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ON AFFINE INTEREST RATE MODELS

PAUL LESCOT

ABSTRACT. Bernstein processes are Brownian diffusions that appear in Euclidean Quantum Mechanics. The consideration of the symmetries of the associated Hamilton-Jacobi-Bellman equation allows one to obtain various relations between stochastic processes (Lescot-Zambrini, **Progress in Probability**, vols 58 and 59). More recently it has appeared that each *one-factor affine interest rate model* (in the sense of Leblanc-Scaillet) could be described using such a Bernstein process.

MSC 91G30, 60H10

Keywords : interest rate models, isovectors, Bessel processes.

1. INTRODUCTION

The relationship between Bernstein processes and Mathematical Finance was first glimpsed in §5 of [9], where the Alili–Patie family of transformations $S^{\alpha,\beta}$ (see [1], §2, for the particular case $\alpha = 1$, and [9], pp.60–61, for the general case) was reinterpreted in the context of isovectors for the (backward) heat equation. Furthermore, Patie’s composition formula ([9], p.60) for these transformations was derived from the commutation relations of the canonical basis of the aforementioned isovector algebra.

As the motivation for Patie’s work had been to compute certain option prices in the framework of an affine interest rate model (see [9], pp. 101–104), it was natural to look for a *direct parametrization* of such a model by a Bernstein process. It turns out that, for any *one-factor interest rate model* in the sense of Leblanc and Scaillet ([4], p. 351), the associated *square root process* coincides, up to some (possibly infinite) random time, with a Bernstein process (§3, Theorem 3.4). The potential $\frac{1}{q^2}$ (that had been considered in [8], p. 220, on physical grounds) appears here naturally (Theorem 3.4) : this is one more example of the deep links between Euclidean Quantum Mechanics and Mathematical Finance.

2. AN ISOVECTOR CALCULATION

We shall work within the context of [8], §3. We consider the Hamilton–Jacobi–Bellman equation

$$(\mathcal{HJB}^V) \quad \frac{\partial S}{\partial t} + \frac{\theta^2}{2} \frac{\partial^2 S}{\partial q^2} - \frac{1}{2} \left(\frac{\partial S}{\partial q} \right)^2 + V = 0$$

with potential

$$V(t, q) = \frac{C}{q^2} + Dq^2 ,$$

where we have replaced (as explained in [5]) $\sqrt{\hbar}$ by θ .

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In order to determine the Lie algebra \mathcal{H}_V of *pure isovectors* for (\mathcal{HJB}^V) , i.e. the algebra \mathfrak{N} of [8], we have to solve the auxiliary equation (3.29) from [8], p. 215, i.e. :

$$-\frac{1}{4}\ddot{T}_N q^2 - \ddot{l}q + \dot{\sigma} + \frac{1}{2}\dot{T}_N q \left(-\frac{C}{q^3} + 2Dq\right) + l \left(-\frac{2C}{q^3} + 2Dq\right) + \dot{T}_N \left(\frac{C}{q^2} + Dq^2\right) - \frac{\theta^2}{4}\ddot{T}_N = 0 ,$$

that is :

$$q^2(2D\dot{T}_N - \frac{1}{4}\ddot{T}_N) + q(-\ddot{l} + 2Dl) + \dot{\sigma} - \frac{\theta^2}{4}\ddot{T}_N - \frac{2Cl}{q^3} = 0 .$$

As T_N , l and σ depend only upon t , the system is equivalent to :

$$\begin{cases} 2Cl = 0 \\ \ddot{l} = 2Dl \\ \dot{\sigma} = \frac{\theta^2}{4}\ddot{T}_N \\ \ddot{T}_N = 8D\dot{T}_N . \end{cases}$$

Two different cases now appear :

1) $C \neq 0$

Then one must have $l = 0$, in which case the second condition holds automatically, and the system reduces itself to :

$$\begin{cases} \dot{\sigma} = \frac{\theta^2}{4}\ddot{T}_N \\ \ddot{T}_N = 8D\dot{T}_N \end{cases}$$

1)a) $D > 0$

Setting $\epsilon = \sqrt{8D}$, we find

$$\dot{T}_N = C_1 e^{\epsilon t} + C_2 e^{-\epsilon t} ,$$

whence

$$T_N = \frac{C_1}{\epsilon} e^{\epsilon t} - \frac{C_2}{\epsilon} e^{-\epsilon t} + C_3$$

and

$$\sigma = \frac{\theta^2}{4}\dot{T}_N + C_4$$

therefore :

$$\sigma = \frac{\theta^2 C_1}{4} e^{\epsilon t} + \frac{\theta^2 C_2}{4} e^{-\epsilon t} + C_4$$

where $(C_j)_{1 \leq j \leq 4}$ denote arbitrary (real) constants. In particular

$$\dim(\mathcal{H}_V) = 4 .$$

1)b) $D = 0$

Then from $\ddot{T}_N = 0$ follows

$$T_N = C_1 t^2 + C_2 t + C_3$$

for constants C_1, C_2, C_3 . Then

$$\dot{\sigma} = \frac{\theta^2}{4} \ddot{T}_N = \frac{\theta^2 C_1}{2}$$

and

$$\sigma = \frac{\theta^2 C_1}{2} t + C_4 .$$

Therefore, here too, $\dim(\mathcal{H}_V) = 4$; furthermore, we get an explicit expression for the isovectors :

$$N^t = T_N = C_1 t^2 + C_2 t + C_3 ,$$

$$\begin{aligned} N^q &= \frac{1}{2} q \dot{T}_N + l \\ &= \frac{1}{2} q (2C_1 t + C_2) \\ &= C_1 t q + \frac{C_2 q}{2} , \end{aligned}$$

and

$$\begin{aligned} N^S &= -\phi \\ &= -\frac{1}{4} q^2 \ddot{T}_N - q \dot{l} + \sigma \\ &= -\frac{1}{4} q^2 \cdot 2C_1 + \frac{\theta^2}{2} C_1 t + C_4 \\ &= \frac{C_1}{2} (\theta^2 t - q^2) + C_4 . \end{aligned}$$

A canonical basis for \mathcal{H}_V is thus given by $(M_i)_{1 \leq i \leq 4}$, where M_i is characterized by $C_j = \delta_{ij}$ (Kronecker's symbol). Using the notation of [7], it appears that

$$M_1 = \frac{1}{2} N_6 ,$$

$$M_2 = \frac{1}{2} N_4 ,$$

$$M_3 = N_1 ,$$

and

$$M_4 = -\frac{1}{\theta^2} N_3 ,$$

therefore \mathcal{H}_V is generated by N_1, N_3, N_4 and N_6 . We thereby recover the result of [8], p. 220, modulo the correction of a misprint. This list ties in nicely with the symmetry properties of certain diffusions related to Bessel processes (see [6] for a detailed explanation).

1)c) $D < 0$

Setting now $\epsilon = \sqrt{-8D}$, we find

$$\dot{T}_N = C_1 \cos(\epsilon t) + C_2 \sin(\epsilon t) ,$$

whence

$$T_N = \frac{C_1}{\epsilon} \sin(\epsilon t) - \frac{C_2}{\epsilon} \cos(\epsilon t) + C_3 ,$$

and, as above :

$$\sigma = \frac{\theta^2}{4} \dot{T}_N + C_4 ,$$

therefore :

$$\sigma = \frac{\theta^2 C_1}{4} \cos(\epsilon t) + \frac{\theta^2 C_2}{4} \sin(\epsilon t) + C_4$$

where $(C_j)_{1 \leq j \leq 4}$ denote arbitrary (real) constants. In particular ,

$$\dim(\mathcal{H}_V) = 4 .$$

2) $C = 0$

Then the system becomes

$$\begin{cases} \ddot{l} = 2Dl \\ \dot{\sigma} = \frac{\theta^2}{4} \ddot{T}_N \\ \ddot{T}_N = 8D\dot{T}_N . \end{cases}$$

The equation for l on the one hand, and the system for (σ, T_N) on the other hand, are independent, and, as above, the first one has a two-dimensional space of solutions and the second one a four-dimensional space of solutions, *i.e.*

$$\dim(\mathcal{H}_V) = 6 .$$

Whence

Theorem 2.1. *The isovector algebra \mathcal{H}_V associated with V has dimension 6 if and only if $C = 0$; in the opposite case, it has dimension 4.*

3. PARAMETRIZATION OF A ONE-FACTOR AFFINE MODEL

As general references we shall use, concerning Bernstein processes, our recent survey ([5]), and, concerning affine models, Hénon's PhD thesis([3]) as well as Leblanc and Scaillet's seminal paper ([4]).

An *one-factor affine interest rate model* is characterized by the instantaneous rate $r(t)$, satisfying the following stochastic differential equation :

$$(3.1) \quad dr(t) = \sqrt{\alpha r(t) + \beta} dw(t) + (\phi - \lambda r(t)) dt$$

under the risk-neutral probability Q ($\alpha = 0$ corresponds to the so-called Vasicek model, and $\beta = 0$ corresponds to the Cox-Ingersoll-Ross model ; cf. [4]).

Assuming $\alpha > 0$, let us set

$$\begin{aligned} \tilde{\phi} &=_{def} \phi + \frac{\lambda\beta}{\alpha} , \\ \delta &=_{def} \frac{4\tilde{\phi}}{\alpha} , \end{aligned}$$

and let us also assume that $\tilde{\phi} \geq 0$.

The following two quantities will play an important role :

$$\begin{aligned} C &:= \frac{\alpha^2}{8} \left(\tilde{\phi} - \frac{\alpha}{4} \right) \left(\tilde{\phi} - \frac{3\alpha}{4} \right) \\ &= \frac{\alpha^4}{128} (\delta - 1)(\delta - 3) \end{aligned}$$

and

$$D := \frac{\lambda^2}{8} .$$

Let us set $X_t = \alpha r(t) + \beta$.

Proposition 3.1. *Let $r_0 \in \mathbf{R}$; then the stochastic differential equation*

$$(3.2) \quad dr(t) = \sqrt{|\alpha r(t) + \beta|} dw(t) + (\phi - \lambda r(t)) dt$$

has a unique strong solution such that $r(0) = r_0$. Furthermore, in case that

$$\alpha r_0 + \beta \geq 0 ,$$

one has $\alpha r(t) + \beta \geq 0$ for all $t \geq 0$; in particular, $r(t)$ satisfies (3.1).

Proof. Let us set $X_t = \alpha r(t) + \beta$; it is easy to see that, in terms of X_t , equation 3.2 becomes:

$$\begin{aligned} dX_t &= \alpha dr(t) \\ &= \alpha(\sqrt{|X_t|} dw(t) + (\phi - \lambda \frac{X_t - \beta}{\alpha}) dt) \\ &= \alpha\sqrt{|X_t|} dw(t) + (\alpha\tilde{\phi} - \lambda X_t) dt . \end{aligned}$$

We are therefore in the situation of (1), p.313, in [2], with $c = \alpha$, $a = \alpha\tilde{\phi}$, and $b = -\lambda$; the result follows.

In case $\lambda \neq 0$, one may also refer to [3],p.55, Proposition 12.1, with $\sigma = \alpha$, $\kappa = \lambda$ and $a = \frac{\alpha\tilde{\phi}}{\lambda}$. □

We shall henceforth assume all of the hypotheses of Proposition 3.1 to be satisfied.

Corollary 3.2. *One has*

$$\begin{cases} X_t = e^{-\lambda t} Y\left(\frac{\alpha^2(e^{\lambda t}-1)}{4\lambda}\right) \text{ for } \lambda \neq 0, \text{ and} \\ X_t = Y\left(\frac{\alpha^2 t}{4}\right) \text{ for } \lambda = 0 \end{cases}$$

where Y is a $BESQ^\delta$ (squared Bessel process with parameter δ) having initial value $Y_0 = \alpha r_0 + \beta$.

Proof. In case $\lambda \neq 0$, one applies the result of [3], p. 314. For $\lambda = 0$, let

$$Z_t := \frac{4}{\alpha^2} X_t ;$$

it appears that

$$dZ_t = 2\sqrt{|Z_t|}dw(t) + \delta dt ,$$

whence Z_t is a $BESQ^\delta$ -process. As

$$X_t := \frac{\alpha^2}{4} Z_t ,$$

the scaling property of Bessel processes yields the result. \square

Theorem 3.3. *If $\delta \geq 2$ one has, almost surely :*

$$\forall t > 0 \quad X_t > 0 ;$$

on the other hand, if $\delta < 2$, almost surely there is a $t > 0$ such that $X_t = 0$.

Proof. We apply Corollary 1, p. 317, from [4](12.2) yielding that

$$(3.3) \quad \forall t > 0 \quad X_t > 0 .$$

One may also use [3], p.56, from which follows that (3.3) is equivalent to

$$\frac{2a\kappa}{\sigma^2} \geq 1 ;$$

but, according to the above identifications,

$$\frac{2a\kappa}{\sigma^2} = \frac{2\frac{\alpha\tilde{\phi}}{\lambda}\lambda}{\alpha^2} = \frac{2\tilde{\phi}}{\alpha} = \frac{\delta}{2} .$$

\square

Our main result is the following :

Theorem 3.4. *Let us define the process*

$$z(t) = \sqrt{X_t}$$

and the stopping time

$$T = \inf\{t > 0 | X_t = 0\};$$

as seen in Theorem 3.3, $T = +\infty$ a.s. for $\delta \geq 2$, and $T < +\infty$ a.s. for $\delta < 2$. Then there exists a Bernstein process $y(t)$ for

$$\theta = \frac{\alpha}{2}$$

and the potential

$$V(t, q) = \frac{C}{q^2} + Dq^2 .$$

such that

$$\forall t \in [0, T[\quad z(t) = y(t) .$$

In particular, for $\delta \geq 2$, z itself is a Bernstein process.

Proof. One has (cf. Proposition 3.1 and its proof)

$$\begin{aligned} dX_t &= \alpha \sqrt{X_t} dw(t) + (\alpha \tilde{\phi} - \lambda X_t) dt \\ &= \alpha z(t) dw(t) + (\alpha \tilde{\phi} - \lambda z(t)^2) dt . \end{aligned}$$

Taking now $f(x) = \sqrt{x}$, we have

$$\forall x > 0 \quad f'(x) = \frac{1}{2\sqrt{x}} \quad \text{and} \quad f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} ,$$

therefore, for all $t \in]0, T[$, $f'(X_t) = \frac{1}{2z(t)}$ and $f''(X_t) = -\frac{1}{4}z(t)^{-3}$. The application of Itô's formula now gives :

$$\begin{aligned} dz(t) &= d(f(X_t)) \\ &= f'(X_t) dX_t + \frac{1}{2} f''(X_t) (dX_t)^2 \\ &= \frac{1}{2z(t)} (\alpha z(t) dw(t) + (\alpha \tilde{\phi} - \lambda z(t)^2) dt) - \frac{1}{8} z(t)^{-3} \alpha^2 z(t)^2 dt \\ &= \frac{\alpha}{2} dw(t) + \frac{1}{8z(t)} (4\alpha \tilde{\phi} - 4\lambda z(t)^2 - \alpha^2) dt . \end{aligned}$$

Let us now define η by

$$\begin{aligned} \eta(t, q) &:= e^{\frac{\lambda \tilde{\phi} t}{\alpha} - \frac{\lambda q^2}{\alpha^2} \frac{2\tilde{\phi}}{q} - \frac{1}{2}} \\ &= e^{\frac{\lambda \delta t}{4} - \frac{\lambda q^2}{\alpha^2} \frac{\delta - 1}{2}} . \end{aligned}$$

It is easy to check that η solves the equation

$$\theta^2 \frac{\partial \eta}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \eta}{\partial q^2} + V \eta$$

for

$$V = \frac{C}{q^2} + Dq^2 ;$$

in other words,

$$\begin{aligned} S &:= -\theta^2 \ln(\eta) \\ &= -\theta^2 \left(\frac{\lambda \delta t}{4} - \frac{\lambda q^2}{\alpha^2} + \frac{\delta - 1}{2} \ln(q) \right) \\ &= -\frac{\alpha^2 \lambda \delta t}{16} + \frac{\lambda q^2}{4} - \alpha^2 \left(\frac{\delta - 1}{8} \right) \ln(q) \end{aligned}$$

satisfies (\mathcal{HJB}^V) . Furthermore we have :

$$\begin{aligned}
\tilde{B} &:= \theta^2 \frac{\frac{\partial \eta}{\partial q}}{\eta} \\
&= -\frac{\partial S}{\partial q} \\
&= -\frac{\lambda q}{2} + \frac{\alpha^2(\delta - 1)}{8q} \\
&= \frac{1}{8q}(\alpha^2\delta - \alpha^2 - 4\lambda q^2)
\end{aligned}$$

whence

$$\tilde{B}(t, z(t)) = \frac{1}{8z(t)}(4\alpha\tilde{\phi} - \alpha^2 - 4\lambda z(t)^2)$$

and z satisfies the stochastic differential equation associated with η :

$$\forall t \in]0, T[\quad dz(t) = \theta dw(t) + \tilde{B}(t, z(t))dt$$

(as in [5], §1, equation (B)) ; the result follows. \square

Proposition 3.5. *The isovector algebra \mathcal{H}_V associated with V has dimension 6 if and only if $\tilde{\phi} \in \{\frac{\alpha}{4}, \frac{3\alpha}{4}\}$, i.e. $\delta \in \{1, 3\}$; in the opposite case, it has dimension 4.*

Proof. It is enough to apply Theorem 2.1, observing that the condition $C = 0$ is equivalent to $\tilde{\phi} \in \{\frac{\alpha}{4}, \frac{3\alpha}{4}\}$. \square

In the context of Hénon's already mentioned PhD thesis ([3],p.55) we have $\phi = \kappa a$, $\lambda = \kappa$, $\alpha = \sigma^2$ et $\beta = 0$, whence $\tilde{\phi} = \kappa a$ and the condition $C = 0$ is equivalent to

$$\kappa a \in \left\{ \frac{\sigma^2}{4}, \frac{3\sigma^2}{4} \right\} .$$

Let us analyze more closely the situation in which $C = 0$; the general case will be commented upon in [6].

$$\mathbf{1)} \tilde{\phi} = \frac{\alpha}{4}, \text{ i.e. } \delta = 1 .$$

Then $y(t)$ is a solution of

$$dy(t) = \frac{\alpha}{2}dw(t) - \frac{\lambda}{2}y(t)dt ,$$

i.e. $y(t)$ is an Ornstein–Uhlenbeck process (it was already known that the Ornstein–Uhlenbeck process was a Bernstein process for a quadratic potential). Therefore $z(t)$ coincides, on the random interval $[0, T[$, with an Ornstein–Uhlenbeck process. Here

$$\eta(t, q) = e^{\frac{\lambda t}{4} - \frac{\lambda q^2}{\alpha^2}} .$$

From

$$\begin{aligned}
y(t) &= e^{-\frac{\lambda t}{2}} \left(y_0 + \frac{\alpha}{2} \int_0^t e^{\frac{\lambda s}{2}} dw(s) \right) \\
&= e^{-\frac{\lambda t}{2}} \left(z_0 + \tilde{w} \left(\frac{\alpha^2(e^{\lambda t} - 1)}{4\lambda} \right) \right)
\end{aligned}$$

(\tilde{w} denoting another Brownian motion), it appears that $y(t)$ follows a normal law with mean $e^{-\frac{\lambda t}{2}} z_0$ and variance $\frac{\alpha^2(1-e^{-\lambda t})}{4\lambda}$. The density $\rho_t(q)$ of $y(t)$ is therefore given by :

$$\rho_t(q) = \frac{2\sqrt{\lambda}}{\alpha\sqrt{2\pi(1-e^{-\lambda t})}} \exp\left(-\frac{2\lambda(q - e^{-\frac{\lambda t}{2}} z_0)^2}{\alpha^2(1-e^{-\lambda t})}\right).$$

Whence

$$\begin{aligned} \forall t > 0 \quad \eta_*(t, q) &= \frac{\rho_t(q)}{\eta(t, q)} \\ &= \frac{1}{\alpha} \sqrt{\frac{\lambda}{\pi \sinh\left(\frac{\lambda t}{2}\right)}} e^{\left(\frac{-\lambda q^2 - \lambda q^2 e^{-\lambda t} + 4\lambda q z_0 e^{-\frac{\lambda t}{2}} - 2\lambda z_0^2 e^{-\lambda t}}{\alpha^2(1-e^{-\lambda t})}\right)} \end{aligned}$$

and one may check that, as was to be expected, η_* satisfies the following equation ($\mathcal{C}_2^{(V)}$ in [5]) :

$$(3.4) \quad -\theta^2 \frac{\partial \eta_*}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \eta_*}{\partial q^2} + V \eta_* .$$

$$\mathbf{2)} \tilde{\phi} = \frac{3\alpha}{4}, \text{ i.e. } \delta = 3.$$

In that case, according to Theorem 3.2, $T = +\infty$ whence $y = z$. Furthermore

$$\eta(t, q) = q e^{\frac{\lambda}{\alpha^2} \left(\frac{3\alpha^2 t}{4} - q^2\right)} .$$

Let us define

$$s(t) = e^{-\frac{\lambda t}{2}} \frac{1}{z(t)} ;$$

then an easy computation, using Itô's formula in the same way as above, shows that

$$ds(t) = -\frac{\alpha}{2} e^{\frac{\lambda t}{2}} s(t)^2 dw(t) ;$$

in particular, $s(t)$ is a martingale.

Referring once more to Proposition 3.1 and its proof, we see that

$$dX_t = \alpha \sqrt{X_t} dw(t) + \left(\frac{3\alpha^2}{4} - \lambda X_t\right) dt .$$

Let us now assume $X_0 = 0$ and $\lambda \neq 0$; then, according to Corollary 3.2,

$$X_t = e^{-\lambda t} Y \left(\frac{\alpha^2(e^{\lambda t} - 1)}{4\lambda} \right)$$

where Y is a $BESQ^3$ (squared Bessel process with parameter 3) such that $Y(0) = 0$. But, for each fixed $t > 0$, Y_t has the same law as tY_1 , and $Y_1 = \|B_1\|^2$ is the square of the norm of a 3-dimensional Brownian motion ; the law of Y_1 is therefore

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{u}{2}} \sqrt{u} \mathbf{1}_{u \geq 0} du .$$

Therefore the density $\rho_t(q)$ of the law of $z(t)$ is given by :

$$\rho_t(q) = \frac{1}{\sqrt{2\pi}} \frac{16\lambda^{\frac{3}{2}}}{\alpha^3(1 - e^{-\lambda t})^{\frac{3}{2}}} q^2 e^{-\frac{2\lambda q^2}{\alpha^2(1 - e^{-\lambda t})}}$$

and

$$\forall t > 0 \quad \eta_*(t, q) = \frac{\rho_t(q)}{\eta(t, q)} = \frac{16\lambda^{\frac{3}{2}}}{\alpha^3\sqrt{2\pi}} (1 - e^{-\lambda t})^{-\frac{3}{2}} q e^{-\frac{3\lambda t}{4} - \frac{\lambda q^2}{\alpha^2 \tanh(\frac{\lambda t}{2})} .$$

Here, too, one may check directly that η_* satisfies equation (3.4) above.

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