



**HAL**  
open science

# Bernstein processes, Euclidean Quantum Mechanics and Interest Rate Models

Paul Lescot

► **To cite this version:**

Paul Lescot. Bernstein processes, Euclidean Quantum Mechanics and Interest Rate Models. 2009. hal-00431260v2

**HAL Id: hal-00431260**

**<https://hal.science/hal-00431260v2>**

Submitted on 26 Oct 2011

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

**BERNSTEIN PROCESSES, EUCLIDEAN QUANTUM  
MECHANICS AND INTEREST RATE MODELS**

PAUL LESCOT

Laboratoire de Mathématiques Raphaël Salem  
UMR 6085 CNRS-Université de Rouen

Bureau M2.24

Avenue de l'Université, B.P.12

Technopôle du Madrillet

F76801 Saint-Etienne-du-Rouvray

Tél. (+ 33) (0)2 32 95 52 24

Fax (+ 33) (0)2 32 95 52 86

Paul.Lescot@univ-rouen.fr

<http://www.univ-rouen.fr/LMRS/Persopage/Lescot>

November 28th, 2009

Typeset by  $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

## ABSTRACT

We give an exposition, following joint works with J.-C. Zambrini, of the link between euclidean quantum mechanics, Bernstein processes and isovectors for the heat equation. A new application to mathematical finance is then discussed.

## 1. EUCLIDEAN QUANTUM MECHANICS

Schrödinger's equation for a (possibly time-dependent) potential  $V(t, q)$  :

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V\psi \equiv H\psi$$

on  $L^2(\mathbf{R}^d, dq)$  can be written, in space dimension  $d = 1$ , and for  $m = 1$ :

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2} \frac{\partial^2 \psi}{\partial q^2} + V\psi .$$

We shall henceforth treat  $\theta = \sqrt{\hbar}$  as a new parameter.

In Zambrini's Euclidean Quantum Mechanics (see e.g. [1]), this equation splits into :

$$\theta^2 \frac{\partial \psi}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \psi}{\partial q^2} + V\psi \quad (\mathcal{C}_1^{(V)})$$

and

$$-\theta^2 \frac{\partial \psi}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \psi}{\partial q^2} + V\psi , \quad (\mathcal{C}_2^{(V)})$$

the probability density being given, not by  $\psi\bar{\psi}$  as in the usual quantum mechanics, but by  $\eta\eta_*$ ,  $\eta$  and  $\eta_*$  denoting respectively an everywhere strictly positive solution of  $(\mathcal{C}_1^{(V)})$  and an everywhere strictly positive solution of  $(\mathcal{C}_2^{(V)})$ . To these data is associated a *Bernstein process*  $z$ , satisfying the stochastic differential equation

$$dz(t) = \theta dw(t) + \tilde{B}(t, z(t))dt \quad ((\mathcal{B}))$$

relatively to the canonical increasing filtration of the brownian  $w$ , and the stochastic differential equation

$$d_* z(t) = \theta d_* w_*(t) + \tilde{B}_*(t, z(t))dt \quad ((\mathcal{B}_*))$$

relatively to the canonical decreasing filtration of another brownian  $w_*$ , where

$$\tilde{B} \equiv_{def} \theta^2 \frac{\frac{\partial \eta}{\partial q}}{\eta}$$

and

$$\tilde{B}_* \equiv_{def} -\theta^2 \frac{\frac{\partial \eta_*}{\partial q}}{\eta_*} .$$

Setting  $S = -\theta^2 \ln(\psi)$ , equation  $(\mathcal{C}_1^{(V)})$  becomes the Hamilton–Jacobi–Bellman equation :

$$\frac{\partial S}{\partial t} = -\frac{\theta^2}{2} \frac{\partial^2 S}{\partial q^2} + \frac{1}{2} \left( \frac{\partial S}{\partial q} \right)^2 - V . \quad (\mathcal{C}_3^{(V)})$$

Modulo the addition of the derivatives  $E = -\frac{\partial S}{\partial t}$  and  $B = -\frac{\partial S}{\partial q}$  as auxiliary unknown functions,  $(\mathcal{C}_3^{(V)})$  is equivalent to the vanishing of the following differential forms :

$$\omega = dS + Edt + Bdq ,$$

$$\Omega = d\omega = dEdt + dBdq ,$$

and

$$\beta = (E + \frac{B^2}{2} - V)dqdt + \frac{\theta^2}{2}dBdt$$

on a 2-dimensional submanifold of  $\mathbf{M} = \mathbf{R}^5$  ( $(t, q, S, E, B)$  being now considered as *independent* variables). Let then  $L = \frac{1}{2}B^2 + V$  denote the *formal lagrangian* ,

$$\omega_{PC} = Edt + Bdq = \omega - dS$$

the *Poincaré–Cartan form* , and  $I$  the ideal of  $\mathcal{A} = \wedge T^*(\mathbf{M})$  generated by  $\omega$ ,  $d\omega$  and  $\beta$ . By an *isovector* we shall mean a vector field  $N$  on  $\mathbf{M}$  such that

$\mathcal{L}_N(I) \subseteq I$  ; because of the linearity of  $(\mathcal{C}_1^{(V)})$  the Lie algebra  $\mathcal{G}_V$  of these isovectors contains an infinite-dimensional abelian ideal  $\mathcal{J}_V$  , that possesses a canonical supplement  $\mathcal{H}_V$ .

In the free case ( $V = 0$ ) this canonical supplement has dimension 6 and admits a natural basis, each element of which corresponds to a symmetry of the underlying physical system.

Let  $\Phi_N = -N(S)$  be the *phase* associated to  $N$ , and

$$D \equiv_{def} \frac{\partial}{\partial t} + B\frac{\partial}{\partial q} + \frac{\hbar}{2}\frac{\partial^2}{\partial q^2}$$

the *formal Ito differential* along the Bernstein process  $z$ . The following purely algebraic results are analogs of well-known theorems of classical analytical mechanics:

**Theorem 1.1.** *For each  $N \in \mathcal{H}_V$  one has :*

- (1)  $\mathcal{L}_N(\omega_{PC}) = d\Phi_N$  ;
- (2)  $\mathcal{L}_N(\Omega) = 0$  ;
- (3)  $\mathcal{L}_N(L) + L\frac{dN^t}{dt} = D\Phi_N$  .

For a detailed proof see [7], and for complete calculations in the free case ( $V = 0$ ) see [6].

## 2. ROSENCRANS' THEOREM

Let  $N$  be an *isovector* (for  $V = 0$ ), let  $\psi$  be a solution of  $\mathcal{C}_1^{(0)}$ , and let

$$S = -\theta^2 \ln(\psi);$$

then  $e^{\alpha N}$  maps  $(t, q, S, E, B)$  to  $(t_\alpha, q_\alpha, S_\alpha, E_\alpha, B_\alpha)$ ; setting

$$e^{-\frac{S_\alpha}{\theta^2}} = \psi_\alpha(t_\alpha, q_\alpha),$$

it follows that  $\psi_\alpha$  is also a solution of  $(\mathcal{C}_1^{(0)})$ . We shall denote

$$e^{\alpha \hat{N}} : \psi \mapsto \psi_\alpha$$

the associated one-parameter group; it is easily seen that, for

$$N = N^t \frac{\partial}{\partial t} + N^q \frac{\partial}{\partial q} - \Phi_N \frac{\partial}{\partial S} + \dots$$

then

$$\hat{N} = -N^t \frac{\partial}{\partial t} - N^q \frac{\partial}{\partial q} + \frac{1}{\theta^2} \Phi_N,$$

and it follows that  $N \mapsto -\hat{N}$  is a homomorphism of Lie algebras.

Let  $\eta_u$  denote the solution de  $(\mathcal{C}_1^{(0)})$  with initial condition  $u$ :

$$\frac{\partial \eta_u}{\partial t} = -\frac{\theta^2}{2} \frac{\partial^2 \eta_u}{\partial q^2},$$

and

$$\eta_u(0, q) = u(q).$$

Let us set :

$$\rho_N(\alpha, t, q) = (e^{\alpha \hat{N}} \eta_u)(t, q)$$

and

$$\psi^N(\alpha, q) \equiv_{def} \rho_N(\alpha, 0, q).$$

Then

**Theorem 2.1.** ([6], pp.321-322)

$\psi^N$  satisfies :

$$\frac{\partial \psi^N}{\partial \alpha} = -N^t(0, q) \left( -\frac{\theta^2}{2} \frac{\partial^2 \psi^N}{\partial \alpha^2} \right) - N^q(0, q) \frac{\partial \psi^N}{\partial q} + \frac{1}{\theta^2} \Phi_N(0, q) \psi^N$$

and

$$\psi^N(0, q) = u(q).$$

Whence :

**Corollary 2.2.** *Let  $N$  be chosen so that  $N^t(0, q) = -1$ ,  $N^q(0, q) = -\frac{1}{\theta^2}(aq + b)$  and  $\Phi_N(0, q) = cq^2 + dq + f$ , where  $a, b, c, d, f$  denote real constants, then*

$$\eta_u^V(t, q) \equiv_{def} \psi^N(t, q)$$

*satisfies the “backwards heat equation with drift term  $D(q) = aq + b$  and quadratic potential  $V(q) = cq^2 + dq + f + \frac{1}{2\hbar^2}(D(q))^2 - \frac{a}{2}$ ”, corresponding to a vector potential*

$$A = \frac{aq + b}{\theta^2} :$$

$$\theta^2 \frac{\partial \eta_u^V}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \eta_u^V}{\partial q^2} + (aq + b) \frac{\partial \eta_u^V}{\partial q} + (cq^2 + dq + f) \eta_u^V \quad (\mathcal{C}_4^{(V)})$$

and

$$\eta_u^V(0, q) = u(q) .$$

*(In the case  $D(q) = 0$ , the potential is given by  $V(q) = cq^2 + dq + f$  and  $\eta_u^{(V)}$  satisfies  $\mathcal{C}_1^{(V)}$ ; for the general case, cf. [1], pp.71–72).*

## 3. THE CASE OF A LINEAR POTENTIAL

Here  $V(q) = \lambda q$  ; it appears that :

$$\eta_u^V(t, q) = e^{-\frac{\lambda^2}{6\theta^2}t^3} e^{\frac{\lambda tq}{\theta^2}} \eta_u(t, q - \lambda \frac{t^2}{2}) .$$

Then  $\eta_u^V$  satisfies  $(\mathcal{C}_1^{(V)})$  ; the drift term can be written :

$$\begin{aligned} \tilde{B}_V(t, q) &= \theta^2 \frac{\partial}{\partial q} (\ln(\eta_u^V)(t, q)) \\ &= \theta^2 \frac{\partial}{\partial q} \left( -\frac{\lambda^2}{6\theta^2}t^3 + \frac{\lambda tq}{\theta^2} + \ln(\eta_u)(t, q - \lambda \frac{t^2}{2}) \right) \\ &= \lambda t + \theta^2 \frac{\partial}{\partial q} (\ln(\eta_u))(t, q - \lambda \frac{t^2}{2}) \\ &= \lambda t + \tilde{B}(t, q - \lambda \frac{t^2}{2}) . \end{aligned}$$

Therefore, we have :

$$dz_V(t) = \theta dw(t) + \lambda t dt + \tilde{B}(t, z_V(t) - \lambda \frac{t^2}{2}) dt .$$

Let us set  $y(t) \equiv_{def} z_V(t) - \lambda \frac{t^2}{2}$  ; then

$$\begin{aligned} dy(t) &= dz_V(t) - \lambda t dt \\ &= \theta dw(t) + \tilde{B}(t, y(t)) dt , \end{aligned}$$

*i.e.*  $y(t)$  is a Bernstein process  $z(t)$  associated to solution  $\eta_u$  of the free equation  $(\mathcal{C}_1^{(0)})$ , and

$$z_V(t) = z(t) + \lambda \frac{t^2}{2} .$$

In other terms, the ‘‘perturbation’’ by a constant force  $\lambda$  produces a deterministic translation by  $\lambda \frac{t^2}{2}$ , which is logical on physical grounds.

Details are given in [7].



## 4. THE CASE OF A QUADRATIC POTENTIAL

For  $V(t, q) = \frac{\omega^2 q^2}{2}$ , one finds :

$$\eta_u^V(t, q) = \cosh(\omega t)^{-\frac{1}{2}} e^{\frac{\omega q^2}{2\theta^2} \tanh(\omega t)} \eta_u\left(\frac{\tanh(\omega t)}{\omega}, \frac{q}{\cosh(\omega t)}\right).$$

Whence

$$\tilde{B}_V(t, q) = \omega q \tanh(\omega t) + \frac{1}{\cosh(\omega t)} \tilde{B}\left(\frac{\tanh(\omega t)}{\omega}, \frac{q}{\cosh(\omega t)}\right).$$

Details are exposed in [9], §5, and a more general formula is proved in [5].

5. AN EXAMPLE WITH  $D \neq 0$ 

Here, we take  $a = \theta^2 \beta$ , and  $b = c = d = f = 0$ . With the notations of [1], pp. 71–72 (but, of course, replacing  $\mathbf{R}^3$  with  $\mathbf{R}$ ),  $A = \beta q$ , and

$$V(q) = \frac{\beta^2 q^2}{2} - \frac{\beta \theta^2}{2} .$$

Then  $\eta_u^V$  satisfies

$$\frac{\partial \eta_u^V}{\partial t} = -\frac{\theta^2}{2} \frac{\partial^2 \eta_u^V}{\partial q^2} + \beta q \frac{\partial \eta_u^V}{\partial q} .$$

It is easy to see that :

$$\eta_u^V(t, q) = \eta_u \left( \frac{1}{2\beta} (e^{2\beta t} - 1), e^{\beta t} q \right) .$$

The drift term (cf. [1], p.72) is given by :

$$\begin{aligned} \tilde{B}_V(t, q) &= \theta^2 \frac{\partial}{\partial q} (\ln(\eta_u^V)(t, q)) - A(t, q) \\ &= e^{\beta t} \tilde{B} \left( \frac{1}{2\beta} (e^{2\beta t} - 1), e^{\beta t} q \right) - \beta q . \end{aligned}$$

In particular, for  $\eta = 1$ , one finds  $\eta_u = 1$ ,  $\tilde{B}_V(t, q) = -\beta q$ ,  $z(t) = \theta w(t)$  and

$$dz_V(t) = \theta dw(t) - \beta z_V(t) dt ,$$

*i.e.*  $z_V(t)$  is an Ornstein–Uhlenbeck process, as expected.

## 6. ONE-FACTOR AFFINE INTEREST RATE MODELS

Such a model is characterized by the instantaneous rate  $r(t)$ , satisfying

$$dr(t) = \sqrt{\alpha r(t) + \beta} dw(t) + (\phi - \lambda r(t)) dt$$

(cf. [3]).

Let us set  $\tilde{\phi} = \phi + \frac{\lambda\beta}{\alpha}$ ; then :

**Theorem 6.1.** *Let*

$$z(t) = \sqrt{\alpha r(t) + \beta};$$

*then  $z(t)$  is a Bernstein process for*

$$\theta = \frac{\alpha}{2}$$

*and the potential*

$$V(t, q) = \frac{A}{q^2} + Bq^2$$

*where :*

$$A = \frac{\alpha^2}{8} \left( \tilde{\phi} - \frac{\alpha}{4} \right) \left( \tilde{\phi} - \frac{3\alpha}{4} \right)$$

*and*

$$B = \frac{\lambda^2}{8}.$$

**Corollary 6.2.** *The isovector algebra  $\mathcal{H}_V$  associated with  $V$  has dimension 6 if and only if  $A = 0$ ; in the opposite case, it has dimension 4.*

The condition  $A = 0$  is equivalent to  $\tilde{\phi} \in \left\{ \frac{\alpha}{4}, \frac{3\alpha}{4} \right\}$ . I am now able to explain this in terms of Bessel processes (see [4] and [5]).

ACKNOWLEDGEMENTS

This work is an updated version of the text of my lecture given at ISMANS on October 24th, 2008. I am happy to thank the organizers Alain LE MEHAUTE and Alexandre WANG for their kind invitation and their hospitality.

### References

1. A.B.Cruzeiro and J.-C.Zambrini, *Malliavin calculus and Euclidean Quantum Mechanics.I.Functional Calculus*, J.Funct.Anal. **96(1)** (1991), 62–95.
2. S.Hénon, *Un modèle de taux avec volatilité stochastique*, Ph D thesis (2005).
3. B. Leblanc , O. Scaillet, *Path dependent options on yields in the affine term structure model*, Finance and Stochastics **2(4)** (1998), 349–367.
4. P. Lescot, *A note on affine interest rate models*, submitted (2009).
5. P. Lescot, P. Patie, in preparation.
6. P.Lescot, J.-C. Zambrini, *Isoectors for the Hamilton–Jacobi–Bellman Equation, Formal Stochastic Differentials and First Integrals in Euclidean Quantum Mechanics*, Proceedings of the Ascona conference (2002), Birkhäuser (Progress in Probability, vol 58), 2004, pp. 187–202.
7. P.Lescot, J.-C. Zambrini, *Probabilistic deformation of contact geometry, diffusion processes and their quadratures*, Seminar on Stochastic Analysis, Random Fields and applications V, Birkhäuser(Progress in Probability, vol. 59), 2008, pp. 203-226.
8. S.I.Rosencrans, *Perturbation Algebra of an Elliptic Operator*, Journal of Mathematical Analysis and Applications **56** (1976), 317–329.
9. J.-C. Zambrini, *From the geometry of parabolic PDE to the geometry of SDE*, Mathematical Analysis of Random Phenomena, World Sci., Hackensack, NJ, 2007, pp. 213–230.

©Paul Lescot, November 2009