Uniting two Control Lyapunov Functions for affine systems (full version)

Vincent Andrieu, Christophe Prieur

Uniting two Control Lyapunov Functions for affine systems (full version)

Vincent Andrieu and Christophe Prieur

Abstract

The problem of piecing together two Control Lyapunov Functions (CLFs) is addressed. The first CLF characterizes a local asymptotic controllability property toward the origin, whereas the second CLF is related to a global asymptotic controllability property with respect to a compact set. A sufficient condition is expressed to obtain an explicit solution. This sufficient condition is shown to be always satisfied for a linear second order controllable system. In a second part, it is shown how this uniting CLF problem can be used to solve the problem of piecing together two stabilizing control laws. Finally, this framework is applied on a numerical example to improve local performance of a globally stabilizing state feedback.

I. Introduction

Smooth Control Lyapunov Functions (CLFs) are instrumental in many feedback control designs and can be traced back to Artstein who introduced this Lyapunov characterization of asymptotic controllability in [3]. For instance, one of the useful characteristic of smooth CLFs is the existence of *universal formulas* for stabilization of non-linear affine (in the control) systems (see [5], [7]). Numerous tools for the design of global CLF are now available (for instance by backstepping [6], or by forwarding [9], [14]). On another hand, via linearization (or other local approaches), one may design local CLF yielding locally stabilizing controllers. This leads to the idea of uniting a local CLF with a global CLF.

The two authors are with CNRS; LAAS ; 7, avenue du Colonel Roche, F-31077 Toulouse, France and Université de Toulouse; UPS, INSA, INP, ISAE ; LAAS ; F-31077 Toulouse, France. Vincent Andrieu is also with Universit de Lyon, Lyon, F-69003, France; Universit Lyon 1; CNRS UMR 5007 LAGEP (Laboratoire dAutomatique et de GEnie des Procds), 43 bd du 11 novembre, 69100 Villeurbanne, France. vincent.andrieu@gmail.com, cprieur@laas.fr

This issue is closely related to the ability to piece together a local controller and a global one. This problem of unification of control laws was introduced in [17]. It has been subsequently developed in [11] where this problem has been solved by considering controllers with continuous and discrete dynamics (namely hybrid controller). As shown in Section IV below, solving the uniting CLF problem provides a simple solution to the uniting control problem without employing discrete dynamics. Some related results concerning the unification of different controllers can be found in [13], [18] where hybrid controllers are used, or in [1] where the patchy feedbacks design has been studied.

The problem of piecing together two CLFs seems to be challenging. Indeed, in [11], it is shown threw a topological obstruction that it may be impossible to piece two arbitrary controllers when restricting to continuous stabilizing feedbacks. Thus, with the converse Lyapunov theory, this implies that the uniting CLF problem may have no solution either. This obstruction is a motivation to look for a sufficient condition guaranteeing the existence of a solution to the uniting CLF problem.

In Section II a sufficient condition to piece together a pair of CLFs is given. For linear systems, this sufficient condition can be formalized (in a stronger version) as a Linear Matrix Inequality (LMI). This provides a simple and efficient test to show that the proposed algorithm provides a solution to the uniting control problem when dealing with linear systems.

This result on linear systems is interesting also for non-linear systems since it may be helpful to change the local behavior of the trajectories based on the first order approximation. A numerical example is given in Section V showing how this framework can be used to modify the local behavior of the trajectories of a non-linear system in order to minimize a cost function. Again a hybrid controller may be used (see e.g. [12] for the design of a stabilizer being locally optimal).

The paper is organized as follows. In Section II, the uniting CLF problem is precisely stated and a sufficient condition guaranteeing its solvability is given. In Section III, the linear case is investigated through a simple example and a sufficient condition in terms of LMI is provided. Section IV is devoted to the uniting control problem. In this section it is shown how a solution can be obtained once the uniting CLF problem is solved. An illustration of the proposed result on a non-linear example, in which a prescribed local optimality is obtained is given in Section V. Finally Section VI contains some concluding remarks.

Notation: $L_f V$ denotes the Lie derivative of a differentiable function V with respect to the vector

field f. Given a symmetric matrix Q, the notation $Q < 0$ means that it is negative definite.

II. Problem statement and main result

A. Problem formulation

In this section it is considered a control affine nonlinear system described by:

$$\dot{x} \;=\; f(x) \,+\, g(x)\, u \;, \tag{1}$$

where x in $\mathbb{R}^n$ is the state, u in $\mathbb{R}^p$ is the control input, and $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g \;: \mathbb{R}^n \to \mathbb{R}^p$ are locally Lipschitz functions such that $f(0) = 0$.

For system (1), two CLFs V_0 and V_∞ satisfying the Artstein condition (see [3]) on specific sets are given. More precisely, the following assumption holds.

Assumption 1: *There exist a positive definite and continuously differentiable function* $V_0 : \mathbb{R}^n \to \mathbb{R}_+$*, a positive semi-definite, proper and continuously differentiable function* $V_\infty \;: \mathbb{R}^n \to \mathbb{R}_+$*, and positive values* R_0 *and* r_∞ *such that :*

- Local CLF: $\{x \,:\, 0 \,<\, V_0(x) \,\leq\, R_0, L_g V_0(x) = 0\} \;\subseteq\; \{x \,:\, L_f V_0(x) \;<\; 0\}$; (2)
- Global set-CLF: $\{x \,:\, V_\infty(x) \,\geq\, r_\infty,\; L_g V_\infty(x) = 0\} \;\subseteq\; \{x \,:\, L_f V_\infty(x) \;<\; 0\}$; (3)
- Covering assumption: $\{x \,:\, V_\infty(x) \,>\, r_\infty\} \,\cup\, \{x \,:\, V_0(x) \,<\, R_0\} \;=\; \mathbb{R}^n$.

The function V_∞ characterizes the global asymptotic controllability toward the set $\{x : V_\infty \leq R_0\}$ for system (1). Hence, this function is proper but not necessarily positive definite.

Roughly speaking the Covering assumption means that the two sets in which the asymptotic controllability property holds (the two sets in which each CLF satisfies Artstein condition) overlap and cover the entire domain.

The problem addressed in this paper can be formalized as follows:

Uniting CLF problem: The uniting CLF problem is to find a proper, positive definite and continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that:

- *Global CLF:* $\{x : x \;\neq\; 0,\; L_g V(x) \;=\; 0\} \;\subseteq \{x : L_f V(x) \;<\; 0\}$; (4)
- *Local property:* $\{x : V_\infty(x) \leq r_\infty\} \;\subseteq\; \{x : V(x) \;=\; r_\infty\, V_0(x)\}$; (5)
- *Global property:* $\{x : V_0(x) \geq R_0\} \;\subseteq\; \{x : V(x) \;=\; R_0\, V_\infty(x)\}$. (6)

If the local CLF V_0 satisfies the small control property (see [15]), then, in view of property (5), the same holds for the function V. In this case, the so-called *universal formulas* (see [15], [7], [5]) can be used to compute a controller which renders the origin a globally asymptotically stable equilibrium. Furthermore, as shown in Section IV, solving the uniting CLF problem provides a way to piece together some specific stabilizing controllers.

B. A sufficient condition and a constructive theorem

The first result establishes that, with the following additional assumption, the existence of a solution to the uniting CLF problem is obtained.

Assumption 2: *Given two positive values r_∞ and R_0 and two functions $V_0 : \mathbb{R}^n \to \mathbb{R}_+$ and $V_\infty : \mathbb{R}^n \to \mathbb{R}_+$, for all x in $\{x : V_\infty(x) > r_\infty\,,\ V_0(x) < R_0\}$, the following implication holds:*

$$\exists\, \lambda_x > 0 : L_g V_0(x) = -\lambda_x L_g V_\infty(x) \Rightarrow L_f V_0(x)|L_g V_\infty(x)| + L_f V_\infty(x)|L_g V_0(x)| < 0\ . \quad (7)$$

The first result can now be stated.

Theorem 2.1: *Under Assumptions 1 and 2, there exists a solution to the uniting CLF problem. More precisely, the function $V : \mathbb{R}^n \to \mathbb{R}_+$ defined, for all x in $\mathbb{R}^n$, by*

$$V(x) = R_0 \Big[\varphi_0(V_0(x)) + \varphi_\infty(V_\infty(x))\Big] V_\infty(x) + r_\infty \Big[1 - \varphi_0(V_0(x)) - \varphi_\infty(V_\infty(x))\Big] V_0(x)\ , \quad (8)$$

where $\varphi_0 : \mathbb{R}_+ \to [0,1]$ and $\varphi_\infty : \mathbb{R}_+ \to [0,1]$ are two continuously differentiable non decreasing functions satisfying[1]*:*

$$\varphi_0(s) \begin{cases} = 0 & \forall\, s \le r_0 \\ > 0 & \forall\, r_0 < s < R_0 \\ = \frac{1}{2} & \forall\, s \ge R_0 \end{cases}, \ \varphi_\infty(s) \begin{cases} = 0 & \forall\, s \le r_\infty \\ > 0 & \forall\, r_\infty < s < R_\infty \\ = \frac{1}{2} & \forall\, s \ge R_\infty \end{cases}, \quad (11)$$

and where $r_0 = \max_{\{x : V_\infty(x) \le r_\infty\}} V_0(x)$ and[2] *$R_\infty = \min_{\{x : V_0(x) \ge R_0\}} V_\infty(x)$, is a proper, positive definite continuously differentiable function satisfying (4), (5), and (6).*

[1] For instance, φ_0 and φ_∞ can be defined as:

$$\varphi_0(s) = \frac{3}{2}\left(\frac{s - r_0}{R_0 - r_0}\right)^2 - \left(\frac{s - r_0}{R_0 - r_0}\right)^3 , \quad s \in [r_0, R_0]\ , \quad (9)$$

$$\varphi_\infty(s) = \frac{3}{2}\left(\frac{s - r_\infty}{R_\infty - r_\infty}\right)^2 - \left(\frac{s - r_\infty}{R_\infty - r_\infty}\right)^3 , \quad s \in [r_\infty, R_\infty]\ . \quad (10)$$

[2] In the case where $\{x : V_0(x) \ge R_0\} = \emptyset$ let R_∞ be such that $R_\infty > r_\infty$.

The structure of the function V is inspired by the construction given in [2].

Proof: The first part of the proof is devoted to show that the positive real numbers r_0 and R_∞ are properly defined. Indeed, the function V_∞ being positive semi-definite and proper, the set $\{x : V_\infty(x) \leq r_\infty\}$ is a non empty compact subset and r_0 can be properly defined. For R_∞, two cases need to be considered:

- If $\{x : V_0(x) \geq R_0\} \neq \emptyset$, pick any element x^* in $\{x : V_0(x) \geq R_0\}$. Since the function V_∞ is proper, it yields that $\{x : V_\infty(x) \leq V(x^*)\}$ is a compact set and $\min_{\{x\,:\,V_0(x)\,\geq\,R_0\}} V_\infty(x) = \min_{\{x\,:\,V_0(x)\,\geq\,R_0, V_\infty(x)\,\leq\,V(x^*)\}} V_\infty(x)$. Therefore in this case, R_∞ can be defined.
- In the case where $\{x : V_0(x) \geq R_0\} = \emptyset$ let R_∞ be any positive real number such that $R_\infty > r_\infty$.

Note that with the Covering assumption, it yields that:

$$r_0 < R_0 \quad , \qquad r_\infty < R_\infty \ . \tag{12}$$

Indeed if one of two inequalities in (12) is not satisfied then this would imply the existence of x^* in $\mathbb{R}^n$ such that $V_\infty(x^*) \leq r_\infty$ and $V_0(x^*) \geq R_0$ and consequently x^* is not in the set $\{x : V_\infty(x) > r_\infty\} \cup \{x : V_0(x) < R_0\}$ which contradicts the Covering assumption.

The function V_0 being positive definite and the function V_∞ being proper, it can be checked that V is positive definite and proper. Moreover it satisfies the local and asymptotic properties given in Equations (5) and (6). It remains to show that V satisfies the Artstein condition for all x in $\mathbb{R}^n \setminus \{0\}$.

Note that the functions V_0 and V_∞ satisfying the implications (2) and (3), it yields that the function V satisfies the Artstein condition on the set $\{x : V_0(x) \geq R_0\} \cup \{x \neq 0 : V_\infty(x) \leq r_\infty\}$. It remains to show that the Artstein condition is also satisfied on the set $\{x : V_0(x) < R_0\,,\ V_\infty(x) > r_\infty\}$. First of all, note that in this set, the following inequality holds:

$$R_0\, V_\infty(x) - r_\infty\, V_0(x) \ > \ 0 \ . \tag{13}$$

Furthermore,

$$\begin{aligned} L_f V(x) &= A(x)\, L_f V_0(x) + B(x)\, L_f V_\infty(x) \ , \\ L_g V(x) &= A(x)\, L_g V_0(x) + B(x)\, L_g V_\infty(x) \ , \end{aligned}$$

where the continuous functions $A : \mathbb{R}^n \to \mathbb{R}_+$ and $B : \mathbb{R}^n \to \mathbb{R}_+$ are defined as, for all x in $\mathbb{R}^n$,

$$A(x) \ = \ [R_0\, V_\infty(x) - r_\infty\, V_0(x)]\, \varphi_0'(V_0(x)) \ + \ r_\infty\ [1 \ - \ \varphi_0(V_0(x)) \ - \ \varphi_\infty(V_\infty(x))] \ ,$$

$$B(x) \;=\; [R_0\, V_\infty(x) - r_\infty\, V_0(x)]\, \varphi'_\infty(V_\infty(x)) + R_0\, [\varphi_0(V_0(x)) \,+\, \varphi_\infty(V_\infty(x))] \;.$$

In the set $\{x \,:\, V_0(x) \,<\, R_0\,,\; V_\infty(x) \,>\, r_\infty\}$ it holds that $A(x) \,>\, 0$ and $B(x) \,>\, 0$. Suppose there exists x^* in this set such that $L_g V(x^*) = 0$. Two cases have to be considered:

- If $L_g V_0(x^*) \,=\, 0$, then $L_g V_\infty(x^*) \,=\, 0$, and since V_0 and V_∞ satisfy the Artstein condition, this implies that $L_f V(x^*) \,<\, 0$;
- If $L_g V_0(x^*) \,\neq\, 0$, this implies:

$$L_g V_0(x^*) \;=\; -\frac{B(x^*)}{A(x^*)}\, L_g V_\infty(x^*) \;, \tag{14}$$

and

$$A(x^*) \;=\; \frac{B(x^*)\, |L_g V_\infty(x^*)|}{|L_g V_0(x^*)|} \;.$$

Consequently,

$$L_f V(x^*) \;=\; \frac{B(x^*)}{|L_g V_0(x^*)|} \Big[L_f V_0(x^*)\, |L_g V_\infty(x^*)| \;+\; L_f V_\infty(x)\, |L_g V_0(x^*)| \Big] \;,$$

and with (14) and Assumption 2, it yields $L_f V(x^*) \,<\, 0$.

This concludes the proof of Theorem 2.1. □

Note that this result is applied for linear systems in Section III below and on a non-linear example to give prescribed optimal behavior of the trajectories around the equilibrium in Section V.

C. About Assumption 2

Note that a way to relax this assumption is to restrict the sufficient condition in Theorem 2.1 to $\lambda_x \,=\, \frac{B(x)}{A(x)}$, where A and B are the continuous functions defined in the proof of Theorem 2.1.

Another way to relax Assumption 2 is to suppose that the implication (7) is valid only for all x in $\{x \,:\, V_\infty(x) \,>\, \tilde{r}_\infty, V_0(x) \,<\, \tilde{R}_0\}$ where $\tilde{R}_0$ and $\tilde{r}_\infty$ are two positive real numbers satisfying:

$$\tilde{r}_\infty \,\geq\, r_\infty \quad , \qquad \tilde{R}_0 \,\leq\, r_0 \;,$$

and

$$\{x \,:\, V_\infty(x) \,>\, \tilde{r}_\infty\} \,\cup\, \{x \,:\, V_0(x) \,<\, \tilde{R}_0\} \;=\; \mathbb{R}^n \;.$$

In this case the positive real numbers r_0 and R_∞ and the functions φ_0 and φ_∞ involved in the construction of the global CLF V have to be redefined accordingly (i.e. with $\tilde{R}_0$ instead of R_0 and $\tilde{r}_\infty$ instead of r_∞).

Another formulation of this assumption can be given in terms of existence for each x of a same control u_x rendering negative the time derivative of both V_0 and V_∞. This is expressed in the following proposition.

Proposition 2.2: *Given two continuously differentiable functions $V_0 : \mathbb{R}^n \to \mathbb{R}_+$ and $V_\infty : \mathbb{R}^n \to \mathbb{R}_+$, and a state x in $\mathbb{R}^n \setminus \{0\}$ such that Artstein condition is satisfied for both functions, the implication (7) is equivalent to the existence of a control u_x in $\mathbb{R}^p$ such that:*

$$L_f V_0(x) \,+\, L_g V_0(x)\, u_x < 0 \;,\; L_f V_\infty(x) \,+\, L_g V_\infty(x)\, u_x < 0 \;. \tag{15}$$

Proof: Proof of (15) $\Rightarrow$ (7): Let x^* in $\mathbb{R}^n \setminus \{0\}$ and λ_{x^*} in $\mathbb{R}_+$ be such that $L_g V_0(x^*) = -\lambda_x\, L_g V_\infty(x^*)$, and suppose there exists u_{x^*} in $\mathbb{R}^p$ such that (15) is satisfied with $x = x^*$ and $u = u_{x^*}$. This implies:

$$\begin{aligned} L_f V_0(x^*) \,<\, -L_g V_0(x^*) u_{x^*} \;&=\; \lambda_{x^*}\, L_g V_\infty(x^*) u_{x^*} \;, \\ &<\; -\lambda_{x^*}\, L_f V_\infty(x^*) \;. \end{aligned}$$

Since $\lambda_{x^*} \,=\, \frac{|L_g V_0(x^*)|}{|L_g V_\infty(x^*)|}$ it yields (7).

Proof of (7) $\Rightarrow$ (15): For the converse, suppose (7) is satisfied. Several cases need to be distinguished. If $L_g V_0(x^*) = 0$, since $x \neq 0$ and the function V_0 satisfies the Artstein condition, it yields $L_f V_0(x^*) \,<\, 0$. Consequently each control input u_{x^*} such that $L_f V_\infty(x^*) \,+\, L_g V_\infty(x^*)\, u_{x^*} < 0$ ensures that (15) is satisfied. The case $L_g V_\infty(x^*) = 0$ can be dealt with in a similar way. Hence, suppose that $L_g V_0(x^*) \neq 0$ and $L_g V_\infty(x^*) \neq 0$ and let u_{x^*} be defined by:

$$u_{x^*} \,=\, -k \left(\frac{L_g V_0(x^*)^T}{|L_g V_0(x^*)|} \,+\, \frac{L_g V_\infty(x^*)^T}{|L_g V_\infty(x^*)|} \right) \;,$$

where k is a positive real number. Using the fact that

$$L_g V_0(x^*)^T L_g V_\infty(x^*) \,=\, |L_g V_0(x^*)|\, |L_g V_\infty(x^*)| \cos(L_g V_0(x^*), L_g V_\infty(x^*)) \;,$$

it yields:

$$L_f V_0(x^*) \,+\, L_g V_0(x^*)\, u_{x^*} = L_f V_0(x^*) \,-\, k(1 \,+\, \cos(L_g V_0(x^*), L_g V_\infty(x^*))\, |L_g V_0(x^*)| \;,$$

$$L_f V_\infty(x^*) \,+\, L_g V_\infty(x^*)\, u_{x^*} = L_f V_\infty(x^*) \,-\, k(1 \,+\, \cos(L_g V_0(x^*), L_g V_\infty(x^*))\, |L_g V_\infty(x^*)| \;.$$

Suppose $\cos(L_gV_0(x^*), L_gV_\infty(x^*)) > -1$. In this case, the result is obtained taking k sufficiently large. When $\cos(L_gV_0(x^*), L_gV_\infty(x^*)) = -1$ (i.e the upper condition in (7) is satisfied), by Assumption 2 a real number μ_{x^*} can be selected such that:

$$\frac{L_fV_\infty(x^*)}{|L_gV_\infty(x^*)|^2} < \mu_{x^*} < -\frac{L_fV_0(x^*)}{|L_gV_0(x^*)|\,|L_gV_\infty(x^*)|} . \tag{16}$$

If the control input u_{x^*} is defined as:

$$u_{x^*} = -\mu_{x^*} L_gV_\infty(x^*)^T ,$$

the second inequality of (16) yields

$$\begin{aligned} L_fV_\infty(x^*) + L_gV_\infty(x^*)\,u_{x^*} &= L_fV_\infty(x^*) - \mu_{x^*}\,|L_gV_\infty(x^*)|^2 , \\ &< 0 . \end{aligned}$$

Employing the first inequality of (16), it yields:

$$\begin{aligned} L_fV_0(x^*) + L_gV_0(x^*)\,u_{x^*} &= L_fV_0(x^*) - \mu_{x^*}\,L_gV_0(x^*)L_gV_\infty(x^*)^T , \\ &= L_fV_0(x^*) + \mu_{x^*}\,|L_gV_0(x^*)|\,|L_gV_\infty(x^*)| , \\ &< 0 . \end{aligned}$$

This concludes the proof of Proposition 2.2. □

Remark 2.3: *Note that if Assumption 2 is satisfied for system (1) then it is also satisfied for any system which can be written as :*

$$\dot{x} = \tilde{f}(x) + g(x)\,u , \tag{17}$$

with $\tilde{f}(x) = f(x) + g(x)k(x)$ *where* $k : \mathbb{R}^n \to \mathbb{R}^p$ *is any locally Lipschitz function.*

Indeed, for x *in* $\mathbb{R}^n$ *the following equality holds :*

$$\begin{aligned} &L_{\tilde{f}}V_0(x)|L_gV_\infty(x)| + L_{\tilde{f}}V_\infty(x)|L_gV_0(x)| = \\ &\qquad L_fV_0(x)|L_gV_\infty(x)| + L_fV_\infty(x)|L_gV_0(x)| + N(x)k(x) , \end{aligned}$$

with

$$N(x) = L_gV_0(x)|L_gV_\infty(x)| + L_gV_\infty(x)|L_gV_0(x)| .$$

It can be checked that for all x^* such that there exists $\lambda_{x^*} > 0$ satisfying

$$L_g V_0(x^*) \,=\, -\lambda_{x^*}\, L_g V_\infty(x^*)\ ,$$

it yields

$$N(x^*) = 0\ .$$

Consequently if Assumption 2 is satisfied for system (1), then

$$L_{\tilde{f}} V_0(x^*)|L_g V_\infty(x^*)| + L_{\tilde{f}} V_\infty(x^*)|L_g V_0(x^*)| < 0\ ,$$

and Assumption 2 is satisfied for system (17).

Moreover note that each CLF for (1) is also a CLF for (17) and thus Assumption 1 for system (1) is equivalent to Assumption 1 for system (17).

This remark will be useful for the proof of Propositions 3.1 and 3.2 below.

In the next section the sufficient condition is expressed when considering linear systems. This study will be useful for the non-linear example of Section V below when considering first order approximation.

III. Uniting two CLFs in the linear case

In this section, the system (1) is supposed to be linear, i.e. there exist two matrices F in $\mathbb{R}^{n\times n}$ and G in $\mathbb{R}^{n\times p}$ such that the system (1) can be rewritten as:

$$\dot{x} \,=\, Fx \,+\, G\,u\ . \tag{18}$$

In the linear framework, the CLFs are defined as $V_0(x) \,=\, x^T P_0 x$ and $V_\infty(x) \,=\, x^T P_\infty x$ where P_0 and P_∞ are symmetric positive definite matrices in $\mathbb{R}^{n\times n}$ such that:

$$\begin{array}{l} x^T P_0 G \,=\, 0\,, \quad x \,\neq\, 0 \quad \Rightarrow \quad x^T \left(F^T P_0 \,+\, P_0 F\right) x \,<\, 0\ , \\ x^T P_\infty G \,=\, 0\,, \quad x \,\neq\, 0 \quad \Rightarrow \quad x^T \left(F^T P_\infty \,+\, P_\infty F\right) x \,<\, 0\ . \end{array} \tag{19}$$

Despite the fact that for linear systems all local quadratic CLFs are global, for robustness issue or qualitative behavior, it may be interesting to unit a pair of CLFs (see Section IV for an illustration).

A. Case of a second order system

As a first illustration of Theorem 2.1, system (18) is supposed to be controllable with $n = 2$ and $m = 1$ (systems of higher dimension are considered in Section III-B). By a change of coordinates, the system can be written in canonical controllability form with F and G given as

$$F = \begin{bmatrix} 0 & 1 \\ a_F & b_F \end{bmatrix} \quad , \qquad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ , \tag{20}$$

where a_F and b_F are two real numbers. Let P_0 and P_∞ be two symmetric matrices in $\mathbb{R}^{2\times 2}$ with entries $P_0 = \begin{bmatrix} a_0 & b_0 \\ \star & c_0 \end{bmatrix}$, $P_\infty = \begin{bmatrix} a_\infty & b_\infty \\ \star & c_\infty \end{bmatrix}$.

For $i = 0$ and $i = \infty$, the function $x \mapsto x^T P_i x$ defines a quadratic CLF if and only if

$$a_i > 0 \ , \ b_i > 0 \ , \ c_i > 0 \ , \ a_i\, c_i - b_i^2 > 0 \ . \tag{21}$$

The interest of this system is that Assumptions 1 and 2 of Theorem 2.1 hold provided both real numbers R_0 and r_∞ are selected in an appropriate way. Indeed, for this particular system, the following result holds.

Proposition 3.1: *Consider system (18) when $n = 2$, $m = 1$ and with (20). For all matrices P_0 and P_∞ in $\mathbb{R}^{2\times 2}$ satisfying (21), and for all positive real numbers R_0, r_∞ satisfying*

$$r_\infty P_0 - R_0 P_\infty \leq 0 \ , \tag{22}$$

Assumption 2 holds for the functions $V_0(x) = x^T P_0 x$ and $V_\infty(x) = x^T P_\infty x$, and a solution to the uniting CLF problem can be computed.

Proof: As noticed in Remark 2.3, without loss of generality, it can be assumed for the system (18) the two real numbers a_F and b_F are zero since it is equivalent to work on the system $\dot{x} = \tilde{F}x + Gu$, with $\tilde{F} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. where $\tilde{F} = F - G[\ a_F \quad b_F\ ]$. So, in the following, $a_F = 0$ and $b_F = 0$ but the result will hold for the more general case (with the same CLF V). Let $V_0(x) = x^T P_0 x$ and $V_\infty(x) = x^T P_\infty x$. Using the $\mathcal{S}$-procedure (see [4, Chapter 2]), Condition (22) is equivalent to the Covering assumption in Assumption 1.

A simple computation gives, for all $x = (x_1, x_2)$ in $\mathbb{R}^2$, $L_g V_0(x) L_g V_\infty(x) = (x_1 b_0 + x_2 c_0)(x_1 b_\infty + x_2 c_\infty)$. If $\frac{b_0}{c_0} - \frac{b_\infty}{c_\infty} = 0$ then $L_g V_0(x) L_g V_\infty(x) \geq 0$. Hence, for all x, there does not exist any positive λ_x such that $L_g V_0(x) = -\lambda_x L_g V_\infty(x)$ and thus Assumptions 1 and 2 of Theorem 2.1 are satisfied.

Suppose now that $\frac{b_0}{c_0} - \frac{b_\infty}{c_\infty} \neq 0$. it is first proved that for all $x = (x_1, x_2)$ in $\mathbb{R}^2 \setminus \{0\}$, $L_gV_0(x)L_gV_\infty(x) < 0$ is equivalent to the fact that there exists $0 < \mu_x < 1$ such that

$$x_2 = -\left(\mu_x \frac{b_\infty}{c_\infty} + (1-\mu_x)\frac{b_0}{c_0}\right) x_1 \ . \tag{23}$$

Indeed, if there exists $0 < \mu_x < 1$ such that (23) is satisfied, then it yields directly:

$$L_gV_0(x)L_gV_\infty(x) = -c_0\, c_\infty \mu_x(1-\mu_x)\left|\frac{b_\infty}{c_\infty} - \frac{b_0}{c_0}\right| x_1^2 < 0 \ .$$

Conversely, suppose x is such that $L_gV_0(x)L_gV_\infty(x) < 0$ then $-\frac{b_0}{c_0}x_1 < x_2 < -\frac{b_\infty}{c_\infty}x_1$ or $-\frac{b_0}{c_0}x_1 > x_2 > -\frac{b_\infty}{c_\infty}x_1$ and $x_1 \neq 0$. Note also that if μ_x is selected as:

$$\mu_x = \frac{x_2 + \frac{b_0}{c_0}x_1}{x_1\left(\frac{b_0}{c_0} - \frac{b_\infty}{c_\infty}\right)} \ ,$$

then (23) is satisfied. Since $0 < \mu_x < 1$ the equivalence is obtained.

Consequently, considering $x^* = (x_1^*, x_2^*) \neq (0,0)$ such that

$$x_2^* = -\left(\mu_x \frac{b_\infty}{c_\infty} + (1-\mu_x)\frac{b_0}{c_0}\right) x_1^* \quad , \qquad 0 < \mu_x < 1 \ ,$$

it yields:

$$\begin{aligned}
L_gV_0(x^*) &= \frac{1-\mu_x}{c_0}|c_0b_\infty - b_0c_\infty|\,|x_1^*| \ , \\
L_gV_\infty(x^*) &= \frac{\mu_x}{c_\infty}|c_0b_\infty - b_0c_\infty|\,|x_1^*| \ , \\
L_fV_0(x^*) &= 2\frac{x_1^{*2}}{c_0^2c_\infty^2}(\mu_x b_\infty c_0 + (1-\mu_x)b_0c_\infty)[-a_0c_0c_\infty + b_0(\mu_x b_\infty c_0 + (1-\mu_x)b_0c_\infty)] \ , \\
L_fV_\infty(x^*) &= 2\frac{x_1^{*2}}{c_0^2c_\infty^2}(\mu_x b_\infty c_0 + (1-\mu_x)b_0c_\infty)[-a_\infty c_0c_\infty + b_\infty(\mu_x b_\infty c_0 + (1-\mu_x)b_0c_\infty)] \ .
\end{aligned}$$

Hence, it gives:

$$\begin{aligned}
&L_fV_0(x^*)|L_gV_\infty(x^*)| + L_fV_\infty(x^*)|L_gV_0(x^*)| \\
&\qquad = -\tfrac{2}{c_0^2c_\infty^2}|x_1^*|^3(\mu_x b_\infty c_0 + (1-\mu_x)b_0c_\infty)\,|c_0b_\infty - b_0c_\infty|\,M
\end{aligned}$$

where

$$M = (1-\mu_x)a_0c_\infty - 2(1-\mu_x)\mu_x b_0b_\infty - (1-\mu_x)^2b_0^2\frac{c_\infty}{c_0} + \mu_x a_\infty c_0 - \mu_x^2 b_\infty^2\frac{c_0}{c_\infty} \ .$$

But with (21), it yields that:

$$M > \mu_x(1-\mu_x)\left(a_0c_\infty - 2\sqrt{a_0c_\infty}\sqrt{a_\infty c_0} + a_\infty c_0\right) \geq 0 \ .$$

Consequently, for all x such that $L_g V_0(x) L_g V_\infty(x) < 0$,

$$L_f V_0(x)|L_g V_\infty(x)| + L_f V_\infty(x)|L_g V_0(x)| < 0 ,$$

and therefore Assumptions 1 and 2 are satisfied. Applying Theorem 2.1 a uniting CLF which satisfies (4), (5) and (6) is obtained. This concludes the proof of Proposition 3.1. □

B. System of higher dimension

For system of higher dimension, Assumption 2 might be difficult to check. Nevertheless, a stronger sufficient condition can be expressed in terms of LMI.

Proposition 3.2: *Consider system (18). Let P_0 and P_∞ be two symmetric positive definite matrices defining two quadratic CLFs. If there exists a matrix K in $\mathbb{R}^{n\times p}$ such that the following LMIs are satisfied*

$$\begin{cases} (F + GK)^T P_0 + P_0(F + GK) < 0 , \\ (F + GK)^T P_\infty + P_\infty(F + GK) < 0 , \end{cases} \tag{24}$$

then, for all positive real numbers R_0 and r_∞ satisfying (22), Assumption 2 holds for the functions $V_0(x) = x^T P_0 x$ and $V_\infty(x) = x^T P_\infty x$, and a solution to the uniting CLF problem can be computed.

Proof: The proof of this result follows from Proposition 2.2. Indeed, if the matrix inequality (24) is satisfied, for all x in $\mathbb{R}^n$, taking $u_x = K\,x$ gives inequalities (15). Consequently (7) is satisfied. For any R_0 and r_∞ satisfying (22), the Covering assumption is satisfied and Theorem 2.1 applies. □

As an illustration of Proposition 3.2, consider system (18) when it is controllable with $n = 3$ and $m = 1$. Without loose of generalities (up to a change of coordinates), this system can be supposed to be in canonical controllable form:

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_F & b_F & c_F \end{bmatrix} , \quad G = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} , \tag{25}$$

where a_F, b_F and c_F are three real values. For this system, a pair of symmetric positive definite matrices defining two quadratic CLFs can be selected as

$$P_0 = \begin{bmatrix} 5 & 8 & 1 \\ \star & 37 & 7 \\ \star & \star & 11 \end{bmatrix}, \; P_\infty = \begin{bmatrix} 2 & 2.5 & 2 \\ \star & 4 & 3 \\ \star & \star & 3 \end{bmatrix} .$$

Employing the *Yalmip* package ([8]) in *Matlab* in combination with the solver *Sedumi* ([16]), it can be checked that for all a_F, b_F and c_F (recall Remark 2.3), the functions $x \mapsto x^T P_0 x$ and $x \mapsto x^T P_\infty x$ are two CLFs for (18) such that the LMIs (24) are satisfied with the vector

$$K = \begin{pmatrix} -0.6621 - a_F & -2.0181 - b_F & -2.0702 - c_F \end{pmatrix} ,$$

where a_F, b_F and c_F define the matrix F. Therefore, for all R_0 and r_∞ satisfying (22) (pick e.g. $R_0 \geq \nu$ and $r_\infty \leq 81\,\nu$ for all positive real number ν), Assumption 2 holds and associated uniting CLF problem is solved by applying Proposition 3.2. Analogous examples can be found for systems of order 4.

However, considering the following pair of symmetric positive definite matrices

$$P_0 = \begin{bmatrix} 73 & -70 & 30 \\ \star & 121 & 10 \\ \star & \star & 48 \end{bmatrix}, \quad P_\infty = \begin{bmatrix} 3 & 1 & 1 \\ \star & 5 & 3 \\ \star & \star & 2 \end{bmatrix} . \tag{26}$$

Defining two CLFs for the system (18) with F and G defined in (25), the LMIs (24) fails to be satisfied for any matrix K. Consequently Proposition 3.2 cannot be applied.

Moreover, for this pair (26) of matrices, Assumption 2 does not hold for any R_0 and r_∞ satisfying (22). Indeed, for all real values a_F, b_F and c_F, picking $f(x) = Fx$ and $g(x) = Gx$ for all x in $\mathbb{R}^3$, it can be checked that for all $x^* = \nu\,(-1.5, 1, -0.5)^T$ where ν is a real number, the following inequalities hold: $L_g V_\infty(x^*) L_g V_0(x^*) < 0$ and

$$L_f V_\infty(x^*)\,|L_g V_0(x^*)| \; + \; L_f V_0(x^*)\,|L_g V_\infty(x^*)| > 0 .$$

Therefore, given any R_0 and r_∞ satisfying (22), Assumption 2 is not satisfied for the pair of quadratic CLFs defined with (26).

$$F = \begin{bmatrix} 1.1 & -0.76 & -1.1 \\ 1.6 & 0.44 & 0.20 \\ 1.4 & 0.91 & 0.76 \end{bmatrix}, \quad G = \begin{bmatrix} -1.3 \\ -0.95 \\ 0.78 \end{bmatrix} .$$

However, for all controllable third order linear systems, there exists a pair of CLFs such that, for all $R_0 > 0$ and $r_\infty > 0$, Assumption 2 is not satisfied. Indeed consider the following symmetric positive definite matrices:

$$P_0 = \begin{bmatrix} 73 & -70 & 30 \\ \star & 121 & 10 \\ \star & \star & 48 \end{bmatrix}, \quad P_\infty = \begin{bmatrix} 3 & 1 & 1 \\ \star & 5 & 3 \\ \star & \star & 2 \end{bmatrix}.$$

Due to Remark 2, P_0,P_infty.

for third order controllable systems, Proposition 3.2 cannot be applied for all pairs of quadratic CLFs, since Assumption 2 may not be satisfied. For instance, the following symmetric positive definite matrices:

$$P_0 = \begin{bmatrix} 73 & -70 & 30 \\ \star & 121 & 10 \\ \star & \star & 48 \end{bmatrix}, \quad P_\infty = \begin{bmatrix} 3 & 1 & 1 \\ \star & 5 & 3 \\ \star & \star & 2 \end{bmatrix},$$

are such that $x \mapsto x^T P_0 x$ and $x \mapsto x^T P_\infty x$ are two CLFs[3]. It can be checked that for $x^* = (-1.5, 1, -0.5)^T$, $L_g V_\infty(x^*) L_g V_0(x^*) < 0$ and that:

$$L_f V_\infty(x^*) \, |L_g V_0(x^*)| \; + \; L_f V_0(x^*) \, |L_g V_\infty(x^*)| > 0 \; .$$

Therefore Assumption 2 is not satisfied.

$$F = \begin{bmatrix} 1.1 & -0.76 & -1.1 \\ 1.6 & 0.44 & 0.20 \\ 1.4 & 0.91 & 0.76 \end{bmatrix}, \quad G = \begin{bmatrix} -1.3 \\ -0.95 \\ 0.78 \end{bmatrix}.$$

IV. Application to the design of a uniting controller

Theorem 2.1 can be used to design stabilizing controllers with a prescribed behavior around the equilibrium, and another behavior for large values of the state. In other words Theorem 2.1 gives a solution to the uniting control problem. This problem has been introduced in [17] and further developed in [11]. In the present context, the following theorem is obtained.

[3]Both are computed as solution of a Riccati equation.

Theorem 4.1: *Consider two functions $V_0 : \mathbb{R}^n \to \mathbb{R}_+$ and $V_\infty : \mathbb{R}^n \to \mathbb{R}_+$ and two positive real numbers R_0 and r_∞ satisfying Assumptions 1 and 2. Assume that V_0 is proper. For any continuous function $\phi_0 : \mathbb{R}^n \to \mathbb{R}^p$ satisfying, for all x in $\{x : 0 < V_0(x) \leq R_0\}$,*

$$L_f V_0(x) \; + \; L_g V_0(x) \, \phi_0(x) \; < \; 0 \; , \tag{27}$$

and any continuous function $\phi_\infty : \mathbb{R}^n \to \mathbb{R}^p$ satisfying for all x in $\{x : V_\infty(x) \geq r_\infty\}$

$$L_f V_\infty(x) \; + \; L_g V_\infty(x) \, \phi_\infty(x) \; < \; 0 \; , \tag{28}$$

there exists a continuous function $\phi : \mathbb{R}^n \to \mathbb{R}^p$ which solves the uniting controller problem, i.e. such that

1) $\phi(x) \; = \; \phi_0(x)$ *for all x such that $V_\infty(x) \leq r_\infty$;*
2) $\phi(x) \; = \; \phi_\infty(x)$ *for all x such that $V_0(x) \geq R_0$;*
3) *the origin of the system $\dot{x} \; = \; f(x) \; + \; g(x) \, \phi(x)$ is a globally asymptotically stable equilibrium.*

The function $\phi : \mathbb{R}^n \to \mathbb{R}^m$ obtained from Theorem 4.1 and which is a solution to the uniting controller problem is defined as

$$\phi(x) \; = \; \mathcal{H}(x) \; - \; k \, c(x) \, L_g V(x) \; , \forall x \; \in \; \mathbb{R}^n \; , \tag{29}$$

where $V : \mathbb{R}^n \to \mathbb{R}_+$ is the Control Lyapunov Function obtained from Theorem 2.1, and with $\mathcal{H}(x) \; = \; \gamma(x) \, \phi_0(x) \; + \; [1 - \gamma(x)] \, \phi_\infty(x)$ where γ is any continuous function[4] such that

$$\gamma(x) \; = \; \begin{cases} 1 & \text{if } V_\infty(x) \; \leq \; r_\infty \; , \\ 0 & \text{if } V_0(x) \; \geq \; R_0 \; , \end{cases}$$

and the function c is any continuous function such that[5]

$$c(x) \begin{cases} \; = \; 0 & \text{if } V_0(x) \; \geq \; R_0 \;\; \text{or } V_\infty(x) \; \leq \; r_\infty \; , \\ \; > \; 0 & \text{if } V_0(x) \; < \; R_0 \;\; \text{and } V_\infty(x) \; > \; r_\infty \; , \end{cases}$$

and k is a positive real number sufficiently large.

[4]For instance, a possible choice is:

$$\gamma(s) = \min\left\{1, \max\left\{\frac{R_\infty - s}{R_\infty - r_\infty}, 0\right\}\right\} \tag{30}$$

with R_∞ is the positive real number defined previously.

[5]For instance, a possible choice is $c(x) = \max\{0, (R_0 - V_0(x))(V_\infty(x) - r_\infty)\}$ (31)

Proof: Note that the function ϕ satisfies item 1) and 2) of Theorem 4.1. It remains to show item 3). Taking the function V as a candidate Lyapunov function obtained in (8), the continuous function $\dot{V}$ can be introduced as, for all x in $\mathbb{R}^n$,

$$\dot{V}(x) \;=\; \frac{\partial V}{\partial x}(x) f(x) \;+\; \frac{\partial V}{\partial x}(x) g(x) \mathcal{H}(x) \;-\; k\, c(x) \left| \frac{\partial V}{\partial x}(x) g(x) \right|^2 \;.$$

With the local and global properties of the function V (as stated in (5) and (6) respectively), for all x in $\{x \neq 0 \,:\, V_\infty(x) \;\leq\; r_\infty$ or $V_0(x) \;\geq\; R_0\}$:

$$\dot{V}(x) < 0 \;. \tag{32}$$

It is now shown that if k is selected sufficiently large then this control law ensures the negativeness of $\dot{V}$ on the whole domain. To prove that, suppose the assertion is wrong and suppose for each k in $\mathbb{N}$, there exists x_k in $\mathbb{R}^n \setminus \{0\}$ such that

$$\dot{V}(x_k) \geq 0 \quad , \qquad \forall\, k \;\in\; \mathbb{N} \;. \tag{33}$$

With (32), $(x_k)_{k\in\mathbb{N}}$ is a sequence living in a compact set $\{x \,:\, V_\infty(x) \;\geq\; r_\infty\} \cap \{x \,:\, V_0(x) \leq R_0\}$. This subset being compact there exists a converging subsequence $(x_{k_\ell})_{\ell\in\mathbb{N}}$ which converges to a point denoted x^*. The function $x \mapsto \dot{V}(x)$ being continuous, it yields $\dot{V}(x^*) \geq 0$. With (32), this yields that x^* is in $\{x \,:\, V_\infty(x) \;>\; r_\infty\} \cap \{x \,:\, V_0(x) < R_0\}$, and by the definition of the function c it gives $c(x^*) > 0$. Consequently, this implies that, $\frac{\partial V}{\partial x}(x^*) g(x^*) \;=\; 0$. The function V being a CLF, this contradicts Assertion (33) and establishes that (32) is satisfied for all $x \neq 0$. Hence, item 3) is also satisfied. □

This theorem shows that as soon as the uniting CLF problem is solved, a continuous solution to the uniting controller problem is obtained. Note also, that if discontinuous controllers with discrete dynamics (not only continuous static controllers) are allowed, the existence of a hybrid controller solving the problem is obtained under Assumption 1 only (see [11], [13]).

Combining Proposition 3.1 and Theorem 4.1, it yields that for all linear second order controllable systems, each pair of linear stabilizing controller can be united, as stated in the following

Proposition 4.2: *Consider system (18) when $n = 2$, $m = 1$ and with F and G given in (20). Let K_0 and K_∞ in $\mathbb{R}^{1\times 2}$ be such that the origin of the systems:*

$$\dot{x} \;=\; (F \;+\; G\, K_0)\, x \;, \quad \dot{x} \;=\; (F \;+\; G\, K_\infty)\, x \;.$$

is globally asymptotically stable. Then there exists a continuous function $\phi : \mathbb{R}^n \to \mathbb{R}^p$ *such that the origin of the system* $\dot{x} = Fx + G\,\phi(x)$ *is globally asymptotically stable and such that* $\phi(x) = \phi_0(x)$ *in a neighborhood of the origin and* $\phi(x) = \phi_\infty(x)$ *outside of a compact set.*

V. Illustration on an example

To illustrate the interest of the uniting controller solution developed in this paper, a numerical example is provided in this section. Consider the non-linear system (1) when $n = 3$, $p = 1$, and the vector fields f and g defined by:

$$f(x) = \begin{bmatrix} -x_1 + x_3 \\ x_1^2 - x_2 - 2x_1x_3 + x_3 \\ -x_2 \end{bmatrix} \quad , \qquad g(x) = Gx \quad , \qquad G = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ , \tag{34}$$

for all $x = (x_1, x_2, x_3)$ in $\mathbb{R}^3$.

Let V_∞ be the continuously differentiable positive definite and proper function defined by

$$V_\infty(x_1, x_2, x_3) = \frac{1}{2}\left[x_1^2 + \left(x_1^2 + x_2\right)^2 + x_3^2\right] \ , \ \forall x \in \mathbb{R}^3 \ .$$

Along the vector fields f and g defined in (34), the Lie derivatives of the function V_∞ is

$$L_fV_\infty(x) = -x_1^2 - \left(x_1^2 + x_2\right)^2 + x_3\left(x_1^2 + x_1\right) \ , \ L_gV_\infty(x) = x_3 \ , \ \forall x \in \mathbb{R}^3 \ .$$

Note that for all x in $\mathbb{R}^3 \setminus \{0\}$ Artstein condition is satisfied (i.e. $L_gV_\infty(x) = 0 \Rightarrow L_fV_\infty < 0$). Consequently, V_∞ is a global CLF and the control law $u = \phi_\infty(x)$ with

$$\phi_\infty(x) = -x_1^2 - x_1 - x_3 \quad , \qquad \forall x \in \mathbb{R}^3 \ , \tag{35}$$

is such that, along the trajectories of the system (1) in closed-loop with ϕ_∞,

$$\dot{\overline{V_\infty(x)}} \leq -x_1^2 - \left(x_1^2 + x_2\right)^2 - x_3^2 \quad , \qquad \forall x \in \mathbb{R}^3 \ .$$

Hence the function ϕ_∞ defined in (35) ensures global asymptotic stability of the origin of the system defined in (1) and (34).

Note however that despite the global asymptotic stability of the origin is obtained with this control law, there is no guarantee that the performance obtained is satisfactory. For instance, it may be interesting that the controller locally minimizes a criterium defined as the limit, when

$t \to \infty$, of the operator $J : L^2(\mathbb{R}_+;\mathbb{R}^3) \times L^2(\mathbb{R}_+;\mathbb{R}) \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by, for all (x,u,t) in $L^2(\mathbb{R}_+;\mathbb{R}^3) \times L^2(\mathbb{R}_+;\mathbb{R}) \times \mathbb{R}_+$,

$$J(x,u,t) \;=\; \int_0^t x(s)^T Q x(s) + R u(s)^2 \, ds \;, \tag{36}$$

where Q is a symmetric positive definite matrix in $\mathbb{R}^3$ and R is a positive real number.

The techniques developed in this paper may be instrumental to modify the stabilizing controller $u = \phi_\infty$ such that the criterium J is minimized around the origin. A similar problem has been addressed in [10] where a general cost function depending on exogenous disturbances is considered. In [10], using a backstepping approach for upper triangular systems, a controller, which matches the optimal control law up to a desired order, is extended to a global stabilizer. In the uniting CLF approach, the global controller is computed independently from the optimal problem and an upper triangular structure is not required. However an assumption (namely Assumption 2) is needed. Using the first order approximation, this assumption can be rewritten in terms of an LMI (see Proposition 5.1 below).

The first order approximation around the origin of system (1) with f and g defined in (34) is

$$\dot{x} \;=\; F\,x \;+\; G\,u \quad , \qquad F \;=\; \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \;. \tag{37}$$

The system (37) being linear, an LQ controller minimizing the criterium defined in (36), is given by $\phi_0(x) = R^{-1} G^T P_0 \, x$, where P_0 is the symmetric positive definite solution of the Riccati equation:

$$P_0 F \;+\; F^T P_0 \;-\; P_0 G R^{-1} G^T P_0 \;+\; Q \;=\; 0 \;. \tag{38}$$

The tools developed in this paper provides a sufficient condition guaranteeing the existence of a continuous state feedback $u = \phi(x)$ which unites the optimal local controller ϕ_0 and the global one ϕ_∞ while ensuring global asymptotic stability of the origin:

Proposition 5.1: *Assume there exists a matrix K in $\mathbb{R}^3$ satisfying the following LMI:*

$$\begin{cases} (F \;+\; GK)^T P_0 \;+\; P_0 (F \;+\; GK) & < \;\; 0 \;, \\ (F \;+\; GK)^T P_\infty \;+\; P_\infty (F \;+\; GK) & < \;\; 0 \;, \end{cases} \tag{39}$$

where $P_\infty = \frac{1}{2} diag(1,1,1)$. Then there exists a continuous function $\phi \;: \mathbb{R}^3 \to \mathbb{R}$ such that the control law $u = \phi(x)$ makes the origin of the system (1) a globally asymptotically stable equilibrium and such that $\phi(x) = \phi_0(x)$ in a neighborhood of the origin.

Proof: The matrix P_0 being the solution of the Riccati equation (38), it yields, along the trajectories of system (37) with the control law $u = \phi_0$,

$$\dot{\widetilde{V_0(x)}} = -x^T Q x \ ,$$

where $V_0(x) = x^T P_0 x$ for all $x \in \mathbb{R}^3$. The matrix Q being symmetric positive definite and system (37) being the first order approximation of the system defined in (1) and (34), it implies that, for all R_0 sufficiently small V_0 is a local CLF. Moreover, V_∞ being a (global) CLF, it yields that for all R_0 sufficiently small, Assumption 1 is satisfied provided r_∞ is selected sufficiently small to guarantee that the covering assumption is satisfied.

Also, note that the function V_∞ and P_∞ are such that

$$\begin{aligned} V_\infty(x) &= x^T P_\infty x + x_1^2 x_2 + \frac{1}{2} x_1^4 \ , \\ &= x^T P_\infty x + o\left(|x|^2\right) \ . \end{aligned}$$

In other words, $x^T P_\infty x$ is the quadratic approximation around the origin of the global CLF V_∞. Moreover the Lie derivative of V_∞ along the vector fields f and g satisfy :

$$L_f V_\infty(x) = x^T \left(P_\infty F + F^T P_\infty\right) x + o\left(|x|^2\right) \ ,$$

and

$$L_g V_\infty(x) K x = 2 x^T P_\infty G K x + o\left(|x|^2\right) \ ,$$

where K is the obtained solution of the LMI (39). Consequently, with (39), the time derivative of the function V_∞ along the trajectory of the system with $u = Kx$ satisfies :

$$\dot{\widetilde{V_\infty(x)}} = x^T S_\infty x + o\left(|x|^2\right) \ ,$$

where the matrix S_∞, defined as

$$S_\infty = (F + GK)^T P_\infty + P_\infty (F + GK) \ ,$$

is symmetric negative definite due to (39).

Hence, the control law $u = Kx$ renders the time derivative of the CLF V_∞ and the local CLF V_0 negative definite for all x sufficiently small. With Proposition 2.2, this implies that Assumption 2 is satisfied provided R_0 and r_∞ are selected sufficiently small. Hence, with Theorem 4.1, taking R_0 and r_∞ sufficiently small and satisfying (22), the control law $u = \phi(x)$ with ϕ defined in (29) is such that

1) $\phi(x) = \phi_0(x)$ for all x such that $V_\infty(x) \leq r_\infty$;
2) $\phi(x) = \phi_\infty(x)$ for all x such that $V_0(x) \geq R_0$;
3) the origin of the system $\dot{x} = f(x) + g(x)\,\phi(x)$ is a globally asymptotically stable equilibrium.

This concludes the proof of Proposition 5.1. □

For the numerical illustration, the matrix Q is randomly selected as :

$$Q = \begin{bmatrix} 0.8 & 0.6 & 0.3 \\ \star & 0.6 & 0.5 \\ \star & \star & 1 \end{bmatrix} , \tag{40}$$

and $R = 1$. The matrix P_0 and the optimal local controller $\phi_0 = K_0\,x$ obtained solving the associated Riccati equation can be computed employing the *care* routine in *Matlab* :

$$P_0 = \begin{bmatrix} 0.3389 & 0.1412 & 0.3496 \\ \star & 0.3870 & -0.0912 \\ \star & \star & 1.2316 \end{bmatrix} , \qquad K_0 = \begin{bmatrix} 0.3496 \\ -0.0912 \\ 1.2316 \end{bmatrix} . \tag{41}$$

Employing the *Matlab* package *Yalmip* ([8]) in combination with the solver *Sedumi* ([16]), it can be checked[6] that the LMI condition (39) is satisfied for a particular K. Consequently, Proposition 5.1 applies and a controller which unites the optimal local one ϕ_0 and the global one ϕ_∞ can be constructed.

The uniting controller is given in (29) where the uniting CLF V is obtained from Theorem 2.1, and the functions φ_0, φ_∞, γ, and c are respectively defined by (9), (10), (30) and (31), with the following tuning parameters

$$R_0 = 0.88 , \quad r_\infty = 0.35 , \quad r_0 = 0.4739 , \quad R_\infty = 0.65 , \quad k = 1 .$$

Figure 1 compares the time-evolution of the cost J defined in (36) when considering the nominal control law $u = \phi_\infty$ and the uniting one $u = \phi$, with the initial condition $x(0) = [1\ 1\ 1]^T$. Figure 2 shows the time-evolution of the control values $u = \phi$.

On Figures 1 and 2 the times the state x crosses the level sets $\{x : V_0(x) = R_0\}$ and $\{x : V_\infty(x) = r_\infty\}$ are shown. These time instants define the time interval in which the

[6]The Matlab file can be downloaded from `http://homepages.laas.fr/~vandrieu/Publication.htm`

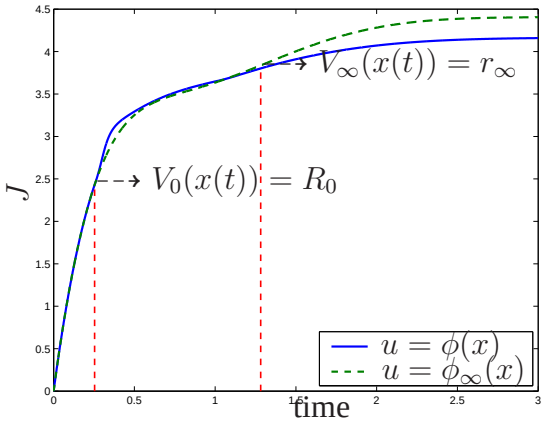

Fig. 1. Time-evolution of the cost function J with the controls ϕ (in plain line) and ϕ_∞ (in dashed line).

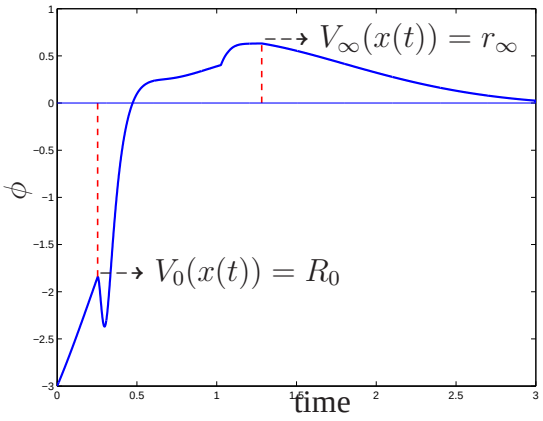

Fig. 2. Time-evolution of the uniting controller ϕ.

interpolation between both controllers occurs. In other words, before this time interval, the controller ϕ equals ϕ_∞ and after the local controller is employed.

However, there is no guarantee that, for all initial conditions, the cost obtained employing the uniting controller will be lower than the one obtained using the global one. More precisely, there exist initial conditions for which the use of the interpolation between both controllers affects too strongly the cost. To check if the uniting controller is statistically better than the global one, a set of initial conditions uniformly distributed on a sphere have been considered. For 77% of initial conditions the cost is lower with the uniting controller. On this particular example, the uniting controller seems to give better result (in terms of the considered cost) than the global one.

VI. Conclusion

In this paper, the problem of piecing together two Control Lyapunov Functions is considered. Solving this one provides a simple solution to the uniting controllers problem. Two characterizations of a sufficient condition guaranteeing the solvability of the united CLF problem are given. Moreover, this sufficient condition is always satisfied in the case of a second order linear controllable system. When dealing with linear systems, a stronger version of this sufficient condition can be formulated in terms of LMIs. As shown on a numerical illustration, it allows to exhibit a sufficient condition to improve the qualitative behavior of the trajectories of a non-linear

system around the equilibrium.

ACKNOWLEDGEMENT

The authors are deeply grateful to Laurent Praly and Alessandro Astolfi for helpful suggestions.

REFERENCES

[1] F. Ancona and A. Bressan. Flow stability of patchy vector fields and robust feedback stabilization. *SIAM Journal on Control and Optimization*, 41(5):1455–1476, 2003.

[2] V. Andrieu, L. Praly, and A. Astolfi. Homogeneous approximation, recursive observer design and output feedback. *SIAM Journal on Control and Optimization*, 47(4):1814–1850, 2008.

[3] Z. Arststein. Stabilization with relaxed controls. *Nonlinear analysis*, 7(11):1163–1173, 1983.

[4] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, volume 15 of *Studies in Applied Mathematics*. SIAM, Philadelphia, PA, June 1994.

[5] R. A. Freeman and P. V. Kokotovic. Inverse optimality in robust stabilization. *SIAM Journal on Control Optimization*, 34(4):1365–1391, 1996.

[6] M. Krstic, P.V. Kokotovic, and I. Kanellakopoulos. *Nonlinear and Adaptive Control Design*. John Wiley & Sons, Inc. New York, NY, USA, 1995.

[7] Y. Lin and E.D. Sontag. A universal formula for stabilization with bounded controls. *Systems & Control Letters*, 16(6):393–397, 1991.

[8] J. Lofberg. YALMIP: Yet Another LMI Parser, August 2001. URL:`http:// www.control.isy.liu.se/ ~johanl/ yalmip.html`.

[9] F. Mazenc and L. Praly. Adding integrations, saturated controls, and stabilization forfeedforward systems. *IEEE Transactions on Automatic Control*, 41(11):1559–1578, 1996.

[10] Z. Pan, K. Ezal, A.J. Krener, and P.V. Kokotovic. Backstepping design with local optimality matching. *IEEE Transactions on Automatic Control*, 46(7):1014–1027, 2001.

[11] C. Prieur. Uniting local and global controllers with robustness to vanishing noise. *Mathematics of Control, Signals, and Systems*, 14(2):143–172, 2001.

[12] C. Prieur and E. Trélat. Quasi-optimal robust stabilization of control systems. *SIAM Journal on Control and Optimization*, 45(5):1875–1897, 2006.

[13] R. G. Sanfelice, A. R. Teel, and R. Goebel. Supervising a family of hybrid controllers for robust global asymptotic stabilization. In *Proc. 47th IEEE Conference on Decision and Control*, pages 4700–4705, Cancun, Mexico, 2008.

[14] R. Sepulchre, M. Janković, and P. V. Kokotović. *Constructive nonlinear control*. Communications and Control Engineering Series. Springer-Verlag, 1997.

[15] E.D. Sontag. A ”universal” construction of Artstein’s theorem on nonlinear stabilization. *Systems & control letters*, 13(2):117–123, 1989.

[16] J.F. Sturm. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11-12:625–653, 1999. URL:`http://fewcal.kub.nl/ sturm/software/sedumi.html`.

[17] A. R. Teel, O. E. Kaiser, and R. M. Murray. Uniting local and global controllers for the Caltech ducted fan. In *16th American Control Conference*, pages 539 – 1543, Albuquerque NM, 1997.

[18] Q. Zheng and F. Wu. Improving nonlinear control performance with a hybrid control strategy. In *Proc. 46th IEEE Conference on Decision and Control*, pages 3208–3213, New Orleans, LA, 2007.