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ON AFFINE INTEREST RATE MODELS

PAUL LESCOT

ABSTRACT. Bernstein processes are Brownian diffusions that appear in Euclidean Quantum Mechanics. The consideration of the symmetries of the associated Hamilton-Jacobi-Bellman equation allows one to obtain various relations between stochastic processes (Lescot-Zambrini, **Progress in Probability**, vols 58 and 59). More recently it has appeared that each *one-factor affine interest rate model* (in the sense of Leblanc-Scaillet) could be described using such a Bernstein process.

MSC 91G30, 60H10

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1. AN ISOVECTOR CALCULATION

We shall work within the context of [7], §3. We consider the Hamilton–Jacobi–Bellman equation

$$(\mathcal{HJB}^V) \quad \frac{\partial S}{\partial t} + \frac{\theta^2}{2} \frac{\partial^2 S}{\partial q^2} - \frac{1}{2} \left(\frac{\partial S}{\partial q} \right)^2 + V = 0$$

with potential

$$V(t, q) = \frac{C}{q^2} + Dq^2 ;$$

where we have replaced (as explained in [4]) $\sqrt{\hbar}$ by θ .

In order to determine the Lie algebra \mathcal{H}_V of *pure isovectors* for (\mathcal{HJB}^V) , *i.e.* the algebra \mathfrak{N} of [7], we have to solve the auxiliary equation (3.29) from [7], p. 215, *i.e.* :

$$-\frac{1}{4} \ddot{T}_N q^2 - \ddot{l} q + \dot{\sigma} + \frac{1}{2} \dot{T}_N q \left(-\frac{C}{q^3} + 2Dq \right) + l \left(-\frac{2C}{q^3} + 2Dq \right) + \dot{T}_N \left(\frac{C}{q^2} + Dq^2 \right) - \frac{\theta^2}{4} \ddot{T}_N = 0 ,$$

that is :

$$q^2 (2D\dot{T}_N - \frac{1}{4} \ddot{T}_N) + q(-\ddot{l} + 2Dl) + \dot{\sigma} - \frac{\theta^2}{4} \ddot{T}_N - \frac{2Cl}{q^3} = 0 .$$

As T_N , l and σ depend only upon t , the system is equivalent to :

$$\begin{cases} 2Cl = 0 \\ \ddot{l} = 2Dl \\ \dot{\sigma} = \frac{\theta^2}{4} \ddot{T}_N \\ \ddot{T}_N = 8D\dot{T}_N . \end{cases}$$

Two different cases now appear :

1) $C \neq 0$

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Then one must have $l = 0$, in which case the second condition holds automatically, and the system reduces itself to :

$$\begin{cases} \dot{\sigma} = \frac{\theta^2}{4} \ddot{T}_N \\ \ddot{T}_N = 8D\dot{T}_N \end{cases}$$

1)a) $D > 0$

Setting $\epsilon = \sqrt{8D}$, we find

$$\dot{T}_N = C_1 e^{\epsilon t} + C_2 e^{-\epsilon t} ,$$

whence

$$T_N = \frac{C_1}{\epsilon} e^{\epsilon t} - \frac{C_2}{\epsilon} e^{-\epsilon t} + C_3$$

and

$$\sigma = \frac{\theta^2}{4} \dot{T}_N + C_4$$

therefore :

$$\sigma = \frac{\theta^2 C_1}{4} e^{\epsilon t} + \frac{\theta^2 C_2}{4} e^{-\epsilon t} + C_4$$

where $(C_j)_{1 \leq j \leq 4}$ denote arbitrary (real) constants. In particular

$$\dim(\mathcal{H}_V) = 4 .$$

1)b) $D = 0$

Then from $\ddot{T}_N = 0$ follows

$$T_N = C_1 t^2 + C_2 t + C_3$$

for constants C_1, C_2, C_3 . Then

$$\dot{\sigma} = \frac{\theta^2}{4} \ddot{T}_N = \frac{\theta^2 C_1}{2}$$

and

$$\sigma = \frac{\theta^2 C_1}{2} t + C_4 .$$

Therefore, here too, $\dim(\mathcal{H}_V) = 4$; furthermore, we get an explicit expression for the isovectors :

$$N^t = T_N = C_1 t^2 + C_2 t + C_3 ,$$

$$\begin{aligned} N^q &= \frac{1}{2} q \dot{T}_N + l \\ &= \frac{1}{2} q (2C_1 t + C_2) \\ &= C_1 t q + \frac{C_2 q}{2} , \end{aligned}$$

and

$$\begin{aligned}
N^S &= -\phi \\
&= -\frac{1}{4}q^2 T_N^{\ddot{}} - q\dot{l} + \sigma \\
&= -\frac{1}{4}q^2 \cdot 2C_1 + \frac{\theta^2}{2}C_1 t + C_4 \\
&= \frac{C_1}{2}(\theta^2 t - q^2) + C_4 .
\end{aligned}$$

A canonical basis for \mathcal{H}_V is thus given by $(M_i)_{1 \leq i \leq 4}$, where M_i is characterized by $C_j = \delta_{ij}$ (Kronecker's symbol). Using the notation of [6], it appears that

$$M_1 = \frac{1}{2}N_6 ,$$

$$M_2 = \frac{1}{2}N_4 ,$$

$$M_3 = N_1 ,$$

and

$$M_4 = -\frac{1}{\theta^2}N_3 ,$$

therefore \mathcal{H}_V is generated by N_1, N_3, N_4 and N_6 . We thereby recover the result of [7], p. 220, modulo the correction of a misprint. This list ties in nicely with the symmetry properties of certain diffusions related to Bessel processes (see [5] for a detailed explanation).

1)c) $D < 0$

Setting now $\epsilon = \sqrt{-8D}$, we find

$$\dot{T}_N = C_1 \cos(\epsilon t) + C_2 \sin(\epsilon t) ,$$

whence

$$T_N = \frac{C_1}{\epsilon} \sin(\epsilon t) - \frac{C_2}{\epsilon} \cos(\epsilon t) + C_3 ,$$

and, as above :

$$\sigma = \frac{\theta^2}{4} \dot{T}_N + C_4 ,$$

therefore :

$$\sigma = \frac{\theta^2 C_1}{4} \cos(\epsilon t) + \frac{\theta^2 C_2}{4} \sin(\epsilon t) + C_4$$

where $(C_j)_{1 \leq j \leq 4}$ denote arbitrary (real) constants. In particular ,

$$\dim(\mathcal{H}_V) = 4 .$$

2) $C = 0$

Then the system becomes

$$\begin{cases} \ddot{l} = 2Dl \\ \dot{\sigma} = \frac{\theta^2}{4} \ddot{T}_N \\ \ddot{T}_N = 8D\dot{T}_N . \end{cases}$$

The equation for l on the one hand, and the system for (σ, T_N) on the other hand, are independent, and, as above, the first one has a two-dimensional space of solutions and the second one a four-dimensional space of solutions, *i.e.*

$$\dim(\mathcal{H}_V) = 6 .$$

Whence

Theorem 1.1. *The isovector algebra \mathcal{H}_V associated with V has dimension 6 if and only if $C = 0$; in the opposite case, it has dimension 4.*

2. PARAMETRIZATION OF A ONE-FACTOR AFFINE MODEL

As general references we shall use, concerning Bernstein processes, our recent survey ([4]), and, concerning affine models, Hénon's PhD thesis([2]) as well as Leblanc and Scaillet's seminal paper ([3]). An *one-factor affine interest rate model* is characterized by the instantaneous rate $r(t)$, satisfying the following stochastic differential equation :

$$dr(t) = \sqrt{\alpha r(t) + \beta} dw(t) + (\phi - \lambda r(t)) dt (*)$$

under the risk-neutral probability Q ($\alpha = 0$ corresponds to the so-called Vasicek model, and $\beta = 0$ corresponds to the Cox-Ingersoll-Ross model ; cf. [3]).

Assuming $\alpha > 0$, let us set

$$\tilde{\phi} =_{def} \phi + \frac{\lambda\beta}{\alpha} ,$$

$$\delta =_{def} \frac{4\tilde{\phi}}{\alpha} ,$$

and let us also assume $\lambda > 0$ and $\tilde{\phi} \geq 0$.

The following two quantities will play an important role :

$$\begin{aligned} C &:= \frac{\alpha^2}{8} \left(\tilde{\phi} - \frac{\alpha}{4} \right) \left(\tilde{\phi} - \frac{3\alpha}{4} \right) \\ &= \frac{\alpha^4}{128} (\delta - 1)(\delta - 3) \end{aligned}$$

and

$$D := \frac{\lambda^2}{8} .$$

Let us set $X_t = \alpha r(t) + \beta$.

Proposition 2.1. *Let $r_0 \in \mathbf{R}$; then the stochastic differential equation*

$$dr(t) = \sqrt{|\alpha r(t) + \beta|} dw(t) + (\phi - \lambda r(t)) dt (**)$$

has a unique strong solution such that $r(0) = r_0$. Furthermore, in case that

$$\alpha r_0 + \beta \geq 0 ,$$

one has $\alpha r(t) + \beta \geq 0$ for all $t \geq 0$; in particular, $r(t)$ satisfies ().*

Proof. Let us set $X_t = \alpha r(t) + \beta$; it is easy to see that, in terms of X_t , equation (**) becomes:

$$\begin{aligned} dX_t &= \alpha dr(t) \\ &= \alpha(\sqrt{|X_t|}dw(t) + (\phi - \lambda \frac{X_t - \beta}{\alpha})dt) \\ &= \alpha\sqrt{|X_t|}dw(t) + (\alpha\tilde{\phi} - \lambda X_t)dt . \end{aligned}$$

We are therefore in the situation of [2],p.55, Proposition 12.1, with $\sigma = \alpha$, $\kappa = \lambda$ and $a = \frac{\alpha\tilde{\phi}}{\lambda}$; the result follows. \square

We shall henceforth assume all of the hypotheses of Proposition 2.1 to be satisfied.

Theorem 2.2. *If $\delta \geq 2$ one has, almost surely :*

$$\forall t > 0 \quad X_t > 0 ;$$

on the other hand, if $\delta < 2$, almost surely there is a $t > 0$ such that $X_t = 0$.

Proof. We apply (12.2) in [2], p.56, yielding that

$$\forall t > 0 \quad X_t > 0$$

is equivalent to $\frac{2a\kappa}{\sigma^2} \geq 1$; but, according to the above identifications,

$$\frac{2a\kappa}{\sigma^2} = \frac{2\frac{\alpha\tilde{\phi}}{\lambda}\lambda}{\alpha^2} = \frac{2\tilde{\phi}}{\alpha} = \frac{\delta}{2} .$$

\square

Our main result is the following :

Theorem 2.3. *Let us define the process*

$$z(t) = \sqrt{X_t}$$

and the stopping time

$$T = \inf\{t > 0 | X_t = 0\};$$

as seen in Theorem 2.2, $T = +\infty$ a.s. for $\delta \geq 2$, and $T < +\infty$ a.s. for $\delta < 2$.

Then there exists a Bernstein process $y(t)$ for

$$\theta = \frac{\alpha}{2}$$

and the potential

$$V(t, q) = \frac{C}{q^2} + Dq^2 .$$

such that

$$\forall t \in [0, T[\quad z(t) = y(t) .$$

In particular, for $\delta \geq 2$, z itself is a Bernstein process.

Proof. One has (cf. Proposition 2.1 and its proof)

$$\begin{aligned} dX_t &= \alpha \sqrt{X_t} dw(t) + (\alpha \tilde{\phi} - \lambda X_t) dt \\ &= \alpha z(t) dw(t) + (\alpha \tilde{\phi} - \lambda z(t)^2) dt . \end{aligned}$$

Taking now $f(x) = \sqrt{x}$, we have

$$\forall x > 0 \quad f'(x) = \frac{1}{2\sqrt{x}} \quad \text{and} \quad f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} ,$$

therefore, for all $t \in]0, T[$, $f'(X_t) = \frac{1}{2z(t)}$ and $f''(X_t) = -\frac{1}{4}z(t)^{-3}$. The application of Itô's formula now gives :

$$\begin{aligned} dz(t) &= d(f(X_t)) \\ &= f'(X_t) dX_t + \frac{1}{2} f''(X_t) (dX_t)^2 \\ &= \frac{1}{2z(t)} (\alpha z(t) dw(t) + (\alpha \tilde{\phi} - \lambda z(t)^2) dt) - \frac{1}{8} z(t)^{-3} \alpha^2 z(t)^2 dt \\ &= \frac{\alpha}{2} dw(t) + \frac{1}{8z(t)} (4\alpha \tilde{\phi} - 4\lambda z(t)^2 - \alpha^2) dt . \end{aligned}$$

Let us now define η by

$$\begin{aligned} \eta(t, q) &:= e^{\frac{\lambda \tilde{\phi} t}{\alpha} - \frac{\lambda q^2}{\alpha^2} \frac{2\tilde{\phi}}{q} - \frac{1}{2}} \\ &= e^{\frac{\lambda \delta t}{4} - \frac{\lambda q^2}{\alpha^2} \frac{\delta - 1}{2}} . \end{aligned}$$

It is easy to check that η solves the equation

$$\theta^2 \frac{\partial \eta}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \eta}{\partial q^2} + V \eta$$

for

$$V = \frac{C}{q^2} + Dq^2 ;$$

in other words,

$$\begin{aligned} S &:= -\theta^2 \ln(\eta) \\ &= -\theta^2 \left(\frac{\lambda \delta t}{4} - \frac{\lambda q^2}{\alpha^2} + \frac{\delta - 1}{2} \ln(q) \right) \\ &= -\frac{\alpha^2 \lambda \delta t}{16} + \frac{\lambda q^2}{4} - \alpha^2 \left(\frac{\delta - 1}{8} \right) \ln(q) \end{aligned}$$

satisfies (\mathcal{HJB}^V) . Furthermore we have :

$$\begin{aligned}
\tilde{B} &:= \theta^2 \frac{\frac{\partial \eta}{\partial q}}{\eta} \\
&= -\frac{\partial S}{\partial q} \\
&= -\frac{\lambda q}{2} + \frac{\alpha^2(\delta - 1)}{8q} \\
&= \frac{1}{8q}(\alpha^2\delta - \alpha^2 - 4\lambda q^2)
\end{aligned}$$

whence

$$\tilde{B}(t, z(t)) = \frac{1}{8z(t)}(4\alpha\tilde{\phi} - \alpha^2 - 4\lambda z(t)^2)$$

and z satisfies the stochastic differential equation associated with η :

$$\forall t \in]0, T[\quad dz(t) = \theta dw(t) + \tilde{B}(t, z(t))dt$$

(as in [4], §1, equation (B)) ; the result follows. \square

Corollary 2.4. *Let us now assume $X_0 = 0$; then*

$$X_t = e^{-\lambda t} Y \left(\frac{\alpha^2(e^{\lambda t} - 1)}{4\lambda} \right)$$

where Y is a BESQ $^\delta$ (squared Bessel process with parameter δ).

Proof. One applies the result of [1], p. 314. \square

Proposition 2.5. *The isovector algebra \mathcal{H}_V associated with V has dimension 6 if and only if $\tilde{\phi} \in \{\frac{\alpha}{4}, \frac{3\alpha}{4}\}$, i.e. $\delta \in \{1, 3\}$; in the opposite case, it has dimension 4.*

Proof. It is enough to apply Theorem 1.1, observing that the condition $C = 0$ is equivalent to $\tilde{\phi} \in \{\frac{\alpha}{4}, \frac{3\alpha}{4}\}$. \square

In the context of Hénon's already mentioned PhD thesis ([2],p.55) we have $\phi = \kappa a$, $\lambda = \kappa$, $\alpha = \sigma^2$ et $\beta = 0$, whence $\tilde{\phi} = \kappa a$ and the condition $C = 0$ is equivalent to

$$\kappa a \in \left\{ \frac{\sigma^2}{4}, \frac{3\sigma^2}{4} \right\} .$$

Let us analyze more closely the case $C = 0$; the general case will be commented upon in [5].

$$1) \tilde{\phi} = \frac{\alpha}{4}, \text{ i.e. } \delta = 1 .$$

Then $y(t)$ is a solution of

$$dy(t) = \frac{\alpha}{2} dw(t) - \frac{\lambda}{2} y(t) dt ,$$

i.e. $y(t)$ is an Ornstein–Uhlenbeck process (it was already known that the Ornstein–Uhlenbeck process was a Bernstein process for a quadratic potential). Therefore $z(t)$ coincides, on the random interval $[0, T[$, with an Ornstein–Uhlenbeck process. Here

$$\eta(t, q) = e \frac{\lambda t}{4} - \frac{\lambda q^2}{\alpha^2} .$$

From

$$\begin{aligned} y(t) &= e^{-\frac{\lambda t}{2}} \left(y_0 + \frac{\alpha}{2} \int_0^t e^{\frac{\lambda s}{2}} dw(s) \right) \\ &= e^{-\frac{\lambda t}{2}} \left(z_0 + \tilde{w} \left(\frac{\alpha^2 (e^{\lambda t} - 1)}{4\lambda} \right) \right) \end{aligned}$$

(\tilde{w} denoting another Brownian motion), it appears that $y(t)$ follows a normal law with mean $e^{-\frac{\lambda t}{2}} z_0$ and variance $\frac{\alpha^2 (1 - e^{-\lambda t})}{4\lambda}$. The density $\rho_t(q)$ of $y(t)$ is therefore given by :

$$\rho_t(q) = \frac{2\sqrt{\lambda}}{\alpha\sqrt{2\pi(1 - e^{-\lambda t})}} \exp\left(-\frac{2\lambda(q - e^{-\frac{\lambda t}{2}} z_0)^2}{\alpha^2(1 - e^{-\lambda t})}\right).$$

Whence

$$\begin{aligned} \eta_*(t, q) &= \frac{\rho_t(q)}{\eta(t, q)} \\ &= \frac{1}{\alpha} \sqrt{\frac{\lambda}{\pi \sinh\left(\frac{\lambda t}{2}\right)}} e^{\left(\frac{-\lambda q^2 - \lambda q^2 e^{-\lambda t} + 4\lambda q z_0 e^{-\frac{\lambda t}{2}} - 2\lambda z_0^2 e^{-\lambda t}}{\alpha^2(1 - e^{-\lambda t})}\right)} \end{aligned}$$

and one may check that, as was to be expected, η_* satisfies the equation

$$-\theta^2 \frac{\partial \eta_*}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \eta_*}{\partial q^2} + V \eta_* .$$

$$\mathbf{2)} \tilde{\phi} = \frac{3\alpha}{4}, \text{ i.e. } \delta = 3.$$

In that case, according to Theorem 2.2, $T = +\infty$ whence $y = z$. Furthermore

$$\eta(t, q) = q e^{\frac{\lambda}{\alpha^2} \left(\frac{3\alpha^2 t}{4} - q^2 \right)} .$$

Let us define

$$s(t) = e^{-\frac{\lambda t}{2}} \frac{1}{z(t)} ;$$

then an easy computation, using Itô's formula in the same way as above, shows that

$$ds(t) = -\frac{\alpha}{2} e^{\frac{\lambda t}{2}} s(t)^2 dw(t) ;$$

in particular, $s(t)$ is a martingale.

Referring once more to Proposition 2.1 and its proof, we see that

$$dX_t = \alpha \sqrt{X_t} dw(t) + \left(\frac{3\alpha^2}{4} - \lambda X_t \right) dt .$$

Let us now assume $X_0 = 0$; then, according to Corollary 2.4,

$$X_t = e^{-\lambda t} Y \left(\frac{\alpha^2 (e^{\lambda t} - 1)}{4\lambda} \right)$$

where Y is a $BESQ^3$ (squared Bessel process with parameter 3). But, for each fixed $t > 0$, Y_t has the same law as tY_1 , and $Y_1 = \|B_1\|^2$ is the square of the norm of a 3-dimensional Brownian motion ; the law of Y_1 is therefore

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{u}{2}} \sqrt{u} \mathbf{1}_{u \geq 0} du .$$

Therefore the density $\rho_t(q)$ of the law of $z(t)$ is given by :

$$\rho_t(q) = \frac{1}{\sqrt{2\pi}} \frac{16\lambda^{\frac{3}{2}}}{\alpha^3(1 - e^{-\lambda t})^{\frac{3}{2}}} q^2 e^{-\frac{2\lambda q^2}{\alpha^2(1 - e^{-\lambda t})}}$$

and

$$\eta_*(t, q) = \frac{\rho_t(q)}{\eta(t, q)} = \frac{16\lambda^{\frac{3}{2}}}{\alpha^3\sqrt{2\pi}} (1 - e^{-\lambda t})^{-\frac{3}{2}} q e^{-\frac{3\lambda t}{4} - \frac{\lambda q^2}{\alpha^2 \tanh(\frac{\lambda t}{2})} }.$$

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