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► **To cite this version:**

Mathieu Bargès, Hélène Cossette, Etienne Marceau. TVaR-based capital allocation with copulas. 2009. hal-00431265

HAL Id: hal-00431265

<https://hal.science/hal-00431265>

Preprint submitted on 11 Nov 2009

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TVaR-based capital allocation with copulas

Mathieu Bargès^{*‡} Hélène Cossette[‡] Étienne Marceau[‡]

Abstract

Because of regulation projects from control organizations such as the European solvency II reform and recent economic events, insurance companies need to consolidate their capital reserve with coherent amounts allocated to the whole company and to each line of business. The present study considers an insurance portfolio consisting of several lines of risk which are linked by a copula and aims to evaluate not only the capital allocation for the overall portfolio but also the contribution of each risk over their aggregation. We use the tail value at risk (TVaR) as risk measure. The handy form of the FGM copula permits an exact expression for the TVaR of the sum of the risks and for the TVaR-based allocations when claim amounts are exponentially distributed and distributed as a mixture of exponentials. We first examine the bivariate model and then the multivariate case. We also show how to approximate the TVaR of the aggregate risk and the contribution of each risk when using any copula.

Keywords: Capital allocation; Tail value at risk; Dependence models; Copulas; Discretization methods

1 Introduction

In recent years, a lot of research has focused on insurance capital allocation. Indeed the European Solvency II project and the recent events encourage insurance companies to consolidate their financial reserves and investments. Risk measures are well-known tools to determine the capital amount that has to be allocated to a risk portfolio. Artzner et al. (1999) proposed an axiomatic definition of a coherent risk measure that can be used for allocation issues. This coherence property has also been discussed in Wang (2002). Using their definition, Artzner et al. (1999) proposed the tail value at risk (TVaR), also called expected shortfall (ES), as a coherent alternative to the non-coherent risk measure value at risk (VaR). Applied to continuous random variables, the TVaR can identically be defined as the conditional tail expectation (CTE). But these two risk measures differ in discrete contexts where the CTE is no longer coherent. The differences between these definitions and properties have been highlighted in Acerbi et al. (2001) and Acerbi and Tasche (2002).

In the literature on capital allocation, continuous situations are widely studied in contrast with discrete cases. That is why most of the references speak in terms of CTE. The capital

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allocation principle has first been introduced by [Tasche \(1999\)](#) where the capital allocated to each risk is expressed in terms of the CTE of the aggregate risk. This top down allocation method has then been used to provide several closed formulae and approximations of the CTE and the CTE-based allocations for different types of multivariate continuous distributions. The first multivariate top down model was considered by [Panjer \(2002\)](#) where the risks have a multivariate normal distribution. This work has been extended to a multivariate elliptical distribution in [Landsman and Valdez \(2003\)](#) and in [Dhaene et al. \(2008\)](#). A multivariate gamma distribution for risks has been studied in [Furman and Landsman \(2005\)](#) as well as a multivariate Tweedie distribution in [Furman and Landsman \(2007\)](#). In these papers, explicit expressions for the CTE and the CTE-based allocation are derived. Other closed form expressions for the CTE of the sum of multivariate phase-type distributed risks and the contribution of one risk to the portfolio have been given in [Cai and Li \(2005\)](#). More recently, [Chiragiev and Landsman \(2007\)](#) found a CTE and CTE-based allocation for multivariate Pareto risks. Further information on the CTE-based allocation of risk capital can be found in [Kim \(2007\)](#).

In most papers mentioned above, the dependence between the different lines of business of the insurance company is due to the construction of a multivariate distribution. In the present paper, we propose introducing dependence with a copula. Copulas are currently seen as effective and flexible tools to represent dependence between random variables. Furthermore, in order to have the risk measure coherence property in every continuous and discrete situation, we propose using the TVaR as defined in [Acerbi et al. \(2001\)](#) and [Acerbi and Tasche \(2002\)](#) to develop a top down approach of the capital allocation. Indeed, the Committee of European Insurance and Occupational Pensions Supervisors (CEIOPS) advises in the Solvency II context the use of the TVaR for the evaluation of the Solvency Capital Requirement (SCR); see [CEIOPS \(2006\)](#) and [CEIOPS \(2007\)](#).

First, closed form expressions for the TVaR and then the TVaR-based contribution of one risk over the aggregation of all risks are obtained when the Farlie-Gumbel-Morgenstern copula describes the dependence between the risk marginals. With most copulas introducing dependence between different risks however, we are not able to reach closed form expressions. Consequently, we also present approximation methods to evaluate the TVaR and the TVaR-based allocation by the use of different discretization methods of continuous random variables which are applicable with any copula and any marginals.

In the first section, we give the general definitions for the tail value at risk of the aggregate risk and the contribution of one of the risks. The second section deals with the application of the TVaR-based allocation rule using the FGM copula and exponential distributed risks. We first consider two lines of business and then pursue to a multivariate context. We widen our results to risks that are distributed as mixture of exponentials in section 3. For these two last sections, we are able to have closed form expressions for both the TVaR and the individual risk contribution based on it. Then, we expose approximation methods for the TVaR and TVaR-based allocation when the dependence structure is defined by any copula. The results are illustrated with numerical applications.

2 Definition of the TVaR and the TVaR-based allocation

In this section, we define the tail value at risk (TVaR) for the aggregate risk and the TVaR-based allocation rule. We consider the aggregate claim amount (or loss) S of a portfolio of n risks. The claim amount (or loss) for risk i is denoted by X_i . Thus we have $S = X_1 + \dots + X_n$ where all X_i 's are non-negative random variables.

The value at risk at level κ , $0 < \kappa < 1$, of S is defined by

$$VaR_\kappa(S) = \inf(x \in \mathbb{R}, F_S(x) \geq \kappa).$$

It is well known that the VaR is a risk measure that is not coherent. Thus we choose to work with the tail value at risk of S as introduced in [Acerbi and Tasche \(2002\)](#), [Schmock and Straumann \(1999\)](#) and [Schmock \(2006\)](#) at level κ , for $\kappa \in (0, 1)$. Its definition is

$$\begin{aligned} TVaR_\kappa(S) &= \frac{1}{1-\kappa} \int_\kappa^1 VaR_u(S) du \\ &= \frac{E[S1_{\{S > VaR_\kappa(S)\}}] + VaR_\kappa(S)(Pr(S \leq VaR_\kappa(S)) - \kappa)}{1-\kappa}, \end{aligned}$$

which is a coherent risk measure. When S is continuous, $Pr(S \leq VaR_\kappa(S)) = \kappa$ which implies that the TVaR is exactly the conditional tail expectation (CTE) meaning $TVaR_\kappa(S) = E[S|S > VaR_\kappa(S)] = CTE(S)$. In financial risk management, the TVaR is called the expected shortfall.

The additivity of the expectation allows the decomposition of the TVaR (CTE) into the sum of TVaR contributions as follows

$$TVaR_\kappa(S) = \sum_{i=1}^n TVaR_\kappa(X_i; S),$$

where the TVaR contribution of the i th risk to the total risk represents the part of the capital that is allocated to risk i . For $\kappa \in (0, 1)$, it can be expressed as

$$TVaR_\kappa(X_i; S) = \frac{E[X_i \times 1_{\{S > VaR_\kappa(S)\}}] + \beta_S E[X_i \times 1_{\{S = VaR_\kappa(S)\}}]}{1-\kappa},$$

with

$$\beta_S = \begin{cases} \frac{Pr(S \leq VaR_\kappa(S)) - \kappa}{Pr(S = VaR_\kappa(S))}, & \text{if } Pr(S = VaR_\kappa(S)) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

For continuous distributions, we have

$$\begin{aligned}
TVaR_\kappa(X_i; S) &= \frac{1}{1-\kappa} E[X_i \times 1_{\{S > VaR_\kappa(S)\}}] \\
&= E[X_i | S > VaR_\kappa(S)] \\
&= CTE_\kappa(X_i|S).
\end{aligned}$$

That means that the TVaR-based contribution of one risk is equal to the CTE-based contribution of the same risk when all the marginals are continuous.

Note that

$$\begin{aligned}
E[X_i | S = s] &= E[X_i | X_1 + \dots + X_n = s] \\
&= \int_0^s x f_{X_i|S}(x | S = s) dx \\
&= \int_0^s x \frac{f_{X_i,S}(x, s)}{f_S(s)} dx.
\end{aligned}$$

Then, the CTE-based contribution can be expressed as

$$\begin{aligned}
CTE_\kappa(X_i|S) &= \int_{VaR_\kappa(S)}^\infty E[X_i | S = s] f_{S|S > VaR_\kappa(S)}(s) ds \\
&= \frac{1}{Pr(S > VaR_\kappa(S))} \int_{VaR_\kappa(S)}^\infty E[X_i | S = s] f_S(s) ds \\
&= \frac{1}{1-\kappa} \int_{VaR_\kappa(S)}^\infty \int_0^s x f_{X_i,S}(x, s) dx ds. \tag{1}
\end{aligned}$$

3 TVaR and the TVaR-based allocation with exponential marginals and the FGM copula

In this section, we derive the expression for the TVaR and the TVaR-based allocation for two exponentially distributed risks joined by a FGM copula. We also extend the results obtained with this bivariate model to a multivariate model. The exponential distribution is a classical distribution for the risk random variables. Its convenient and practical mathematic properties permit to develop explicit results. We are aware that the FGM copula introduces only light dependence. However, it admits positive as well as negative dependence between a set of random variables. As said in [Yeo and Valdez \(2006\)](#) where the FGM copula is used to link claim variables in a credibility model, even if it can model only weak dependence, the FGM copula permits to assign a unique dependence parameter for each pair or group of risks and allows a more complex dependence structure than most of the copulas which use only one or few parameters. Furthermore, its handy form allows explicit calculus and thus exact results. This copula was also used to describe different correlation relations on the financial markets in [Gatfaoui \(2005\)](#) and [Gatfaoui \(2007\)](#).

3.1 The bivariate case

Let X_1 and X_2 be two exponentially distributed random variables representing the claim amounts of two insurance risks. Their cumulative distribution functions (cdf) and probability density functions (pdf) are given by

$$\begin{aligned} F_{X_i}(x_i) &= 1 - e^{-\lambda_i x_i}, \\ f_{X_i}(x_i) &= \lambda_i e^{-\lambda_i x_i}, \text{ for } i = 1, 2. \end{aligned}$$

In order to simplify our presentation, we restrain our study to the constraints $\lambda_1 \neq \lambda_2, \lambda_1 \neq 2\lambda_2, \lambda_2 \neq 2\lambda_1$. It is possible to find adjusted results without these constraints by applying a similar method as the one exposed below.

A dependence structure for (X_1, X_2) based on the bivariate FGM copula is introduced. The FGM copula is defined by

$$C_\theta^{FGM}(u_1, u_2) = u_1 u_2 + \theta u_1 u_2 (1 - u_1)(1 - u_2)$$

for $u_i \in [0, 1], i = 1, 2$, and dependence parameter $\theta \in [-1, 1]$.

The density of the bivariate FGM copula is

$$\begin{aligned} c_\theta^{FGM}(u_1, u_2) &= \frac{\partial^2 C_\theta^{FGM}(u_1, u_2)}{\partial u_1 \partial u_2} \\ &= 1 + \theta(1 - 2u_1)(1 - 2u_2) \\ &= 1 + \theta(2\bar{u}_1 - 1)(2\bar{u}_2 - 1), \end{aligned}$$

where $\bar{u}_i = 1 - u_i, i = 1, 2$.

The TVaR of the aggregate risk $S = X_1 + X_2$ is given in the following proposition.

Proposition 1 *Let X_1 and X_2 be two exponentially distributed random variables with joint cdf defined by a bivariate FGM copula as follows*

$$F_{X_1, X_2}(x_1, x_2) = C_\theta^{FGM}(F_{X_1}(x_1), F_{X_2}(x_2)),$$

with $\theta \in [-1, 1]$. Then, the TVaR of the aggregate risk $S = X_1 + X_2$ at level $\kappa, 0 < \kappa < 1$, is

$$\begin{aligned} TVaR_\kappa(S) &= \frac{1}{1 - \kappa} \left[(1 + \theta)\zeta(VaR_\kappa(S); \lambda_1; \lambda_2) - \theta\zeta(VaR_\kappa(S); 2\lambda_1; \lambda_2) \right. \\ &\quad \left. - \theta\zeta(VaR_\kappa(S); \lambda_1; 2\lambda_2) + \theta\zeta(VaR_\kappa(S); 2\lambda_1; 2\lambda_2) \right], \end{aligned} \quad (2)$$

where $\zeta(x; \gamma_1, \gamma_2) = \frac{\gamma_2}{\gamma_2 - \gamma_1} e^{-\gamma_1 x} \left(x + \frac{1}{\gamma_1}\right) + \frac{\gamma_1}{\gamma_1 - \gamma_2} e^{-\gamma_2 x} \left(x + \frac{1}{\gamma_2}\right)$.

Proof. The joint pdf of (X_1, X_2) is given by

$$\begin{aligned}
f_{X_1, X_2}(x_1, x_2) &= c_\theta^{FGM}(F_{X_1}(x_1), F_{X_2}(x_2))f_{X_1}(x_1)f_{X_2}(x_2) \\
&= f_{X_1}(x_1)f_{X_2}(x_2) + \theta f_{X_1}(x_1)f_{X_2}(x_2)(1 - 2F_{X_1}(x_1))(1 - 2F_{X_2}(x_2)) \\
&= (1 + \theta)\lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2} - \theta 2\lambda_1 e^{-2\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2} \\
&\quad - \theta \lambda_1 e^{-\lambda_1 x_1} 2\lambda_2 e^{-2\lambda_2 x_2} + \theta 2\lambda_1 e^{-2\lambda_1 x_1} 2\lambda_2 e^{-2\lambda_2 x_2}.
\end{aligned}$$

Let $h(x, \lambda_1, \lambda_2)$ be the distribution function of a generalized Erlang random variable X

$$h(x, \lambda_1, \lambda_2) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 x} + \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} e^{-\lambda_2 x}.$$

Then, the pdf of S can be expressed as a combination of generalized Erlang pdf's

$$\begin{aligned}
f_S(s) &= \int_0^s f_{X_1, S}(x, s) dx \\
&= \int_0^s f_{X_1, X_2}(x, s-x) dx \\
&= (1 + \theta)h(s; \lambda_1; \lambda_2) - \theta h(s; 2\lambda_1; \lambda_2) - \theta h(s; \lambda_1; 2\lambda_2) + \theta h(s; 2\lambda_1; 2\lambda_2), \tag{3}
\end{aligned}$$

where the first component of $f_S(s)$ is

$$\begin{aligned}
\int_0^s (1 + \theta)\lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2(s-x)} dx &= (1 + \theta)\lambda_1 \lambda_2 e^{-\lambda_2 s} \int_0^s e^{x(\lambda_2 - \lambda_1)} dx \\
&= (1 + \theta) \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 s} + (1 + \theta) \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} e^{-\lambda_2 s} \\
&= (1 + \theta)h(s; \lambda_1; \lambda_2).
\end{aligned}$$

The three other components can be found similarly.

Thus, the TVaR of S takes the form

$$\begin{aligned}
TVaR_\kappa(S) &= E[S | S > VaR_\kappa(S)] \\
&= \int_{VaR_\kappa}^\infty s \frac{f_S(s)}{Pr(S > VaR_\kappa(S))} ds \\
&= \frac{1}{1 - F_S(VaR_\kappa(S))} \int_{VaR_\kappa(S)}^\infty s f_S(s) ds \\
&= \frac{1}{1 - \kappa} \int_{VaR_\kappa(S)}^\infty s \left[(1 + \theta)h(s; \lambda_1; \lambda_2) - \theta h(s; 2\lambda_1; \lambda_2) \right. \\
&\quad \left. - \theta h(s; \lambda_1; 2\lambda_2) + \theta h(s; 2\lambda_1; 2\lambda_2) \right] ds. \tag{4}
\end{aligned}$$

Define

$$\begin{aligned}
\int_{VaR_\kappa(S)}^{\infty} sh(s; \lambda_1; \lambda_2) ds &= \int_{VaR_\kappa(S)}^{\infty} s \left(\frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 s} + \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} e^{-\lambda_2 s} \right) ds \\
&= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \left\{ \left[-s \frac{e^{-\lambda_1 s}}{\lambda_1} \right]_{VaR_\kappa(S)}^{\infty} + \int_{VaR_\kappa(S)}^{\infty} \frac{e^{-\lambda_1 s}}{\lambda_1} ds \right\} \\
&\quad + \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} \left\{ \left[-s \frac{e^{-\lambda_2 s}}{\lambda_2} \right]_{VaR_\kappa(S)}^{\infty} + \int_{VaR_\kappa(S)}^{\infty} \frac{e^{-\lambda_2 s}}{\lambda_2} ds \right\} \\
&= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \left(VaR_\kappa(S) \frac{e^{-\lambda_1 VaR_\kappa(S)}}{\lambda_1} + \frac{e^{-\lambda_1 VaR_\kappa(S)}}{\lambda_1^2} \right) \\
&\quad + \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} \left(VaR_\kappa(S) \frac{e^{-\lambda_2 VaR_\kappa(S)}}{\lambda_2} + \frac{e^{-\lambda_2 VaR_\kappa(S)}}{\lambda_2^2} \right) \\
&= \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 VaR_\kappa(S)} \left(VaR_\kappa(S) + \frac{1}{\lambda_1} \right) + \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{-\lambda_2 VaR_\kappa(S)} \left(VaR_\kappa(S) + \frac{1}{\lambda_2} \right) \\
&= \zeta(VaR_\kappa(S); \lambda_1, \lambda_2). \tag{5}
\end{aligned}$$

Inserting (5) in (4), we obtain

$$\begin{aligned}
TVaR_\kappa(S) &= \frac{1}{1 - \kappa} \left[(1 + \theta) \zeta(VaR_\kappa(S); \lambda_1; \lambda_2) - \theta \zeta(VaR_\kappa(S); 2\lambda_1; \lambda_2) \right. \\
&\quad \left. - \theta \zeta(VaR_\kappa(S); \lambda_1; 2\lambda_2) + \theta \zeta(VaR_\kappa(S); 2\lambda_1; 2\lambda_2) \right],
\end{aligned}$$

with

$$\zeta(x; \gamma_1, \gamma_2) = \frac{\gamma_2}{\gamma_2 - \gamma_1} e^{-\gamma_1 x} \left(x + \frac{1}{\gamma_1} \right) + \frac{\gamma_1}{\gamma_1 - \gamma_2} e^{-\gamma_2 x} \left(x + \frac{1}{\gamma_2} \right).$$

■

A closed form expression for the TVaR-based capital attributed to risk i , $i = 1, 2$, is given in the next proposition.

Proposition 2 *Let X_1 and X_2 be two exponentially distributed random variables with joint cdf defined by a bivariate FGM copula. Then, the TVaR-based contribution of risk i , $i = 1, 2$, to the aggregate risk $S = X_1 + X_2$ at level κ , $0 < \kappa < 1$, is*

$$\begin{aligned}
TVaR_\kappa(X_i; S) &= \frac{1}{1 - \kappa} \left[(1 + \theta) \xi(VaR_\kappa(S); \lambda_i; \lambda_j) - \theta \xi(VaR_\kappa(S); 2\lambda_i; \lambda_j) \right. \\
&\quad \left. - \theta \xi(VaR_\kappa(S); \lambda_i; 2\lambda_j) + \theta \xi(VaR_\kappa(S); 2\lambda_i; 2\lambda_j) \right], \tag{6}
\end{aligned}$$

where $\xi(x; \gamma_i; \gamma_j) = \frac{\gamma_j e^{-\gamma_i x} (x + \frac{1}{\gamma_i})}{\gamma_j - \gamma_i} - \frac{\gamma_j e^{-\gamma_i x} - \gamma_i e^{-\gamma_j x}}{(\gamma_j - \gamma_i)^2}$ and $i \neq j$.

Proof. Let $i = 1$ and $j = 2$. Recall that for continuous random variables the TVaR-based allocation is equal to the CTE-based allocation. From (1), we have

$$\begin{aligned} TVaR_\kappa(X_1; S) &= E[X_1 | S > VaR_\kappa(S)] \\ &= \frac{1}{Pr(S > VaR_\kappa(S))} \int_{VaR_\kappa(S)}^\infty \int_0^s x f_{X_1, S}(x, s) dx ds, \end{aligned}$$

where

$$\begin{aligned} \int_0^s x f_{X_1, S}(x, s) dx &= \int_0^s x f_{X_1, X_2}(x, s-x) dx \\ &= \int_0^s x \left((1+\theta)\lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2(s-x)} - \theta 2\lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2(s-x)} \right. \\ &\quad \left. - \theta \lambda_1 e^{-\lambda_1 x} 2\lambda_2 e^{-\lambda_2(s-x)} + \theta 2\lambda_1 e^{-\lambda_1 x} 2\lambda_2 e^{-\lambda_2(s-x)} \right) dx \\ &= (1+\theta)\lambda_1 \lambda_2 \left(\frac{se^{-\lambda_1 s}}{\lambda_2 - \lambda_1} - \frac{e^{-\lambda_1 s} - e^{-\lambda_2 s}}{(\lambda_2 - \lambda_1)^2} \right) \\ &\quad - \theta 2\lambda_1 \lambda_2 \left(\frac{se^{-2\lambda_1 s}}{\lambda_2 - 2\lambda_1} - \frac{e^{-2\lambda_1 s} - e^{-\lambda_2 s}}{(\lambda_2 - 2\lambda_1)^2} \right) \\ &\quad - \theta \lambda_1 2\lambda_2 \left(\frac{se^{-\lambda_1 s}}{2\lambda_2 - \lambda_1} - \frac{e^{-\lambda_1 s} - e^{-2\lambda_2 s}}{(2\lambda_2 - \lambda_1)^2} \right) \\ &\quad + \theta 2\lambda_1 2\lambda_2 \left(\frac{se^{-2\lambda_1 s}}{2\lambda_2 - 2\lambda_1} - \frac{e^{-2\lambda_1 s} - e^{-2\lambda_2 s}}{(2\lambda_2 - 2\lambda_1)^2} \right). \end{aligned}$$

The first component on the right-hand side of the last equality is given by

$$\begin{aligned} \int_0^s x(1+\theta)\lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2(s-x)} dx &= (1+\theta)\lambda_1 \lambda_2 e^{-\lambda_2 s} \int_0^s x e^{x(\lambda_2 - \lambda_1)} dx \\ &= (1+\theta)\lambda_1 \lambda_2 e^{-\lambda_2 s} \left(\left[x \frac{e^{x(\lambda_2 - \lambda_1)}}{\lambda_2 - \lambda_1} \right]_0^s - \int_0^s \frac{e^{x(\lambda_2 - \lambda_1)}}{\lambda_2 - \lambda_1} dx \right) \\ &= (1+\theta)\lambda_1 \lambda_2 e^{-\lambda_2 s} \left(s \frac{e^{s(\lambda_2 - \lambda_1)}}{\lambda_2 - \lambda_1} - \left[\frac{e^{x(\lambda_2 - \lambda_1)}}{(\lambda_2 - \lambda_1)^2} \right]_0^s \right) \\ &= (1+\theta)\lambda_1 \lambda_2 e^{-\lambda_2 s} \left(s \frac{e^{s\lambda_2} e^{-s\lambda_1}}{\lambda_2 - \lambda_1} - \frac{e^{s\lambda_2} e^{-s\lambda_1} - 1}{(\lambda_2 - \lambda_1)^2} \right) \\ &= (1+\theta)\lambda_1 \lambda_2 \left(\frac{se^{-\lambda_1 s}}{\lambda_2 - \lambda_1} - \frac{e^{-\lambda_1 s} - e^{-\lambda_2 s}}{(\lambda_2 - \lambda_1)^2} \right). \end{aligned}$$

Denoting $V = VaR_\kappa(S)$, we have

$$\begin{aligned}
TVaR_\kappa(X_1; S) &= \frac{1}{1 - F_S(V)} \int_V^\infty \left\{ (1 + \theta) \lambda_1 \lambda_2 \left(\frac{se^{-\lambda_1 s}}{\lambda_2 - \lambda_1} - \frac{e^{-\lambda_1 s} - e^{-\lambda_2 s}}{(\lambda_2 - \lambda_1)^2} \right) \right. \\
&\quad - \theta 2\lambda_1 \lambda_2 \left(\frac{se^{-2\lambda_1 s}}{\lambda_2 - 2\lambda_1} - \frac{e^{-2\lambda_1 s} - e^{-\lambda_2 s}}{(\lambda_2 - 2\lambda_1)^2} \right) - \theta \lambda_1 2\lambda_2 \left(\frac{se^{-\lambda_1 s}}{2\lambda_2 - \lambda_1} - \frac{e^{-\lambda_1 s} - e^{-2\lambda_2 s}}{(2\lambda_2 - \lambda_1)^2} \right) \\
&\quad \left. + \theta 2\lambda_1 2\lambda_2 \left(\frac{se^{-2\lambda_1 s}}{2\lambda_2 - 2\lambda_1} - \frac{e^{-2\lambda_1 s} - e^{-2\lambda_2 s}}{(2\lambda_2 - 2\lambda_1)^2} \right) \right\} ds \\
&= \frac{1}{1 - F_S(V)} \left[(1 + \theta) \left(\frac{\lambda_2 e^{-\lambda_1 V} (V + \frac{1}{\lambda_1})}{\lambda_2 - \lambda_1} - \frac{\lambda_2 e^{-\lambda_1 V} - \lambda_1 e^{-\lambda_2 V}}{(\lambda_2 - \lambda_1)^2} \right) \right. \\
&\quad - \theta \left(\frac{\lambda_2 e^{-2\lambda_1 V} (V + \frac{1}{2\lambda_1})}{\lambda_2 - 2\lambda_1} - \frac{\lambda_2 e^{-2\lambda_1 V} - 2\lambda_1 e^{-\lambda_2 V}}{(\lambda_2 - 2\lambda_1)^2} \right) \\
&\quad - \theta \left(\frac{2\lambda_2 e^{-\lambda_1 V} (V + \frac{1}{\lambda_1})}{2\lambda_2 - \lambda_1} - \frac{2\lambda_2 e^{-\lambda_1 V} - \lambda_1 e^{-2\lambda_2 V}}{(2\lambda_2 - \lambda_1)^2} \right) \\
&\quad \left. + \theta \left(\frac{2\lambda_2 e^{-2\lambda_1 V} (V + \frac{1}{2\lambda_1})}{2\lambda_2 - 2\lambda_1} - \frac{2\lambda_2 e^{-2\lambda_1 V} - 2\lambda_1 e^{-2\lambda_2 V}}{(2\lambda_2 - 2\lambda_1)^2} \right) \right].
\end{aligned}$$

We finally obtain

$$\begin{aligned}
TVaR_\kappa(X_1; S) &= \frac{1}{1 - \kappa} \left[(1 + \theta) \xi (VaR_\kappa(S); \lambda_1; \lambda_2) - \theta \xi (VaR_\kappa(S); 2\lambda_1; \lambda_2) \right. \\
&\quad \left. - \theta \xi (VaR_\kappa(S); \lambda_1; 2\lambda_2) + \theta \xi (VaR_\kappa(S); 2\lambda_1; 2\lambda_2) \right],
\end{aligned}$$

where

$$\xi(x; \gamma_1; \gamma_2) = \frac{\gamma_2 e^{-\gamma_1 x} (x + \frac{1}{\gamma_1})}{\gamma_2 - \gamma_1} - \frac{\gamma_2 e^{-\gamma_1 x} - \gamma_1 e^{-\gamma_2 x}}{(\gamma_2 - \gamma_1)^2}.$$

The TVaR-based allocation for the second risk is symmetrically given by

$$\begin{aligned}
TVaR_\kappa(X_2; S) &= \frac{1}{1 - \kappa} \left[(1 + \theta) \xi (VaR_\kappa(S); \lambda_2; \lambda_1) - \theta \xi (VaR_\kappa(S); 2\lambda_2; \lambda_1) \right. \\
&\quad \left. - \theta \xi (VaR_\kappa(S); \lambda_2; 2\lambda_1) + \theta \xi (VaR_\kappa(S); 2\lambda_2; 2\lambda_1) \right],
\end{aligned}$$

where

$$\xi(x; \gamma_2; \gamma_1) = \frac{\gamma_1 e^{-\gamma_2 x} (x + \frac{1}{\gamma_2})}{\gamma_1 - \gamma_2} - \frac{\gamma_1 e^{-\gamma_2 x} - \gamma_2 e^{-\gamma_1 x}}{(\gamma_1 - \gamma_2)^2}.$$

■

Remark 3 *It can be verified that the TVaR of S is the sum of the risk contributions*

$$TVaR_\kappa(S) = \sum_{i=1}^2 TVaR_\kappa(X_i; S).$$

Explicit expressions for the TVaR and the one risk TVaR-based contribution cannot only be obtained in the bivariate case but even for an undefined number of risks.

3.2 The multivariate case

Suppose now that there are n different exponential risks joined by a multivariate FGM copula characterized by

$$C(u_1, u_2, \dots, u_n) = u_1 u_2 \dots u_n \left(1 + \sum_{k=2}^n \sum_{1 \leq j_1 < \dots < j_k \leq n} \theta_{j_1 j_2 \dots j_k} \bar{u}_{j_1} \bar{u}_{j_2} \dots \bar{u}_{j_k} \right),$$

where $\bar{u}_{(\cdot)} = 1 - u_{(\cdot)}$ (see [Nelsen \(2006\)](#) p.108). We have here $2^n - n - 1$ copula parameters to describe the dependence between each pair or group of risks.

Its density can be written as

$$c(u_1, u_2, \dots, u_n) = 1 + \sum_{k=2}^n \sum_{1 \leq j_1 < \dots < j_k \leq n} \theta_{j_1 j_2 \dots j_k} (2\bar{u}_{j_1} - 1)(2\bar{u}_{j_2} - 1) \dots (2\bar{u}_{j_k} - 1).$$

As in the bivariate case, we suppose that the parameters of the n exponential risks satisfy the conditions below

$$\lambda_i \neq \lambda_j \text{ and } \lambda_i \neq 2\lambda_j \text{ for } i \neq j. \quad (7)$$

Then, the following proposition holds.

Proposition 4 *Let $S_n = X_1 + X_2 + \dots + X_n$ be the sum of n dependent exponential random variables with joint cdf defined by a multivariate FGM copula as follows*

$$F_{X_1, \dots, X_n}(x_1, \dots, x_n) = C_\theta^{FGM}(F_{X_1}(x_1), \dots, F_{X_n}(x_n))$$

with $\theta \in [-1, 1]$. Then, the TVaR of S_n at level κ , $0 < \kappa < 1$, is

$$TVaR_\kappa(S_n) = \frac{1}{1 - \kappa} \times \left[1 + \sum_{k=2}^n \sum_{1 \leq j_1 < \dots < j_k \leq n} \theta_{j_1 j_2 \dots j_k} \times \left(\sum_{l=0}^k \sum_{(a_1, \dots, a_k) \in A_{l,k}} (-1)^l \zeta(VaR_\kappa(S_n); 2^{i_1} \lambda_1, \dots, 2^{i_k} \lambda_k, \lambda_{i_{k+1}}, \dots, \lambda_{i_n}) \right) \right], \quad (8)$$

where $\zeta(x; \gamma_1, \dots, \gamma_n) = \sum_{i=1}^n \left(\prod_{j=1, j \neq i}^n \frac{\gamma_j}{\gamma_j - \gamma_i} \right) e^{-\gamma_i x} \left(x + \frac{1}{\gamma_i} \right)$, i_{k+1}, \dots, i_n are the missing indexes of j_1, \dots, j_k to complete $1, \dots, n$ and $A_{l,k}$ are the sets of k -tuples composed of l zeros and $(k-l)$ ones, for $l = 0, 1, \dots, k$ and $k = 2, \dots, n$.

In fact, the $A_{l,k}$ are defined by $A_{0,k} = \{(1, 1, \dots, 1)_{1 \times k}\}$, $A_{1,k} = \{(1, 1, \dots, 0)_{1 \times k}, \dots, (0, 1, \dots, 1)_{1 \times k}\}$, $A_{2,k} = \{(1, 1, \dots, 0, 0)_{1 \times k}, \dots, (0, 0, \dots, 1)_{1 \times k}\}$, \dots , $A_{k,k} = \{(0, 0, \dots, 0)_{1 \times k}\}$.

Proof. The joint pdf of (X_1, X_2, \dots, X_n) is

$$\begin{aligned} f_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n) &= c(F_{X_1}(x_1), F_{X_2}(x_2), \dots, F_{X_n}(x_n)) f_{X_1}(x_1) f_{X_2}(x_2) \dots f_{X_n}(x_n) \\ &= f_{X_1}(x_1) f_{X_2}(x_2) \dots f_{X_n}(x_n) \\ &\quad \times \left[1 + \sum_{k=2}^n \sum_{1 \leq j_1 < \dots < j_k \leq n} \theta_{j_1 j_2 \dots j_k} (1 - F_{X_{j_1}}(x_{j_1})) (1 - F_{X_{j_2}}(x_{j_2})) \dots (1 - F_{X_{j_k}}(x_{j_k})) \right]. \end{aligned}$$

As the n risk random variables are exponentially distributed with parameters λ_i , $i = 1, \dots, n$ with constraint (7), we have

$$\begin{aligned} f_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n) &= \lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2} \dots \lambda_n e^{-\lambda_n x_n} \\ &\quad \times \left[1 + \sum_{k=2}^n \sum_{1 \leq j_1 < \dots < j_k \leq n} \theta_{j_1 j_2 \dots j_k} (2e^{-\lambda_{j_1} x_{j_1}} - 1) (2e^{-\lambda_{j_2} x_{j_2}} - 1) \dots (2e^{-\lambda_{j_n} x_{j_n}} - 1) \right] \\ &= \nu(x_1, x_2, \dots, x_n; \lambda_1, \lambda_2, \dots, \lambda_n) + \sum_{k=2}^n \sum_{1 \leq j_1 < \dots < j_k \leq n} \theta_{j_1 j_2 \dots j_k} \\ &\quad \times \left[\sum_{l=0}^k \sum_{(a_1, \dots, a_k) \in A_{l,k}} (-1)^l \nu(x_{j_1}, x_{j_2}, \dots, x_{j_k}, x_{i_{k+1}}, \dots, x_{i_n}; 2^{a_1} \lambda_{j_1}, 2^{a_2} \lambda_{j_2}, \dots, 2^{a_k} \lambda_{j_k}, \lambda_{i_{k+1}}, \dots, \lambda_{i_n}) \right], \quad (9) \end{aligned}$$

where i_{k+1}, \dots, i_n and $A_{l,k}$ are defined as in the proposition and

$$\nu(x_1, \dots, x_n; \gamma_1, \dots, \gamma_n) = \gamma_1 e^{-\gamma_1 x_1} \times \gamma_2 e^{-\gamma_2 x_2} \times \dots \times \gamma_n e^{-\gamma_n x_n}.$$

Given that

$$\begin{aligned} f_{S_n}(s) &= \int_0^s \int_0^{s-x_1} \dots \int_0^{s-x_1-\dots-x_{n-1}} f_{X_1, X_2, \dots, X_{n-1}, X_n}(x_1, x_2, \dots, x_{n-1}, s - x_1 - x_2 - \dots - x_{n-1}) dx_1 dx_2 \dots dx_{n-1} \\ &= \int_0^s \int_0^{s-x_1} \dots \int_0^{s-x_1-\dots-x_{n-2}} f_{X_1, X_2, \dots, X_{n-2}, X_{n-1}+X_n}(x_1, x_2, \dots, x_{n-2}, s - x_1 - x_2 - \dots - x_{n-2}) dx_1 dx_2 \dots dx_{n-2} \\ &= \dots \\ &= \int_0^s f_{X_1, X_2 + \dots + X_n}(x_1, s - x_1) dx_1 \end{aligned}$$

and that

$$\begin{aligned} \int_0^s \int_0^{s-x_1} \dots \int_0^{s-x_1-\dots-x_{n-1}} \nu(x_1, x_2, \dots, x_n; \gamma_1, \gamma_2, \dots, \gamma_n) dx_1 dx_2 \dots dx_{n-1} &= \sum_{i=1}^n \left(\prod_{j=1, j \neq i}^n \frac{\gamma_j}{\gamma_j - \gamma_i} \right) \gamma_i e^{-\gamma_i s} \\ &= h(s; \gamma_1, \gamma_2, \dots, \gamma_n), \end{aligned}$$

one can write

$$\begin{aligned} f_{S_n}(s) &= h(s; \lambda_1, \dots, \lambda_n) \\ &+ \sum_{k=2}^n \sum_{1 \leq j_1 < \dots < j_k \leq n} \theta_{j_1 j_2 \dots j_k} \left(\sum_{l=0}^k \sum_{(a_1, \dots, a_k) \in A_{l,k}} (-1)^l h(s; 2^{a_1} \lambda_{j_1}, 2^{a_2} \lambda_{j_2}, \dots, 2^{a_k} \lambda_{j_k}, \lambda_{i_{k+1}}, \dots, \lambda_{i_n}) \right). \end{aligned}$$

Then, the TVaR of S_n for $n \geq 2$ and $0 < \kappa < 1$ is

$$\begin{aligned} TVaR_\kappa(S_n) &= E[S_n | S_n > VaR_\kappa(S_n)] \\ &= \int_{VaR_\kappa}^{\infty} s \frac{f_{S_n}(s)}{Pr(S_n > VaR_\kappa(S_n))} ds \\ &= \frac{1}{1-\kappa} \int_{VaR_\kappa(S_n)}^{\infty} s f_{S_n}(s) ds \\ &= \frac{1}{1-\kappa} \times \left[1 + \sum_{k=2}^n \sum_{1 \leq j_1 < \dots < j_k \leq n} \theta_{j_1 j_2 \dots j_k} \times \right. \\ &\quad \left. \left(\sum_{l=0}^k \sum_{(a_1, \dots, a_k) \in A_{l,k}} (-1)^l \zeta(VaR_\kappa(S_n); 2^{a_1} \lambda_1, \dots, 2^{a_k} \lambda_k, \lambda_{i_{k+1}}, \dots, \lambda_{i_n}) \right) \right], \end{aligned}$$

where

$$\zeta(x; \gamma_1, \dots, \gamma_n) = \sum_{i=1}^n \left(\prod_{j=1, j \neq i}^n \frac{\gamma_j}{\gamma_j - \gamma_i} \right) e^{-\gamma_i x} \left(x + \frac{1}{\gamma_i} \right).$$

■

As in the bivariate case, the capital allocation for risk i can also be explicitly given. In order to find that expression, we first need to introduce the n th order divided difference of a function f as in [Chiragiev and Landsman \(2007\)](#). Consider $x_1, x_2, \dots, x_n, x_{n+1}$ arbitrary points such that $x_i \neq x_j$ for $i \neq j$. The n th order divided difference of f on $x_1, x_2, \dots, x_n, x_{n+1}$ is defined by

$$f(x_1, x_2, \dots, x_n, x_{n+1}) = \sum_{i=1}^{n+1} \frac{f(x_i)}{\prod_{j \neq i} (x_i - x_j)}.$$

The following proposition gives the expression of the TVaR-based allocation.

Proposition 5 Let X_1, \dots, X_n be n exponentially distributed random variables with joint cdf defined by a multivariate FGM copula. Then, the TVaR-based contribution of risk i , $i = 1, \dots, n$, to the sum $S_n = X_1 + \dots + X_n$ at level κ , $0 < \kappa < 1$, is

$$\begin{aligned}
TVaR_\kappa(X_i; S_n) &= \frac{(-1)^{n-1} \Lambda}{1 - \kappa} \left[\overline{H}_i(VaR_\kappa(S_n); \lambda_1; \dots; \lambda_n) + \sum_{k=2}^n \sum_{1 \leq j_1 < \dots < j_k \leq n \cap i \in \{j_1, \dots, j_k\}} \theta_{j_1 \dots j_k} \times \right. \\
&\quad \left(\sum_{l=0}^{k-1} \sum_{(a_1, \dots, a_k)_{-i} \in A_{l, k-1}} \left\{ (-1)^{l+1} 2^{k-1-l} \overline{H}_i(VaR_\kappa(S_n); 2^{a_1} \lambda_{j_1}; \dots; 2^{a_k} \lambda_{j_k}; \lambda_{i_{k+1}}; \dots; \lambda_{i_n}) \right. \right. \\
&\quad \left. \left. + (-1)^l 2^{k-1-l} \overline{G}_i(VaR_\kappa(S_n); 2^{a_1} \lambda_{j_1}; \dots; 2^{a_k} \lambda_{j_k}; \lambda_{i_{k+1}}; \dots; \lambda_{i_n}) \right\} \right) \\
&\quad + \sum_{k=2}^{n-1} \sum_{1 \leq j_1 < \dots < j_k \leq n \cap i \notin \{j_1, \dots, j_k\}} \theta_{j_1 \dots j_k} \times \\
&\quad \left. \left(\sum_{l=0}^k \sum_{(a_1, \dots, a_k) \in A_{l, k}} (-1)^l 2^{k-l} \overline{H}_i(VaR_\kappa(S_n); 2^{a_1} \lambda_{j_1}; \dots; 2^{a_k} \lambda_{j_k}; \lambda_{i_{k+1}}; \dots; \lambda_{i_n}) \right) \right], \quad (10)
\end{aligned}$$

where $\overline{H}_i(x; \gamma_1, \dots, \gamma_{i-1}, \gamma_{i+1}, \dots, \gamma_n)$ and $\overline{G}_i(x; \gamma_1, \dots, \gamma_{i-1}, \gamma_{i+1}, \dots, \gamma_n)$ are the $(n-2)$ -th order divided differences of respectively $\overline{H}_i(x; \gamma) = \frac{x e^{-\gamma_i x}}{\gamma_i(\gamma_i - \gamma)} + \frac{e^{-\gamma_i x}}{\gamma_i^2(\gamma_i - \gamma)} + \frac{\gamma e^{-\gamma_i x} - \gamma_i e^{-\gamma x}}{\gamma \gamma_i(\gamma_i - \gamma)^2}$ and $\overline{G}_i(x; \gamma) = 2 \left(\frac{x e^{-2\gamma_i x}}{2\gamma_i(2\gamma_i - \gamma)} + \frac{e^{-2\gamma_i x}}{(2\gamma_i)^2(2\gamma_i - \gamma)} + \frac{\gamma e^{-2\gamma_i x} - 2\gamma_i e^{-\gamma x}}{\gamma 2\gamma_i(2\gamma_i - \gamma)^2} \right)$, $\Lambda = \lambda_1 \times \dots \times \lambda_n$ is the product of the parameters of exponential distributions, i_{k+1}, \dots, i_n are the missing indexes of j_1, \dots, j_k to complete $1, \dots, n$ and $A_{l, k}$ are the sets of k -tuples composed of l zeros and $(k-l)$ ones, for $l = 0, 1, \dots, k$ and $k = 2, \dots, n$.

Proof. The capital attributed to the continuous distributed risk i can be expressed as

$$\begin{aligned}
TVaR_\kappa(X_i; S_n) &= CTE_\kappa(X_i | S_n) \\
&= \frac{1}{\overline{F}_{S_n}(VaR_\kappa(S_n))} \int_{VaR_\kappa(S_n)}^{\infty} \int_0^s x_i f_{X_i, S_n}(x_i, s) dx ds. \quad (11)
\end{aligned}$$

A recursive formula for $f_{X_i, S_n}(x_i, s) = f_{X_i, S_n - X_i}(x_i, s - x_i)$ is needed to evaluate this expression. Given that the risk random variables here are not independent, we cannot directly separate $f_{X_i, S_n - X_i}(x_i, s - x_i)$ into the product of $f_{X_i}(x_i)$ and $f_{S_n - 1}(s - x_i)$.

First, we have

$$\begin{aligned}
f_{X_i, S_n - X_i}(x_i, s - x_i) &= f_{X_i, X_1 + X_2 + \dots + X_{i-1} + X_{i+1} + \dots + X_n}(x_i, s - x_i) \\
&= \int_0^{s-x_i} \int_0^{s-x_i-x_1} \int_0^{s-x_i-x_1-x_2} \dots \int_0^{s-x_1-\dots-x_{n-1}} f_{X_1, X_2, \dots, X_{n-1}, X_n}(x_1, x_2, \dots, x_{n-1}, s - x_1 - \dots - x_{n-1}) \\
&\quad \times dx_1 dx_2 \dots dx_{i-1} dx_{i+1} \dots dx_{n-1}
\end{aligned}$$

with $f_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n)$ defined as in (9) and which can be extended to

$$\begin{aligned}
f_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n) &= f_{X_i}(x_i) \left(\nu_{-i}(x_1; x_2; \dots; x_n; \lambda_1; \lambda_2; \dots; \lambda_n) \right. \\
&+ \sum_{k=2}^n \sum_{1 \leq j_1 < \dots < j_k \leq n \cap i \in \{j_1, \dots, j_k\}} \theta_{j_1 j_2 \dots j_k} (1 - 2F_{X_i}(x_i)) \\
&\times \left[\sum_{l=0}^k \sum_{(a_1, \dots, a_k)_{-i} \in A_{l, k-1}} (-1)^l \nu_{-i}(x_{j_1}; x_{j_2}; \dots; x_{j_k}; x_{i_{k+1}}; \dots; x_{i_n}; 2^{a_1} \lambda_{j_1}; 2^{a_2} \lambda_{j_2}; \dots; 2^{a_k} \lambda_{j_k}; \lambda_{i_{k+1}}; \dots; \lambda_{i_n}) \right] \\
&+ \sum_{k=2}^n \sum_{1 \leq j_1 < \dots < j_k \leq n \cap i \notin \{j_1, \dots, j_k\}} \theta_{j_1 j_2 \dots j_k} \\
&\times \left[\sum_{l=0}^k \sum_{(a_1, \dots, a_k)_{-i} \in A_{l, k-1}} (-1)^l \nu_{-i}(x_{j_1}; x_{j_2}; \dots; x_{j_k}; x_{i_{k+1}}; \dots; x_{i_n}; 2^{a_1} \lambda_{j_1}; 2^{a_2} \lambda_{j_2}; \dots; 2^{a_k} \lambda_{j_k}; \lambda_{i_{k+1}}; \dots; \lambda_{i_n}) \right],
\end{aligned}$$

where i_{k+1}, \dots, i_n are the missing indexes of j_1, \dots, j_k to complete $1, \dots, n$, $A_{0,k} = \{(1, 1, \dots, 1)_{1 \times k}\}$, $A_{1,k} = \{(1, 1, \dots, 0)_{1 \times k}, \dots, (0, 1, \dots, 1)_{1 \times k}\}$, $A_{2,k} = \{(1, 1, \dots, 0, 0)_{1 \times k}, \dots, (0, 0, \dots, 1)_{1 \times k}\}$, \dots , $A_{k,k} = \{(0, 0, \dots, 0)_{1 \times k}\}$ and

$$\nu_{-i}(x_1; x_2; \dots; x_n; \gamma_1; \gamma_2; \dots; \gamma_n) = \gamma_1 e^{-\gamma_1 x_1} \times \gamma_2 e^{-\gamma_2 x_2} \times \dots \times \gamma_{i-1} e^{-\gamma_{i-1} x_{i-1}} \times \gamma_{i+1} e^{-\gamma_{i+1} x_{i+1}} \times \dots \times \gamma_n e^{-\gamma_n x_n}.$$

As in the proof of the Proposition 4, we use the fact that

$$\begin{aligned}
&\int_0^{s-x_i} \int_0^{s-x_i-x_1} \int_0^{s-x_i-x_1-x_2} \dots \int_0^{s-x_1-\dots-x_{n-1}} \nu_{-i}(x_1; x_2; \dots; x_n; \gamma_1; \gamma_2; \dots; \gamma_n) dx_1 dx_2 \dots dx_{i-1} dx_{i+1} \dots dx_{n-1} \\
&= \sum_{\substack{j=1 \\ j \neq i}}^n \left(\prod_{\substack{k=1, k \neq j \\ k \neq i}}^n \frac{\gamma_k}{\gamma_k - \gamma_j} \right) \gamma_j e^{-\gamma_j (s-x_i)} \\
&= h_{-i}(s-x_i; \gamma_1, \gamma_2, \dots, \gamma_n),
\end{aligned}$$

and we obtain

$$\begin{aligned}
f_{X_i, S_n - X_i}(x_i, s - x_i) &= f_{X_i}(x_i) \left[h_{-i}(s - x_i; \lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_n) \right. \\
&+ \sum_{k=2}^n \sum_{1 \leq j_1 < \dots < j_k \leq n \cap i \in \{j_1, \dots, j_k\}} \theta_{j_1 j_2 \dots j_k} (1 - 2F_{X_i}(x)) \\
&\times \left(\sum_{l=0}^k \sum_{(a_1, \dots, a_k)_{-i} \in A_{l, k-1}} (-1)^l h_{-i}(s - x_i; 2^{a_1} \lambda_{j_1}, \dots, 2^{a_k} \lambda_{j_k}, \lambda_{i_{k+1}}, \dots, \lambda_{i_n}) \right) \\
&+ \sum_{k=2}^n \sum_{1 \leq j_1 < \dots < j_k \leq n \cap i \notin \{j_1, \dots, j_k\}} \theta_{j_1 j_2 \dots j_k} \\
&\times \left. \left(\sum_{l=0}^k \sum_{(a_1, \dots, a_k) \in A_{l, k}} (-1)^l h_{-i}(s - x_i; 2^{a_1} \lambda_{j_1}, \dots, 2^{a_k} \lambda_{j_k}, \lambda_{i_{k+1}}, \dots, \lambda_{i_n}) \right) \right]. \quad (12)
\end{aligned}$$

Using the divided difference as in [Chiragiev and Landsman \(2007\)](#), notice that

$$\begin{aligned}
h_{-i}(x; \gamma_1, \dots, \gamma_{i-1}, \gamma_{i+1}, \dots, \gamma_n) &= (-1)^{n-2} \times \gamma_1 \times \dots \times \gamma_{i-1} \times \gamma_{i+1} \times \dots \times \gamma_n \times \bar{F}(x; \gamma_1; \dots; \gamma_{i-1}; \gamma_{i+1}; \dots; \gamma_n), \\
\text{where } \bar{F}(x; \gamma_1; \dots; \gamma_{i-1}; \gamma_{i+1}; \dots; \gamma_n) &\text{ is the } (n-2)\text{-th order divided difference of } \bar{F}(x; \gamma) = e^{-\gamma x}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\int_0^s x_i f_{X_i}(x_i) h_{-i}(s - x_i; \lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_n) dx_i &= \int_0^s x_i f_{X_i}(x_i) (-1)^{n-2} \Lambda_{-i} \bar{F}(x_i; \lambda_1; \dots; \lambda_{i-1}; \lambda_{i+1}; \dots; \lambda_n) dx_i \\
&= (-1)^{n-1} \Lambda H_i(s; \lambda_1; \dots; \lambda_{i-1}; \lambda_{i+1}; \dots; \lambda_n), \quad (13)
\end{aligned}$$

where $\Lambda_{-i} = \lambda_1 \times \dots \times \lambda_{i-1} \times \lambda_{i+1} \times \dots \times \lambda_n$, $\Lambda = \lambda_1 \times \lambda_2 \times \dots \times \lambda_n$ and

$$H_i(s; \lambda) = - \int_0^s x_i e^{-\lambda x_i} \bar{F}(s - x_i; \lambda) dx_i.$$

Let also

$$G_i(s; \lambda) = - \int_0^s 2x_i e^{-2\lambda x_i} \bar{F}(s - x_i; \lambda) dx_i.$$

Given that the risks here are exponentially distributed, H_i and G_i take the form

$$\begin{aligned}
H_i(s; \lambda) &= - \int_0^s x_i e^{-\lambda x_i} \bar{F}(s - x_i; \lambda) dx_i \\
&= -e^{-\lambda s} \int_0^s x_i e^{-(\lambda_i - \lambda)x_i} ds \\
&= \frac{se^{-\lambda_i s}}{\lambda_i - \lambda} + \frac{e^{-\lambda_i s}}{(\lambda_i - \lambda)^2} - \frac{e^{-\lambda s}}{(\lambda_i - \lambda)^2}
\end{aligned}$$

and

$$\begin{aligned}
G_i(s; \lambda) &= - \int_0^s 2x_i e^{-2\lambda_i x_i} \bar{F}(s - x_i; \lambda) dx_i \\
&= -2e^{-\lambda s} \int_0^s x_i e^{-(2\lambda_i - \lambda)x_i} ds \\
&= 2 \left(\frac{se^{-2\lambda_i s}}{2\lambda_i - \lambda} + \frac{e^{-2\lambda_i s}}{(2\lambda_i - \lambda)^2} - \frac{e^{-\lambda s}}{(2\lambda_i - \lambda)^2} \right).
\end{aligned}$$

Using (13), we can express $\int_0^s x_i f_{X_i, S_n}(x_i, s) dx_i$ as

$$\begin{aligned}
\int_0^s x f_{X_i, S_n}(x, s) dx &= (-1)^{n-1} \Lambda \left[H_i(s; \lambda_1; \dots; \lambda_n) + \sum_{k=2}^n \sum_{1 \leq j_1 < \dots < j_k \leq n \cap i \in \{j_1, \dots, j_k\}} \theta_{j_1 \dots j_k} \times \right. \\
&\quad \left\{ \sum_{l=0}^{k-1} \sum_{(a_1, \dots, a_k)_{-i} \in A_{l, k-1}} \left((-1)^{l+1} 2^{k-1-l} H_i(s; 2^{a_1} \lambda_{j_1}; \dots; 2^{a_k} \lambda_{j_k}; \lambda_{i_{k+1}}; \dots; \lambda_{i_n}) \right) \right. \\
&\quad \left. + (-1)^l 2^{k-1-l} G_i(s; 2^{a_1} \lambda_{j_1}; \dots; 2^{a_k} \lambda_{j_k}; \lambda_{i_{k+1}}; \dots; \lambda_{i_n}) \right\} \\
&\quad + \sum_{k=2}^{n-1} \sum_{1 \leq j_1 < \dots < j_k \leq n \cap i \notin \{j_1, \dots, j_k\}} \theta_{j_1 \dots j_k} \times \\
&\quad \left. \left\{ \sum_{l=0}^k \sum_{(a_1, \dots, a_k) \in A_{l, k}} (-1)^l 2^{k-l} H_i(s; 2^{a_1} \lambda_{j_1}; \dots; 2^{a_k} \lambda_{j_k}; \lambda_{i_{k+1}}; \dots; \lambda_{i_n}) \right\} \right]. \quad (14)
\end{aligned}$$

To calculate the risk contribution as in (11), the $H_i(s)$ and $G_i(s)$ terms in (14) must be integrated on s as follows

$$\begin{aligned}
\bar{H}_i(V; \lambda) &= \int_V^\infty H_i(s; \lambda) ds \\
&= \int_V^\infty \left(\frac{se^{-\lambda_i s}}{\lambda_i - \lambda} + \frac{e^{-\lambda_i s}}{(\lambda_i - \lambda)^2} - \frac{e^{-\lambda s}}{(\lambda_i - \lambda)^2} \right) ds \\
&= \frac{Ve^{-\lambda_i V}}{\lambda_i(\lambda_i - \lambda)} + \frac{e^{-\lambda_i V}}{\lambda_i^2(\lambda_i - \lambda)} + \frac{\lambda e^{-\lambda_i V} - \lambda_i e^{-\lambda V}}{\lambda \lambda_i(\lambda_i - \lambda)^2}
\end{aligned}$$

and

$$\begin{aligned}
\bar{G}_i(V; \lambda) &= \int_V^\infty G_i(s; \lambda) ds \\
&= \int_V^\infty 2 \left(\frac{se^{-2\lambda_i s}}{2\lambda_i - \lambda} + \frac{e^{-2\lambda_i s}}{(2\lambda_i - \lambda)^2} - \frac{e^{-\lambda s}}{(2\lambda_i - \lambda)^2} \right) ds \\
&= 2 \left(\frac{Ve^{-2\lambda_i V}}{2\lambda_i(2\lambda_i - \lambda)} + \frac{e^{-2\lambda_i V}}{(2\lambda_i)^2(2\lambda_i - \lambda)} + \frac{\lambda e^{-2\lambda_i V} - 2\lambda_i e^{-\lambda V}}{\lambda 2\lambda_i(2\lambda_i - \lambda)^2} \right).
\end{aligned}$$

Finally, expression (11) for $TVaR_\kappa(X_i; S_n)$ is obtained

$$\begin{aligned}
TVaR_\kappa(X_i; S_n) &= \frac{(-1)^{n-1} \Lambda}{1 - \kappa} \left[\overline{H}_i(VaR_\kappa(S_n); \lambda_1; \dots; \lambda_n) + \sum_{k=2}^n \sum_{1 \leq j_1 < \dots < j_k \leq n \cap i \in \{j_1, \dots, j_k\}} \theta_{j_1 \dots j_k} \times \right. \\
&\quad \left(\sum_{l=0}^{k-1} \sum_{(a_1, \dots, a_k)_{-i} \in A_{l, k-1}} \left\{ (-1)^{l+1} 2^{k-1-l} \overline{H}_i(VaR_\kappa(S_n); 2^{a_1} \lambda_{j_1}; \dots; 2^{a_k} \lambda_{j_k}; \lambda_{i_{k+1}}; \dots; \lambda_{i_n}) \right. \right. \\
&\quad \left. \left. + (-1)^l 2^{k-1-l} \overline{G}_i(VaR_\kappa(S_n); 2^{a_1} \lambda_{j_1}; \dots; 2^{a_k} \lambda_{j_k}; \lambda_{i_{k+1}}; \dots; \lambda_{i_n}) \right\} \right) \\
&\quad + \sum_{k=2}^{n-1} \sum_{1 \leq j_1 < \dots < j_k \leq n \cap i \notin \{j_1, \dots, j_k\}} \theta_{j_1 \dots j_k} \times \\
&\quad \left. \left(\sum_{l=0}^k \sum_{(a_1, \dots, a_k) \in A_{l, k}} (-1)^l 2^{k-l} \overline{H}_i(VaR_\kappa(S_n); 2^{a_1} \lambda_{j_1}; \dots; 2^{a_k} \lambda_{j_k}; \lambda_{i_{k+1}}; \dots; \lambda_{i_n}) \right) \right].
\end{aligned}$$

■

Remark 6 To obtain the TVaR-based contribution (6) in the bivariate case from (10), we just have to use the equalities

$$\begin{aligned}
\overline{H}_1(x; \lambda_2) &= -\frac{1}{\lambda_1 \lambda_2} \xi(x; \lambda_1; \lambda_2), \\
\overline{H}_1(x; 2\lambda_2) &= -\frac{1}{\lambda_1 2\lambda_2} \xi(x; \lambda_1; 2\lambda_2), \\
\overline{G}_1(x; \lambda_2) &= -2 \frac{1}{2\lambda_1 \lambda_2} \xi(x; 2\lambda_1; \lambda_2), \\
\overline{G}_1(x; 2\lambda_2) &= -2 \frac{1}{2\lambda_1 2\lambda_2} \xi(x; 2\lambda_1; 2\lambda_2).
\end{aligned}$$

3.3 Numerical application

We illustrate here our results with a numerical example for the bivariate exponential case. Suppose that the parameters of the distributions of X_1 and X_2 are respectively $\lambda_1 = 1/2$ and $\lambda_2 = 1/3$. Let us calculate the VaR, TVaR and TVaR-based allocations for X_1 and X_2 for different risk levels κ and different FGM copula parameters θ . We write bellow the cumulative distribution function of S which can be expressed in the current case as a combination of generalized Erlang cdf's

$$F_S(s) = (1 + \theta)H(s; \lambda_1; \lambda_2) - \theta H(s; 2\lambda_1; \lambda_2) - \theta H(s; \lambda_1; 2\lambda_2) + \theta H(s; 2\lambda_1; 2\lambda_2),$$

where $H(s; \lambda_1; \lambda_2) = \frac{\lambda_2}{\lambda_2 - \lambda_1} (1 - e^{-\lambda_1 s}) + \frac{\lambda_1}{\lambda_1 - \lambda_2} (1 - e^{-\lambda_2 s})$ is the cdf of a 2-generalized Erlang distribution with parameters (λ_1, λ_2) . The numerical results for the VaR, TVaR and TVaR-based allocations are displayed in Tables 1, 2 and 3.

$\theta = -1$	$\kappa = 0.5$	$\kappa = 0.75$	$\kappa = 0.95$	$\kappa = 0.99$	$\kappa = 0.995$
$VaR_\kappa(S)$	4.3188	6.5053	11.0436	15.5235	17.4860
$TVaR_\kappa(S)$	7.3270	9.3394	13.8369	18.3810	20.3716
$TVaR_\kappa(X_1; S)$	2.7244	3.1489	3.5085	3.2649	3.0613
$TVaR_\kappa(X_2; S)$	4.6026	6.1905	10.3283	15.1161	17.3103

Table 1: Bivariate exponential example with $\theta = -1$.

$\theta = 0$	$\kappa = 0.5$	$\kappa = 0.75$	$\kappa = 0.95$	$\kappa = 0.99$	$\kappa = 0.995$
$VaR_\kappa(S)$	4.1589	6.7187	11.9994	16.9914	19.1073
$TVaR_\kappa(S)$	7.6589	9.9967	15.0984	20.0310	22.1324
$TVaR_\kappa(X_1; S)$	2.9206	3.5756	4.6115	5.2234	5.4002
$TVaR_\kappa(X_2; S)$	4.7383	6.4211	10.4869	14.8075	16.7323

Table 2: Bivariate exponential example with $\theta = 0$.

$\theta = 1$	$\kappa = 0.5$	$\kappa = 0.75$	$\kappa = 0.95$	$\kappa = 0.99$	$\kappa = 0.995$
$VaR_\kappa(S)$	3.9328	6.9975	12.8673	18.0635	20.2236
$TVaR_\kappa(S)$	7.9817	10.6369	16.0906	21.1529	23.2818
$TVaR_\kappa(X_1; S)$	3.1066	3.9947	5.4022	6.2662	6.5272
$TVaR_\kappa(X_2; S)$	4.8750	6.6422	10.6883	14.8867	16.7546

Table 3: Bivariate exponential example with $\theta = 1$.

4 TVaR and the TVaR-based allocation with mixtures of exponential marginals and the FGM copula

Let us consider now that we have two risks X_1 and X_2 which are distributed as a mixture of exponentials. Their cdf's and pdf's can be written as

$$\begin{aligned} F_{X_1}(x) &= \alpha_{11}(1 - e^{-\lambda_{11}x}) + \alpha_{12}(1 - e^{-\lambda_{12}x}) \\ F_{X_2}(x) &= \alpha_{21}(1 - e^{-\lambda_{21}x}) + \alpha_{22}(1 - e^{-\lambda_{22}x}) \\ f_{X_1}(x) &= \alpha_{11}\lambda_{11}e^{-\lambda_{11}x} + \alpha_{12}\lambda_{12}e^{-\lambda_{12}x} \\ f_{X_2}(x) &= \alpha_{21}\lambda_{21}e^{-\lambda_{21}x} + \alpha_{22}\lambda_{22}e^{-\lambda_{22}x}, \end{aligned}$$

where we restrict our model to $\lambda_{1i} \neq \lambda_{2j}$, $\lambda_{1i} \neq 2\lambda_{2j}$, $\lambda_{11} + \lambda_{12} \neq \lambda_{2j}$, $\lambda_{11} + \lambda_{12} \neq 2\lambda_{2j}$ and $\lambda_{11} + \lambda_{12} \neq \lambda_{21} + \lambda_{22}$. As for the exponential distribution case, the calculations can be done without these constraints but the results are not presented here. Mixtures of exponential distributions, also called hyper-exponential distributions, belong to the phase-type distribution family, see [Neuts \(1981\)](#) and [Asmussen \(2000\)](#). They can be used to approximate light- or heavy-tailed distributions with completely monotone pdf's and decreasing failure rates as shown in [Feldmann and Whitt \(1998\)](#), [Keatinge \(1999\)](#) and [Khayari et al. \(2003\)](#). The practical form of the mixture of exponential distributions also permits explicit results.

The following two propositions can be proven similarly as Proposition 1 and Proposition 2 in the previous section given that a mixture of exponentials is just an extension of the exponential distribution.

Proposition 7 *Let X_1 and X_2 be two random variables with mixture of exponential distributions and a joint cdf defined by a bivariate FGM copula as follows*

$$F_{X_1, X_2}(x_1, x_2) = C_{\theta}^{FGM}(F_{X_1}(x_1), F_{X_2}(x_2))$$

with $\theta \in [-1, 1]$. Then, the TVaR of the aggregate risk $S = X_1 + X_2$ at level κ , $0 < \kappa < 1$, is

$$TVaR_{\kappa}(S) = (I + J + K) \times \frac{1}{1 - \kappa},$$

where

$$\begin{aligned} I = \sum_{i=1}^2 \sum_{j=1}^2 \left\{ \left[\alpha_{1i}\alpha_{2j} + \theta(\alpha_{1i}\alpha_{2j} - 2\alpha_{1i}^2\alpha_{2j} - 2\alpha_{11}\alpha_{12}\alpha_{2j} - 2\alpha_{1i}\alpha_{2j}^2 - 2\alpha_{1i}\alpha_{21}\alpha_{22} \right. \right. \\ \left. \left. + 4\alpha_{1i}^2\alpha_{2j}^2 + 4\alpha_{1i}^2\alpha_{21}\alpha_{22} + 4\alpha_{11}\alpha_{12}\alpha_{2j}^2 + 4\alpha_{11}\alpha_{12}\alpha_{21}\alpha_{22} \right] \zeta(VaR_{\kappa}(S); \lambda_{1i}; \lambda_{2j}) \right\}, \end{aligned}$$

$$\begin{aligned}
J = & \theta \sum_{i=1}^2 \sum_{j=1}^2 \left\{ [\alpha_{1i}^2 \alpha_{2j} - 2\alpha_{1i}^2 \alpha_{2j}^2 - 2\alpha_{1i}^2 \alpha_{21} \alpha_{22}] \zeta(\text{VaR}_\kappa(S); 2\lambda_{1i}; \lambda_{2j}) \right. \\
& \left. + [2\alpha_{11} \alpha_{12} \alpha_{2j} - 4\alpha_{11} \alpha_{12} \alpha_{2j}^2 - 4\alpha_{11} \alpha_{12} \alpha_{21} \alpha_{22}] \frac{\lambda_{1i}}{\lambda_{11} + \lambda_{12}} \zeta(\text{VaR}_\kappa(S); \lambda_{11} + \lambda_{12}; \lambda_{2j}) \right\}
\end{aligned}$$

and

$$\begin{aligned}
K = & \theta \sum_{i=1}^2 \sum_{j=1}^2 \left\{ [\alpha_{1i} \alpha_{2j}^2 - 2\alpha_{1i}^2 \alpha_{2j}^2 - 2\alpha_{11} \alpha_{12} \alpha_{2j}^2] \zeta(\text{VaR}_\kappa(S); \lambda_{1i}; 2\lambda_{2j}) \right. \\
& + \alpha_{1i}^2 \alpha_{2j}^2 \zeta(\text{VaR}_\kappa(S); 2\lambda_{1i}; 2\lambda_{2j}) \\
& + [2\alpha_{1i} \alpha_{21} \alpha_{22} - 4\alpha_{1i}^2 \alpha_{21} \alpha_{22} - 4\alpha_{11} \alpha_{12} \alpha_{21} \alpha_{22}] \frac{\lambda_{2j}}{\lambda_{21} + \lambda_{22}} \zeta(\text{VaR}_\kappa(S); \lambda_{1i}; \lambda_{21} + \lambda_{22}) \\
& + 2\alpha_{1i}^2 \alpha_{21} \alpha_{22} \frac{\lambda_{2j}}{\lambda_{21} + \lambda_{22}} \zeta(\text{VaR}_\kappa(S); 2\lambda_{11}; \lambda_{21} + \lambda_{22}) + 2\alpha_{11} \alpha_{12} \alpha_{2j}^2 \frac{\lambda_{1i}}{\lambda_{11} + \lambda_{12}} \zeta(\text{VaR}_\kappa(S); \lambda_{11} + \lambda_{12}; 2\lambda_{2j}) \\
& \left. + 4\alpha_{11} \alpha_{12} \alpha_{21} \alpha_{22} \frac{\lambda_{1i} \lambda_{2j}}{(\lambda_{11} + \lambda_{12})(\lambda_{21} + \lambda_{22})} \zeta(\text{VaR}_\kappa(S); \lambda_{11} + \lambda_{12}; \lambda_{21} + \lambda_{22}) \right\}.
\end{aligned}$$

Proposition 8 Let X_1 and X_2 be two mixture of exponentials distributed random variables with joint cdf defined by a bivariate FGM copula. Then, the TVaR-based contribution of risk i , $i = 1, 2$, to the aggregate risk $S = X_1 + X_2$ at level κ , $0 < \kappa < 1$, is

$$\text{TVaR}_\kappa(X_i; S) = (L + M + N) \times \frac{1}{1 - \kappa},$$

where

$$\begin{aligned}
L = & \sum_{i=1}^2 \sum_{j=1}^2 \left\{ [\alpha_{1i} \alpha_{2j} + \theta(\alpha_{1i} \alpha_{2j} - 2\alpha_{1i}^2 \alpha_{2j} - 2\alpha_{11} \alpha_{12} \alpha_{2j} - 2\alpha_{1i} \alpha_{2j}^2 - 2\alpha_{1i} \alpha_{21} \alpha_{22}) \right. \\
& \left. + 4\alpha_{1i}^2 \alpha_{2j}^2 + 4\alpha_{1i}^2 \alpha_{21} \alpha_{22} + 4\alpha_{11} \alpha_{12} \alpha_{2j}^2 + 4\alpha_{11} \alpha_{12} \alpha_{21} \alpha_{22}] \xi(\text{VaR}_\kappa(S); \lambda_{1i}; \lambda_{2j}) \right\},
\end{aligned}$$

$$\begin{aligned}
M = & \theta \sum_{i=1}^2 \sum_{j=1}^2 \left\{ [\alpha_{1i}^2 \alpha_{2j} - 2\alpha_{1i}^2 \alpha_{2j}^2 - 2\alpha_{1i}^2 \alpha_{21} \alpha_{22}] \zeta(\text{VaR}_\kappa(S); 2\lambda_{1i}; \lambda_{2j}) \right. \\
& \left. + [2\alpha_{11} \alpha_{12} \alpha_{2j} - 4\alpha_{11} \alpha_{12} \alpha_{2j}^2 - 4\alpha_{11} \alpha_{12} \alpha_{21} \alpha_{22}] \frac{\lambda_{1i}}{\lambda_{11} + \lambda_{12}} \zeta(\text{VaR}_\kappa(S); \lambda_{11} + \lambda_{12}; \lambda_{2j}) \right\},
\end{aligned}$$

$$\begin{aligned}
N &= \theta \sum_{i=1}^2 \sum_{j=1}^2 \left\{ [\alpha_{1i}\alpha_{2j}^2 - 2\alpha_{1i}^2\alpha_{2j}^2 - 2\alpha_{11}\alpha_{12}\alpha_{2j}^2] \xi(\text{VaR}_\kappa(S); \lambda_{1i}; 2\lambda_{2j}) \right. \\
&\quad + \alpha_{1i}^2\alpha_{2j}^2 \xi(\text{VaR}_\kappa(S); 2\lambda_{1i}; 2\lambda_{2j}) \\
&\quad + [2\alpha_{1i}\alpha_{21}\alpha_{22} - 4\alpha_{1i}^2\alpha_{21}\alpha_{22} - 4\alpha_{11}\alpha_{12}\alpha_{21}\alpha_{22}] \frac{\lambda_{2j}}{\lambda_{21} + \lambda_{22}} \xi(\text{VaR}_\kappa(S); \lambda_{1i}; \lambda_{21} + \lambda_{22}) \\
&\quad + 2\alpha_{1i}^2\alpha_{21}\alpha_{22} \frac{\lambda_{2j}}{\lambda_{21} + \lambda_{22}} \xi(s; 2\lambda_{1i}; \lambda_{21} + \lambda_{22}) + 2\alpha_{11}\alpha_{12}\alpha_{2j}^2 \frac{\lambda_{1i}}{\lambda_{11} + \lambda_{12}} \xi(\text{VaR}_\kappa(S); \lambda_{11} + \lambda_{12}; +2\lambda_{2j}) \\
&\quad \left. + 4\alpha_{11}\alpha_{12}\alpha_{21}\alpha_{22} \frac{\lambda_{1i}\lambda_{2j}}{(\lambda_{11} + \lambda_{12})(\lambda_{21} + \lambda_{22})} \xi(\text{VaR}_\kappa(S); \lambda_{11} + \lambda_{12}; \lambda_{21} + \lambda_{22}) \right\}.
\end{aligned}$$

5 Approximation methods for TVaR-based allocation

We have seen in the previous sections that it is possible to have an exact expression for the TVaR of a sum of several dependent random variables and the contribution of each random variable to the aggregate TVaR for some specific situations, in particular when using the FGM copula. For most copulas, it is more complicated to directly calculate this risk measure. [Embrechts and Puccetti \(2007\)](#) proposed an algorithm to compute numerically the cdf of the sum of two random variables joined by a copula. They used an approximation of the set $\{(x_1, x_2) \in [0, +\infty)^2 : x_1 + x_2 \leq s\}$ by a countable union of disjoint rectangles to obtain an evaluation of $F_S(s)$ with $S = X_1 + X_2$. In the present paper, we expose a simple alternative to approximate this cumulative distribution function with the use of common discretization methods that can be found in [Klugman et al. \(2008\)](#). Then we evaluate the TVaR and its contributions when the random variables are linked by any copula. The method is here exposed for two random variables but can be expanded to more random variables as shown in the numerical applications.

5.1 Discretization methods

We use three discretization methods in our study that are defined just below. For these three methods, we suppose that X is a continuous random variable with cdf F_X and that h is the discretization span.

Definition 9 (Lower method) *The lower method provides a probability mass function of the discretized random variable \tilde{X} given by*

$$\begin{cases} f_{\tilde{X}}(0) = 0 \\ f_{\tilde{X}}(jh) = F_X(jh) - F_X((j-1)h), \quad \text{for } j = 1, 2, \dots \end{cases}$$

Definition 10 (Upper method) *The upper method provides a probability mass function of the*

discretized random variable \tilde{X} given by

$$\begin{cases} f_{\tilde{X}}(0) = F_X(h) \\ f_{\tilde{X}}(jh) = F_X((j+1)h) - F_X(jh), \quad \text{for } j = 1, 2, \dots \end{cases}$$

Definition 11 (Mean preserving method) *The mean preserving method provides a probability mass function of the discretized random variable \tilde{X} given by*

$$\begin{cases} f_{\tilde{X}}(0) = 1 - \frac{E[X \wedge h]}{h} \\ f_{\tilde{X}}(jh) = \frac{2E[X \wedge jh] - E[X \wedge (j-1)h] - E[X \wedge (j+1)h]}{h}, \quad \text{for } j = 1, 2, \dots \end{cases}$$

This method ensures that the mean of the discretized distribution is the same as the original distribution.

Remark 12 *It is shown e.g. in Müller and Stoyan (2002) and Denuit et al. (2005) that $X \leq_{sd} \tilde{X}$ under the lower method, $\tilde{X} \leq_{sd} X$ under the upper method, and $X \leq_{icx} \tilde{X}$ under the mean preserving method, where \leq_{sd} and \leq_{icx} designate the stochastic dominance order and the increasing convex order respectively (see the same references for the definitions).*

5.2 The bivariate case

Suppose that we have two continuous distributed risks X_1 and X_2 . The joint cdf F_{X_1, X_2} is defined by a fixed copula C which introduces a dependence structure between the risks. We discretize X_1 and X_2 with one of the three methods described before. We denote by \tilde{X}_1, \tilde{X}_2 the discretized random variables obtained and keep the same dependence relation between these two new random variables with the copula C . Then, we define $\tilde{S} = \tilde{X}_1 + \tilde{X}_2$ that we use two approximate $S = X_1 + X_2$.

For a constant discretization span h , the cdf of $(\tilde{X}_1, \tilde{X}_2)$ for $k \geq 0$ and $l \geq 0$ is

$$F_{\tilde{X}_1, \tilde{X}_2}(kh, lh) = \sum_{i=0}^k \sum_{j=0}^l Pr(\tilde{X}_1 = ih, \tilde{X}_2 = jh).$$

The cdf of \tilde{S} for $j \geq 0$ is given by

$$F_{\tilde{S}}(jh) = \sum_{i=0}^j Pr(\tilde{S} = ih),$$

where the probability mass function (pmf) of \tilde{S} is

$$\begin{aligned} Pr(\tilde{S} = 0) &= Pr(\tilde{X}_1 = 0; \tilde{X}_2 = 0), \text{ and} \\ Pr(\tilde{S} = jh) &= \sum_{i=0}^j Pr(\tilde{X}_1 = ih; \tilde{X}_2 = (j-i)h), \text{ for } j = 1, 2, \dots \end{aligned}$$

The joint pmf of $(\tilde{X}_1, \tilde{X}_2)$ is obtained with the copula as follows

$$\begin{aligned}
Pr(\tilde{X}_1 = 0, \tilde{X}_2 = 0) &= C\left(F_{\tilde{X}_1}(0), F_{\tilde{X}_2}(0)\right), \\
Pr(\tilde{X}_1 = 0, \tilde{X}_2 = jh) &= C\left(F_{\tilde{X}_1}(0), F_{\tilde{X}_2}(jh)\right) - C\left(F_{\tilde{X}_1}(0), F_{\tilde{X}_2}((j-1)h)\right), \\
Pr(\tilde{X}_1 = ih, \tilde{X}_2 = 0) &= C\left(F_{\tilde{X}_1}(ih), F_{\tilde{X}_2}(0)\right) - C\left(F_{\tilde{X}_1}((i-1)h), F_{\tilde{X}_2}(0)\right), \\
Pr(\tilde{X}_1 = ih, \tilde{X}_2 = jh) &= C\left(F_{\tilde{X}_1}(ih), F_{\tilde{X}_2}(jh)\right) - C\left(F_{\tilde{X}_1}((i-1)h), F_{\tilde{X}_2}(jh)\right) \\
&\quad - C\left(F_{\tilde{X}_1}(ih), F_{\tilde{X}_2}((j-1)h)\right) + C\left(F_{\tilde{X}_1}((i-1)h), F_{\tilde{X}_2}((j-1)h)\right).
\end{aligned}$$

Then, the TVaR of S can be approximated by the TVaR of \tilde{S} which is given by

$$\begin{aligned}
TVaR_\kappa(\tilde{S}) &= \frac{E[\tilde{S} \times 1_{\{\tilde{S} > VaR_\kappa(\tilde{S})\}}] + VaR_\kappa(\tilde{S}) \left(Pr(\tilde{S} \leq VaR_\kappa(\tilde{S})) - \kappa\right)}{1 - \kappa} \\
&= \frac{E[\tilde{S} \times 1_{\{\tilde{S} > k_0h\}}] + k_0h \left(Pr(\tilde{S} \leq k_0h) - \kappa\right)}{1 - \kappa},
\end{aligned}$$

where $VaR_\kappa(\tilde{S}) = k_0h$.

The TVaR-based allocation of risk X_i over the global risk S can be approximated by the TVaR-based allocation of risk \tilde{X}_i over the global discretized risk \tilde{S} where

$$\begin{aligned}
TVaR_\kappa(\tilde{X}_i; \tilde{S}) &= \frac{E\left[\tilde{X}_i \times 1_{\{\tilde{S} > VaR_\kappa(\tilde{S})\}}\right] + \beta_{\tilde{S}} E\left[\tilde{X}_i \times 1_{\{\tilde{S} = VaR_\kappa(\tilde{S})\}}\right]}{1 - \kappa} \\
&= \frac{E\left[\tilde{X}_i \times 1_{\{\tilde{S} > k_0h\}}\right] + \beta_{\tilde{S}} E\left[\tilde{X}_i \times 1_{\{\tilde{S} = k_0h\}}\right]}{1 - \kappa},
\end{aligned}$$

$$\text{where } \beta_{\tilde{S}} = \begin{cases} \frac{Pr(\tilde{S} \leq k_0h) - \kappa}{Pr(\tilde{S} = k_0h)}, & \text{if } Pr(\tilde{S} = k_0h) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 13 In corollary 4.6 of Müller and Scarsini (2001), it is shown that if (X_1, X_2) and $(\tilde{X}_1, \tilde{X}_2)$ are random vectors with a common conditionally increasing copula and if $X_i \leq_{cx} \tilde{X}_i$ for $i = 1, 2$, then for all non-negative scalars a_1 and a_2 we have

$$a_1X_1 + a_2X_2 \leq_{cx} a_1\tilde{X}_1 + a_2\tilde{X}_2.$$

This result also holds for the increasing convex order since it is implied by the convex order. From Bäuerle and Müller (2006), it follows that

$$TVaR_\kappa(a_1X_1 + a_2X_2) \leq TVaR_\kappa(a_1\tilde{X}_1 + a_2\tilde{X}_2)$$

for $\kappa \in (0, 1)$.

5.3 Numerical applications

5.3.1 Bivariate case

Suppose that X_1 and X_2 are exponentially distributed with parameters $\lambda_1 = 1/2$ and $\lambda_2 = 1/3$ respectively. The dependence between these two risks is defined by the bivariate FGM copula with parameter $\theta^{FGM} = 0.8$. This value of θ^{FGM} implicates a correlation of 0.2 between X_1 and X_2 . Figure 1 in the appendix illustrates the accuracy of the approximations of the cdf of S_2 when using the three discretization methods. Indeed, the decrease of the discretization span h implicates a convergence of the discretized cdf to the real one.

The discretization methods allow an approximation of the cdf of S_2 with any copula. We illustrate the use of the FGM, the Clayton, the Frank and the Gumbel copula. The copulas' parameters are chosen such that we have the same coefficient of correlation between X_1 and X_2 . Figure 2 in the appendix shows the impact of the copulas on the cdf of S_2 which is discretized with the mean preserving method when the correlation coefficient is fixed at 0.2. The first graph displays the drawing of the approximated cdf's of S_2 . We then do the difference between a dependent cdf and the independent cdf and trace this difference for the FGM, the Frank, the Clayton and the Gumbel copula in the second graph of Figure 2. These two graphs highlight the fact that the Clayton copula introduces dependence in lower values and that the Gumbel copula permits dependence in the upper queue.

In Figure 3 in the appendix, we trace the differences between the TVaR of S_2 using dependent risks with one of the copulas discussed before and the TVaR of S_2 using the independent copula against the risk level κ . This is done for three increasing values of correlation between X_1 and X_2 . Given that the FGM copula only allows weak dependence it just appears on the first graph. The graphs confirm the fact that the Gumbel copula introduces dependence in high values.

Tables 4 and 5 expose the numerical results for the TVaR and the TVaR-based allocation for X_1 and X_2 with the four copulas discussed above and confidence levels equal to 0.99 and 0.995. The calculations are done with the mean preserving discretization method with span $h = 0.05$ for the four copulas and also with the exact expression for the FGM copula. The tables attest the good precision of the approximation method and confirm the high dependence values inserted by the Gumbel copula.

5.3.2 Trivariate case

We illustrate here the difference between the exact and the approximated methods with three risk variables dependent through a trivariate FGM copula. Suppose that X_1 , X_2 and X_3 are exponentially distributed with parameters $\lambda_1 = 1/2$, $\lambda_2 = 1/3$ and $\lambda_3 = 1/5$ respectively. The copula parameters θ_i^{FGM} , θ_{ij}^{FGM} and θ_{ijk}^{FGM} , for $i = 1, 2, 3$, $j = 1, 2, 3$, $k = 1, 2, 3$, $j \neq i$, $k \neq i$

Copula	$TVaR_{0.99}(S_2)$	$TVaR_{0.99}(X_1; S_2)$	% of $TVaR_{0.99}(S_2)$	$TVaR_{0.99}(X_2; S_2)$	% of $TVaR_{0.99}(S_2)$
FGM exact	20.9561	6.0998	29.1%	14.8563	70.9%
FGM M.P.	20.9574	6.1003	29.1%	14.8571	70.9%
Clayton M.P.	20.7918	5.9419	28.6%	14.8499	71.4%
Frank M.P.	21.0612	6.2158	29.5%	14.8454	70.5%
Gumbel M.P.	22.9669	7.7988	34.0%	15.1682	66%

Table 4: TVaR and TVaR-based allocation for S_2 with $\kappa = 0.99$.

Copula	$TVaR_{0.995}(S_2)$	$TVaR_{0.995}(X_1; S_2)$	% of $TVaR_{0.995}(S_2)$	$TVaR_{0.995}(X_2; S_2)$	% of $TVaR_{0.995}(S_2)$
FGM exact	23.0839	6.3523	27.5%	16.7316	72.5%
FGM M.P.	23.0859	6.3530	27.5%	16.7329	72.5%
Clayton M.P.	22.9135	6.1776	27.0%	16.7359	73.0%
Frank M.P.	23.2014	6.4953	28.0%	16.7061	72.0%
Gumbel M.P.	26.0088	8.9850	34.5%	17.0237	65.5%

Table 5: TVaR and TVaR-based allocation for S_2 with $\kappa = 0.995$.

and $k \neq j$ are all fixed to 1. As for the bivariate case, we show in Figure 4 of the appendix the convergence to the real cdf of S_3 of the discretized cdf's when using the three discretization methods.

Tables 6 and 7 compare the numerical results for the TVaR and the TVaR-based allocation for the three risks with confidence levels equal to 0.99 and 0.995 between exact expressions and approximated results using the mean preserving discretization method with span $h = 0.3$. As for the bivariate case, they show a satisfying accuracy of the approximation method.

Method	$TVaR_{0.99}(S_3)$	$TVaR_{0.99}(X_1; S_3)$	$TVaR_{0.99}(X_2; S_3)$	$TVaR_{0.99}(X_3; S_3)$
Exact	37.5988	4.1726	8.4033	25.0230
Mean P.	37.6062	4.1760	8.4060	25.0243

Table 6: TVaR and TVaR-based allocation for S_3 with $\kappa = 0.99$.

6 Conclusion

This paper introduces the use of copulas in TVaR-based capital allocation. We obtain explicit expressions for the TVaR and TVaR-based allocation for risks that have exponential and mixture of exponentials distributions linked by a FGM copula. The handy form of this copula permits a direct calculation of the coherent risk measure and its decomposition when we suppose only two different risks. In the multivariate situation, we use divided differences as in [Chiragiev and Landsman \(2007\)](#). For other copulas, we present approximations for the TVaR and the TVaR-based allocation using three discretization methods for continuous distributions.

Method	$TVaR_{0.995}(S_3)$	$TVaR_{0.995}(X_1; S_3)$	$TVaR_{0.995}(X_2; S_3)$	$TVaR_{0.995}(X_3; S_3)$
Exact	41.1177	4.2044	8.6536	28.2597
Mean P.	41.1262	4.2080	8.6567	28.2616

Table 7: TVaR and TVaR-based allocation for S_3 with $\kappa = 0.995$.

Acknowledgements

The authors acknowledge the Natural Sciences and Engineering Research Council of Canada and the Chaire d'actuariat de l'Universit Laval for their support. This work has also been partially supported by the French Research National Agency (ANR) under the reference ANR-08-BLAN-0314-01. The authors would like to thank an anonymous referee for her/his comments.

APPENDIX

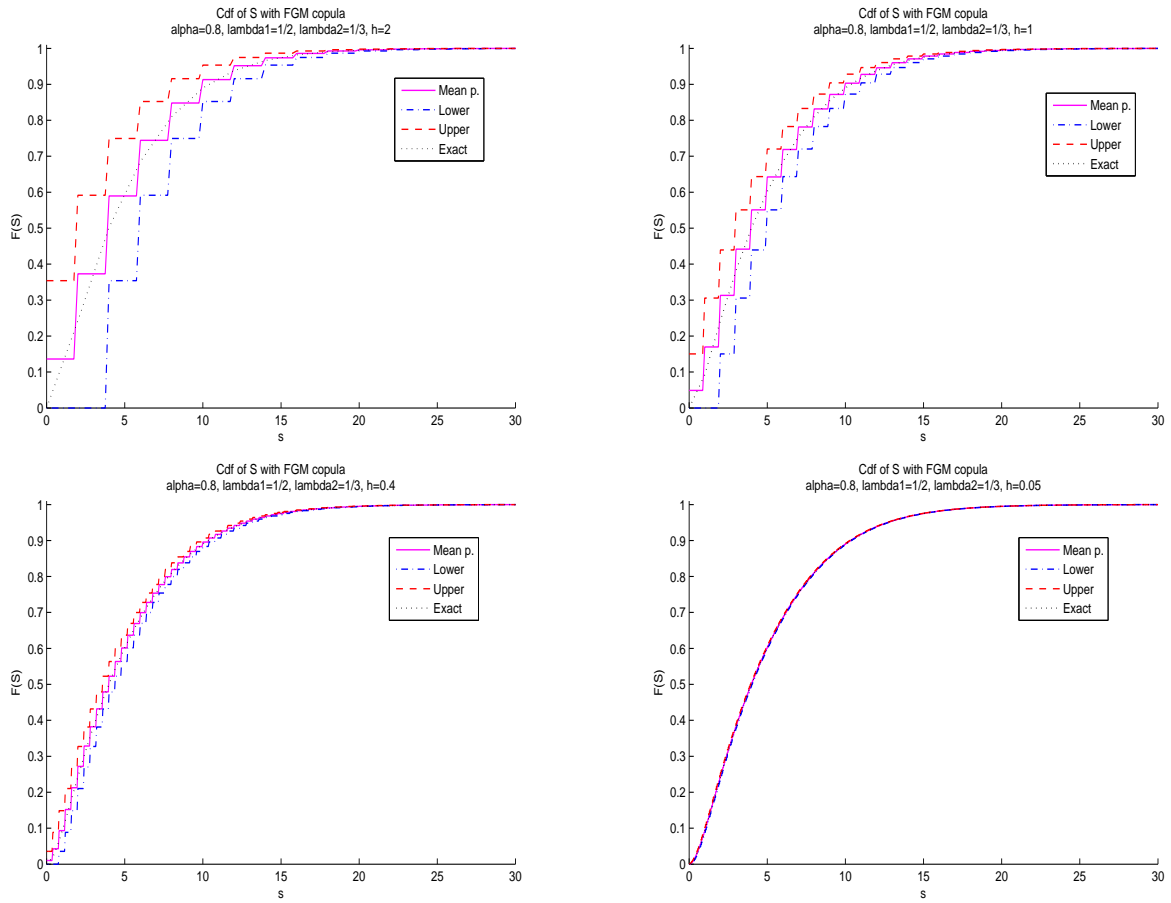


Figure 1: Discretized cdf's vs exact cdf for 2 risks

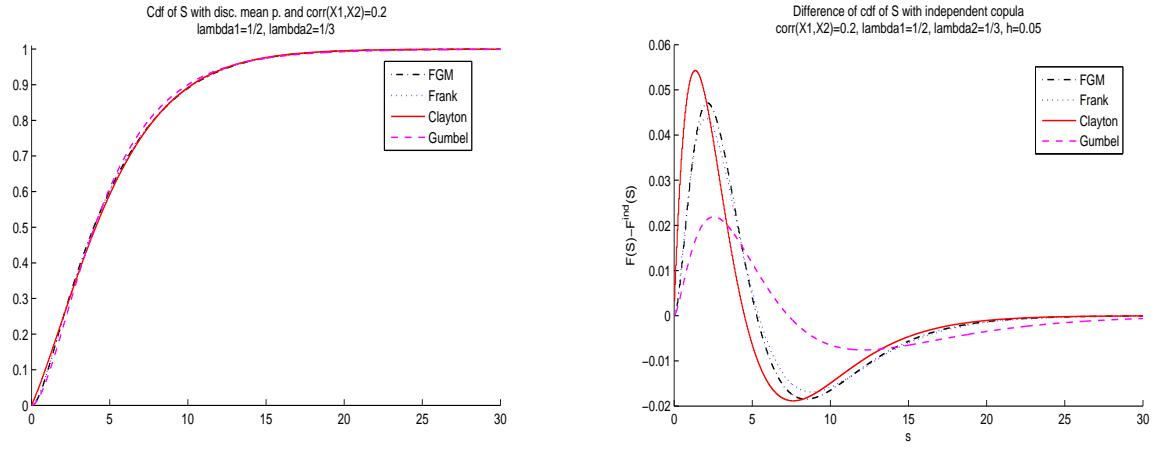


Figure 2: Comparison of copulas with $Corr(X_1, X_2) = 0.2$

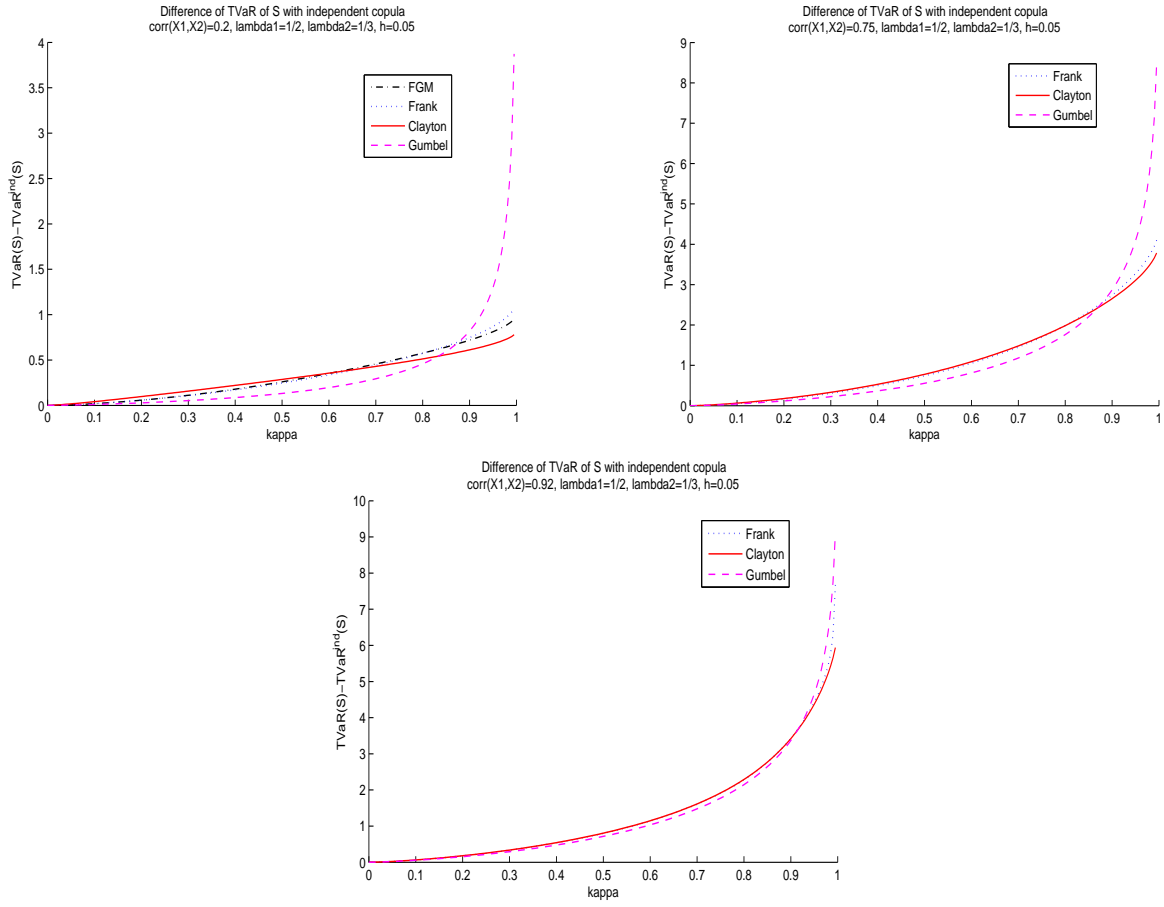


Figure 3: TVaR of S with different copulas and correlation coefficients

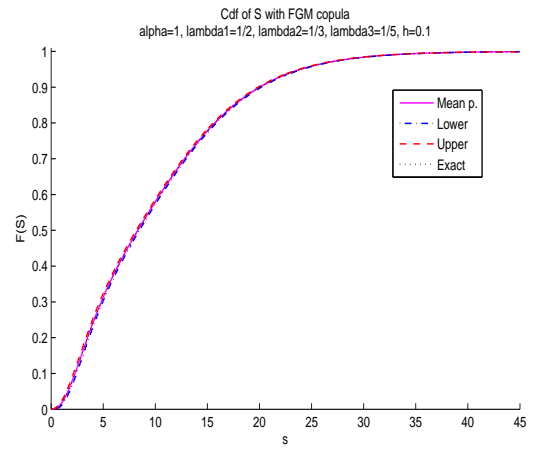
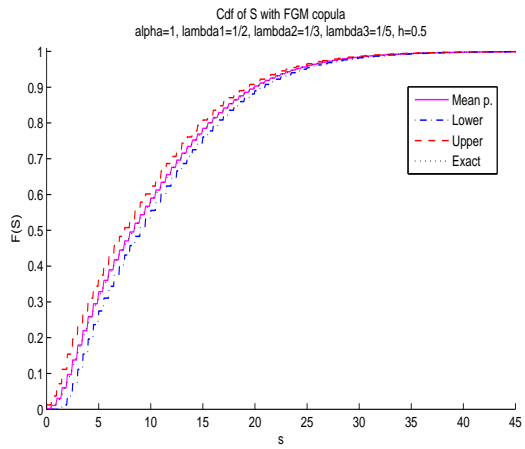
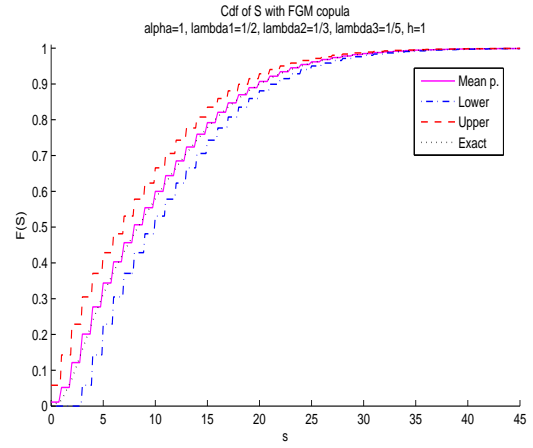
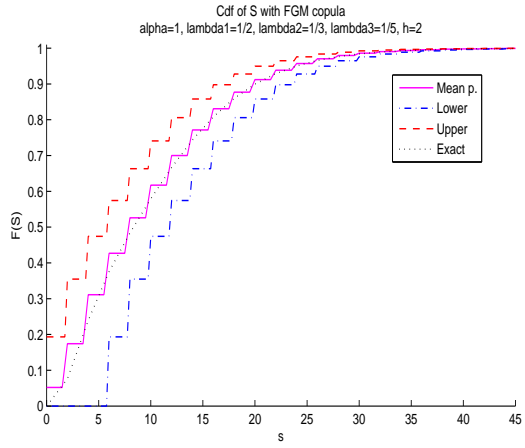


Figure 4: Discretized cdf's vs exact cdf for 3 risks

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