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Olivier Bodini. On the Minimum Size of a Contraction-Universal Tree. 2003. hal-00432221

## HAL Id: hal-00432221 https://hal.science/hal-00432221

Preprint submitted on 14 Nov 2009

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## On the Minimum Size of a Contraction-Universal Tree

Olivier Bodini

LIP, École Normale Supérieure de Lyon, 46 Allée d'Italie, 69364 Lyon Cedex 05, France.

**Abstract.** A tree  $T_{uni}$  is *m*-universal for the class of trees if for every tree *T* of size *m*, *T* can be obtained from  $T_{uni}$  by successive contractions of edges. We prove that a *m*-universal tree for the class of trees has at least  $m\ln(m) + (\gamma - 1)m + O(1)$  edges where  $\gamma$  is the Euler's constant and we build such a tree with less than  $m^c$  edges for a fixed constant c = 1.984...

#### 1 Introduction

What is the minimum size of an object in which every object of size m embeds? Issued from the category theory, questions of this kind appeared in graph theory. For instance, R. Rado [1] proved the existence of an "initial countable graph". Recently, Z. Füredi and P. Komjàth [2] studied a connected question.

We use here the following definition : given a sub-class C of graphs (trees, planar graphs, etc.), a graph  $G_{uni}$  is *m*-universal for C if for every graph G of size m in C, G is a minor of  $G_{uni}$ , i.e. it can be obtained from  $G_{uni}$  by successive contractions or deletions of edges.

Inspired by the Robertson and Seymour work [3] on graph minors, P. Duchet asked whether a polynomial bound in m could be found for the size of a m-universal tree for the class of trees. We give here a positive sub-quadratic answer.

From an applied point of view, such an object would possibly allows us to define a tree from the representation of its contraction.

The main results of this paper are the following theorems which give bounds for the minimum size of a m-universal tree for the class of trees :

**Theorem 1.** A m-universal tree for the class of trees has at least  $m \ln(m) + (\gamma - 1)m + O(1)$  edges where  $\gamma$  is the Euler's constant.

**Theorem 2.** There exists a m-universal tree  $T_{uni}$  for the class of trees with less than  $m^c$  edges for a fixed constant c = 1.984...

Our proof follows a recursive construction where large trees are obtained by some amalgamation process involving simpler trees. With this method, the constant c could be reduced to 1.88... but it seems difficult to improve this value.

We conclude the paper with related open questions.

#### 2 Terminology

Our graphs are undirected and simple (with neither loops nor multiple edges). We denote by G(V, E) a graph (its vertex set is V(G) and its edge set is E(G) (a subset of the family of all the V(G)-subsets of cardinality 2)). Referring to C. Thomassen [4], we recall some basic definitions that are useful for our purpose:

We denote by  $P_n$  the path of size n.

If x is a vertex then d(x), the *degree* of x, is the number of edges incident to x.

Let e be an edge of E(G), the graph denoted by G - e is the graph on the vertex set of G, whose edge set is the edge set of G without e. We call classically this operation *deletion*.

Let  $e = \{a, b\}$  be an edge of G(V, E), we name contraction of G along e, the graph denoted by G/e = H(V', E'), with  $V' = (V/\{a, b\}) \cup \{c\}$  where c is a new vertex and E' the edge set which contains all the edges of the sub-graph  $G_1$  on V/e and all the edges of the form  $\{c, x\}$  for  $\{a, x\}$  or  $\{b, x\}$  belonging to E.

We say that H is a *minor* of G if and only if we can obtain it from G by successively deleting and /or contracting edges, in an other way, we can define the set M(G) of minors of G by the recursive formula :

$$M(G) = G \cup \left(\bigcup_{e \in E(G)} M(G/e)\right) \cup \left(\bigcup_{e \in E(G)} M(G-e)\right)$$

The notion of minor induces a partial order on graphs. We write  $A \preceq B$  to mean "A is a minor of B".

For technical reasons, we prefer to use the size of a tree (edge number) rather than its order (vertex number).

Finally, let us recall that, a graph  $G_{uni}$  is *m*-universal for a sub-class C of graphs if for every element G of C with m edges, G is a minor of  $G_{uni}$ .

#### 3 A Lower Bound

In this section, we prove that a *m*-universal tree  $T_{uni}$  for the trees has asymptotically at least  $m \ln(m)$  edges. We use the fact that  $T_{uni}$  has to contain all spiders of size *m* as minors. A spider *S* on a vertex *w* is a tree such that  $\forall v \in V(S) \setminus \{w\}, d(v) \leq 2$ . We denote the spider constituted by paths of lengths  $1 \leq m_1 \leq \ldots \leq m_k$  by  $Sp(m_1, \ldots, m_k)$  (Fig.1).

**Definition 1.** Let T be a tree, we denote by  $\partial T$  the subtree of T with  $V(\partial T) = V(T) \setminus A$ , where A is the set of the leaves of T. Also, we denote by  $\partial^k$  the k-th iteration of  $\partial$ .

**Lemma 1.**  $Sp(m_1, ..., m_k) \leq T$  involves that  $\partial Sp(m_1, ..., m_k) \leq \partial T$ . Moreover, if for all  $i, m_i = 1$  then  $\partial Sp(m_1, ..., m_k)$  is a vertex. Otherwise, put a the first value such that  $m_a > 1$ , we have  $\partial Sp(m_1, ..., m_k) = Sp(m_a - 1, ..., m_k - 1)$  excepted for k = 1, in this last case we have  $\partial Sp(m_1) = Sp(m_1 - 2)$ .

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Fig.1. Sp(2, 2, 2, 3, 3)

*Proof.* This just follows from an observation.

**Lemma 2.** For every tree T,  $Sp(m_1, ..., m_k) \preceq T \Rightarrow T$  has at least k leaves.

Proof. Trivial.

**Theorem 3.** A m-universal tree  $T_{uni}$  for the class of trees has at least  $\sum_{i=1,i\neq 2}^{m} \lfloor \frac{m}{i} \rfloor$  edges.

Proof. A *m*-universal tree  $T_{uni}$  for the class of trees has to contain as minors all spiders of size *m*. So, for all *p* it contains as minors the spiders Sp(p, ..., p) where we have  $\left\lfloor \frac{m}{p} \right\rfloor$  times the letter *p*. By the lemma 1, for all  $p \leq \frac{m}{2}$ ,  $Sp(1, ..., 1) \leq \partial^{p-1}T_{uni}$  and if *m* is odd,  $Sp(1) \leq \partial^{\left\lfloor \frac{m}{2} \right\rfloor - 1}T_{uni}$ . Moreover, it is clear that the terminal edges of the  $\partial^p T_{uni}$  constitute a partition of  $T_{uni}$ . By the lemma 2, this involves that  $T_{uni}$  has at least  $\sum_{i=1}^{\left\lfloor \frac{m}{i} \right\rfloor} edges$  if *m* is even and  $1 + \sum_{i=1}^{\left\lfloor \frac{m}{i} \right\rfloor} \left\lfloor \frac{m}{i} \right\rfloor$  edges if *m* is odd. An easy calculation proves that these values are always equal to  $\sum_{i=1,i\neq 2}^{m} \left\lfloor \frac{m}{i} \right\rfloor$ .

*Proof.* (of the theorem 1) it follows from the usual estimate  $\sum_{i=1}^{n} \frac{1}{i} \sim \ln(n) + \gamma + O\left(\frac{1}{n}\right)$  and the inequality  $\sum_{i=1,i\neq 2}^{m} \left\lfloor \frac{m}{i} \right\rfloor \ge 1 + \sum_{i=1,i\neq 2}^{m-1} \left(\frac{m}{i} - 1\right).$ 

What the above proof shows, in fact, is the following :

**Corollary 1.** A minimum m-universal spider for the class of spiders has  $\sum_{i=1,i\neq 2}^{m} \lfloor \frac{m}{i} \rfloor$  edges.

*Proof.* The spider  $Sp\left(\left\lfloor \frac{m}{m} \right\rfloor, \left\lfloor \frac{m}{m-1} \right\rfloor, ..., \left\lfloor \frac{m}{2} \right\rfloor, \left\lceil \frac{m}{2} \right\rceil\right)$  is clearly a *m*-universal spider of size  $\sum_{i=1, i \neq 2}^{m} \left\lfloor \frac{m}{i} \right\rfloor$  for the class of spiders, and by theorem 3 it is a minimum value.

#### 4 The Main Stem

In the sequel, we deal with rooted graph, i.e. graph G where we can distinguish a special vertex denoted by r(G), called the root. Conventionally, any contracted graph G' of same rooted graph G will be rooted at the unique vertex which is the image of the root under the contraction mapping, we say in this case that the rooted graph G' is a rooted contraction of G. Note that, the contraction operator suffices to obtain all minor trees of a tree. So, we can now define the following new notion for sub-classes of rooted trees : a rooted tree  $T_{uni}$  is strongly m-universal for a sub-classes C of rooted trees if for every rooted tree T in C of size m, T is a rooted contraction of  $T_{uni}$ . The concept of root is introduced to avoid problems with graph isomorphisms that, otherwise would greatly impede our inductive proof.

For every edge e of a tree T, the forest  $T \setminus e$  has two connected components. We call *e*-branch, denoted by  $B_e$ , the connected component of T' which does not contain r(T), we define the root of  $B_e$  as  $e \cap V(B_e)$ .

A main stem of a rooted tree of size m is defined as a path P which is issued from the root and such that for all e-branches  $B_e$  with  $e \notin E(C)$ , we have  $|E(B_e)| < \lfloor \frac{m}{2} \rfloor$  (Fig.2).



Fig.2. A main stem in bold

The following lemma suggests the procedure which will be used to find a subquadratic upper bound for universal trees. Roughly speaking, it endows every tree with some recursive structure constructed with the help of main stems.

#### Lemma 3. Every rooted tree has a main stem.

*Proof.* By induction on the size of the rooted tree. Let T be a rooted tree, if T has one or two edges, it is trivial. Otherwise let us consider the sub-graph  $T \setminus r(T)$ , which is a forest. We choose a connected component  $T_1$  with maximum size and we denote by  $b_1$  the unique vertex of  $T_1$  which is adjacent to r(T). Tree  $T_1$ , rooted in  $b_1$ , has, by the induction hypothesis, a main stem B. Then the path  $(V(B) \cup \{r(T)\}, E(B) \cup \{r(T), b_1\})$  is a main stem of T.

*Remark 1.* A tree may possess in general several main stems. Let us notice also that a main stem is not necessarily one of the longest paths which contain the root.

WG (2002)

#### 5 The Upper Bound

We need some new definitions. A rooted brush (Fig.3) is a rooted tree such that the vertices of degree greater than 2 are on a same path P issued from the root.



Fig.3. A rooted brush

A rooted comb X (Fig.4) is a rooted brush with  $d(r(X)) \leq 2$  and  $\forall v \in V(X)$ ,  $d(v) \leq 3$ .



Fig.4. A rooted comb

The *length of a rooted comb* corresponds to the length of the longest path P issued from the root which contains all vertices of degree greater than 2.

To obtain an upper bound, we consider two building processes : the first one, a brushing  $M_B$ , maps rooted trees with a main stem into rooted brushes, the second one, a ramifying  $M_T$ , consists in obtaining a sequence of rooted trees, assuming that we have an increasing sequence of rooted combs. We note  $M_T^k$  the *k*-th element of the sequence. These building processes will possess the following fundamental property:

Property 1. Let  $(T, \sigma)$  a rooted tree with a main stem  $\sigma$  and  $(X_n)_{n \in \mathbb{N}}$  a sequence of rooted combs :

$$\left(\forall T' \preceq T, M_B\left(T', \sigma\right) \preceq X_{|E(T')|}\right) \Rightarrow T \preceq M_T^{|E(T)|}\left((X_n)_{n \in \mathbb{N}}\right).$$

**Lemma 4.** If building processes verify the property 1 and if for all *i*, the rooted comb  $X_i$  is strongly *i*-universal for the class of rooted brushes then the rooted tree  $M_T^m((X_n)_{n\in\mathbb{N}})$  is strongly *m*-universal for the class of rooted trees.

*Proof.* It is just an interpretation of the property.

We now establish the existence of building processes which satisfy property 1.

**Brushing**  $M_B$  (Fig.5). Let T be a rooted tree with a main stem  $\sigma$ . We are going to associate a rooted brush B with it, denoted  $M_B(T, \sigma)$  of the same size built from the same main stem  $\sigma$  with the following process: every e-branch  $B_e$  connected to the main stem by edge e is replaced by a path of length  $|E(B_e)|$  connected by the same edge.



**Ramifying**  $M_T^k$ . For the second building process we work in two steps : **First step.** Given rooted trees  $T_1, ..., T_k$  with disjoint vertex sets, we build another rooted tree T, denoted  $[T_1, ..., T_k]$ , in the following way :

$$V(T) = \bigcup_{i=1}^{k} V(T_i) \cup \{v_1, ..., v_{k+1}\},\$$

$$E(T) = \bigcup_{i=1}^{k} E(T_i) \cup \{\{v_1, r(T_1)\}, ..., \{v_k, r(T_k)\}\} \cup \{\{v_1, v_2\}, ..., \{v_k, v_{k+1}\}\},\$$

and  $r(T) = v_1$ .

If  $T_i = \emptyset$ , conventionally  $\{v_i, r(T_i)\} = \emptyset$ .

Prosaically, from a path  $P_k = [v_1, ..., v_{k+1}]$  of size k and from k rooted trees  $T_1, ..., T_k$ , we build a rooted tree joining a branch  $T_i$  to the vertex  $v_i$  of P (Fig.6). Second step. By convention,  $P_{-1} = \emptyset$ .

We are going to construct rooted trees  $T_k$  in the following way :

 $T_{-1} = \emptyset, T_0 = X_0, \text{ and } \forall i, 1 \le i \le k, T_i = \left[ T_{\min(u_1, i-1)}, ..., T_{\min(u_{n_i}, i-1)} \right] \text{ if } X_i = \left[ P_{u_1}, ..., P_{u_{n_i}} \right].$ 



Fig.6. A rooted comb  $[T_1, T_2, T_3]$ 

We can now define  $M_T^k$ :

$$M_T^k\left((X_n)_{n\in\mathbb{N}}\right) = T_k$$

Lemma 5. The building processes described above verify the property 1.

*Proof.* First, note that  $M_T((X_n)_{n\in\mathbb{N}})$  is an increasing sequence. We prove the lemma by recurrence on the size *m* of *T*. When *m* = 0 or *m* = 1, this is trivial. We suppose the property is verified for *T* with size *m* < *m*<sub>0</sub>. Let *T* be a rooted tree of size *m*<sub>0</sub> with a stem *σ*, we note *e*<sub>1</sub>, ..., *e*<sub>k</sub> the edges of *T* issued from *σ* which do not belong to *σ*. To each *e*-branch of *T* with *e* ∈ {*e*<sub>1</sub>, ..., *e*<sub>k</sub>} the edges of *T* issued from *σ* which do not belong to *σ*. To each *e*-branch of *T* with *e* ∈ {*e*<sub>1</sub>, ..., *e*<sub>k</sub>} corresponds by *M*<sub>B</sub> a *e*-branch (it is a path of same size) in *M*<sub>B</sub>(*T*,*σ*). So there exists *k* distinct *e*-branch with *e* = *e*<sub>1</sub>, ..., *e*<sub>k</sub> in *M*<sub>B</sub>(*T*,*σ*). By recurrence hypothesis, we have for  $1 \le i \le k, B_{e_i} \le M_T^{|E(B_{e_i})|}((X_n)_{n\in\mathbb{N}})$  and we have also  $M_T^{|E(B_{e_i})|}((X_n)_{n\in\mathbb{N}}) \le M_T^{|E(R_i)|}((X_n)_{n\in\mathbb{N}})$ . So each *e*-branch of *T* is a minor contraction of  $M_T^{|E(R_i)|}((X_n)_{n\in\mathbb{N}})$ . By associativity of contraction map, we have  $T \le M_T^{|E(T)|}((X_n)_{n\in\mathbb{N}})$ .

In this phase, we determine a sequence of rooted combs  $(X_i)_{i \in \mathbb{N}}$  such that the rooted combs  $X_i$  are strongly *i*-universal for the rooted brushes.

In order to achieve this result, we define  $F_p$  as the set of functions  $f : \{1, ..., p\} \rightarrow \{1, ..., \lfloor \frac{p}{2} \rfloor\}$  satisfying the following property :

$$(\forall n \in \{1, ..., p\}) \left(\forall i \le \left\lfloor \frac{n}{2} \right\rfloor\right) (\exists k \in \mathbb{N}) (n - i + 1 \le k \le n \text{ and } f(k) \ge i)$$

**Lemma 6.**  $F_p$  is not empty, it contains the following function  $\varphi_p$ , defined for  $1 \le i \le p$  by :

$$\varphi_p(i) = \min\left(2^{\upsilon_2(i)+1} - 1, \left\lfloor \frac{p}{2} \right\rfloor, i-1\right)$$

where  $v_2(k)$  is the 2-valuation of k (i.e. the greatest power of 2 dividing k).

*Proof.* The verification is obvious.

**Lemma 7.** For every sequence  $F = (f_1, f_2, ...)$  of functions such that  $f_i \in F_i$  for  $i \ge 1$  and  $f_i(k) \le f_{i+1}(k)$  for all  $i \ge 1$  and  $1 \le k \le i$ , the rooted comb defined by  $Comb_m^F = [Pf_1^m, ..., Pf_m^m]$  where  $Pf_i^m$  designs the path of size  $f_m(m+1-i)-1$ , for  $1 \le i \le m$  is strongly m-universal for the rooted brushes.

*Proof.* By induction on  $m : Comb_1^F$  is strongly 1-universal for the rooted brushes. Suppose that  $Comb_i^F$  has all rooted brushes with i - 1 edges as rooted contractions.

We consider two cases depending on the shape of a rooted brush B of size i : case 1 case 2



Brushes of case 1 are clearly rooted contractions of the rooted comb  $Comb_i^F$  $(B' \leq Comb_{i-1}^F)$ , so  $B \leq [P_0, Pf_1^{i-1}, ..., Pf_{i-1}^{i-1}] \leq Comb_i^F)$ . Let us study case 2: B' is by induction hypothesis a rooted contraction of the rooted comb  $Comb_{i-j}^F$ , moreover  $Comb_{i-j}^F \leq [Pf_{j+1}^i, ..., Pf_i^i]$ . Finally, by the property of  $f_i$ , there exists  $1 \leq \alpha \leq j$ , such that  $Pf_{\alpha}^i$  has more than j edges. Linking these two points, we can conclude that the rooted brush B is always a rooted contraction of the rooted comb  $Comb_i^F$ .

The rooted comb built as in lemma 7 will be said to be associated to the sequence F and denoted by  $Comb_m^F$ .

**Theorem 4.** A minimum strongly m-universal rooted brush for the rooted brushes has  $O(m \ln(m))$  edges.

*Proof.* Proceeding as for theorem 1, we obtain, mutatis mutandis, that a *m*-universal brush for the brushes has at least  $m \ln(m) + O(m)$  edges. This order of magnitude is precisely the size of the strongly *m*-universal rooted comb  $Comb_m^F$  for the class of rooted brushes.

We have this immediate corollary :

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**Corollary 2.** A minimum m-universal brush for the brushes has  $O(m \ln(m))$ edges.

By convention, we put  $Comb_0^F = P_0$  (tree reduced in a vertex)

We define  $Tree_m^F = M_T^m \left( \left( Comb_n^F \right)_{n \in \mathbb{N}} \right)$ . As before, we will say that the tree built in such a way is *recursively associated* to the sequence F and denoted by  $Tree_m^F$ .

Thus, we have :

**Theorem 5.** The rooted tree  $Tree_m^F$  is strongly m-universal for the class of rooted trees.

We now analyze the size of  $Tree_m^F$ .

**Proposition 1.** Let  $F = (f_1, f_2, ...)$  be a sequence of functions such that  $f_i \in F_i$ for  $i \geq 1$ . The size of a m-universal tree constructed from the sequence is given by the following recursive formula :

$$u_{-1} = -1, u_0 = 0$$
 and  $u_k = 2k - 1 + \sum_{i=1}^k u_{f_k(i)-1}$ 

*Proof.* It derives from the following observation :

m edges constitute the main stem, we have to add m-1 edges to link branches to the main stem and  $\sum_{i=1}^{k} u_{f_k(i)-1}$  edges for the branches. 

**Theorem 6.** There is a sequence of functions  $G = (g_1, g_2, ...)$  such that  $g_i \in F_i$ 

and  $|E(Tree_m^G)| < (2m)^c$  where c = 1.984... is the unique positive solution of the equation  $\frac{1}{2^c} + \frac{1}{2^{2c}} + \frac{1}{2^{(c-1)}-1} - \frac{1}{2^c-1} = 1.$ 

*Proof.* We take the following sequence of functions :

 $g_m(i) = \min\left(2^{v_2(i)+1}, i\right)$  if i < m and i even,  $g_m(i) = 1$  if i odd and  $g_m(m) = 1$  $\left\lfloor \frac{m}{4} \right\rfloor$ . It is clear that, if m is a power of 2, the comb  $Comb_m^G$  is strongly muniversal for the brushes.

In fact, the function  $g_m$  takes the value  $2^{v_2(i)+1}$  when *i* is not a power of 2, otherwise it is equal to *i*. Thanks to this remark and with  $u_m < m + \sum_{i=1}^m u_{f_m(i)}$ , (the sequence of sizes is increasing), we obtain  $u_{2^n} < 2^n + 2^{n-1} + \sum_{i=2}^{n-1} 2^{n-i} u_{2^i} -$ 

 $\sum_{i=2}^{n-1} u_{2^i} + u_{2^{n-1}} + u_{2^{n-2}}$ . Thus, in evaluating the sums and reorganizing the terms, we obtain :

$$u_{2^n} < \alpha_n + 2^{nc}\beta$$

with

$$\alpha_n = 2^{n-1} + 1 + 2^c + \frac{1}{2^c - 1} - \left(\frac{2^n}{2^{(c-1)} - 1} + 2^{n(c-1)}\right)$$

$$\beta = \frac{1}{2^c} + \frac{1}{2^{2c}} + \frac{1}{2^{(c-1)} - 1} - \frac{1}{2^c - 1}$$

Now  $\alpha_n < 0$  when m > 1 and  $\beta \le 1$  by definition of c. So  $u_{2^n} < 2^{nc}$ , hence  $u_m < (2m)^c$ .

Remark 2. We observe that  $c = \frac{\ln(x)}{\ln(2)}$ , where x is the positive root of  $X^4 - 5X^3 + 4X^2 + X - 2 = 0$ .

Theorem 2 then follows since any rooted tree which is strongly m-universal for the rooted trees is also clearly m-universal for the class of trees.

#### 6 Conclusion and Related Questions

When using the sequence  $\Phi = (\varphi_1, \varphi_2, ...)$  of lemma 7, the induction step leads to involved expressions that do not allow us to find the asymptotic behavior of the corresponding term  $u_m$ . A computer simulation gives that such a *m*-universal tree for the trees has less than  $m^{1.88}$  edges. In any case, the constructive approach we proposed here, seems to be hopeless to reach the asymptotic best size of a *m*-universal tree for the trees.

Conjecture 1. The minimal size of a *m*-universal tree for the trees is  $m^{1+o(1)}$ .

As a possible way to prove such a conjecture, it would be interesting to obtain an explicit effective coding of a tree of size m using a list of contracted edges taken in a m-universal tree for the trees.

A variant of our problem consists in determining a minimum tree which contains as a subtree every tree of size m. This is closely related to a well known still open conjecture due to Erdös and Sös (see [5]).

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