



HAL
open science

On the Minimum Size of a Contraction-Universal Tree

Olivier Bodini

► **To cite this version:**

| Olivier Bodini. On the Minimum Size of a Contraction-Universal Tree. 2003. hal-00432221

HAL Id: hal-00432221

<https://hal.science/hal-00432221>

Preprint submitted on 14 Nov 2009

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On the Minimum Size of a Contraction-Universal Tree

Olivier Bodini

LIP, École Normale Supérieure de Lyon, 46 Allée d'Italie, 69364 Lyon Cedex 05, France.

Abstract. A tree T_{uni} is m -universal for the class of trees if for every tree T of size m , T can be obtained from T_{uni} by successive contractions of edges. We prove that a m -universal tree for the class of trees has at least $m \ln(m) + (\gamma - 1)m + O(1)$ edges where γ is the Euler's constant and we build such a tree with less than m^c edges for a fixed constant $c = 1.984\dots$

1 Introduction

What is the minimum size of an object in which every object of size m embeds? Issued from the category theory, questions of this kind appeared in graph theory. For instance, R. Rado [1] proved the existence of an "initial countable graph". Recently, Z. Füredi and P. Komjáth [2] studied a connected question.

We use here the following definition : given a sub-class C of graphs (trees, planar graphs, etc.), a graph G_{uni} is m -universal for C if for every graph G of size m in C , G is a minor of G_{uni} , i.e. it can be obtained from G_{uni} by successive contractions or deletions of edges.

Inspired by the Robertson and Seymour work [3] on graph minors, P. Duchet asked whether a polynomial bound in m could be found for the size of a m -universal tree for the class of trees. We give here a positive sub-quadratic answer.

From an applied point of view, such an object would possibly allows us to define a tree from the representation of its contraction.

The main results of this paper are the following theorems which give bounds for the minimum size of a m -universal tree for the class of trees :

Theorem 1. *A m -universal tree for the class of trees has at least $m \ln(m) + (\gamma - 1)m + O(1)$ edges where γ is the Euler's constant.*

Theorem 2. *There exists a m -universal tree T_{uni} for the class of trees with less than m^c edges for a fixed constant $c = 1.984\dots$*

Our proof follows a recursive construction where large trees are obtained by some amalgamation process involving simpler trees. With this method, the constant c could be reduced to 1.88... but it seems difficult to improve this value.

We conclude the paper with related open questions.

2 Terminology

Our graphs are undirected and simple (with neither loops nor multiple edges). We denote by $G(V, E)$ a graph (its vertex set is $V(G)$ and its edge set is $E(G)$ (a subset of the family of all the $V(G)$ -subsets of cardinality 2)). Referring to C. Thomassen [4], we recall some basic definitions that are useful for our purpose:

We denote by P_n the path of size n .

If x is a vertex then $d(x)$, the *degree* of x , is the number of edges incident to x .

Let e be an edge of $E(G)$, the graph denoted by $G - e$ is the graph on the vertex set of G , whose edge set is the edge set of G without e . We call classically this operation *deletion*.

Let $e = \{a, b\}$ be an edge of $G(V, E)$, we name *contraction of G along e* , the graph denoted by $G/e = H(V', E')$, with $V' = (V/\{a, b\}) \cup \{c\}$ where c is a new vertex and E' the edge set which contains all the edges of the sub-graph G_1 on V/e and all the edges of the form $\{c, x\}$ for $\{a, x\}$ or $\{b, x\}$ belonging to E .

We say that H is a *minor* of G if and only if we can obtain it from G by successively deleting and /or contracting edges, in an other way, we can define the set $M(G)$ of minors of G by the recursive formula :

$$M(G) = G \cup \left(\bigcup_{e \in E(G)} M(G/e) \right) \cup \left(\bigcup_{e \in E(G)} M(G - e) \right)$$

The notion of minor induces a partial order on graphs. We write $A \preceq B$ to mean "A is a minor of B".

For technical reasons, we prefer to use the size of a tree (edge number) rather than its order (vertex number).

Finally, let us recall that, a graph G_{uni} is *m-universal for a sub-class C* of graphs if for every element G of C with m edges, G is a minor of G_{uni} .

3 A Lower Bound

In this section, we prove that a m -universal tree T_{uni} for the trees has asymptotically at least $m \ln(m)$ edges. We use the fact that T_{uni} has to contain all spiders of size m as minors. A *spider S on a vertex w* is a tree such that $\forall v \in V(S) \setminus \{w\}, d(v) \leq 2$. We denote the spider constituted by paths of lengths $1 \leq m_1 \leq \dots \leq m_k$ by $Sp(m_1, \dots, m_k)$ (Fig.1).

Definition 1. Let T be a tree, we denote by ∂T the subtree of T with $V(\partial T) = V(T) \setminus A$, where A is the set of the leaves of T . Also, we denote by ∂^k the k -th iteration of ∂ .

Lemma 1. $Sp(m_1, \dots, m_k) \preceq T$ involves that $\partial Sp(m_1, \dots, m_k) \preceq \partial T$. Moreover, if for all $i, m_i = 1$ then $\partial Sp(m_1, \dots, m_k)$ is a vertex. Otherwise, put a the first value such that $m_a > 1$, we have $\partial Sp(m_1, \dots, m_k) = Sp(m_a - 1, \dots, m_k - 1)$ excepted for $k = 1$, in this last case we have $\partial Sp(m_1) = Sp(m_1 - 2)$.

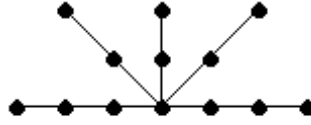


Fig.1. $Sp(2, 2, 2, 3, 3)$

Proof. This just follows from an observation. □

Lemma 2. For every tree T , $Sp(m_1, \dots, m_k) \preceq T \Rightarrow T$ has at least k leaves.

Proof. Trivial. □

Theorem 3. A m -universal tree T_{uni} for the class of trees has at least $\sum_{i=1, i \neq 2}^m \lfloor \frac{m}{i} \rfloor$ edges.

Proof. A m -universal tree T_{uni} for the class of trees has to contain as minors all spiders of size m . So, for all p it contains as minors the spiders $Sp(p, \dots, p)$ where we have $\lfloor \frac{m}{p} \rfloor$ times the letter p . By the lemma 1, for all $p \leq \frac{m}{2}$, $Sp(1, \dots, 1) \preceq \partial^{p-1} T_{uni}$ and if m is odd, $Sp(1) \preceq \partial^{\lfloor \frac{m}{2} \rfloor - 1} T_{uni}$. Moreover, it is clear that the terminal edges of the $\partial^p T_{uni}$ constitute a partition of T_{uni} . By the lemma 2, this involves that T_{uni} has at least $\sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} \lfloor \frac{m}{i} \rfloor$ edges if m is even and $1 + \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} \lfloor \frac{m}{i} \rfloor$ edges if m is odd. An easy calculation proves that these values are always equal to $\sum_{i=1, i \neq 2}^m \lfloor \frac{m}{i} \rfloor$. □

Proof. (of the theorem 1) it follows from the usual estimate $\sum_{i=1}^n \frac{1}{i} \sim \ln(n) + \gamma + O(\frac{1}{n})$ and the inequality $\sum_{i=1, i \neq 2}^m \lfloor \frac{m}{i} \rfloor \geq 1 + \sum_{i=1, i \neq 2}^{m-1} (\frac{m}{i} - 1)$. □

What the above proof shows, in fact, is the following :

Corollary 1. A minimum m -universal spider for the class of spiders has $\sum_{i=1, i \neq 2}^m \lfloor \frac{m}{i} \rfloor$ edges.

Proof. The spider $Sp\left(\lfloor \frac{m}{m} \rfloor, \lfloor \frac{m}{m-1} \rfloor, \dots, \lfloor \frac{m}{2} \rfloor, \lceil \frac{m}{2} \rceil\right)$ is clearly a m -universal spider of size $\sum_{i=1, i \neq 2}^m \lfloor \frac{m}{i} \rfloor$ for the class of spiders, and by theorem 3 it is a minimum value. □

4 The Main Stem

In the sequel, we deal with *rooted graph*, i.e. graph G where we can distinguish a special vertex denoted by $r(G)$, called the *root*. Conventionally, any contracted graph G' of same rooted graph G will be rooted at the unique vertex which is the image of the root under the contraction mapping, we say in this case that the rooted graph G' is a *rooted contraction* of G . Note that, the contraction operator suffices to obtain all minor trees of a tree. So, we can now define the following new notion for sub-classes of rooted trees : a rooted tree T_{uni} is *strongly m -universal for a sub-classes C of rooted trees* if for every rooted tree T in C of size m , T is a rooted contraction of T_{uni} . The concept of root is introduced to avoid problems with graph isomorphisms that, otherwise would greatly impede our inductive proof.

For every edge e of a tree T , the forest $T \setminus e$ has two connected components. We call *e -branch*, denoted by B_e , the connected component of T' which does not contain $r(T)$, we define the root of B_e as $e \cap V(B_e)$.

A *main stem* of a rooted tree of size m is defined as a path P which is issued from the root and such that for all e -branches B_e with $e \notin E(C)$, we have $|E(B_e)| < \lfloor \frac{m}{2} \rfloor$ (Fig.2).

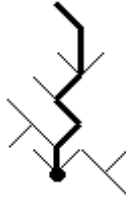


Fig.2. A main stem in bold

The following lemma suggests the procedure which will be used to find a sub-quadratic upper bound for universal trees. Roughly speaking, it endows every tree with some recursive structure constructed with the help of main stems.

Lemma 3. *Every rooted tree has a main stem.*

Proof. By induction on the size of the rooted tree. Let T be a rooted tree, if T has one or two edges, it is trivial. Otherwise let us consider the sub-graph $T \setminus r(T)$, which is a forest. We choose a connected component T_1 with maximum size and we denote by b_1 the unique vertex of T_1 which is adjacent to $r(T)$. Tree T_1 , rooted in b_1 , has, by the induction hypothesis, a main stem B . Then the path $(V(B) \cup \{r(T)\}, E(B) \cup \{\{r(T), b_1\}\})$ is a main stem of T . \square

Remark 1. A tree may possess in general several main stems. Let us notice also that a main stem is not necessarily one of the longest paths which contain the root.

5 The Upper Bound

We need some new definitions. A *rooted brush* (Fig.3) is a rooted tree such that the vertices of degree greater than 2 are on a same path P issued from the root.

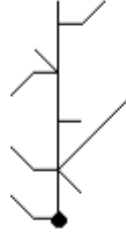


Fig.3. A rooted brush

A *rooted comb* X (Fig.4) is a rooted brush with $d(r(X)) \leq 2$ and $\forall v \in V(X), d(v) \leq 3$.

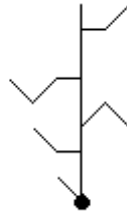


Fig.4. A rooted comb

The *length of a rooted comb* corresponds to the length of the longest path P issued from the root which contains all vertices of degree greater than 2.

To obtain an upper bound, we consider two building processes : the first one, a brushing M_B , maps rooted trees with a main stem into rooted brushes, the second one, a ramifying M_T , consists in obtaining a sequence of rooted trees, assuming that we have an increasing sequence of rooted combs. We note M_T^k the k -th element of the sequence. These building processes will possess the following fundamental property:

Property 1. Let (T, σ) a rooted tree with a main stem σ and $(X_n)_{n \in \mathbb{N}}$ a sequence of rooted combs :

$$(\forall T' \preceq T, M_B(T', \sigma) \preceq X_{|E(T')|}) \Rightarrow T \preceq M_T^{|E(T)|}((X_n)_{n \in \mathbb{N}}).$$

Lemma 4. *If building processes verify the property 1 and if for all i , the rooted comb X_i is strongly i -universal for the class of rooted brushes then the rooted tree $M_T^m((X_n)_{n \in \mathbb{N}})$ is strongly m -universal for the class of rooted trees.*

Proof. It is just an interpretation of the property. \square

We now establish the existence of building processes which satisfy property 1.

Brushing M_B (Fig.5). Let T be a rooted tree with a main stem σ . We are going to associate a rooted brush B with it, denoted $M_B(T, \sigma)$ of the same size built from the same main stem σ with the following process: every e -branch B_e connected to the main stem by edge e is replaced by a path of length $|E(B_e)|$ connected by the same edge.

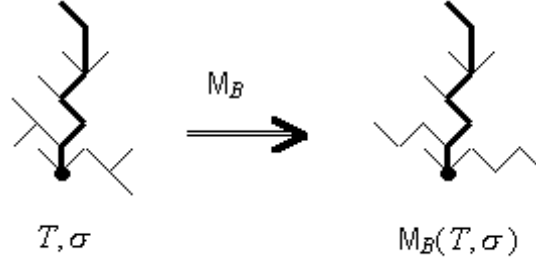


Fig.5.

Ramifying M_T^k . For the second building process we work in two steps :

First step. Given rooted trees T_1, \dots, T_k with disjoint vertex sets, we build another rooted tree T , denoted $[T_1, \dots, T_k]$, in the following way :

$$V(T) = \bigcup_{i=1}^k V(T_i) \cup \{v_1, \dots, v_{k+1}\},$$

$$E(T) = \bigcup_{i=1}^k E(T_i) \cup \{\{v_1, r(T_1)\}, \dots, \{v_k, r(T_k)\}\} \cup \{\{v_1, v_2\}, \dots, \{v_k, v_{k+1}\}\},$$

and $r(T) = v_1$.

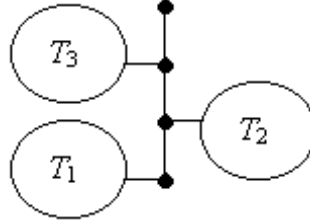
If $T_i = \emptyset$, conventionally $\{v_i, r(T_i)\} = \emptyset$.

Prosaically, from a path $P_k = [v_1, \dots, v_{k+1}]$ of size k and from k rooted trees T_1, \dots, T_k , we build a rooted tree joining a branch T_i to the vertex v_i of P (Fig.6).

Second step. By convention, $P_{-1} = \emptyset$.

We are going to construct rooted trees T_k in the following way :

$T_{-1} = \emptyset$, $T_0 = X_0$, and $\forall i, 1 \leq i \leq k, T_i = [T_{\min(u_1, i-1)}, \dots, T_{\min(u_{n_i}, i-1)}]$ if $X_i = [P_{u_1}, \dots, P_{u_{n_i}}]$.

Fig.6. A rooted comb $[T_1, T_2, T_3]$

We can now define M_T^k :

$$M_T^k((X_n)_{n \in \mathbb{N}}) = T_k.$$

Lemma 5. *The building processes described above verify the property 1.*

Proof. First, note that $M_T((X_n)_{n \in \mathbb{N}})$ is an increasing sequence. We prove the lemma by recurrence on the size m of T . When $m = 0$ or $m = 1$, this is trivial. We suppose the property is verified for T with size $m < m_0$. Let T be a rooted tree of size m_0 with a stem σ , we note e_1, \dots, e_k the edges of T issued from σ which do not belong to σ . To each e -branch of T with $e \in \{e_1, \dots, e_k\}$ corresponds by M_B a e -branch (it is a path of same size) in $M_B(T, \sigma)$. So there exists k distinct e -branches R_1, \dots, R_k in X_{m_0} that we can respectively contract to obtain each e -branch with $e = e_1, \dots, e_k$ in $M_B(T, \sigma)$. By recurrence hypothesis, we have for $1 \leq i \leq k$, $B_{e_i} \preceq M_T^{|E(B_{e_i})|}((X_n)_{n \in \mathbb{N}})$ and we have also $M_T^{|E(B_{e_i})|}((X_n)_{n \in \mathbb{N}}) \preceq M_T^{|E(R_i)|}((X_n)_{n \in \mathbb{N}})$. So each e -branch of T is a minor contraction of $M_T^{|E(R_i)|}((X_n)_{n \in \mathbb{N}})$. By associativity of contraction map, we have $T \preceq M_T^{|E(T)|}((X_n)_{n \in \mathbb{N}})$. \square

In this phase, we determine a sequence of rooted combs $(X_i)_{i \in \mathbb{N}}$ such that the rooted combs X_i are strongly i -universal for the rooted brushes.

In order to achieve this result, we define F_p as the set of functions $f : \{1, \dots, p\} \rightarrow \{1, \dots, \lfloor \frac{p}{2} \rfloor\}$ satisfying the following property :

$$(\forall n \in \{1, \dots, p\}) \left(\forall i \leq \left\lfloor \frac{n}{2} \right\rfloor \right) (\exists k \in \mathbb{N}) (n - i + 1 \leq k \leq n \text{ and } f(k) \geq i)$$

Lemma 6. F_p is not empty, it contains the following function φ_p , defined for $1 \leq i \leq p$ by :

$$\varphi_p(i) = \min \left(2^{v_2(i)+1} - 1, \left\lfloor \frac{p}{2} \right\rfloor, i - 1 \right)$$

where $v_2(k)$ is the 2-valuation of k (i.e. the greatest power of 2 dividing k).

Proof. The verification is obvious. \square

Corollary 2. *A minimum m -universal brush for the brushes has $O(m \ln(m))$ edges.*

By convention, we put $Comb_0^F = P_0$ (tree reduced in a vertex)

We define $Tree_m^F = M_T^m \left((Comb_n^F)_{n \in \mathbb{N}} \right)$.

As before, we will say that the tree built in such a way is *recursively associated to the sequence F* and denoted by $Tree_m^F$.

Thus, we have :

Theorem 5. *The rooted tree $Tree_m^F$ is strongly m -universal for the class of rooted trees.*

We now analyze the size of $Tree_m^F$.

Proposition 1. *Let $F = (f_1, f_2, \dots)$ be a sequence of functions such that $f_i \in F_i$ for $i \geq 1$. The size of a m -universal tree constructed from the sequence is given by the following recursive formula :*

$$u_{-1} = -1, u_0 = 0 \text{ and } u_k = 2k - 1 + \sum_{i=1}^k u_{f_k(i)-1}$$

Proof. It derives from the following observation :

m edges constitute the main stem, we have to add $m - 1$ edges to link branches to the main stem and $\sum_{i=1}^k u_{f_k(i)-1}$ edges for the branches. \square

Theorem 6. *There is a sequence of functions $G = (g_1, g_2, \dots)$ such that $g_i \in F_i$ and $|E(Tree_m^G)| < (2m)^c$ where $c = 1.984\dots$ is the unique positive solution of the equation $\frac{1}{2^c} + \frac{1}{2^{2^c}} + \frac{1}{2^{(c-1)-1}} - \frac{1}{2^{c-1}} = 1$.*

Proof. We take the following sequence of functions :

$g_m(i) = \min(2^{v_2(i)+1}, i)$ if $i < m$ and i even, $g_m(i) = 1$ if i odd and $g_m(m) = \lfloor \frac{m}{4} \rfloor$. It is clear that, if m is a power of 2, the comb $Comb_m^G$ is strongly m -universal for the brushes.

In fact, the function g_m takes the value $2^{v_2(i)+1}$ when i is not a power of 2, otherwise it is equal to i . Thanks to this remark and with $u_m < m + \sum_{i=1}^m u_{f_m(i)}$,

(the sequence of sizes is increasing), we obtain $u_{2^n} < 2^n + 2^{n-1} + \sum_{i=2}^{n-1} 2^{n-i} u_{2^i} -$

$\sum_{i=2}^{n-1} u_{2^i} + u_{2^{n-1}} + u_{2^{n-2}}$. Thus, in evaluating the sums and reorganizing the terms, we obtain :

$$u_{2^n} < \alpha_n + 2^{nc} \beta$$

with

$$\alpha_n = 2^{n-1} + 1 + 2^c + \frac{1}{2^c - 1} - \left(\frac{2^n}{2^{(c-1)} - 1} + 2^{n(c-1)} \right)$$

$$\beta = \frac{1}{2^c} + \frac{1}{2^{2c}} + \frac{1}{2^{(c-1)} - 1} - \frac{1}{2^c - 1}$$

Now $\alpha_n < 0$ when $m > 1$ and $\beta \leq 1$ by definition of c .

So $u_{2^n} < 2^{n^c}$, hence $u_m < (2m)^c$. \square

Remark 2. We observe that $c = \frac{\ln(x)}{\ln(2)}$, where x is the positive root of $X^4 - 5X^3 + 4X^2 + X - 2 = 0$.

Theorem 2 then follows since any rooted tree which is strongly m -universal for the rooted trees is also clearly m -universal for the class of trees.

6 Conclusion and Related Questions

When using the sequence $\Phi = (\varphi_1, \varphi_2, \dots)$ of lemma 7, the induction step leads to involved expressions that do not allow us to find the asymptotic behavior of the corresponding term u_m . A computer simulation gives that such a m -universal tree for the trees has less than $m^{1.88}$ edges. In any case, the constructive approach we proposed here, seems to be hopeless to reach the asymptotic best size of a m -universal tree for the trees.

Conjecture 1. The minimal size of a m -universal tree for the trees is $m^{1+o(1)}$.

As a possible way to prove such a conjecture, it would be interesting to obtain an explicit effective coding of a tree of size m using a list of contracted edges taken in a m -universal tree for the trees.

A variant of our problem consists in determining a minimum tree which contains as a subtree every tree of size m . This is closely related to a well known still open conjecture due to Erdős and Sös (see [5]).

References

1. R. Rado, Universal graphs and universal functions, *Acta Arith.*, **9** (1964), 331-340.
2. Z. Füredi and P. Komjãth, Nonexistence of universal graphs without some trees, *Combinatorica*, **17**, (2) (1997), 163-171.
3. N. Robertson and P.D. Seymour, series of papers on Graph minors, *Journal of combinatorics, serie B*, (1983-...).
4. C. Thomassen, Embeddings and Minors, chapter 5 in *Handbook of Combinatorics*, (R. Graham, M. Grötschel and L. Lovász, eds.), Elsevier Science B.V., 1995, 301-349.
5. P. Erdős and T. Gallai, On maximal paths and circuits of graphs, *Acta Math. Sci. Hungar.* **10** (1959), 337-356.