  $k : \mathbb{R}^n \rightarrow \mathbb{R}^p$  is any locally Lipschitz function.

Indeed, for  $x$  in  $\mathbb{R}^n$  the following equality holds :

$$\begin{aligned} L_{\tilde{f}} V_0(x) |L_g V_\infty(x)| + L_{\tilde{f}} V_\infty(x) |L_g V_0(x)| &= \\ L_f V_0(x) |L_g V_\infty(x)| + L_f V_\infty(x) |L_g V_0(x)| + N(x) k(x) , \end{aligned}$$

with

$$N(x) = L_g V_0(x) |L_g V_\infty(x)| + L_g V_\infty(x) |L_g V_0(x)| .$$

It can be checked that for all  $x^*$  such that there exists  $\lambda_{x^*} > 0$  satisfying

$$L_g V_0(x^*) = -\lambda_{x^*} L_g V_\infty(x^*) ,$$

it yields

$$N(x^*) = 0 .$$

Consequently if Assumption 2 is satisfied for system (1), then

$$L_{\tilde{f}} V_0(x^*) |L_g V_\infty(x^*)| + L_{\tilde{f}} V_\infty(x^*) |L_g V_0(x^*)| < 0 ,$$

and Assumption 2 is satisfied for system (17).

Moreover note that each CLF for (1) is also a CLF for (17) and thus Assumption 1 for system (1) is equivalent to Assumption 1 for system (17).

This remark will be useful for the proof of Propositions 3.1 and 3.2 below.

In the next section the sufficient condition is expressed when considering linear systems. This study will be useful for the nonlinear example of Section V below when considering first order approximation.

### III. UNITING TWO CLFS IN THE LINEAR CASE

In this section, the system (1) is supposed to be linear, i.e. there exist two matrices  $F$  in  $\mathbb{R}^{n \times n}$  and  $G$  in  $\mathbb{R}^{n \times p}$  such that the system (1) can be rewritten as:

$$\dot{x} = Fx + Gu . \quad (18)$$

In the linear framework, the CLFs are defined as  $V_0(x) = x^T P_0 x$  and  $V_\infty(x) = x^T P_\infty x$  where  $P_0$  and  $P_\infty$  are symmetric positive definite matrices in  $\mathbb{R}^{n \times n}$  such that:

$$\begin{aligned} x^T P_0 G = 0, \quad x \neq 0 &\Rightarrow x^T (F^T P_0 + P_0 F) x < 0 , \\ x^T P_\infty G = 0, \quad x \neq 0 &\Rightarrow x^T (F^T P_\infty + P_\infty F) x < 0 . \end{aligned} \quad (19)$$

Despite the fact that for linear systems all local quadratic CLFs are global, for robustness issue or qualitative behavior, it may be interesting to unit a pair of CLFs (see Section IV for an illustration).

### A. Case of a second order system

As a first illustration of Theorem 2.1, system (18) is supposed to be controllable with  $n = 2$  and  $m = 1$  (systems of higher dimension are considered in Section III-B). By a change of coordinates, the system can be written in canonical controllability form with  $F$  and  $G$  given as

$$F = \begin{bmatrix} 0 & 1 \\ a_F & b_F \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (20)$$

where  $a_F$  and  $b_F$  are two real numbers. Let  $P_0$  and  $P_\infty$  be two symmetric matrices in  $\mathbb{R}^{2 \times 2}$  with entries  $P_0 = \begin{bmatrix} a_0 & b_0 \\ \star & c_0 \end{bmatrix}$ ,  $P_\infty = \begin{bmatrix} a_\infty & b_\infty \\ \star & c_\infty \end{bmatrix}$ .

For  $i = 0$  and  $i = \infty$ , the function  $x \mapsto x^T P_i x$  defines a quadratic CLF if and only if

$$a_i > 0, \quad b_i > 0, \quad c_i > 0, \quad a_i c_i - b_i^2 > 0. \quad (21)$$

The interest of this system is that Assumptions 1 and 2 of Theorem 2.1 hold provided both real numbers  $R_0$  and  $r_\infty$  are selected in an appropriate way. Indeed, for this particular system, the following result holds.

**Proposition 3.1:** *Consider system (18) when  $n = 2$ ,  $m = 1$  and with (20). For all matrices  $P_0$  and  $P_\infty$  in  $\mathbb{R}^{2 \times 2}$  satisfying (21), and for all positive real numbers  $R_0$ ,  $r_\infty$  satisfying*

$$r_\infty P_0 - R_0 P_\infty \leq 0, \quad (22)$$

*Assumption 2 holds for the functions  $V_0(x) = x^T P_0 x$  and  $V_\infty(x) = x^T P_\infty x$ , and a solution to the uniting CLF problem can be computed.*

*Proof:* As noticed in Remark 2.3, without loss of generality, it can be assumed for the system (18) the two real numbers  $a_F$  and  $b_F$  are zero since it is equivalent to work on the system  $\dot{x} = \tilde{F}x + Gu$ , with  $\tilde{F} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ , where  $\tilde{F} = F - G \begin{bmatrix} a_F & b_F \end{bmatrix}$ . So, in the following,  $a_F = 0$  and  $b_F = 0$  but the result will hold for the more general case (with the same CLF  $V$ ). Let  $V_0(x) = x^T P_0 x$  and  $V_\infty(x) = x^T P_\infty x$ . Using the  $\mathcal{S}$ -procedure (see [5, Chapter 2]), Condition (22) is equivalent to the Covering assumption in Assumption 1.

A simple computation gives, for all  $x = (x_1, x_2)$  in  $\mathbb{R}^2$ ,  $L_g V_0(x) L_g V_\infty(x) = (x_1 b_0 + x_2 c_0)(x_1 b_\infty + x_2 c_\infty)$ . If  $\frac{b_0}{c_0} - \frac{b_\infty}{c_\infty} = 0$  then  $L_g V_0(x) L_g V_\infty(x) \geq 0$ . Hence, for all  $x$ , there does not exist any positive  $\lambda_x$  such that  $L_g V_0(x) = -\lambda_x L_g V_\infty(x)$  and thus Assumptions 1 and 2 of Theorem 2.1 are satisfied.

Suppose now that  $\frac{b_0}{c_0} - \frac{b_\infty}{c_\infty} \neq 0$ . it is first proved that for all  $x = (x_1, x_2)$  in  $\mathbb{R}^2 \setminus \{0\}$ ,  $L_g V_0(x) L_g V_\infty(x) < 0$  is equivalent to the fact that there exists  $0 < \mu_x < 1$  such that

$$x_2 = - \left( \mu_x \frac{b_\infty}{c_\infty} + (1 - \mu_x) \frac{b_0}{c_0} \right) x_1 . \quad (23)$$

Indeed, if there exists  $0 < \mu_x < 1$  such that (23) is satisfied, then it yields directly:

$$L_g V_0(x) L_g V_\infty(x) = -c_0 c_\infty \mu_x (1 - \mu_x) \left| \frac{b_\infty}{c_\infty} - \frac{b_0}{c_0} \right| x_1^2 < 0 .$$

Conversely, suppose  $x$  is such that  $L_g V_0(x) L_g V_\infty(x) < 0$  then  $-\frac{b_0}{c_0} x_1 < x_2 < -\frac{b_\infty}{c_\infty} x_1$  or  $-\frac{b_0}{c_0} x_1 > x_2 > -\frac{b_\infty}{c_\infty} x_1$  and  $x_1 \neq 0$ . Note also that if  $\mu_x$  is selected as:

$$\mu_x = \frac{x_2 + \frac{b_0}{c_0} x_1}{x_1 \left( \frac{b_0}{c_0} - \frac{b_\infty}{c_\infty} \right)} ,$$

then (23) is satisfied. Since  $0 < \mu_x < 1$  the equivalence is obtained.

Consequently, considering  $x^* = (x_1^*, x_2^*) \neq (0, 0)$  such that

$$x_2^* = - \left( \mu_x \frac{b_\infty}{c_\infty} + (1 - \mu_x) \frac{b_0}{c_0} \right) x_1^* , \quad 0 < \mu_x < 1 ,$$

it yields:

$$\begin{aligned} L_g V_0(x^*) &= \frac{1 - \mu_x}{c_0} |c_0 b_\infty - b_0 c_\infty| |x_1^*| , \\ L_g V_\infty(x^*) &= \frac{\mu_x}{c_\infty} |c_0 b_\infty - b_0 c_\infty| |x_1^*| , \\ L_f V_0(x^*) &= 2 \frac{x_1^{*2}}{c_0^2 c_\infty^2} (\mu_x b_\infty c_0 + (1 - \mu_x) b_0 c_\infty) [-a_0 c_0 c_\infty + b_0 (\mu_x b_\infty c_0 + (1 - \mu_x) b_0 c_\infty)] , \\ L_f V_\infty(x^*) &= 2 \frac{x_1^{*2}}{c_0^2 c_\infty^2} (\mu_x b_\infty c_0 + (1 - \mu_x) b_0 c_\infty) [-a_\infty c_0 c_\infty + b_\infty (\mu_x b_\infty c_0 + (1 - \mu_x) b_0 c_\infty)] . \end{aligned}$$

Hence, it gives:

$$\begin{aligned} L_f V_0(x^*) |L_g V_\infty(x^*)| + L_f V_\infty(x^*) |L_g V_0(x^*)| \\ = -\frac{2}{c_0^2 c_\infty^2} |x_1^*|^3 (\mu_x b_\infty c_0 + (1 - \mu_x) b_0 c_\infty) |c_0 b_\infty - b_0 c_\infty| M \end{aligned}$$

where

$$M = (1 - \mu_x) a_0 c_\infty - 2(1 - \mu_x) \mu_x b_0 b_\infty - (1 - \mu_x)^2 b_0^2 \frac{c_\infty}{c_0} + \mu_x a_\infty c_0 - \mu_x^2 b_\infty^2 \frac{c_0}{c_\infty} .$$

But with (21), it yields that:

$$M > \mu_x (1 - \mu_x) (a_0 c_\infty - 2\sqrt{a_0 c_\infty} \sqrt{a_\infty c_0} + a_\infty c_0) \geq 0 .$$

Consequently, for all  $x$  such that  $L_g V_0(x) L_g V_\infty(x) < 0$ ,

$$L_f V_0(x) |L_g V_\infty(x)| + L_f V_\infty(x) |L_g V_0(x)| < 0 ,$$

and therefore Assumptions 1 and 2 are satisfied. Applying Theorem 2.1 a uniting CLF which satisfies (4), (5) and (6) is obtained. This concludes the proof of Proposition 3.1. ■

### B. System of higher dimension

For system of higher dimension, Assumption 2 might be difficult to check. Nevertheless, a stronger sufficient condition can be expressed in terms of LMI.

**Proposition 3.2:** *Consider system (18). Let  $P_0$  and  $P_\infty$  be two symmetric positive definite matrices defining two quadratic CLFs. If there exists a matrix  $K$  in  $\mathbb{R}^{n \times p}$  such that the following LMIs are satisfied*

$$\begin{cases} (F + GK)^T P_0 + P_0 (F + GK) < 0 , \\ (F + GK)^T P_\infty + P_\infty (F + GK) < 0 , \end{cases} \quad (24)$$

*then, for all positive real numbers  $R_0$  and  $r_\infty$  satisfying (22), Assumption 2 holds for the functions  $V_0(x) = x^T P_0 x$  and  $V_\infty(x) = x^T P_\infty x$ , and a solution to the uniting CLF problem can be computed.*

*Proof:* The proof of this result follows from Proposition 2.2. Indeed, if the matrix inequality (24) is satisfied, for all  $x$  in  $\mathbb{R}^n$ , taking  $u_x = Kx$  gives inequalities (15). Consequently (7) is satisfied. For any  $R_0$  and  $r_\infty$  satisfying (22), the Covering assumption is satisfied and Theorem 2.1 applies. ■

As an illustration of Proposition 3.2, consider system (18) when it is controllable with  $n = 3$  and  $m = 1$ . Without loose of generalities (up to a change of coordinates), this system can be supposed to be in canonical controllable form:

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_F & b_F & c_F \end{bmatrix} , \quad G = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} , \quad (25)$$

where  $a_F$ ,  $b_F$  and  $c_F$  are three real values. For this system, a pair of symmetric positive definite matrices defining two quadratic CLFs can be selected as

$$P_0 = \begin{bmatrix} 5 & 8 & 1 \\ \star & 37 & 7 \\ \star & \star & 11 \end{bmatrix} , \quad P_\infty = \begin{bmatrix} 2 & 2.5 & 2 \\ \star & 4 & 3 \\ \star & \star & 3 \end{bmatrix} .$$

Employing the *Yalmip* package ([9]) in *Matlab* in combination with the solver *Sedumi* ([17]), it can be checked that for all  $a_F$ ,  $b_F$  and  $c_F$  (recall Remark 2.3), the functions  $x \mapsto x^T P_0 x$  and  $x \mapsto x^T P_\infty x$  are two CLFs for (18) such that the LMIs (24) are satisfied with the vector

$$K = \begin{pmatrix} -0.6621 - a_F & -2.0181 - b_F & -2.0702 - c_F \end{pmatrix},$$

where  $a_F$ ,  $b_F$  and  $c_F$  define the matrix  $F$ . Therefore, for all  $R_0$  and  $r_\infty$  satisfying (22) (pick e.g.  $R_0 \geq \nu$  and  $r_\infty \leq 81\nu$  for all positive real number  $\nu$ ), Assumption 2 holds and associated uniting CLF problem is solved by applying Proposition 3.2. Analogous examples can be found for systems of order 4.

However, considering the following pair of symmetric positive definite matrices

$$P_0 = \begin{bmatrix} 73 & -70 & 30 \\ \star & 121 & 10 \\ \star & \star & 48 \end{bmatrix}, \quad P_\infty = \begin{bmatrix} 3 & 1 & 1 \\ \star & 5 & 3 \\ \star & \star & 2 \end{bmatrix}. \quad (26)$$

Defining two CLFs for the system (18) with  $F$  and  $G$  defined in (25), the LMIs (24) fails to be satisfied for any matrix  $K$ . Consequently Proposition 3.2 cannot be applied.

Moreover, for this pair (26) of matrices, Assumption 2 does not hold for any  $R_0$  and  $r_\infty$  satisfying (22). Indeed, for all real values  $a_F$ ,  $b_F$  and  $c_F$ , picking  $f(x) = Fx$  and  $g(x) = Gx$  for all  $x$  in  $\mathbb{R}^3$ , it can be checked that for all  $x^* = \nu(-1.5, 1, -0.5)^T$  where  $\nu$  is a real number, the following inequalities hold:  $L_g V_\infty(x^*) L_g V_0(x^*) < 0$  and

$$L_f V_\infty(x^*) |L_g V_0(x^*)| + L_f V_0(x^*) |L_g V_\infty(x^*)| > 0.$$

Therefore, given any  $R_0$  and  $r_\infty$  satisfying (22), Assumption 2 is not satisfied for the pair of quadratic CLFs defined with (26).

$$F = \begin{bmatrix} 1.1 & -0.76 & -1.1 \\ 1.6 & 0.44 & 0.20 \\ 1.4 & 0.91 & 0.76 \end{bmatrix}, \quad G = \begin{bmatrix} -1.3 \\ -0.95 \\ 0.78 \end{bmatrix}.$$

However, for all controllable third order linear systems, there exists a pair of CLFs such that, for all  $R_0 > 0$  and  $r_\infty > 0$ , Assumption 2 is not satisfied. Indeed consider the following

symmetric positive definite matrices:

$$P_0 = \begin{bmatrix} 73 & -70 & 30 \\ \star & 121 & 10 \\ \star & \star & 48 \end{bmatrix}, \quad P_\infty = \begin{bmatrix} 3 & 1 & 1 \\ \star & 5 & 3 \\ \star & \star & 2 \end{bmatrix}.$$

Due to Remark 2,  $P_0, P_\infty$  *infty*.

for third order controllable systems, Proposition 3.2 cannot be applied for all pairs of quadratic CLFs, since Assumption 2 may not be satisfied. For instance, the following symmetric positive definite matrices:

$$P_0 = \begin{bmatrix} 73 & -70 & 30 \\ \star & 121 & 10 \\ \star & \star & 48 \end{bmatrix}, \quad P_\infty = \begin{bmatrix} 3 & 1 & 1 \\ \star & 5 & 3 \\ \star & \star & 2 \end{bmatrix},$$

are such that  $x \mapsto x^T P_0 x$  and  $x \mapsto x^T P_\infty x$  are two CLFs<sup>3</sup>. It can be checked that for  $x^* = (-1.5, 1, -0.5)^T$ ,  $L_g V_\infty(x^*) L_g V_0(x^*) < 0$  and that:

$$L_f V_\infty(x^*) |L_g V_0(x^*)| + L_f V_0(x^*) |L_g V_\infty(x^*)| > 0.$$

Therefore Assumption 2 is not satisfied.

$$F = \begin{bmatrix} 1.1 & -0.76 & -1.1 \\ 1.6 & 0.44 & 0.20 \\ 1.4 & 0.91 & 0.76 \end{bmatrix}, \quad G = \begin{bmatrix} -1.3 \\ -0.95 \\ 0.78 \end{bmatrix}.$$

#### IV. APPLICATION TO THE DESIGN OF A UNITING CONTROLLER

Theorem 2.1 can be used to design stabilizing controllers with a prescribed behavior around the equilibrium, and another behavior for large values of the state. In other words Theorem 2.1 gives a solution to the uniting control problem. This problem has been introduced in [18] and further developed in [12]. In the present context, the following theorem is obtained.

**Theorem 4.1:** *Consider two functions  $V_0 : \mathbb{R}^n \rightarrow \mathbb{R}_+$  and  $V_\infty : \mathbb{R}^n \rightarrow \mathbb{R}_+$  and two positive real numbers  $R_0$  and  $r_\infty$  satisfying Assumptions 1 and 2. Assume that  $V_0$  is proper. For any continuous function  $\phi_0 : \mathbb{R}^n \rightarrow \mathbb{R}^p$  satisfying, for all  $x$  in  $\{x : 0 < V_0(x) \leq R_0\}$ ,*

$$L_f V_0(x) + L_g V_0(x) \phi_0(x) < 0, \quad (27)$$

<sup>3</sup>Both are computed as solution of a Riccati equation.

and any continuous function  $\phi_\infty : \mathbb{R}^n \rightarrow \mathbb{R}^p$  satisfying for all  $x$  in  $\{x : V_\infty(x) \geq r_\infty\}$

$$L_f V_\infty(x) + L_g V_\infty(x) \phi_\infty(x) < 0 , \quad (28)$$

there exists a continuous function  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^p$  which solves the uniting controller problem, i.e. such that

- 1)  $\phi(x) = \phi_0(x)$  for all  $x$  such that  $V_\infty(x) \leq r_\infty$  ;
- 2)  $\phi(x) = \phi_\infty(x)$  for all  $x$  such that  $V_0(x) \geq R_0$  ;
- 3) the origin of the system  $\dot{x} = f(x) + g(x) \phi(x)$  is a globally asymptotically stable equilibrium.

The idea of the proof is to design a controller which is a continuous path going from  $\phi_0(x)$  for  $x$  small to  $\phi_\infty(x)$  for larger values of the state. The good behavior of the trajectories in between is ensured by adding a sufficiently large term which depends on the uniting control Lyapunov function. More precisely, the function  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$  obtained from Theorem 4.1 and which is a solution to the uniting controller problem is defined as

$$\phi(x) = \mathcal{H}(x) - k c(x) L_g V(x)^T , \forall x \in \mathbb{R}^n , \quad (29)$$

where  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is the Control Lyapunov Function obtained from Theorem 2.1, and with  $\mathcal{H}(x) = \gamma(x) \phi_0(x) + [1 - \gamma(x)] \phi_\infty(x)$  where  $\gamma$  is any continuous function<sup>4</sup> such that

$$\gamma(x) = \begin{cases} 1 & \text{if } V_\infty(x) \leq r_\infty , \\ 0 & \text{if } V_0(x) \geq R_0 , \end{cases}$$

and the function  $c$  is any continuous function such that<sup>5</sup>

$$c(x) \begin{cases} = 0 & \text{if } V_0(x) \geq R_0 \text{ or } V_\infty(x) \leq r_\infty , \\ > 0 & \text{if } V_0(x) < R_0 \text{ and } V_\infty(x) > r_\infty , \end{cases}$$

and  $k$  is a positive real number sufficiently large to ensure that  $V$  is a Lyapunov function of the closed-loop system. The existence of  $k$  is obtained employing compactness arguments (see analogous arguments in [2, Lemma 2.13]).

<sup>4</sup>For instance, giving  $\varphi_0$  and  $\varphi_\infty$  defined in (11), a possible choice is:  $\gamma(x) = 1 - \varphi_0(V_0(x)) - \varphi_\infty(V_\infty(x))$  . (30)

<sup>5</sup>For instance, a possible choice is  $c(x) = \max \{0, (R_0 - V_0(x))(V_\infty(x) - r_\infty)\}$  (31)



*Proof:* Note that the function  $\phi$  satisfies item 1) and 2) of Theorem 4.1. It remains to show item 3). Taking the function  $V$  as a candidate Lyapunov function obtained in (8), the continuous function  $\dot{V}$  can be introduced as, for all  $x$  in  $\mathbb{R}^n$ ,

$$\dot{V}(x) = \frac{\partial V}{\partial x}(x)f(x) + \frac{\partial V}{\partial x}(x)g(x)\mathcal{H}(x) - k c(x) \left| \frac{\partial V}{\partial x}(x)g(x) \right|^2. \quad (32)$$

With the local and global properties of the function  $V$  (as stated in (5) and (6) respectively), for all  $x$  in  $\{x \neq 0 : V_\infty(x) \leq r_\infty \text{ or } V_0(x) \geq R_0\}$ :

$$\dot{V}(x) < 0. \quad (33)$$

It is now shown that if  $k$  is selected sufficiently large then this control law ensures the negativity of  $\dot{V}$  on the whole domain. To prove that, suppose the assertion is wrong and suppose for each  $k$  in  $\mathbb{N}$ , there exists  $x_k$  in  $\mathbb{R}^n \setminus \{0\}$  such that

$$\dot{V}(x_k) \geq 0, \quad \forall k \in \mathbb{N}. \quad (34)$$

With (33),  $(x_k)_{k \in \mathbb{N}}$  is a sequence living in the set  $\{x : V_\infty(x) \geq r_\infty\} \cap \{x : V_0(x) \leq R_0\}$ . This set is compact since  $V_0$  is proper and  $V_\infty$  is continuous. Thus there exists a converging subsequence  $(x_{k_\ell})_{\ell \in \mathbb{N}}$  which converges to a point denoted  $x^* \neq 0$ . The function  $x \mapsto \dot{V}(x)$  being continuous, it yields  $\dot{V}(x^*) \geq 0$ . With (33), this yields that  $x^*$  is in  $\{x : V_\infty(x) > r_\infty\} \cap \{x : V_0(x) < R_0\}$ , and by the definition of the function  $c$  it gives  $c(x^*) > 0$ . Consequently, with (32), this implies that  $\frac{\partial V}{\partial x}(x^*)g(x^*) = 0$ . The function  $V$  being a CLF, we have  $L_f V(x^*) < 0$  which contradicts  $\dot{V}(x^*) \geq 0$ . Therefore there exists  $k > 0$  such that (33) is satisfied for all  $x \neq 0$ . Hence, item 3) is also satisfied. ■

This theorem shows that as soon as the uniting CLF problem is solved, a continuous solution to the uniting controller problem is obtained. Note also, that if discontinuous controllers with discrete dynamics (not only continuous static controllers) are allowed, the existence of a hybrid controller solving the problem is obtained under Assumption 1 only (see [12], [14]).

Combining Proposition 3.1 and Theorem 4.1, it yields that for all linear second order controllable systems, each pair of linear stabilizing controller can be united, as stated in the following

**Proposition 4.2:** *Consider system (18) when  $n = 2$ ,  $m = 1$  and with  $F$  and  $G$  given in (20). Let  $K_0$  and  $K_\infty$  in  $\mathbb{R}^{1 \times 2}$  be such that the origin of the systems:*

$$\dot{x} = (F + G K_0) x, \quad \dot{x} = (F + G K_\infty) x.$$

is globally asymptotically stable. Then there exists a continuous function  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^p$  such that the origin of the system  $\dot{x} = Fx + G\phi(x)$  is globally asymptotically stable and such that  $\phi(x) = \phi_0(x)$  in a neighborhood of the origin and  $\phi(x) = \phi_\infty(x)$  outside of a compact set.

## V. ILLUSTRATION ON AN EXAMPLE

To illustrate the interest of the uniting controller solution developed in this paper, a numerical example is provided in this section. Consider the nonlinear system (1) when  $n = 3$ ,  $p = 1$ , and the vector fields  $f$  and  $g$  defined by:

$$f(x) = \begin{bmatrix} -x_1 + x_3 \\ x_1^2 - x_2 - 2x_1x_3 + x_3 \\ -x_2 \end{bmatrix}, \quad g(x) = Gx, \quad G = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad (35)$$

for all  $x = (x_1, x_2, x_3)$  in  $\mathbb{R}^3$ .

Let  $V_\infty$  be the continuously differentiable positive definite and proper function defined by

$$V_\infty(x_1, x_2, x_3) = x_1^2 + (x_1^2 + x_2)^2 + x_3^2, \quad \forall x \in \mathbb{R}^3.$$

Along the vector fields  $f$  and  $g$  defined in (35), the Lie derivatives of the function  $V_\infty$  is

$$L_f V_\infty(x) = -2x_1^2 - 2(x_1^2 + x_2)^2 + 2x_3(x_1^2 + x_1), \quad L_g V_\infty(x) = 2x_3, \quad \forall x \in \mathbb{R}^3.$$

Note that, for all  $x$  in  $\mathbb{R}^3 \setminus \{0\}$ , Artstein condition is satisfied (i.e.  $L_g V_\infty(x) = 0 \Rightarrow L_f V_\infty < 0$ ).

Consequently,  $V_\infty$  is a global CLF and the control law  $u = \phi_\infty(x)$  with

$$\phi_\infty(x) = -x_1^2 - x_1 - x_3, \quad \forall x \in \mathbb{R}^3, \quad (36)$$

is such that, along the trajectories of the system (1) in closed loop with  $\phi_\infty$ , for all  $x$  in  $\mathbb{R}^3$ ,  $\overline{V_\infty(x)} \leq -2x_1^2 - 2(x_1^2 + x_2)^2 - 2x_3^2$ . Hence the function  $\phi_\infty$  defined in (36) ensures global asymptotic stability of the origin of the system defined in (1) and (35).

Note however that despite the global asymptotic stability of the origin is obtained with this control law, there is no guarantee that the performance obtained is satisfactory. For instance, it may be interesting that the controller locally minimizes a criterium defined as the limit, when  $t \rightarrow \infty$ , of the operator  $J : L^2(\mathbb{R}_+; \mathbb{R}^3) \times L^2(\mathbb{R}_+; \mathbb{R}) \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  defined by, for all  $(x, u, t)$  in  $L^2(\mathbb{R}_+; \mathbb{R}^3) \times L^2(\mathbb{R}_+; \mathbb{R}) \times \mathbb{R}_+$ ,

$$J(x, u, t) = \int_0^t x(s)^T Q x(s) + R u(s)^2 ds, \quad (37)$$

where  $Q$  is a symmetric positive definite matrix in  $\mathbb{R}^3$  and  $R$  is a positive real number.

The techniques developed in this paper may be instrumental to modify the stabilizing controller  $u = \phi_\infty$  such that the criterium  $J$  is minimized around the origin. A similar problem has been addressed in [11] where a general cost function depending on exogenous disturbances is considered. In [11], using a backstepping approach for upper triangular systems, a controller, which matches the optimal control law up to a desired order, is extended to a global stabilizer. In the uniting CLF approach, the global controller is computed independently from the optimal problem and an upper triangular structure is not required. However an assumption (namely Assumption 2) is needed. Using the first order approximation, this assumption can be rewritten in terms of an LMI (see Proposition 5.1 below).

The first order approximation around the origin of system (1) with  $f$  and  $g$  defined in (35) is

$$\dot{x} = Fx + Gu \quad , \quad F = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix} . \quad (38)$$

The system (38) being linear, an LQ controller minimizing the criterium defined in (37), is given by  $\phi_0(x) = -R^{-1}G^T P_0 x$ , where  $P_0$  is the symmetric positive definite solution of the Riccati equation:

$$P_0 F + F^T P_0 - P_0 G R^{-1} G^T P_0 + Q = 0 . \quad (39)$$

The tools developed in this paper provides a sufficient condition guaranteeing the existence of a continuous state feedback  $u = \phi(x)$  which unites the optimal local controller  $\phi_0$  and the global one  $\phi_\infty$  while ensuring global asymptotic stability of the origin. Indeed, this proposition can be obtained

**Proposition 5.1:** *Assume there exists a matrix  $K$  in  $\mathbb{R}^{1 \times 3}$  satisfying the following LMI:*

$$\begin{cases} (F + GK)^T P_0 + P_0 (F + GK) < 0 , \\ (F + GK)^T P_\infty + P_\infty (F + GK) < 0 , \end{cases} \quad (40)$$

where  $P_\infty = \text{diag}(1, 1, 1)$ . Then there exists a continuous function  $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}$  such that the control law  $u = \phi(x)$  makes the origin of the system (1) a globally asymptotically stable equilibrium and such that  $\phi(x) = \phi_0(x)$  in a neighborhood of the origin.

*Proof:* The matrix  $P_0$  being the solution of the Riccati equation (39), it yields, along the trajectories of system (38) with the control law  $u = \phi_0$ ,

$$\dot{\overline{V_0(x)}} = -x^T Q x ,$$

where  $V_0(x) = x^T P_0 x$  for all  $x \in \mathbb{R}^3$ . The matrix  $Q$  being symmetric positive definite and system (38) being the first order approximation of the system defined in (1) and (35), it implies that, for all  $R_0$  sufficiently small  $V_0$  is a local CLF. Moreover,  $V_\infty$  being a (global) CLF, it yields that for all  $R_0$  sufficiently small, Assumption 1 is satisfied provided  $r_\infty$  is selected sufficiently small to guarantee that the covering assumption is satisfied.

Also, note that the function  $V_\infty$  and  $P_\infty$  are such that

$$\begin{aligned} V_\infty(x) &= x^T P_\infty x + x_1^2 x_2 + \frac{1}{2} x_1^4 , \\ &= x^T P_\infty x + o(|x|^2) . \end{aligned}$$

In other words,  $x^T P_\infty x$  is the quadratic approximation around the origin of the global CLF  $V_\infty$ . Moreover the Lie derivative of  $V_\infty$  along the vector fields  $f$  and  $g$  satisfy :

$$L_f V_\infty(x) = x^T (P_\infty F + F^T P_\infty) x + o(|x|^2) ,$$

and

$$L_g V_\infty(x) K x = 2x^T P_\infty G K x + o(|x|^2) ,$$

where  $K$  is the obtained solution of the LMI (40). Consequently, with (40), the time derivative of the function  $V_\infty$  along the trajectory of the system with  $u = Kx$  satisfies :

$$\dot{\overline{V_\infty(x)}} = x^T S_\infty x + o(|x|^2) ,$$

where the matrix  $S_\infty$ , defined as

$$S_\infty = (F + GK)^T P_\infty + P_\infty (F + GK) ,$$

is symmetric negative definite due to (40).

Hence, the control law  $u = Kx$  renders the time derivative of the CLF  $V_\infty$  and the local CLF  $V_0$  negative definite for all  $x$  sufficiently small. With Proposition 2.2, this implies that Assumption 2 is satisfied provided  $R_0$  and  $r_\infty$  are selected sufficiently small. Hence, with Theorem 4.1, taking  $R_0$  and  $r_\infty$  sufficiently small and satisfying (22), the control law  $u = \phi(x)$  with  $\phi$  defined in (29) is such that

- 1)  $\phi(x) = \phi_0(x)$  for all  $x$  such that  $V_\infty(x) \leq r_\infty$  ;
- 2)  $\phi(x) = \phi_\infty(x)$  for all  $x$  such that  $V_0(x) \geq R_0$  ;
- 3) the origin of the system  $\dot{x} = f(x) + g(x)\phi(x)$  is a globally asymptotically stable equilibrium.

This concludes the proof of Proposition 5.1. ■

For the numerical illustration, the matrix  $Q$  is randomly selected as :

$$Q = \begin{bmatrix} 0.8 & 0.6 & 0.3 \\ \star & 0.6 & 0.5 \\ \star & \star & 1 \end{bmatrix}, \quad (41)$$

and  $R = 1$ . The matrix  $P_0$  and the optimal local controller  $\phi_0 = K_0 x$  obtained solving the associated Riccati equation can be computed employing the *care* routine in *Matlab* :

$$P_0 = \begin{bmatrix} 0.3389 & 0.1412 & 0.3496 \\ \star & 0.3870 & -0.0912 \\ \star & \star & 1.2316 \end{bmatrix}, \quad K_0 = \begin{bmatrix} 0.3496 & -0.0912 & 1.2316 \end{bmatrix}. \quad (42)$$

Employing the *Matlab* package *Yalmip* ([9]) in combination with the solver *Sedumi* ([17]), it can be checked<sup>6</sup> that the LMI condition (40) is satisfied for a particular  $K$  in  $\mathbb{R}^{1 \times 3}$ . Consequently, Proposition 5.1 applies and a controller which unites the optimal local one  $\phi_0$  and the global one  $\phi_\infty$  can be constructed.

The uniting controller is given in (29) where the uniting CLF  $V$  is obtained from Theorem 2.1, and the functions  $\varphi_0$ ,  $\varphi_\infty$ ,  $\gamma$ , and  $c$  are respectively defined by (9), (10), (30) and (31), with the following tuning parameters  $R_0 = 0.88$ ,  $r_\infty = 0.35$ ,  $r_0 = 0.4739$ ,  $R_\infty = 0.65$  and  $k = 1$ .

Figure 1 compares the time-evolution of the cost  $J$  defined in (37) when considering the nominal control law  $u = \phi_\infty$  and the uniting one  $u = \phi$ , with the initial condition  $x(0) = [1 \ 1 \ 1]^T$ . Figure 2 shows the time-evolution of the control values  $u = \phi$ . On Figures 1 and 2 the times the state  $x$  crosses the level sets  $\{x : V_0(x) = R_0\}$  and  $\{x : V_\infty(x) = r_\infty\}$  are shown. These time instants define the time interval in which the interpolation between both controllers occurs. In other words, before this time interval, the controller  $\phi$  equals  $\phi_\infty$  and after the local controller is employed. With this approach, there is no guarantee that, for all initial conditions, the cost

<sup>6</sup>The Matlab files can be downloaded from <http://homepages.laas.fr/~vandrieu/Publication.htm>

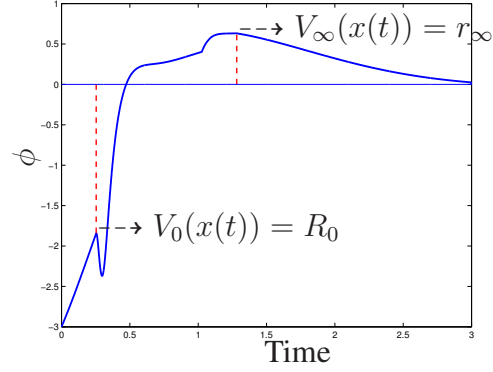
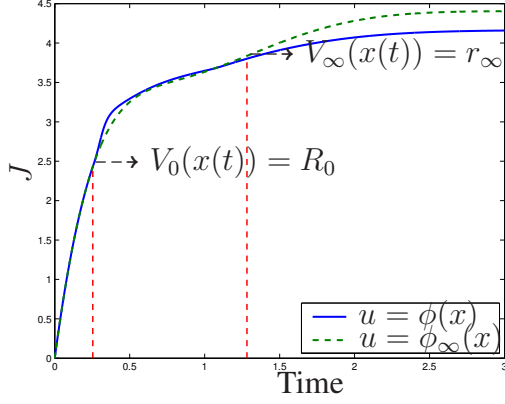


Fig. 1. Time-evolution of the cost function  $J$  with the controls  $\phi$  (in plain line) and  $\phi_\infty$  (in dashed line).

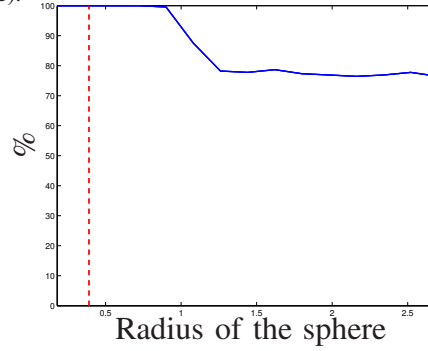


Fig. 3. Percentage of initial conditions for which the cost with the uniting controller  $\phi$  is better than the global controller  $\phi_\infty$ . The left part of the dashed line corresponds is included in the set  $\{x, V_\infty(x) \leq r_\infty\}$ .

obtained employing the uniting controller will be lower than the one obtained using the global one. More precisely, there exist initial conditions for which the use of the interpolation between both controllers affects too strongly the cost.

To check if the uniting controller is statistically better than the global one, a set of initial conditions is considered. This set is uniformly distributed on spheres with different radius. Figure 3 plots the percentage of initial conditions for which the cost has been improved when using the uniting controller. For more than 75% of initial conditions the cost is lower with the uniting controller than with the global controller. Note that for small radius, the corresponding initial conditions are inside the set  $\{x, V_\infty(x) \leq r_\infty\}$  and consequently the uniting controller is exactly the optimal one. Hence, it is not surprising that the percentage of improvement is 100%.

## VI. CONCLUSION

In this paper, the problem of piecing together two Control Lyapunov Functions is considered. Solving this one provides a simple solution to the uniting controllers problem. Two characterizations of a sufficient condition guaranteeing the solvability of the united CLF problem are given. Moreover, this sufficient condition is always satisfied in the case of a second order linear controllable system. When dealing with linear systems, a stronger version of this sufficient condition can be formulated in terms of LMIs. As shown on a numerical illustration, it allows to exhibit a sufficient condition to improve the qualitative behavior of the trajectories of nonlinear systems around the equilibrium.

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