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Stabilization of parabolic nonlinear systems with finite dimensional feedback or dynamical controllers. Application to the Navier–Stokes system

Mehdi Badra, Takéo Takahashi

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Abstract

Let $A : \mathcal{D}(A) \rightarrow \mathcal{X}$ be the generator of an analytic semigroup and $B : \mathcal{U} \rightarrow [\mathcal{D}(A^*)]'$ a quasi-bounded operator. In this paper, we consider the stabilization of the system $y' = Ay + Bu$ where u is the linear combination of a family (v_1, \dots, v_K) . Our main result shows that if (A^*, B^*) satisfies a unique continuation property and if K is greater or equal to the maximum of the geometric multiplicities of the the unstable modes of A , then the system is generically stabilizable with respect to the family (v_1, \dots, v_K) . With the same functional framework, we also prove the stabilizability of a class of nonlinear system when using feedback or dynamical controllers. We apply these results to stabilize the Navier–Stokes equations in 2D and in 3D by using boundary control.

Key words. Navier-Stokes equations, feedback stabilization, Dirichlet boundary control, finite dimensional control, Lyapunov functional, Riccati equation.

AMS subject classifications. 93D15, 35B40, 35Q30, 76D05, 76D07, 76D55, 93B52, 93C20.

1 Introduction

Let Ω be a bounded domain of \mathbb{R}^d , for $d = 2$ or $d = 3$, with a boundary $\partial\Omega$ of class $C^{2,1}$, and let $(w_S, q_S) \in (H^2(\Omega))^d \times H^1(\Omega)$ be a solution of the stationary Navier-Stokes equations:

$$\begin{cases} -\nu\Delta w_S + (w_S \cdot \nabla)w_S + \nabla q_S = f_S & \text{in } \Omega, \\ \nabla \cdot w_S = 0 & \text{in } \Omega, \\ w_S = v_S & \text{on } \partial\Omega. \end{cases} \quad (1)$$

In this setting, $\nu > 0$ is the viscosity coefficient. It is well-known that such a pair (w_S, q_S) exists if $f_S \in (L^2(\Omega))^d$ and $v_S \in (H^{\frac{3}{2}}(\partial\Omega))^d$ satisfies $\int_{\Gamma} v_S \cdot n d\Gamma = 0$ for each connected component Γ of $\partial\Omega$, see for instance [15, Thm. VIII.4.1 and VIII.5.2, p.48]. Here n denotes the unit normal vector to $\partial\Omega$ exterior to Ω . Thus, we consider a trajectory (w_U, q_U) solution of the unstationary Navier-Stokes equations:

$$\begin{cases} \frac{\partial w_U}{\partial t} - \nu\Delta w_U + (w_U \cdot \nabla)w_U + \nabla q_U = f_S & \text{in } (0, \infty) \times \Omega, \\ \nabla \cdot w_U = 0 & \text{in } (0, \infty) \times \Omega, \\ w_U = v_S + u & \text{on } (0, \infty) \times \partial\Omega, \\ w_U(0) = w_S + w_0 & \text{in } \Omega. \end{cases} \quad (2)$$

In the above system, the function u at the boundary is a control used to “reach”, at least asymptotically, the stationary solution (w_S, q_S) . More precisely, if we denote by (w, q) the difference between the solution of (1) and the stationary solution (2) which satisfies

$$\begin{cases} \frac{\partial w}{\partial t} - \nu\Delta w + (w_S \cdot \nabla)w + (w \cdot \nabla)w_S + (w \cdot \nabla)w + \nabla q = 0 & \text{in } (0, \infty) \times \Omega, \\ \nabla \cdot w = 0 & \text{in } (0, \infty) \times \Omega, \\ w = u & \text{on } (0, \infty) \times \partial\Omega, \\ w(0) = w_0 & \text{in } \Omega, \end{cases} \quad (3)$$

then for a prescribed rate of decrease $\sigma > 0$ we want to find a control u of finite dimension:

$$u(t, x) = \sum_{i=1}^K u_i(t) v_i(x), \quad \bar{u} = (u_1, \dots, u_K) \in L^2(\mathbb{R}^K) \quad (4)$$

such that, for some norm $\|\cdot\|$ and at least for w_0 in a neighborhood of 0, the following exponential decay holds:

$$\|w(t)\| = \|w_U(t) - w_S\| \leq C e^{-\sigma t} \|w_0\|.$$

In (4), the number K and the functions v_i , $i = 1, \dots, K$, are fixed and will be determined later on, and \bar{u} is a control function that we want to express with a feedback formulation.

The works in the literature which are most relevant to our present paper are [5, 6, 4, 7, 21]. Indeed, the stabilization of the Navier-Stokes system by means of finite-dimensional feedback controllers is obtained in [4], in the case of an internal control, and in [5, 6, 7, 21], in the case of a boundary control and only for $d = 2$, by using a Riccati strategy. The first (and main) step of the Riccati approach consists in proving the stabilizability of the Oseen system (which is obtained from (3) by linearizing around zero) by means of a finite dimensional control. It allows in a second step to define an auxiliary optimal control problem which provides a feedback controller stabilizing the Oseen system. Such a feedback law is obtained from the solution of an algebraic Riccati equation. Thus, in a third step, it remains to verify that the feedback controller also stabilizes the Navier-Stokes system. The stabilizability of the Oseen system obtained in [5, 6, 7, 4] is deduced from the exact controllability of the finite dimensional projected system related to the unstable eigenvalues of the Oseen operator, and the feedback law is obtained from the solution of an infinite dimensional Riccati equation. We underline that in [5, 6, 7] it is assumed that algebraic and geometric multiplicity of each unstable eigenvalue coincide, or equivalently, that the finite dimensional projected operator is diagonalizable. Such an assumption is referred as the *finite-dimensional spectral assumption* (FDSA) and the authors postulate that the FDSA is also necessary for exact controllability of the projected system, see [6, Rem. 3.6.2 and Rem. 3.6.4]. In [21] the exact controllability of the finite dimensional unstable projected system is avoided, and the stabilizability by means of a finite dimensional control is simply obtained from the stabilizability by means of an infinite dimensional control which is proved in [26]. Moreover, the feedback control obtained in [21] is obtained from the solution of a finite dimensional Riccati equation stated in $\mathbb{R}^{N_u \times N_u}$ where N_u is the dimension of the unstable space of the Oseen operator. In the present paper, we obtain the stabilizability of the Oseen system by proving the exact controllability of the projected system without assuming the FDSA, and we obtain a feedback control from a finite dimensional Riccati equation similarly as in [21].

However, in the above quoted work the question of finding the minimal number K in (4) for which (3),(4) is stabilizable is not really clear. The stabilization result obtained in [4] requires $K = 2m_a$ actuators, where m_a is the greater algebraic multiplicity of the unstable eigenvalues, see [4, Rem. 3.9]. The proof relies on the linear independence of the family $(B^* \varphi_k)$, where (φ_k) is a family of generalized eigenfunctions related to the unstable eigenvalues and B^* is the adjoint of the control operator, which in the present case is the restriction operator to the open subset where the control is acting, see [4, Claim. 3.3]. In [5, 6, 7], because the linear independence of the family $(B^* \varphi_k)$ corresponding to the boundary control is not known, the method of [4] does not apply and the number K is simply the dimension of the unstable space of the Oseen operator. In [21] the number K is the number of linearly independent elements of $(B^* \varphi_k)$, but such a number is not known. In the present paper we show that the minimal value of K is the maximum of the geometric (and not algebraic) multiplicities of the unstable eigenvalues and that the linear independence of $(B^* \varphi_k)$ is not a necessary condition for constructing a minimal and admissible family (v_1, \dots, v_K) .

Although the main part of the paper is written in a general framework, let us summarize our results in the particular case of Oseen and 2-D Navier-Stokes equations. For N given eigenvalues $\lambda_1, \dots, \lambda_N$ of the Oseen operator, we prove that the minimal value K , for which there exists (v_1, \dots, v_K) so that the projected system related to $\lambda_1, \dots, \lambda_N$ is exactly controllable, is equal to the maximum of the geometric multiplicities of the N eigenvalues. We also prove (Theorem 9) that the set of admissible (v_1, \dots, v_K) is an open and dense subspace of $(\mathbf{V}^0(\partial\Omega))^K$ (see Section 2 for the precise definition of this set). Moreover, we show that the exact controllability of the projected system is a consequence of the approximate controllability of the unstationary Oseen system (Theorem 8). Then for a prescribed rate of decrease $\sigma > 0$ and a value K equal to the maximum geometric multiplicity of the eigenvalues with real part strictly greater than $-\sigma$, we obtain the stabilizability of the Oseen system by means of a control of dimension K (Proposition 11). Thus, with a Riccati strategy we obtain the stabilizability of the Oseen and of the 2-D Navier-Stokes system by means of a feedback control of dimension

K (Proposition 24 and Theorem 25). Notice that the Riccati equation is stated in $\mathbb{R}^{N_\sigma \times N_\sigma}$ where N_σ is the dimension of the subspace related to $\lambda_1, \dots, \lambda_N$, and that the feedback law is of size $K \times N_\sigma$.

In the three dimensional case, our feedback stabilization theorem for the Navier-Stokes equations is still valid but for a very specific set of initial data. Indeed, it has been underlined in [2] that a pointwise feedback expression $u(t) = Fw(t)$ where $F : (L^2(\Omega))^d \rightarrow (L^2(\partial\Omega))^d$ is not the good choice because it may create an initial discontinuity of the trace which does not allow to define a strong solution to the 3-D Navier-Stokes equations: for an initial datum w_0 which does not satisfy $w_0|_{\partial\Omega} = Fw_0$ it is not possible to define solution with a fixed-point technique. That is the reason why we shall search \bar{u} as a solution of an ODE

$$\frac{d}{dt}\bar{u} = D_K\bar{u} + \bar{v}, \quad \bar{u}(0) = \bar{u}_0, \quad (5)$$

where D_K is a matrix of order K and \bar{v} is a control function that we shall search as a feedback control for the coupled system (3), (4), (5):

$$\bar{v} = F(w, \bar{u}). \quad (6)$$

Notice that to have the continuity of the trace, and then to be able to define a 3-D fixed-point solution, we must impose an initial trace compatibility conditions $u(0) = w_0|_{\partial\Omega}$, or equivalently:

$$\sum_{i=1}^K u_i(0)v_i = w_0 \quad \text{on} \quad \partial\Omega.$$

Then to obtain a stabilization theorem for the 3-D Navier-Stokes System we also assume that $w_0|_{\partial\Omega} = 0$ and we consider (5) with the following initial condition:

$$\bar{u}_0 = 0. \quad (7)$$

Our results concerning the stabilization of the 3-D Navier-Stokes equations can be summarized as follows. Let $\sigma > 0$ be a prescribed rate of decrease, let N_σ be the dimension of the subspace related to the eigenvalues $\lambda_1, \dots, \lambda_N$ of the Oseen operator with real part greater than $-\sigma$ and let K be the maximum of their geometric multiplicities. We prove that if we choose D_K with spectrum composed of K distinct eigenvalues $(\mu_i)_{i=1, \dots, K}$ such that $\mu_i \notin \{\lambda_1, \dots, \lambda_N\}$ for all $i = 1, \dots, K$, and if $\bar{g} \in \mathbb{R}^K$ is such that $(\bar{g}|\bar{d})_{\mathbb{R}^K} \neq 0$ for all eigenvectors \bar{d} of tD_K , then the Oseen system coupled with (5) is stabilizable for a rate of decrease σ by means of a one dimensional control $\bar{v} = \alpha\bar{g}$. Here $\alpha \in L^2(\mathbb{R})$ is the control function (Proposition 18). Thus, with a Riccati strategy we prove a stabilization result for (3), (4), (5),(6), (7) with a feedback law F of size $1 \times (N_\sigma + K)$ (Theorem 28).

The present paper provides general abstract stabilization theorems for linear and nonlinear systems with finite-dimensional feedback or dynamical control (Propositions 11, 18 and Theorems 16, 19). The results are obtained by constructing Lyapunov functions in a way similar, but slightly different, of the approach in [1, 3]. Such a construction is inspired from [23].

The rest of the paper is organized as follows. In Section 2 we recall the notations and the functional framework related to the Oseen and Navier-Stokes systems. Section 3 is devoted to general infinite dimensional parabolic systems: we characterize the families (v_1, \dots, v_K) for which the finite dimensional projected system on the unstable modes is exactly controllable, and then we apply a Riccati strategy to obtain a feedback or a dynamical control stabilizing the projected system. In Section 4 we apply the finite dimensional feedback or dynamical law obtained in Section 3 to the complete abstract evolution equation and we prove stabilization theorems with a Lyapunov approach. In Section 5 we apply the abstract stabilization results of Section 4 in the particular case of the Oseen and Navier-Stokes systems.

2 Notations and Functional Framework

Throughout the following, the letter C denotes a generic positive constant that may change from line to line. When particular positive constants are needed, we use a subscript: C_0, C_1 , etc.

If \mathcal{H} is a Hilbert space, we denote by $\|\cdot\|_{\mathcal{H}}$ its corresponding norm, we denote by \mathcal{H}' its dual space and by $\langle \cdot | \cdot \rangle_{\mathcal{H}', \mathcal{H}}$ the \mathcal{H}' - \mathcal{H} duality pairing. For two Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 we use the notation $\mathcal{H}_1 \hookrightarrow \mathcal{H}_2$ to say that \mathcal{H}_1 is continuously embedded into \mathcal{H}_2 , we denote by $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ the space of all bounded linear operators from \mathcal{H}_1 into \mathcal{H}_2 and we use the shorter expression $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}, \mathcal{H})$. For $0 < T \leq \infty$, the space $L^2(0, T; \mathcal{H})$ is the

usual vector-valued Lebesgue space, $H^s(0, T; \mathcal{H})$ for $s \geq 0$ is the usual vector-valued Sobolev space, $H_0^s(0, T; \mathcal{H})$ denotes the closure in $H^s(0, T; \mathcal{H})$ of the space of infinitely differentiable and compactly supported functions of $t \in]0, T[$ with values in \mathcal{H} , and $H^{-s}(0, T; \mathcal{H}')$ denotes the dual space of $H_0^s(0, T; \mathcal{H})$. We also define:

$$W(0, T; \mathcal{H}_1, \mathcal{H}_2) \stackrel{\text{def}}{=} L^2(0, T; \mathcal{H}_1) \cap H^1(0, T; \mathcal{H}_2),$$

and when $T = +\infty$ we use the shorter expressions $L^2(\mathcal{H}) = L^2(0, +\infty; \mathcal{H})$, $H^s(\mathcal{H}) = H^s(0, +\infty; \mathcal{H})$, $W(\mathcal{H}_1, \mathcal{H}_2) = W(0, +\infty; \mathcal{H}_1, \mathcal{H}_2)$. Moreover, $L^\infty(\mathcal{H})$ (resp. $C_b(\mathcal{H})$) is the space of bounded (resp. continuous and bounded) functions of $t \in [0, \infty[$ with values in \mathcal{H} . For $\theta \in (0, 1)$ the space $[\mathcal{H}_1, \mathcal{H}_2]_\theta$ is the interpolation space obtained from \mathcal{H}_1 and \mathcal{H}_2 with the complex interpolation method [27, p.55]. We recall that if $\mathcal{H}_1 \hookrightarrow \mathcal{H}_2$, then $W(\mathcal{H}_1, \mathcal{H}_2)$ is continuously embedded in $C_b([\mathcal{H}_1, \mathcal{H}_2]_{1/2})$ [27, 1.8 (2), p.44 and Rem.3 p.143]. Finally, we denote by $L_{loc}^2(\mathcal{H})$ the space of functions belonging to $L^2(0, T; \mathcal{H})$ for all $T > 0$, and we define $L_{loc}^\infty(\mathcal{H})$, $H_{loc}^s(\mathcal{H})$ etc, analogously.

If L is a closed linear mapping on \mathcal{H} , we denote its domain by $\mathcal{D}(L)$, and L^* denotes the adjoint of L . We recall that if $-L$ the infinitesimal generator of an analytic semigroup on \mathcal{H} of negative type [8, Part. II, Chap 1, Par. 2.2. p.91 and Par. 2.7. p.108] then $L^{-1} \in \mathcal{L}(\mathcal{H})$ [8, Part. II, Chap 1, Prop. 2.9. p.120] and the fractional powers L^θ for $\theta \in (0, 1)$ are well-defined [8, Part. II, Chap 1, Par. 5. p.167]. Moreover, if we also suppose that the interpolation equality $[\mathcal{D}(L), \mathcal{H}]_{\frac{1}{2}} = \mathcal{D}(L^{\frac{1}{2}})$ is true then the mapping $z \mapsto (z' - Lz, z(0))$ is an isomorphism from $W(\mathcal{D}(L), \mathcal{H})$ onto $L^2(\mathcal{H}) \times \mathcal{D}(L^{\frac{1}{2}})$ [8, Part. II, Chap 1, Thm. 3.1. p.143 and Par. 6 eq. (6.4)]. More generally, for $r \geq 0$ one set $\mathcal{H}^r = \mathcal{D}(L^r)$ and $\mathcal{H}^{-r} = [\mathcal{D}(L^{*r})]'$ and from a change of variable $y = L^{-r}z$ one easily obtain that the mapping

$$y \mapsto (y' - Ly, y(0)) : W(\mathcal{H}^{r+1}, \mathcal{H}^{r-1}) \rightarrow L^2(\mathcal{H}^{r-1}) \times \mathcal{H}^r \quad \text{is an isomorphism for all } r \in \mathbb{R}. \quad (8)$$

Next, we recall that Ω is an open subset of \mathbb{R}^d , $d = 2, 3$, with a boundary $\partial\Omega$ of class $C^{2,1}$, we denote by $L^2(\Omega)$, $L^2(\partial\Omega)$, $H^s(\Omega)$, $H^s(\partial\Omega)$, $H_0^s(\Omega)$ and $H^{-s}(\Omega) = (H_0^s(\Omega))'$ for $s \geq 0$, the usual Lebesgue and Sobolev spaces of scalar functions in Ω or in $\partial\Omega$, and we write in bold the spaces of vector-valued functions: $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$, $\mathbf{L}^2(\partial\Omega) = (L^2(\partial\Omega))^d$, etc. We also introduce different spaces of free divergence functions and some corresponding trace spaces:

$$\begin{aligned} \mathbf{V}^s(\Omega) &\stackrel{\text{def}}{=} \left\{ y \in \mathbf{H}^s(\Omega) ; \nabla \cdot y = 0 \quad \text{in } \Omega, \quad \langle y \cdot n, 1 \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} = 0 \right\} \quad \text{for } s \geq 0, \\ \mathbf{V}_n^s(\Omega) &\stackrel{\text{def}}{=} \left\{ y \in \mathbf{H}^s(\Omega) ; \nabla \cdot y = 0 \quad \text{in } \Omega, \quad y \cdot n = 0 \quad \text{on } \partial\Omega \right\} \quad \text{for } s \geq 0, \\ \mathbf{V}_0^s(\Omega) &\stackrel{\text{def}}{=} \left\{ y \in \mathbf{H}^s(\Omega) ; \nabla \cdot y = 0 \quad \text{in } \Omega, \quad y = 0 \quad \text{on } \partial\Omega \right\} \quad \text{for } s > \frac{1}{2}, \\ \mathbf{V}^s(\partial\Omega) &\stackrel{\text{def}}{=} \left\{ y \in \mathbf{H}^s(\partial\Omega) ; \langle y \cdot n, 1 \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} = 0 \right\} \quad \text{for } s \geq -\frac{1}{2}. \end{aligned}$$

Moreover, we also define $\mathbf{V}_0^s(\Omega)$ for $s \in [0, 1/2[$ by $\mathbf{V}_0^s(\Omega) = \mathbf{V}_n^s(\Omega)$ and for $s = 1/2$ by $\mathbf{V}_0^{1/2}(\Omega) = \{ y \in \mathbf{V}_n^{1/2}(\Omega) \mid y \in \mathbf{L}_{-1/2}^2(\Omega) \}$ where $\mathbf{L}_{-1/2}^2(\Omega)$ is the space of functions $y \in \mathbf{L}^2(\Omega)$ such that $\int_\Omega \text{dist}(x, \partial\Omega)^{-1} |y|^2 dx < +\infty$. Notice that the subscript 0 in $\mathbf{V}_0^s(\Omega)$ only means that one may have a vanishing Dirichlet boundary condition.

We denote by P the orthogonal projector from $\mathbf{L}^2(\Omega)$ onto $\mathbf{V}_n^0(\Omega)$ [14, Chap. III, Thm. 1.1]. Notice that since Ω is of class $C^{2,1}$, by studying its related Neumann problem, see [14, Chap.III, Lem 1.2] and [16, Chap. I, Thm. 1.10], we also have for all $s \in [0, 2]$:

$$P \in \mathcal{L}(\mathbf{H}^s(\Omega), \mathbf{V}_n^s(\Omega)). \quad (9)$$

Moreover, P can also be continuously extended to a bounded linear operator from $\mathbf{H}^{-1}(\Omega)$ onto $\mathbf{V}_0^{-1}(\Omega)$ [6, App. A].

In order to rewrite (3) as an evolution equation by using a semigroup approach, let us introduce the Oseen operator A on $\mathbf{V}_n^0(\Omega)$:

$$\begin{cases} \mathcal{D}(A) &\stackrel{\text{def}}{=} \{ y \in \mathbf{V}_n^1(\Omega) \mid P(\nu \Delta y - (w_S \cdot \nabla)y - (y \cdot \nabla)w_S) \in \mathbf{V}_n^0(\Omega) \}, \\ Ay &\stackrel{\text{def}}{=} P(\nu \Delta y - (w_S \cdot \nabla)y - (y \cdot \nabla)w_S). \end{cases} \quad (10)$$

Following [25], there exists $\lambda_0 > 0$ such that $\lambda_0 \in \rho(A)$ and

$$((\lambda_0 I - A)y, y)_{\mathbf{V}_n^0(\Omega)} \geq \frac{\nu}{2} \|y\|_{\mathbf{V}_n^1(\Omega)}^2 \quad \forall y \in \mathcal{D}(A), \quad (11)$$

and the following theorem holds.

Theorem 1. *The Oseen operator (10) obeys $\mathcal{D}(A) = \mathbf{V}_0^2(\Omega)$, its adjoint is given by*

$$\mathcal{D}(A^*) = \mathbf{V}_0^2(\Omega) \quad \text{and} \quad A^*y = P(\nu\Delta y + (w_S \cdot \nabla)y - {}^t(\nabla w_S)y),$$

and the resolvent of $(\mathcal{D}(A), A)$ (resp. $(\mathcal{D}(A^), A^*)$) is compact. In the above setting t denotes the transpose. Moreover, $(\mathcal{D}(A), A)$ (resp. $(\mathcal{D}(A^*), A^*)$) is the infinitesimal generator of an analytic semigroup on $\mathbf{V}_n^0(\Omega)$ and the characterization below holds:*

$$\mathcal{D}((\lambda_0 - A)^\theta) = \mathcal{D}((\lambda_0 - A^*)^\theta) = [\mathbf{V}_0^2(\Omega), \mathbf{V}_n^0(\Omega)]_{1-\theta} = \mathbf{V}_0^{2\theta}(\Omega) \quad \text{for all } \theta \in [0, 1]. \quad (12)$$

Proof. First, since regularity result for the Stokes system [14, Chap. IV, Thm. 6.1, p. 231] ensures that the domain of the Stokes operator (defined by (10) for $w_S = 0$) is equal to $\mathbf{V}_0^2(\Omega)$, then $\mathcal{D}(A) = \mathbf{V}_0^2(\Omega)$ follows from a perturbation argument [24, Chap. 3, Cor. 2.4 p. 81]. The characterization of A^* follows analogously with an integration by parts. Thus, the compactness of the resolvent of $(\mathcal{D}(A), A)$ and of $(\mathcal{D}(A^*), A^*)$ is a consequence of the compact embedding $\mathbf{V}_0^2(\Omega) \subset\subset \mathbf{V}_n^0(\Omega)$. Next, the fact that A generates an analytic semigroup is a consequence of (11) with [8, Part. II, Chap. 1, Thm 2.12], and a duality argument yields that A^* also generates an analytic semigroup on $\mathbf{V}_n^0(\Omega)$. Finally, (12) follows by interpolation: since $\lambda_0 - A$ is accretive and boundedly invertible with domain $\mathbf{V}_0^2(\Omega)$ we have $\mathcal{D}((\lambda_0 - A)^\theta) = [\mathbf{V}_0^2(\Omega), \mathbf{V}_n^0(\Omega)]_{1-\theta}$ and $\mathcal{D}((\lambda_0 - A^*)^\theta) = [\mathbf{V}_0^2(\Omega), \mathbf{V}_n^0(\Omega)]_{1-\theta}$ [8, Part. II, Chap. 1, Thm 2.12]. Finally, since we can construct a projection operator $P_0 : \mathbf{L}^2(\Omega) \rightarrow \mathbf{V}_n^0(\Omega)$ which is also continuous from $\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$ onto $\mathbf{V}_0^2(\Omega)$ (for instance by setting $P_0 = A_0^{-1}P\Delta$ where A_0 is the Stokes operator, see [13]) then by [27, Thm. 1.17.1.1, p.118] we deduce that $[\mathbf{V}_0^2(\Omega), \mathbf{V}_n^0(\Omega)]_{1-\theta} = [\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega), \mathbf{L}^2(\Omega)]_{1-\theta} \cap \mathbf{V}_n^0(\Omega)$ and the characterization of $[\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega), \mathbf{L}^2(\Omega)]_{1-\theta}$ obtained from [27, Thm. 4.3.3.1, p.321], or from [17], allows to conclude. \square

Next, we consider the Dirichlet operator $D : \mathbf{V}^0(\partial\Omega) \rightarrow \mathbf{V}^0(\Omega)$ defined as follows: for $v \in \mathbf{V}^0(\partial\Omega)$, $w_v = Dv$ is the unique solution of

$$\begin{aligned} \lambda_0 w_v - \nu\Delta w_v + (w_v \cdot \nabla)w_S + (w_S \cdot \nabla)w_v + \nabla q_v &= 0 \quad \text{in } \Omega, \\ \nabla \cdot w_v &= 0 \quad \text{in } \Omega, \\ w_v &= v \quad \text{on } \partial\Omega. \end{aligned}$$

The operator D defined as above satisfies the following properties, see [25].

Lemma 2. *The following results hold.*

- *The operator D is bounded from $\mathbf{V}^s(\partial\Omega)$ onto $\mathbf{V}^{s+1/2}(\Omega)$ for all $s \in [0, 3/2]$.*
- *The adjoint of D is defined by*

$$D^*f = \chi n - \nu \frac{\partial \varepsilon}{\partial n} \quad \text{for all } \varepsilon \in \mathbf{V}^0(\Omega),$$

where $(\varepsilon, \chi) \in \mathbf{V}_0^2(\Omega) \times H^1(\Omega)$ is the solution of

$$\begin{aligned} \lambda_0 \varepsilon - \nu\Delta \varepsilon + {}^t(\nabla w_S)\varepsilon - (w_S \cdot \nabla)\varepsilon + \nabla \chi &= f \quad \text{in } \Omega, \\ \nabla \cdot \varepsilon &= 0 \quad \text{in } \Omega, \\ \varepsilon &= 0 \quad \text{on } \partial\Omega, \\ \int_{\partial\Omega} \chi \, d\Gamma &= 0. \end{aligned}$$

Moreover, to deal with a boundary control which is localized in an open subset of $\partial\Omega$ we follow the idea of [26]: we introduce a weight function $m \in C^2(\partial\Omega)$ with values in $[0, 1]$, with support in $\Gamma_m \subset \partial\Omega$ and equal to 1 in Γ_1 , where Γ_1 is an open subset of Γ_m , and we define a localization operator as follows:

$$M(v)(x) \stackrel{\text{def}}{=} m(x)v(x) - \frac{m(x)}{\int_{\partial\Omega} m \, d\Gamma} \left(\int_{\partial\Omega} m v \cdot n \, d\Gamma \right) n(x). \quad (13)$$

We observe that M is self-adjoint and that $m \in C^2(\partial\Omega)$ ensures that for all $s \in [0, 2]$:

$$M \in \mathcal{L}(\mathbf{V}^s(\partial\Omega)). \quad (14)$$

Remark 3. *Theorem 1 and Lemma 2 only require the assumptions that Ω is of class $C^{1,1}$ and $w_S \in \mathbf{V}^1(\Omega)$. The stronger assumption that Ω is of class $C^{2,1}$ is required to have (9) and (14) for $s \in [1, 2]$. In particular, it implies that $n \in \mathbf{V}^2(\partial\Omega)$ in (14). The stronger assumption $w_S \in \mathbf{V}^2(\Omega)$ is needed later in Section 5, see Remark 21.*

We then consider Oseen ($\kappa = 0$) or Navier-Stokes ($\kappa = 1$) system with boundary condition localized on $\partial\Omega_m$:

$$\left\{ \begin{array}{ll} \frac{\partial w}{\partial t} - \nu \Delta w + (w_S \cdot \nabla)w + (w \cdot \nabla)w_S + \kappa(w \cdot \nabla)w + \nabla p = 0 & \text{in } (0, \infty) \times \Omega, \\ \nabla \cdot w = 0 & \text{in } (0, \infty) \times \Omega, \\ w = M(u) & \text{on } (0, \infty) \times \partial\Omega, \\ w(0) = w_0 & \text{in } \Omega. \end{array} \right. \quad (15)$$

We shall say that a pair (w, p) is a solution of (15) if $w \in L^2_{loc}((0, \infty); \mathbf{H}^1(\Omega)) \cap C([0, \infty); \mathbf{L}^2(\Omega))$, if p is a distribution in $\mathcal{D}'((0, \infty); L^2(\Omega))$, and if (w, p) satisfies the first equation in (15) as an equality in the distribution space $\mathcal{D}'(0, \infty; \mathbf{H}^{-1}(\Omega))$, and the second, third and fourth equations in (15) as equalities in $L^2_{loc}(L^2(\Omega))$, in $L^2(\mathbf{L}^2(\partial\Omega))$ and in $\mathbf{L}^2(\Omega)$ respectively. Notice that such a definition is meaningful because when $w \in L^2_{loc}((0, \infty); \mathbf{H}^1(\Omega))$ and $w_S \in \mathbf{V}^1(\Omega)$ the convective terms $(w_S \cdot \nabla)w$, $(w \cdot \nabla)w_S$, and in particular the nonlinear term $(w \cdot \nabla)w$, are well-defined in $L^2_{loc}(\mathbf{H}^{-1}(\Omega))$. More precisely, if s_1, s_2 and s_3 are real nonnegative numbers such that $s_1 + s_2 + s_3 \geq \frac{d}{2}$ if $s_i \neq \frac{d}{2}$, $i = 1, 2, 3$ or $s_1 + s_2 + s_3 > \frac{d}{2}$ if $s_i = \frac{d}{2}$, for at least one i then from [10, Prop. 6.1, (6.10)] we have that the following estimate holds for $(y_1, y_2, y_3) \in \mathbf{H}^{s_1}(\Omega) \times \mathbf{H}^{1+s_2}(\Omega) \times \mathbf{H}^{s_3}(\Omega)$:

$$\int_{\Omega} (y_1 \cdot \nabla) y_2 \cdot y_3 \, dx \leq C \|y_1\|_{\mathbf{H}^{s_1}(\Omega)} \|y_2\|_{\mathbf{H}^{1+s_2}(\Omega)} \|y_3\|_{\mathbf{H}^{s_3}(\Omega)}. \quad (16)$$

Then we deduce that the nonlinear map $w \mapsto (w \cdot \nabla)w$ obeys the following boundedness and Lipschitz property: if $s \in [\frac{d-2}{2}, 1]$ then for w, w_1, w_2 in $\mathbf{V}^{s+1}(\Omega)$ we have

$$\|(w \cdot \nabla)w\|_{\mathbf{H}^{s-1}(\Omega)} \leq C \|w\|_{\mathbf{V}^s(\Omega)} \|w\|_{\mathbf{V}^{s+1}(\Omega)}, \quad (17)$$

$$\begin{aligned} \|(w_1 \cdot \nabla)w_1 - (w_2 \cdot \nabla)w_2\|_{\mathbf{H}^{s-1}(\Omega)} &\leq C (\|w_1 - w_2\|_{\mathbf{V}^s(\Omega)} (\|w_1\|_{\mathbf{V}^{s+1}(\Omega)} + \|w_2\|_{\mathbf{V}^{s+1}(\Omega)}) \\ &\quad + \|w_1 - w_2\|_{\mathbf{V}^{s+1}(\Omega)} (\|w_1\|_{\mathbf{V}^s(\Omega)} + \|w_2\|_{\mathbf{V}^s(\Omega)})). \end{aligned} \quad (18)$$

Notice that to obtain (17)-(18) in the limit case $d = 2$ and $s = 0$, one must combine the antisymmetry property $\int_{\Omega} (y_1 \cdot \nabla) y_2 \cdot y_3 \, dx = - \int_{\Omega} (y_1 \cdot \nabla) y_3 \cdot y_2 \, dx$ for $(y_1, y_2, y_3) \in \mathbf{V}^1(\Omega) \times \mathbf{V}^1(\Omega) \times \mathbf{H}_0^1(\Omega)$ with (16) for $(s_1, s_2, s_3) = (1/2, 0, 1/2)$ and the interpolation inequality $\|\cdot\|_{\mathbf{V}^{\frac{1}{2}}(\Omega)} \leq C \|\cdot\|_{\mathbf{V}^0(\Omega)}^{\frac{1}{2}} \|\cdot\|_{\mathbf{V}^1(\Omega)}^{\frac{1}{2}}$.

Next, to rewrite system (15) as an abstract dynamical system, we introduce the linear control operator:

$$B : \mathbf{V}^0(\partial\Omega) \rightarrow [\mathcal{D}(A^*)]', \quad v \mapsto (\lambda_0 - A)PDM(v). \quad (19)$$

In the above setting $[\mathcal{D}(A^*)]'$ is the dual space of $\mathcal{D}(A^*)$ with respect to the pivot space $\mathbf{V}_n^0(\Omega)$. It corresponds to the space X_{-1} in the terminology of [28]. Notice that for $3/4 < \gamma < 1$, by combining (14) with $s = 0$, the first point of Lemma 2 with $s = 0$, (9) with $s = 2(1 - \gamma)$ and (12) with $\theta = 1 - \gamma$, we obtain $PDM \in \mathcal{L}(\mathbf{V}^0(\partial\Omega), \mathcal{D}((\lambda_0 - A)^{1-\gamma}))$, which means that B is quasi-bounded with respect to $\lambda_0 - A$:

$$(\lambda_0 - A)^{-\gamma} B = (\lambda_0 - A)^{1-\gamma} PDM \in \mathcal{L}(\mathbf{V}^0(\partial\Omega), \mathbf{V}_n^0(\Omega)) \quad \text{for } 3/4 < \gamma < 1. \quad (20)$$

According to [25] system (15) can be written under the form

$$Pw' = APw + Bu - \kappa P(w \cdot \nabla)w \in [\mathcal{D}(A^*)]', \quad w(0) = w_0, \quad (21)$$

$$(I - P)w = (I - P)DM(u). \quad (22)$$

The following proposition, which is taken from [3, Prop. 4], gives a precise statement of the equivalence between formulations (21)-(22) and (15). For it, we need to define spaces of pressures with free mean:

$$L_{\sharp}^2(\Omega) \stackrel{\text{def}}{=} \left\{ p \in L^2(\Omega) \mid \int_{\Omega} p(x) \, dx = 0 \right\} \quad \text{and} \quad H_{\sharp}^s(\Omega) \stackrel{\text{def}}{=} H^s(\Omega) \cap L_{\sharp}^2(\Omega), \quad s \geq 0.$$

Proposition 4. *Let $\kappa \in \{0, 1\}$ and assume that $u \in L^2_{loc}(\mathbf{V}^{s+\frac{1}{2}}(\partial\Omega))$ with $s \in [0, 1]$.*

1. If $w \in L_{loc}^2(\mathbf{V}^{s+1}\Omega) \cap H_{loc}^{\frac{s}{2}+\frac{1}{2}}(\mathbf{V}^0(\Omega))$ obeys (21)-(22) then there exists a unique pressure function $p \in H_{loc}^{-1/2+s/2}(H_{\sharp}^s(\Omega))$ such that (w, p) satisfies (15).
2. Conversely, if $(w, p) \in [L_{loc}^2(\mathbf{V}^{s+1}\Omega) \cap H_{loc}^{\frac{s}{2}+\frac{1}{2}}(\mathbf{V}^0(\Omega))] \times H_{loc}^{-1/2+s/2}(H_{\sharp}^s(\Omega))$ obeys (15), then w satisfies (21)-(22).

3 Controllability and stabilization of the linear projected system

3.1 Complex projected systems

Let \mathcal{Z} be a complex Hilbert space and let $A : \mathcal{D}(A) \subset \mathcal{Z} \rightarrow \mathcal{Z}$ be a closed linear operator with compact resolvent and which is the infinitesimal generator of an strongly continuous semigroup on \mathcal{Z} . The spectrum Σ of A then consists entirely of eigenvalues which are isolated points of the complex plane \mathbb{C} [22, III. Thm 6.29] and which are located in a left half-plane: there is a complex sequence $(\lambda_k)_{k \in \mathbb{N}^*}$ such that $\Sigma = \{\lambda_k \mid k \in \mathbb{N}^*\}$ and for all $k \in \mathbb{N}^*$ we have $\Re \lambda_k < \omega$ for some $\omega \in \mathbb{R}$. Thus, for a given $N \in \mathbb{N}^*$ we set

$$\Sigma_N \stackrel{\text{def}}{=} \{\lambda_k \mid k = 1, \dots, N\}, \quad (23)$$

and we decompose the space \mathcal{Z} as the direct sum of two invariant subspaces of A related to Σ_N and $\Sigma \setminus \Sigma_N$. Let Γ_N be a positively-oriented curve enclosing Σ_N but no other point of the spectrum of A , and let $P_N : \mathcal{Z} \rightarrow \mathcal{Z}$ be the projection operator defined by

$$P_N \stackrel{\text{def}}{=} -\frac{1}{2\pi i} \int_{\Gamma_N} (\xi - A)^{-1} d\xi. \quad (24)$$

It is well known that P_N provides the following decomposition of \mathcal{Z} :

$$\mathcal{Z} = \mathcal{Z}_N \oplus \mathcal{Z}_N^- \quad \text{where} \quad \mathcal{Z}_N \stackrel{\text{def}}{=} P_N(\mathcal{Z}) \quad \text{and} \quad \mathcal{Z}_N^- \stackrel{\text{def}}{=} (I - P_N)(\mathcal{Z}),$$

that \mathcal{Z}_N and \mathcal{Z}_N^- are invariant subspaces under A , and that the spectra of the restricted operators $A|_{\mathcal{Z}_N}$ and $A|_{\mathcal{Z}_N^-}$ are Σ_N and $\Sigma \setminus \Sigma_N$ respectively [22, III. Thm 6.17]. In what follows, we use the notations:

$$A_N \stackrel{\text{def}}{=} A|_{D(A) \cap \mathcal{Z}_N} : D(A) \cap \mathcal{Z}_N \rightarrow \mathcal{Z}_N \quad \text{and} \quad A_N^- \stackrel{\text{def}}{=} A|_{D(A) \cap \mathcal{Z}_N^-} : D(A) \cap \mathcal{Z}_N^- \rightarrow \mathcal{Z}_N^-.$$

Moreover, since Γ_N is a bounded closed curve, P_N is a bounded operator from \mathcal{Z} into $\mathcal{D}(A)$ and the continuous embedding $\mathcal{Z}_N \hookrightarrow \mathcal{D}(A)$ holds. As a consequence, P_N is a compact projection operator and then its range \mathcal{Z}_N is necessarily finite dimensional. Next, since the spectrum of $A^* : \mathcal{D}(A^*) \subset \mathcal{Z} \rightarrow \mathcal{Z}$ (the adjoint of A) is exactly composed of the complex conjugates of the eigenvalues of A [22, III. Thm 6.22], then we have another decomposition of \mathcal{Z} as a direct sum of two invariant subspaces of A^* :

$$\mathcal{Z} = \mathcal{Z}_N^* \oplus \mathcal{Z}_N^{*-}, \quad \mathcal{Z}_N^* \stackrel{\text{def}}{=} P_N^*(\mathcal{Z}) \quad \text{and} \quad \mathcal{Z}_N^{*-} \stackrel{\text{def}}{=} (I - P_N^*)(\mathcal{Z}), \quad \text{where} \quad P_N^* \stackrel{\text{def}}{=} -\frac{1}{2\pi i} \int_{\bar{\Gamma}_N} (\xi - A^*)^{-1} d\xi.$$

In the above setting, $P_N^* : \mathcal{Z} \rightarrow \mathcal{Z}$ is a projection operator with finite dimensional range \mathcal{Z}_N^* included in $\mathcal{D}(A^*)$, and since P_N^* is the adjoint of P_N we deduce that their ranges have the same dimension [22, III. Thm 4.13]:

$$\dim(\mathcal{Z}_N) = \dim(\mathcal{Z}_N^*).$$

Moreover, if we denote by \mathcal{Z}_N^\perp (resp. $\mathcal{Z}_N^{*\perp}$) the orthogonal space of \mathcal{Z}_N (resp. \mathcal{Z}_N^*) in \mathcal{Z} then we verify that:

$$\mathcal{Z}_N^\perp = \mathcal{Z}_N^{*-} \quad \text{and} \quad \mathcal{Z}_N^{*\perp} = \mathcal{Z}_N^-. \quad (25)$$

Next, let us introduce a complex Hilbert space of control \mathcal{U} and a linear control operator $B : \mathcal{U} \rightarrow [\mathcal{D}(A^*)]'$, where $[\mathcal{D}(A^*)]'$ denotes the dual space of $\mathcal{D}(A^*)$ with respect to the pivot space \mathcal{Z} . We assume that B is admissible i.e.

$$u \mapsto \int_0^t e^{A(t-\tau)} B u(\tau) d\tau : L^2(0, \infty; \mathcal{U}) \longrightarrow \mathcal{X} \quad \text{is bounded for all } t \geq 0. \quad (26)$$

Then for $y_0 \in \mathcal{Z}$ and a control function $u \in L^2(0, T; \mathcal{U})$ the following infinite dimensional dynamical system:

$$y' = Ay + Bu \in [\mathcal{D}(A^*)]' \quad (27)$$

admits a unique solution $y \in C([0, T]; \mathcal{Z})$ such that $y(0) = y_0$, see [28, Def. 4.1.1 and Prop.4.2.5]. Moreover, if we write $y = y_N + y_{\bar{N}}$ where $y_N \stackrel{\text{def}}{=} P_N y$ and $y_{\bar{N}} \stackrel{\text{def}}{=} (I - P_N)y$ then there is a corresponding splitting of (27) in two equations satisfied by y_N and $y_{\bar{N}}$ respectively. In order to apply P_N and $I - P_N$ to (27) we first need to extend their definition. Since P_N^* is bounded from \mathcal{Z} into $\mathcal{D}(A^*)$, the projection operator P_N can be uniquely extended as a bounded operator from $[\mathcal{D}(A^*)]'$ into \mathcal{Z}_N as follows:

$$(P_N e|\varepsilon)_{\mathcal{Z}} = \langle e|P_N^* \varepsilon \rangle_{[\mathcal{D}(A^*)]', \mathcal{D}(A^*)} \quad \forall (e, \varepsilon) \in [\mathcal{D}(A^*)]' \times \mathcal{D}(A^*).$$

Notice that if $e \in [\mathcal{D}(A^*)]'$ then $P_N e$ belongs \mathcal{Z}_N because $((I - P_N)P_N e|\varepsilon)_{\mathcal{Z}} = \langle e|P_N^*(I - P_N^*)\varepsilon \rangle_{[\mathcal{D}(A^*)]', \mathcal{D}(A^*)} = 0$ for all $\varepsilon \in \mathcal{D}(A^*)$. Moreover, as in [21] one can prove that $[\mathcal{D}(A^*)]' = \mathcal{Z}_N \oplus [\mathcal{D}(A^*) \cap \mathcal{Z}_N^*]'$ and since for all $(e, \varepsilon) \in [\mathcal{D}(A^*)]' \times (\mathcal{D}(A^*) \cap \mathcal{Z}_N^*)$ we have:

$$\langle (I - P_N)e|\varepsilon \rangle_{[\mathcal{D}(A^*)]', \mathcal{D}(A^*)} = \langle e|(I - P_N^*)\varepsilon \rangle_{[\mathcal{D}(A^*)]', \mathcal{D}(A^*)} = \langle e|\varepsilon \rangle_{[\mathcal{D}(A^*)]', \mathcal{D}(A^*)},$$

we deduce that the projection operator $(I - P_N)$ can be identified as a bounded operator from $[\mathcal{D}(A^*)]'$ into $[\mathcal{D}(A^*) \cap \mathcal{Z}_N^*]'$. As a consequence, for $u \in L^2(0, T; \mathcal{U})$ we have $P_N B u \in L^2(0, T; \mathcal{Z}_N)$ and $(I - P_N)B u \in L^2(0, T; [\mathcal{D}(A^*) \cap \mathcal{Z}_N^*]')$, and by applying P_N and $(I - P_N)$ to (27) we deduce that $(y_N, y_{\bar{N}})$ obeys:

$$y'_N = A_N y_N + P_N B u \in \mathcal{Z}_N \quad \text{and} \quad y'_{\bar{N}} = A_{\bar{N}} y_{\bar{N}} + (I - P_N)B u \in [\mathcal{D}(A^*) \cap \mathcal{Z}_N^*]'. \quad (28)$$

To go further, we introduce the spaces

$$\mathcal{U}_N \stackrel{\text{def}}{=} \{B^* \varepsilon \mid \varepsilon \in \mathcal{Z}_N^*\} \quad \text{and} \quad \mathcal{U}_{\bar{N}} \stackrel{\text{def}}{=} \{B^* \varepsilon \mid \varepsilon \in \mathcal{D}(A^*) \cap \mathcal{Z}_N^*\},$$

as well as the orthogonal projection operators $p_N : \mathcal{U} \rightarrow \mathcal{U}_N$ and $p_{\bar{N}} : \mathcal{U} \rightarrow \mathcal{U}_{\bar{N}}$ and we recall that their respective adjoints $p_N^* : \mathcal{U}_N \rightarrow \mathcal{U}$ and $p_{\bar{N}}^* : \mathcal{U}_{\bar{N}} \rightarrow \mathcal{U}$ are the injection operators. Thus we introduce the projected control operators:

$$B_N \stackrel{\text{def}}{=} P_N B p_N^* : \mathcal{U}_N \longrightarrow \mathcal{Z}_N \quad \text{and} \quad B_{\bar{N}} \stackrel{\text{def}}{=} (I - P_N)B p_{\bar{N}}^* : \mathcal{U}_{\bar{N}} \longrightarrow [\mathcal{D}(A^*) \cap \mathcal{Z}_N^*]',$$

which obey:

$$B_N p_N = P_N B \quad \text{and} \quad B_{\bar{N}} p_{\bar{N}} = (I - P_N)B. \quad (29)$$

Indeed, the first above equality is obtained by making the following calculations for all $(v, \varepsilon) \in \mathcal{U} \times \mathcal{Z}$:

$$(P_N B v|\varepsilon)_{\mathcal{Z}} = (v|B^* P_N^* \varepsilon)_{\mathcal{U}} = (p_N v|p_N B^* P_N^* \varepsilon)_{\mathcal{U}} = (P_N B p_N^* p_N v|\varepsilon)_{\mathcal{Z}} = (B_N p_N v|\varepsilon)_{\mathcal{Z}},$$

where we have used that p_N is an orthogonal projection. The second equality of (29) is obtained by making the following calculations for all $(v, z) \in \mathcal{U} \times \mathcal{D}(A^*)$:

$$\begin{aligned} \langle (I - P_N)B v|\varepsilon \rangle_{[\mathcal{D}(A^*)]', \mathcal{D}(A^*)} &= (v|B^*(I - P_N^*)\varepsilon)_{\mathcal{U}} = (p_{\bar{N}} v|p_{\bar{N}} B^*(I - P_N^*)\varepsilon)_{\mathcal{U}} \\ &= \langle (I - P_N)B p_{\bar{N}}^* p_{\bar{N}} v|\varepsilon \rangle_{[\mathcal{D}(A^*)]', \mathcal{D}(A^*)} = \langle B_{\bar{N}} p_{\bar{N}} v|\varepsilon \rangle_{[\mathcal{D}(A^*)]', \mathcal{D}(A^*)}. \end{aligned}$$

Finally, by combining (28) and (29) we deduce that the pair $(y_N, y_{\bar{N}})$ obeys:

$$y'_N = A_N y_N + B_N p_N u \in \mathcal{Z}_N, \quad (30)$$

$$y'_{\bar{N}} = A_{\bar{N}} y_{\bar{N}} + B_{\bar{N}} p_{\bar{N}} u \in [\mathcal{D}(A^*) \cap \mathcal{Z}_N^*]'. \quad (31)$$

Notice that conversely, if $(y_N, y_{\bar{N}}) \in \mathcal{Z}_N \times \mathcal{Z}_{\bar{N}}$ obeys (30)-(31), then by adding (30) to (31) and by using (29) we recover the fact that $y = y_N + y_{\bar{N}}$ obeys (27). Then we have proved that formulations (27) and (30)-(31) are equivalent, and that the \mathcal{Z}_N -component (resp. $\mathcal{Z}_{\bar{N}}$ -component) of the state y is entirely determined by the \mathcal{U}_N -component (resp. $\mathcal{U}_{\bar{N}}$ -component) of the control u .

3.2 Controllability of complex finite dimensional projected system

We are now interested in constructing a control space of minimal dimension such that the finite dimensional system (30) is exactly controllable. We suppose that we have a family of K vectors v_1, \dots, v_K of \mathcal{U} and we set

$$\mathcal{C}_K \stackrel{\text{def}}{=} \text{span}_{\mathbb{C}}\{v_j \mid j = 1, \dots, K\}. \quad (32)$$

We introduce the orthogonal projection $\pi_K : \mathcal{U} \rightarrow \mathcal{C}_K$ and its adjoint $\pi_K^* : \mathcal{C}_K \rightarrow \mathcal{U}$, which is the injection operator from \mathcal{C}_K into \mathcal{U} , and we define the control operator :

$$B_{N,K} \stackrel{\text{def}}{=} B_N p_N \pi_K^* : \mathcal{C}_K \longrightarrow \mathcal{Z}_N.$$

Notice that from the first equality in (29) we have the simplified expression $B_{N,K} = P_N B \pi_K^*$. It allows to rewrite (30) for a control function $u \in L^2(0, T; \mathcal{C}_K)$ as follows:

$$y'_N = A_N y_N + B_{N,K} u \in \mathcal{Z}_N. \quad (33)$$

Our main objective is to characterize the families $(v_j)_{1 \leq j \leq K}$ for which (33) is controllable. According to (30), which states that y_N is only determined by $p_N u$, if an admissible family $(v_j)_{1 \leq j \leq K}$ exists then it can be chosen in $(\mathcal{U}_N)^K$. More precisely, we are going to prove that each v_j can be chosen in $B^*(\mathcal{E}_N)$ where:

$$\mathcal{E}_N \stackrel{\text{def}}{=} \bigoplus_{k=1}^N \text{Ker}(A^* - \bar{\lambda}_k) = \text{span}_{\mathbb{C}}\{\varepsilon_k^i, k = 1, \dots, N, i = 1, \dots, \ell_k\},$$

and that we can choose K equal to the maximal geometric multiplicity of the eigenvalues of A_N . In the above setting $\{\varepsilon_k^i, i = 1, \dots, \ell_k\}$ denotes a basis of $\text{Ker}(A^* - \bar{\lambda}_k)$ and ℓ_k is the geometric multiplicity of $\bar{\lambda}_k$. Using those notations, the following theorem provides a first necessary and sufficient condition for (33) to be exactly controllable.

Theorem 5. *For all $k = 1, \dots, N$, let $\{\varepsilon_k^i, i = 1, \dots, \ell_k\}$ be a basis of $\text{Ker}(A^* - \bar{\lambda}_k)$, let $(v_j)_{1 \leq j \leq K}$ be a family of \mathcal{U}^K and let define:*

$$W_k \stackrel{\text{def}}{=} \left((v_j \mid B^* \varepsilon_k^i)_{\substack{1 \leq i \leq \ell_k, 1 \leq j \leq K}} \right) = \begin{pmatrix} (v_1 \mid B^* \varepsilon_k^1)_{\mathcal{U}} & \dots & (v_j \mid B^* \varepsilon_k^1)_{\mathcal{U}} & \dots & (v_K \mid B^* \varepsilon_k^1)_{\mathcal{U}} \\ \vdots & & \vdots & & \vdots \\ (v_1 \mid B^* \varepsilon_k^i)_{\mathcal{U}} & \dots & (v_j \mid B^* \varepsilon_k^i)_{\mathcal{U}} & \dots & (v_K \mid B^* \varepsilon_k^i)_{\mathcal{U}} \\ \vdots & & \vdots & & \vdots \\ (v_1 \mid B^* \varepsilon_k^{\ell_k})_{\mathcal{U}} & \dots & (v_j \mid B^* \varepsilon_k^{\ell_k})_{\mathcal{U}} & \dots & (v_K \mid B^* \varepsilon_k^{\ell_k})_{\mathcal{U}} \end{pmatrix}.$$

Then the system (33) is exactly controllable, if and only if, for all $k = 1, \dots, N$ $\text{Rank}(W_k) = \ell_k$.

Proof. Because \mathcal{Z}_N is finite dimensional, (33) is exactly controllable, if and only if, the following Hautus test is satisfied, see [28, Prop. 1.5.5]:

$$\forall k \in \{1, \dots, N\}, \quad \text{Rank}[A_N - \lambda_k \mid B_{N,K}] = \dim(\mathcal{Z}_N),$$

or equivalently:

$$\forall k \in \{1, \dots, N\}, \quad \mathcal{Z}_N = \{(A_N - \lambda_k)e + B_{N,K}u \mid (e, u) \in \mathcal{Z}_N \times \mathcal{C}_K\}. \quad (34)$$

Thus, we have that (33) is exactly controllable, if and only if, for all $\varepsilon \in \mathcal{Z}_N^*$ and all $k = 1, \dots, N$ we have :

$$\forall (e, u) \in \mathcal{Z}_N \times \mathcal{C}_K, \quad ((A_N - \lambda_k)e + B_{N,K}u \mid \varepsilon)_{\mathcal{Z}} = 0 \implies \varepsilon = 0. \quad (35)$$

Indeed, the fact that (34) implies (35) follows from the first equality in (25) which guarantees that $\mathcal{Z}_N^\perp \cap \mathcal{Z}_N^* = \{0\}$. Conversely, if (34) is not satisfied, then there exists a nonzero $e \in \mathcal{Z}_N$ which is orthogonal to the space define at the left of the equality in (34), and we verify that (35) is false for $\varepsilon = P_N^* e \in \mathcal{Z}_N^*$ (which is also non null because it obeys $(e \mid \varepsilon)_{\mathcal{Z}} = \|e\|_{\mathcal{Z}}^2$). Next, by replacing $e \in \mathcal{Z}_N$ by $P_N e$ for $e \in \mathcal{Z}$ in (35) and since $B_{N,K}u = P_N B u$ for all $u \in \mathcal{C}_K$, we obtain that for $\varepsilon \in \mathcal{Z}_N^*$, (35) is also equivalent to:

$$\forall (e, u) \in \mathcal{Z} \times \mathcal{C}_K, \quad (e \mid (A^* - \bar{\lambda}_k)\varepsilon)_{\mathcal{Z}} = 0 \quad \text{and} \quad (u \mid B^* \varepsilon)_{\mathcal{U}} = 0 \implies \varepsilon = 0.$$

Then we have proved that (33) is exactly controllable, if and only if, for all $k \in \{1, \dots, N\}$ we have

$$\forall j \in \{1, \dots, K\}, \quad \forall \varepsilon \in \text{Ker}(A^* - \bar{\lambda}_k), \quad (v_j \mid B^* \varepsilon)_{\mathcal{U}} = 0 \implies \varepsilon = 0.$$

Finally, because the above implication means that $\text{Ker}({}^t W_k) = \{0\}$, the desired result is obtained. \square

The two first obvious consequences of the above theorem is that K must be greater or equal to all the geometric multiplicities ℓ_k , and that there is no eigenvector related to an eigenvalue $\bar{\lambda}_k$ of A^* which is in the kernel of B^* . Otherwise there would be a linear combination of line of W_k equal to zero for some $k \in \{1, \dots, N\}$.

Corollary 6. *The following conditions are necessary for the exact controllability of (33):*

1. Any $\varepsilon \in \mathcal{D}(A^*)$ which obeys $A^*\varepsilon = \bar{\lambda}\varepsilon$ and $B^*\varepsilon = 0$ for $\lambda \in \Sigma_N$ is necessarily equal to zero.
2. The dimension K of the control space \mathcal{C}_K is necessarily greater or equal to the maximum of the geometric multiplicities of λ_k , $k = 1, \dots, N$:

$$K \geq \max\{\ell_k \mid k = 1, \dots, N\}.$$

In fact, Conditions 1 and 2 of Corollary 6 are also sufficient to construct K vectors v_1, \dots, v_K of $B^*(\mathcal{E}_N)$ such that (33) is exactly controllable.

Theorem 7. *If Conditions 1 and 2 of Corollary 6 hold then there exists a family $(v_j)_{1 \leq j \leq K}$ of $B^*(\mathcal{E}_N)$ such that (33) is exactly controllable. Moreover, the families $(v_j)_{1 \leq j \leq K}$ such that (33) is exactly controllable form an open and dense subset of \mathcal{U}^K .*

Proof. First, for all $k \in \{1, \dots, N\}$ let $\mathcal{M}_{\ell_k, K}(\mathbb{C})$ be the space of complex matrix of size $\ell_k \times K$ and let us consider the subset \mathcal{R}_k of $\mathcal{M}_{\ell_k, K}(\mathbb{C})$ composed by the matrices of rank ℓ_k . We also consider the continuous linear maps

$$\Theta_k : \mathcal{U}^K \rightarrow \mathcal{M}_{\ell_k, K}(\mathbb{C}), \quad (v_1, \dots, v_K) \mapsto W_k(v_1, \dots, v_K) \stackrel{\text{def}}{=} \left((v_j \mid B^* \varepsilon_k^i)_{\mathcal{U}} \right)_{1 \leq i \leq \ell_k, 1 \leq j \leq K}.$$

By using Theorem 5, we know that the set of admissible families

$$\mathcal{A} \stackrel{\text{def}}{=} \{(v_j)_{1 \leq j \leq K} \in \mathcal{U}^K \mid (33) \text{ is exactly controllable} \}$$

is characterized by

$$\mathcal{A} = \bigcap_{k=1}^N \Theta_k^{-1}(\mathcal{R}_k).$$

Since \mathcal{R}_k is an open subset of $\mathcal{M}_{\ell_k, K}(\mathbb{C})$, we deduce that $\Theta_k^{-1}(\mathcal{R}_k)$ is an open subset of \mathcal{U}^K and thus \mathcal{A} is also an open subset of \mathcal{U}^K .

Second, let set $\ell \stackrel{\text{def}}{=} \sum_{r=1}^N \ell_r$, let denote by $v(z) \stackrel{\text{def}}{=} \sum_{k=1}^N \sum_{i=1}^{\ell_k} z_{k,i} B^* \varepsilon_k^i$ where $z \stackrel{\text{def}}{=} (z_{k,i})_{k=1, \dots, N, i=1, \dots, \ell_k} \in \mathbb{C}^\ell$ the elements of $B^*(\mathcal{E}_N)$:

$$B^*(\mathcal{E}_N) = \{v(z) \mid z \in \mathbb{C}^\ell\},$$

and let denote by $U(Z) \stackrel{\text{def}}{=} (v(z^{(1)}), \dots, v(z^{(K)})) \in (B^*(\mathcal{E}_N))^K$ where $Z \stackrel{\text{def}}{=} (z^{(1)}, \dots, z^{(K)}) \in \mathbb{C}^{\ell \times K}$ the elements of $(B^*(\mathcal{E}_N))^K$:

$$(B^*(\mathcal{E}_N))^K = \{U(Z) \mid Z \in \mathbb{C}^{\ell \times K}\}.$$

If we denote by $B^*(\mathcal{E}_N)^\perp$ the orthogonal of $B^*(\mathcal{E}_N)$ in \mathcal{U} , then the following bounded linear mapping is onto:

$$\Psi : (B^*(\mathcal{E}_N)^\perp)^K \times \mathbb{C}^{\ell \times K} \mapsto \mathcal{U}^K, \quad \Psi(U_\perp, Z) \stackrel{\text{def}}{=} U_\perp + U(Z).$$

Moreover, Condition 2 of Corollary 6 ensures that for $k = 1, \dots, N$ and $Z \in \mathbb{C}^{\ell \times K}$ the following determinant is well defined:

$$Q_k(Z) \stackrel{\text{def}}{=} \det \begin{pmatrix} (v(z^{(1)}) \mid B^* \varepsilon_k^1)_{\mathcal{U}} & \dots & (v(z^{(j)}) \mid B^* \varepsilon_k^1)_{\mathcal{U}} & \dots & (v(z^{(\ell_k)}) \mid B^* \varepsilon_k^1)_{\mathcal{U}} \\ \vdots & & \vdots & & \vdots \\ (v(z^{(1)}) \mid B^* \varepsilon_k^i)_{\mathcal{U}} & \dots & (v(z^{(j)}) \mid B^* \varepsilon_k^i)_{\mathcal{U}} & \dots & (v(z^{(\ell_k)}) \mid B^* \varepsilon_k^i)_{\mathcal{U}} \\ \vdots & & \vdots & & \vdots \\ (v(z^{(1)}) \mid B^* \varepsilon_k^{\ell_k})_{\mathcal{U}} & \dots & (v(z^{(j)}) \mid B^* \varepsilon_k^{\ell_k})_{\mathcal{U}} & \dots & (v(z^{(\ell_k)}) \mid B^* \varepsilon_k^{\ell_k})_{\mathcal{U}} \end{pmatrix},$$

and we set

$$Q(Z) \stackrel{\text{def}}{=} \prod_{k=1}^N Q_k(Z). \quad (36)$$

Thus, we denote by \mathbb{O} the set of roots of Q and by \mathbb{O}^c its complementary part in $\mathbb{C}^{\ell \times K}$. Then we have:

$$\mathcal{A} \supset \{U_{\perp} + U(Z) \mid (U_{\perp}, Z) \in (B^*(\mathcal{E}_N)^{\perp})^K \times \mathbb{C}^{\ell \times K}, Q(Z) \neq 0\} = \Psi((B^*(\mathcal{E}_N)^{\perp})^K \times \mathbb{O}^c).$$

We observe that for all $k = 1, \dots, N$ the coefficient of $z_{k,1}^{(1)} z_{k,2}^{(2)} \dots z_{k,\ell_k}^{(\ell_k)}$ in $Q_k(Z)$ is equal to

$$\det \begin{pmatrix} (B^* \varepsilon_k^1 | B^* \varepsilon_k^1)_{\mathcal{U}} & \dots & (B^* \varepsilon_k^j | B^* \varepsilon_k^1)_{\mathcal{U}} & \dots & (B^* \varepsilon_k^{\ell_k} | B^* \varepsilon_k^1)_{\mathcal{U}} \\ \vdots & & \vdots & & \vdots \\ (B^* \varepsilon_k^1 | B^* \varepsilon_k^i)_{\mathcal{U}} & \dots & (B^* \varepsilon_k^j | B^* \varepsilon_k^i)_{\mathcal{U}} & \dots & (B^* \varepsilon_k^{\ell_k} | B^* \varepsilon_k^i)_{\mathcal{U}} \\ \vdots & & \vdots & & \vdots \\ (B^* \varepsilon_k^1 | B^* \varepsilon_k^{\ell_k})_{\mathcal{U}} & \dots & (B^* \varepsilon_k^j | B^* \varepsilon_k^{\ell_k})_{\mathcal{U}} & \dots & (B^* \varepsilon_k^{\ell_k} | B^* \varepsilon_k^{\ell_k})_{\mathcal{U}} \end{pmatrix}. \quad (37)$$

By remarking that Condition 1 of Corollary 6 ensures that $\{B^* \varepsilon_k^i \mid i = 1, \dots, \ell_k\}$ is a linearly independent family, we deduce that the above determinant is not zero and then that the polynomials Q_k for all $k = 1, \dots, N$ are not identically zero. As a consequence, $Q \neq 0$ and the set of its roots \mathbb{O} is a dense subset of $\mathbb{C}^{\ell \times K}$. Then the density of $\Psi((B^*(\mathcal{E}_N)^{\perp})^K \times \mathbb{O}^c)$ in \mathcal{U}^K follows from the continuity and the surjectivity of the linear map Ψ . It implies the density of \mathcal{A} . Finally, the existence of a family $(v_j)_{1 \leq j \leq K}$ of $B^*(\mathcal{E}_N)$ such that (33) is exactly controllable can be proved by considering $\Psi((0, Z))$ where Z is such that $Q(Z) \neq 0$. \square

Finally, let us give a sufficient condition for Condition 1 of Corollary 6.

Theorem 8. *If system (27) is approximately controllable, then any $\varepsilon \in \mathcal{D}(A^*)$ which obeys $A^* \varepsilon = \lambda \varepsilon$ and $B^* \varepsilon = 0$ for $\lambda \in \mathbb{C}$ is necessarily equal to zero.*

Proof. Since the proof is obvious for λ in the resolvent set of A^* , we can suppose that λ is an eigenvalue of A^* . By following the lines of Subsection 3.1 we define a projection operator $P_{\bar{\lambda}} : \mathcal{Z} \rightarrow \mathcal{Z}$ by:

$$P_{\bar{\lambda}} \stackrel{\text{def}}{=} -\frac{1}{2\pi i} \int_{\Gamma_{\bar{\lambda}}} (\xi - A)^{-1} d\xi,$$

where $\Gamma_{\bar{\lambda}}$ is now a positively-oriented curve enclosing $\bar{\lambda}$ but no other point of the spectrum of A . We then define the invariant subspace $\mathcal{Z}_{\bar{\lambda}} \stackrel{\text{def}}{=} P_{\bar{\lambda}}(\mathcal{Z})$ of A as well as the finite dimensional restricted operator $A_{\bar{\lambda}} \stackrel{\text{def}}{=} A|_{\mathcal{Z}_{\bar{\lambda}}}$, with spectrum reduced to $\{\bar{\lambda}\}$. We also introduce $B_{\bar{\lambda}} \stackrel{\text{def}}{=} P_{\bar{\lambda}} B p_{\bar{\lambda}}^*$, where $p_{\bar{\lambda}}$ is the orthogonal projection operator from \mathcal{U} onto $\mathcal{U}_{\bar{\lambda}} \stackrel{\text{def}}{=} \{B^* \varepsilon \mid \varepsilon \in \mathcal{Z}_{\bar{\lambda}}^*\}$ and by applying $P_{\bar{\lambda}}$ to (27) we obtain that $y_{\bar{\lambda}} \stackrel{\text{def}}{=} P_{\bar{\lambda}} y$ and $u_{\bar{\lambda}} \stackrel{\text{def}}{=} p_{\bar{\lambda}} u$ obey:

$$y_{\bar{\lambda}} = A_{\bar{\lambda}} y_{\bar{\lambda}} + B_{\bar{\lambda}} u_{\bar{\lambda}}. \quad (38)$$

Because the approximate controllability of (27) implies the approximate controllability of (38) and because (38) is a finite dimensional system, we deduce that (38) is exactly controllable. As a consequence, the Hautus test $\text{Rank}[A_{\bar{\lambda}} - \bar{\lambda} \mid B_{\bar{\lambda}}] = \dim(\mathcal{Z}_{\bar{\lambda}})$ is satisfied, which means that for all $\varepsilon \in \mathcal{Z}_{\bar{\lambda}}^*$:

$$\forall (e, u) \in \mathcal{Z}_{\bar{\lambda}} \times \mathcal{U}_{\bar{\lambda}} \quad ((A_{\bar{\lambda}} - \bar{\lambda})e + B_{\bar{\lambda}} u | \varepsilon)_{\mathcal{Z}} = 0 \implies \varepsilon = 0,$$

or equivalently

$$\forall (e, u) \in \mathcal{Z}_{\bar{\lambda}} \times \mathcal{U}_{\bar{\lambda}} \quad (e | (A^* - \lambda)\varepsilon)_{\mathcal{Z}} = 0 \quad \text{and} \quad (u | B^* \varepsilon)_{\mathcal{U}} = 0 \implies \varepsilon = 0.$$

Finally, by replacing $(e, u) \in \mathcal{Z}_{\bar{\lambda}} \times \mathcal{U}_{\bar{\lambda}}$ by $(P_{\bar{\lambda}} e, p_{\bar{\lambda}} u) \in \mathcal{Z} \times \mathcal{U}$ in the above implication we obtain

$$\forall (e, u) \in \mathcal{Z} \times \mathcal{U} \quad (e | (A^* - \lambda)\varepsilon)_{\mathcal{Z}} = 0 \quad \text{and} \quad (u | B^* \varepsilon)_{\mathcal{U}} = 0 \implies \varepsilon = 0,$$

which gives the desired result. \square

3.3 Controllability of real finite dimensional projected system

Let us now suppose that $A : \mathcal{D}(A) \subset \mathcal{X} \rightarrow \mathcal{X}$ and $B : \mathcal{V} \rightarrow [\mathcal{D}(A^*)]'$ where \mathcal{X} and \mathcal{V} are real Hilbert spaces, and let us suppose that their extensions $A : \mathcal{D}(A) \subset \mathcal{Z} \rightarrow \mathcal{Z}$ and $B : \mathcal{U} \rightarrow [\mathcal{D}(A^*)]'$ to the complexified spaces $\mathcal{Z} \stackrel{\text{def}}{=} \mathcal{X} + \iota\mathcal{X}$ and $\mathcal{U} \stackrel{\text{def}}{=} \mathcal{V} + \iota\mathcal{V}$ obey the properties stated in Subsection 3.1. Since the spectrum of A now admits the real line as a symmetry axis, the positively-oriented curve Γ_N in (24) can be defined so that it obeys $\Gamma_N = \bar{\Gamma}_N$, and an easy calculation shows that it implies $P_N = \bar{P}_N$. As a consequence, the projection operator P_N maps \mathcal{X} to itself and it provides the following decompositions of \mathcal{X} :

$$\mathcal{X} = \mathcal{X}_N \oplus \mathcal{X}_N^-, \quad \mathcal{X}_N \stackrel{\text{def}}{=} P_N(\mathcal{X}) \quad \text{and} \quad \mathcal{X}_N^- \stackrel{\text{def}}{=} (I - P_N)(\mathcal{X}),$$

and

$$\mathcal{X} = \mathcal{X}_N^* \oplus \mathcal{X}_N^{-*}, \quad \mathcal{X}_N^* \stackrel{\text{def}}{=} P_N^*(\mathcal{X}) \quad \text{and} \quad \mathcal{X}_N^{-*} \stackrel{\text{def}}{=} (I - P_N^*)(\mathcal{X}).$$

Analogously as in Subsection 3.1, we introduce the real control spaces

$$\mathcal{V}_N \stackrel{\text{def}}{=} \{B^*x \mid x \in \mathcal{X}_N^*\} \quad \text{and} \quad \mathcal{V}_N^- \stackrel{\text{def}}{=} \{B^*x \mid x \in \mathcal{D}(A^*) \cap \mathcal{X}_N^{-*}\},$$

with their related orthogonal projections $p_N : \mathcal{V} \rightarrow \mathcal{V}_N$ and $p_N^- : \mathcal{V} \rightarrow \mathcal{V}_N^-$ and control operators:

$$B_N \stackrel{\text{def}}{=} P_N B p_N^* : \mathcal{V}_N \longrightarrow \mathcal{X}_N \quad \text{and} \quad B_N^- \stackrel{\text{def}}{=} (I - P_N) B p_N^{-*} : \mathcal{V}_N^- \longrightarrow [\mathcal{D}(A^*) \cap \mathcal{X}_N^{-*}]'.$$

We recall that the adjoints $p_N^* : \mathcal{V}_N \rightarrow \mathcal{V}$ and $p_N^{-*} : \mathcal{V}_N^- \rightarrow \mathcal{V}$ are simply injection operators. Thus, by following the steps of Section 3.1 we show that for $u \in L^2(0, T; \mathcal{V})$, y is solution of (27) if and only if the pair $(y_N, y_N^-) \stackrel{\text{def}}{=} (P_N y, (I - P_N)y)$ obeys the system:

$$y_N' = A_N y_N + B_N p_N u \in \mathcal{X}_N, \tag{39}$$

$$y_N^{-'} = A_N^- y_N^- + B_N^- p_N^- u \in [\mathcal{D}(A^*) \cap \mathcal{X}_N^{-*}]'. \tag{40}$$

Next, we denote:

$$\mathcal{F}_N \stackrel{\text{def}}{=} \text{span}_{\mathbb{R}} \{ \Re \varepsilon_k^i, \Im \varepsilon_k^i, k = 1, \dots, N, i = 1, \dots, \ell_k \},$$

and the following theorem states that $(v_j)_{1 \leq j \leq K}$ provided by Theorem 7 can be chosen in $(B^*(\mathcal{F}_N))^K \subset (\mathcal{V}_N)^K$.

Theorem 9. *If Conditions 1 and 2 of Corollary 6 hold then there exists a family $(v_j)_{1 \leq j \leq K}$ of $B^*(\mathcal{F}_N)$ such that (33) is exactly controllable. Moreover, the families $(v_j)_{1 \leq j \leq K}$ such that (33) is exactly controllable form an open and dense subset of \mathcal{V}^K .*

Proof. The proof that

$$\mathcal{A} \stackrel{\text{def}}{=} \{(v_j)_{1 \leq j \leq K} \in \mathcal{V}^K \mid (33) \text{ is exactly controllable} \}$$

is an open subset of \mathcal{V}^K is similar to the proof of Theorem 7. For the density of \mathcal{A} , we also follow the same arguments than in the proof of Theorem 7: we set $\ell \stackrel{\text{def}}{=} \sum_{k=1}^N \ell_k$ and we define

$$v(x, y) \stackrel{\text{def}}{=} \sum_{k,i} x_{k,i} B^* \Re \varepsilon_k^i + \sum_{k,i} y_{k,i} B^* \Im \varepsilon_k^i, \quad (x, y) \stackrel{\text{def}}{=} (x_{k,i}, y_{k,i})_{k=1, \dots, N, i=1, \dots, \ell_k} \in \mathbb{C}^{2\ell},$$

and

$$U(X, Y) \stackrel{\text{def}}{=} (v(x^{(1)}, y^{(1)}), \dots, v(x^{(K)}, y^{(K)})), \quad (X, Y) \stackrel{\text{def}}{=} (x^{(1)}, \dots, x^{(K)}, y^{(1)}, \dots, y^{(K)}) \in \mathbb{C}^{2\ell \times K}.$$

In particular, we have

$$B^*(\mathcal{F}_N) = \{v(x, y) \mid (x, y) \in \mathbb{R}^{2\ell}\},$$

and the linear and continuous map

$$\Phi : (B^*(\mathcal{F}_N)^\perp)^K \times \mathbb{R}^{2\ell \times K} \rightarrow \mathcal{V}^K : \Phi(V^\perp, X, Y) \stackrel{\text{def}}{=} V^\perp + U(X, Y)$$

is onto.

Next, Condition 2 of Corollary 6 ensures that for $k = 1, \dots, N$ the following determinant is well defined:

$$R_k(X, Y) \stackrel{\text{def}}{=} \det \begin{pmatrix} (v(x^{(1)}, y^{(1)})|B^* \varepsilon_k^1)_U & \dots & (v(x^{(j)}, y^{(j)})|B^* \varepsilon_k^1)_U & \dots & (v(x^{(\ell_k)}, y^{(\ell_k)})|B^* \varepsilon_k^1)_U \\ \vdots & & \vdots & & \vdots \\ (v(x^{(1)}, y^{(1)})|B^* \varepsilon_k^i)_U & \dots & (v(x^{(j)}, y^{(j)})|B^* \varepsilon_k^i)_U & \dots & (v(x^{(\ell_k)}, y^{(\ell_k)})|B^* \varepsilon_k^i)_U \\ \vdots & & \vdots & & \vdots \\ (v(x^{(1)}, y^{(1)})|B^* \varepsilon_k^{\ell_k})_U & \dots & (v(x^{(j)}, y^{(j)})|B^* \varepsilon_k^{\ell_k})_U & \dots & (v(x^{(\ell_k)}, y^{(\ell_k)})|B^* \varepsilon_k^{\ell_k})_U \end{pmatrix},$$

and we set

$$R(X, Y) \stackrel{\text{def}}{=} \prod_{k=1}^N R_k(X, Y).$$

We denote by \mathbb{O}_r the set of real roots of R and by \mathbb{O}_r^c its complementary part in $\mathbb{R}^{2\ell \times K}$. By applying Theorem 5, we have

$$\mathcal{A} \supset \Phi((B^*(\mathcal{F}_N)^\perp)^K \times \mathbb{O}_r^c).$$

Next, we remark that for $Z \in \mathbb{C}^\ell$ we have $R(Z, \iota Z) = Q(Z)$, where Q is defined by (36). By using Condition 1 of Corollary 6, $Q \neq 0$ and thus $R \neq 0$. Thus \mathbb{O}_r is dense in $\mathbb{R}^{2\ell \times K}$ and we deduce that $\Phi((B^*(\mathcal{F}_N)^\perp)^K \times \mathbb{O}_r^c)$ is dense in \mathcal{V}^K . \square

If $(v_j)_{1 \leq j \leq K}$ is a real family provided by Theorem 9 then we define the real control space:

$$\mathcal{G}_K \stackrel{\text{def}}{=} \text{span}_{\mathbb{R}}\{v_j \mid j = 1, \dots, K\},$$

and we verify that the control operator $B_{N,K} \stackrel{\text{def}}{=} B_N p_N \pi_K^* = P_N B \pi_K^*$ maps \mathcal{G}_K to \mathcal{X}_N . As a consequence, equation (39) with $u \in L^2(0, T; \mathcal{G}_K)$ becomes the following dynamical system:

$$y'_N = A_N y_N + B_{N,K} u \in \mathcal{X}_N, \quad (41)$$

and the exact controllability of (41) follows from the exact controllability of (33), which is provided by Theorem 9. Indeed, since the spaces $\mathcal{Z}_N \stackrel{\text{def}}{=} P_N(\mathcal{Z})$ and \mathcal{C}_K defined by (32) are also given by $\mathcal{Z}_N = \mathcal{X}_N + \iota \mathcal{X}_N$ and $\mathcal{C}_K = \mathcal{G}_K + \iota \mathcal{G}_K$, and since $A_N \in \mathcal{L}(\mathcal{X}_N)$ and $B_{N,K} \in \mathcal{L}(\mathcal{X}_N, \mathcal{G}_K)$, we have:

$$\forall (y_N, u) \in \mathcal{Z}_N \times \mathcal{C}_K, \quad \Re(A_N y_N) = A_N \Re y_N \quad \text{and} \quad \Re(B_{N,K} u) = B_{N,K} \Re u.$$

It implies that if $(y_N, u) \in L^2(0, T; \mathcal{Z}_N) \times L^2(0, T; \mathcal{C}_K)$ satisfies (33) then $(\Re y_N, \Re u) \in L^2(0, T; \mathcal{X}_N) \times L^2(0, T; \mathcal{G}_K)$ satisfies (41). As a consequence, for a prescribed pair $(y_0, y_T) \in \mathcal{X}_N \times \mathcal{X}_N$, if $\hat{u} \in L^2(0, T; \mathcal{C}_K)$ is a control sending the solution of (33) from y_0 to y_T , then by taking the real part of the equation we deduce that $\Re \hat{u} \in L^2(0, T; \mathcal{G}_K)$ is a control sending the solution of (41) from y_0 to y_T .

Next, let us introduce the linear map:

$$V : \mathbb{R}^K \longrightarrow [\mathcal{D}(A^*)]', \quad V \bar{w} \stackrel{\text{def}}{=} \sum_{j=1}^K w_j B v_j \quad \text{where} \quad \bar{w} = (w_1, \dots, w_K) \in \mathbb{R}^K, \quad (42)$$

with adjoint given by:

$$V^* : \mathcal{D}(A^*) \longrightarrow \mathbb{R}^K, \quad V^* \varepsilon = ((v_j | B^* \varepsilon)_V)_{1 \leq j \leq K} = ((v_1 | B^* \varepsilon)_V, \dots, (v_K | B^* \varepsilon)_V). \quad (43)$$

If explicitly express u in (41) as a linear combination of v_1, \dots, v_K :

$$u = \sum_{j=1}^K u_j v_j \quad \text{where} \quad \bar{u} = (u_1, \dots, u_K) \in L^2(\mathbb{R}^K),$$

then we have $B_{N,K} u = P_N B u = P_N V \bar{u}$, and by setting

$$V_N \stackrel{\text{def}}{=} P_N V : \mathbb{R}^K \rightarrow \mathcal{X}_N$$

system (41) becomes:

$$y'_N = A_N y_N + V_N \bar{u} \in \mathcal{X}_N. \quad (44)$$

Moreover, the exact controllability of (41) obviously implies the exact controllability of (44).

3.4 Feedback stabilization of real finite-dimensional projected system

Let us assume that Conditions 1 and 2 of Corollary 6 hold and let us consider the real family $(v_j)_{1 \leq j \leq K}$ given by Theorem 9. The main goal of this subsection is to prove that for any arbitrary chosen rate $\sigma > 0$ system (44) is stabilizable with a rate of decrease $-\sigma$, by means of a linear feedback controller. First, from (44) the change of variable $(y_{N,\sigma}(t), \bar{u}_\sigma(t)) = e^{\sigma t}(y_N(t), \bar{u}(t))$ yields the system:

$$y'_{N,\sigma} = (A_N + \sigma)y_{N,\sigma} + V_N \bar{u}_\sigma \in \mathcal{X}_N, \quad (45)$$

which is exactly controllable since (44) is exactly controllable. Thus, the following optimal control problem is well-posed:

$$\inf \left\{ \frac{1}{2} \int_0^\infty \|y_{N,\sigma}\|_{\mathcal{X}}^2 dt + \frac{1}{2} \int_0^\infty \|\bar{u}_\sigma\|_{\mathbb{R}^K}^2 dt ; (y_{N,\sigma}, \bar{u}_\sigma) \text{ obeys (45) and } y_{N,\sigma}(0) = h \right\},$$

and from Theorem 2.9 p. 44, Theorem 1.4 p. 136 and Theorem 1.5 p. 138 in [30] we know that the solution to the above optimal control problem is exponentially stable and obeys the feedback equality $\bar{u}_\sigma(t) = -V_N^* \tilde{\Pi} y_{N,\sigma}(t)$ where $V_N^* : \mathcal{X}_N \rightarrow \mathbb{R}^K$ is the adjoint of V_N and $\tilde{\Pi} \in \mathcal{L}(\mathcal{X}_N)$ is the unique solution of the Riccati equation: for all $(\xi, \zeta) \in \mathcal{X}_N \times \mathcal{X}_N$,

$$\begin{aligned} (\xi|\zeta)_{\mathcal{X}} + ((A_N + \sigma)\xi|\tilde{\Pi}\zeta)_{\mathcal{X}} + (\tilde{\Pi}\xi|(A_N + \sigma)\zeta)_{\mathcal{X}} - (V_N^* \tilde{\Pi}\xi|V_N^* \tilde{\Pi}\zeta)_{\mathbb{R}^K} &= 0, \\ (\tilde{\Pi}\xi|\zeta)_{\mathcal{X}} = (\xi|\tilde{\Pi}\zeta)_{\mathcal{X}} \quad \text{and} \quad (\tilde{\Pi}\xi|\xi)_{\mathcal{X}} > 0 \text{ if } \xi \neq 0. \end{aligned}$$

Notice that as it is underlined in [21, Par. 5], the solution $\tilde{\Pi}$ belongs to $\mathcal{L}(\mathcal{X}_N)$ because when we have applied the theory of [30] we have identified \mathcal{X}_N with its dual. However, we remark that $\tilde{\Pi}$ is simply the orthogonal projection of $\Pi \stackrel{\text{def}}{=} P_N^* \tilde{\Pi} \in \mathcal{L}(\mathcal{X}_N, \mathcal{X}_N^*)$ on \mathcal{X}_N : $(\Pi\xi|\zeta)_{\mathcal{X}} = (\tilde{\Pi}\xi|\zeta)_{\mathcal{X}}$ for all $(\xi, \zeta) \in \mathcal{X}_N \times \mathcal{X}_N$. Then by observing that we have $V_N^* \tilde{\Pi} = V^* P_N^* \tilde{\Pi} = V^* \Pi$, where $V^* : \mathcal{D}(A^*) \rightarrow \mathbb{R}^K$ is the adjoint of V given in (43), we deduce that Π is the unique element of $\mathcal{L}(\mathcal{X}_N, \mathcal{X}_N^*)$ which obeys: for all $(\xi, \zeta) \in \mathcal{X}_N \times \mathcal{X}_N$,

$$(\xi|\zeta)_{\mathcal{X}} + ((A_N + \sigma)\xi|\Pi\zeta)_{\mathcal{X}} + (\Pi\xi|(A_N + \sigma)\zeta)_{\mathcal{X}} - \sum_{j=1}^K (B^* \Pi\xi|v_j)_{\mathcal{V}} (B^* \Pi\zeta|v_j)_{\mathcal{V}} = 0, \quad (46)$$

$$(\Pi\xi|\zeta)_{\mathcal{X}} = (\xi|\Pi\zeta)_{\mathcal{X}} \quad \text{and} \quad (\Pi\xi|\xi)_{\mathcal{X}} > 0 \text{ if } \xi \neq 0. \quad (47)$$

The uniqueness of the solution of the above Riccati equation follows by remarking that every $\Pi \in \mathcal{L}(\mathcal{X}_N, \mathcal{X}_N^*)$ with orthogonal projection $\tilde{\Pi}$ on \mathcal{X}_N necessarily obeys $\Pi = P_N^* \tilde{\Pi}$. Thus, because we have $V_N^* = V^* \Pi$, the optimal pair obeys $\bar{u}_\sigma(t) = -V^* \Pi y_{N,\sigma}(t)$ and then the optimal state is solution to:

$$y'_{N,\sigma} = (A_N + \sigma)y_{N,\sigma} - V_N (V^* \Pi) y_{N,\sigma}, \quad y_{N,\sigma}(0) = h. \quad (48)$$

Moreover, we have the exponential decrease

$$\|y_{N,\sigma}(t)\|_{\mathcal{X}_N} \leq C e^{-at} \|h\|_{\mathcal{X}_N} \quad a > 0. \quad (49)$$

Thus, with the change of variable $(y_N(t), \bar{u}(t)) = e^{-\sigma t}(y_{N,\sigma}(t), \bar{u}_\sigma(t))$ we obtain that the solution to:

$$y'_N = A_N y_N - V_N (V^* \Pi) y_N, \quad y_N(0) = h, \quad (50)$$

obeys the exponential decrease:

$$\|y_N(t)\|_{\mathcal{X}_N} \leq C e^{-(\sigma+a)t} \|h\|_{\mathcal{X}_N} \quad a > 0, \quad (51)$$

and the feedback relation $(u_1, \dots, u_K) = \bar{u} = -V^* \Pi y_N$ ensures that the optimal control $u = \sum_{j=1}^K u_j v_j$ obeys:

$$\|u(t)\|_{\mathcal{V}} \leq C e^{-(\sigma+a)t} \quad a > 0. \quad (52)$$

Finally by recalling the expression of V^* given in (43) we deduce that u is expressed as follows:

$$u(t) = - \sum_{j=1}^K (v_j | B^* \Pi y_N(t))_{\mathcal{V}} v_j, \quad t \geq 0. \quad (53)$$

3.5 Dynamical stabilization of real finite-dimensional projected system

Let us assume that Conditions 1 and 2 of Corollary 6 hold and let us consider the real family $(v_j)_{1 \leq j \leq K}$ given by Theorem 9. The main goal of this subsection is to prove that for any arbitrary chosen rate $\sigma > 0$ system (44) is stabilizable with a rate of decrease $-\sigma$, by means of $u = \sum_{j=1}^K u_j v_j$ where $\bar{u} = (u_1, \dots, u_K)$ is solution to:

$$\bar{u}' = D_K \bar{u} + \bar{g} \alpha, \quad (54)$$

and where $D_K \in \mathcal{L}(\mathbb{R}^K)$, $\bar{g} \in \mathbb{R}^K$ and $\alpha \in L^2(\mathbb{R})$. Here α is the control of (44), (54). By setting $Y_N = {}^t(y_N, \bar{u})$, system (44), (54) can be rewritten as:

$$Y_N' = \mathbb{A}_N Y_N + \mathbb{B}_N \alpha \quad \text{where} \quad \mathbb{A}_N \stackrel{\text{def}}{=} \begin{pmatrix} A_N & V_N \\ 0 & D_K \end{pmatrix} \quad \text{and} \quad \mathbb{B}_N \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ \bar{g} \end{pmatrix}. \quad (55)$$

The following theorem states that one can choose D_K and \bar{g} such that (55) is exactly controllable.

Theorem 10. *Suppose that Conditions 1 and 2 of Corollary 6 hold, let $D_K : \mathbb{R}^K \rightarrow \mathbb{R}^K$ be a linear operator with spectrum composed of K distinct eigenvalues and that have no common eigenvalues with A_N , and let $\bar{g} \in \mathbb{R}^K$ be such that $(\bar{g}|\bar{d})_{\mathbb{R}^K} \neq 0$ for all eigenvectors \bar{d} of ${}^t D_K$. Then (55) is exactly controllable.*

Proof. Let denote $\Sigma(D_K)$ the spectrum of D_K . Since $\Sigma(D_K) \cup \Sigma_N$ is exactly the spectrum of \mathbb{A}_N , the controllability of (55) follows from the following Hautus rank condition:

$$\text{Rank} \left(\begin{array}{cc|c} A_N - \lambda & V_N & 0 \\ 0 & D_K - \lambda & \bar{g} \end{array} \right) = \dim(\mathcal{X}_N) + K \quad \forall \lambda \in \Sigma(D_K) \cup \Sigma_N,$$

which is an easy consequence of the assumptions. Indeed, the controllability of (44) implies $\text{Rank}(A_N - \lambda|V_N) = \dim(\mathcal{X}_N)$ for all $\lambda \in \Sigma_N$, the fact that \bar{g} obeys $(\bar{g}|\bar{d})_{\mathbb{R}^K} \neq 0$ for all eigenvectors \bar{d} of ${}^t D_K$ implies

$$\text{Ker} \left(\begin{array}{c} {}^t D_K - \bar{\lambda} \\ \bar{g} \end{array} \right) = \{0\} \quad \forall \lambda \in \Sigma(D_K),$$

or equivalently $\text{Rank}(D_K - \lambda|\bar{g}) = K$ for all $\lambda \in \Sigma(D_K)$, and we conclude since $\Sigma(D_K) \cap \Sigma_N = \emptyset$. \square

We suppose that the hypotheses of Theorem 10 are verified and we consider the system

$$Y_{N,\sigma}' = (\mathbb{A}_N + \sigma) Y_{N,\sigma} + \mathbb{B}_N \alpha_\sigma, \quad (56)$$

which is also exactly controllable. For $H \in \mathcal{X}_N \times \mathbb{R}^K$ we define the following optimal control problem:

$$\inf \left\{ \frac{1}{2} \int_0^\infty \|Y_{N,\sigma}\|_{\mathcal{X}_N \times \mathbb{R}^K}^2 dt + \frac{1}{2} \int_0^\infty |\alpha_\sigma|^2 dt ; (Y_{N,\sigma}, \alpha_\sigma) \text{ obeys (56) and } Y_{N,\sigma}(0) = H \right\}. \quad (57)$$

According to Theorem 2.9 p. 44, Theorem 1.4 p. 136 and Theorem 1.5 p. 138 in [30] the above minimization problem admits a unique solution $(Y_{N,\sigma}, \alpha_\sigma)$ which is exponentially stable:

$$\|Y_{N,\sigma}(t)\|_{\mathcal{X} \times \mathbb{R}^K} + |\alpha_\sigma(t)| \leq C e^{-at} \|H\|_{\mathcal{X} \times \mathbb{R}^K}, \quad a > 0, \quad (58)$$

and which obeys $\alpha_\sigma(t) = -\mathbb{B}_N^* \mathbf{\Pi} Y_{N,\sigma}(t)$ where $\mathbf{\Pi} \in \mathcal{L}(\mathcal{X}_N \times \mathbb{R}^K)$ is the unique solution of the Riccati equation: for all $(\xi, \zeta) \in (\mathcal{X}_N \times \mathbb{R}^K) \times (\mathcal{X}_N \times \mathbb{R}^K)$

$$(\xi|\zeta)_{\mathcal{X} \times \mathbb{R}^K} + ((\mathbb{A}_N + \sigma)\xi|\mathbf{\Pi}\zeta)_{\mathcal{X} \times \mathbb{R}^K} + (\mathbf{\Pi}\xi|(\mathbb{A}_N + \sigma)\zeta)_{\mathcal{X} \times \mathbb{R}^K} - (\mathbb{B}_N^* \mathbf{\Pi}\xi|\mathbb{B}_N^* \mathbf{\Pi}\zeta)_{\mathcal{X} \times \mathbb{R}^K} = 0, \quad (59)$$

$$(\mathbf{\Pi}\xi|\zeta)_{\mathcal{X} \times \mathbb{R}^K} = (\xi|\mathbf{\Pi}\zeta)_{\mathcal{X} \times \mathbb{R}^K} \quad \text{and} \quad (\mathbf{\Pi}\xi|\xi)_{\mathcal{X} \times \mathbb{R}^K} > 0 \text{ if } \xi \neq 0. \quad (60)$$

Then from (56) with $\alpha_\sigma(t) = -\mathbb{B}_N^* \mathbf{\Pi} Y_{N,\sigma}(t)$ and from (58), we obtain that the pair $(Y_N, \alpha) = e^{-\sigma t} (Y_{N,\sigma}, \alpha_\sigma)$ obeys

$$Y_N' = \mathbb{A}_N Y_N - \mathbb{B}_N \mathbb{B}_N^* \mathbf{\Pi} Y_N, \quad Y_N(0) = H, \quad \alpha(t) = -\mathbb{B}_N^* \mathbf{\Pi} Y_N(t),$$

and

$$\|Y_N(t)\|_{\mathcal{X} \times \mathbb{R}^K} + |\alpha(t)| \leq C e^{-(\sigma+a)t} \|H\|_{\mathcal{X} \times \mathbb{R}^K}, \quad a > 0.$$

Thus, by recalling the expression of \mathbb{A}_N and \mathbb{B}_N given in (55), and by writing

$$\mathbf{\Pi} = \begin{pmatrix} \Pi_1 & \Pi_2^* \\ \Pi_2 & \Pi_3 \end{pmatrix} \quad \text{with} \quad \Pi_1 \in \mathcal{L}(\mathcal{X}_N), \quad \Pi_2 \in \mathcal{L}(\mathcal{X}_N, \mathbb{R}^K) \quad \text{and} \quad \Pi_3 \in \mathcal{L}(\mathbb{R}^K), \quad (61)$$

we deduce that the closed-loop system satisfied by $Y_N = {}^t(y_N, \bar{u})$ (with $H = {}^t(h, \bar{v})$) can be equivalently written as follows:

$$\begin{cases} y'_N &= A_N y_N + V_N \bar{u}, & y_N(0) &= h \\ \bar{u}' &= D_K \bar{u} - (\bar{g} | \Pi_2 y_N)_{\mathbb{R}^K} \bar{g} - (\bar{g} | \Pi_3 \bar{u})_{\mathbb{R}^K} \bar{g}, & \bar{u}(0) &= \bar{c} \end{cases}$$

and that its solution (y_N, \bar{u}) obeys:

$$\|y_N(t)\|_{\mathcal{X}} + \|\bar{u}(t)\|_{\mathbb{R}^K} \leq C e^{-(\sigma+a)t} (\|h\|_{\mathcal{X}} + \|\bar{c}\|_{\mathbb{R}^K}). \quad (62)$$

4 Stabilization of the infinite-dimensional system

In this section we use the framework developed in Section 3 and we make some additional assumptions. We recall that $A : \mathcal{D}(A) \subset \mathcal{X} \rightarrow \mathcal{X}$ is a closed linear operator with compact resolvent in a real Hilbert space \mathcal{X} , and we now assume that A is the infinitesimal generator of an analytic semigroup on \mathcal{X} . Then the spectrum $\Sigma = \{\lambda_k \mid k \in \mathbb{N}^*\}$ of A is located in a sector of the complex left half-plane [8, Part. II, Chap. 1, Thm. 2.11 p. 112] and for a given $\sigma > 0$ there is only a finite number N of eigenvalues with real part strictly greater than $-\sigma$. More precisely we can reorder $(\lambda_k)_{k \in \mathbb{N}^*}$ so that the sequence $(\Re \lambda_k)_{k \in \mathbb{N}^*}$ is non increasing and we suppose that $\sigma > 0$ and $N \in \mathbb{N}^*$ are such that:

$$\cdots \leq \Re \lambda_{N+1} < -\sigma < \Re \lambda_N \leq \cdots \leq \Re \lambda_2 \leq \Re \lambda_1.$$

We also set $\widehat{A} \stackrel{\text{def}}{=} \lambda_0 - A$ for $\lambda_0 > 0$ large enough so that the fractional powers of \widehat{A} are well-defined and we suppose that B is quasi-bounded:

$$\widehat{A}^{-\gamma} B : \mathcal{V} \rightarrow \mathcal{X} \quad \text{is bounded for some } 0 \leq \gamma < 1. \quad (63)$$

Notice that the above assumption combined with the analyticity of $(e^{At})_{t \geq 0}$ implies $\|e^{-\widehat{A}t} B\|_{\mathcal{L}(\mathcal{X})} \leq Ct^{\gamma-1}$ [24, Chap. 2, Thm. 6.13, p.74] and then guarantees the admissibility of B (see (26)). We also assume that the following complex interpolation equalities hold:

$$\mathcal{D}(\widehat{A}^\theta) = [\mathcal{D}(A), \mathcal{X}]_{1-\theta} \quad \text{and} \quad \mathcal{D}(\widehat{A}^{*\theta}) = [\mathcal{D}(A^*), \mathcal{X}]_{1-\theta} \quad \forall \theta \in [0, 1], \quad (64)$$

and that there is a Hilbert space \mathcal{O} and a bounded operator $G \in \mathcal{L}(\mathcal{X}, \mathcal{O})$ such that:

$$G \text{ admits a bounded inverse and } G^* G \in \mathcal{L}(\mathcal{D}(\widehat{A}^{\frac{1}{2}}), \mathcal{D}(\widehat{A}^{*\frac{1}{2}})). \quad (65)$$

Finally, $(v_j)_{j=1, \dots, K}$ is a family given by Theorem 9 which is such that (41) is exactly controllable, and we consider system (27) with a finite dimensional control $u = \sum_{j=1}^K u_j v_j$ for $\bar{u} = (u_1, \dots, u_K) \in L^2(\mathbb{R}^K)$ and for an initial datum $y_0 \in \mathcal{X}$. With the notations of Subsection 3.4 it can be rewritten as follows:

$$y' = Ay + V \bar{u} \in [\mathcal{D}(A^*)]', \quad y(0) = y_0. \quad (66)$$

4.1 Feedback stabilization with finite-dimensional controllers

Let us consider system (66) with the feedback control $\bar{u}(t) = -V^* \Pi P_N y(t)$ obtained in Subsection 3.4:

$$y' = Ay - V V^* \Pi P_N y \in [\mathcal{D}(A^*)]', \quad y(0) = y_0. \quad (67)$$

Notice that the above system corresponds to system (27) with the finite dimensional feedback control given by

$$u(t) = - \sum_{j=1}^K (v_j | B^* \Pi P_N y(t))_{\mathcal{V}} v_j. \quad (68)$$

Proposition 11. For $y_0 \in \mathcal{X}$, (67) admits a unique solution given by $y(t) = e^{-A_\Pi t} y_0$, where $(e^{-A_\Pi t})_{t \geq 0}$ is an analytic semigroup on \mathcal{X} with infinitesimal generator $-A_\Pi$ defined by

$$\mathcal{D}(A_\Pi) \stackrel{\text{def}}{=} \{\xi \in \mathcal{X} \mid A\xi - VV^*\Pi P_N \xi \in \mathcal{X}\} \quad \text{and} \quad A_\Pi \xi \stackrel{\text{def}}{=} VV^*\Pi P_N \xi - A\xi. \quad (69)$$

The semigroup $(e^{-A_\Pi t})_{t \geq 0}$ is exponentially stable with an exponential rate strictly lower than $-\sigma$:

$$\inf\{\Re \lambda \mid \lambda \in \sigma(A_\Pi)\} > \sigma. \quad (70)$$

The adjoint of A_Π is given by:

$$\mathcal{D}(A_\Pi^*) = \mathcal{D}(A^*) \quad \text{and} \quad A_\Pi^* \xi = P_N^*(V^*\Pi)^*V^*\xi - A^*\xi. \quad (71)$$

Proof. Definition (69) means that $\xi \in \mathcal{D}(A_\Pi)$, if and only if, $\zeta \in \mathcal{D}(A^*) \mapsto \langle A\xi - VV^*\Pi P_N \xi \mid \zeta \rangle_{[\mathcal{D}(A^*)]', \mathcal{D}(A^*)}$ is bounded for the \mathcal{X} -topology, and since $\langle A\xi - VV^*\Pi P_N \xi \mid \zeta \rangle_{[\mathcal{D}(A^*)]', \mathcal{D}(A^*)} = (\xi \mid P_N^*(V^*\Pi)^*V^*\zeta - A^*\zeta)_{\mathcal{X}}$ it exactly means that $(\mathcal{D}(A_\Pi), A_\Pi)$ is the adjoint of $(\mathcal{D}(A^*), P_N^*(V^*\Pi)^*V^* - A^*)$. Then (71) is proved. Thus, since $P_N^*(V^*\Pi)^*V^*$ belongs to $\mathcal{L}(\widehat{A}^{*\gamma}, \mathcal{X})$, a perturbation argument ensures that $(\mathcal{D}(A_\Pi^*), A_\Pi^*)$ is the infinitesimal generator of an analytic semigroup on \mathcal{X} [24, Chap. 3, Cor. 2.4], and the analyticity of A_Π follows from a duality argument. As a consequence, if $y_0 \in \mathcal{X}$ then $y(t) = e^{-A_\Pi t} y_0$ obeys $y' = -A_\Pi y \in [\mathcal{D}(A_\Pi^*)]'$ and $y(0) = y_0$ i.e. y is the solution of (67). Let us now verify that $(e^{-A_\Pi t})_{t \geq 0}$ is also exponentially stable. Since system (67) corresponds to (27) with (68), by following the steps of Subsection 3.3 (see (40) and (41)) it can be rewritten as:

$$\begin{aligned} y_N' &= A_N y_N + B_{N,K} u \in \mathcal{X}_N, & y_N(0) &= P_N y_0 \\ y_N^- &= A_N^- y_N^- + B_N^- p_N^- \pi_K^* u \in [\mathcal{D}(A^*) \cap \mathcal{X}_N^{-*}]', & y_N(0) &= (I - P_N) y_0, \end{aligned}$$

where $(y_N, y_N^-) = (P_N y, (I - P_N) y)$ and u is given by (68). Moreover, because the first above equality can be equivalently rewritten as (50) with $h = P_N y_0$, then from (51) we already know that y_N is exponentially stable with a rate of decrease $-(\sigma + a)$, and to obtain (70) it remains to prove the exponential decay of $y_N^-(t)$ with a rate $-(\sigma + a)$. We start by using the Duhamel representation:

$$y_N^-(t) = e^{A_N^- t} (I - P_N) y_0 - \int_0^t e^{A_N^- (t-\tau)} B_N^- p_N^- \pi_K^* u(\tau) d\tau. \quad (72)$$

According to [8, Part. II, Chap. 1, Cor. 2.1 p. 92 and Prop. 2.9, p.120], the fact that $\Re \lambda_{N+1} = \sup\{\Re \lambda \mid \lambda \in \sigma(A_N^-)\} < -\sigma - \epsilon < 0$ for some $\epsilon > 0$ combined with the analyticity of $(e^{A_N^- t})_{t \geq 0}$ on \mathcal{X}_N^- implies that the exponential decrease of $(e^{A_N^- t})_{t \geq 0}$ has a rate $-\sigma - \epsilon$. Moreover, we set $\widehat{A}_N^- = \lambda_0 - A_N^-$, we remark that we have $(\widehat{A}_N^-)^{-\gamma} = (I - P_N) \widehat{A}^{-\gamma}$ and $B_N^- = (I - P_N) B = (\widehat{A}_N^-)^{\gamma} (I - P_N) \widehat{A}^{-\gamma} B$, and the analyticity of $(e^{A_N^- t})_{t \geq 0}$ with (63) gives:

$$\|e^{A_N^- t} B_N^- u\|_{\mathcal{X}} = \|(\widehat{A}_N^-)^{\gamma} e^{A_N^- t} (I - P_N) \widehat{A}^{-\gamma} B u\|_{\mathcal{X}} \leq C \frac{e^{-(\sigma+\epsilon)t}}{t^\gamma} \|u\|_{\mathcal{V}}, \quad \forall t > 0.$$

The above estimate combined with (72) and (52) yields:

$$\begin{aligned} \|y_N^-(t)\|_{\mathcal{X}} &\leq C e^{-(\sigma+\epsilon)t} \|(I - P_N) y_0\|_{\mathcal{X}} + C \int_0^t \frac{e^{-(\sigma+\epsilon)(t-\tau)}}{(t-\tau)^\gamma} e^{-(\sigma+a)\tau} \|P_N y_0\|_{\mathcal{X}} d\tau, \\ &\leq C \left(e^{-(\sigma+\epsilon)t} + e^{-(\sigma+a)t} \int_0^t \frac{e^{(a-\epsilon)(t-\tau)}}{(t-\tau)^\gamma} d\tau \right) \|y_0\|_{\mathcal{X}}, \end{aligned}$$

and thus up to decreasing a so that $a < \epsilon$ we obtain that $\|y_N^-(t)\|_{\mathcal{X}} \leq C \|y_0\|_{\mathcal{X}} e^{-(\sigma+a)t}$. \square

Remark 12. Notice that assumption (64) is not used in the proof of Proposition 11.

Let us now explain how to construct a Lyapunov function for the closed-loop system (67). First, since $-A_\Pi$ generates an analytic and exponentially stable semigroup on \mathcal{X} , it is boundedly invertible [8, Part. II, Chap. 1, Prop. 2.9, p. 120], the fractional powers of A_Π are well defined [8, Part II, Chap. 1, Par. 5 p. 167] and (64) combined with a perturbation argument [11, Prop. 2.7] ensures that the complex interpolation equalities

$\mathcal{D}(A_{\Pi}^{\theta}) = [\mathcal{D}(A_{\Pi}), \mathcal{X}]_{1-\theta}$ and $\mathcal{D}(A_{\Pi}^{*\theta}) = [\mathcal{D}(A_{\Pi}^*), \mathcal{X}]_{1-\theta}$ hold for all $\theta \in [0, 1]$. Then for $\theta \in [0, 1]$ we introduce the spaces:

$$\mathcal{X}^{2\theta} \stackrel{\text{def}}{=} \mathcal{D}(A_{\Pi}^{\theta}), \quad \mathcal{X}_*^{2\theta} \stackrel{\text{def}}{=} \mathcal{D}(A_{\Pi}^{*\theta}), \quad \mathcal{X}^{-2\theta} \stackrel{\text{def}}{=} [\mathcal{X}_*^{2\theta}]' \quad \text{and} \quad \mathcal{X}_*^{-2\theta} \stackrel{\text{def}}{=} [\mathcal{X}^{2\theta}]',$$

respectively normed with:

$$\|\cdot\|_{\mathcal{X}^{2\theta}} \stackrel{\text{def}}{=} \|A_{\Pi}^{\theta} \cdot\|_{\mathcal{X}}, \quad \|\cdot\|_{\mathcal{X}_*^{2\theta}} \stackrel{\text{def}}{=} \|A_{\Pi}^{*\theta} \cdot\|_{\mathcal{X}}, \quad \|\cdot\|_{\mathcal{X}^{-2\theta}} \stackrel{\text{def}}{=} \|A_{\Pi}^{-\theta} \cdot\|_{\mathcal{X}} \quad \text{and} \quad \|\cdot\|_{\mathcal{X}_*^{-2\theta}} \stackrel{\text{def}}{=} \|A_{\Pi}^{*- \theta} \cdot\|_{\mathcal{X}}. \quad (73)$$

Notice that because $A_{\Pi}^{-\theta} = (A_{\Pi}^{\theta})^{-1}$ and $A_{\Pi}^{*- \theta} = (A_{\Pi}^{*\theta})^{-1}$, the above dual norm's definition can be obtained from [28, Chap. 2, Prop. 2.10.2]. We introduce the following linear and bounded operator:

$$T \stackrel{\text{def}}{=} I + \widehat{A}^{-1}V(V^*\Pi)P_N : \mathcal{X} \rightarrow \mathcal{X}, \quad (74)$$

and the next proposition gives a characterization of the spaces $\mathcal{X}^{2\theta}$ and $\mathcal{X}_*^{2\theta}$ by using T .

Proposition 13. *Let $\theta \in [0, 1]$. Then the following results hold:*

1. T is an isomorphism from $\mathcal{D}(A_{\Pi}^{\theta})$ onto $\mathcal{D}(\widehat{A}^{\theta})$.
2. T^* is an isomorphism from $\mathcal{D}(\widehat{A}^{*\theta})$ onto itself.
3. The following characterizations hold:

$$\mathcal{D}(A_{\Pi}^{\theta}) = \{\xi \in \mathcal{X} \mid T\xi \in \mathcal{D}(\widehat{A}^{\theta})\} \quad \text{and} \quad \mathcal{D}(A_{\Pi}^{*\theta}) = \mathcal{D}(\widehat{A}^{*\theta}). \quad (75)$$

Proof. First, since we obviously have $T \in \mathcal{L}(\mathcal{X})$, and since from (69) we deduce that:

$$\mathcal{D}(A_{\Pi}) = \{\xi \in \mathcal{X} \mid \xi + \widehat{A}^{-1}V(V^*\Pi)P_N\xi \in \mathcal{D}(A)\} = \{\xi \in \mathcal{X} \mid T\xi \in \mathcal{D}(A)\}, \quad (76)$$

which also means that $T \in \mathcal{L}(\mathcal{D}(A_{\Pi}), \mathcal{D}(A))$, then $T \in \mathcal{L}([\mathcal{D}(A_{\Pi}), \mathcal{X}]_{1-\theta}, [\mathcal{D}(A), \mathcal{X}]_{1-\theta}) = \mathcal{L}(\mathcal{D}(A_{\Pi}^{\theta}), \mathcal{D}(\widehat{A}^{\theta}))$ follows by interpolation. Next, to prove that $T : \mathcal{X} \rightarrow \mathcal{X}$ is boundedly invertible we first remark that it is a compact perturbation of the identity, and then that it suffices to prove its injectivity. Let us suppose that $\xi \in \mathcal{X}$ obeys the equality $T\xi = \xi + \widehat{A}^{-1}V(V^*\Pi)P_N\xi = 0$. By multiplying by $\widehat{A}^*\Pi P_N\xi$ we obtain $\lambda_0(P_N\xi \mid \Pi P_N\xi) - (A_N P_N\xi \mid \Pi P_N\xi)_{\mathcal{X}} + \|(V^*\Pi)P_N\xi\|_{\mathbb{R}^K}^2 = 0$ and, with (46) applied to $(P_N\xi, P_N\xi)$, we obtain:

$$(\lambda_0 + \sigma)(P_N\xi \mid \Pi P_N\xi) + \frac{1}{2}\|P_N\xi\|_{\mathcal{X}}^2 + \frac{1}{2}\sum_{j=1}^K (B^*\Pi P_N\xi \mid v_j)_{\mathbb{V}}^2 = 0,$$

which guarantees that $P_N\xi = 0$. Thus, since $\xi + \widehat{A}^{-1}V(V^*\Pi)P_N\xi = 0$ we finally deduce that $\xi = 0$. Then we have proved that T is an isomorphism from \mathcal{X} onto \mathcal{X} . Moreover, since (76) exactly means that T^{-1} maps $\mathcal{D}(A)$ to $\mathcal{D}(A_{\Pi})$, we obtain $T^{-1} \in \mathcal{L}([\mathcal{D}(A), \mathcal{X}]_{1-\theta}, [\mathcal{D}(A_{\Pi}), \mathcal{X}]_{1-\theta}) = \mathcal{L}(\mathcal{D}(\widehat{A}^{\theta}), \mathcal{D}(A_{\Pi}^{\theta}))$ by interpolation.

Next, we have $T^* = I + P_N^*(V^*\Pi)^*V^*\widehat{A}^{*-1}$ and since $P_N^* \in \mathcal{L}(\mathcal{X}, \mathcal{D}(A^*))$ we deduce that $T^* \in \mathcal{L}(\mathcal{D}(\widehat{A}^{*\theta}))$. Moreover, since $T : \mathcal{X} \rightarrow \mathcal{X}$ is boundedly invertible then so does T^* , and we obtain that $T^{*-1} \in \mathcal{L}(\mathcal{D}(\widehat{A}^{*\theta}))$ from $P_N^* \in \mathcal{L}(\mathcal{X}, \mathcal{D}(A^*))$ by remarking that $T^{*-1} = I - P_N^*(V^*\Pi)^*V^*\widehat{A}^{*-1}T^{*-1}$. Finally, the second equality in (75) is a direct consequence of (71) \square

An obvious consequence of the above proposition and of (65) is the following corollary:

Corollary 14. *The operator T^*G^*GT is an isomorphism from \mathcal{X} onto itself as well as from \mathcal{X}^1 onto \mathcal{X}_*^1 .*

By following an approach which is inspired from [23], we introduce a linear operator L_s depending on A_{Π} and on T^*G^*GT which will define a Lyapunov function for system (67):

Proposition 15. *Let T be given by (74) and $G \in \mathcal{L}(\mathcal{X}, \mathcal{O})$ be an operator obeying (65). Then the operator*

$$L_s \stackrel{\text{def}}{=} \int_0^{\infty} e^{2\sigma t} e^{-A_{\Pi}^* t} A_{\Pi}^{*\frac{1}{2} + \frac{s}{2}} T^* G^* G T A_{\Pi}^{\frac{1}{2} + \frac{s}{2}} e^{-A_{\Pi} t} dt$$

is bounded from \mathcal{X}^s to \mathcal{X}_^{-s} as well as from \mathcal{X}^{1+s} to \mathcal{X}_*^{1-s} . Moreover, the following equalities hold*

$$\langle L_s \xi \mid \xi \rangle_{\mathcal{X}_*^{-s}, \mathcal{X}^s} = \int_0^{\infty} e^{2\sigma t} \|G T A_{\Pi}^{\frac{1}{2} + \frac{s}{2}} e^{-A_{\Pi} t} \xi\|_{\mathcal{O}}^2 dt \quad \forall \xi \in \mathcal{X}^s, \quad (77)$$

$$\langle L_s \xi \mid A_{\Pi} \xi \rangle_{\mathcal{X}_*^{1-s}, \mathcal{X}^{s+1}} = \frac{1}{2} \|G T A_{\Pi}^{\frac{1}{2} + \frac{s}{2}} \xi\|_{\mathcal{O}}^2 + \sigma \langle L_s \xi \mid \xi \rangle_{\mathcal{X}_*^{-s}, \mathcal{X}^s} \quad \forall \xi \in \mathcal{X}^{s+1}, \quad (78)$$

and define norms on \mathcal{X}^s and \mathcal{X}_^{s+1} respectively. These norms are equivalent to the norms defined by (73).*

In the following, we will use the notations:

$$\|\cdot\|_s \stackrel{\text{def}}{=} (\langle L_s \cdot | \cdot \rangle_{\mathcal{X}_*^{-s}, \mathcal{X}^s})^{1/2} \quad \text{and} \quad \mathcal{N}_{s+1}(\cdot) \stackrel{\text{def}}{=} \|GT A_{\Pi}^{\frac{1}{2} + \frac{s}{2}} \cdot\|_{\mathcal{O}}.$$

Proof. First, since $y = A_{\Pi}^{\frac{1}{2} + \frac{s}{2}} e^{-(A_{\Pi} - \sigma)(\cdot)} \xi$ is the solution of $y' + (A_{\Pi} - \sigma)y = 0$ and $y(0) = A_{\Pi}^{\frac{1}{2} + \frac{s}{2}} \xi$, from maximal regularity results for analytic semigroup (8) we deduce that

$$A_{\Pi}^{\frac{1}{2} + \frac{s}{2}} e^{-(A_{\Pi} - \sigma)(\cdot)} \in \mathcal{L}(\mathcal{X}^s, L^2(\mathcal{X})) \cap \mathcal{L}(\mathcal{X}^{s+1}, L^2(\mathcal{X}^1)),$$

and by using Corollary 14 we obtain:

$$T^* G^* GT A_{\Pi}^{\frac{1}{2} + \frac{s}{2}} e^{-(A_{\Pi} - \sigma)(\cdot)} \in \mathcal{L}(\mathcal{X}^s, L^2(\mathcal{X})) \cap \mathcal{L}(\mathcal{X}^{s+1}, L^2(\mathcal{X}_*^1)). \quad (79)$$

Next, let us introduce $\mathcal{K} \in \mathcal{L}(L^2(\mathcal{X}))$ as follows:

$$\mathcal{K}(f)(t) = \int_0^t e^{-(A_{\Pi} - \sigma)(t-\tau)} f(\tau) d\tau.$$

An easy calculation shows that the adjoint of \mathcal{K} with respect to $L^2(\mathcal{X})$ is given by

$$\mathcal{K}^*(f)(t) = \int_t^{\infty} e^{-(A_{\Pi}^* - \sigma)(\tau-t)} f(\tau) d\tau.$$

Moreover, since $y = \mathcal{K}(f)$ is the solution of $y' + (A_{\Pi} - \sigma)y = f$ and $y(0) = 0$, maximal regularity results for analytic semigroup (8) apply and we have:

$$\mathcal{K} \in \mathcal{L}(L^2(\mathcal{X}^{-2}), L^2(\mathcal{X})) \cap \mathcal{L}(L^2(\mathcal{X}^{-3}), L^2(\mathcal{X}^{-1})) \quad \text{and} \quad \mathcal{K} A_{\Pi} \in \mathcal{L}(L^2(\mathcal{X})) \cap \mathcal{L}(L^2(\mathcal{X}^{-1})).$$

Thus, a duality argument gives:

$$\mathcal{K}^* \in \mathcal{L}(L^2(\mathcal{X}), L^2(\mathcal{X}_*^2)) \cap \mathcal{L}(L^2(\mathcal{X}_*^1), L^2(\mathcal{X}_*^3)) \quad \text{and} \quad A_{\Pi}^* \mathcal{K}^* \in \mathcal{L}(L^2(\mathcal{X})) \cap \mathcal{L}(L^2(\mathcal{X}_*^1)),$$

and since $\frac{d}{dt} \mathcal{K}^* = -(A_{\Pi}^* - \sigma) \mathcal{K}^* + I$, we obtain:

$$\mathcal{K}^* \in \mathcal{L}(L^2(\mathcal{X}), W(\mathcal{X}_*^2, \mathcal{X})) \cap \mathcal{L}(L^2(\mathcal{X}_*^1), W(\mathcal{X}_*^3, \mathcal{X}_*^1)). \quad (80)$$

Finally, with the continuous embeddings $W(\mathcal{X}_*^2, \mathcal{X}) \hookrightarrow C_b(\mathcal{X}_*^1)$ and $W(\mathcal{X}_*^3, \mathcal{X}_*^1) \hookrightarrow C_b(\mathcal{X}_*^2)$, and by remarking that $L_s \xi = A_{\Pi}^{*\frac{1}{2} + \frac{s}{2}} \mathcal{K}^*(T^* G^* GT A_{\Pi}^{\frac{1}{2} + \frac{s}{2}} e^{-(A_{\Pi} - \sigma)(\cdot)} \xi)(0)$, the first part of the proposition follows from (79) and (80).

Next, from (8) we deduce that square root of $\int_0^{\infty} \|A_{\Pi} e^{-(A_{\Pi} - \sigma)t}\|_{\mathcal{O}}^2 dt$ defines a norm equivalent to $\|\cdot\|_{\mathcal{X}^1}$, and then so does the square root of $\int_0^{\infty} \|GT A_{\Pi} e^{-(A_{\Pi} - \sigma)t}\|_{\mathcal{O}}^2 dt$ since $GT \in \mathcal{L}(\mathcal{X}, \mathcal{O})$ is boundedly invertible. Thus, by using the change of variable $\xi \mapsto A_{\Pi}^{\frac{s}{2} - \frac{1}{2}} \xi$ one obtains that (77) defines a norm equivalent to $\|\cdot\|_{\mathcal{X}^s}$. Finally, (78) follows from an integration by parts and the fact that it defines a norm on \mathcal{X}^{s+1} follows from the bounded invertibility of $GT \in \mathcal{L}(\mathcal{X}, \mathcal{O})$. \square

If we now multiply the first equality in (67) by $L_s y$ we obtain:

$$\frac{d}{dt} \|y(t)\|_s^2 + 2\sigma \|y(t)\|_s^2 + \mathcal{N}_{s+1}(y(t))^2 = 0,$$

which means that $\|\cdot\|_s^2$ is a Lyapunov function associated to (67): $t \mapsto \|y(t)\|_s^2$ decreases to zero and

$$\|y(t)\|_s \leq e^{-(\sigma + \beta_s)t} \|y(0)\|_s \quad \text{where} \quad \beta_s \stackrel{\text{def}}{=} \inf_{0 \neq \xi \in \mathcal{X}^{s+1}} \frac{\mathcal{N}_{s+1}(\xi)^2}{2\|\xi\|_s^2}.$$

We use now $\|\cdot\|_s^2$ as a Lyapunov function for a nonlinear system of type:

$$y' + A_{\Pi} y = N(y), \quad y(0) = y_0, \quad (81)$$

provided that the nonlinear mapping $N(\cdot) : \mathcal{X}^{1+s} \rightarrow \mathcal{X}^{s-1}$ satisfies:

$$\|N(\xi)\|_{\mathcal{X}^{s-1}} \leq C \|\xi\|_{\mathcal{X}^s} \|\xi\|_{\mathcal{X}^{s+1}}, \quad (82)$$

$$\|N(\xi) - N(\zeta)\|_{\mathcal{X}^{s-1}} \leq C(\|\xi - \zeta\|_{\mathcal{X}^s} (\|\xi\|_{\mathcal{X}^{s+1}} + \|\zeta\|_{\mathcal{X}^{s+1}}) + \|\xi - \zeta\|_{\mathcal{X}^{s+1}} (\|\xi\|_{\mathcal{X}^s} + \|\zeta\|_{\mathcal{X}^s})). \quad (83)$$

It is the main subject of the following theorem:

Theorem 16. Assume (82),(83) for some $s \in [0, 1]$ and $y_0 \in \mathcal{X}^s$. There exist $\rho > 0$ and $\mu > 0$ such that, if $\|y_0\|_s < \mu$, then system (81) admits a solution $y_{y_0} \in W(\mathcal{X}^{s+1}, \mathcal{X}^{s-1})$ such that $\|y_{y_0}\|_{W(\mathcal{X}^{s+1}, \mathcal{X}^{s-1})} \leq \rho \|y_0\|_s$, which is unique within the class of functions in $L_{loc}^\infty(\mathcal{X}^s) \cap L_{loc}^2(\mathcal{X}^{s+1})$. Moreover, every solution with an initial datum obeying:

$$\|y_0\|_s < K_s \quad \text{where} \quad \frac{1}{2K_s} \stackrel{\text{def}}{=} \sup_{0 \neq \xi \in \mathcal{X}^{1+s}} \frac{\langle L_s \xi | N(\xi) \rangle_{\mathcal{X}^{1-s}, \mathcal{X}^{s-1}}}{\|\xi\|_s \mathcal{N}_{s+1}(\xi)^2},$$

is such that $t \mapsto \|y_{y_0}(t)\|_s$ is decreasing and we have:

$$\|y_{y_0}(t)\|_s \leq \|y_0\|_s e^{-(\sigma + \beta_s(1 - \|y_0\|_s/K_s))t} \quad \text{where} \quad \beta_s \stackrel{\text{def}}{=} \inf_{0 \neq \xi \in \mathcal{X}^{s+1}} \frac{\mathcal{N}_{s+1}(\xi)^2}{2\|\xi\|_s^2}, \quad (84)$$

$$\int_0^\infty e^{2\sigma t} \mathcal{N}_{s+1}(y_{y_0}(t))^2 dt \leq \frac{\|y_0\|_s^2}{(1 - \|y_0\|_s/K_s)}. \quad (85)$$

Proof. First, suppose that $\|y_0\|_s < K_s$ and that $y \in L_{loc}^\infty(\mathcal{X}^s) \cap L_{loc}^2(\mathcal{X}^{s+1})$ is a solution of (81) and let us prove that $y \in W(\mathcal{X}^{s+1}, \mathcal{X}^{s-1})$ as well as estimates (84) and (85). First, since (82) ensures that $N(y) \in L_{loc}^2(\mathcal{X}^{s-1})$, from (81) we obtain $y \in W_{loc}(\mathcal{X}^{s+1}, \mathcal{X}^{s-1})$, and by multiplying the first equality in (81) by $L_s y(t)$ and using (78) and (82), we obtain:

$$\frac{d}{dt} \|y(t)\|_s^2 + 2\sigma \|y(t)\|_s^2 + (1 - \|y(t)\|_s/K_s) \mathcal{N}_{s+1}(y(t))^2 \leq 0.$$

Thus, because $\|y_0\|_s < K_s$, the mapping $t \mapsto \|y(t)\|_s$ is a nonincreasing function lower than K_s and:

$$\frac{d}{dt} \|y(t)\|_s^2 + 2\sigma \|y(t)\|_s^2 + (1 - \|y_0\|_s/K_s) \mathcal{N}_{s+1}(y(t))^2 \leq 0.$$

Then (84) follows from $2\beta_s \|y(t)\|_s^2 \leq \mathcal{N}_{s+1}(y(t))^2$ and by multiplying the inequality by $e^{\sigma t}$ and integrating over $(0, \infty)$ gives (85). Moreover, let us also notice that combining the first equation in (81) with (82) yields $\|y'(t)\|_{\mathcal{X}^{s-1}}^2 \leq C \|y(t)\|_{\mathcal{X}^{s+1}}^2$, and with (85) it provides $C_s > 0$ such that:

$$\|y\|_{W(\mathcal{X}^{s+1}, \mathcal{X}^{s-1})}^2 \leq \frac{C_s}{1 - \|y_0\|_s/K_s} \|y_0\|_s^2. \quad (86)$$

Next, let us set $W_s \stackrel{\text{def}}{=} W(\mathcal{X}^{s+1}, \mathcal{X}^{s-1})$ for readability convenience. To prove existence and uniqueness of a solution to (81), we consider the mapping

$$\Psi : z \in W_s \mapsto y_z \in W_s \quad \text{where} \quad y'_z + A_\Pi y_z + N(z) = 0, \quad y_z(0) = y_0.$$

First, by combining the continuity of the mapping $y \in W_s \mapsto (y' - A_\Pi y, y(0)) \in L^2(\mathcal{X}^{s-1}) \times \mathcal{X}^s$, which follows from (8), and the fact that $-A_\Pi$ generates an analytic semigroup on \mathcal{X} , with (82),(83) we obtain

$$\|\Psi(z)\|_{W_s} \leq C_0(\|z\|_{W_s}^2 + \|y_0\|_s) \quad \text{and} \quad \|\Psi(z_1) - \Psi(z_2)\|_{W_s} \leq C_1(\|z_1\|_{W_s} + \|z_2\|_{W_s})\|z_1 - z_2\|_{W_s}. \quad (87)$$

Then if we assume that

$$\rho \geq 2C_0, \quad \text{and} \quad \|y_0\|_s < \min\left(\frac{1}{2C_0\rho}, \frac{1}{2C_1\rho}\right),$$

then we deduce from (87) that the ball $B_\rho \stackrel{\text{def}}{=} \{z \in W_s \mid \|z\|_{W_s} \leq \rho \|y_0\|_s\}$ is invariant by Ψ and that the restriction of Ψ to B_ρ is a contraction. As a consequence, (81) admits a unique solution in B_ρ . If we impose moreover that

$$\rho \geq \sqrt{2C_s} \quad \text{and} \quad \|y_0\|_s \leq \frac{K_s}{2}, \quad (88)$$

we notice that (86) ensures that every solution in $L_{loc}^\infty(\mathcal{X}^s) \cap L_{loc}^2(\mathcal{X}^{s+1})$ belongs to B_ρ , and then must coincide with the fixed-point solution. It means that for $\rho = \max(\sqrt{2C_s}, 2C_0)$ and $\|y_0\|_s < \mu = \min(\frac{1}{2C_0\rho}, \frac{1}{2C_1\rho})$ there exists a fixed point solution of (81) which is unique within the class of functions in $L_{loc}^\infty(\mathcal{X}^s) \cap L_{loc}^2(\mathcal{X}^{s+1})$. \square

4.2 Feedback stabilization with finite-dimensional dynamical controllers

We are now interested in stabilizing the solution to (66) by means of a control $\bar{u} = (u_1, \dots, u_K) \in L^2(\mathbb{R}^K)$ which is solution to the dynamical system:

$$\bar{u}' = D_K \bar{u} + \bar{g}\alpha, \quad \bar{u}(0) = \bar{u}_0, \quad (89)$$

where $D_K \in \mathcal{L}(\mathbb{R}^K)$ and $\bar{g} \in \mathbb{R}^K$ satisfy the hypothesis of Theorem 10:

$$D_K \in \mathcal{L}(\mathbb{R}^K) \text{ has a spectrum composed of } K \text{ distinct eigenvalues } \mu_i \notin \{\lambda_1, \dots, \lambda_N\}, \quad (90)$$

and

$$\bar{g} \in \mathbb{R}^K \text{ is such that } (\bar{g}|\bar{d})_{\mathbb{R}^K} \neq 0 \text{ for all eigenvectors } \bar{d} \text{ of } {}^t D_K. \quad (91)$$

We introduce the new state $Y = {}^t(y, \bar{u})$, the new initial datum $Y_0 = {}^t(y_0, \bar{u}_0) \in \mathcal{X} \times \mathbb{R}^K$ and the following extended linear operator on $\mathcal{X} \times \mathbb{R}^K$:

$$\mathbb{A} \stackrel{\text{def}}{=} \begin{pmatrix} A & V \\ 0 & D_K \end{pmatrix} \quad \text{and} \quad \mathcal{D}(\mathbb{A}) \stackrel{\text{def}}{=} \{ Y \in \mathcal{X} \times \mathbb{R}^K \mid \mathbb{A}Y \in \mathcal{X} \times \mathbb{R}^K \}, \quad (92)$$

so that system (66), (89) can be rewritten as follows:

$$Y' = \mathbb{A}Y + \mathbb{B}\alpha, \quad Y(0) = Y_0 \quad \text{where} \quad \mathbb{B} \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ \bar{g} \end{pmatrix}. \quad (93)$$

Proposition 17. *The unbounded operator $(\mathcal{D}(\mathbb{A}), \mathbb{A})$ is a closed linear operator with dense domain in $\mathcal{X} \times \mathbb{R}^K$ and it is the infinitesimal generator of an analytic semigroup on $\mathcal{X} \times \mathbb{R}^K$. The adjoint of \mathbb{A} is given by*

$$\mathbb{A}^* = \begin{pmatrix} A^* & 0 \\ V^* & D_K^* \end{pmatrix} \quad \text{and} \quad \mathcal{D}(\mathbb{A}^*) = \mathcal{D}(A^*) \times \mathbb{R}^K,$$

and there exists $\lambda_0 > 0$ such that $\widehat{\mathbb{A}} \stackrel{\text{def}}{=} \lambda_0 - \mathbb{A}$ obeys:

$$\mathcal{D}(\widehat{\mathbb{A}}^\theta) = [\mathcal{D}(\mathbb{A}), \mathcal{X}]_{1-\theta} = \{ Y = {}^t(y, \bar{u}) \in \mathcal{X} \times \mathbb{R}^K \mid y + \widehat{A}^{-1}V\bar{u} \in \mathcal{D}(\widehat{A}^\theta) \}, \quad \forall \theta \in [0, 1], \quad (94)$$

$$\mathcal{D}(\widehat{\mathbb{A}}^{*\theta}) = [\mathcal{D}(\mathbb{A}^*), \mathcal{X}]_{1-\theta} = \mathcal{D}(\widehat{A}^{*\theta}) \times \mathbb{R}^K, \quad \forall \theta \in [0, 1]. \quad (95)$$

Proof. First, the closedness of \mathbb{A} is an easy consequence of the closedness of A and of D_K . Thus, to prove that $\mathcal{D}(\mathbb{A})$ is dense in $\mathcal{X} \times \mathbb{R}^K$, we first remark that $\mathcal{D}(\mathbb{A})$ is composed with ${}^t(y, \bar{u}) \in \mathcal{X} \times \mathbb{R}^K$ such that $Ay + V\bar{u} \in \mathcal{X}$. Then we have

$$\mathcal{D}(\mathbb{A}) = \{ {}^t(y, \bar{u}) \in \mathcal{X} \times \mathbb{R}^K \mid y + A^{-1}V\bar{u} \in \mathcal{D}(A) \}, \quad (96)$$

and since for any ${}^t(y, \bar{u}) \in \mathcal{X} \times \mathbb{R}^K$ the denseness of $\mathcal{D}(A)$ in \mathcal{X} ensures that there is a sequence (z_n) of $\mathcal{D}(A)$ which converges to $y + A^{-1}V\bar{u}$ in \mathcal{X} , it follows that $(z_n - A^{-1}V\bar{u}, \bar{u})$ defines a sequence of $\mathcal{D}(\mathbb{A})$ which converges to (y, \bar{u}) in $\mathcal{X} \times \mathbb{R}^K$. Next, to characterize the adjoint of \mathbb{A} let us show the inclusion $\mathcal{D}(\mathbb{A}^*) \subset \mathcal{D}(A^*) \times \mathbb{R}^K$ which is the only non obvious fact to prove. If $Y = {}^t(y, \bar{u}) \in \mathcal{D}(\mathbb{A}^*)$ then $Z \in \mathcal{D}(\mathbb{A}) \mapsto (\mathbb{A}Y|Z)_{\mathcal{X}}$ is continuous for the topology of $\mathcal{X} \times \mathbb{R}^K$, and by remarking that $\mathcal{D}(A) \times \{0\} \subset \mathcal{D}(\mathbb{A})$ we deduce that $z \in \mathcal{D}(A) \mapsto (Ay|z)_{\mathcal{X}}$ is continuous for the topology of \mathcal{X} . Then it means that $y \in \mathcal{D}(A^*)$ and the desired inclusion is proved. Notice that the second equality in (95) follows from (64) by interpolation. Next, by remarking that for $\lambda \in \mathbb{C}$ the equality $(\lambda - \mathbb{A}){}^t(y, \bar{u}) = {}^t(f, \bar{h})$ is equivalent to

$$(\lambda - A)y = V\bar{u} + f \quad \text{and} \quad (\lambda - D_K)\bar{u} = \bar{h}, \quad (97)$$

we deduce that the resolvent set of \mathbb{A} is exactly the union of the resolvent sets of A and of D_K . Thus, to prove that \mathbb{A} generates an analytic semigroup we have to prove that there exists $M > 0$ such that for all $F = {}^t(f, \bar{h}) \in \mathcal{X} \times \mathbb{R}^K$ and for all λ in an open sector of the complex plane, symmetric with respect to the axis \mathbb{R} and with an opening angle greater than π [8, Chap. II-1, Thm. 2.10], we have:

$$\|(\lambda - \mathbb{A})^{-1}F\|_{\mathcal{X} \times \mathbb{R}^K} \leq \frac{M}{|\lambda|} \|F\|_{\mathcal{X} \times \mathbb{R}^K}.$$

The above estimate can be obtained by remarking that for λ in the resolvent set of \mathbb{A} , (97) is equivalent to

$$y = (\lambda - A)^{-1} \widehat{A}(\widehat{A}^{-1}V)\bar{u} + (\lambda - A)^{-1}f \quad \text{and} \quad \bar{u} = (\lambda - D_K)^{-1}\bar{h},$$

and by using the boundedness of $\widehat{A}^{-1}V$ as well as resolvent estimates related to the analyticity of $(e^{At})_{t \geq 0}$ and of $(e^{D_K t})_{t \geq 0}$. Thus, we can choose $\lambda_0 > 0$ large enough so that the positive halfaxis \mathbb{R}^+ is contained in the resolvent set of $\widehat{A} \stackrel{\text{def}}{=} \lambda_0 - A$ and of $\widehat{D}_K \stackrel{\text{def}}{=} \lambda_0 - D_K$. Let us recall that if L is a closed linear mapping on a Hilbert space \mathcal{H} such that $-L$ is the infinitesimal generator of an analytic semigroup of negative type then the interpolation equalities $[\mathcal{D}(L), \mathcal{X}]_{1-\theta} = \mathcal{D}(L^\theta)$ and $[\mathcal{D}(L^*), \mathcal{X}]_{1-\theta} = \mathcal{D}(L^{*\theta})$ are equivalent to the fact that the operators defined for $z \in \mathbb{C}$ such that $\Re z > 0$ as follows

$$L^{-z} = \frac{\sin \pi z}{\pi} \int_0^{+\infty} t^{-z} (t + L)^{-1} dt,$$

can be extended to strongly continuous functions from $\{z \in \mathbb{C} \mid \Re z \geq 0\}$ to $\mathcal{L}(\mathcal{H})$, see [8, Chap. II-1, Thm. 6.1] or [29]. Then, since we have the complex interpolation equalities $[\mathcal{D}(\widehat{A}), \mathcal{X}]_{1-\theta} = \mathcal{D}(\widehat{A}^\theta)$ and $[\mathcal{D}(\widehat{D}_K), \mathcal{X}]_{1-\theta} = \mathcal{D}(\widehat{D}_K^\theta)$ for $\theta \in (0, 1)$, and since easy calculations give

$$(t + \widehat{\mathbb{A}})^{-1} = \begin{pmatrix} (t + \widehat{A})^{-1} & -(t + \widehat{A})^{-1}V(t + \widehat{D}_K)^{-1} \\ 0 & (t + \widehat{D}_K)^{-1} \end{pmatrix}$$

and

$$\widehat{\mathbb{A}}^{-z} = \begin{pmatrix} \widehat{A}^{-z} & -\beta(z) \\ 0 & \widehat{D}_K^{-z} \end{pmatrix} \quad \text{where} \quad \beta(z) = \frac{\sin \pi z}{\pi} \int_0^{+\infty} t^{-z} (t + \widehat{A})^{-1}V(t + \widehat{D}_K)^{-1} dt,$$

then to prove the first equalities in (94),(95) it remains to prove that we can extend $\beta(z)$ to a strongly continuous function from $\{z \in \mathbb{C} \mid \Re(z) \geq 0\}$ in $\mathcal{L}(\mathbb{R}^K, \mathcal{X})$. From $\widehat{A}(t + \widehat{A})^{-1} = I - t(t + \widehat{A})^{-1}$ and $\|(t + \widehat{A})^{-1}\|_{\mathcal{L}(\mathcal{X})} \leq C_0(1+t)^{-1}$ we deduce that $\|(t + \widehat{A})^{-1}\|_{\mathcal{L}(\mathcal{X}, \mathcal{D}(\widehat{A}))} = \|\widehat{A}(t + \widehat{A})^{-1}\|_{\mathcal{L}(\mathcal{X})} \leq 1 + C_0$ and an interpolation argument with $[\mathcal{D}(\widehat{A}), \mathcal{X}]_{1-\gamma} = \mathcal{D}(\widehat{A}^\gamma)$ gives $\|(t + \widehat{A})^{-1}\|_{\mathcal{L}(\mathcal{X}, \mathcal{D}(\widehat{A}^\gamma))} = \|\widehat{A}^\gamma(t + \widehat{A})^{-1}\|_{\mathcal{L}(\mathcal{X})} \leq (1 + C_0)^\gamma (C_0(1+t)^{-1})^{1-\gamma}$. Then $(1+t)^{1-\gamma} \widehat{A}^\gamma(t + \widehat{A})^{-1}$ is bounded independently of t and with $\widehat{A}^{-\gamma}V \in \mathcal{L}(\mathcal{X})$ we can bound the term under the integral and obtain that $\beta(z)$ is bounded independently on $z \in \{z \in \mathbb{C} \mid \Re(z) > 0\}$ in a neighborhood of 0. Then by [19, Ch. 17, Thm. 17.9.1] one can extend $\beta(z)$ to a strongly continuous function from $\{z \in \mathbb{C} \mid \Re(z) \geq 0\}$ in $\mathcal{L}(\mathbb{R}^K, \mathcal{X})$. Finally, since we have $[\mathcal{D}(\widehat{A}), \mathcal{X}]_{1-\theta} = \mathcal{D}(\widehat{A}^\theta)$ for $\theta \in (0, 1)$, the second equality in (94) follows from (96) with an interpolation argument. \square

We now consider system (66) with a control function solution to the dynamical system:

$$u' = D_K \bar{u} - (\bar{g} | \Pi_2 y_N)_{\mathbb{R}^K} \bar{g} - (\bar{g} | \Pi_3 \bar{u})_{\mathbb{R}^K} \bar{g}, \quad \bar{u}(0) = \bar{u}_0,$$

where Π_2 and Π_3 are components of the Riccati operator $\mathbf{\Pi}$ introduced in Subsection 3.5. Such a system can be rewritten as:

$$Y' = \mathbb{A}Y - \mathbb{B}\mathbb{B}^* \mathbf{\Pi} \mathbb{P}_N Y, \quad Y(0) = Y_0 \quad \text{where} \quad \mathbb{P}_N = \begin{pmatrix} P_N & 0 \\ 0 & \text{Id} \end{pmatrix}. \quad (98)$$

Proposition 18. *Assume that (90), (91) are satisfied. For $Y_0 \in \mathcal{X} \times \mathbb{R}^K$ the solution of (98) is given by $Y(t) = e^{-\mathbb{A}\mathbf{\Pi}t} Y_0$, where $(e^{-\mathbb{A}\mathbf{\Pi}t})_{t \geq 0}$ is an analytic semigroup on $\mathcal{X} \times \mathbb{R}^K$ with infinitesimal generator $-\mathbb{A}\mathbf{\Pi}$ defined by*

$$\mathcal{D}(\mathbb{A}\mathbf{\Pi}) = \mathcal{D}(\mathbb{A}) \quad \text{and} \quad \mathbb{A}\mathbf{\Pi}\xi = \mathbb{B}\mathbb{B}^* \mathbf{\Pi} \mathbb{P}_N \xi - \mathbb{A}\xi. \quad (99)$$

The semigroup $(e^{-\mathbb{A}\mathbf{\Pi}t})_{t \geq 0}$ is exponentially stable with an exponential rate strictly lower than $-\sigma$:

$$\inf\{\Re \lambda \mid \lambda \in \sigma(\mathbb{A}\mathbf{\Pi})\} > \sigma. \quad (100)$$

Moreover, the adjoint of $\mathbb{A}\mathbf{\Pi}$ is given by

$$\mathcal{D}(\mathbb{A}\mathbf{\Pi}^*) = \mathcal{D}(\mathbb{A}^*) \quad \text{and} \quad \mathbb{A}\mathbf{\Pi}^* \xi = \mathbb{P}_N^* (\mathbb{B}^* \mathbf{\Pi})^* \mathbb{B}^* \xi - \mathbb{A}^* \xi, \quad (101)$$

and for all $\theta \in [0, 1]$ the following equalities hold:

$$\mathcal{D}(\mathbb{A}\mathbf{\Pi}^\theta) = \mathcal{D}(\widehat{\mathbb{A}}^\theta) \quad \text{and} \quad \mathcal{D}(\mathbb{A}\mathbf{\Pi}^{*\theta}) = \mathcal{D}(\widehat{\mathbb{A}}^{*\theta}). \quad (102)$$

Proof. First, the boundedness of $\mathbb{B}\mathbb{B}^*\mathbf{IIP}_N$ ensures that $\mathbb{A}_{\mathbf{II}} - \mathbb{B}\mathbb{B}^*\mathbf{IIP}_N$ and \mathbb{A} have the same domain and the analyticity of $(e^{-\mathbb{A}\mathbf{II}t})_{t \geq 0}$ is a consequence of the analyticity of $(e^{\mathbb{A}t})_{t \geq 0}$, see [24, Chap. 3, Cor. 2.2, p. 81]. Thus, similarly as in Subsection 4.1 we can prove that there is $a > 0$ such that

$$\|y_{\bar{N}}(t)\|_{\mathcal{X}} \leq C (\|y_0\|_{\mathcal{X}} + \|\bar{u}_0\|_{\mathbb{R}^K}) e^{-(\sigma+a)t},$$

which, with (62), gives the exponential stability of $(e^{-\mathbb{A}\mathbf{II}t})_{t \geq 0}$. Finally, from (94) and (95) with a perturbation argument [11, Prop. 2.7] we deduce the interpolation equalities $\mathcal{D}(\mathbb{A}_{\mathbf{II}}^{\theta}) = [\mathcal{D}(\mathbb{A}_{\mathbf{II}}), \mathcal{X}]_{1-\theta}$ and $\mathcal{D}(\mathbb{A}_{\mathbf{II}}^{*\theta}) = [\mathcal{D}(\mathbb{A}_{\mathbf{II}}^*), \mathcal{X}]_{1-\theta}$, and (102) follows from (94),(95) and (99),(101). \square

Next, we recall that $G \in \mathcal{L}(\mathcal{X}, \mathcal{O})$ is an operator satisfying (65) and we define an analogue extended operator $\mathbb{G} \in \mathcal{L}(\mathcal{X} \times \mathbb{R}^K, \mathcal{O} \times \mathbb{R}^K)$ which is boundedly invertible and which obeys $\mathbb{G}^*\mathbb{G} \in \mathcal{L}(\mathcal{D}(\mathbb{A}_{\mathbf{II}}^{\frac{1}{2}}), \mathcal{D}(\mathbb{A}_{\mathbf{II}}^{*\frac{1}{2}}))$ as follows:

$$\mathbb{G} \begin{pmatrix} y \\ \bar{u} \end{pmatrix} = \begin{pmatrix} G(y + \hat{A}^{-1}V\bar{u}) \\ \bar{u} \end{pmatrix}.$$

The bounded invertibility of \mathbb{G} is an easy consequence of the bounded invertibility of G . To prove the second fact, we first notice $\mathcal{D}(\hat{\mathbb{A}}^{\frac{1}{2}}) = \mathcal{D}(\mathbb{A}_{\mathbf{II}}^{\frac{1}{2}})$ and $\mathcal{D}(\hat{\mathbb{A}}^{*\frac{1}{2}}) = \mathcal{D}(\mathbb{A}_{\mathbf{II}}^{*\frac{1}{2}})$. Then, the fact that $\mathbb{G}^*\mathbb{G} \in \mathcal{L}(\mathcal{D}(\hat{\mathbb{A}}^{\frac{1}{2}}), \mathcal{D}(\hat{\mathbb{A}}^{*\frac{1}{2}}))$ is a consequence of $G^*G \in \mathcal{L}(\mathcal{D}(\hat{A}^{\frac{1}{2}}), \mathcal{D}(\hat{A}^{*\frac{1}{2}}))$ and of the expression:

$$\mathbb{G}^*\mathbb{G} \begin{pmatrix} y \\ \bar{u} \end{pmatrix} = \begin{pmatrix} G^*G(y + \hat{A}^{-1}V\bar{u}) \\ V^*\hat{A}^{*-1}G^*G(y + \hat{A}^{-1}V\bar{u}) + \bar{u} \end{pmatrix}.$$

Then we can proceed as in Subsection 4.1: we define the following equivalent norms on $\mathcal{D}(\hat{\mathbb{A}}^{\frac{s}{2}})$ and on $\mathcal{D}(\hat{\mathbb{A}}^{\frac{s}{2}+\frac{1}{2}})$ for $s \in [0, 1]$

$$\|\cdot\|_s^2 \stackrel{\text{def}}{=} \int_0^\infty e^{2\sigma t} \|\mathbb{G}\mathbb{A}_{\mathbf{II}}^{\frac{1}{2}+\frac{s}{2}} e^{-\mathbb{A}\mathbf{II}t} \cdot\|_{\mathcal{O} \times \mathbb{R}^K}^2 dt = \langle \mathbb{L}_s \cdot | \cdot \rangle_{[\mathcal{D}(\hat{\mathbb{A}}^{\frac{s}{2}})]', \mathcal{D}(\hat{\mathbb{A}}^{\frac{s}{2}})},$$

and

$$\mathcal{N}_{s+1}(\cdot)^2 \stackrel{\text{def}}{=} \|\mathbb{G}\mathbb{A}_{\mathbf{II}}^{\frac{1}{2}+\frac{s}{2}} \cdot\|_{\mathcal{O} \times \mathbb{R}^K}^2 = 2\langle \mathbb{L}_s \cdot | \mathbb{A}_{\mathbf{II}} \cdot \rangle_{\mathcal{D}(\hat{\mathbb{A}}^{*\frac{1}{2}-\frac{s}{2}}), \mathcal{D}(\hat{\mathbb{A}}^{\frac{s}{2}+\frac{1}{2}})} - 2\sigma \|\cdot\|_s^2.$$

Thus, we prove a stabilization theorem for a nonlinear system of type:

$$Y' + \mathbb{A}_{\mathbf{II}}Y = N(Y), \quad Y(0) = Y_0 \tag{103}$$

provided that the nonlinear mapping $N(\cdot) : \mathcal{D}(\hat{\mathbb{A}}^{\frac{s}{2}+\frac{1}{2}}) \rightarrow [\mathcal{D}(\hat{\mathbb{A}}^{*(\frac{1}{2}-\frac{s}{2})})'$ satisfies

$$\begin{cases} \|N(\xi)\|_{[\mathcal{D}(\hat{\mathbb{A}}^{*(\frac{1}{2}-\frac{s}{2})})]'} \leq C \|\xi\|_{\mathcal{D}(\hat{\mathbb{A}}^{\frac{s}{2}})} \|\xi\|_{\mathcal{D}(\hat{\mathbb{A}}^{\frac{s}{2}+\frac{1}{2}})}, \\ \|N(\xi) - N(\zeta)\|_{[\mathcal{D}(\hat{\mathbb{A}}^{*(\frac{1}{2}-\frac{s}{2})})]'} \leq C \|\xi - \zeta\|_{\mathcal{D}(\hat{\mathbb{A}}^{\frac{s}{2}})} \left(\|\xi\|_{\mathcal{D}(\hat{\mathbb{A}}^{\frac{s}{2}+\frac{1}{2}})} + \|\zeta\|_{\mathcal{D}(\hat{\mathbb{A}}^{\frac{s}{2}+\frac{1}{2}})} \right) \\ \quad + \left(\|\xi\|_{\mathcal{D}(\hat{\mathbb{A}}^{\frac{s}{2}})} + \|\zeta\|_{\mathcal{D}(\hat{\mathbb{A}}^{\frac{s}{2}})} \right) \|\xi - \zeta\|_{\mathcal{D}(\hat{\mathbb{A}}^{\frac{s}{2}+\frac{1}{2}})}. \end{cases} \tag{104}$$

Theorem 19. *Assume that (90), (91) are satisfied and assume (104) for some $s \in [0, 1]$ and $y_0 \in \mathcal{D}(\hat{\mathbb{A}}^{\frac{s}{2}})$. There exist $\rho > 0$ and $\mu > 0$ such that, if $\|Y_0\|_s < \mu$, then system (103) admits a solution $Y_{Y_0} \in W(\mathcal{D}(\hat{\mathbb{A}}^{\frac{s}{2}+\frac{1}{2}}), [\mathcal{D}(\hat{\mathbb{A}}^{*(\frac{1}{2}-\frac{s}{2})})]')$ such that*

$$\|Y_{Y_0}\|_{W(\mathcal{D}(\hat{\mathbb{A}}^{\frac{s}{2}+\frac{1}{2}}), [\mathcal{D}(\hat{\mathbb{A}}^{*(\frac{1}{2}-\frac{s}{2})})]')} \leq \rho \|Y_0\|_s,$$

which is unique within the class of functions in $L_{loc}^\infty(\mathcal{D}(\hat{\mathbb{A}}^{\frac{s}{2}})) \cap L_{loc}^2(\mathcal{D}(\hat{\mathbb{A}}^{\frac{s}{2}+\frac{1}{2}}))$. Moreover, every solution with an initial datum obeying:

$$\|Y_0\|_s < K_s \quad \text{where} \quad \frac{1}{2K_s} \stackrel{\text{def}}{=} \sup_{0 \neq \xi \in \mathcal{D}(\hat{\mathbb{A}}^{\frac{s}{2}+\frac{1}{2}})} \frac{\langle \mathbb{L}_s \xi | N(\xi) \rangle_{[\mathcal{D}(\hat{\mathbb{A}}^{\frac{s}{2}})]', \mathcal{D}(\hat{\mathbb{A}}^{\frac{s}{2}})}}{\|\xi\|_s \mathcal{N}_{s+1}(\xi)^2},$$

is such that $t \mapsto \|Y_{Y_0}(t)\|_s$ is decreasing and we have:

$$\|Y_{Y_0}(t)\|_s \leq \|Y_0\|_s e^{-(\sigma + \beta_s(1 - \|Y_0\|_s/K_s))t} \quad \text{where} \quad \beta_s \stackrel{\text{def}}{=} \inf_{0 \neq \xi \in \mathcal{D}(\hat{\mathbb{A}}^{\frac{s}{2}+\frac{1}{2}})} \frac{\mathcal{N}_{s+1}(\xi)^2}{2\|\xi\|_s^2}, \tag{105}$$

$$\int_0^\infty e^{2\sigma t} \mathcal{N}_{s+1}(Y_{Y_0}(t))^2 dt \leq \frac{\|Y_0\|_s^2}{(1 - \|Y_0\|_s/K_s)}. \tag{106}$$

5 Stabilization of the Navier-Stokes system

5.1 Stabilization of the Navier-Stokes system by means of a finite dimensional feedback control

Here, we come back to the question of stabilizing the solutions of the Navier-Stokes system $(15)_{\kappa=1}$. At a first step, we consider the Oseen system $(15)_{\kappa=0}$ which can be rewritten as

$$y' = Ay + Bu \in [\mathcal{D}(A^*)]', \quad y(0) = Pw_0, \quad (107)$$

where A, B are defined by (10) and by (19) and where $y = Pw$. To fit the framework of Section 3 we set $\mathcal{X} \stackrel{\text{def}}{=} \mathbf{V}_n^0(\Omega)$ and $\mathcal{V} \stackrel{\text{def}}{=} \mathbf{V}^0(\partial\Omega)$ and we recall that the complexified of $A : \mathcal{D}(A) \subset \mathcal{X} \rightarrow \mathcal{X}$ and of $B : \mathcal{V} \rightarrow [\mathcal{D}(A^*)]'$ must satisfy

- (a) A has a compact resolvent;
- (b) A is the generator of an analytic semigroup on \mathcal{X} ;
- (c) $\widehat{A}^{-\gamma}B : \mathcal{V} \rightarrow \mathcal{X}$ is bounded for some $0 \leq \gamma < 1$;
- (d) Conditions 1 and 2 of Corollary 6.

Conditions (a) and (b) follow from Theorem 1 and Condition (c) is given in (20). Condition (d) is obtained by Theorem 8 by using the exact controllability result stated in [12] (see also [20] and [18]). Indeed, although the previous cited work only deals with a distributed control localized in an open part of the domain, the controllability of $(15)_{\kappa=0}$ is obtained with a classical extension procedure (see for instance the appendix of [2]).

Notice that from [3, Prop. 3] we deduce that the adjoint of B is given by

$$B^*\varepsilon = m\left(\chi(\varepsilon)n - \nu\frac{\partial\varepsilon}{\partial n}\right) \quad \forall \varepsilon \in \mathbf{V}_0^2(\Omega),$$

where $\chi(\varepsilon) \in H^1(\Omega)$ is a pressure function solution to $\nabla\chi(\varepsilon) = (I - P)(\nu\Delta\varepsilon - {}^t(\nabla w_S)\varepsilon + (w_S \cdot \nabla)\varepsilon)$ and $\int_{\partial\Omega} m\chi(\varepsilon)d\Gamma = 0$, which also means that:

$$\begin{cases} \Delta\chi(\varepsilon) = \nabla \cdot ((w_S \cdot \nabla)\varepsilon - {}^t(\nabla w_S)\varepsilon) & \text{in } \Omega, \quad \int_{\partial\Omega} m\chi(\varepsilon)d\Gamma = 0, \\ \frac{\partial\chi(\varepsilon)}{\partial n} = (\nu\Delta\varepsilon + (w_S \cdot \nabla)\varepsilon - {}^t(\nabla w_S)\varepsilon) \cdot n & \text{on } \partial\Omega. \end{cases} \quad (108)$$

In particular, if $\varepsilon \in \text{Ker}(A^* - \lambda)$ then $B^*\varepsilon = m\left(\chi(\varepsilon)n - \nu\frac{\partial\varepsilon}{\partial n}\right)$ where $(\varepsilon, \chi(\varepsilon))$ satisfies:

$$\begin{aligned} \lambda\varepsilon - \nu\Delta\varepsilon - (w_S \cdot \nabla)\varepsilon + {}^t(\nabla w_S)\varepsilon + \nabla\chi(\varepsilon) &= 0 & \text{in } \Omega, & \int_{\partial\Omega} m\chi(\varepsilon)d\Gamma = 0, \\ \nabla \cdot \varepsilon &= 0 & \text{in } \Omega, & \\ \varepsilon &= 0 & \text{on } \partial\Omega, & \end{aligned} \quad (109)$$

and the unique continuation property which is obtain from [12] by Theorem 8 can be stated as follows:

Theorem 20. *For all $\lambda \in \mathbb{C}$, every solution of (109) satisfying*

$$m\left(\chi(\varepsilon)n - \nu\frac{\partial\varepsilon}{\partial n}\right) = 0 \quad \text{on } \partial\Omega,$$

is necessarily equal to zero.

Moreover, to fit the framework of Section 4 the following additional statements are required:

- (e) For all $\theta \in [0, 1]$ we have the equalities $\mathcal{D}(\widehat{A}^\theta) = [\mathcal{D}(A), \mathcal{X}]_{1-\theta}$ and $\mathcal{D}(\widehat{A}^{*\theta}) = [\mathcal{D}(A^*), \mathcal{X}]_{1-\theta}$.
- (f) There is a Hilbert space \mathcal{O} and an isomorphism mapping $G : \mathcal{X} \rightarrow \mathcal{O}$ such that $G^*G \in \mathcal{L}(\mathcal{D}(\widehat{A}^{\frac{1}{2}}), \mathcal{D}(\widehat{A}^{*\frac{1}{2}}))$.

Condition (e) follows from (12) and since $\mathcal{D}(\widehat{A}^{\frac{1}{2}}) = \mathcal{D}(\widehat{A}^{*\frac{1}{2}})$ Condition (f) can be obtained by choosing G as the identity in $\mathbf{V}_n^0(\Omega)$.

Now let us recall the notations of Sections 3 and 4.

- $\sigma > 0$ is an expected rate of decrease, $\Sigma = \{\lambda_k \mid k \in \mathbb{N}^*\}$ is the spectrum of A and:

$$\cdots \leq \Re \lambda_{N+1} < -\sigma < \Re \lambda_N \leq \cdots \leq \Re \lambda_2 \leq \Re \lambda_1.$$

- $\mathcal{X}_N = P_N(\mathbf{V}_n^0(\Omega))$ where $P_N : \mathbf{V}_n^0(\Omega) \rightarrow \mathcal{X}_N$ is the projection operator related to $\lambda_1, \dots, \lambda_N$ defined by:

$$P_N \stackrel{\text{def}}{=} -\frac{1}{2\pi i} \int_{\Gamma_N} (\xi - A)^{-1} d\xi,$$

where Γ_N is a positively-oriented curve obeying $\Gamma_N = \bar{\Gamma}_N$, and enclosing Σ_N but no other point of Σ . We have $\bar{P}_N = P_N$ and \mathcal{X}_N is a real finite dimensional subspace of $\mathbf{V}_n^0(\Omega)$.

- $\mathcal{X}_N^* = P_N^*(\mathbf{V}_n^0(\Omega))$ where P_N^* is the adjoint of P_N .
- $\{\varepsilon_k^i, i = 1, \dots, \ell_k\}$ a basis of $\text{Ker}(A^* - \bar{\lambda}_k)$, ℓ_k is the geometric multiplicity of $\bar{\lambda}_k$ and we set

$$\mathcal{F}_N = \text{span}_{\mathbb{R}} \{ \Re \varepsilon_k^i, \Im \varepsilon_k^i, k = 1, \dots, N, i = 1, \dots, \ell_k \}.$$

Notice that Ω of class $C^{2,1}$ and $w_S \in \mathbf{V}^2(\Omega)$ ensure that $\mathcal{F}_N \subset \mathbf{V}^{\frac{3}{2}}(\partial\Omega)$, see Remark 21 below.

- $K \geq \ell_k$ for all $k = 1, \dots, N$, $(v_1, \dots, v_K) \in B^*(\mathcal{F}_N)^K$ is the family given by Theorem 9 and

$$\mathcal{C}_K = \text{span}\{v_1, \dots, v_K\} \subset (\mathbf{V}^{\frac{3}{2}}(\partial\Omega))^K.$$

- $\Pi \in \mathcal{L}(\mathcal{X}_N, \mathcal{X}_N^*)$ is the Riccati operator solution to (46),(47).
- $V \in \mathcal{L}(\mathbb{R}^K, [\mathcal{D}(A^*)]')$ is defined by (42) and A_{Π} is the closed-loop operator defined by (69).

Remark 21. (i) The assumption $w_S \in \mathbf{V}^2(\Omega)$ is only required for the inclusion $\mathcal{F}_N \subset \mathbf{V}^{\frac{3}{2}}(\partial\Omega)$. Indeed, suppose that $\varepsilon \in \text{Ker}(A^* - \lambda)$. Then ε belongs to $\mathcal{D}(A^*) \subset \mathbf{H}^2(\Omega)$ and with $w_S \in \mathbf{H}^2(\Omega)$ and Sobolev embeddings we obtain $(w_S \cdot \nabla)\varepsilon + {}^t(\nabla w_S)\varepsilon \in \mathbf{H}^1(\Omega)$. As a consequence, since ε satisfies (109) and since Ω is of class $C^{2,1}$, regularity result for Stokes system [9, Thm. III.3.16 p.148] yields $(\varepsilon, \chi(\varepsilon)) \in \mathbf{H}^3(\Omega) \times H^2(\Omega)$ and $B^*\varepsilon = m(\chi(\varepsilon)n - \nu \frac{\partial \varepsilon}{\partial n}) \in \mathbf{V}^{\frac{3}{2}}(\partial\Omega)$. In fact, with an analogue local argument one can prove that it is sufficient to have $w_S \in \mathbf{V}^1(\Omega) \cap \mathbf{H}^2(\omega)$ where ω is a neighborhood of the part of the boundary where the control is acting: $\Gamma_m \subset \partial\Omega \cap \partial\omega$.

(ii) Since the assumption that Ω of class $C^{2,1}$ also ensures that (9), (14) are true for $s = 2$ we have:

$$PDM(v_j) \in \mathbf{V}_n^2(\Omega) \quad \forall j = 1, \dots, K. \quad (110)$$

Proposition 22. For all $\theta \in [0, 1]$ the following characterization holds:

$$\mathcal{D}(A_{\Pi}^{\theta}) = \left\{ \xi \in \mathbf{V}_n^{2\theta}(\Omega) \mid \xi + PD \sum_{j=1}^K (v_j | (V^* \Pi) P_N \xi)_{\mathbf{V}^0(\partial\Omega)} M(v_j) \in \mathbf{V}_0^{2\theta}(\Omega) \right\}. \quad (111)$$

Proof. Without loss of generality we can assume that the values λ_0 in (11) and in Section 4 are the same. Then from (75) with (12), and with the expression of V and of B given by (42) and (19), we first deduce that:

$$\mathcal{D}(A_{\Pi}^{\theta}) = \left\{ \xi \in \mathbf{V}_n^0(\Omega) \mid \xi + \sum_{j=1}^K (v_j | (V^* \Pi) P_N \xi)_{\mathbf{V}^0(\partial\Omega)} PDM(v_j) \in \mathbf{V}_0^{2\theta}(\Omega) \right\}.$$

Thus with (110) we deduce that $\mathcal{D}(A_{\Pi}^{\theta})$ is the closed subspace of $\mathbf{V}_n^{2\theta}(\Omega)$ defined by (111). \square

Corollary 23. For $s \in [0, 2]$ the space defined by

$$\begin{aligned} \mathbf{V}_{\Pi}^s(\Omega) &= \mathbf{V}^s(\Omega) \quad \text{if } s < \frac{1}{2}, \\ \mathbf{V}_{\Pi}^{\frac{1}{2}}(\Omega) &= \left\{ y \in \mathbf{V}^s(\Omega) \mid \int_{\Omega} \text{dist}(x, \partial\Omega)^{-1} (y(x) + D[\sum_{j=1}^K (v_j |(V^* \Pi) P_N y)_{\mathbf{V}^0(\partial\Omega)} M(v_j)](x)) dx < +\infty \right\}, \\ \mathbf{V}_{\Pi}^s(\Omega) &= \left\{ y \in \mathbf{V}^s(\Omega) \mid y + \sum_{j=1}^K (v_j |(V^* \Pi) P_N y)_{\mathbf{V}^0(\partial\Omega)} M(v_j) \text{ on } \partial\Omega \right\} \quad \text{if } s > \frac{1}{2}, \end{aligned}$$

obeys $P\mathbf{V}_{\Pi}^s(\Omega) = \mathcal{D}(A_{\Pi}^{\frac{s}{2}})$.

Using Proposition 11 the solution of the system

$$\left\{ \begin{array}{ll} \frac{\partial w}{\partial t} - \nu \Delta w + (w_S \cdot \nabla) w + (w \cdot \nabla) w_S + \nabla p = 0 & \text{in } (0, \infty) \times \Omega, \\ \nabla \cdot w = 0 & \text{in } (0, \infty) \times \Omega, \\ w = \sum_{j=1}^K (v_j | B^* \Pi P_N P w) M(v_j) & \text{on } (0, \infty) \times \partial\Omega, \\ w(0) = w_0 & \text{in } \Omega, \end{array} \right. \quad (112)$$

are exponentially stable, more precisely:

Proposition 24. Let $s \in [0, 1]$ and $w_0 \in \mathbf{V}_{\Pi}^s(\Omega)$. Then (112) admits a unique solution

$$(w, p) \in W(\mathbf{V}^{1+s}(\Omega), \mathbf{V}_0^{s-1}(\Omega)) \times H^{-1/2+s/2}(H_{\sharp}^s(\Omega))$$

which obeys:

$$\|w\|_{\mathbf{V}^s(\Omega)} \leq C \|Pw_0\|_{\mathbf{V}^s(\Omega)} e^{-\sigma t} \quad \text{for all } t \geq 0. \quad (113)$$

Proof. Since $W(\mathbf{V}^{1+s}(\Omega), \mathbf{V}_0^{s-1}(\Omega)) \hookrightarrow L^2(\mathbf{V}^{s+1}(\Omega)) \cap H^{\frac{s}{2}+\frac{1}{2}}(\mathbf{V}^0(\Omega))$, then by Proposition 4 it suffices to prove that system (21) _{$\kappa=0$} , (22) with $u = \sum_{j=1}^K (v_j | B^* \Pi P_N P w) v_j$ admits a unique solution in $W(\mathbf{V}^{1+s}(\Omega), \mathbf{V}_0^{s-1}(\Omega))$ which obeys (113). Let us define $J \in (\mathbf{V}_n^0(\Omega), \mathbf{V}^0(\Omega))$ by:

$$J(y) \stackrel{\text{def}}{=} \sum_{i=1}^K (v_i | B^* \Pi P_N y) u [(I - P) D M(v_i)]. \quad (114)$$

From $P_N \in \mathcal{L}(\mathbf{V}_0^{s-1}(\Omega), \mathbf{V}_0^2(\Omega))$ and from (110) we deduce that $J \in \mathcal{L}(\mathbf{V}_0^{s-1}(\Omega), \mathbf{V}^2(\Omega))$ and then with $\mathcal{D}(A_{\Pi}^{\frac{1}{2}+\frac{s}{2}}) \hookrightarrow [\mathcal{D}(A_{\Pi}^{*\frac{1}{2}-\frac{s}{2}})]' = \mathbf{V}_0^{s-1}(\Omega)$:

$$J \in \mathcal{L}(W(\mathcal{D}(A_{\Pi}^{\frac{1}{2}+\frac{s}{2}}), [\mathcal{D}(A_{\Pi}^{*\frac{1}{2}-\frac{s}{2}})]'), H^1(\mathbf{V}^2(\Omega))).$$

Thus, we verify that $y \in W(\mathcal{D}(A_{\Pi}^{\frac{1}{2}+\frac{s}{2}}), [\mathcal{D}(A_{\Pi}^{*\frac{1}{2}-\frac{s}{2}})]')$ obeys:

$$y' + A_{\Pi} y = 0, \quad y(0) = Pw_0,$$

if and only if, $w = y + J(y) \in W(\mathbf{V}^{1+s}(\Omega), \mathbf{V}_0^{s-1}(\Omega))$ obeys (21) _{$\kappa=0$} , (22) with $u = \sum_{j=1}^K (v_j | B^* \Pi P_N P w) M(v_j)$. In particular $w_0 \in \mathbf{V}_{\Pi}^s(\Omega)$ implies $w(0) = Pw_0 + J(Pw_0) = w_0$. Then since by Corollary 23 we have $Pw_0 \in \mathcal{D}(A_{\Pi}^{\frac{s}{2}})$, the conclusion follows from Proposition 11 with maximal regularity result for analytic semigroup (8). \square

Using Theorem 16, the solution of the nonlinear system

$$\left\{ \begin{array}{ll} \frac{\partial w}{\partial t} - \nu \Delta w + (w_S \cdot \nabla) w + (w \cdot \nabla) w_S + \nabla p = 0 & \text{in } (0, \infty) \times \Omega, \\ \nabla \cdot w = 0 & \text{in } (0, \infty) \times \Omega, \\ w = \sum_{j=1}^K (v_j | B^* \Pi P_N P w) M(v_j) & \text{on } (0, \infty) \times \partial\Omega, \\ w(0) = w_0 & \text{in } \Omega, \end{array} \right. \quad (115)$$

are locally exponentially stable, more precisely:

Theorem 25. *Let $s \in [\frac{d-2}{2}, 2]$ and $w_0 \in \mathbf{V}_\Pi^s(\Omega)$. There exists $\mu > 0$ such that if $\|Pw_0\|_{\mathbf{V}_n^s(\Omega)} \leq \mu$ then (115) admits a solution $(w, p) \in W(\mathbf{V}^{1+s}(\Omega), \mathbf{V}_0^{s-1}(\Omega)) \times H^{-1/2+s/2}(H_\sharp^s(\Omega))$, which is unique within the class of function of $(L_{loc}^2(\mathbf{V}^{s+1}(\Omega)) \cap L_{loc}^\infty(\mathbf{V}^s(\Omega))) \times H_{loc}^{-1/2+s/2}(H_\sharp^s(\Omega))$, and which obeys:*

$$\|w\|_{\mathbf{V}^s(\Omega)} \leq C\|w_0\|_{\mathbf{V}^s(\Omega)}e^{-\sigma t} \quad \text{for all } t \geq 0. \quad (116)$$

Proof. Since $W(\mathbf{V}^{1+s}(\Omega), \mathbf{V}_0^{s-1}(\Omega)) \hookrightarrow L^2(\mathbf{V}^{s+1}(\Omega)) \cap H^{\frac{s}{2}+\frac{1}{2}}(\mathbf{V}^0(\Omega))$, then by Proposition 4 it suffices to prove that there is $\mu > 0$ such that for $\|Pw_0\|_{\mathbf{V}_n^s(\Omega)} \leq \mu$ system (21) $_{\kappa=1}$,(22) with $u = \sum_{j=1}^K (v_j | B^* \Pi P_N P w) v_j$ admits a solution in $W(\mathbf{V}^{1+s}(\Omega), \mathbf{V}_0^{s-1}(\Omega))$ which is unique within the class of function of $L_{loc}^2(\mathbf{V}^{s+1}(\Omega)) \cap L_{loc}^\infty(\mathbf{V}^s(\Omega))$ and which obeys (116). Thus, we set $N(y) \stackrel{\text{def}}{=} ((y + J(y)) \cdot \nabla)(y + J(y))$ where J is given by (114) and, as in the proof of Proposition 24, we verify that $y \in W(\mathcal{D}(A_\Pi^{\frac{1}{2}+\frac{s}{2}}), [\mathcal{D}(A_\Pi^{*\frac{1}{2}-\frac{s}{2}}])'$ obeys:

$$y' + A_\Pi y = N(y), \quad y(0) = Pw_0,$$

if and only if, $w = y + J(y)$ obeys (21) $_{\kappa=1}$,(22) with $u = \sum_{j=1}^K (v_j | B^* \Pi P_N P w) v_j$. Then by combining estimates (17),(18) and the fact that $J \in (\mathbf{V}_n^s(\Omega), \mathbf{V}^s(\Omega))$ for all $s \in [0, 2]$ we deduce that N obeys (82),(83) and, since $w_0 \in \mathbf{V}_\Pi^s(\Omega)$ implies $Pw_0 \in \mathcal{D}(A_\Pi^{\frac{s}{2}})$, the conclusion follows from Theorem 16. \square

5.2 Stabilization of the Navier–Stokes system by means of a finite dimensional dynamical feedback control

Let us recall the additional notations introduced in Subsection 3.5:

- $\Pi_2 \in \mathcal{L}(\mathcal{X}_N, \mathcal{C}_K)$ and $\Pi_3 \in \mathcal{L}(\mathcal{C}_K)$ are defined by (59), (60) and (61).
- $D_K : \mathbb{R}^K \rightarrow \mathbb{R}^K$ is a linear operator with spectrum composed of K distinct eigenvalues $\mu_i \notin \{\lambda_1, \dots, \lambda_N\}$, and $\bar{g} \in \mathbb{R}^K$ is such that $(\bar{g} | \bar{d})_{\mathbb{R}^K} \neq 0$ for all eigenvectors \bar{d} of ${}^t D_K$.
- \mathbb{A}_Π is the closed-loop operator defined by (99).

Proposition 26. *For all $\theta \in [0, 1]$ the following characterization holds:*

$$\mathcal{D}(\mathbb{A}_\Pi^\theta) = \left\{ Y = {}^t(y, \bar{u}) \in \mathbf{V}_n^{2\theta}(\Omega) \times \mathbb{R}^K \mid y + PD \sum_{j=1}^K u_j M(v_j) \in \mathbf{V}_0^{2\theta}(\Omega) \right\}, \quad (117)$$

$$\mathcal{D}(\mathbb{A}_\Pi^{*\theta}) = \mathbf{V}_0^{2\theta}(\Omega) \times \mathbb{R}^K. \quad (118)$$

Proof. Since we have (110), then (117) follows from (102), (94), (12) by recalling (42), (19). Moreover, (118) follows from (102),(95) and (12). \square

Using Proposition 18 the solution of the system:

$$\left\{ \begin{array}{ll} \frac{\partial w}{\partial t} - \nu \Delta w + (w_S \cdot \nabla)w + (w \cdot \nabla)w_S + \nabla p = 0 & \text{in } (0, \infty) \times \Omega, \\ \nabla \cdot w = 0 & \text{in } (0, \infty) \times \Omega, \\ w = \sum_{j=1}^K u_j v_j & \text{on } (0, \infty) \times \partial\Omega, \\ w(0) = w_0 & \text{in } \Omega. \\ (u_1, \dots, u_K) = \bar{u}, \\ \bar{u}' - D_K \bar{u} + (\bar{g} | \Pi_2 P_N w) \bar{g} + (\bar{g} | \Pi_3 \bar{u}) \bar{g} = 0, \\ \bar{u}(0) = 0, \end{array} \right. \quad (119)$$

is exponentially stable, more precisely:

Proposition 27. *Let $s \in [0, 1]$ and $w_0 \in \mathbf{V}_0^s(\Omega)$. Then (119) admits a unique solution*

$$(w, p, \bar{u}) \in W(\mathbf{V}^{s+1}(\Omega), \mathbf{V}_0^{s-1}(\Omega)) \times H^{-1/2+s/2}(H_\sharp^s(\Omega)) \times H^1(\mathbb{R}^K)$$

which obeys:

$$\|w(t)\|_{\mathbf{V}^s(\Omega)} + \|\bar{u}(t)\|_{\mathbb{R}^K} \leq C\|w_0\|_{\mathbf{V}_0^s(\Omega)}e^{-\sigma t} \quad \text{for all } t \geq 0. \quad (120)$$

Proof. Let us introduce the operator $\mathcal{M} : \mathbb{R}^K \rightarrow \mathbf{V}^2(\Omega)$ defined by

$$\mathcal{M}(\bar{u}) = \sum_{j=1}^K u_j [(I - P)DM(v_j)]. \quad (121)$$

By Proposition 4 it suffices to prove that system

$$\begin{aligned} {}^t(Pw, \bar{u}) &= Y, \\ Y' + \mathbb{A}_{\Pi} Y &= 0, \quad Y(0) = {}^t(Pw_0, 0), \\ (I - P)w &= \mathcal{M}(\bar{u}) \end{aligned}$$

admits a unique solution $(w, \bar{u}) \in W(\mathbf{V}^{1+s}(\Omega), \mathbf{V}_0^{s-1}(\Omega)) \times H^1(\mathbb{R}^K)$ which obeys (120). Moreover, according to (117) for $w_0 \in \mathbf{V}_0^s(\Omega)$ we have ${}^t(Pw_0, 0) = {}^t(w_0, 0) \in \mathcal{D}(\mathbb{A}_{\Pi}^{\frac{s}{2}})$ and, according to Proposition 18 with maximal regularity result for analytic semigroup (8), the corresponding solution $Y = {}^t(y, \bar{u})$ belongs to $W(\mathcal{D}(\mathbb{A}_{\Pi}^{\frac{1}{2} + \frac{s}{2}}), [\mathcal{D}(\mathbb{A}_{\Pi}^{*\frac{s}{2} - \frac{1}{2}})]') \hookrightarrow W(\mathbf{V}_n^{s+1}(\Omega) \times \mathbb{R}^K, \mathbf{V}_0^{s-1}(\Omega) \times \mathbb{R}^K)$. Then we deduce that $\bar{u} \in H^1(\mathbb{R}^K)$, $\mathcal{M}(\bar{u}) \in H^1(\mathbf{V}^2(\Omega))$ and $w = y + \mathcal{M}(\bar{u}) \in W(\mathbf{V}^{s+1}(\Omega), \mathbf{V}_0^{s-1}(\Omega))$ which allows to conclude. Finally, (120) is a consequence of (100). \square

Using Theorem 19 the solution of the system:

$$\left\{ \begin{array}{ll} \frac{\partial w}{\partial t} - \nu \Delta w + (w_S \cdot \nabla)w + (w \cdot \nabla)w_S + (w \cdot \nabla)w + \nabla p = 0 & \text{in } (0, \infty) \times \Omega, \\ \nabla \cdot w = 0 & \text{in } (0, \infty) \times \Omega, \\ w = \sum_{j=1}^K u_j v_j & \text{on } (0, \infty) \times \partial\Omega, \\ w(0) = w_0 & \text{in } \Omega, \\ (u_1, \dots, u_K) = \bar{u}, \\ \bar{u}' - D_K \bar{u} + (\bar{g}|\Pi_2 P_N w)\bar{g} + (\bar{g}|\Pi_3 \bar{u})\bar{g} = 0, \\ \bar{u}(0) = 0, \end{array} \right. \quad (122)$$

is exponentially stable, more precisely:

Theorem 28. *Let $s \in [\frac{d-2}{2}, 1]$ and $w_0 \in \mathbf{V}_0^s(\Omega)$. There exists $\mu > 0$ such that if $\|w_0\|_{\mathbf{V}_0^s(\Omega)} \leq \mu$ then (122) admits a solution $(w, p, \bar{u}) \in W(\mathbf{V}^{s+1}(\Omega), \mathbf{V}_0^{s-1}(\Omega)) \times H^{-1/2+s/2}(H_{\sharp}^s(\Omega)) \times H^1(\mathbb{R}^K)$ which is unique within the class of function of $(L^2(\mathbf{V}^{s+1}(\Omega)) \cap L^\infty(\mathbf{V}^s(\Omega))) \times H^{-1/2+s/2}(H_{\sharp}^s(\Omega)) \times H^1(\mathbb{R}^K)$ and which obeys:*

$$\|w(t)\|_{\mathbf{V}^s(\Omega)} + \|\bar{u}(t)\|_{\mathbb{R}^K} \leq C \|w_0\|_{\mathbf{V}_0^s(\Omega)} e^{-\sigma t} \quad \text{for all } t \geq 0. \quad (123)$$

Proof. With $\mathcal{M} : \mathbb{R}^K \rightarrow \mathbf{V}^2(\Omega)$ given by (121) we set:

$$N \begin{pmatrix} y \\ \bar{u} \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} P((y + \mathcal{M}(\bar{u})) \cdot \nabla)(y + \mathcal{M}(\bar{u})) \\ 0 \end{pmatrix},$$

and by Proposition 4 it suffices to prove that for some $\mu > 0$ and $\|w_0\|_{\mathbf{V}_0^s(\Omega)} \leq \mu$ system

$$\begin{aligned} {}^t(Pw, \bar{u}) &= Y, \\ Y' + \mathbb{A}_{\Pi} Y &= N(Y), \quad Y(0) = {}^t(Pw_0, 0), \\ (I - P)w &= \mathcal{M}(\bar{u}) \end{aligned}$$

admits a solution $(w, \bar{u}) \in W(\mathbf{V}^{1+s}(\Omega), \mathbf{V}_0^{s-1}(\Omega)) \times H^1(\mathbb{R}^K)$, unique within the class of function of $(L^2(\mathbf{V}^{s+1}(\Omega)) \cap L^\infty(\mathbf{V}^s(\Omega))) \times H^1(\mathbb{R}^K)$, which obeys (123). Then since N obeys (104), which is an easy consequence of (17), (18) and of (117),(118), it suffices to apply Theorem 19. \square

References

- [1] Mehdi Badra. Local stabilization of the Navier-Stokes equations with a feedback controller localized in an open subset of the domain. *Numer. Funct. Anal. Optim.*, 28(5-6):559–589, 2007.
- [2] Mehdi Badra. Feedback stabilization of the 2-D and 3-D navier-stokes equations based on an extended system. *ESAIM: Control, Optimisation and Calculus of Variations*, 15(4):934–968, 2009.
- [3] Mehdi Badra. Lyapunov function and local feedback boundary stabilization of the Navier-Stokes equations. *SIAM J. Control Optim.*, 48(3):1797–1830, 2009.
- [4] V. Barbu and R.L. Triggiani. Internal stabilization of Navier-Stokes equations with finite-dimensional controllers. *Indiana Univ. Math. J.*, 53(5):1443–1494, 2004.
- [5] Viorel Barbu, Irena Lasiecka, and Roberto Triggiani. Abstract settings for tangential boundary stabilization of Navier-Stokes equations by high- and low-gain feedback controllers. *Nonlinear Anal.*, 64(12):2704–2746, 2006.
- [6] Viorel Barbu, Irena Lasiecka, and Roberto Triggiani. Tangential boundary stabilization of Navier-Stokes equations. *Mem. Amer. Math. Soc.*, 181(852):x+128, 2006.
- [7] Viorel Barbu, Irena Lasiecka, and Roberto Triggiani. Local exponential stabilization strategies of the Navier-Stokes equations, $d = 2, 3$, via feedback stabilization of its linearization. In *Control of coupled partial differential equations*, volume 155 of *Internat. Ser. Numer. Math.*, pages 13–46. Birkhäuser, Basel, 2007.
- [8] Alain Bensoussan, Giuseppe Da Prato, Michel C. Delfour, and Sanjoy K. Mitter. *Representation and control of infinite dimensional systems*. Systems & Control: Foundations & Applications Birkhäuser Boston Inc., Boston, MA, second edition, 2007.
- [9] Franck Boyer and Pierre Fabrie. *Éléments d’analyse pour l’étude de quelques modèles d’écoulements de fluides visqueux incompressibles*, volume 52 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 2006.
- [10] P. Constantin and C. Foias. *Navier-Stokes equations*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1988.
- [11] R. Denk, M. Hieber, and J. Prüss. \mathcal{R} -boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Mem. Amer. Math. Soc.*, 166(788):viii+114, 2003.
- [12] E. Fernández-Cara, S. Guerrero, O. Yu. Imanuvilov, and J.-P. Puel. Local exact controllability of the Navier-Stokes system. *J. Math. Pures Appl. (9)*, 83(12):1501–1542, 2004.
- [13] Hiroshi Fujita and Hiroko Morimoto. On fractional powers of the Stokes operator. *Proc. Japan Acad.*, 46:1141–1143, 1970.
- [14] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I. Linearized steady problems*, volume 38 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York, 1994.
- [15] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. II, Nonlinear steady problems*, volume 39 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York, 1994.
- [16] V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations. Theory and algorithms*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986.
- [17] P. Grisvard. Caractérisation de quelques espaces d’interpolation. *Arch. Rational Mech. Anal.*, 25:40–63, 1967.
- [18] Sergio Guerrero. Controllability of systems of Stokes equations with one control force: existence of insensitizing controls. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 24(6):1029–1054, 2007.
- [19] E. Hille and R. S. Phillips. *Functional analysis and semi-groups*. American Mathematical Society Colloquium Publications, vol. 31. American Mathematical Society, Providence, R. I., 1957. rev. ed.

- [20] Oleg Yu. Imanuvilov. Remarks on exact controllability for the Navier-Stokes equations. *ESAIM Control Optim. Calc. Var.*, 6:39–72 (electronic), 2001.
- [21] L. Thevenet J-P. Raymond. Boundary feedback stabilization of the two dimensional Navier-Stokes equations with finite dimensional controllers. preprint.
- [22] Tosio Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.
- [23] Cătălin Lefter. Feedback stabilization of 2D Navier-Stokes equations with Navier slip boundary conditions. *Nonlinear Anal.*, 70(1):553–562, 2009.
- [24] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.
- [25] J.-P. Raymond. Stokes and Navier-Stokes equations with nonhomogeneous boundary conditions. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 24(6):921–951, 2007.
- [26] Jean-Pierre Raymond. Feedback boundary stabilization of the two-dimensional Navier-Stokes equations. *SIAM J. Control Optim.*, 45(3):790–828 (electronic), 2006.
- [27] H. Triebel. *Interpolation theory, function spaces, differential operators*. Johann Ambrosius Barth, Heidelberg, second edition, 1995.
- [28] Marius Tucsnak and George Weiss. *Observation and control for operator semigroups*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2009.
- [29] A. Yagi. Coïncidence entre des espaces d’interpolation et des domaines de puissances fractionnaires d’opérateurs. *C. R. Acad. Sci. Paris Sér. I Math.*, 299(6):173–176, 1984.
- [30] Jerzy Zabczyk. *Mathematical control theory*. Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, 2008. An introduction, Reprint of the 1995 edition.