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► **To cite this version:**

Alain Pham Ngoc Dinh, Dang Duc Trong, Pham Hoang Quan, Nguyen Huy Tuan. A new stability results for the backward heat equation. 2009. hal-00431878

**HAL Id: hal-00431878**

**<https://hal.science/hal-00431878v1>**

Preprint submitted on 13 Nov 2009

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# A new stability results for the backward heat equation

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November 13, 2009

## Abstract

In this paper, we regularize the nonlinear inverse time heat problem in the unbounded region by Fourier method. Some new convergence rates are obtained. Meanwhile, some quite sharp error estimates between the approximate solution and exact solution are provided. Especially, the optimal convergence of the approximate solution at  $t = 0$  is also proved. This work extends to many earlier results in [7, 8, 10, 15, 17, 18, 21, 24].

**Keywords and phrases:** Backward heat problem, Ill-posed problem, Fourier Transform, Contraction principle.

**Mathematics subject Classification 2000:** 35K05, 35K99, 47J06, 47H10.

# 1 Introduction

Transient heat conduction phenomena are generally governed by the parabolic heat conduction equation and if the initial temperature distribution and boundary conditions are specified, then this, in general, leads to a well posed problem which may easily be solved numerically by using various methods.

However, in many practical situations when dealing with a heat conducting body it is not always possible to specify the boundary conditions or the initial temperature. For example, in practice, one may have to investigate the temperature distribution and the heat flux history from the known data at a particular time. In other words, it may be possible to specify the temperature distribution at a particular time, say  $t = T > 0$ , and from this data the question arises as to whether the temperature distribution at any earlier time  $t < T$  can be retrieved. This is usually referred to as the backward heat conduction problem (BHCP), or the final boundary value problem. The systematic study of the backward heat conduction problem is of rather recent origin, although isolated considerations have been given to such problems for several hundred years. In general no solution which satisfies the heat conduction equation, the final data and the boundary conditions exists. Further, even if a solution did exist, it would not be continuously dependent on the boundary and the final data, see Payne [2]. Thus the BHCP is an example of an ill-posed problem that is impossible to solve using classical numerical methods and requires special techniques to be employed, see Hadamard [3]. Conditions for which the BHCP becomes well-posed have been investigated by Miranker [4] and Cannon and Douglas [5]. These studies introduced additional hypotheses which restrict the class of functions to which the solution must belong, and which are seldom satisfied. Therefore numerical methods of solution appear more useful. Thus regularization techniques, see for example Cannon [6] and Han et al. [7], have been developed for solving the BHCP. Different methods, based on a perturbation of the original parabolic heat equation were proposed by Lattes and Lions [8] and Lesnic et al. [9]. These methods consist of replacing the operator by a perturbed higher order one that holds better invertibility features.

In this paper, we consider the following problem

$$\begin{cases} u_t - u_{xx} = 0, & (x, t) \in R \times (0, T), \\ u(x, T) = \varphi(x), & x \in R, \end{cases} \quad (1)$$

where  $T > 0$  be a given and  $\varphi(x)$  are given. We want to retrieve the temperature dis-

tribution  $u(x, t)$  for  $0 \leq t < T$ . Of course, since the data  $\varphi(\cdot)$  is based on (physical) observations, there will be measurement errors, and we would actually have as data some function  $\varphi_\epsilon \in L^2(R)$ , for which  $\|\varphi - \varphi_\epsilon\| \leq \epsilon$ , where  $\|\cdot\|$  denotes the  $L^2$ -norm, the constant  $\epsilon > 0$  represents a bound on the measurement error. That is to say, practically, we need to consider the following problem,

$$\begin{cases} u_t - u_{xx} = 0, & (x, t) \in R \times (0, T), \\ u(x, T) = \varphi_\epsilon(x), & x \in R, \end{cases} \quad (2)$$

Notice the reader that the problem (1) is investigated in some recent papers of ChuLiFu [7, 8, 24] and of other authors such as Lien[21], Murniz [14], et al . To the authors knowledge, so far there are many papers on the backward heat equation, but theoretically the error estimates of most regularization methods in the literature are Holder type, i.e.,

$$\|u(\cdot, t) - v^\epsilon(\cdot, t)\| \leq C\epsilon^k, \quad k > 0. \quad (3)$$

where  $C$  is the constant depend on  $u$ ,  $k$  is a constant is not depend on  $t, u$  . As we know,  $\epsilon^k$  converges to zero more quickly than the logarithmic term. So, the major object of this paper is to provide new regularization method to established the Holder estimates such as (3). We give a new approximation problem and investigate the error estimate between the regularization solution and the exact one.

The remainder of the paper is divided into three sections. In Section 2, we establish the approximated problem and show that it is well posed. Then, we also estimate the error between an exact solution  $u$  of Problem (1) and the approximation solution  $v^\epsilon(\cdot, t)$  with the Holder type. Finally, a numerical experiment will be given in Section 3.

## 2 Regularization and error estimates.

Let

$$\hat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(x) e^{-i\xi x} dx$$

be the Fourier transform of the function  $\varphi \in L^2(R)$ . We assume  $u(x, t)$  is the unique solution of (1). Using the Fourier transform technique to problem (1) with respect to the variable  $x$ , we can get the Fourier transform  $\hat{u}(\xi, t)$  of the exact solution  $u(x, t)$  of problem (1)

$$\begin{cases} \hat{u}_t(\xi, t) = (i\xi)^2 \hat{u}(\xi, t), \\ \hat{u}(\xi, T) = \hat{\varphi}(\xi), \quad \xi \in R, \end{cases} \quad (4)$$

The solution to equation (4) is given by

$$\hat{u}(\xi, t) = e^{(T-t)\xi^2} \hat{\varphi}(\xi). \quad (5)$$

or equivalently,

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{(T-t)\xi^2} \hat{\varphi}(\xi) e^{i\xi x} d\xi. \quad (6)$$

Since  $t < T$ , we know from (6) that, when  $\xi$  becomes large,  $\exp(T-t)\xi^2$  increases rather quickly. Thus for  $\hat{u}(\xi, t) \in L^2(R)$  with respect to  $\xi$ , the exact data function  $\hat{\varphi}(\xi)$  must decay rapidly as  $|\xi| \rightarrow \infty$ . Small errors in high-frequency components can blow up and completely destroy the solution for  $0 \leq t < T$ . As for the measured data  $\varphi_\epsilon(x)$ , its Fourier transform  $\hat{\varphi}_\epsilon(\xi)$  is merely in  $L^2(R)$ .

Let  $p > 1$  be a constant number. We approximated problem (3) by perturbing the Fourier transform of final value  $\varphi$  as follows

$$\begin{cases} \frac{\partial \hat{w}_\epsilon(\xi, t)}{\partial t} = (i\xi)^2 \hat{w}_\epsilon(\xi, t), \\ \hat{w}_\epsilon(\xi, T) = \frac{e^{-T\xi^2}}{\beta e^{(p-1)T\xi^2} + e^{-T\xi^2}} \hat{\varphi}_\epsilon(\xi), \quad \xi \in R, \end{cases} \quad (7)$$

The formal solution of (7) is also easily seen to be

$$\hat{w}_\epsilon(\xi, t) = \frac{e^{-t\xi^2}}{\beta e^{(p-1)T\xi^2} + e^{-T\xi^2}} \hat{\varphi}_\epsilon(\xi), \quad (8)$$

or

$$w_\epsilon(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-t\xi^2}}{\beta e^{(p-1)T\xi^2} + e^{-T\xi^2}} \hat{\varphi}_\epsilon(\xi) e^{i\xi x} d\xi, \quad (9)$$

where  $\beta$  is a positive number such that  $\beta > 0$ .

Let  $v_\epsilon(\cdot, t)$  be the approximated solution given by

$$v_\epsilon(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-t\xi^2}}{\beta e^{(p-1)T\xi^2} + e^{-T\xi^2}} \hat{\varphi}_\epsilon(\xi) e^{i\xi x} d\xi, \quad (10)$$

Note that if  $\beta$  is chosen small, then for small  $|\xi|$ ,  $\frac{e^{-t\xi^2}}{\beta e^{(p-1)T\xi^2} + e^{-T\xi^2}}$  in (8) is close to  $e^{(T-t)\xi^2}$  in (5).

For  $\xi, x, \beta > 0, 0 \leq a \leq b$ , we prove the following inequality

$$i). \frac{e^{a\xi^2}}{1 + \beta e^{b\xi^2}} \leq \beta^{-\frac{a}{b}}. \quad (11)$$

$$ii). \frac{e^{a\xi^2}}{\xi^2(1 + \beta e^{b\xi^2})} \leq \frac{b}{\ln(\frac{1}{\beta})} \beta^{-\frac{a}{b}} \quad (12)$$

Thus, we have

$$\begin{aligned} \frac{e^{a\xi^2}}{1 + \beta e^{b\xi^2}} &= \frac{e^{a\xi^2}}{(1 + \beta e^{b\xi^2})^{\frac{a}{b}} (1 + \beta e^{b\xi^2})^{1-\frac{a}{b}}} \\ &\leq \frac{e^{a\xi^2}}{(1 + \beta e^{b\xi^2})^{\frac{a}{b}}} \\ &\leq \beta^{-\frac{a}{b}}. \end{aligned}$$

For  $0 \leq t \leq s \leq T$ , denote

$$\begin{aligned} A(\xi, t) &= \frac{\exp\{-t\xi^2\}}{\beta \exp\{(p-1)T\xi^2\} + \exp\{-T\xi^2\}}, \\ B(\xi, s, t) &= \frac{\exp\{(s-t)\xi^2\}}{\beta \exp\{(p-1)T\xi^2\} + \exp\{-T\xi^2\}}. \end{aligned}$$

Using the inequality (11), we obtain

$$A(\xi, t) = \frac{e^{(T-t)\xi^2}}{1 + \beta e^{pT\xi^2}} \leq \beta^{\frac{t-T}{pT}} \quad (13)$$

$$B(\xi, s, t) = \frac{e^{(s-t)\xi^2}}{1 + \beta e^{pT\xi^2}} \leq \beta^{\frac{t-s}{pT}}. \quad (14)$$

**Theorem 1.** *Let  $\varphi \in L^2(R)$ . Then unique solution of the problem (6) depends on the final value  $\varphi$ , i.e., if  $w, v$  are the solution of the problem (6) corresponding to the final values  $\varphi$  and  $\omega$ , then*

$$\|w(\cdot, t) - v(\cdot, t)\| \leq \beta^{\frac{t-T}{pT}} \|\varphi - \omega\|.$$

**Proof of Theorem 1.**

Let  $w$  and  $v$  be two solution of the problem (6) corresponding to the final values  $\varphi$  and  $\omega$ .

Using Parseval inequality, we have

$$\begin{aligned}
\|w(., t) - v(., t)\|^2 &= \|\hat{w}(., t) - \hat{v}(., t)\|^2 \\
&\leq \int_{-\infty}^{+\infty} \left| A(\xi, t) \left( \hat{\varphi}(\xi) - \hat{\phi}(\xi) \right) \right|^2 d\xi \\
&\leq \beta^{\frac{2t-2T}{pT}} \int_{-\infty}^{+\infty} \left| \left( \hat{\varphi}(\xi) - \hat{\phi}(\xi) \right) \right|^2 d\xi \\
&\leq \beta^{\frac{2t-2T}{pT}} \|\hat{\varphi} - \hat{\phi}\|^2 \\
&\leq \beta^{\frac{2t-2T}{pT}} \|\varphi - \phi\|^2.
\end{aligned}$$

Hence

$$\|w(., t) - v(., t)\| \leq \beta^{\frac{t-T}{pT}} \|\varphi - \phi\|.$$

**Remark 1.**

In [4, 19], the stability of magnitude is  $e^{\frac{T}{\epsilon}}$ . In [7, 15, 20], the stability estimate is of order  $\epsilon^{\frac{t}{T}-1}$ .

In our paper, we give a better estimation of the stability order, which is  $C\beta^{\frac{t}{T}-1} \left( \frac{T}{1+\ln(\frac{T}{\beta})} \right)^{1-\frac{t}{T}}$ .

It is easy to see that the order of the error, introduced by small changes in the final value  $g$ , is less than the order given in [15, 20]. This is among of the advantages of our method.

**Theorem 2.** *Let  $w_\epsilon$  and  $v_\epsilon$  defined by (9) and (10). Then one has*

$$\|w_\epsilon(., t) - v_\epsilon(., t)\| \leq \beta^{\frac{t-T}{pT}} \epsilon. \quad (15)$$

**Proof of Theorem 2.** Using Theorem 1, we get

$$\|w_\epsilon(., t) - v_\epsilon(., t)\| \leq \beta^{\frac{t-T}{pT}} \|\varphi_\epsilon - \varphi\| \leq \beta^{\frac{t-T}{pT}} \epsilon. \quad (16)$$

**Theorem 3.** *Let  $u$  be the exact solution of problem such that  $\|u(., 0)\| \leq E_1$ . Then one has*

$$\|u(., t) - v_\epsilon(., t)\| \leq \epsilon^{\frac{t}{T}} (E_1 + 1).$$

**Proof of Theorem 2.** Using (11), we have

$$\begin{aligned}
\|u(\cdot, t) - w_\epsilon(\cdot, t)\|^2 &= \int_{-\infty}^{+\infty} |\hat{u}(\xi, t) - \hat{w}_\beta(\xi, t)|^2 d\xi \\
&= \int_{-\infty}^{+\infty} \left| (e^{(T-t)\xi^2} - A(\xi, t)) \hat{\varphi}(\xi) \right|^2 d\xi \\
&= \int_{-\infty}^{+\infty} \left| \frac{\beta e^{pT\xi^2} e^{(T-t)\xi^2}}{(1 + \beta e^{pT\xi^2})} \hat{\varphi}(\xi) \right|^2 d\xi \\
&= \int_{-\infty}^{+\infty} \left| \frac{\beta e^{(pT-t)\xi^2}}{(1 + \beta e^{pT\xi^2})} \hat{u}(\xi, 0) \right|^2 d\xi \\
&\leq \beta^2 \beta^{\frac{2t-2T}{pT}} \int_{-\infty}^{+\infty} |\hat{u}(\xi, 0)|^2 d\xi \\
&\leq \beta^{\frac{2t}{pT}} E_1^2.
\end{aligned}$$

Therefore

$$\|u(\cdot, t) - w_\epsilon(\cdot, t)\| \leq \beta^{\frac{t}{pT}} E_1.$$

From  $\beta = \epsilon^p$  and using the inequality, we obtain

$$\begin{aligned}
\|u(\cdot, t) - v_\epsilon(\cdot, t)\| &\leq \|u(\cdot, t) - w_\epsilon(\cdot, t)\| + \|w_\epsilon(\cdot, t) - v_\epsilon(\cdot, t)\| \\
&\leq \beta^{\frac{t}{pT}} E_1 + \beta^{\frac{t-T}{pT}} \epsilon \\
&\leq \epsilon^{\frac{t}{T}} (E_1 + 1).
\end{aligned}$$

**Remark 2.** Notice that the convergence estimate in Theorem 1 does not give any useful information on the continuous dependence of the solution at  $t = 0$ . This is common in the theory of ill-posed problems, if we do not have additional conditions on the smoothness of the solution. To retain the continuous dependence of the solution at  $t = 0$ , instead of (2.5), one has to introduce a stronger a priori assumption.

We denote  $\|\cdot\|_k$  be the norm in Sobolev space  $H^k(R)$ ,  $k > 0$  defined by

$$\|u(\cdot, 0)\|_k := \left( \int_{-\infty}^{+\infty} (1 + \xi^2)^k |\hat{u}(\xi, 0)|^2 d\xi \right)^{\frac{1}{2}}.$$



**Theorem 3.** Let  $u$  be the exact solution of problem such that  $\|u(\cdot, 0)\|_2 \leq E_2$ . Let  $\beta = \epsilon$ , then one has

$$\|u(\cdot, t) - w_\epsilon(\cdot, t)\| \leq \frac{pT}{\ln(\frac{1}{\epsilon})} \epsilon^{\frac{t}{pT}} E_2 + \epsilon^{\frac{t-T+pT}{pT}}. \quad (17)$$

**Proof of Theorem 3.**

$$\begin{aligned} \|u(\cdot, t) - w_\epsilon(\cdot, t)\|^2 &= \int_{-\infty}^{+\infty} \left| \frac{\beta e^{(pT-t)\xi^2}}{\xi^2(1 + \beta e^{pT\xi^2})} \xi^2 \hat{u}(\xi, 0) \right|^2 d\xi \\ &\leq \int_{-\infty}^{+\infty} \xi^4 \hat{u}^2(\xi, 0) d\xi \\ &\leq \left( \frac{pT\beta}{\ln(\frac{1}{\beta})} \beta^{\frac{t-pT}{pT}} \right)^2 \int_{-\infty}^{+\infty} \xi^4 \hat{u}^2(\xi, 0) d\xi \\ &\leq \left( \frac{pT\beta}{\ln(\frac{1}{\beta})} \beta^{\frac{t-pT}{pT}} \right)^2 \|u(\cdot, 0)\|_2^2 \\ &\leq \frac{pT}{\ln(\frac{1}{\beta})} \beta^{\frac{t}{pT}} E_2. \end{aligned}$$

From  $\beta = \epsilon$  and using the inequality, we obtain

$$\begin{aligned} \|u(\cdot, t) - v_\epsilon(\cdot, t)\| &\leq \|u(\cdot, t) - w_\epsilon(\cdot, t)\| + \|w_\epsilon(\cdot, t) - v_\epsilon(\cdot, t)\| \\ &\leq \frac{pT}{\ln(\frac{1}{\epsilon})} \epsilon^{\frac{t}{pT}} E_2 + \epsilon^{\frac{t-T+pT}{pT}}. \end{aligned}$$

**Remark 3.**

1. In  $t = 0$ , the error (17) becomes

$$\|u(\cdot, 0) - v_\epsilon(\cdot, 0)\| \leq \frac{pT}{\ln(\frac{1}{\epsilon})} E_2 + \epsilon^{\frac{-T+pT}{pT}}. \quad (18)$$

It follows from  $p > 1$  that the right hand side of (18) converges to zero when  $\epsilon \rightarrow 0$ . This error is the same order as Theorem 3.1 in paper [8] (see page 566) and Theorem 2.1 in paper [9].

2. Since (17), the first term of the right hand side of (17) is the logarithmic form, and the second term is a power, so the order of (17) is also logarithmic order. It is the same order as some results which is mentioned in Remark 2. This often occurs in the boundary

error estimate for ill-posed problems. To retain the Holder order in  $[0, T]$ , we introduce a different priori assumption.

**Theorem 4.**

Assume that there exist a positive number  $\gamma \in (0, pT)$  such that

$$\int_{-\infty}^{+\infty} e^{2\gamma\xi^2} \hat{u}(\xi, 0) d\xi < E_3^2.$$

Let  $\beta = \epsilon$  and  $h = \min\{\gamma, (p-1)T\}$ . Then, one has

$$\|u(\cdot, t) - w_\epsilon(\cdot, t)\| \leq \epsilon^{\frac{t+h}{pT}} (E_3 + 1). \quad (19)$$

**Proof of Theorem 4.**

From , we get

$$\begin{aligned} \|u(\cdot, t) - w_\epsilon(\cdot, t)\|^2 &= \int_{-\infty}^{+\infty} |\hat{u}(\xi, t) - \hat{w}_\beta(\xi, t)|^2 d\xi \\ &= \int_{-\infty}^{+\infty} \left| \frac{\beta e^{pT\xi^2} e^{(T-t)\xi^2}}{(1 + \beta e^{pT\xi^2})} \hat{\varphi}(\xi) \right. \\ &= \int_{-\infty}^{+\infty} \left| \frac{\beta e^{pT\xi^2}}{1 + \beta e^{pT\xi^2}} \hat{u}(\xi, t) \right|^2 d\xi \\ &= \int_{-\infty}^{+\infty} \left| \frac{\beta e^{(pT-t-\gamma)\xi^2}}{1 + \beta e^{pT\xi^2}} e^{\gamma\xi^2} \hat{u}(\xi, 0) \right|^2 d\xi. \end{aligned} \quad (20)$$

Using the inequality (11), we have

$$\frac{\beta e^{(pT-t-\gamma)\xi^2}}{1 + \beta e^{pT\xi^2}} \leq \beta \beta^{\frac{t+\gamma-pT}{pT}}. \quad (21)$$

Combining (20) and (21), we get

$$\|u(\cdot, t) - w_\epsilon(\cdot, t)\|^2 \leq \beta^{\frac{2t+2\gamma}{pT}} \int_{-\infty}^{+\infty} e^{2\gamma\xi^2} \hat{u}(\xi, 0) d\xi \leq \beta^{\frac{2t+2\gamma}{pT}} E_3^2.$$

From  $\beta = \epsilon$  and using the inequality, we obtain

$$\begin{aligned} \|u(\cdot, t) - v_\epsilon(\cdot, t)\| &\leq \|u(\cdot, t) - w_\epsilon(\cdot, t)\| + \|w_\epsilon(\cdot, t) - v_\epsilon(\cdot, t)\| \\ &\leq \beta^{\frac{t+\gamma}{pT}} E_3 + \beta^{\frac{t-T}{pT}} \epsilon \\ &\leq \epsilon^{\frac{t+h}{pT}} (E_3 + 1). \end{aligned}$$

**Remark 4.**

1. In  $t = 0$ , the error (19) becomes

$$\|u(\cdot, t) - w_\epsilon(\cdot, t)\| \leq \epsilon^{\frac{h}{pT}} (E_3 + 1). \quad (22)$$

2. Suppose that  $E_\epsilon = \|v^\epsilon - u\|$  be the error of the exact solution and the approximate solution. In most of results concerning the backward heat, then optimal error between is of the logarithmic form. It means that

$$E_\epsilon \leq C \left( \ln \frac{T}{\epsilon} \right)^{-m}$$

where  $m > 0$ . The error order of logarithmic form is investigated in many recent papers, such as [4, 8, 5, 19, 20, 22].

To illustrate this, we can enumerate some more recent papers considering the errors of logarithmic order.

Let  $u$  and  $v^\epsilon$  be exact solution and the approximated solution respectively.

In [20, 22], the error is given the form

$$\|v^\epsilon(\cdot, t) - u(\cdot, t)\| \leq \frac{C_1}{1 + \ln \frac{T}{\epsilon}}.$$

In recently, Feng Xiao-Li and coauthors [6] gave the error estimates as follows

$$\|v^{\alpha, \delta}(\cdot) - u(\cdot)\| \leq \frac{\delta}{2\sqrt{\alpha}} + \max\left\{\left(\frac{4T}{\ln \frac{1}{\alpha}}\right)^{\frac{p}{2}}, \alpha^{\frac{1}{2}}\right\}.$$

From the discussed error, we can see that most of recent regularization methods established the logarithmic stability. The convergence rates in here is very slowly.

In Tautenhahn and Schroter [18], the authors proved that the best possible worst case error for identifying  $u(t)$  is given by  $E_\epsilon = D\epsilon^{\frac{1}{T}}$ .

In 2007, Zhi Qian et al. [8] (See Remark 3.6, p.570) gave the error estimation in  $t = 0$  for (2) as follows

$$\|u(\cdot, 0) - v(\cdot, 0)\| \leq \frac{1}{(\ln(1/\epsilon))^{\frac{8}{5}}} + \max\{1, T\} \beta^{\frac{1}{5}} E.$$

where  $\beta = \frac{T}{\ln\left(\frac{1}{\epsilon} \left(\ln \frac{1}{\epsilon}\right)^{-\frac{8}{5}}\right)}$ .

In [9], ChuLiFu and his coauthors established the logarithmic order of the form

$$\begin{aligned} & \|u(\cdot, t) - u_{\delta, \xi_{\max}}\| \\ & \leq E^{1-\frac{t}{T}} \left( \ln \frac{E}{\delta} \right)^{\frac{(t-T)s}{2T}} \left( 1 + \left( \frac{\ln \frac{E}{\delta}}{\frac{1}{T} \ln \frac{E}{\delta} + \ln(\ln \frac{E}{\delta})^{\frac{-s}{2T}}} \right)^{\frac{s}{2}} \right). \end{aligned}$$

Comparing (22) with the discussed error, we can see (22) is the optimal error.

### 3 Numerical results.

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