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**BERNSTEIN PROCESSES, EUCLIDEAN QUANTUM
MECHANICS AND INTEREST RATE MODELS**

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ABSTRACT

We give an exposition, following joint works with J.-C. Zambrini, of the link between euclidean quantum mechanics, Bernstein processes and isovectors for the heat equation. A new application to mathematical finance is then discussed.

1. EUCLIDEAN QUANTUM MECHANICS

Schrödinger's equation for a (possibly time-dependent) potential $V(t, q)$:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V\psi \equiv H\psi$$

on $L^2(\mathbf{R}^d, dq)$ can be written, in space dimension $d = 1$, and for $m = 1$:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2} \frac{\partial^2 \psi}{\partial q^2} + V\psi .$$

We shall henceforth treat $\theta = \sqrt{\hbar}$ as a new parameter.

In Zambrini's Euclidean Quantum Mechanics (see e.g. [1]), this equation splits into :

$$\theta^2 \frac{\partial \psi}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \psi}{\partial q^2} + V\psi \quad (\mathcal{C}_1^{(V)})$$

and

$$-\theta^2 \frac{\partial \psi}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \psi}{\partial q^2} + V\psi , \quad (\mathcal{C}_2^{(V)})$$

the probability density being given, not by $\psi\bar{\psi}$ as in the usual quantum mechanics, but by $\eta\eta_*$, η and η_* denoting respectively an everywhere strictly positive solution of $(\mathcal{C}_1^{(V)})$ and an everywhere strictly positive solution of $(\mathcal{C}_2^{(V)})$. To these data is associated a *Bernstein process* z , satisfying the stochastic differential equation

$$dz(t) = \theta dw(t) + \tilde{B}(t, z(t))dt \quad ((\mathcal{B}))$$

relatively to the canonical increasing filtration of the brownian w , and the stochastic differential equation

$$d_* z(t) = \theta d_* w_*(t) + \tilde{B}_*(t, z(t))dt \quad ((\mathcal{B}_*))$$

relatively to the canonical decreasing filtration of another brownian w_* , where

$$\tilde{B} \equiv_{def} \theta^2 \frac{\frac{\partial \eta}{\partial q}}{\eta}$$

and

$$\tilde{B}_* \equiv_{def} -\theta^2 \frac{\frac{\partial \eta_*}{\partial q}}{\eta_*} .$$

Setting $S = -\theta^2 \ln(\psi)$, equation $(\mathcal{C}_1^{(V)})$ becomes the Hamilton–Jacobi–Bellman equation :

$$\frac{\partial S}{\partial t} = -\frac{\theta^2}{2} \frac{\partial^2 S}{\partial q^2} + \frac{1}{2} \left(\frac{\partial S}{\partial q} \right)^2 - V . \quad (\mathcal{C}_3^{(V)})$$

Modulo the addition of the derivatives $E = -\frac{\partial S}{\partial t}$ and $B = -\frac{\partial S}{\partial q}$ as auxiliary unknown functions, $(\mathcal{C}_3^{(V)})$ is equivalent to the vanishing of the following differential forms :

$$\omega = dS + Edt + Bdq ,$$

$$\Omega = d\omega = dEdt + dBdq ,$$

and

$$\beta = (E + \frac{B^2}{2} - V)dqdt + \frac{\theta^2}{2}dBdt$$

on a 2-dimensional submanifold of $\mathbf{M} = \mathbf{R}^5$ ((t, q, S, E, B) being now considered as *independent* variables). Let then $L = \frac{1}{2}B^2 + V$ denote the *formal lagrangian* ,

$$\omega_{PC} = Edt + Bdq = \omega - dS$$

the *Poincaré–Cartan form* , and I the ideal of $\mathcal{A} = \wedge T^*(\mathbf{M})$ generated by ω , $d\omega$ and β . By an *isovector* we shall mean a vector field N on \mathbf{M} such that

$\mathcal{L}_N(I) \subseteq I$; because of the linearity of $(\mathcal{C}_1^{(V)})$ the Lie algebra \mathcal{G}_V of these isovectors contains an infinite-dimensional abelian ideal \mathcal{J}_V , that possesses a canonical supplement \mathcal{H}_V .

In the free case ($V = 0$) this canonical supplement has dimension 6 and admits a natural basis, each element of which corresponds to a symmetry of the underlying physical system.

Let $\Phi_N = -N(S)$ be the *phase* associated to N , and

$$D \equiv_{def} \frac{\partial}{\partial t} + B\frac{\partial}{\partial q} + \frac{\hbar}{2}\frac{\partial^2}{\partial q^2}$$

the *formal Ito differential* along the Bernstein process z . The following purely algebraic results are analogs of well-known theorems of classical analytical mechanics:

Theorem 1.1. *For each $N \in \mathcal{H}_V$ one has :*

- (1) $\mathcal{L}_N(\omega_{PC}) = d\Phi_N$;
- (2) $\mathcal{L}_N(\Omega) = 0$;
- (3) $\mathcal{L}_N(L) + L\frac{dN^t}{dt} = D\Phi_N$.

For a detailed proof see [7], and for complete calculations in the free case ($V = 0$) see [6].

2. ROSENCRANS' THEOREM

Let N be an *isovector* (for $V = 0$), let ψ be a solution of $\mathcal{C}_1^{(0)}$, and let

$$S = -\theta^2 \ln(\psi);$$

then $e^{\alpha N}$ maps (t, q, S, E, B) to $(t_\alpha, q_\alpha, S_\alpha, E_\alpha, B_\alpha)$; setting

$$e^{-\frac{S_\alpha}{\theta^2}} = \psi_\alpha(t_\alpha, q_\alpha),$$

it follows that ψ_α is also a solution of $(\mathcal{C}_1^{(0)})$. We shall denote

$$e^{\alpha \hat{N}} : \psi \mapsto \psi_\alpha$$

the associated one-parameter group; it is easily seen that, for

$$N = N^t \frac{\partial}{\partial t} + N^q \frac{\partial}{\partial q} - \Phi_N \frac{\partial}{\partial S} + \dots$$

then

$$\hat{N} = -N^t \frac{\partial}{\partial t} - N^q \frac{\partial}{\partial q} + \frac{1}{\theta^2} \Phi_N,$$

and it follows that $N \mapsto -\hat{N}$ is a homomorphism of Lie algebras.

Let η_u denote the solution de $(\mathcal{C}_1^{(0)})$ with initial condition u :

$$\frac{\partial \eta_u}{\partial t} = -\frac{\theta^2}{2} \frac{\partial^2 \eta_u}{\partial q^2},$$

and

$$\eta_u(0, q) = u(q).$$

Let us set :

$$\rho_N(\alpha, t, q) = (e^{\alpha \hat{N}} \eta_u)(t, q)$$

and

$$\psi^N(\alpha, q) \equiv_{def} \rho_N(\alpha, 0, q).$$

Then

Theorem 2.1. ([6], pp.321-322)

ψ^N satisfies :

$$\frac{\partial \psi^N}{\partial \alpha} = -N^t(0, q) \left(-\frac{\theta^2}{2} \frac{\partial^2 \psi^N}{\partial \alpha^2} \right) - N^q(0, q) \frac{\partial \psi^N}{\partial q} + \frac{1}{\theta^2} \Phi_N(0, q) \psi^N$$

and

$$\psi^N(0, q) = u(q).$$

Whence :

Corollary 2.2. *Let N be chosen so that $N^t(0, q) = -1$, $N^q(0, q) = -\frac{1}{\theta^2}(aq + b)$ and $\Phi_N(0, q) = cq^2 + dq + f$, where a, b, c, d, f denote real constants, then*

$$\eta_u^V(t, q) \equiv_{def} \psi^N(t, q)$$

satisfies the “backwards heat equation with drift term $D(q) = aq + b$ and quadratic potential $V(q) = cq^2 + dq + f + \frac{1}{2\hbar^2}(D(q))^2 - \frac{a}{2}$ ”, corresponding to a vector potential

$$A = \frac{aq + b}{\theta^2} :$$

$$\theta^2 \frac{\partial \eta_u^V}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \eta_u^V}{\partial q^2} + (aq + b) \frac{\partial \eta_u^V}{\partial q} + (cq^2 + dq + f) \eta_u^V \quad (\mathcal{C}_4^{(V)})$$

and

$$\eta_u^V(0, q) = u(q) .$$

(In the case $D(q) = 0$, the potential is given by $V(q) = cq^2 + dq + f$ and $\eta_u^{(V)}$ satisfies $\mathcal{C}_1^{(V)}$; for the general case, cf. [1], pp.71–72).

3. THE CASE OF A LINEAR POTENTIAL

Here $V(q) = \lambda q$; it appears that :

$$\eta_u^V(t, q) = e^{-\frac{\lambda^2}{6\theta^2}t^3} e^{\frac{\lambda tq}{\theta^2}} \eta_u(t, q - \lambda \frac{t^2}{2}) .$$

Then η_u^V satisfies $(\mathcal{C}_1^{(V)})$; the drift term can be written :

$$\begin{aligned} \tilde{B}_V(t, q) &= \theta^2 \frac{\partial}{\partial q} (\ln(\eta_u^V)(t, q)) \\ &= \theta^2 \frac{\partial}{\partial q} \left(-\frac{\lambda^2}{6\theta^2}t^3 + \frac{\lambda tq}{\theta^2} + \ln(\eta_u)(t, q - \lambda \frac{t^2}{2}) \right) \\ &= \lambda t + \theta^2 \frac{\partial}{\partial q} (\ln(\eta_u))(t, q - \lambda \frac{t^2}{2}) \\ &= \lambda t + \tilde{B}(t, q - \lambda \frac{t^2}{2}) . \end{aligned}$$

Therefore, we have :

$$dz_V(t) = \theta dw(t) + \lambda t dt + \tilde{B}(t, z_V(t) - \lambda \frac{t^2}{2}) dt .$$

Let us set $y(t) \equiv_{def} z_V(t) - \lambda \frac{t^2}{2}$; then

$$\begin{aligned} dy(t) &= dz_V(t) - \lambda t dt \\ &= \theta dw(t) + \tilde{B}(t, y(t)) dt , \end{aligned}$$

i.e. $y(t)$ is a Bernstein process $z(t)$ associated to solution η_u of the free equation $(\mathcal{C}_1^{(0)})$, and

$$z_V(t) = z(t) + \lambda \frac{t^2}{2} .$$

In other terms, the ‘‘perturbation’’ by a constant force λ produces a deterministic translation by $\lambda \frac{t^2}{2}$, which is logical on physical grounds.

Details are given in [7].

4. THE CASE OF A QUADRATIC POTENTIAL

For $V(t, q) = \frac{\omega^2 q^2}{2}$, one finds :

$$\eta_u^V(t, q) = \cosh(\omega t)^{-\frac{1}{2}} e^{\frac{\omega q^2}{2\theta^2} \tanh(\omega t)} \eta_u\left(\frac{\tanh(\omega t)}{\omega}, \frac{q}{\cosh(\omega t)}\right).$$

Whence

$$\tilde{B}_V(t, q) = \omega q \tanh(\omega t) + \frac{1}{\cosh(\omega t)} \tilde{B}\left(\frac{\tanh(\omega t)}{\omega}, \frac{q}{\cosh(\omega t)}\right).$$

Details are exposed in [9], §5, and a more general formula is proved in [5].

5. AN EXAMPLE WITH $D \neq 0$

Here, we take $a = \theta^2\beta$, and $b = c = d = f = 0$. With the notations of [1], pp. 71–72 (but, of course, replacing \mathbf{R}^3 with \mathbf{R}), $A = \beta q$, and

$$V(q) = \frac{\beta^2 q^2}{2} - \frac{\beta\theta^2}{2} .$$

Then η_u^V satisfies

$$\frac{\partial \eta_u^V}{\partial t} = -\frac{\theta^2}{2} \frac{\partial^2 \eta_u^V}{\partial q^2} + \beta q \frac{\partial \eta_u^V}{\partial q} .$$

It is easy to see that :

$$\eta_u^V(t, q) = \eta_u \left(\frac{1}{2\beta} (e^{2\beta t} - 1), e^{\beta t} q \right) .$$

The drift term (cf. [1], p.72) is given by :

$$\begin{aligned} \tilde{B}_V(t, q) &= \theta^2 \frac{\partial}{\partial q} (\ln(\eta_u^V)(t, q)) - A(t, q) \\ &= e^{\beta t} \tilde{B} \left(\frac{1}{2\beta} (e^{2\beta t} - 1), e^{\beta t} q \right) - \beta q . \end{aligned}$$

In particular, for $\eta = 1$, one finds $\eta_u = 1$, $\tilde{B}_V(t, q) = -\beta q$, $z(t) = \theta w(t)$ and

$$dz_V(t) = \theta dw(t) - \beta z_V(t) dt ,$$

i.e. $z_V(t)$ is an Ornstein–Uhlenbeck process, as expected.

6. ONE-FACTOR AFFINE INTEREST RATE MODELS

Such a model is characterized by the instantaneous rate $r(t)$, satisfying

$$dr(t) = \sqrt{\alpha r(t) + \beta} dw(t) + (\phi - \lambda r(t)) dt$$

(cf. [3]).

Let us set $\tilde{\phi} = \phi + \frac{\lambda\beta}{\alpha}$; then :

Theorem 6.1. *Let*

$$z(t) = \sqrt{\alpha r(t) + \beta};$$

then $z(t)$ is a Bernstein process for

$$\theta = \frac{\alpha}{2}$$

and the potential

$$V(t, q) = \frac{A}{q^2} + Bq^2$$

where :

$$A = \frac{\alpha^2}{8} \left(\tilde{\phi} - \frac{\alpha}{4} \right) \left(\tilde{\phi} - \frac{3\alpha}{4} \right)$$

and

$$B = \frac{\lambda^2}{8}.$$

Corollary 6.2. *The isovector algebra \mathcal{H}_V associated with V has dimension 6 if and only if $A = 0$; in the opposite case, it has dimension 4.*

The condition $A = 0$ is equivalent to $\tilde{\phi} \in \left\{ \frac{\alpha}{4}, \frac{3\alpha}{4} \right\}$. I am now able to explain this in terms of Bessel processes (see [4] and [5]).

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