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Well-posedness of the equations of a viscoelastic fluid with a free boundary

Hervé VJ. Le Meur

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Abstract

In this article, we prove the local well-posedness, for arbitrary initial data with certain regularity assumptions, of the equations of a Viscoelastic Fluid of Johnson-Segalman type with a free surface. More general constitutive laws can be easily managed in the same way. The geometry is defined by a solid fixed bottom and an upper free boundary submitted to surface tension. The proof relies on a Lagrangian formulation. First we solve two intermediate problems through a fixed point using mainly [4] for the Navier-Stokes part. Then we solve the whole Lagrangian problem on $[0, T_0]$ *for* T_0 *small enough through a contraction mapping. Since the Lagrangian solution is smooth, we can come back to an Eulerian one.*

1 Introduction

This article deals with the equations modeling the flow of a viscoelastic fluid with a free surface. The fluid is assumed to be incompressible, viscous and its stress tensor contains both a viscous and a viscoelastic part. For the proof, the latter obeys a Johnson-Segalman constitutive law and more general models are studied in an appendix. The geometry is 2D, horizontally infinite and vertically bounded by a rigid bottom and a free boundary.
Gravity and surface tension are the only external forces.

Unless specified, all the articles mentionned hereafter consider viscous fluids obeying the Navier-Stokes equations, but not viscoelastic fluids obeying more complex laws.
A similar problem in a *bounded* geometry (drop of a fluid) was dealt with in numerous articles by Solonnikov. He wrote an article [28] in 1977 in which he proved the local in time unique solvability in the Hölder space $\mathcal{C}^{2+\alpha,1+\alpha/2}$ $(1/2 < \alpha < 1)$ (no surface tension). He also proved the global existence with no source term and sufficiently small initial data in [27] in the space $W^{2,1}_p$ with $p > n$ (no surface tension). More recently Shibata and Shimizu [25] improved this result in $W^{2,1}_{q,p}$ $(= L^p(0,T; W^{2,q}(\Omega)) \bigcap W^{1,p}(0,T; L^q(\Omega)))$ and still with no surface tension. The latter article also contains an interesting review and related problems. Surface tension was included in other articles by Solonnikov where he proved local existence in time and uniqueness

(in $W_2^{2+\alpha,1+\alpha/2}$ with $1/2 < \alpha < 1$ for any initial data in [26]) and global existence for initial data sufficiently close to equilibrium in [27].

An other direction of research is the flow of a fluid down an inclined plane on which Teramoto proved the local in time unique solvability in 3D without surface tension in [33] and with surface tension in [34]. Nishida, Teramoto and Win [19] proved the global in time unique existence in a periodic 2D domain and with sufficiently small initial data. In [8], Bresch and Noble derived a shallow water model and obtained estimates $\varepsilon \to 0$ (still for a periodic in x flow).

Here, we investigate an other direction in which we must quote the pioneer article of Beale in 1981 who proved existence and uniqueness in small time in 3D without surface tension in [6], using Sobolev-Slobodetskiĭ spaces. He also proved in [7] the global existence for sufficiently small initial data, thanks to gravity *and* surface tension. Fujita-Yashima proved in 1985 the existence of a stationnary and a time-periodic solution in the same geometry with surface tension in [10] by perturbation methods. Allain adapted the articles of Beale to the 2D geometry with surface tension. She proved well-posedness in [4] and [3]. In [29], Sylvester had large time existence even without surface tension. Tani extended these results to the 3D case in [31]. In a joint article with Tanaka, he gave a proof of large time existence for sufficiently small initial data whether there is surface tension or not in [32]. In [1], Abels generalized these results to the L^q spaces ($N < q < \infty$) in 2 or 3 dimensions (the velocity is in $W_q^{2,1}$). In [30], Tanaka and Tani proved in 2003 the local existence for arbitrary initial data and global existence for sufficiently small initial data of a compressible Navier-Stokes flow with heat and surface tension taken into account.

Recent articles were published on the numerical resolution of the latter problem especially in view of the simulation of surface waves. We may quote Gutiérrez and Bermejo [14], Audusse et al. [5], Guidorzi and Padula [13] and in 2009 Fang et al. [9].

Our main result is the well-posedness of the equations of a viscoelastic fluid with a free boundary for arbitrary large initial data sufficiently regular. Most of the results of the present article are announced in [17] and extend those of the author's PhD thesis [16].

Up to some minor modifications, more complex constitutive laws can be dealt with. Basically, it relies on the fact that the constitutive law improves the regularity of its source term which is the velocity gradient. The nonlinear terms are easy to handle because of an algebra property. So, provided one may have estimates of the extra stress in an algebra, the same theorem applies to more general viscoelastic fluids.

Below, we give the Eulerian (Subsection 2.1), and then Lagrangian equations (Subsection 2.2). The Equations given, we set the spaces and operators to be inverted and give the sketch of the proof (Subsection 2.3). Section 3 is devoted to solving an auxiliary problem which proof is sketched in Subsection 3.1. Then, one solves a second nonlinear auxiliary problem specific to the viscoelastic fluids in Section 4. Section 5 is devoted to estimates of the error terms and we use all the preceding results in a fixed point in Section 6.

2 The equations and the sketch of the proof

Starting from the dimensionless equations in Eulerian coordinates (Subsection 2.1), we derive the dimensionless equations in Lagrangian coordinates (Subsection 2.2), state the operators, spaces and give the sketch of the proof (Subsection 2.3).

2.1 Eulerian equations

The dimensioned variables and fields are tilded. After exhibiting the dimensioned system of equations, we make it dimensionless.
The domain of the flow is denoted $\Omega(t) \subset \mathbb{R}^2$. Its bottom S_B is given independent of time, and represented by a depth function $b(\tilde{x}_1)$ such that at the bottom $\tilde{z} = -b$. Initially $\Omega(\tilde{t} = 0)$ is denoted Ω. The upper boundary is a free surface $S_F(\tilde{t})$ at time $\tilde{t}$. At $\tilde{t} = 0$, it is denoted S_F and represented by a height function ζ. We assume that these surfaces do not cross (the bottom does not dry even at infinity). A typical domain can be seen on Figure 1.

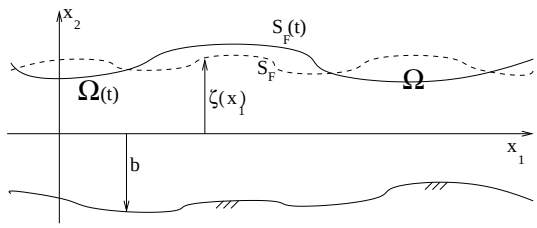

Figure 1: Domain of the flow.

The functions b and ζ are the initial $H^{5/2+r}$ $(0 < r < 1/2)$ height functions of the bottom S_B and of the free surface S_F respectively. We assume that ζ tends to zero at $\pm\infty$ and b to some limit at $\pm\infty$. So the domain is unbounded but of finite depth in the vertical direction. If we denote $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$ a current point in Ω (so at $\tilde{t} = 0$), then $\Omega = \{\tilde{x}/b(\tilde{x}_1) < \tilde{x}_2 < \zeta(\tilde{x}_1)\}$.
Hereafter, vectors and tensors are written in bold letters. Their components are in non-bold letters and with corresponding indices. We use the summation convention and indices after a comma designate a differentiation with respect to the variable: $\partial_{\tilde{x}_j}\tilde{u}_i = \tilde{u}_{i,j}$. We denote $\tilde{\mathbf{v}}$ the velocity, $\tilde{p}$ the pressure, and $\tilde{\boldsymbol{\tau}}$ the extra stress tensor due to the viscoelasticity in the full system:

$$\left\{\begin{array}{lll}
\rho\left(\partial_{\tilde{t}}\tilde{\mathbf{v}} + \tilde{\mathbf{v}}.\tilde{\nabla}\tilde{\mathbf{v}}\right) - \mu_{sol}\tilde{\boldsymbol{\Delta}}\tilde{\mathbf{v}} + \tilde{\boldsymbol{\nabla}}\tilde{p} & = \tilde{\mathrm{div}}\,\tilde{\boldsymbol{\tau}} - \rho\tilde{g}_0\vec{\mathrm{J}} & \text{in } \Omega(t)\times(0,T) \\
\tilde{\mathrm{div}}\,\tilde{\mathbf{v}} & = 0 & \text{in } \Omega(t)\times(0,T), \\
\tilde{\boldsymbol{\tau}} + \lambda\dfrac{\mathcal{D}_a[\tilde{\mathbf{v}}]}{\mathcal{D}\tilde{t}}\tilde{\boldsymbol{\tau}} & = 2\mu_{pol}\tilde{\mathbf{D}}[\tilde{\mathbf{v}}] & \text{in } \Omega(t)\times(0,T), \\
\tilde{\boldsymbol{\tau}}.\mathbf{n} - \tilde{p}\mathbf{n} + 2\mu_{sol}\tilde{\mathbf{D}}[\tilde{\mathbf{v}}].\mathbf{n} - \tilde{\alpha}\tilde{H}\mathbf{n} & = -P_{atm}\mathbf{n} & \text{on } S_F(t)\times(0,T), \\
\tilde{\mathbf{v}} & = 0 & \text{on } S_B, \\
\tilde{\mathbf{v}}(\tilde{x},\tilde{t}=0) & = \tilde{\mathbf{u}}_0(\tilde{x}) & \text{in } \Omega, \\
\tilde{\boldsymbol{\tau}}(\tilde{x},\tilde{t}=0) & = \tilde{\boldsymbol{\tau}}_0(\tilde{x}) & \text{in } \Omega.
\end{array}\right. \tag{1}$$

In this system, ρ is the density of the fluid, μ_{sol} the solvent viscosity, $\tilde{g}_0$ is the acceleration of gravity, λ the relaxation time, μ_{pol} the polymeric viscosity, $\tilde{\alpha}$ the surface tension coefficient, P_{atm} the atmospheric pressure, $\mathbf{D}[\tilde{\mathbf{v}}]$ the symmetric part of the velocity gradient (rate of strain tensor) and

$$\begin{aligned} &\frac{\mathcal{D}_a[\tilde{\mathbf{v}}]}{\mathcal{D}\tilde{t}}\tilde{\boldsymbol{\tau}} = \partial_{\tilde{t}}\tilde{\boldsymbol{\tau}} + \tilde{\mathbf{v}}.\tilde{\boldsymbol{\nabla}}\tilde{\boldsymbol{\tau}} - g_a(\tilde{\boldsymbol{\nabla}}\tilde{\mathbf{v}}, \tilde{\boldsymbol{\tau}}) \\ &\text{where } g_a(\boldsymbol{\nabla}\mathbf{v}, \boldsymbol{\tau}) = \tfrac{a-1}{2}\left(\boldsymbol{\nabla}\mathbf{v}^T\boldsymbol{\tau} + \boldsymbol{\tau}\,\boldsymbol{\nabla}\mathbf{v}\right) + \tfrac{a+1}{2}\left(\boldsymbol{\tau}\,\boldsymbol{\nabla}\mathbf{v}^T + \boldsymbol{\nabla}\mathbf{v}\,\boldsymbol{\tau}\right), \end{aligned} \tag{2}$$

is the interpolated ($a \in [-1, 1]$) Johnson-Segalman time derivative of tensors designed to let the tensors remain frame-invariant. The viscoelastic fluid is supposed to have its total stress tensor equal to the sum of a diagonal pressure matrix, a viscous term and an extra stress tensor $\tilde{\boldsymbol{\tau}}$. The constitutive equation $(1)_3$ satisfied by $\tilde{\boldsymbol{\tau}}$ describes the behavior of complex fluids that have a memory (see [15]). Here we write the Oldroyd B model but other models could be treated the same way.
To get dimensionless equations, we define a characteristic length L (uniform in every dimension) and a characteristic velocity U_0. They enable to define new dimensionless variables untilded:

$$\tilde{x} = (\tilde{x}_1, \tilde{x}_2) = Lx = L(x_1, x_2), \quad \tilde{t} = \frac{L}{U_0}\,t, \tag{3}$$

new dimensionless fields untilded:

$$\begin{aligned} \tilde{\mathbf{v}}(\tilde{x}, \tilde{t}) = U_0\mathbf{v}(x, t),\ \tilde{p}(\tilde{x}, \tilde{t}) = P_{atm} - \rho\tilde{g}_0 L x_2 + (\mu_{sol} + \mu_{pol})\frac{U_0}{L}p(x, t), \\ \tilde{\boldsymbol{\tau}}(\tilde{x}, \tilde{t}) = \frac{(\mu_{sol} + \mu_{pol})U_0}{L}\boldsymbol{\tau}(x, t), \end{aligned} \tag{4}$$

and some dimensionless numbers:

$$\mathrm{Re} = \frac{\rho L U_0}{\mu_{sol} + \mu_{pol}},\ \mathrm{We} = \frac{\lambda U_0}{L},\ \varepsilon = \frac{\mu_{pol}}{\mu_{sol} + \mu_{pol}},\ \alpha = \frac{\tilde{\alpha}}{U_0(\mu_{sol} + \mu_{pol})},\ g_0 = \frac{\rho\tilde{g}_0\,L^2}{(\mu_{sol} + \mu_{pol})U_0}. \tag{5}$$

So we denote Re the Reynold's number, We the Weissenberg number, and α the dimensionless surface tension. All these changes enable to make (1) dimensionless:

$$\begin{cases} \mathrm{Re}\,(\partial_t\mathbf{v} + \mathbf{v}.\boldsymbol{\nabla}\,\mathbf{v}) - (1-\varepsilon)\Delta\mathbf{v} + \boldsymbol{\nabla}p - \mathrm{div}\,\boldsymbol{\tau} & = 0 & \text{in } \Omega(t)\times(0,T) \\ \mathrm{div}\,\mathbf{v} & = 0 & \text{in } \Omega(t)\times(0,T) \\ \boldsymbol{\tau} + \mathrm{We}\dfrac{\mathcal{D}_a}{\mathcal{D}t}[\mathbf{v}]\boldsymbol{\tau} - 2\varepsilon\mathbf{D}[\mathbf{v}] & = 0 & \text{in } (\Omega(t)\times(0,T)), \\ -p\mathbf{n} + 2(1-\varepsilon)\mathbf{D}[\mathbf{v}]\cdot\mathbf{n} + \boldsymbol{\tau}\cdot\mathbf{n} - \alpha H\mathbf{n} + g_0x_2\,\mathbf{n} & = 0 & \text{on } S_F(t)\times(0,T), \\ \mathbf{v}(x,t) & = 0 & \text{on } S_B, \\ \mathbf{v}(x,0) & = \mathbf{u}_0(x) & \text{in } \Omega, \\ \boldsymbol{\tau}(x,0) & = \boldsymbol{\sigma}_0(x) & \text{in } \Omega. \end{cases} \tag{6}$$

This system of partial differential equations is supposed to describe, in the Eulerian coordinates, the flow of a viscoelastic fluid submited to surface tension and gravity. We have explicitely used that g_a is bilinear (see (2)), but more complex constitutive laws without any higher order derivative of $\mathbf{u}$ and of $\boldsymbol{\tau}$ can be included in the proof with minor changes. This will be depicted in Appendix A.

2.2 Lagrangian equations

We need to define the function

$$\begin{array}{rccl} \overline{\boldsymbol{\eta}}(.,t): & \Omega & \rightarrow & \Omega(t) \\ & X & \mapsto & \overline{\boldsymbol{\eta}}(X,t), \end{array} \tag{7}$$

which gives the location at time t of the point that used to be at $X \in \Omega$ at time $t = 0$. In small times, $\overline{\boldsymbol{\eta}}$ will be close to identity and we write

$$\overline{\boldsymbol{\eta}}(X,t) = X + \eta(X,t). \tag{8}$$

In a sense that will be made clearer later, the displacement η is "small". Let us then define the fields in the Lagrangian coordinates:

$$\mathbf{u}(X,t) = \mathbf{v}(\overline{\boldsymbol{\eta}}(X,t),t);\ q(X,t) = p(\overline{\boldsymbol{\eta}}(X,t),t);\ \boldsymbol{\sigma}(X,t) = \boldsymbol{\tau}(\overline{\boldsymbol{\eta}}(X,t),t), \tag{9}$$

where these fields are respectively the velocity, the pressure and the extra stress tensor (modeling the polymer). We need also to define some geometrical quantities:

$$\begin{array}{c} (\mathrm{d}\overline{\boldsymbol{\eta}})_{ij} = \partial_{X_j}\overline{\boldsymbol{\eta}}_i(X,t) = \overline{\boldsymbol{\eta}}_{i,j}(X,t);\ (\overline{\boldsymbol{\xi}}) = (\mathbf{d}\overline{\boldsymbol{\eta}})^{-1}(X,t); \\ \mathbf{N}(X,t) = (-\zeta'(X_1),1)/\sqrt{1+\zeta'^2};\ \boldsymbol{\mathcal{N}} = (N_1 - \partial_{\mathbf{T}}\eta_2, N_2 + \partial_{\mathbf{T}}\eta_1), \end{array} \tag{10}$$

where $\partial_{\mathbf{T}} = (1+\zeta'^2)^{-1/2}\partial_{X_1}$ is the tangential derivative on S_F (applied to functions that depend only on X_1), $\mathbf{N}$ is the unit normal to S_F pointing upward, and $\boldsymbol{\mathcal{N}}$ is a vector that will appear later.

In (6) we must transform the terms to the Lagrangian coordinates on S_F. The most difficult term is $\alpha H(x,t)\mathbf{n}(x,t)$ which is defined on $x \in S_F(t)$. We will make explicit this transformation and let the reader check the other terms.

Simple differential geometry gives us the unit vector $\mathbf{N}$ normal to S_F. In a similar vein, the coordinates of a current point on $S_F(t)$ are $(X_1 + \eta_1(X_1,\zeta(X_1),t), \zeta(X_1) + \eta_2(X_1,\zeta(_1),t))$. If we denote

$$\begin{array}{rcl} \eta_{1,X_1} & = & \partial_{X_1}(\eta_1(X_1,\zeta(X_1),t)), \\ \eta_{2,X_1} & = & \partial_{X_1}(\eta_2(X_1,\zeta(X_1),t)), \end{array} \tag{11}$$

a unit tangent vector to $S_F(t)$ is

$$\mathbf{T} = (1+\eta_{1,X_1}, \zeta'(X_1) + \eta_{2,X_1})/\sqrt{(1+\eta_{1,X_1})^2 + (\zeta'(X_1)+\eta_{2,X_1})^2}.$$

¿From the Frenet-Serret formula, one may write:

$$H\mathbf{n} = \frac{1}{\sqrt{(1+\eta_{1,X_1})^2 + (\zeta'(X_1)+\eta_{2,X_1})^2}}\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}X_1} = \frac{\sqrt{1+\zeta'^2}}{\sqrt{(1+\eta_{1,X_1})^2 + (\zeta'(X_1)+\eta_{2,X_1})^2}}\partial_{\mathbf{T}}\mathbf{T},$$

where $\partial_{\mathbf{T}}(.)$ is the tangential derivative along S_F. Since it is applied on functions only of X_1 we have $\partial_{\mathbf{T}}(.) = \partial_{X_1}(.)/\sqrt{1+\zeta'^2}$ and this derivative is more convenient since it is geometrical. We may simplify further this formula by defining:

$$\Phi(X_1,t) = \frac{\zeta'(X_1) + \eta_{2,X_1}(X_1,t)}{1+\eta_{1,X_1}(X_1,t)} - \zeta'(X_1) \tag{12}$$

as is done in [4] (this article refers to [3] where the reader will find some further details). This definition enables to write

$$H\mathbf{n}(X_1,t) = \frac{\sqrt{1+\zeta'^2}}{\sqrt{(1+\eta_{1,X_1})^2+(\zeta'(X_1)+\eta_{2,X_1})^2}}\partial_{\mathbf{T}}\left(\begin{pmatrix} 1 \\ \Phi(X_1,t)+\zeta' \end{pmatrix}/\sqrt{1+(\Phi+\zeta')^2}\right). \tag{13}$$

In the Lagrangian coordinates, we will need also an evolution equation for $\Phi(X_1,t)$. Since

$$\partial_t\overline{\boldsymbol{\eta}}(X,t) = \mathbf{u}(X,t),$$

one finds easily the time derivative of Φ:

$$\Phi_t(X,t) = \frac{\partial_{\mathbf{T}}(\mathbf{u}(X_1,\zeta(X_1),t)).\mathcal{N}(X_1,t)}{\mathcal{N}_2^2(X_1,t)}.$$

Easy computations enable to complete the derivation of the Lagrangian system of equations that may also be found in classical books ([15], [22], ...):

$$\begin{cases} \mathrm{Re}\, u_{i,t} - (1-\varepsilon)\overline{\xi}_{kj}(\overline{\xi}_{lj}u_{i,l})_{,k} + \overline{\xi}_{ki}q_{,k} - \sigma_{ij,k}\overline{\xi}_{kj} & = 0 \quad \text{in } \Omega\times(0,T), \\ \overline{\xi}_{kj}u_{j,k} & = 0 \quad \text{in } \Omega\times(0,T), \\ \sigma_{ij} + \mathrm{We}\Big(\partial_t\sigma_{ij} - \frac{a-1}{2}(\overline{\xi}_{li}u_{k,l}\sigma_{kj} + \sigma_{ik}u_{k,l}\overline{\xi}_{lj}) & \\ \qquad -\frac{a+1}{2}(\sigma_{ik}\overline{\xi}_{lk}u_{j,l} + u_{i,l}\overline{\xi}_{lk}\sigma_{kj})\Big) & \\ \qquad -\varepsilon(u_{i,k}\overline{\xi}_{kj} + u_{j,k}\overline{\xi}_{ki}) & = 0 \quad \text{in } \Omega\times(0,T), \\ \sigma_{ij}\underline{\mathcal{N}}_j - q\underline{\mathcal{N}}_i + (1-\varepsilon)(\overline{\xi}_{kj}u_{i,k} + \overline{\xi}_{ki}u_{j,k})\underline{\mathcal{N}}_j + & \\ \qquad +g_0(\zeta(X_1)+\eta_2(X_1,t))\underline{\mathcal{N}}_i - & \\ \qquad \alpha\left(\partial_{\mathbf{T}}\left((1+(\Phi+\zeta')^2)^{\frac{-1}{2}}\begin{pmatrix} 1 \\ \Phi+\zeta' \end{pmatrix}\right)\right)_i & = 0 \quad \text{on } S_F\times(0,T), \\ \Phi_t - \dfrac{(\partial_{\mathbf{T}}\mathbf{u})\cdot\boldsymbol{\mathcal{N}}}{\mathcal{N}_2^2} & = 0 \quad \text{on } S_F\times(0,T), \\ \Phi(t=0) & = 0 \quad \text{on } S_F, \\ \mathbf{u}(X,0) & = \mathbf{u}_0(X) \quad \text{in } \Omega, \\ \boldsymbol{\sigma}(X,0) & = \boldsymbol{\sigma}_0(X) \quad \text{in } \Omega, \\ \mathbf{u} & = 0 \quad \text{on } S_B\times(0,T). \end{cases} \tag{14}$$

Here, $\varepsilon \in [0,1[$ is the dimensionless polymeric viscosity, g_0 the acceleration of gravity, $\mathbf{N}$ the outward unit normal to S_F, and $\boldsymbol{\mathcal{N}} = (N_1 - \partial_{\mathbf{T}}\eta_2, N_2 + \partial_{\mathbf{T}}\eta_1)$ a (non-unit) vector normal to $S_F(t)$ at the point $\overline{\boldsymbol{\eta}}(X_1,\zeta(X_1),t)$ convected back on Ω, defined on (X_1,t) and geometrically defined with $\partial_{\mathbf{T}}$ instead of ∂_{X_1}.

In system (14), there are equations inside Ω ($(14)_1$ to $(14)_3$), equations at the free boundary of Ω defined on $X_1 \in \mathbb{R}$ ($(14)_4$ and $(14)_5$), initial conditions ($(14)_6$ to $(14)_8$) and a Dirichlet condition on the bottom $(14)_9$.

¿From now on, the physical domain is the initial one and we consider only the Lagrangian formulation.

2.3 Operators, spaces and sketch of the proof

2.3.1 The operators

Let us remember that for short time, $\overline{\boldsymbol{\eta}}(X,t) = X + \eta(X,t) \simeq X$. So the displacement η will be small (in small time) in a sense to be defined. For the same reason, if we define $\boldsymbol{\xi}$:

$$\left((\mathrm{d}\overline{\boldsymbol{\eta}})^{-1} =\right) \overline{\boldsymbol{\xi}} = \mathbf{Id} + \boldsymbol{\xi},$$

we see that $\boldsymbol{\xi} = (\mathbf{Id} + \mathrm{d}\boldsymbol{\eta})^{-1} - \mathbf{Id}$ will be small.
If we define the operator $P(\boldsymbol{\xi}, \mathbf{u}, q, \phi, \boldsymbol{\sigma})$ as the left-hand side of (14) for equations $(14)_1$ to $(14)_5$ and the initial conditions $(14)_7$ to $(14)_8$, then (14) amounts to solving

$$P(\boldsymbol{\xi}, \mathbf{u}, q, \phi, \boldsymbol{\sigma}) = (0, 0, 0, 0, 0, \mathbf{u}_0, \boldsymbol{\sigma}_0),$$

for $\mathbf{u}$ vanishing on S_B (see $(14)_9$) and $\Phi(t=0) = 0$ (see $(14)_6$).
The orders of magnitude of various terms must now be identified so as to see (14) as a perturbation of an invertible system.
In system (14), there are some source terms : gravity and the initial curvature that are not small even for small times. They are of zeroth order and will be put in $P(0,0,0,0,0)$.
All the terms containing at least one $\boldsymbol{\xi}$ are stored in E (for "Error"). The occurrence of a $\boldsymbol{\xi}$ will enforce smallness.
Proceeding as in [4], we collect the remaining terms in a new operator acting on $(\mathbf{u}, q, \phi, \boldsymbol{\sigma})$ with the zeroth order in $\boldsymbol{\xi}$ (we put $\boldsymbol{\xi}$ in E). Moreover, we keep only the linear in Φ operator from (14) and put the h.o.t. in Φ terms in E^5 (since Φ is initially vanishing these nonlinear terms will be small in small time). We will denote $P_1(\mathbf{u}, q, \phi, \boldsymbol{\sigma})$ this new operator with a minor change $\phi = (1+\zeta'^2)^{-1}\Phi$. So an auxiliary problem to be solved is $P_1(\mathbf{u}, q, \phi, \boldsymbol{\sigma}) = (\mathbf{f}, a, \mathbf{m}, g, k, \mathbf{u}_0, \boldsymbol{\sigma}_0)$ and writes:

$$\left\{\begin{array}{lll}
\mathrm{Re}\partial_t \mathbf{u} - (1-\varepsilon)\Delta\mathbf{u} + \boldsymbol{\nabla} q - \mathrm{div}\boldsymbol{\sigma} & = \mathbf{f} & \text{in } \Omega_0 \times (0,T), \\
\mathrm{div}\ \mathbf{u} & = a & \text{in } \Omega_0 \times (0,T), \\
\boldsymbol{\sigma} + \mathrm{We}\,(\partial_t \boldsymbol{\sigma} - g_a(\boldsymbol{\nabla}\mathbf{u}, \boldsymbol{\sigma})) - 2\varepsilon \mathbf{D}[\mathbf{u}] & = \mathbf{m} & \text{in } \Omega_0 \times (0,T), \\
\boldsymbol{\sigma}\cdot\mathbf{N} - q\mathbf{N} + 2(1-\varepsilon)\mathbf{D}[\mathbf{u}]\cdot\mathbf{N} - \alpha\partial_{\mathbf{T}}(\phi\mathbf{N}) & = g & \text{on } S_F \times (0,T), \\
\phi_t - (\partial_{\mathbf{T}}\mathbf{u})\cdot\mathbf{N} & = k & \text{on } S_F \times (0,T), \\
\mathbf{u}(t=0) & = \mathbf{u}_0(X) & \text{in } \Omega, \\
\boldsymbol{\sigma}(t=0) & = \boldsymbol{\sigma}_0(X) & \text{in } \Omega,
\end{array}\right. \tag{15}$$

for initially vanishing ϕ, and $\mathbf{u} = 0$ on S_B. It is a viscoelastic Stokes flow. Notice that passing from (14) to (15), the gravity term $g_0(\zeta(X_1) + \eta_2)\boldsymbol{\mathcal{N}}$ and the zeroth order surface tension disappear because they are in the zeroth order term $P(0,0,0,0,0)$. In the Navier-Stokes case P_1 is linear (Stokes) but because of the nonlinearity of g_a, our operator P_1 is nonlinear.
With the preceding definitions, (14) is equivalent to

$$\begin{array}{rcl}
P(\boldsymbol{\xi}, \mathbf{u}, q, \phi, \boldsymbol{\sigma}) & = & P(0,0,0,0,0) + P_1(\mathbf{u}, q, \phi, \boldsymbol{\sigma}) + E(\boldsymbol{\xi}, \mathbf{u}, q, \phi, \boldsymbol{\sigma}) \\
& = & (0,0,0,0,0,\mathbf{u}_0, \boldsymbol{\sigma}_0),
\end{array} \tag{16}$$

for a velocity vanishing on the bottom and $\Phi(t=0)=\phi(t=0)=0$. In the remaining, we will need also to define the nonlinear auxiliary operator P_2 which gives the *nonlinear* behaviour close to $(\mathbf{u}_1, q_1, \phi_1, \boldsymbol{\sigma}_1)$:

$$\begin{aligned} P_2[\mathbf{u}_1,\boldsymbol{\sigma}_1](\mathbf{u},q,\phi,\boldsymbol{\sigma}) := \; & P_1(\mathbf{u}_1+\mathbf{u}, q_1+q, \phi_1+\phi, \boldsymbol{\sigma}_1+\boldsymbol{\sigma}) - P_1(\mathbf{u}_1,q_1,\phi_1,\boldsymbol{\sigma}_1) \\ = \; & \begin{pmatrix} \mathrm{Re}\,\partial_t\mathbf{u} - (1-\varepsilon)\Delta\mathbf{u} + \boldsymbol{\nabla} q - \mathrm{div}\boldsymbol{\sigma} \\ \mathrm{div}\mathbf{u} \\ \boldsymbol{\sigma} + \mathrm{We}\,(\partial_t\boldsymbol{\sigma} - g_a(\boldsymbol{\nabla}\mathbf{u},\boldsymbol{\sigma}) - g_a(\boldsymbol{\nabla}\mathbf{u}_1,\boldsymbol{\sigma}) - \\ \qquad\qquad g_a(\boldsymbol{\nabla}\mathbf{u},\boldsymbol{\sigma}_1)) - 2\varepsilon\mathbf{D}[\mathbf{u}] \\ \boldsymbol{\sigma}\cdot\mathbf{N} - q\mathbf{N} + 2(1-\varepsilon)\mathbf{D}[\mathbf{u}]\cdot\mathbf{N} - \alpha\partial_{\mathbf{T}}(\phi\mathbf{N}) \\ \phi_t - \partial_{\mathbf{T}}\mathbf{u}\cdot\mathbf{N} \\ \mathbf{u}(t=0) \\ \boldsymbol{\sigma}(t=0) \end{pmatrix}, \end{aligned} \tag{17}$$

for $\mathbf{u}$ vanishing on S_B and $\phi(t=0)=0$. In the Navier-Stokes case $P_2 \equiv P_1$ because P_1 is linear.

2.3.2 The functionnal spaces

Following J.T. Beale [6] and G. Allain [4], we define anisotropic Sobolev-Slobodetskiĭ spaces for any s:

$$\begin{aligned} K^s(\Omega\times(0,T)) &= L^2(0,T;H^s(\Omega)) \bigcap H^{s/2}(0,T;L^2(\Omega)), \\ K^s(S_F\times(0,T)) &= L^2(0,T;H^s(S_F)) \bigcap H^{s/2}(0,T;L^2(S_F)), \end{aligned}$$

where H^s is the classical Sobolev space whose norm is denoted $|\,.\,|_s$. We denote $|\,.\,|_{K^s}$ the norm of the space K^s. The properties of these spaces, namely embedding and trace theorems, can be found for instance in [18].
One may define the space of $(\mathbf{u}, q, \phi, \boldsymbol{\sigma})$ for $0<r<1/2$:

$$\begin{aligned} X_T^r(\Omega\times(0,T)) = \{ \; & (\mathbf{u},q,\phi,\boldsymbol{\sigma})/ \\ & \mathbf{u}\in K^{r+2} \text{ and } \mathbf{u}=0 \text{ on } S_B\times(0,T); \\ & \boldsymbol{\nabla} q \in K^r \text{ and } q|_{S_F} \in H^{\frac{r}{2}+\frac{1}{4}}(0,T;L^2(S_F)); \\ & \partial_{\mathbf{T}}\phi \in L^2(0,T;H^{r+\frac{1}{2}}(S_F))\,;\; \phi_t \in K^{r+\frac{1}{2}}(S_F\times(0,T)); \phi(0)=0; \\ & \boldsymbol{\sigma} \in H^{\frac{1+r}{2}}(0,T;H^{1+r}(\Omega)) \;\}. \end{aligned} \tag{18}$$

This space X_T^r is basically a regularity space with the condition at the bottom for $\mathbf{u}$ and the initial condition of ϕ. The $H^{\frac{1+r}{2}}(0,T;H^{1+r}(\Omega))$ regularity on $\boldsymbol{\sigma}$ is needed to have an algebra so as to manage the nonlinear terms of a wide variety of constitutive equations.
Indeed, the $\boldsymbol{\nabla}\mathbf{u}\,\boldsymbol{\sigma}$ terms of the constitutive law must be estimated in $L^2(0,T;H^{1+r}) \bigcap H^{\frac{1+r}{2}}(0,T;L^2)$.
For $L^2(0,T;H^{1+r})$, if we use the $L^2(0,T;H^{1+r})$ for $\boldsymbol{\nabla}\mathbf{u}$, we need a $L^\infty(0,T;H^{1+r})$ estimate for $\boldsymbol{\sigma}$. For $H^{\frac{1+r}{2}}(0,T;L^2)$, if we use the $H^{\frac{1+r}{2}}(0,T;L^2)$ estimate for $\boldsymbol{\nabla}\mathbf{u}$, we need a $H^{\frac{1+r}{2}}(0,T;L^\infty)$ for $\boldsymbol{\sigma}$. So estimates of $\boldsymbol{\sigma}$ in $H^{\frac{1+r}{2}}(0,T;H^{1+r})$ are natural.

The image space Y_T^r is

$$
\begin{aligned}
Y_T^r(\Omega\times(0,T)) = \{ \quad & (\mathbf{f}, a, \mathbf{m}, g, k, \mathbf{u}_0, \boldsymbol{\sigma}_0)/ \\
& \mathbf{f} \in K^r(\Omega\times(0,T)), \\
& a \in L^2(0,T; H^{r+1}(\Omega)) \bigcap H^{\frac{r}{2}+1}(0,T,\, {}_0H^{-1}(\Omega)), \\
& \mathbf{m} \in K^{r+1}(\Omega\times(0,T)), \\
& g, k \in K^{r+\frac{1}{2}}(S_F\times(0,T)), \\
& \mathbf{u}_0, \boldsymbol{\sigma}_0 \in H^{r+1}(\Omega)\},
\end{aligned}
\tag{19}
$$

where ${}_0H^{-1}(\Omega)$ is the dual space of ${}^0H^1 = \{p \in H^1/p \equiv 0 \text{ sur } S_F\}$. We will need to keep the space $H^{\frac{r}{2}+1}(0,T;\, {}_0H^{-1}(\Omega))$ so as to have a lift field $\mathbf{u} \in H^{\frac{r}{2}+1}(0,T,L^2(\Omega))$.

2.3.3 Main result and sketch of the proof

Our main result of well-posedness is the following:

Theorem 2.1. *Let $0 < r < 1/2$, and the initial height functions of the free boundary ζ and of the bottom b be such that $(-b - \lim_{\pm\infty} b) \in H^{r+5/2}(I\!R)$. Let also $(\mathbf{u}_0, \boldsymbol{\sigma}_0)$ be in $H^{r+1} \times H^{r+1}_{sym}$ and the compatibility conditions*

$$
\begin{gathered}
div\ \mathbf{u}_0 = 0 \ in\ \Omega, \\
\mathbf{u}_0 = 0 \ on\ S_B, \\
\boldsymbol{\sigma}_0.\mathbf{N}.\mathbf{T} + (1-\varepsilon)(\mathbf{D}[\mathbf{u}_0]).\mathbf{N}.\mathbf{T} - \alpha\partial_{\mathbf{T}}(\mathbf{T}).\mathbf{N}.\mathbf{T} = 0,
\end{gathered}
$$

where $\mathbf{N}$ is the unit normal and $\mathbf{T}$ is the unit tangent vector, be satisfied. Then there exists $T_0 > 0$ depending on the data ; $r, \Omega, \mathbf{u}_0, \boldsymbol{\sigma}_0, \zeta, b, We, \varepsilon, a$ and there exists a unique $(\mathbf{u}, q, \phi, \boldsymbol{\sigma}) \in X^r_{T_0}$ solution to the Lagrangian system (14). Under the same hypothesis, the Eulerian system (6) admits a solution with $\overline{\boldsymbol{\eta}} \in H^1(0,T_0; H^{2+r}(\Omega)) \bigcap H^{2+\frac{r}{2}}(0,T_0; L^2(\Omega))$. The solution depends continuously on the initial conditions provided they are in a bounded subset of $(H^{r+1}(\Omega))^2$.

Remark 2.2. *One could even prove that the solution depends continuously on the parameters (ε, Re, We, a) in the domain $[0,1)\times\mathbb{R}^{+2}\times[-1,1]$. To that end, we should track the occurrence of the parameters in the constants. Since the constants in the estimates depend only polynomialy on the parameters we can have the same estimates on any parameter in an open neighborhood of the given parameter. This continuity is very unlikely to be extendable to the whole domain at least since $\varepsilon = 1$ would not have the regularizing effect of the Navier-Stokes equation.*

In a first part (Section 3) we solve (15) also written $P_1(\mathbf{u}, q, \phi, \boldsymbol{\sigma}) = (\mathbf{f}, a, \mathbf{m}, g, k, \mathbf{u}_0, \boldsymbol{\sigma}_0)$. For that purpose, we first introduce a reduced auxilliary problem $P_2[\mathbf{u}_1, \boldsymbol{\sigma}_1](\mathbf{u}, q, \phi, \boldsymbol{\sigma}) = (\mathbf{f}, 0, \mathbf{m}, 0, 0, 0, 0)$ with zero initial conditions (lifted in the $\mathbf{u}_1, \boldsymbol{\sigma}_1$) that we solve through a fixed point between the Navier-Stokes part (solved in [4]) and the constitutive equation. This step needs uniform (Subsection 3.3) and contraction (Subsection 3.4) estimates. Using this auxilliary problem, we solve (15) and so invert P_1 in Subsection 3.6. In Subsection 3.7 we show the continuous dependence with respect to the right-hand side (rhs) of (15) which comprises the initial conditions. To show that the final mapping P_1^{-1} is Lipschitz, we have to check that

this inverted operator is bounded (independently of $T < T_0$).

In a second part (Section 4), for given $(\mathbf{u}_1, \boldsymbol{\sigma}_1)$ in a bounded ball of $K^{2+r} \times K^{r+1} \bigcap L^\infty(0, T_0; H^{1+r})$, we prove there exists a unique solution in X_T^* ($= X_T^r$ with zero initial conditions) to the problem $P_2[\mathbf{u}_1, \boldsymbol{\sigma}_1](\mathbf{u}, q, \phi, \boldsymbol{\sigma}) = (\mathbf{f}, a, \mathbf{m}, g, k, 0, 0)$ and prove boundedness of $P_2[\mathbf{u}_1, \boldsymbol{\sigma}_1]^{-1}$ with a constant independent of $(\mathbf{u}_1, \boldsymbol{\sigma}_1)$ in a given ball.

In the third part of the proof (Section 5), we derive some estimates on the "error" terms E insuring that they are small and contractant for T_0 small enough.

Last, in Section 6, we gather these results to build a functional F involving P_1, P_2 and E that is contracting. The regularity found ensures then the existence and uniqueness of a solution to the eulerian problem (14). This completes the proof.

3 Inversion of the operator P_1

In the present section, we prove the following Theorem of well-posedness:

Theorem 3.1. *Let $0 < r < 1/2$, $B > 0$ be given and the compatibility conditions:*

$$\begin{gathered} div\ \mathbf{u}_0 = 0 \ in\ \Omega, \\ \mathbf{u}_0 = 0 \ on\ S_B, \\ (\boldsymbol{\sigma}_{0\,i,j}).\mathbf{N}.\mathbf{T} + (1-\varepsilon)(\mathbf{D}[\mathbf{u}_0]).\mathbf{N}.\mathbf{T} - \alpha\partial_{\mathbf{T}}(\mathbf{T}).\mathbf{N}.\mathbf{T} = 0, \end{gathered}$$

be satisfied. There exists a time T_0 such that if $T \leq T_0$, for any $(\mathbf{f}, a, \mathbf{m}, g, k, \mathbf{u}_0, \boldsymbol{\sigma}_0)$ in the ball in Y_T^r of radius B denoted $B_{Y_T^r}(0, B)$, there exists a unique $(\mathbf{u}, p, \phi, \boldsymbol{\sigma}) \in X_T^r$ solution of (15). Moreover the (nonlinear) operator $(\mathbf{u}, p, \phi, \boldsymbol{\sigma}) = P_1^{-1}(\mathbf{f}, a, \mathbf{m}, g, k, \mathbf{u}_0, \boldsymbol{\sigma}_0)$ is such that

$$\begin{aligned} &\mid \mathbf{u} - \mathbf{u}', p - p', \phi - \phi', \boldsymbol{\sigma} - \boldsymbol{\sigma}' \mid_{X_T^r} \leq \\ &C \mid \mathbf{f} - \mathbf{f}', a - a', \mathbf{m} - \mathbf{m}', g - g', k - k', \mathbf{u}_0 - \mathbf{u}_0', \boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0' \mid_{Y_T^r}, \end{aligned} \tag{20}$$

where $C = C(\varepsilon, We, B, T_0, Re)$ is a constant that does not depend on $T \leq T_0$.

Proof. To prove this Theorem, we need to solve a restricted problem $P_2[\mathbf{u}_1, \boldsymbol{\sigma}_1](\mathbf{u}, q, \phi, \boldsymbol{\sigma}) = (\mathbf{f}, 0, \mathbf{m}, 0, 0, 0, 0)$ with vanishing initial conditions in which $\mathbf{u}_1$ and $\boldsymbol{\sigma}_1$ lift the initial conditions. As a consequence, notice that the continuity with respect to the initial conditions means continuity also with respect to those fields. Subsections 3.1 to 3.5 are devoted to this, while Subsection 3.6 proves existence of P_1^{-1} and Subsection 3.7 its boundedness (20). Local continuity with respect to the initial conditions is a mere consequence of (20).

3.1 A reduced second auxiliary problem

Let $(\mathbf{u}_1, \boldsymbol{\sigma}_1)$ be in $K^{r+2} \times K^{r+1} \bigcap L^\infty(0, T_0; H^{1+r})$. To prove Theorem 3.1, we need to solve

$$P_2[\mathbf{u}_1, \boldsymbol{\sigma}_1](\mathbf{u}, q, \Phi, \boldsymbol{\sigma}) = (\mathbf{f}, 0, \mathbf{m}, 0, 0, 0, 0). \tag{21}$$

for $(\mathbf{u}, q, \Phi, \boldsymbol{\sigma}) \in X_T^r$.
The resolution of the non-reduced problem $P_2[\mathbf{u}_1, \boldsymbol{\sigma}_1](\mathbf{u}, q, \phi, \boldsymbol{\sigma}) = (\mathbf{f}, a, \mathbf{m}, g, k, 0, 0)$ is postponed to Section 4 and will use the resolution of (21).
To prove Theorem 3.1, in a first step, we solve (21) in Subsections 3.3 and 3.4. In Subsection 3.6, we invert P_1 and in Subsection 3.7, we prove that its inverse is bounded (20).
Let us begin by a very brief sketch of the proof, then a more precise justification of the required estimates. We report the beginning of the proof to Subsection 3.3 after recalling some lemmas in Subsection 3.2.

In the system (21), where P_2 is defined in (17), we will split the equations with the Stokes/Navier-Stokes part on the one hand and the constitutive law on the other hand. This splitting is done by numerous authors (see *e.g.* Guillopé-Saut [12] and Renardy [23]). Then, given $\mathbf{u}^n, q^n, \phi^n$, we will find an extra stress $\boldsymbol{\sigma}^{n+1}$. This extra stress may be carried forward in the Stokes-like system solved by [4] and will provide $\mathbf{u}^{n+1}, q^{n+1}, \phi^{n+1}$.
Then we will prove that this sequence $\mathbf{u}^n, q^n, \phi^n, \boldsymbol{\sigma}^n$ is a Cauchy sequence in a Banach space.

Let us be more precise below on the required estimates for the full proof. Let $\mathbf{u}^0 = 0, q^0 = 0, \phi^0 = 0, \boldsymbol{\sigma}^0 = 0$. We assume we know $(\mathbf{u}^n, q^n, \phi^n, \boldsymbol{\sigma}^n) \in X_{T_0}^r$ and we look for $\boldsymbol{\sigma}^{n+1}$ such that:

$$\left\{ \begin{array}{l} \boldsymbol{\sigma}^{n+1} + \mathrm{We}\,(\partial_t \boldsymbol{\sigma}^{n+1} - g_a(\boldsymbol{\nabla}\mathbf{u}^n, \boldsymbol{\sigma}^{n+1}) - g_a(\boldsymbol{\nabla}\mathbf{u}_1, \boldsymbol{\sigma}^{n+1}) - g_a(\boldsymbol{\nabla}\mathbf{u}^n, \boldsymbol{\sigma}_1)) = 2\varepsilon \mathbf{D}[\mathbf{u}^n] + \mathbf{m} \\ \boldsymbol{\sigma}^{n+1}(t=0) = 0. \end{array} \right. \tag{22}$$

Such an equation is a simple ODE in the space K^{1+r} with no loss of regularity thanks to the Lagrangian simplification. Even, there is a gain of regularity of $\boldsymbol{\sigma}^{n+1}$ with respect to $\boldsymbol{\nabla}\mathbf{u}^n$ because there is no time derivative on $\boldsymbol{\nabla}\mathbf{u}^n$ and no gradient on $\boldsymbol{\sigma}^{n+1}$. The nonlinearity will be managed with uniform bounds on $\mid \boldsymbol{\sigma}^n \mid_{H^{\frac{1+r}{2}}(0,T;H^{1+r})}$. Thus, the velocity will occur only to the source term $\boldsymbol{\nabla}\mathbf{u}^n$. This enables to gain one level of time regularity and prove estimates of $\boldsymbol{\sigma}^{n+1}$ in $H^{\frac{1+r}{2}}(0,T;H^{1+r})$. Indeed, since $\boldsymbol{\nabla}\mathbf{u}_1, \boldsymbol{\sigma}_1, \boldsymbol{\nabla}\mathbf{u}^n, \mathbf{m}$ are in $L^2(0,T_0;H^{1+r})$, we have $\boldsymbol{\sigma}^{n+1} \in H^1(0,T_0;H^{1+r}) \subset H^{\frac{1+r}{2}}(0,T_0;H^{1+r})$. Notice that this is true until time T_0. This gain will enable to prove that $(\mathbf{u}^n, q^n, \phi^n, \boldsymbol{\sigma}^n) \mapsto (\mathbf{u}^{n+1}, q^{n+1}, \phi^{n+1}, \boldsymbol{\sigma}^{n+1})$ is a contraction for T_0 sufficiently small in $X_{T_0}^r$.
Then we look for $(\mathbf{u}^{n+1}, q^{n+1}, \phi^{n+1}, 0) \in X_{T_0}^r$ solution of the Stokes like problem:

$$\left\{ \begin{array}{lcl} \mathrm{Re}\,\partial_t \mathbf{u}^{n+1} - (1-\varepsilon)\Delta\mathbf{u}^{n+1} + \boldsymbol{\nabla} q^{n+1} & = & \mathbf{f} + \mathrm{div}\boldsymbol{\sigma}^{n+1} \ \text{ in } \Omega \times (0,T), \\ \mathrm{div}\mathbf{u}^{n+1} & = & 0 \ \text{ in } \Omega \times (0,T), \\ -q^{n+1}\mathbf{N} + 2(1-\varepsilon)\mathbf{D}[\mathbf{u}^{n+1}]\cdot\mathbf{N} - \alpha\partial_{\mathbf{T}}(\phi^{n+1}\mathbf{N}) & = & -\boldsymbol{\sigma}^{n+1}\cdot\mathbf{N} \ \text{ on } S_F \times (0,T), \\ \phi_t^{n+1} - (\partial_{\mathbf{T}}\mathbf{u}^{n+1})\cdot\mathbf{N} & = & 0 \ \text{ on } S_F \times (0,T), \\ \phi^{n+1}(0) & = & 0 \ \text{ on } S_F, \\ \mathbf{u}^{n+1}(0) & = & 0 \ \text{ in } \Omega, \\ \mathbf{u}^{n+1} & = & 0 \ \text{ on } S_B \ \ \forall t. \end{array} \right. \tag{23}$$

Theorem 4.1 of [4] in the case $\mathbf{f} = \mathbf{f} + \mathrm{div}\boldsymbol{\sigma}^{n+1}, a = 0, g = -\boldsymbol{\sigma}^{n+1}\cdot\mathbf{N}, k = 0, \mathbf{u}_0 = 0$ states that for given rhs and T_0 (even though it is not obvious in the way it is written, one may state

also this result thanks to the regularizing effect of the Stokes operator), there exists a unique $(\mathbf{u}^{n+1}, q^{n+1}, \phi^{n+1})$ in $X^r_{T_0}$ if $\mathbf{f} + \mathrm{div}\boldsymbol{\sigma}^{n+1} \in K^r(\Omega \times (0, T_0))$ and $\boldsymbol{\sigma}^{n+1} \cdot \mathbf{N} \in K^{r+\frac{1}{2}}(S_F \times (0, T_0))$. Thanks to Theorem 4.1 of [4], we have

$$| \mathbf{u}^{n+1}, q^{n+1}, \phi^{n+1}, 0 |_{X^r_T} \leq C \mid \mathbf{f} + \mathrm{div}\boldsymbol{\sigma}^{n+1} |_{K^r} + C \mid \boldsymbol{\sigma}^{n+1} |_{K^{r+\frac{1}{2}}(S_F \times (0,T))},$$

where C will denote a constant independent of $T \leq T_0$ until the end of this article unless otherwise mentioned. From the assumptions and the definition of $\boldsymbol{\sigma}^{n+1}$, we have that $\mathbf{f} + \mathrm{div}\boldsymbol{\sigma}^{n+1} \in K^r$. Still thanks to the fact that $\boldsymbol{\sigma}^{n+1} \in H^1(0, T; H^{1+r})$ we know that $\boldsymbol{\sigma}^{n+1} \cdot \mathbf{N} \in K^{r+1/2}(S_F \times (0, T_0))$. So

$$| \mathbf{u}^{n+1}, q^{n+1}, \phi^{n+1}, 0 |_{X^r_T} \leq C \mid \mathbf{f} |_{K^r} + C \mid \boldsymbol{\sigma}^{n+1} |_{L^2(0,T;H^{1+r}) \bigcap H^{\frac{r}{2}}(0,T;H^1) \bigcap H^{\frac{r}{2}+\frac{1}{4}}(0,T;H^{\frac{1}{2}})} .$$

So, for $(\mathbf{u}^{n+1}, q^{n+1}, \phi^{n+1}, 0)$, we will need uniform estimates of $\boldsymbol{\sigma}^{n+1}$ in various spaces.
For $(0, 0, 0, \boldsymbol{\sigma}^{n+1})$, we will even need to estimate $\boldsymbol{\sigma}^{n+1}$ in $H^{\frac{1+r}{2}}(0, T; H^{1+r})$ which is even stronger. So we will prove only the latter.

Concerning convergence, since the Stokes-like system is linear (it is not the case of our constitutive equation !), we even have thanks to the same Theorem 4.1 of [4]:

$$| \mathbf{u}^{n+1} - \mathbf{u}^n, q^{n+1} - q^n, \phi^{n+1} - \phi^n, 0 |_{X^r_T} \leq C \mid \mathrm{div}(\boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n) |_{K^r} + C \mid \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n |_{K^{r+\frac{1}{2}}(S_F \times 0,T)} .$$

Thanks to the continuity, injection and trace theorems found in [18] (Chapter 4) or in [6] (Lemma 2.1):

$$\begin{aligned} | \mathrm{div}(\boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n) |_{L^2(0,T;H^r)} &\leq C \mid \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n |_{L^2(0,T;H^{1+r})}, \\ | \mathrm{div}(\boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n) |_{H^{\frac{r}{2}}(0,T;L^2)} &\leq C \mid \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n |_{H^{\frac{r}{2}}(0,T;H^1)}, \\ | \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n |_{K^{r+\frac{1}{2}}(S_F \times (0,T))} &\leq C \mid \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n |_{L^2(0,T;H^{1+r}(\Omega)) \bigcap H^{\frac{r}{2}+\frac{1}{4}}(0,T;H^{\frac{1}{2}})} . \end{aligned}$$

So

$$| \mathbf{u}^{n+1} - \mathbf{u}^n, q^{n+1} - q^n, \phi^{n+1} - \phi^n, 0 |_{X^r_T} \leq C \mid \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n |_{K^{1+r} \bigcap H^{\frac{r}{2}}(0,T;H^1) \bigcap H^{\frac{r}{2}+\frac{1}{4}}(0,T;H^{\frac{1}{2}})} . \quad (24)$$

To prove that the $(\mathbf{u}^{n+1}, q^{n+1}, \phi^{n+1}, 0)$ part of our sequence is of Cauchy type in a Banach space (and we will always need C independent of $T \leq T_0$), it suffices to prove thanks to the interpolation Lemma 3.5:

$$\begin{aligned} | \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n |_{L^2(0,T;H^{1+r})} &\leq CT^{\epsilon'} \mid \mathbf{u}^n - \mathbf{u}^{n-1} |_{K^{r+2}}, \\ | \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n |_{H^{\frac{1+r}{2}}(0,T;L^2)} &\leq CT^{\epsilon'} \mid \mathbf{u}^n - \mathbf{u}^{n-1} |_{K^{r+2}}, \\ | \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n |_{H^{\frac{r}{2}}(0,T;H^1)} &\leq CT^{\epsilon'} \mid \mathbf{u}^n - \mathbf{u}^{n-1} |_{K^{r+2}}, \\ | \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n |_{H^{\frac{r}{2}+\frac{1}{4}}(0,T;H^{\frac{1}{2}})} &\leq CT^{\epsilon'} \mid \mathbf{u}^n - \mathbf{u}^{n-1} |_{K^{r+2}} . \end{aligned}$$

where ϵ' is a positive constant not depending on T_0 nor n.
In addition, for $(0, 0, 0, \boldsymbol{\sigma}^{n+1})$, we need to prove also

$$| \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n |_{H^{\frac{1+r}{2}}(0,T;H^{1+r})} \leq CT^{\epsilon'} | \mathbf{u}^n - \mathbf{u}^{n-1} |_{K^{r+2}},$$

which contains all the others. So we will prove only the latter.

As a first step for all the preceding estimates, we need uniform in n and in $T \leq T_0$ estimates so as to manage nonlinearities:

$$\exists T, V, S \,/\, | \boldsymbol{\sigma}^n |_{H^{1+r}}(t) \leq S \quad \forall n, 0 < t < T \tag{25}$$

$$| \mathbf{u}^n, q^n, \phi^n, 0 |_{X_T^r} \leq V \quad \forall n. \tag{26}$$

This will be done in Subsection 3.3 after some lemmas stated in Subsection 3.2.
As a second step, we prove in Subsection 3.4 that the sequence is of Cauchy type thanks to the fact that C is independent of $T \leq T_0$ and to the $T^{\epsilon'}$ term:

$$\bullet \; | \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n |_{H^{\frac{1+r}{2}}(0,T;H^{1+r})} \leq CT^{\epsilon'} | \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n |_{H^1(0,T;H^{1+r})}, \tag{27}$$

$$\leq CT^{\epsilon'} | \boldsymbol{\nabla}\mathbf{u}^n - \boldsymbol{\nabla}\mathbf{u}^{n-1} |_{L^2(0,T;H^{1+r})} \tag{28}$$

$$\leq CT^{\epsilon'} | \mathbf{u}^n - \mathbf{u}^{n-1} |_{K^{r+2}}, \tag{29}$$

Among these inequalities, (29) is obvious. The inequality (27) will be a simple consequence of Lemma 3.4 proved hereafter that relies on $(\boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n)(t = 0) = 0$. The last inequality will be proved in Subsection 3.4. After lifting some rhs, this will solve (21).

Coming back to the proof of Theorem 3.1, we will lift the initial conditions with $\mathbf{u}_1, q_1, \phi_1, \boldsymbol{\sigma}_1$ in Section 3.6. We will also justify that $\boldsymbol{\sigma}_1 \in K^{1+r} \bigcap L^\infty(0,T;H^{1+r})$ to apply Theorem 4.1. Then the field $(\mathbf{u}_1+\mathbf{u}, q_1+q, \phi_1+\phi, \boldsymbol{\sigma}_1+\boldsymbol{\sigma}) \in X_T^r$ will solve (15) with non-vanishing initial conditions.

In Subsection 3.7, we prove uniqueness of the solution of (15). The estimates enable to prove that the inverse of P_1 is bounded independently of $T < T_0$. This will complete the proof of Theorem 3.1

3.2 Some useful lemmas

Lemma 3.2. *Let X be a Hilbert space, $0 \leq s \leq 2$, such that $s - \frac{1}{2}$ is not integer.*
There exists a bounded extension operator from $\left\{\mathbf{u} \in H^s(0,T;X), \partial_t^k \mathbf{u}(0) = 0,\ 0 \leq k < s - \frac{1}{2}\right\}$ in $H^s(\mathbb{R}^+;X)$. The constant C of its boundedness does not depend on $T \leq T_0$.

The proof of this fundamental lemma based on reflections is classical and can be found either in [18] p. 13 or in [6] (Lemma 2.2).

Remark 3.3. *The initial vanishing conditions (even up to some derivatives) are needed whatever the extension operator (see remark 5.7). This is the reason why we have to solve the system with vanishing initial conditions. For such functions the constants do not depend on $T < T_0$. Then we lift the initial conditions to solve the full system.*

Lemma 3.4. *Let X be a Hilbert space, $1/2 < r \leq 1$, $0 \leq s \leq r$ and $s \neq \frac{1}{2}$. If $f \in H^r(0,T;X)$ and $f(0) = 0$, then ;*

$$| f |_{H^s(0,T;X)} \leq CT^{r-s} | f |_{H^r(0,T;X)} .$$

The constant C does not depend on $T \leq T_0$.

This lemma is proved in [3] (Lemma 5.3). The proof is based on extension and interpolation. We need to have $f(0) = 0$ to have C independent of $T \leq T_0$. Indeed, $|e^{-t}|_{L^2(0,T)}/|e^{-t}|_{H^1(0,T)}$ can be bounded by a constant that does even not depend on $T < T_0$.
We will repeatedly need the following lemma about identity operator from K^r to the space $H^p(0,T;H^{r-2p}(\Omega))$:

Lemma 3.5. *Suppose $0 \leq r \leq 4$ and $p \leq \frac{r}{2}$.*

(i) The identity extends to a bounded operator $K^r(\Omega \times (0,T)) \to H^p(0,T;H^{r-2p}(\Omega))$.

(ii) If r is not an odd integer, the restriction of this operator to the subspace:

$$\left\{ v \in K^r(\Omega \times (0,T)) \ / \ \partial_t^k v(0) = 0, \ \forall k \ 0 \leq k < \frac{r-1}{2} \right\}$$

is bounded independently of $T < T_0$.

Proof.

(i) From the definition of K^r, identity can be defined from K^r to $L^2(0,T;H^r)$ and to $H^{\frac{r}{2}}(0,T;L^2)$. It is bounded. So identity can be defined on the interpolate space between K^r and K^r (which is K^r !) to:

$$\left[L^2(0,T;H^r), H^{\frac{r}{2}}(0,T;L^2)\right]_\theta = H^{\frac{\theta r}{2}}(0,T;H^{r(1-\theta)}),$$

from Proposition 4,2.3 of [18]. It is the announced result with $p = \theta r/2$.

(ii) Lemma 3.2 guarantees that the extension to $t > T$ is bounded independently of $T < T_0$. Then one may apply the same proof as above on $[0,+\infty] \times \Omega$. The norms are bounded independent of $T < T_0$.

□

3.3 Uniform estimates

We want to prove (25, 26) by induction. Indeed we will prove a more general result. The inductive hypothesis writes for $(\mathbf{u}^n, q^n, \phi^n, \boldsymbol{\sigma}^n)$ defined by (22, 23):

$$\exists T_0, V < 1, S \text{ positive such that } \forall n \begin{cases} | \boldsymbol{\sigma}^n |_{H^{1+r}} (t) \leq S, \\ | \mathbf{u}^n, q^n, \phi^n, 0 |_{X_T^r} \leq V, \\ | \boldsymbol{\nabla} \mathbf{u}^n |_{L^2(0,T_0;H^{1+r})} \leq V, \\ C \ | \boldsymbol{\nabla} \mathbf{u}_1 |_{L^2(0,T_0;H^{1+r})} \leq 1, \\ C\sqrt{T_0}(V + | \boldsymbol{\nabla} \mathbf{u}_1 |_{L^2(0,T_0;H^{1+r})}) < 1, \\ C(V + | \mathbf{m} |_{L^2(0,T_0;H^{1+r})}) < S, \\ C \ | \mathbf{f} |_{H^{r,\frac{r}{2}}(0,T_0)} < V/5 \\ CT^{\epsilon'}(V + | \mathbf{m} |_{L^2(0,T_0;H^{1+r})}) < V/5. \end{cases} \tag{30}$$

Notice that the five last inequalities do not depend on n. So it suffices to determine a T_0 at $n=0$ to satisfy these properties. Notice also that the $L^\infty(0,T;H^{1+r})$ estimate on $\boldsymbol{\sigma}^{n+1}$ cannot be omited in the hypothesis because of the nonlinear terms in the constitutive equation which will require even more regularity. Notice also that $(30)_3$ is included in $(30)_2$, but more useful.

Obviously the property at $n=0$ is true if T_0 is small enough.
Let us assume property (30) be true for n and let us try to deduce that it is true for $n+1$. We must be careful that the constants do depend neither on $T<T_0$ nor on n.

In a first step, we want to prove (25) or $(30)_1$ from the H^{1+r} scalar product of (22) with $\boldsymbol{\sigma}^{n+1}$. Thanks to the fact that g_a is only quadratic, it provides (with a C independent of $T<T_0$ and of n):

$$\begin{aligned} \mid \boldsymbol{\sigma}^{n+1} \mid_{1+r}^2 + \frac{\mathrm{We}}{2}\frac{\mathrm{d}}{\mathrm{d}\,t} \mid \boldsymbol{\sigma}^{n+1} \mid_{1+r}^2 \leq C \Big[2\varepsilon \mid \boldsymbol{\nabla}\mathbf{u}^n \mid_{1+r} + \mid \mathbf{m} \mid_{1+r} + \\ + C\mathrm{We}(\mid \boldsymbol{\nabla}\mathbf{u}^n \mid_{1+r} \mid \boldsymbol{\sigma}^{n+1} \mid_{1+r} + \mid \boldsymbol{\nabla}\mathbf{u}_1 \mid_{1+r} \mid \boldsymbol{\sigma}^{n+1} \mid_{1+r} + \mid \boldsymbol{\nabla}\mathbf{u}^n \mid_{1+r} \mid \boldsymbol{\sigma}_1 \mid_{1+r}) \Big] \mid \boldsymbol{\sigma}^{n+1} \mid_{1+r}, \end{aligned} \tag{31}$$

and so

$$\begin{aligned} (1 - C\mathrm{We}(\mid \boldsymbol{\nabla}\mathbf{u}^n \mid_{1+r} + \mid \boldsymbol{\nabla}\mathbf{u}_1 \mid_{1+r})) \mid \boldsymbol{\sigma}^{n+1} \mid_{1+r}^2 + \frac{\mathrm{We}}{2}\frac{\mathrm{d}}{\mathrm{d}\,t} \mid \boldsymbol{\sigma}^{n+1} \mid_{1+r}^2 \leq \\ C\left[(2\varepsilon + C\mathrm{We} \mid \boldsymbol{\sigma}_1 \mid_{1+r}) \mid \boldsymbol{\nabla}\mathbf{u}^n \mid_{1+r} + \mid \mathbf{m} \mid_{1+r}\right] \mid \boldsymbol{\sigma}^{n+1} \mid_{1+r} . \end{aligned}$$

If we choose not to simplify $\mid \boldsymbol{\sigma}^{n+1} \mid_{1+r}$, Hölder inequality enables to state:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\,t}\left(e^{2\int_0^t \frac{(1/2 - C\mathrm{We}(\mid \boldsymbol{\nabla}\mathbf{u}^n \mid_{1+r} + \mid \boldsymbol{\nabla}\mathbf{u}_1 \mid_{1+r}))}{\mathrm{We}} ds} \mid \boldsymbol{\sigma}^{n+1} \mid_{1+r}^2 (t) \right) \leq \\ C(\varepsilon, \boldsymbol{\sigma}_1, \mathrm{We}) \left(\mid \boldsymbol{\nabla}\mathbf{u}^n \mid_{1+r}^2 + \mid \mathbf{m} \mid_{1+r}^2 \right) e^{2\int_0^t \frac{(1/2 - C\mathrm{We}(\mid \boldsymbol{\nabla}\mathbf{u}^n \mid_{1+r} + \mid \boldsymbol{\nabla}\mathbf{u}_1 \mid_{1+r}))}{\mathrm{We}} \mathrm{d}s} . \end{aligned}$$

Thanks to the assumption that $\boldsymbol{\sigma}_1 \in L^\infty(0,T;H^{1+r})$ and that $\boldsymbol{\sigma}^{n+1}(t=0)=0$, one may deduce:

$$\begin{aligned} &\mid \boldsymbol{\sigma}^{n+1} \mid_{1+r}^2 (t) \leq \\ &\int_0^t (e^{-2\int_s^t \frac{(1/2 - C\mathrm{We}(\mid \boldsymbol{\nabla}\mathbf{u}^n \mid_{1+r} + \mid \boldsymbol{\nabla}\mathbf{u}_1 \mid_{1+r}))}{\mathrm{We}} dt'} (C \mid \boldsymbol{\nabla}\mathbf{u}^n \mid_{1+r}^2 + \mid \mathbf{m} \mid_{1+r}^2)) ds. \end{aligned}$$

We have to estimate the term in the exponential. Since $s \leq t$ and thanks to (30) for n:

$$\begin{aligned} -2\int_s^t \frac{1/2 - C\mathrm{We}(\mid \boldsymbol{\nabla}\mathbf{u}^n \mid_{1+r} + \mid \boldsymbol{\nabla}\mathbf{u}_1 \mid_{1+r})}{\mathrm{We}} ds' \leq C\int_s^t \mid \boldsymbol{\nabla}\mathbf{u}^n \mid_{1+r} + \mid \boldsymbol{\nabla}\mathbf{u}_1 \mid_{1+r} ds' \\ \leq C\sqrt{T}\left(\mid \boldsymbol{\nabla}\mathbf{u}^n \mid_{L^2(0,T;H^{1+r})} + \mid \boldsymbol{\nabla}\mathbf{u}_1 \mid_{L^2(0,T;H^{1+r})} \right) < 1. \end{aligned}$$

This enables to state:

$$\mid \boldsymbol{\sigma}^{n+1} \mid_{1+r} (t) \leq C\left(\mid \nabla\mathbf{u}^n \mid_{L^2(0,T;H^{1+r})} + \mid \mathbf{m} \mid_{L^2(0,T;H^{1+r})} \right) \leq C\left(V + \mid \mathbf{m} \mid_{L^2(0,T_0;H^{1+r})} \right), \tag{32}$$

and we deduce the announced (25) or $(30)_1$ for suitable S not depending on n. Moreover we have:

$$| \boldsymbol{\sigma}^{n+1} |_{L^2(0,T;H^{1+r})} \leq CT^{\epsilon'}(V + | \mathbf{m} |_{L^2(0,T;H^{1+r})}) < S. \tag{33}$$

Remark 3.6. *The proof above relies on the fact that the C found at $n+1^{th}$ iteration does not depend on the C assumed at the nth iteration.*

In a second step, so as to have uniform in n estimates on $| \mathbf{u}^n |_{K^{2+r}}$ (denoted $(30)_2$ and $(30)_3$), we need more regular in time estimates on $\boldsymbol{\sigma}$. More precisely we need $| \boldsymbol{\sigma}^{n+1} |_{H^1(0,T;H^{1+r})}$. We take back (22), isolate the time derivative and take the $H^{1+r}(\Omega)$ norm:

$$\begin{aligned} |\partial_t \boldsymbol{\sigma}^{n+1}|_{1+r}(t) \leq \frac{2\varepsilon}{\mathrm{We}} | \boldsymbol{\nabla}\mathbf{u}^n |_{1+r} + \frac{1}{\mathrm{We}} | \mathbf{m} |_{1+r} + \frac{1}{\mathrm{We}} | \boldsymbol{\sigma}^{n+1} |_{1+r} + \\ + C\left(| \boldsymbol{\nabla}\mathbf{u}^n |_{1+r} | \boldsymbol{\sigma}^{n+1} |_{1+r} + | \boldsymbol{\nabla}\mathbf{u}_1 |_{1+r} | \boldsymbol{\sigma}^{n+1} |_{1+r} + | \boldsymbol{\nabla}\mathbf{u}^n |_{1+r} | \boldsymbol{\tau}_1 |_{1+r}\right). \end{aligned}$$

One may then take the L^2_t norm and it writes thanks to (30), (32) and (33):

$$\begin{aligned} |\partial_t \boldsymbol{\sigma}^{n+1}|_{L^2(0,T;H^{1+r})} \leq \; & C(\varepsilon, \mathrm{We})\left(| \boldsymbol{\nabla}\mathbf{u}^n |_{L^2(0,T;H^{1+r})} + | \mathbf{m} |_{L^2(0,T;H^{1+r})}\right) + \\ & + C|\boldsymbol{\sigma}^{n+1}|_{L^2(0,T;H^{1+r})} + C|\boldsymbol{\nabla}\mathbf{u}^n|_{L^2(0,T;H^{1+r})}|\boldsymbol{\sigma}^{n+1}|_{L^\infty(0,T;H^{1+r})} + \\ & + C|\boldsymbol{\nabla}\mathbf{u}_1|_{L^2(0,T;H^{1+r})}|\boldsymbol{\sigma}^{n+1}|_{L^\infty(0,T;H^{1+r})} + \\ & + C|\boldsymbol{\nabla}\mathbf{u}^n|_{L^2(0,T;H^{1+r})}|\boldsymbol{\sigma}_1|_{L^\infty(0,T;H^{1+r})}. \end{aligned}$$

It is then easy to use (33) to prove

$$| \boldsymbol{\sigma}^{n+1} |_{H^1(0,T;H^{1+r})} \leq C(\varepsilon, \mathrm{We}, \mathbf{u}_1, \boldsymbol{\sigma}_1, S)(| \boldsymbol{\nabla}\mathbf{u}^n |_{L^2(0,T,H^{1+r})} + | \mathbf{m} |_{L^2(0,T,H^{1+r})}). \tag{34}$$

This estimate is even stronger than the estimates required in $L^2(0,T;H^{1+r})$, $H^{\frac{r}{2}}(0,T;H^1)$, and $H^{\frac{r}{2}+\frac{1}{4}}(0,T;H^{\frac{1}{2}})$. Because $\boldsymbol{\sigma}^{n+1}(t=0) = 0$ we may use Lemma 3.4 and Lemma 3.5 which imply that the constant C below is independent of $T \leq T_0$:

$$| \boldsymbol{\sigma}^{n+1} |_{H^{\frac{r}{2}}(0,T;H^1)} \leq CT^{\epsilon'} | \boldsymbol{\sigma}^{n+1} |_{H^1(0,T;H^1)} \leq CT^{\epsilon'}(| \boldsymbol{\nabla}\mathbf{u}^n |_{L^2(0,T,H^{1+r})} + | \mathbf{m} |_{L^2(0,T,H^{1+r})}). \tag{35}$$

Similarly, Lemma 3.4 and Lemma 3.5 enable to get:

$$| \boldsymbol{\sigma}^{n+1} |_{H^{\frac{r}{2}+\frac{1}{4}}(0,T;H^{\frac{1}{2}})} \leq CT^{\epsilon'} | \boldsymbol{\sigma}^{n+1} |_{H^1(0,T;H^{\frac{1}{2}})} \leq CT^{\epsilon'}(| \boldsymbol{\nabla}\mathbf{u}^n |_{L^2(0,T;H^{1+r})} + | \mathbf{m} |_{L^2(0,T;H^{1+r})}) \tag{36}$$

We are in position to get estimates on $| \mathbf{u}^{n+1} |_{K^{2+r}}$ independently of n and in particular $(30)_2$. Indeed we use the Theorem 4.1 of G. Allain [4] with $g = -\boldsymbol{\sigma}^{n+1} \cdot \mathbf{N}$, $k = \mathbf{u}_0 = a = 0$, $\mathbf{f} = \mathbf{f} + \mathrm{div}\, \boldsymbol{\sigma}^{n+1}$ to find solutions to (23). Of course we use also the already proved estimates (33, 35, 36) together with classical trace theorems and the induction hypothesis (30). If $0 < r < 1/2$, it enables:

$$\begin{aligned} & | \mathbf{u}^{n+1}, q^{n+1}, \phi^{n+1}, 0 |_{X^r_T} \leq C | \mathbf{f} + \mathrm{div}\boldsymbol{\sigma}^{n+1} |_{K^r} + C | \boldsymbol{\sigma}^{n+1} |_{K^{r+\frac{1}{2}}(S_F \times (0,T))} \\ & \leq C | \mathbf{f} |_{K^r} + C | \boldsymbol{\sigma}^{n+1} |_{L^2(0,T;H^{1+r}) \bigcap H^{\frac{r}{2}}(0,T;H^1)} + C | \boldsymbol{\sigma}^{n+1} |_{L^2(0,T;H^{1+r}(\Omega)) \bigcap H^{\frac{r}{2}+\frac{1}{4}}(0,T;H^{\frac{1}{2}}(\Omega))} \\ & \leq C | \mathbf{f} |_{K^r} + CT^{\epsilon'}(V + | \mathbf{m} |_{L^2(0,T;H^{1+r})}) + CT^{\epsilon'}(V + | \mathbf{m} |_{L^2(0,T;H^{1+r})}) + \\ & \qquad + CT^{\epsilon'}(V + | \mathbf{m} |_{L^2(0,T;H^{1+r})}) + CT^{\epsilon'}(V + | \mathbf{m} |_{L^2(0,T;H^{1+r}(\Omega))}) \\ & \leq V. \end{aligned} \tag{37}$$

In the inductive hypothesis we assumed $V < 1$ and S whatsoever. Should we take a smaller T_0 (the constants C above depend on $\varepsilon, V, S, \mathrm{We}, \mathrm{Re}, \boldsymbol{\sigma}_1, T_0$ but not on $T \leq T_0$ nor on n) we may derive $(30)_2$ and $(30)_3$ for $n+1$ assuming the inductive hypothesis for n.
Since the inequalities $(30)_4$ to $(30)_8$ even do not depend on n, they are satisfied from the fact that they are satisfied at $n = 0$. So the inductive proof of (30) is complete and (25, 26) are also proved.
The case of more general constitutive laws is studied in Appendix A.

3.4 Convergence of the sequence

We intend to prove that the sequence $(\mathbf{u}^n, q^n, \phi^n, \boldsymbol{\sigma}^n)$ is a Cauchy sequence in the Banach space X_T^r. Although we proved (25-26) and even (30), we will only use:

$$\begin{aligned} &\mid \mathbf{u}^n \mid_{K^{2+r}(\Omega\times(0,T))} \leq V \ \forall n, \\ &\mid \boldsymbol{\sigma}^n \mid_{L^\infty(0,T;H^{1+r})} \leq S \ \forall n. \end{aligned}$$

We still need the estimate (28) that will be proved hereafter. Notice that our domain is unboudned. This would make compactness argument very difficult to use.
In order to prove convergence, we take the difference of (22) at the nth and $n+1^{th}$ iteration :

$$\begin{aligned} \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n + \mathrm{We}\Big(\partial_t(\boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n) - g(\boldsymbol{\nabla}\mathbf{u}^n, \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n) - g(\boldsymbol{\nabla}\mathbf{u}^n - \boldsymbol{\nabla}\mathbf{u}^{n-1}, \boldsymbol{\sigma}^n) - \\ g(\boldsymbol{\nabla}\mathbf{u}_1, \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n) - g(\boldsymbol{\nabla}\mathbf{u}^n - \boldsymbol{\nabla}\mathbf{u}^{n-1}, \boldsymbol{\sigma}_1)\Big) = 2\varepsilon\mathbf{D}[\mathbf{u}^n - \mathbf{u}^{n-1}]. \end{aligned} \tag{38}$$

We want to prove (28). The uniform estimates (25-26) and assumptions on $\boldsymbol{\sigma}_1, \boldsymbol{\nabla}\mathbf{u}_1$ enable to simplify the scalar product in H^{1+r} of the equation (38) with $\boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n$:

$$\begin{aligned} &\mid \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n \mid_{1+r}^2 + \frac{\mathrm{We}}{2}\frac{\mathrm{d}}{\mathrm{d}\,t} \mid \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n \mid_{1+r}^2 \leq 2\varepsilon \mid \boldsymbol{\nabla}\mathbf{u}^n - \boldsymbol{\nabla}\mathbf{u}^{n-1} \mid_{1+r} \mid \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n \mid_{1+r} + \\ &+C\mathrm{We}\Big[\mid \boldsymbol{\nabla}\mathbf{u}^n \mid_{1+r} \mid \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n \mid_{1+r}^2 + \mid \boldsymbol{\nabla}\mathbf{u}^n - \boldsymbol{\nabla}\mathbf{u}^{n-1} \mid_{1+r} \mid \boldsymbol{\sigma}^n \mid_{1+r} \mid \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n \mid_{1+r} + \\ &+ \mid \boldsymbol{\nabla}\mathbf{u}_1 \mid_{1+r} \mid \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n \mid_{1+r}^2 + \mid \boldsymbol{\nabla}\mathbf{u}^n - \boldsymbol{\nabla}\mathbf{u}^{n-1} \mid_{1+r} \mid \boldsymbol{\sigma}_1 \mid_{1+r} \mid \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n \mid_{1+r}\Big] \\ &\Rightarrow (1/2 - C\mathrm{We}(\mid \boldsymbol{\nabla}\mathbf{u}^n \mid_{1+r} + \mid \boldsymbol{\nabla}\mathbf{u}_1 \mid_{1+r})) \mid \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n \mid_{1+r}^2 + \frac{\mathrm{We}}{2}\frac{\mathrm{d}}{\mathrm{d}\,t} \mid \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n \mid_{1+r}^2 \leq \\ &\qquad C(\varepsilon, \mathrm{We}, V, S, \boldsymbol{\sigma}_1) \mid \boldsymbol{\nabla}\mathbf{u}^n - \boldsymbol{\nabla}\mathbf{u}^{n-1} \mid_{1+r}^2 . \end{aligned}$$

As in the proof of (33) from (31) one may use (30) to have:

$$\mid \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n \mid_{1+r} (t) \leq C(\varepsilon, \mathrm{We}, V, S, \boldsymbol{\sigma}_1) \mid \boldsymbol{\nabla}\mathbf{u}^n - \boldsymbol{\nabla}\mathbf{u}^{n-1} \mid_{L^2(0,T;H^{1+r})} . \tag{39}$$

We want to prove (28). Since we already know that $\mid \boldsymbol{\sigma}^n \mid_{L^\infty(0,T;H^{1+r})} \leq S$, and $\mid \boldsymbol{\sigma}_1 \mid_{L^\infty(0,T_0;H^{1+r})}$ and $\mid \boldsymbol{\nabla}\mathbf{u}_1 \mid_{L^2(0,T_0;H^{1+r}) \bigcap H^{\frac{1+r}{2}}(0,T_0;L^2)}$ are in a bounded ball (here we need this to have continuity for bounded initial conditions because $\mathbf{u}_1, \boldsymbol{\sigma}_1$ will depend on them), we may use the inequality (39) and come back to (38) whose $L^2(0,T;H^{1+r})$ norm provides:

$$
\begin{aligned}
|\partial_t(\boldsymbol{\sigma}^{n+1}-\boldsymbol{\sigma}^n)|_{L^2(0,T;H^{1+r})} \leq & \mid \boldsymbol{\sigma}^{n+1}-\boldsymbol{\sigma}^n \mid_{L^2(0,T;H^{1+r})} +2\varepsilon \mid \boldsymbol{\nabla}\mathbf{u}^n-\boldsymbol{\nabla}\mathbf{u}^{n-1} \mid_{L^2(0,T;H^{1+r})} + \\
& +C \mid \boldsymbol{\nabla}\mathbf{u}^n \mid_{L^2(0,T;H^{1+r})} \mid \boldsymbol{\sigma}^{n+1}-\boldsymbol{\sigma}^n \mid_{L^\infty(0,T;H^{1+r})} + \\
& +C(S) \mid \boldsymbol{\nabla}\mathbf{u}^n-\boldsymbol{\nabla}\mathbf{u}^{n-1} \mid_{L^2(0,T;H^{1+r})} + \\
& +C \mid \boldsymbol{\nabla}\mathbf{u}_1 \mid_{L^2(0,T;H^{1+r})} \mid \boldsymbol{\sigma}^{n+1}-\boldsymbol{\sigma}^n \mid_{L^\infty(0,T;H^{1+r})} + \\
& +C \mid \boldsymbol{\sigma}_1 \mid_{L^\infty(0,T;H^{1+r})} \mid \boldsymbol{\nabla}\mathbf{u}^n-\boldsymbol{\nabla}\mathbf{u}^{n-1} \mid_{L^2(0,T;H^{1+r})}, \\
\leq & C(\varepsilon, \mathrm{We}, V, S, \mathbf{u}_1, \boldsymbol{\sigma}_1) \mid \boldsymbol{\nabla}\mathbf{u}^n-\boldsymbol{\nabla}\mathbf{u}^{n-1} \mid_{L^2(0,T;H^{1+r})} .
\end{aligned}
$$

Notice that the constant $C(\varepsilon, \mathrm{We}, V, S, \mathbf{u}_1, \boldsymbol{\sigma}_1)$ depends on $\mathbf{u}_1, \boldsymbol{\sigma}_1$ only through a bound of their norm. The latter can be combined with (39) to get the requested inequality:

$$
\left|\boldsymbol{\sigma}^{n+1}-\boldsymbol{\sigma}^n\right|_{H^1(0,T;H^{1+r})} \leq C \mid \boldsymbol{\nabla}\mathbf{u}^n-\boldsymbol{\nabla}\mathbf{u}^{n-1} \mid_{L^2(0,T;H^{1+r})} . \tag{40}
$$

3.5 Passing to the limit

Since estimates (27) to (29) are proved for $(0,0,0,\boldsymbol{\sigma}^n)$ and estimate (24) for $(\mathbf{u}^n, q^n, \Phi^n, 0)$, the whole sequence $(\mathbf{u}^n, q^n, \phi^n, \boldsymbol{\sigma}^n)$ is of Cauchy type, should it be for sufficiently small T_0. So it converges in X_T^r to some $(\mathbf{u}, q, \phi, \boldsymbol{\sigma})$.

Can we take the limit $n \to +\infty$? When the terms are linear it is straightforward. When they are of the shape $\nabla\mathbf{u}^n\ \boldsymbol{\sigma}^n$ in the constitutive law, we remember that $\boldsymbol{\nabla}\mathbf{u}^n$ converges in K^{1+r} and $\boldsymbol{\sigma}^n$ in $H^{\frac{1+r}{2}}(0,T;H^{1+r})$. So

$$
\begin{aligned}
\mid \boldsymbol{\nabla}\mathbf{u}^n\boldsymbol{\sigma}^n-\boldsymbol{\nabla}\mathbf{u}\boldsymbol{\sigma} \mid_{K^{1+r}} \leq & \ C \mid \boldsymbol{\nabla}\mathbf{u}^n-\boldsymbol{\nabla}\mathbf{u} \mid_{K^{1+r}} \mid \boldsymbol{\sigma}^n \mid_{H^{\frac{1+r}{2}}(0,T;H^{1+r})} + \\
& +C \mid \boldsymbol{\nabla}\mathbf{u} \mid_{K^{1+r}} \mid \boldsymbol{\sigma}^n-\boldsymbol{\sigma} \mid_{H^{\frac{1+r}{2}}(0,T;H^{1+r})} .
\end{aligned}
$$

Notice that *a priori* we use the bound C of the embedding $H^{\frac{1+r}{2}}(0,T) \hookrightarrow L^\infty(0,T)$ that depends on $T < T_0$. But we prove further Lemma 5.9 which enables to state that the bound can be taken independent of $T < T_0$ because $(\boldsymbol{\nabla}\mathbf{u}^n-\boldsymbol{\nabla}\mathbf{u})(t=0)=0$ and $\boldsymbol{\sigma}^n(t=0)=0$. So $\boldsymbol{\nabla}\mathbf{u}^n\boldsymbol{\sigma}^n$ tends to $\boldsymbol{\nabla}\mathbf{u}\,\boldsymbol{\sigma}$ in K^{1+r}. Even if more complex nonlinearities of $\boldsymbol{\nabla}\mathbf{u}$ and $\boldsymbol{\sigma}$ had been chosen, they could pass to the limit provided uniform $H^{\frac{1+r}{2}}(0,T;H^{1+r})$ estimates of $\boldsymbol{\sigma}^n$ can be derived. So the most difficult step for more complex constitutive laws is the uniform in n estimate. It is derived for some laws in Appendix A.
As a conclusion, the volumic and boundary equations of $P_2[\mathbf{u}_1, \boldsymbol{\sigma}_1](\mathbf{u}, q, \phi, \boldsymbol{\sigma}) = (\mathbf{f}, 0, \mathbf{m}, 0, 0, 0, 0)$ (see (17) for the definition of P_2) are satisfied by the limit of the sequence.

Concerning the initial velocity, $\mathbf{u}^n$ converges to $\mathbf{u}$ in $H^{\frac{1}{2}+\frac{\theta}{2}}(0,T;H^{1+r-\theta})$ with $0 < \theta < r$ (cf. Lemma 3.5). So we have proved that $\mathbf{u}(t=0)=0$ in $H^{1+r-\theta}$ and so also in H^{1+r}.
Concerning the initial extra stress, $\boldsymbol{\sigma}^n$ converges to $\boldsymbol{\sigma}$ in $H^{\frac{1+r}{2}}(0,T;H^{1+r})$. So $\boldsymbol{\sigma}(0)=0$ in H^{1+r}.
Up to now we have proved that (21) has a solution. By lifting the initial conditions, we will solve (15). This is done in the next subsection.

3.6 Lift the initial conditions to solve P_1

To solve the linearized problem about zero deformation ($\boldsymbol{\xi} = 0$), with general rhs, we introduced and solved the reduced P_2 problem (21). We want now to solve P_1 (or the equation (15)). So let $(\mathbf{f}, a, \mathbf{m}, g, k, \mathbf{u}_0, \boldsymbol{\sigma}_0)$ in a closed ball of radius R in Y_T^r. We also take $\mathbf{f}_1 \in K^r$ and $\mathbf{m}_1 \in K^{1+r}$. We start by lifting the initial extra stress. So let $\boldsymbol{\sigma}_1$ be such that:

$$\begin{cases} \boldsymbol{\sigma}_1 + \mathrm{We}\dfrac{\partial \boldsymbol{\sigma}_1}{\partial t} = \mathbf{m}_1 \\ \boldsymbol{\sigma}_1(0, X) = \boldsymbol{\sigma}_0(X) \ \ \forall X. \end{cases} \tag{41}$$

One may easily check that $\boldsymbol{\sigma}_1 \in L^\infty(0, T_0; H^{1+r})$ for any T_0. This $\boldsymbol{\sigma}_1$ and its special regularity has been repeatedly used in the estimates because of the nonlinearities.
Then applying Theorem 4.1 of [4] to the rhs $(\mathbf{f}_1, a, 0, g - \boldsymbol{\sigma}_1.\mathbf{N}, k, \mathbf{u}_0, 0)$ provides $(\mathbf{u}_1, q_1, \phi_1, 0)$ in X_T^r such that

$$\begin{cases} \mathrm{Re}\mathbf{u}_{1,t} - (1-\varepsilon)\Delta u_1 + \boldsymbol{\nabla} q_1 &= \mathbf{f}_1 & \text{in } \Omega, \\ \mathrm{div}\mathbf{u}_1 &= a & \text{in } \Omega, \\ -q_1\mathbf{N} + 2(1-\varepsilon)\mathbf{D}[\mathbf{u}_1]\cdot\mathbf{N} - \alpha\partial_{\mathbf{T}}(\phi_1\mathbf{N}) &= g - \boldsymbol{\sigma}_1\cdot\mathbf{N} & \text{on } S_F \times (0,T) \\ \phi_{1,t} - \partial_{\mathbf{T}}\mathbf{u}_1\cdot\mathbf{N} &= k & \text{on } S_F \times (0,T) \\ \phi_1(t=0) &= 0 & \text{on } S_F \\ \mathbf{u}_1 &= 0 & \text{on } S_B \times (0,T), \\ \mathbf{u}_1(t=0) &= \mathbf{u}_0(X) & \text{in } \Omega. \end{cases} \tag{42}$$

The just found $\mathbf{u}_1, \boldsymbol{\sigma}_1$ are those denoted the same way as in (21). To summarize, these fields satisfy

$$P_1(\mathbf{u}_1, q_1, \phi_1, \boldsymbol{\sigma}_1) = (\mathbf{f}_1 - \ \mathrm{div}\ \boldsymbol{\sigma}_1, a, \mathbf{m}_1 - 2\varepsilon\mathbf{D}[\mathbf{u}_1] - \mathrm{We}g_a(\boldsymbol{\nabla}\mathbf{u}_1, \boldsymbol{\sigma}_1), g, k, \mathbf{u}_0, \boldsymbol{\sigma}_0).$$

and so $(\mathbf{u}_1 + \mathbf{u}, q_1 + q, \phi_1 + \phi, \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}) \in X_T^r$ is such that

$$\begin{aligned} &P_2[\mathbf{u}_1, \boldsymbol{\sigma}_1](\mathbf{u}, q, \phi, \boldsymbol{\sigma}) + P_1(\mathbf{u}_1, q_1, \phi_1, \boldsymbol{\sigma}_1) = \\ &P_1(\mathbf{u}_1 + \mathbf{u}, q_1 + q, \phi_1 + \phi, \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}) = \\ &\qquad (\mathbf{f} + \mathbf{f}_1 - \ \mathrm{div}\ \boldsymbol{\sigma}_1, a, \mathbf{m} + \mathbf{m}_1 - 2\varepsilon\mathbf{D}[\mathbf{u}_1] - \mathrm{We}g_a(\boldsymbol{\nabla}\mathbf{u}_1, \boldsymbol{\sigma}_1), g, k, \mathbf{u}_0, \boldsymbol{\sigma}_0). \end{aligned}$$

So, by choosing $\mathbf{f}_1 = \ \mathrm{div}\ \boldsymbol{\sigma}_1$ and $\mathbf{m} = 2\varepsilon\mathbf{D}[\mathbf{u}_1] + \mathrm{We}g_a(\boldsymbol{\nabla}\mathbf{u}_1, \boldsymbol{\sigma}_1)$, we provide a solution $(\mathbf{u}_1 + \mathbf{u}, q_1 + q, \phi_1 + \phi, \boldsymbol{\sigma}_1 + \boldsymbol{\sigma})$ of (15) as announced in Theorem 3.1. Notice that we cannot set $\mathbf{m}_1 := 2\varepsilon\mathbf{D}[\mathbf{u}_1] + \mathrm{We}g_a(\boldsymbol{\nabla}\mathbf{u}_1, \boldsymbol{\sigma}_1)$ since $\mathbf{u}_1, \boldsymbol{\sigma}_1$ depend on $\mathbf{m}_1$. It would be only an implicit equation. Then it suffices to let $\mathbf{m}_1$ be whatsoever and determine $\mathbf{m}$ as a function of $\mathbf{u}_1, \boldsymbol{\sigma}_1$ as done here.
To complete the proof of Theorem 3.1, we still must prove the inverse of P_1 is bounded.

3.7 Uniqueness and boundedness

To prove uniqueness of the solutions to (15) and the boundedness of P_1^{-1}, we start from the equation (15) satisfied by $(\mathbf{u}, q, \phi, \boldsymbol{\sigma}) \in X_{T_0}^r$ for given $(\mathbf{f}, a, \mathbf{m}, g, k, \mathbf{u}_0, \boldsymbol{\sigma}_0)$ in a closed ball B in Y_T^r. First we prove uniform estimates and then boundedness of P_1^{-1}.

3.7.1 Uniform estimates

To have uniform estimates, we may pass to the limit in (30) and recover the estimate for the velocity

$$|(\mathbf{u}, q, \Phi, 0)|_{X^r_{T_0}} \leq C, \quad |\boldsymbol{\nabla}\mathbf{u}|_{L^2(0,T;H^{1+r})} \leq C. \tag{43}$$

and the extra stress

$$|\boldsymbol{\sigma}|_{1+r}(t) \leq S. \tag{44}$$

These bounds are true only for initially vanishing fields. But then it suffices to add a lift function to have the same bounds for general $(\mathbf{u}, q, \phi, \boldsymbol{\sigma})$ solution of the system (15).

It is not difficult to mimic the proof of the inequality (28) made in Subsection 3.4 to prove even a stronger result. The bound (44) enables to prove that the partial derivative in time of $\boldsymbol{\sigma}$ is also bounded in $L^2(0,T;H^{1+r})$. Indeed, the nonlinearities can be managed thanks to (32). Similarly to (34), it is then easy to get that:

$$\mid \boldsymbol{\sigma} \mid_{H^1(0,T;H^{1+r})} \leq C \left(\mid \mathbf{m} \mid_{L^2(0,T;H^{1+r})} + \mid \boldsymbol{\nabla}\mathbf{u} \mid_{L^2(0,T;H^{1+r})} + S \right).$$

3.7.2 Boundedness

Let us assume (15) is satisfied by $(\mathbf{u}, p, \phi, \boldsymbol{\sigma})$ and $(\mathbf{u}', p', \phi', \boldsymbol{\sigma}')$ with two right-hand sides.

$$\begin{array}{ll}
\mathrm{Re}\,\partial_t(\mathbf{u}-\mathbf{u}') - (1-\varepsilon)\Delta(\mathbf{u}-\mathbf{u}') + \boldsymbol{\nabla}(q-q') = \mathbf{f}-\mathbf{f}' + \mathrm{div}(\boldsymbol{\sigma}-\boldsymbol{\sigma}') & \text{in } \Omega\times(0,T), \\
\mathrm{div}(\mathbf{u}-\mathbf{u}') = a - a' & \text{in } \Omega\times(0,T), \\
\boldsymbol{\sigma}-\boldsymbol{\sigma}' + \mathrm{We}\,(\partial_t(\boldsymbol{\sigma}-\boldsymbol{\sigma}') - g_a(\boldsymbol{\nabla}\mathbf{u}-\boldsymbol{\nabla}\mathbf{u}', \boldsymbol{\sigma}) - g_a(\boldsymbol{\nabla}\mathbf{u}', \boldsymbol{\sigma}-\boldsymbol{\sigma}')) = & \\
\qquad = 2\varepsilon\mathbf{D}[\mathbf{u}-\mathbf{u}'] + \mathbf{m}-\mathbf{m}' & \text{in } \Omega\times(0,T) \\
(\boldsymbol{\sigma}-\boldsymbol{\sigma}')\cdot\mathbf{N} - (q-q')\mathbf{N} + 2(1-\varepsilon)\mathbf{D}[\mathbf{u}-\mathbf{u}']\cdot\mathbf{N} - \alpha\partial_{\mathbf{T}}((\phi-\phi')\mathbf{N}) = & \\
\qquad = g - g' & \text{in } S_F\times(0,T), \\
(\phi-\phi')_t - \partial_{\mathbf{T}}(\mathbf{u}-\mathbf{u}')\cdot\mathbf{N} = k-k' & \text{in } S_F\times(0,T), \\
(\phi-\phi')(t=0) = 0 & \text{on } S_F, \\
(\mathbf{u}-\mathbf{u}')(t=0) = (\mathbf{u}_0-\mathbf{u}_0')(X) & \text{in } \Omega, \\
(\boldsymbol{\sigma}-\boldsymbol{\sigma}')(t=0) = (\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0')(X) & \text{in } \Omega, \\
\mathbf{u}-\mathbf{u}' = 0 & \text{on } S_B \ \ \forall t.
\end{array} \tag{45}$$

Theorem 4.1 of [4] gives:

$$\begin{array}{l}
\mid \mathbf{u}-\mathbf{u}', q-q', \phi-\phi', 0 \mid_{X^r_T} \leq \\
\quad \leq C \mid \mathbf{f}-\mathbf{f}' + \mathrm{div}(\boldsymbol{\sigma}-\boldsymbol{\sigma}'), a-a', 0, g-g'-(\boldsymbol{\sigma}-\boldsymbol{\sigma}')\cdot\mathbf{N}, k-k', \mathbf{u}_0-\mathbf{u}_0', 0 \mid_{Y^r_T} \\
\quad \leq C \mid \mathbf{f}-\mathbf{f}', a-a', 0, g-g', k-k', \mathbf{u}_0-\mathbf{u}_0', 0 \mid_{Y^r_T} + \\
\qquad\qquad + C \mid \boldsymbol{\sigma}-\boldsymbol{\sigma}' \mid_{L^2(0,T;H^{1+r})\bigcap H^{\frac{r}{2}}(0,T;H^1)} + C \mid \boldsymbol{\sigma}-\boldsymbol{\sigma}' \mid_{K^{r+\frac{1}{2}}(S_F\times(0,T))}.
\end{array} \tag{46}$$

Therefore we must estimate $\boldsymbol{\sigma}-\boldsymbol{\sigma}'$ in three norms $L^2(0,T;H^{1+r})$, $H^{r/2}(0,T;H^1)$ and $H^{\frac{r}{2}+\frac{1}{4}}(0,T;H^{1/2})$. We also need estimates of $\boldsymbol{\sigma}-\boldsymbol{\sigma}'$ in $H^{\frac{1+r}{2}}(0,T;H^{1+r})$ to complete the norm of X^r_T. The last

norm is the strongest but *a priori* no bound can be found for initially non-vanishing functions. Nevertheless we will establish such bounds because we can lift the initial conditions and then use classical results.

Remark 3.7. *By various estimates, we can bound $|\boldsymbol{\sigma}-\boldsymbol{\sigma}'|$ with terms including $|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0'|$ but also $|\boldsymbol{\nabla}\mathbf{u}-\boldsymbol{\nabla}\mathbf{u}'|$. The latter term (and not the former) needs to have a coefficient sufficiently small for T_0 small enough. This will enable these terms to be absorbed by the left-hand sides of (46) and so lead to the expected bound (20).*

From now on, we assume $(\mathbf{u}, q, \phi, \boldsymbol{\sigma})$ solution of (15) and so initially non-vanishing. If the functions vanish initially, the bounds in $L^2(0,T;H^{1+r})$, $H^{r/2}(0,T;H^1)$ and $H^{\frac{r}{2}+\frac{1}{4}}(0,T;H^{1/2})$ are included in the bound in $H^{\frac{1+r}{2}}(0,T;H^{1+r})$. So the hierarchy of these norms is *a priori* not obvious. Yet we prove below that the $H^{\frac{1+r}{2}}(0,T;H^{1+r})$ bound ensures the others. To apply Lemma 3.5, we need to lift the initial values. Then we apply the process that led us to (40) to the new initially vanishing functions. Moreover, we will even derive an estimate in $H^1(0,T;H^{1+r})$.
Let $\boldsymbol{\tau}$ be such that

$$\begin{aligned}\boldsymbol{\tau}+\mathrm{We}\,\partial_t\boldsymbol{\tau} &= 0,\\ \boldsymbol{\tau}(0) &= \boldsymbol{\sigma}_0,\end{aligned}$$

and in the same manner $\boldsymbol{\tau}'(t,X)=e^{\frac{-t}{\mathrm{We}}}\boldsymbol{\sigma}_0'(X)$. Instead of $(45)_3$, by renaming $\boldsymbol{\sigma}:=\hat{\boldsymbol{\sigma}}+\boldsymbol{\tau}$ and $\boldsymbol{\sigma}':=\hat{\boldsymbol{\sigma}}'+\boldsymbol{\tau}'$, we have:

$$\begin{aligned}&\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}'+\mathrm{We}\left(\partial_t(\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}')-g_a(\boldsymbol{\nabla}\mathbf{u}-\boldsymbol{\nabla}\mathbf{u}',\hat{\boldsymbol{\sigma}})-g_a(\boldsymbol{\nabla}\mathbf{u}',\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}')\right)=\\ &\quad=2\varepsilon\mathbf{D}[\mathbf{u}-\mathbf{u}']+\mathbf{m}-\mathbf{m}'+\mathrm{We}\,g_a(\boldsymbol{\nabla}\mathbf{u}-\boldsymbol{\nabla}\mathbf{u}',\boldsymbol{\tau})+\mathrm{We}\,g_a(\boldsymbol{\nabla}\mathbf{u}',\boldsymbol{\tau}-\boldsymbol{\tau}')\ \text{ in } \Omega\times(0,T),\end{aligned}\tag{47}$$

and mainly $(\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}')(t=0)=0$. Moreover, all the uniform bounds stated in Subsubsection 3.7.1 apply to $\hat{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\sigma}}'$. Now, we take the H^{1+r} scalar product of (47) with $\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}'$, then simplify by $|\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}'|_{1+r}$ and we have:

$$\begin{aligned}|\,\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}'\,|_{1+r}+\mathrm{We}\frac{\mathrm{d}}{\mathrm{d}\,t}\,|\,\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}'\,|_{1+r}\leq 2\varepsilon\,|\,\boldsymbol{\nabla}\mathbf{u}-\boldsymbol{\nabla}\mathbf{u}'\,|_{1+r}+|\,\mathbf{m}-\mathbf{m}'\,|_{1+r}+\\ +C\mathrm{We}\left[|\,\boldsymbol{\nabla}\mathbf{u}-\boldsymbol{\nabla}\mathbf{u}'\,|_{1+r}\,(|\,\hat{\boldsymbol{\sigma}}\,|_{1+r}+|\,\boldsymbol{\tau}\,|_{1+r})+|\,\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}'\,|_{1+r}|\,\boldsymbol{\nabla}\mathbf{u}'\,|_{1+r}+\right.\\ \left.+|\,\boldsymbol{\nabla}\mathbf{u}'\,|_{1+r}|\,\boldsymbol{\tau}-\boldsymbol{\tau}'\,|_1\right].\end{aligned}$$

This equation writes also, thanks to the fact that $|\boldsymbol{\tau}-\boldsymbol{\tau}'|_{1+r}(t)\leq|\hat{\boldsymbol{\sigma}}_0-\hat{\boldsymbol{\sigma}}_0'\,|_{1+r}$, $|\boldsymbol{\tau}|_{1+r}(t)\leq|\boldsymbol{\sigma}_0|_{1+r}$ and the uniform bound (44):

$$\begin{aligned}(1-C\mathrm{We}|\boldsymbol{\nabla}\mathbf{u}'|_{1+r})|\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}'\,|_{1+r}+\mathrm{We}\frac{\mathrm{d}}{\mathrm{d}\,t}|\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}'\,|_{1+r}\leq\ & C(|\boldsymbol{\nabla}\mathbf{u}-\boldsymbol{\nabla}\mathbf{u}'|_{1+r}+|\mathbf{m}-\mathbf{m}'|_{1+r})+\\ &+C|\boldsymbol{\nabla}\mathbf{u}'|_{1+r}|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0'\,|_{1+r}\,.\end{aligned}$$

and so

$$\begin{aligned}|\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}'\,|_{1+r}\leq\ & |\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0'\,|_{1+r}\,e^{-\frac{1}{\mathrm{We}}\int_0^t(1-C\mathrm{We}|\boldsymbol{\nabla}\mathbf{u}'|_{1+r})\mathrm{d}s}+\\ &+C\int_0^t e^{-\frac{1}{\mathrm{We}}\int_s^t(1-C\mathrm{We}|\boldsymbol{\nabla}\mathbf{u}'|_{1+r})\mathrm{d}s'}(|\boldsymbol{\nabla}\mathbf{u}-\boldsymbol{\nabla}\mathbf{u}'|_{1+r}(t)+\\ &+|\mathbf{m}-\mathbf{m}'|_{1+r}+|\boldsymbol{\nabla}\mathbf{u}'|_{1+r}(t)|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0'\,|_{1+r})\mathrm{d}s.\end{aligned}$$

Since the term in the exponential can be bounded thanks to (43):

$$-\frac{1}{\mathrm{We}}\int_s^t (1-C\mathrm{We}|\boldsymbol{\nabla}\mathbf{u}'|_{1+r})\mathrm{d}s' \leq C\int_s^t |\boldsymbol{\nabla}\mathbf{u}'|_{1+r} \leq C\sqrt{T_0}|\boldsymbol{\nabla}\mathbf{u}'|_{L^2(0,T;H^{1+r})} \leq C\sqrt{T_0},$$

we prove in $L^\infty(0,T;H^{1+r})$:

$$|\ \hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}'\ |_{1+r}\ (t) \leq C\left(|\ \boldsymbol{\nabla}\mathbf{u}-\boldsymbol{\nabla}\mathbf{u}'\ |_{L^2(0,T;H^1)} + |\ \mathbf{m}-\mathbf{m}'\ |_{L^2(0,T;H^1)} + |\ \boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0'\ |_1\right), \tag{48}$$

where $C=C(\varepsilon,B,\mathrm{We},S,V,\mathrm{Re})$. A similar bound in $L^2(0,T;H^{1+r})$ is easy to derive.
The estimate on the time derivative of $\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}'$ is similar to the one derived in (40) (still because $(\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}')(t=0)=0$). By taking the $H^{1+r}(\Omega)$ norm of the time derivative in (47), we get:

$$\begin{aligned}\left|\partial_t(\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}')\right|_{1+r} \leq\ & \tfrac{1}{\mathrm{We}}|\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}'|_{1+r} + C\left(|\boldsymbol{\nabla}\mathbf{u}-\boldsymbol{\nabla}\mathbf{u}'|_{1+r}|\hat{\boldsymbol{\sigma}}|_{1+r} + |\boldsymbol{\nabla}\mathbf{u}'|_{1+r}|\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}'|_{1+r}+\right.\\ & \left.+|\boldsymbol{\nabla}\mathbf{u}-\boldsymbol{\nabla}\mathbf{u}'|_{1+r}|\boldsymbol{\tau}|_{1+r} + |\boldsymbol{\nabla}\mathbf{u}'|_{1+r}|\boldsymbol{\tau}-\boldsymbol{\tau}'|_{1+r}\right)+\\ & +\tfrac{2\varepsilon}{\mathrm{We}}|\boldsymbol{\nabla}\mathbf{u}-\boldsymbol{\nabla}\mathbf{u}'|_{1+r} + \tfrac{1}{\mathrm{We}}|\mathbf{m}-\mathbf{m}'|_{1+r}.\end{aligned}$$

Then we take the L^2 in time norm of this inequality and have:

$$\left|\partial_t(\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}')\right|_{L^2(0,T;H^{1+r})} \leq C(|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0'|_{1+r} + |\boldsymbol{\nabla}\mathbf{u}-\boldsymbol{\nabla}\mathbf{u}'|_{L^2(0,T;H^{1+r})} + |\mathbf{m}-\mathbf{m}'|_{L^2(0,T;H^{1+r})}), \tag{49}$$

and finally we gather all the results in

$$|\ \hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}'\ |_{H^1(0,T;H^{1+r})} \leq C\left(|\ \boldsymbol{\nabla}\mathbf{u}-\boldsymbol{\nabla}\mathbf{u}'\ |_{L^2(0,T;H^{1+r})} + |\ \mathbf{m}-\mathbf{m}'\ |_{L^2(0,T;H^{1+r})} + |\ \boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0'\ |_{1+r}\right).$$

Now, since $|\ e^{-\frac{t}{\mathrm{We}}}\ |_{H^s(0,T)} \leq C$ for $s\geq 0$, we can state the following properties on $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$:

$$\begin{aligned}|\ \boldsymbol{\sigma}-\boldsymbol{\sigma}'\ |_{L^2(0,T;H^{1+r})} & \leq |\boldsymbol{\tau}-\boldsymbol{\tau}'|_{L^2(0,T;H^{1+r})} + |\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}'|_{L^2(0,T;H^{1+r})}\\ & \leq C\ |\ \boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0'\ |_{1+r} + CT^{\epsilon'}|\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}'|_{H^1(0,T;H^{1+r})}\\ & \leq C\ |\ \boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0'\ |_{1+r} +\\ & \quad +CT^{\epsilon'}\left(|\ \boldsymbol{\nabla}\mathbf{u}-\boldsymbol{\nabla}\mathbf{u}'\ |_{L^2(0,T;H^{1+r})} + |\ \mathbf{m}-\mathbf{m}'\ |_{L^2(0,T;H^{1+r})}\right).\end{aligned} \tag{50}$$

The same can be done in $H^{r/2}(0,T;H^1)$:

$$\begin{aligned}|\ \boldsymbol{\sigma}-\boldsymbol{\sigma}'\ |_{H^{\frac{r}{2}}(0,T;H^1)} & \leq |\boldsymbol{\tau}-\boldsymbol{\tau}'|_{H^{\frac{r}{2}}(0,T;H^1)} + |\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}'|_{H^{\frac{r}{2}}(0,T;H^1)}\\ & \leq C\ |\ \boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0'\ |_1 + CT^{\epsilon'}|\hat{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}'|_{H^1(0,T;H^1)}\\ & \leq C\ |\ \boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0'\ |_1 +\\ & \quad +CT^{\epsilon'}\left(|\ \boldsymbol{\nabla}\mathbf{u}-\boldsymbol{\nabla}\mathbf{u}'\ |_{L^2(0,T;H^{1+r})} + |\ \mathbf{m}-\mathbf{m}'\ |_{L^2(0,T;H^{1+r})}\right).\end{aligned} \tag{51}$$

In order to bound $|\ \boldsymbol{\sigma}-\boldsymbol{\sigma}'\ |_{H^{\frac{r}{2}+\frac{1}{4}}(0,T;H^{\frac{1}{2}}(\Omega))}$, we can do in the same way as above by lifting the initial conditions and by replacing the scalar product in H^1 by the one in $H^{\frac{1}{2}}$. We may then use Lemma 3.2 and Lemma 3.5 because the initial fields, after lifting the initial conditions, vanish. Accurate estimates in $H^1(0,T;H^{1/2})$ enable:

$$\begin{aligned}|\ \boldsymbol{\sigma}-\boldsymbol{\sigma}'\ |_{H^{\frac{r}{2}+\frac{1}{4}}(0,T;L^2(S_F))} \leq\ & CT^{\epsilon'}\left(|\ \boldsymbol{\nabla}\mathbf{u}-\boldsymbol{\nabla}\mathbf{u}'\ |_{L^2(0,T;H^{\frac{1}{2}})} + |\ \mathbf{m}-\mathbf{m}'\ |_{L^2(0,T;H^{\frac{1}{2}})}\right)+\\ & +C\ |\ \boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0'\ |_{\frac{1}{2}}.\end{aligned} \tag{52}$$

In order to bound $\mid \boldsymbol{\sigma} - \boldsymbol{\sigma}' \mid_{H^{\frac{1+r}{2}}(0,T;L^2(\Omega))}$, we lift the non-zero initial conditions and then process as above with the L^2 scalar product. Since T_0 is such that $C\sqrt{T_0} \mid \boldsymbol{\nabla}\mathbf{u}' \mid_{L^2(0,T_0;H^{1+r})} < 1$ (cf. (43) proved in Subsection 3.7.1), for T_0 small enough, we have $(r < 1/2 < 1)$:

$$\mid \boldsymbol{\sigma} - \boldsymbol{\sigma}' \mid_{H^{\frac{1+r}{2}}(0,T;L^2)} \leq C \mid \boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0' \mid + CT^{\epsilon'} \left(\mid \boldsymbol{\nabla}\mathbf{u} - \boldsymbol{\nabla}\mathbf{u}' \mid_{L^2(0,T;L^2)} + \mid \mathbf{m} - \mathbf{m}' \mid_{L^2(0,T;L^2)} \right). \quad (53)$$

The inequalities (50), (53) on $\boldsymbol{\sigma} - \boldsymbol{\sigma}'$ can be mixed into:

$$\mid \boldsymbol{\sigma} - \boldsymbol{\sigma}' \mid_{K^{1+r}} \leq C \mid \boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0' \mid_{1+r} + CT^{\epsilon'} (\mid \boldsymbol{\nabla}\mathbf{u} - \boldsymbol{\nabla}\mathbf{u}' \mid_{K^{2+r}} + \mid \mathbf{m} - \mathbf{m}' \mid_{L^2(0,T;H^{1+r})}). \quad (54)$$

Concerning $\mid \boldsymbol{\sigma} - \boldsymbol{\sigma}' \mid_{H^{\frac{1+r}{2}}(0,T;H^{1+r})}$ we already have the estimate in $L^2(0,T;H^{1+r})$ (see (50)) and $L^\infty(0,T;H^{1+r})$. By lifting the initial conditions, one may, as for (51), bound the $H^1(0,T;H^{1+r})$ norm and get:

$$\begin{aligned} \mid \boldsymbol{\sigma} - \boldsymbol{\sigma}' \mid_{H^{\frac{1+r}{2}}(0,T;H^{1+r})} \leq \; & C \mid \boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0' \mid_{1+r} + \\ & + CT^{\epsilon'} \left(\mid \boldsymbol{\nabla}\mathbf{u} - \boldsymbol{\nabla}\mathbf{u}' \mid_{L^2(0,T;H^{1+r})} + \mid \mathbf{m} - \mathbf{m}' \mid_{L^2(0,T;H^{1+r})} \right). \end{aligned} \quad (55)$$

We are in position now to use the results above in the intermediate (46):

$$\begin{aligned} \mid \mathbf{u} - \mathbf{u}', q - q', \phi - \phi', 0 \mid_{X_T^r} \leq C \mid \mathbf{f} - \mathbf{f}', a - a', 0, g - g', k - k', \mathbf{u}_0 - \mathbf{u}_0', 0 \mid_{Y_T^r} + \\ + CT^{\epsilon'} \mid \mathbf{u} - \mathbf{u}' \mid_{K^{2+r}} + C \mid \mathbf{m} - \mathbf{m}' \mid_{L^2(0,T;H^{1+r})} + C \mid \boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0' \mid_{1+r}. \end{aligned} \quad (56)$$

For T_0 sufficiently small (and the constants do not depend on $T < T_0$), the $|\mathbf{u} - \mathbf{u}'|_{K^{2+r}}$ in the rhs can be absorbed by the lhs. So we have:

$$\mid \mathbf{u} - \mathbf{u}', q - q', \phi - \phi', 0 \mid_{X_T^r} \leq C \mid \mathbf{f} - \mathbf{f}', a - a', \mathbf{m} - \mathbf{m}', g - g', k - k', \mathbf{u}_0 - \mathbf{u}_0', \boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0' \mid_{Y_T^r},$$

which may be combined with (55) to get the continuity (20) of P_1^{-1} in Theorem 3.1. This completes the proof of this Theorem.

□

4 Solving the second auxiliary problem

We want to solve $P_2[\mathbf{u}_1, \boldsymbol{\sigma}_1](\mathbf{u}, q, \phi, \boldsymbol{\sigma}) = (\mathbf{f}, a, \mathbf{m}, g, k, 0, 0)$ (see (17) for the definition of P_2) with $\mathbf{u}_1, \boldsymbol{\sigma}_1$ given and $(\mathbf{u}, q, \Phi, \boldsymbol{\sigma}) \in X_T^* = \{(\mathbf{u}, q, \Phi, \boldsymbol{\sigma}) \in X_T^r / \mathbf{u}(0) = 0\,, \boldsymbol{\sigma}(0) = 0\}$. Indeed, we prove the following well-posedness Theorem.

Theorem 4.1. *Let $0 < r < 1/2$, $B > 0$ be given and $(\mathbf{u}_1, \boldsymbol{\sigma}_1)$ in a bounded subset of $K^{r+2} \times H^{\frac{1+r}{2}}(0, T_0; H^{1+r})$. For any $(\mathbf{f}, a, \mathbf{m}, g, k, 0, 0)$ in a ball $B_{Y_T^r}(0, B)$, there exists a unique $(\mathbf{u}, q, \Phi, \boldsymbol{\sigma}) \in X_T^r$ solution of*

$$P_2[\mathbf{u}_1, \boldsymbol{\sigma}_1](\mathbf{u}, q, \Phi, \boldsymbol{\sigma}) = (\mathbf{f}, a, \mathbf{m}, g, k, 0, 0). \quad (57)$$

Moreover the $P_2[\mathbf{u}_1, \boldsymbol{\sigma}_1]^{-1}$ operator is bounded and even continuous in $B_{Y_T^r}(0, B)$ and its boundedness constant does not depend on $T < T_0$ nor on $\mathbf{u}_1, \boldsymbol{\sigma}_1$ in a given ball of $K^{2+r} \times H^{\frac{1+r}{2}}(0, T_0; H^{1+r})$.

Proof. Incidentally, while solving the first auxiliary problem, we found a solution of (21) which happens to have the same lhs with some vanishing rhs. So we are going to lift the rhs of conditions $(17)_2$, $(17)_4$, $(17)_5$ so as to solve $P_2[\mathbf{u}_1, \boldsymbol{\sigma}_1](\mathbf{u}, q, \phi, \boldsymbol{\sigma}) = (\mathbf{f}, a, \mathbf{m}, g, k, 0, 0)$ which is the same as (57).
It is proved in [4] that if $(0, a, 0, g, k, 0, 0) \in Y_T^r$, there exists $(\mathbf{u}_2, q_2, \phi_2, 0) \in X_T^r$ with continuous dependence such that:

$$\begin{cases} \mathrm{Re}\ \mathbf{u}_{2,t} - (1-\varepsilon)\Delta \mathbf{u}_2 + \boldsymbol{\nabla} q_2 & = 0 \quad \text{in } \Omega \times (0,T), \\ \mathrm{div}\, \mathbf{u}_2 & = a \quad \text{in } \Omega \times (0,T), \\ -q_2 \mathbf{N} + 2(1-\varepsilon)\mathbf{D}[\mathbf{u}_2] \cdot \mathbf{N} - \alpha \partial_{\mathbf{T}}(\phi_2 \mathbf{N}) & = g \quad \text{on } S_F \times (0,T), \\ \phi_{2,t} - (\partial_{\mathbf{T}}(\mathbf{u}_2)) \cdot \mathbf{N} & = k \quad \text{on } S_F \times (0,T), \\ \mathbf{u}_2(t=0) & = 0. \end{cases}$$

Instead of solving (21), we could have solved the same system with $\mathbf{u}_1$ replaced by $\mathbf{u}_1 + \mathbf{u}_2$ and with an other rhs:

$$P_2[\mathbf{u}_1 + \mathbf{u}_2, \boldsymbol{\sigma}_1](\mathbf{u}, q, \phi, \boldsymbol{\sigma}) = (\mathbf{f}, 0, \mathbf{m} + 2\varepsilon \mathbf{D}[\mathbf{u}_2] + \mathrm{We} g_a(\boldsymbol{\nabla} \mathbf{u}_2, \boldsymbol{\sigma}_1), 0, 0, 0, 0).$$

Such an equation can be rewritten

$$P_2[\mathbf{u}_1, \boldsymbol{\sigma}_1](\mathbf{u}, q, \phi, \boldsymbol{\sigma}) = (\mathbf{f}, 0, \mathbf{m} + 2\varepsilon \mathbf{D}[\mathbf{u}_2] + \mathrm{We}\left(g_a(\boldsymbol{\nabla} \mathbf{u}_2, \boldsymbol{\sigma}) + g_a(\boldsymbol{\nabla} \mathbf{u}_2, \boldsymbol{\sigma}_1)\right), 0, 0, 0, 0).$$

Let us write explicitely the Stokes part of this system:

$$\begin{cases} \mathrm{Re}\ \partial_t \mathbf{u} - (1-\varepsilon)\Delta \mathbf{u} + \boldsymbol{\nabla} q - \mathrm{div} \boldsymbol{\sigma} & = \mathbf{f} \quad \text{in } \Omega \times (0,T) \\ \mathrm{div}\ \mathbf{u} & = 0 \quad \text{in } \Omega \times (0,T) \\ \boldsymbol{\sigma} \cdot \mathbf{N} - q \mathbf{N} + 2(1-\varepsilon)\mathbf{D}[\mathbf{u}] \cdot \mathbf{N} - \alpha \partial_{\mathbf{T}}(\phi \mathbf{N}) & = 0 \quad \text{on } S_F \times (0,T), \\ \phi_t - (\partial_{\mathbf{T}} \mathbf{u}) \cdot \mathbf{N} & = 0 \quad \text{on } S_F \times (0,T), \\ \phi(t=0) & = 0 \quad \text{in } \Omega, \\ \mathbf{u}(t=0) & = 0 \quad \text{in } \Omega, \\ \mathbf{u} & = 0 \quad \text{on } S_B \times (0,T), \end{cases}$$

that can be rewritten thanks to the definition of $(\mathbf{u}_2, q_2, \phi_2)$:

$$\begin{cases} \mathrm{Re}\ \partial_t(\mathbf{u}_2 + \mathbf{u}) - (1-\varepsilon)\Delta(\mathbf{u}_2 + \mathbf{u}) + \boldsymbol{\nabla}(q_2 + q) - \mathrm{div}(\boldsymbol{\sigma}) & = \mathbf{f} \quad \text{in } \Omega \times (0,T) \\ \mathrm{div}(\mathbf{u}_2 + \mathbf{u}) & = a \quad \text{in } \Omega \times (0,T) \\ \boldsymbol{\sigma} \cdot \mathbf{N} - (q_2 + q)\mathbf{N} + 2(1-\varepsilon)\mathbf{D}[\mathbf{u}_2 + \mathbf{u}] \cdot \mathbf{N} - \alpha \partial_{\mathbf{T}}((\phi_2 + \phi)\mathbf{N}) & = g \quad \text{on } S_F \times (0,T), \\ (\phi_2 + \phi)_t - (\partial_{\mathbf{T}}(\mathbf{u}_2 + \mathbf{u})) \cdot \mathbf{N} & = k \quad \text{on } S_F \times (0,T), \\ (\phi_2 + \phi)(t=0) & = 0 \quad \text{in } \Omega, \\ (\mathbf{u}_2 + \mathbf{u})(t=0) & = 0 \quad \text{in } \Omega, \\ \mathbf{u}_2 + \mathbf{u} & = 0 \quad \text{on } S_B \times (0,T). \end{cases} \tag{58}$$

Now we must write the viscoelastic part of the system

$$\begin{aligned} \boldsymbol{\sigma} + \mathrm{We}\left(\partial_t \boldsymbol{\sigma} - g_a(\boldsymbol{\nabla} \mathbf{u}, \boldsymbol{\sigma}) - g_a(\boldsymbol{\nabla} \mathbf{u}_1, \boldsymbol{\sigma}) - g_a(\boldsymbol{\nabla} \mathbf{u}, \boldsymbol{\sigma}_1)\right) - 2\varepsilon \mathbf{D}[\mathbf{u}] = \\ = \mathbf{m} + 2\varepsilon \mathbf{D}[\mathbf{u}_2] + \mathrm{We}\left(g_a(\boldsymbol{\nabla} \mathbf{u}_2, \boldsymbol{\sigma}) + g_a(\boldsymbol{\nabla} \mathbf{u}_2, \boldsymbol{\sigma}_1)\right) \text{in } \Omega \times (0,T), \end{aligned}$$

that can be rewritten

$$\boldsymbol{\sigma}+\mathrm{We}\,(\partial_t\boldsymbol{\sigma} - g_a(\boldsymbol{\nabla}\mathbf{u}_2 + \boldsymbol{\nabla}\mathbf{u}, \boldsymbol{\sigma}) - g_a(\boldsymbol{\nabla}\mathbf{u}_1, \boldsymbol{\sigma}) - g_a(\boldsymbol{\nabla}\mathbf{u}_2 + \boldsymbol{\nabla}\mathbf{u}, \boldsymbol{\sigma}_1))-2\varepsilon\mathbf{D}[\mathbf{u}_2+\mathbf{u}] = \mathbf{m}. \quad (59)$$

Equations (58) and (59) can be rewritten

$$P_2[\mathbf{u}_1, \boldsymbol{\sigma}_1](\mathbf{u}_2 + \mathbf{u}, q_2 + q, \phi_2 + \phi, \boldsymbol{\sigma}) = (\mathbf{f}, a, \mathbf{m}, g, k, 0, 0),$$

which is exactly (57) in Theorem 4.1 for $(\mathbf{u}_2 + \mathbf{u}, q_2 + q, \phi_2 + \phi, \boldsymbol{\sigma}) \in X_T^r$.
Uniqueness of the solution and boundedness of this operator $P_2^{-1}[\mathbf{u}_1, \boldsymbol{\sigma}_1]$ may be proved in the same way as for the operator P_1^{-1} and even its continuity (it is nonlinear). The proof relies on the boundedness of $\mathbf{u}_1, \boldsymbol{\sigma}_1$ in the appropriate spaces.
This completes the proof of Theorem 4.1.

□

5 Estimates of the error terms

In (60) we expand P in the neighborhood of $\boldsymbol{\xi} \equiv 0$. If we keep only zeroth order terms of $\boldsymbol{\xi}$ for $P(0,0,0,0,0)$ and P_1, we put the "error" terms in E. Thanks to the fact that ϕ is initially vanishing (which is not the case for the other fields like the velocity), we linearize in ϕ, put apart zeroth order terms in $P(0,0,0,0,0)$, put the linear (in ϕ) term in $P_1(\mathbf{u}, q, \phi, \boldsymbol{\sigma})$ and higher order terms in E. The operator P_1 is nonlinear because of our nonlinear constitutive law. So, one may write the full system of equations to be solved:

$$\begin{aligned} P(\boldsymbol{\xi}, \mathbf{u}, q, \phi, \boldsymbol{\sigma}) &= \overbrace{P(0,0,0,0,0)}^{\text{Order 0}} + P_1(\mathbf{u}, q, \phi, \boldsymbol{\sigma}) + \overbrace{E(\boldsymbol{\xi}, \mathbf{u}, q, \phi, \boldsymbol{\sigma})}^{O(\boldsymbol{\xi})+O(\phi^2)} \\ &= (0,0,0,0,0,\mathbf{u}_0, \boldsymbol{\sigma}_0), \end{aligned} \quad (60)$$

where the E terms are fully written below with usual summation convention and $\overline{\xi}_{ij} = \delta_{ij} + \xi_{ij}$:

$$\begin{aligned} E_i^1(\boldsymbol{\xi}, \mathbf{u}, q, \phi, \boldsymbol{\sigma}) &= -(1-\varepsilon)\xi_{kj}\partial_k[\overline{\xi}_{lj}u_{i,l}] - (1-\varepsilon)\partial_j(\xi_{lj}u_{i,l}) + \xi_{ki}\partial_k q - \sigma_{ij,k}\xi_{kj} \\ E^2(\boldsymbol{\xi}, \mathbf{u}, q, \phi, \boldsymbol{\sigma}) &= \xi_{kj}u_{j,k} \\ E_{ij}^3(\boldsymbol{\xi}, \mathbf{u}, q, \phi, \boldsymbol{\sigma}) &= -\mathrm{We}\left[\frac{(a-1)}{2}\left(\xi_{li}u_{k,l}\sigma_{kj} + \sigma_{ik}u_{k,l}\xi_{lj}\right) + \right. \\ &\qquad \left. + \frac{(a+1)}{2}\left(\sigma_{ik}\xi_{lk}u_{j,l} + u_{i,l}\xi_{lk}\sigma_{kj}\right)\right] - \varepsilon(u_{i,k}\xi_{kj} + u_{j,k}\xi_{ki}) \\ E_i^4(\boldsymbol{\xi}, \mathbf{u}, q, \phi, \boldsymbol{\sigma}) &= -q(\mathcal{N}_i - N_i) + (1-\varepsilon)\left[\xi_{kj}u_{i,k} + \xi_{ki}u_{j,k}\right]N_j + \\ &\quad +(1-\varepsilon)\left[\overline{\xi}_{kj}u_{i,k} + \overline{\xi}_{ki}u_{j,k}\right](\mathcal{N}_j - N_j) + \\ &\quad +g\zeta(X_1)(\mathcal{N}_i - N_i) + g\eta_2\mathcal{N}_i - \alpha\partial_{\mathbf{T}}Q_i + \sigma_{ij}(\mathcal{N}_j - N_j) \\ \text{where } Q_1 &= (1+h'^2)^{\frac{-1}{2}}\left[\left(1 + \frac{2\zeta'\phi + \phi^2}{1+h'^2}\right)^{\frac{-1}{2}} - 1 + \frac{\zeta'\phi}{1+h'^2}\right] \\ \text{and } Q_2 &= -\zeta'\phi^2(1+h'^2)^{\frac{-3}{2}} + (\phi + \zeta')Q_1 \\ E^5(\boldsymbol{\xi}, \mathbf{u}, q, \phi, \boldsymbol{\sigma}) &= -\partial_{\mathbf{T}}(\mathbf{u}) \cdot \mathbf{N}\left[(N_2 + \partial_{\mathbf{T}}\eta_1)^{-2} - N_2^{-2}\right] - \\ &\qquad (N_2 + \partial_{\mathbf{T}}\eta_1)^{-2}\left(\partial_{\mathbf{T}}u_2\partial_{\mathbf{T}}\eta_1 - \partial_{\mathbf{T}}u_1\partial_{\mathbf{T}}\eta_2\right) \\ E^6(\boldsymbol{\xi}, \mathbf{u}, q, \phi, \boldsymbol{\sigma}) &= 0 = E^7(\boldsymbol{\xi}, \mathbf{u}, q, \phi, \boldsymbol{\sigma}). \end{aligned} \quad (61)$$

Lastly, we will lift the initial conditions with $(\mathbf{u}_1, q_1, \phi_1, \boldsymbol{\sigma}_1)$ in $X^r_{T_0}$. We will also use the space of initially vanishing fields $X^*_T = \{(\mathbf{u}, q, \phi, \boldsymbol{\sigma}) \in X^r_T / \mathbf{u}(0) = 0, \boldsymbol{\sigma}(0) = 0\}$. We will denote $B_{X^*_T}(0, R)$ the ball of center 0 and of radius R in X^*_T.
Actually we need to bound the "error" terms. This is done in the following theorem.

Theorem 5.1. *Let $0 < r < 1/2$, and $(\mathbf{u}_1, q_1, \phi_1, \boldsymbol{\sigma}_1) \in B_{X^r_{T_0}}(0, R)$. There exists $\epsilon' > 0$ and $0 < T'_0 \leq T_0$ depending on $(\mathbf{u}_1, q_1, \phi_1, \boldsymbol{\sigma}_1)$ and R, such that if $0 < T < T'_0$ and $(\mathbf{u}, q, \phi, \boldsymbol{\sigma}) \in B_{X^*_T}(0, R)$, then $E(\mathbf{u}_1 + \mathbf{u}, q_1 + q, \phi_1 + \phi, \boldsymbol{\sigma}_1 + \boldsymbol{\sigma})$ is in the space $Y^r_T(\Omega)$, and the following holds:*

$$| E^i(\mathbf{u}_1 + \mathbf{u}, q_1 + q, \phi_1 + \phi, \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}) |_{(Y^r_T)_i} \leq CT^{\epsilon'}. \tag{62}$$

If in addition $(\mathbf{u}', q', \phi', \boldsymbol{\sigma}') \in B_{X^*_T}(0, R)$, the operator E is contracting:

$$\begin{aligned} | E(\mathbf{u}_1 + \mathbf{u}, q_1 + q, \phi_1 + \phi, \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}) - E(\mathbf{u}_1 + \mathbf{u}', q_1 + q', \phi_1 + \phi', \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}') |_{Y^r_T} \\ \leq CT^{\epsilon'} | \mathbf{u} - \mathbf{u}', q - q', \phi - \phi', \boldsymbol{\sigma} - \boldsymbol{\sigma}' |_{X^r_T}, \end{aligned} \tag{63}$$

with a constant C that depends on $\varepsilon, a, We, r, R, (\mathbf{u}_1, q_1, \phi_1, \boldsymbol{\sigma}_1)$, but not on T provided $T \leq T'_0$.

We provide the proof for the first and third components more specific to viscoelastic fluids. In the fourth component (E^4), we study only the term added by viscoelasticity of the fluid. The other components (E^2, E^4, E^5) are studied in [4] or in [6].

5.1 Various lemmas

We want to give a meaning to the fact that $\boldsymbol{\xi}$ is small in small time (used in Theorem 5.1). So we need some lemmas stated hereafter.

Lemma 5.2. *Let $0 < T \leq T_0, 0 \leq s < \frac{1}{2}$, $0 \leq \epsilon' \leq s$ and X be a Hilbert space. The linear mapping $v \mapsto V(t) = \int_0^t v(s)\,\mathrm{d}s$, is a bounded operator from $H^s(0, T; X)$ to $H^{s+1-\epsilon'}(0, T; X)$ and*

$$| V |_{s+1-\epsilon'} \leq CT^{\epsilon'} | v |_s,$$

with C independent of $0 < T \leq T_0$.

Proof. The proof is through double interpolation. For $s = 0$, Cauchy-Schwarz inequality gives;

$$| V |_X (t) \leq t^{\frac{1}{2}} | v |_{H^0(0,T;X)} \Rightarrow | V |_0 \leq CT | v |_0,$$

where $|.|_0 = |.|_{H^0(0,T;X)}$. So $| V |_1 \leq (1 + CT^2) | v |_0 \leq C | v |_0$ where $|.|_1 = |.|_{H^1(0,T;X)}$. Then, by a classical interpolation inequality ($1 - \epsilon' \geq 0$), the result is obvious for $s = 0$:

$$| V |_{1-\epsilon'} \leq C | V |_1^{1-\epsilon'} | V |_0^{\epsilon'} \leq CT^{\epsilon'} | v |_0 .$$

In the same way if $v \in H^1(0, T)$

$$| V |_1^2 = | v |_0^2 + | V |_0^2 \leq C | v |_0^2 \quad \text{if } T \leq T_0.$$

Still if $v \in H^1 : \mid V \mid_2 \leq C \mid v \mid_1$. Moreover if $v(0) = 0$ (this assumption is discussed below), since v is the integral of $\partial_t v$, then $\mid v \mid_0 \leq CT \mid v' \mid_0$ and $|V|_1 \leq CT|v|_1$. The same interpolation inequality provides:

$$\mid V \mid_{2-\epsilon'} \leq C \mid V \mid_2^{1-\epsilon'} \mid V \mid_1^{\epsilon'} \leq CT^{\epsilon'} \mid v \mid_1^{\epsilon'} \mid v \mid_1^{1-\epsilon'} \leq CT^{\epsilon'} \mid v \mid_1 .$$

By re-interpolating between the two inequalities, one completes the proof.
Notice that the assumption $v(0) = 0$ disappears since we restrict the regularity to $s < 1/2$. Such an assumption is meaningless for non-regular functions. So this assumption, also done by J.T. Beale (in his Lemma 2.4), does not limit the proof.

□

Let us remind that we define ${}_0H^{-1}(\Omega)$ as the dual space of ${}^0H^1(\Omega) = \{p \in H^1(\Omega),\ p = 0 \text{ on } S_F\}$. This space is needed for the incompressibility condition. We may then state our next lemma.

Lemma 5.3. *Let Ω be an open subset of $\mathbb{R}^2$.*

(i) Let $r > 1, r \geq s \geq 0$, $v \in H^r(\Omega)$ and $w \in H^s(\Omega)$, then $vw \in H^s(\Omega)$ and

$$\mid vw \mid_s \leq C \mid v \mid_r \mid w \mid_s$$

(ii) Let $v \in H^r(\Omega)$ for $r > 1$, and $w \in {}_0H^{-1}(\Omega)$, then $vw \in {}_0H^{-1}(\Omega)$ and :

$$\mid vw \mid_{-1} \leq C \mid v \mid_r \mid w \mid_{-1}$$

(iii) Let $v, w \in H^1(\Omega)$, then $vw \in L^2(\Omega)$ and

$$\mid vw \mid_0 \leq C \mid v \mid_1 \mid w \mid_1$$

(iv) Let $v \in H^1(\Omega)$ and $w \in L^2(\Omega)$, then $vw \in {}_0H^{-1}(\Omega)$ and

$$\mid vw \mid_{-1} \leq C \mid v \mid_1 \mid w \mid_0$$

This lemma is proved in dimension 3 by J.T. Beale (Lemma 2.5 p. 366 of [6]). The proof in 2-D is very similar. Following J.T. Beale, we state below that the product of two functions in appropriate spaces is a continuous map.

Lemma 5.4. *Let X, Y, Z denote three Hilbert spaces and $M : X \times Y \to Z$, a bounded and bilinear map ("multiplication").*

(i) Suppose $u \in H^s(0, T; X)$ and $v \in H^s(0, T; Y)$ where $s > 1/2$ then $uv = M(u, v) \in H^s(0, T; Z)$ and $\mid uv \mid_s \leq C \mid u \mid_s \mid v \mid_s$.

(ii) If $s \leq 2$ and u, v satisfy in addition to (i) the conditions $\partial_t^k u(0) = 0 = \partial_t^k v(0)$ for $0 \leq k < s - 1/2$ and $s - 1/2$ is not an integer. Then the constant C of (i) does not depend on $T < T_0$.

This lemma is proved p. 366 of [6] (Lemma 2.6) in two lines. The proof is based on an extension to $t \in \mathbb{R}$ and estimates of the transforms of (i). Lemma 3.5 enables to prove (ii).

We will also need the following lemma that ensures there exists a bound independent of $T < T_0$ for fields whose initial value is not zero.

Lemma 5.5. *Let $(\mathbf{u}_1, q_1, \phi_1, \boldsymbol{\sigma}_1) \in X^r_{T_0}$ $(r < 1/2)$ and $(\mathbf{u}, q, \phi, \boldsymbol{\sigma})$ in a ball of radius R in X^r_T with $\mathbf{u}(0) = 0$, $\boldsymbol{\sigma}(0) = 0$ (so in X^*_T). Then the following holds with constants C that depend on $R, \mathbf{u}_1, q_1, \phi_1, \boldsymbol{\sigma}_1, r, \Omega$, but not on $T < T_0$:*

$$| \partial_k \partial_j (\mathbf{u}_1 + \mathbf{u}) |_{K^r(0,T;\Omega)} \leq C \tag{64}$$

$$| \partial_k (\mathbf{u}_1 + \mathbf{u}) |_{K^{r+1}(0,T;\Omega)} \leq C \tag{65}$$

$$| \boldsymbol{\nabla}(q_1 + q) |_{K^r(0,T;\Omega)} \leq C \tag{66}$$

$$| (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma})_{ij,k} |_{K^r(0,T;\Omega)} \leq C \tag{67}$$

The proof relies on two arguments. On the one hand $(\mathbf{u}_1, q_1, \phi_1, \boldsymbol{\sigma}_1)$ does not need to be extended until T_0. On the other hand $\mathbf{u}, q, \phi, \boldsymbol{\sigma}$ are either initially vanishing (for $\mathbf{u}$ and $\boldsymbol{\sigma}$) and so can be extended on $(0, T_0)$ with a bound independent of $T < T_0$, or not very regular (p and ϕ) and so the extension on $(0, T_0)$ does not generate any cost nor need any extra assumption. At that level, we see that improving to $r > 1/2$ our result would modify the whole proof.

Proof. Concerning (64), Lemma 2.1 of [6] (which deals with the boundedness of the derivative and trace operators) and our Lemma 3.2 (which deals with the extension operator from $(0, T)$ to $(0, +\infty)$) enable to state:

$$\begin{aligned} | \partial_k \partial_j (\mathbf{u}_1 + \mathbf{u}) |_{K^r(0,T;\Omega)} &\leq | \partial_k \partial_j \mathbf{u}_1 |_{K^r(0,T_0;\Omega)} + | \partial_k \partial_j \mathbf{u} |_{K^r(0,T;\Omega)} \\ &\leq C | \mathbf{u}_1 |_{K^{r+2}(0,T_0;\Omega)} + C | \mathbf{u} |_{L^2(0,T;H^{2+r}) \bigcap H^{r/2}(0,T;H^2)}, \\ &\leq C | \mathbf{u}_1 |_{K^{r+2}(0,T_0;\Omega)} + C | \mathbf{u} |_{K^{r+2}(0,T_0;\Omega)}, \\ &\leq C. \end{aligned}$$

One may prove in the same way (65).
Concerning the pressure, since $r < 1/2$, we do not need to force unphysical vanishing initial value to have the result thanks to Lemma 2.1 of [6].
The extra stress bound can be written:

$$| (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma})_{,j} |_{K^r} \leq | \boldsymbol{\sigma}_{1,j} |_{K^r(0,T_0)} + | \boldsymbol{\sigma}_{,j} |_{K^r(0,T)} \leq C(T_0) | \boldsymbol{\sigma}_1 |_{K^{r+1}(0,T_0)} + C | \boldsymbol{\sigma} |_{K^{r+1}(0,T_0)} \leq C.$$

□

The following lemma gives a sense to the claim that "$\boldsymbol{\xi}$ is small".

Lemma 5.6. *Let r be such that $0 < r < 1/2$, and let us denote $\mathcal{A}$ the algebra*

$$\mathcal{A} = H^{1+\frac{r}{8}}(0, T; H^{1+\frac{r}{2}}(\Omega)). \tag{68}$$

For any $\mathbf{u}_1 \in K^{2+r}(0,T_0), \mathbf{u}, \mathbf{u}' \in K^{2+r}(0,T)$, $\mathbf{u}(t=0) = 0 = \mathbf{u}'(t=0)$, *and* $\boldsymbol{\xi}$ *defined by* $\boldsymbol{\xi}(\mathbf{u}) = (\mathbf{I} + d\boldsymbol{\eta})^{-1} - \mathbf{I} = (\mathbf{I} + \int_0^t \boldsymbol{\nabla}\mathbf{u})^{-1} - \mathbf{I}$, *there exists* $\epsilon' > 0$ *and constants* C *such that if* $T < T_0$*;*

$$\begin{aligned} \mid \boldsymbol{\xi}(\mathbf{u}_1 + \mathbf{u}) \mid_{H^{\frac{1+r}{2}}(0,T;H^{1+r/2})} \leq \mid \boldsymbol{\xi}(\mathbf{u}_1 + \mathbf{u}) \mid_{\mathcal{A}} &\leq CT^{\epsilon'} \qquad (69) \\ \mid \boldsymbol{\xi}(\mathbf{u}_1 + \mathbf{u}) - \boldsymbol{\xi}(\mathbf{u}_1 + \mathbf{u}') \mid_{H^{\frac{1+r}{2}}(0,T;H^{1+r/2})} \leq \mid \boldsymbol{\xi}(\mathbf{u}_1 + \mathbf{u}) - \boldsymbol{\xi}(\mathbf{u}_1 + \mathbf{u}') \mid_{\mathcal{A}} &\leq CT^{\epsilon'} \mid \mathbf{u} - \mathbf{u}' \mid_{K^{r+2}} \qquad (70) \end{aligned}$$

with C*, dependent on* $\mathbf{u}_1, r, T_0$ *but not on* T *provided* $T < T_0$*.*

The following Remark must be taken into account before we prove this lemma.

Remark 5.7. *The continuity bound of the product in the algebra* $\mathcal{A}$ *depends on* $T < T_0$ *if there is no more condition. If we add the condition that the fields are vanishing initially, Lemma 5.4 (proved with an extension operator whose properties are given in Lemma 3.2) may ensure that the bound does not depend on* $T < T_0$*, but what happens if this assumption is not satisfied ?*
To answer this question, let us write the continuity of the product of two functions that do not depend on time and so do not vanish at $t = 0$*:*

$$\mid 1 \times 1 \mid_{H^{1+\frac{r}{8}}(0,T)} \leq C(\mathcal{A}) \mid 1 \mid_{H^{1+\frac{r}{8}}(0,T)} \mid 1 \mid_{H^{1+\frac{r}{8}}(0,T)} \Rightarrow 1 \leq C(\mathcal{A}) \mid 1 \mid_{H^{1+\frac{r}{8}}(0,T)} .$$

So the constant not only depends on T*, but even tends to* $+\infty$ *when* $T \to 0$*.*
One might wonder whether the Lemma 3.2, needed in the proof of Lemma 5.4, uses a too specific extension which could be improved. Of course, the extension used will not work for instance on constant functions which will not remain in any $H^s(0,\infty)$*. But the above inequality proves that no other extension operator could suit. This explains why in Remark 3.3, we claimed we were forced to lift initial conditions so as to have new initially vanishing fields.*

Proof. We denote $\boldsymbol{\xi} := \boldsymbol{\xi}(\mathbf{u}_1+\mathbf{u})$, $\boldsymbol{\xi}' := \boldsymbol{\xi}(\mathbf{u}_1+\mathbf{u}')$ and $\boldsymbol{\eta}, \boldsymbol{\eta}'$ the associated fields respectively. We repeatedly use the fact that $\mathcal{A}$ is an algebra and that $\mathbf{u}$ is in K^{2+r} and so also in $H^{\frac{1+r}{2}}(0,T;H^1)$. So $\boldsymbol{\xi} \in H^1(0,T;H^{1+r}) \bigcap H^{\frac{3+r}{2}}(0,T;L^2)$.
Let us denote $\mathcal{E}$ the extension to $(0,+\infty)$ operator and $\mathcal{R}$ the restriction to $(0,T)$ operator. From Lemma 3.2 we know that $\mathcal{E}$ is bounded independently of $T < T_0$ for convenient fields. So:

$$\begin{aligned} \mid \mathrm{d}\boldsymbol{\eta}\mathrm{d}\boldsymbol{\eta}' \mid_{H^{1+\frac{r}{8}}(0,T;H^{1+r/2})} &= \mid \mathcal{R}\mathcal{E}\mathrm{d}\boldsymbol{\eta}\mathcal{R}\mathcal{E}\mathrm{d}\boldsymbol{\eta}' \mid_{H^{1+\frac{r}{8}}(0,T;H^{1+r/2})} \\ &\leq C \mid \mathcal{R}\mathcal{E}\mathrm{d}\boldsymbol{\eta} \mid_{H^{1+\frac{r}{8}}(0,T;H^{1+r/2})} \mid \mathcal{R}\mathcal{E}\mathrm{d}\boldsymbol{\eta}' \mid_{H^{1+\frac{r}{8}}(0,T;H^{1+r/2})} \\ &\leq C \mid \mathcal{E}\mathrm{d}\boldsymbol{\eta} \mid_{H^{1+\frac{r}{8}}(\mathbb{R};H^{1+r/2})} \mid \mathcal{E}\mathrm{d}\boldsymbol{\eta}' \mid_{H^{1+\frac{r}{8}}(\mathbb{R};H^{1+r/2})} \\ &\leq C \mid \mathrm{d}\boldsymbol{\eta} \mid_{\mathcal{A}} \mid \mathrm{d}\boldsymbol{\eta}' \mid_{\mathcal{A}}, \end{aligned}$$

where C does not depend on $T < T_0$. As a consequence there exists α_1 such that if $\mid \mathrm{d}\boldsymbol{\eta} \mid_{\mathcal{A}} < \alpha_1$, then, $\boldsymbol{\xi} = (\mathbf{I} + \mathrm{d}\boldsymbol{\eta})^{-1} - \mathbf{I}$ is in $\mathcal{A}$ and it can be expanded in series with:

$$\mid \boldsymbol{\xi} \mid_{\mathcal{A}} \leq C \mid \mathrm{d}\boldsymbol{\eta} \mid_{\mathcal{A}} . \qquad (71)$$

Since $\mathrm{d}\boldsymbol{\eta}(X,t) = \displaystyle\int_0^t \boldsymbol{\nabla}(\mathbf{u}_1 + \mathbf{u})$, $\mathbf{u}_1 \in K^{r+2}(0,T_0)$ and the fact that $\mathbf{u}$ is only in $K^{r+2}(0,T)$ but with vanishing initial condition, Lemmas 5.5 and 5.2 enable to state the announced result (69):

$$| \boldsymbol{\xi} |_{\mathcal{A}} \leq C \mid \mathrm{d}\boldsymbol{\eta} \mid_{H^{1+r/8}(0,T;H^{1+r/2})} \leq CT^{\epsilon'} \mid \boldsymbol{\nabla}(\mathbf{u}_1 + \mathbf{u}) \mid_{H^{\frac{r}{16}}(0,T;H^{1+r/2})} \leq CT^{\epsilon'}.$$

Moreover, since $\mathcal{A}$ is an algebra, if $\mid \mathrm{d}\boldsymbol{\eta} \mid < \alpha_1/2$ and the same for $\mathrm{d}\boldsymbol{\eta}'$;

$$\boldsymbol{\xi} - \boldsymbol{\xi}' = (\mathbf{I} + \boldsymbol{\xi})\mathrm{d}(\boldsymbol{\eta}' - \boldsymbol{\eta})(\mathbf{I} + \boldsymbol{\xi}') = \mathrm{d}(\boldsymbol{\eta} - \boldsymbol{\eta}') + \boldsymbol{\xi}\mathrm{d}(\boldsymbol{\eta} - \boldsymbol{\eta}') + \mathrm{d}(\boldsymbol{\eta} - \boldsymbol{\eta}')\boldsymbol{\xi}' + \boldsymbol{\xi}\mathrm{d}(\boldsymbol{\eta} - \boldsymbol{\eta}')\boldsymbol{\xi}'.$$

Here, all the functions are initially vanishing. So, in the same way as for (71);

$$| \boldsymbol{\xi} - \boldsymbol{\xi}' |_{\mathcal{A}} \leq C \mid \mathrm{d}(\boldsymbol{\eta} - \boldsymbol{\eta}') \mid_{\mathcal{A}} . \tag{72}$$

One may then prove (70) in the same way as (69). □

The previous lemma will not be sufficient in some estimates. So we state the following lemma which will turn to be useful.

Lemma 5.8. *If the velocity* $\mathbf{u}$ *is in* $L^2(0,T;H^{2+r})$*, then the field*

$$\boldsymbol{\xi} = (\mathbf{Id} + \mathrm{d}\boldsymbol{\eta})^{-1} - \mathbf{Id} = (\mathbf{Id} + \int_0^t \boldsymbol{\nabla}\mathbf{u})^{-1} - \mathbf{Id}$$

is in $H^1(0,T;H^{1+r})$ *with*

$$| \boldsymbol{\xi} |_{H^1(0,T;H^{1+r})} \leq C.$$

Moreover, for $0 < r < 1$*, we define the algebra*

$$\mathcal{A}' = H^{\frac{1+r}{2}}(0,T;H^{1+r}(\Omega)), \tag{73}$$

for which

$$| \boldsymbol{\xi} |_{\mathcal{A}'} \leq CT^{\epsilon'}.$$

The proof is similar to the one of Lemma 5.6 for the first estimate. Then one must use Lemma 3.4 to bound $| \boldsymbol{\xi} |_{\mathcal{A}'}$ with a $CT^{\epsilon'}$ and $| \boldsymbol{\xi} |_{H^1(0,T;H^{1+r})}$. The rest is left to the reader.
In the proof of estimates in $L^2(0,T;H^s)$ below, we will use L^∞_t estimates and then $H^{\frac{1+r}{2}}(0,T) \hookrightarrow L^\infty(0,T)$. The constants in Sobolev's inequality must be independent of $T < T_0$. In the general case, it is wrong but for the subspace of initially vanishing functions, it is true as states the following lemma.

Lemma 5.9. *Let* $0 < r \leq 1$*. If* $v \in H^{\frac{1+r}{2}}(0,T)$ *and is initially vanishing then*

$$| v |_{L^\infty(0,T)} \leq C \mid v \mid_{H^{\frac{1+r}{2}}(0,T)}, \tag{74}$$

and the constant C *does not depend on* $T < T_0$*.*

Proof. The proof of $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ is classical. Then, thanks to the properties of the extension operator $\mathcal{E}$ in H^1 (Lemma 3.2 (*ii*)),

$$| v |_{L^\infty(0,T)} \leq | \mathcal{E}(v) |_{L^\infty(\mathbb{R})} \leq | \mathcal{E}(v) |_{H^1(\mathbb{R})} \leq C \mid v \mid_{H^1(0,T)},$$

with C independent of $T < T_0$.
The same can be proved for non-integer Sobolev spaces and completes the proof. □

For the estimates in $H^{\frac{r}{2}}(0,T;L^2)$, we will need a more precise result than Lemma 5.4 to estimate the product of two functions.

Lemma 5.10. *Let X, Y, Z denote three Hilbert spaces and $M : X \times Y \to Z$, a bounded and bilinear map ("multiplication"). Let $1/2 < s < 3/2$ and $0 \leq s' \leq s$.*

(i) Suppose $u \in H^s(0,T;X)$ and $v \in H^{s'}(0,T;Y)$ then $uv := M(u,v) \in H^{s'}(0,T;Z)$ and $\mid uv \mid_{s'} \leq C \mid u \mid_s \mid v \mid_{s'}$.

(ii) Let u, v satisfy the conditions of (i) and in addition $u(t=0) = 0$ and $s' < 1/2$. Then there exists a constant C in (i) that does not depend on $T < T_0$.

Proof. Let us consider the functional $v \mapsto uv$. For $u \in H^s(0,T;X)$, this function can be defined in $H^0(0,T;Y)$ and in $H^s(0,T;Y)$. It satisfies:

$$\begin{array}{ll} \mid uv \mid_{H^0(0,T;Z)} \leq & C \mid u \mid_{H^s(0,T;X)} \mid v \mid_{H^0(0,T;Y)}, \\ \mid uv \mid_{H^s(0,T;Z)} \leq & C \mid u \mid_{H^s(0,T;X)} \mid v \mid_{H^s(0,T;Y)} . \end{array}$$

Then a simple interpolation provides the result (i) for s' between 0 and s.

Then the extension operator enables to exhibit C independent of $T < T_0$:

$$\mid uv \mid_{H^{s'}(0,T)} \leq \mid \mathcal{E}(u)\mathcal{E}(v) \mid_{H^{s'}(\mathbb{R})} \leq C \mid \mathcal{E}(u) \mid_{H^s(\mathbb{R})} \mid \mathcal{E}(v) \mid_{H^{s'}(\mathbb{R})} \leq C \mid u \mid_{H^s(0,T)} \mid v \mid_{H^{s'}(0,T)} .$$

□

5.2 Estimates on E^1

The terms to be estimated are in (61).
For the Navier-Stokes equations, G. Allain [4] sends back to [3] where she indicates the tools to get accurate estimates and refers to J.T. Beale's article [6] for details.
We denote $\xi_{kj}(\mathbf{u}_1 + \mathbf{u}) = \xi_{kj} = ((\mathbf{Id} + \mathrm{d}\boldsymbol{\eta})^{-1} - \mathbf{Id})_{kj}$ because it depends on the velocity $\mathbf{u}_1 + \mathbf{u}$ through $\boldsymbol{\eta}$. For the sake of completeness we consider in detail the first term of E^1 : $\xi_{kj}(\mathbf{u}_1 + \mathbf{u})\partial_k[\overline{\xi}_{lj}(\mathbf{u}_1 + \mathbf{u}) \times (\mathbf{u}_1 + \mathbf{u})_{i,l}]$. We prove it is bounded in $K^r(0,T)$ by $CT^{\epsilon'}$ (see (62)) and contracting (cf (63)).
The main difficulty here is that the constants for the embedding ($H^{\frac{1+r}{2}}(0,T) \hookrightarrow L^\infty(0,T)$) or the property of algebra ($\mid fg \mid_{H^s} \leq C \mid f \mid_{H^{\frac{1+r}{2}}} \mid g \mid_{H^s}$ for $0 \leq s < 1/2$) tend to infinity when T tends to 0 in the general case. Yet, since the fields involved are initially vanishing, we can use our more refined lemmas proved above.
Moreover, for the reader not familiar with the K^r spaces, we prefer to split the estimates in $L^2(0,T;H^r)$ and $H^{\frac{r}{2}}(0,T;L^2(\Omega))$.
Below, all the constants C are independent of $T < T_0$.
<u>In $L^2(0,T;H^r)$.</u>

Here, we can use the algebra $L^\infty(0,T;H^{1+r})$. Then, denoting $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{u}_1+\mathbf{u})$,

$$\begin{aligned} \mid \xi_{kj}\partial_k[\overline{\xi}_{lj}\times(\mathbf{u}_1+\mathbf{u})_{i,l}]\mid_{L^2(0,T;H^r)} &\leq C\mid \xi_{kj}\mid_{L^\infty(0,T;H^{1+r})}\mid \partial_k[\overline{\xi}_{lj}\times(\mathbf{u}_1+\mathbf{u})_{i,l}]\mid_{L^2(0,T;H^r)} \\ &\leq C\mid \xi_{kj}\mid_{H^{\frac{1+r}{2}}(0,T;H^{1+r})}\mid \overline{\xi}_{lj}\times(\mathbf{u}_1+\mathbf{u})_{i,l}\mid_{L^2(0,T;H^{1+r})}, \end{aligned}$$

thanks to Lemma 5.9. Then, one may use Lemma 5.8 and write $\overline{\boldsymbol{\xi}} = \mathbf{Id}+\boldsymbol{\xi}$ to pursue the bound:

$$\leq CT^{\epsilon'}\left(\mid(\mathbf{u}_1+\mathbf{u})_{i,l})\mid_{L^2(0,T;H^{1+r})} + \mid \xi_{lj}\times(\mathbf{u}_1+\mathbf{u})_{i,l})\mid_{L^2(0,T;H^{1+r})}\right).$$

Then one has, with arguments similar to above and with $\xi_{lj}(t=0)=0$:

$$\begin{aligned} &\leq CT^{\epsilon'}\left(\mid \mathbf{u}_1+\mathbf{u}\mid_{L^2(0,T;H^{2+r})} + CT^{\epsilon'}\mid \mathbf{u}_1+\mathbf{u}\mid_{L^2(0,T;H^{2+r})}\right) \\ &\leq CT^{\epsilon'}, \end{aligned}$$

where C does depend on R but not on $T<T_0$.
<u>In $H^{\frac{r}{2}}(0,T;L^2)$.</u>

We can no more take off the $\overline{\boldsymbol{\xi}}$ term by a simple L^∞ bound. So we need algebra properties (in time) and so Lemma 5.10 which is a more refined version of Lemma 5.4. Indeed the latter is useless in $H^{\frac{r}{2}}(0,T;L^2)$ since $r/2<1/2$ and so $H^{r/2}$ is not an algebra.
Because of Lemma 5.10 and $\mathcal{A} = H^{1+\frac{r}{8}}(0,T;H^{1+\frac{r}{2}})$,

$$\begin{aligned} \mid \xi_{kj}\partial_k[\overline{\xi}_{lj}\times(\mathbf{u}_1+\mathbf{u})_{i,l}]\mid_{H^{\frac{r}{2}}(0,T;L^2)} &\leq C\mid \xi_{kj}\mid_{\mathcal{A}}\mid \partial_k[\overline{\xi}_{lj}\times(\mathbf{u}_1+\mathbf{u})_{i,l}]\mid_{H^{r/2}(0,T;L^2)} \\ &\leq CT^{\epsilon'}\mid \overline{\xi}_{lj}\times(\mathbf{u}_1+\mathbf{u})_{i,l}\mid_{H^{r/2}(0,T;H^1)}. \end{aligned}$$

The last inequality uses Lemma 5.6. Then a simple decomposition of $\overline{\boldsymbol{\xi}} = \mathbf{Id}+\boldsymbol{\xi}$ enables to have a $(\mathbf{u}_1+\mathbf{u})_{i,l}$ term not initially vanishing but linear, and a $\boldsymbol{\xi}(\mathbf{u}_1+\mathbf{u})(\mathbf{u}_1+\mathbf{u})_{i,l}$ term which is nonlinear but initially vanishing:

$$\leq CT^{\epsilon'}\left(\mid \mathbf{u}_1+\mathbf{u}\mid_{H^{r/2}(0,T;H^2)} + CT^{\epsilon'}\mid \mathbf{u}_1+\mathbf{u}\mid_{H^{r/2}(0,T;H^2)}\right).$$

The difference with the estimate in $L^2(0,T;H^r)$ is that the equivalent term was $\mid \mathbf{u}_1+\mathbf{u}\mid_{L^2(0,T;H^{2+r})}$. Such a term could be bounded by $\mid \mathbf{u}_1+\mathbf{u}\mid_{K^{2+r}(0,T)}$ with a constant equal to 1. But here we need to use Lemma 3.5 and especially its *(ii)* because we need constants independent of $T<T_0$. So we split $\mathbf{u}_1+\mathbf{u}$, use the fact that $\mathbf{u}_1\in K^{2+r}(0,T_0)$ (costless) and that $\mathbf{u}(t=0)=0$, so that $\mathbf{u}\in K^{2+r}(0,T)$ can be extended without any loss to $K^{2+r}(0,T_0)$. In summary, we have proved the following bound:

$$\mid \xi_{kj}(\mathbf{u}_1+\mathbf{u})\partial_k[\overline{\xi}_{lj}(\mathbf{u}_1+\mathbf{u})\times(\mathbf{u}_1+\mathbf{u})_{i,l}]\mid_{H^{\frac{r}{2}}(0,T;L^2)} \leq CT^{\epsilon'}.$$

The other terms are no more difficult. So (62) is established for E^1.

Remark 5.11. *The argument that $\boldsymbol{\eta}(\mathbf{u}_1+\mathbf{u}) = \int_0^t(\mathbf{u}_1+\mathbf{u})$ is initially vanishing and so that the embedding of $H^{\frac{1+r}{2}}$ in L^∞ has a constant independent of $T<T_0$ is not in p. 380 of [6]. We only added this and Lemmas 5.9 and 5.10 to complete the proof.*

One could prove in the same way that E^1 is lipschitz and so satisfies (63).

5.3 Estimates on E^3

We consider now the error terms of the constitutive equation in $K^{r+1}((0,T)\times\Omega)$:

$$\begin{aligned}&\mid E^3_{ij}(\boldsymbol{\xi}(\mathbf{u}_1+\mathbf{u}),\mathbf{u}_1+\mathbf{u},q_1+q,\phi_1+\phi,\boldsymbol{\sigma}_1+\boldsymbol{\sigma})\mid_{K^{r+1}}\leq\\ &C\mid \xi_{li}(\mathbf{u}_1+\mathbf{u})\times(\mathbf{u}_1+\mathbf{u})_{k,l}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma})_{kj}\mid_{K^{r+1}}+C\mid(\mathbf{u}_1+\mathbf{u})_{i,k}\xi_{kj}(\mathbf{u}_1+\mathbf{u})\mid_{K^{r+1}}.\end{aligned}$$

In order to treat the first term in K^{1+r}, we split K^{1+r} in $L^2(0,T;H^{1+r})$ and $H^{\frac{1+r}{2}}(0,T;L^2(\Omega))$. We apply Lemma 5.5, Lemma 5.8 and Lemma 5.9 to the first term in $L^2(0,T;H^{1+r})$:

$$\begin{aligned}&\mid \xi_{li}(\mathbf{u}_1+\mathbf{u})(\mathbf{u}_1+\mathbf{u})_{k,l}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma})_{kj}\mid_{L^2(0,T;H^{1+r})}\\ &\qquad\leq\mid \xi_{li}(\mathbf{u}_1+\mathbf{u})\mid_{\mathcal{A}'}\mid(\mathbf{u}_1+\mathbf{u})_{k,l}\mid_{L^2(0,T;H^{1+r})}\mid(\boldsymbol{\sigma}_1+\boldsymbol{\sigma})_{kj}\mid_{L^\infty(0,T;H^{1+r})}\\ &\qquad\leq CT^{\epsilon'},\end{aligned}$$

where $\mathcal{A}'$ is defined in (73). One may also bound the first term in $H^{\frac{1+r}{2}}(0,T;L^2)$. Indeed, Lemma 5.6 and Lemma 5.10 enable to write:

$$\begin{aligned}&\mid \xi_{li}(\mathbf{u}_1+\mathbf{u})(\mathbf{u}_1+\mathbf{u})_{k,l}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma})_{kj}\mid_{H^{\frac{1+r}{2}}(0,T;L^2)}\\ &\qquad\leq C\mid \xi_{li}(\mathbf{u}_1+\mathbf{u})\mid_{\mathcal{A}'}\times\\ &\qquad\quad\times(\mid(\mathbf{u}_1)_{k,l}\mid_{H^{\frac{1+r}{2}}(0,T_0;L^2)}+\mid\mathbf{u}_{k,l}\mid_{H^{\frac{1+r}{2}}(0,T;L^2)})\mid\boldsymbol{\sigma}_1+\boldsymbol{\sigma}\mid_{H^{\frac{1+r}{2}}(0,T;H^{1+r})}\\ &\qquad\leq CT^{\epsilon'}(\mid\mathbf{u}_1,q_1,\phi_1,\boldsymbol{\sigma}_1\mid_{X^r_{T_0}}+CR)^2\\ &\qquad\leq CT^{\epsilon'},\end{aligned}$$

with C depending on We, $a,\mathbf{u}_1,q_1,\phi_1,\boldsymbol{\sigma}_1,T_0$, but not on $T\leq T_0$. So we proved:

$$\mid \xi_{li}(\mathbf{u}_1+\mathbf{u})(\mathbf{u}_1+\mathbf{u})_{k,l}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma})_{kj}\mid_{K^{r+1}}\leq CT^{\epsilon'}.$$

In order to treat the second term, the algebra properties of $\mathcal{A}'=H^{\frac{1+r}{2}}(0,T;H^{1+r})$ give:

$$\mid(\mathbf{u}_1+\mathbf{u})_{i,k}\xi_{kj}(\mathbf{u}_1+\mathbf{u})\mid_{K^{r+1}}\leq CT^{\epsilon'}\mid\mathbf{u}_1+\mathbf{u}\mid_{K^{r+2}}\leq CT^{\epsilon'},$$

thanks to Lemma 5.5.

In order to prove the contracting property, we compute the difference:

$$\begin{aligned}&\mid E^3_{ij}(\boldsymbol{\xi}(\mathbf{u}_1+\mathbf{u}),\mathbf{u}_1+\mathbf{u},q_1+q,\phi_1+\phi,\boldsymbol{\sigma}_1+\boldsymbol{\sigma})-\\ &E^3_{ij}(\boldsymbol{\xi}(\mathbf{u}_1+\mathbf{u}'),\mathbf{u}_1+\mathbf{u}',q_1+q',\phi_1+\phi',\boldsymbol{\sigma}_1+\boldsymbol{\sigma}')\mid_{K^{r+1}}\leq\\ &4C\mid(\xi_{li}(\mathbf{u}_1+\mathbf{u})-\xi_{li}(\mathbf{u}_1+\mathbf{u}'))(\mathbf{u}_1+\mathbf{u})_{k,l}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma})_{kj}\mid_{K^{r+1}}+\\ &+4C\mid\xi_{li}(\mathbf{u}_1+\mathbf{u})(\mathbf{u}-\mathbf{u}')_{k,l}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma})_{kj}\mid_{K^{r+1}}+\\ &+4C\mid\xi_{li}(\mathbf{u}_1+\mathbf{u})(\mathbf{u}_1+\mathbf{u}')_{k,l}(\boldsymbol{\sigma}-\boldsymbol{\sigma}')_{kj}\mid_{K^{r+1}}+\\ &+2C\mid(\mathbf{u}-\mathbf{u}')_{i,k}\xi_{kj}(\mathbf{u}_1+\mathbf{u})\mid_{K^{r+1}}+\\ &+2C\mid(\mathbf{u}_1+\mathbf{u}')_{i,k}(\xi_{kj}(\mathbf{u}_1+\mathbf{u})-\xi_{kj}(\mathbf{u}_1+\mathbf{u}'))\mid_{K^{r+1}}.\end{aligned}$$

Every term can be managed in the same way as before by using (70) and not (69) of Lemma 5.6, Lemma 5.8, Lemma 5.9 and Lemma 5.10.

5.4 Estimates on E^4

This operator comes almost only from the Navier-Stokes part. This part has been estimated by G. Allain [3] for a Newtonian fluid. So we only have to estimate the terms $(\boldsymbol{\sigma}_1+\boldsymbol{\sigma})_{ij}(\mathcal{N}_j-N_j)$ in $K^{r+\frac{1}{2}}(S_F\times(0,T))$. We will estimate separately $\boldsymbol{\sigma}_1+\boldsymbol{\sigma}$ and $\mathcal{N}_j-N_j$.
Concerning $\boldsymbol{\sigma}_1+\boldsymbol{\sigma}$, as for Lemma 5.5, we have:

$$\begin{aligned}
\mid \boldsymbol{\sigma}_1+\boldsymbol{\sigma} \mid_{K^{r+\frac{1}{2}}}(S_F\times(0,T)) &\leq \mid \boldsymbol{\sigma}_1 \mid_{K^{r+\frac{1}{2}}(S_F\times(0,T))} + \mid \boldsymbol{\sigma} \mid_{K^{r+\frac{1}{2}}(S_F\times(0,T))} \\
&\leq \mid \boldsymbol{\sigma}_1 \mid_{K^{r+\frac{1}{2}}(S_F\times(0,T_0))} + C \mid \boldsymbol{\sigma} \mid_{K^{r+\frac{1}{2}}(S_F\times(0,T_0))} \\
&\leq \mid \mathbf{u}_1, q_1, \phi_1, \boldsymbol{\sigma}_1 \mid_{X^r_{T_0}} + C \mid \mathbf{u}, q, \phi, \boldsymbol{\sigma} \mid_{X^r_{T_0}} \\
&\leq \mid \mathbf{u}_1, q_1, \phi_1, \boldsymbol{\sigma}_1 \mid_{X^r_{T_0}} + CR,
\end{aligned}$$

thanks to Lemma 3.5 (ii). So $\boldsymbol{\sigma}_1+\boldsymbol{\sigma}$ is bounded in $K^{r+\frac{1}{2}}(S_F\times(0,T))$ by a constant independent of $T<T_0$. Concerning $\mathcal{N}_j-N_j$, we use the following formula:

$$\boldsymbol{\mathcal{N}}-\mathbf{N}=\int_0^t(-\partial_{\mathbf{T}}((\mathbf{u}_1)_2+(\mathbf{u})_2)(s),\partial_{\mathbf{T}}((\mathbf{u}_1)_1+(\mathbf{u})_1)(s))\ \mathrm{d}s.$$

The estimate of $\mathcal{N}_j-N_j$ is done in the algebra $H^{1+\frac{r}{8}}(0,T;H^{\frac{1}{2}+\frac{r}{2}}(S_F))$ so as to conclude. Since $\partial_{\mathbf{T}}(\mathbf{u}_1+\mathbf{u})\in L^2(0,T;H^{\frac{1}{2}+\frac{r}{2}}(S_F))$, Lemmas 5.2 and 5.5 apply and also classical theorems found in [18]:

$$\begin{aligned}
\mid \int_0^t(-\partial_{\mathbf{T}}((\mathbf{u}_1)_2+(\mathbf{u})_2)(s),\partial_{\mathbf{T}}((\mathbf{u}_1)_1+(\mathbf{u})_1)(s))\,ds \mid_{H^{1+\frac{r}{8}}(0,T;H^{\frac{1}{2}+\frac{r}{2}}(S_F))} \\
\leq CT^{\epsilon'} \mid (-\partial_{\mathbf{T}}((\mathbf{u}_1)_2+(\mathbf{u})_2),\partial_{\mathbf{T}}((\mathbf{u}_1)_1+(\mathbf{u})_1)) \mid_{L^2(0,T;H^{\frac{1}{2}+\frac{r}{2}}(S_F))} \\
\leq CT^{\epsilon'} \mid \boldsymbol{\nabla}(\mathbf{u}_1+\mathbf{u}) \mid_{L^2(0,T;H^{1+\frac{r}{2}})} \\
\leq CT^{\epsilon'}.
\end{aligned}$$

Thanks to the fact that $1+\frac{r}{8}>\frac{1}{2}>\frac{1}{4}+\frac{r}{2}$, the term satisfies:

$$\begin{aligned}
\mid (\boldsymbol{\sigma}_1+\boldsymbol{\sigma})_{ij}(\mathcal{N}_j-N_j) \mid_{K^{r+\frac{1}{2}}(S_F\times(0,T))} &\leq C \mid \boldsymbol{\sigma}_1+\boldsymbol{\sigma} \mid_{K^{r+1/2}(S_F)} \mid \mathcal{N}_j-N_j \mid_{H^{1+\frac{r}{8}}(0,T;H^{\frac{1}{2}+\frac{r}{2}}(S_F))} \\
&\leq CT^{\epsilon'} \mid \boldsymbol{\sigma}_1+\boldsymbol{\sigma} \mid_{K^{r+\frac{1}{2}}(S_F)} \leq CT^{\epsilon'}.
\end{aligned}$$

The contracting property of this operator is proved in the same way.

This completes the proof of Theorem 5.1

6 Fixed point

We want to solve the Lagrangian nonlinear system (14) associated to the operator P. Let us remind the reader with the expansion done in the first Section:

$$\begin{aligned}
P(\boldsymbol{\xi},\mathbf{u},q,\phi,\boldsymbol{\sigma}) &= P(0,0,0,0,0)+P_1(\mathbf{u},q,\phi,\boldsymbol{\sigma})+E(\boldsymbol{\xi},\mathbf{u},q,\phi,\boldsymbol{\sigma}) \\
&= (0,0,0,0,0,\mathbf{u}_0,\boldsymbol{\sigma}_0)
\end{aligned} \tag{75}$$

where $P(0,0,0,0,0) = (0,0,0,g_0\zeta(X_1)N_i - \alpha\partial_{\mathbf{T}}(\mathbf{T}),0,0,0)$ contains all the zeroth order terms (gravity and initial surface tension).

First we lift the initial conditions and the zeroth order terms. To that end, we use Theorem 3.1 that states that P_1 is invertible from X_T^r to Y_T^r: $P_1(\mathbf{u},q,\phi,\boldsymbol{\sigma}) = (\mathbf{f},a,\mathbf{m},g,k,\mathbf{u}_0,\boldsymbol{\sigma}_0)$ with continuous dependence on the initial conditions. So let $(\mathbf{u}_1,q_1,\phi_1,\boldsymbol{\sigma}_1) \in X_{T_0}^r$ be such that:

$$P_1(\mathbf{u}_1,q_1,\phi_1,\boldsymbol{\sigma}_1) = (0,0,0,0,0,\mathbf{u}_0,\boldsymbol{\sigma}_0) - P(0,0,0,0,0),$$

which is allowed thanks to the fact that the right-hand side is in $Y_{T_0}^r$ ($\zeta \in H^{\frac{5}{2}+r}$ implies that $\partial_{\mathbf{T}}(\mathbf{T}) \in H^{r+\frac{1}{2}}$).
If we perform a change of variable for the unknown fields ($\mathbf{u}_1+\mathbf{u}$ replaces $\mathbf{u}$ and so on), we are led to find $(\mathbf{u},q,\phi,\boldsymbol{\sigma}) \in X_T^* = \{(\mathbf{u},q,\phi,\boldsymbol{\sigma}) \in X_T^r / \mathbf{u}(0)=0, \boldsymbol{\sigma}(0)=0\}$ such that:

$$\begin{aligned} P_1(\mathbf{u}_1+\mathbf{u},q_1+q,\phi_1+\phi,\boldsymbol{\sigma}_1+\boldsymbol{\sigma}) + E(\boldsymbol{\xi}(\mathbf{u}_1+\mathbf{u}),\mathbf{u}_1+\mathbf{u},q_1+q,\phi_1+\phi,\boldsymbol{\sigma}_1+\boldsymbol{\sigma}) = \\ P_1(\mathbf{u}_1,q_1,\phi_1,\boldsymbol{\sigma}_1) \\ \Leftrightarrow P_2[\mathbf{u}_1,\boldsymbol{\sigma}_1](\mathbf{u},q,\phi,\boldsymbol{\sigma}) = -E(\boldsymbol{\xi}(\mathbf{u}_1+\mathbf{u}),\mathbf{u}_1+\mathbf{u},q_1+q,\phi_1+\phi,\boldsymbol{\sigma}_1+\boldsymbol{\sigma}), \end{aligned}$$

where P_2 is the second auxiliary problem introduced above in (17). Since $P_2[\mathbf{u}_1,\boldsymbol{\sigma}_1]$ is invertible when the rhs is in Y_T^r thanks to Theorem 4.1, we want to solve

$$\begin{aligned} (\mathbf{u},q,\phi,\boldsymbol{\sigma}) &= P_2^{-1}[\mathbf{u}_1,\boldsymbol{\sigma}_1](-E(\boldsymbol{\xi}(\mathbf{u}_1+\mathbf{u}),\mathbf{u}_1+\mathbf{u},q_1+q,\phi_1+\phi,\boldsymbol{\sigma}_1+\boldsymbol{\sigma})) \\ &:= F(\mathbf{u},q,\phi,\boldsymbol{\sigma}). \end{aligned}$$

At that level, G. Allain proves in [3] for a Newtonian fluid that her F lets an invariant ball. But since we use a contraction mapping, we do not need this.
We need now to prove that F is contracting. Thanks to Lemma 5.6 and especially (63):

$$\mid F(\mathbf{u},q,\phi,\boldsymbol{\sigma}) - F(\mathbf{u}',q',\phi',\boldsymbol{\sigma}') \mid_{X_T^*} \leq CT^{\epsilon'} \mid \mathbf{u}-\mathbf{u}',q-q',\phi-\phi',\boldsymbol{\sigma}-\boldsymbol{\sigma}' \mid_{X_T^*},$$

where C depends on the parameters, and also on the functions that lift the intial conditions, but only through a bound of their norm. As before, it does not depend on $T < T_0$, so we have proved that for T sufficiently small, F is contracting. We may apply the contraction mapping principle which provides a solution to the Lagrangian system of equations (14). Owing to the regularity of the Lagrangian velocity $\mathbf{u}$, one has solved also the Eulerian equations (6) by a simple change of variables.
Thanks to the contraction property, uniqueness is obvious.
Last the solution depends on the initial conditions directly in a continuous way and indirectly through the lift functions also continuously.
The proof of our main Theorem 2.1 is complete.

A Appendix

The proof of our main theorem is written for the specific constitutive law of Oldroyd. In order to convince the reader that it can include most reasonnable constitutive laws, we study

successively various models well-known in the literature. We could even treat a new model of an elastoviscoplastic fluid [24], but it would require deeper modifications of the proof.
The most crucial step in the proof is the first one: when we prove uniform (in n and T_0) estimate on $\boldsymbol{\sigma}^n$. Then, using this estimate, it is easy to derive the same inequalities as ours until the conclusion.

A.1 The Giesekus constitutive law

In [11], Giesekus provides a new constitutive law:

$$\boldsymbol{\sigma} + \mathrm{We}\frac{\mathcal{D}_1[\mathbf{v}]}{\mathcal{D}t}\boldsymbol{\sigma} + c_{Giesekus}\boldsymbol{\sigma}^2 = 2\varepsilon\mathbf{D}[\mathbf{u}],$$

where $\mathrm{We}, \varepsilon, c_{Giesekus}$ are positive parameters. We propose to discretize a reduced version of it in which we keep only the new terms:

$$\boldsymbol{\sigma}^{n+1} + \mathrm{We}\partial_t\boldsymbol{\sigma}^{n+1} + c_{Giesekus}\boldsymbol{\sigma}^n\,\boldsymbol{\sigma}^{n+1} = 2\varepsilon\mathbf{D}[\mathbf{u}^n].$$

If we take the H^{1+r} scalar product of this equation with $\boldsymbol{\sigma}^{n+1}$, one has

$$|\boldsymbol{\sigma}^{n+1}|_{1+r}^2 + \frac{\mathrm{We}}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|\boldsymbol{\sigma}^{n+1}|_{1+r}^2) \leq (2\varepsilon|\boldsymbol{\nabla}\mathbf{u}^n|_{1+r} + c_{Giesekus}|\boldsymbol{\sigma}^n|_{1+r})|\boldsymbol{\sigma}^{n+1}|_{1+r}.$$

The same computations as in our proof leads to

$$\begin{aligned}&\mid \boldsymbol{\sigma}^{n+1} \mid_{1+r}^2 (t) \leq \\ &C\int_0^t (e^{-2\int_s^t \frac{(1/2 - c_{Giesekus}\mathrm{We} \mid \boldsymbol{\sigma}^n \mid_{1+r})}{\mathrm{We}}\mathrm{d}t'} \mid \boldsymbol{\nabla}\mathbf{u}^n \mid_{1+r}^2)\mathrm{d}s.\end{aligned}$$

And as above, we may bound the term in the exponential:

$$\begin{aligned}-2\int_s^t \frac{(1/2 - c_{Giesekus}\mathrm{We} \mid \boldsymbol{\sigma}^n \mid_{1+r})}{\mathrm{We}}\mathrm{d}t' &\leq \frac{2c_{Giesekus}}{\mathrm{We}}\int_s^t |\boldsymbol{\sigma}^n|_{1+r}\mathrm{d}t' \\ &\leq \frac{2c_{Giesekus}ST_0}{\mathrm{We}}.\end{aligned}$$

So if we require that T_0 be such that $2c_{Giesekus}ST_0/\mathrm{We} < 1$, we have

$$|\boldsymbol{\sigma}^{n+1}|_{1+r}(t) \leq C(|\boldsymbol{\nabla}\mathbf{u}^n|_{L^2(0,T;H^{1+r})}) \leq CV.$$

It suffice then to require $CV \leq S$ to get the uniform in n estimate on $|\boldsymbol{\sigma}^{n+1}|_{1+r}$.

A.2 The Phan-Thien Tanner constitutive law

In [21], Phan-Thien and Tanner derive new models of viscoelasticity from a molecular argument using a network which is allowed to be non-affine:

$$Y_{\varepsilon_{PTT}}(\mathrm{tr}\boldsymbol{\sigma})\boldsymbol{\sigma} + \mathrm{We}\frac{\mathcal{D}_a[\mathbf{v}]}{\mathcal{D}t}\boldsymbol{\sigma} = 2\varepsilon\mathbf{D}[\mathbf{u}],$$

where $Y_{\varepsilon_{PTT}}(x) = \exp(\varepsilon_{PTT}\mathrm{We}\, x)$ in the exponential model and $Y(x) = 1 + \varepsilon_{PTT}\mathrm{We}\, x$ in the linear model.
We propose to discretize the exponential law in which already treated terms are removed:

$$\boldsymbol{\sigma}^{n+1} + \mathrm{We}\partial_t \boldsymbol{\sigma}^{n+1} = 2\varepsilon \mathbf{D}[\mathbf{u}^n] + (1 - e^{\varepsilon_{PTT}\mathrm{tr}\boldsymbol{\sigma}^n})\boldsymbol{\sigma}^{n+1}.$$

We take the H^{1+r} scalar product of this equation with $\boldsymbol{\sigma}^{n+1}$ and make the same computation as above to have:

$$\begin{array}{l} \mid \boldsymbol{\sigma}^{n+1} \mid_{1+r} (t) \leq \\ C \displaystyle\int_0^t (e^{-\displaystyle\int_{t'}^t \frac{(1- \mid 1 - e^{\varepsilon_{PTT}\mathrm{tr}\boldsymbol{\sigma}^n} \mid_{1+r})}{\mathrm{We}} \mathrm{d}t''} (\mid \boldsymbol{\nabla}\mathbf{u}^n \mid_{1+r}))\mathrm{d}t'. \end{array}$$

One is led to estimate the integral:

$$\begin{array}{rcl} -\displaystyle\int_{t'}^t \frac{(1- \mid 1 - e^{\varepsilon_{PTT}\mathrm{tr}\boldsymbol{\sigma}^n} \mid_{1+r})}{\mathrm{We}} \mathrm{d}t'' & \leq & C \int_{t'}^t \mid 1 - e^{\varepsilon_{PTT}\mathrm{tr}\boldsymbol{\sigma}^n} \mid_{1+r} \mathrm{d}t'' \\ & \leq & C \int_{t'}^t \left| \sum_{k=1}^{+\infty} \varepsilon_{PTT}^k (\mathrm{tr}\boldsymbol{\sigma}^n)^k \right|_{1+r} \mathrm{d}t'' \\ & \leq & C(e^{\varepsilon_{PTT}\, S} - 1)T_0. \end{array}$$

So a condition on the PTT parameter ε_{PTT} enables to ensure the uniform in n (and in T_0) estimate on $|\boldsymbol{\sigma}^{n+1}|_{1+r}$.

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