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VIRIAL IDENTITY AND WEAK DISPERSION FOR THE MAGNETIC DIRAC EQUATION

NABILE BOUSSAID, PIERO D'ANCONA, AND LUCA FANELLI

ABSTRACT. We analyze the dispersive properties of a Dirac system perturbed with a magnetic field. We prove a general virial identity; as applications, we obtain smoothing and endpoint Strichartz estimates which are optimal from the decay point of view. We also prove a Hardy-type inequality for the perturbed Dirac operator.

1. INTRODUCTION

The Dirac equation is one of the fundamental systems of modern physics and mathematics, used to describe a spin 1/2 particle in quantum electrodynamics. Even if the physical interpretation of Dirac fields is not completely unambiguous, the very rich mathematical structure of this system makes it an interesting object of study. We refer to [19] for a thorough treatment of the subject, including the physical validity of the model.

We fix our notations. The Dirac equation is the 4×4 constant coefficient system

$$(1.1) \quad iu_t = m\beta u + \mathcal{D}u, \quad m \in \mathbb{R}$$

where $u : \mathbb{R}_t \times \mathbb{R}_x^3 \rightarrow \mathbb{C}^4$, the operator \mathcal{D} is defined as

$$\mathcal{D} = i^{-1} \sum_{k=1}^3 \alpha_k \partial_k$$

and the 4×4 Dirac matrices can be written

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad k = 1, 2, 3$$

in terms of the Pauli matrices

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The coefficient m is called the mass and we shall distinguish the massless case $m = 0$ from the massive case $m \neq 0$. This distinction is important in relation to the dispersive properties of the equation. Indeed, the commutation rules

$$\alpha_\ell \alpha_k + \alpha_k \alpha_\ell = 2\delta_{kl} I_4$$

imply the identity

$$\mathcal{D}^2 = -\Delta I_4.$$

Thus we have the property

$$(i\partial_t - \mathcal{D})(i\partial_t + \mathcal{D}) = (\Delta - \partial_{tt}^2) I_4$$

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showing the intimate relation between the Dirac and the wave equation. A similar computation in the massive case produces a Klein-Gordon equation with positive mass m^2 :

$$(i\partial_t - \mathcal{D} - m\beta)(i\partial_t + \mathcal{D} + m\beta) = (\Delta - m^2 - \partial_{tt}^2)I_4$$

It is then straightforward to derive dispersive and smoothing properties of the free flows from the corresponding ones for the scalar equations using these identities (see e.g. [8]). Our purpose here is to extend these properties to the case of system perturbed by a magnetostatic potential

$$A = A(x) = (A^1(x), A^2(x), A^3(x)) : \mathbb{R}^3 \rightarrow \mathbb{R}^3.$$

In virtue of the principle of minimal electromagnetic coupling, the influence of the field is introduced in the equation by replacing the standard derivatives with the covariant derivatives

$$\nabla_A := \nabla - iA,$$

so that the operator \mathcal{D} is replaced by

$$(1.2) \quad \mathcal{D}_A = i^{-1} \sum_{k=1}^3 \alpha_k (\partial_k - iA^k)$$

We shall also use the unified notation

$$(1.3) \quad \mathcal{H} = i^{-1} \alpha \cdot \nabla_A + m\beta = \mathcal{D}_A + m\beta$$

for the perturbed operator, which covers both the massive and the massless case. Thus our main goal here is to investigate the dispersive properties of the flow $u = e^{it\mathcal{H}}f$ relative to the Cauchy problem

$$(1.4) \quad iu_t(t, x) + \mathcal{H}u(t, x) = 0, \quad u(0, x) = f(x)$$

where $u(t, x) : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}^4$, $f(x) : \mathbb{R}^3 \rightarrow \mathbb{C}^4$, and $m \in \mathbb{R}$.

It is natural to require that the operator \mathcal{H} be selfadjoint. Several sufficient conditions on the potential A are known; e.g., if the field A is smooth or satisfies

$$|A(x)| \leq \frac{a}{|x|} + b, \quad a < 1, \quad b > 0$$

then \mathcal{H} admits a unique selfadjoint extension. We refer to [19] for a discussion of this problem; here we prefer to make an all-encompassing abstract assumption on the operator:

SELF-ADJOINTNESS ASSUMPTION (A): the operator \mathcal{H} is essentially selfadjoint on $C_c^\infty(\mathbb{R}^n)$, and in addition, for initial data in $C_c^\infty(\mathbb{R}^n)$, the flow $e^{it\mathcal{H}}f$ belongs to $C(\mathbb{R}, H^{3/2})$.

The density condition allows to approximate rough solutions with smoother ones, locally uniformly in time, and is easily verified in concrete cases.

Dispersive, smoothing and Strichartz estimates for a perturbed Dirac equation of the form

$$iu_t = \mathcal{H}_0 u + V(x)u$$

were obtained earlier in [7, 8, 3, 4], for a general potential $V = V^* \in \mathbb{C}^{4 \times 4}$ satisfying suitable smallness and decay conditions. In those works we used a perturbative approach, relying heavily on spectral methods. Here we follow a different approach, based on multiplier methods, with two major advantages. First, we can partially overcome the smallness assumption; and second, the assumptions are more natural from the physical point of view since they are expressed in terms of the magnetic field

$$B = \text{curl } A,$$

which is the physically relevant quantity. Actually, all assumptions are in terms of the quantities

$$B_r = \frac{x}{|x|} \wedge B, \quad \partial_r B = (\partial_r B^1, \partial_r B^2, \partial_r B^3),$$

which are, respectively, the tangential component and the radial derivative of the field B . Moreover, we establish for the first time a virial identity for the perturbed Dirac equation, which has several application not restricted to smoothing properties of the solution. Multiplier methods in relation with weak dispersion properties have a long story, starting from Morawetz [15] for the Klein-Gordon equation and [6], [18], [20] for the Schrödinger equation, and adapted to more general situations in [16], [17]. Potential perturbations for the Schrödinger equation were considered in [1], [2], while the magnetic case was studied in [11]. The perturbed Dirac equation was studied in [7], [8], [3] and [4] using spectral instead of multiplier methods, which are applied for the first time here.

The paper is organized as follows. In the rest of the Introduction we shall describe the main results of the paper, namely a general virial identity and optimal smoothing and Strichartz estimates for the Dirac equation perturbed with a magnetostatic potential. Sections 2, 3 and 4 are devoted to the proofs. In the Appendix we prove a magnetic Hardy inequality (Proposition 1.5) which is elementary but has maybe an independent interest.

1.1. Virial identities. The Dirac operator does not have a definite sign and this is a substantial difficulty for a direct application of multiplier methods. To overcome it we shall resort to the squared Dirac equation

$$(i\partial_t - \mathcal{H})(i\partial_t + \mathcal{H}) = (-\partial_{tt} - \mathcal{H}^2).$$

Thus we are reduced to study a diagonal system of wave (Klein-Gordon) equations of the form

$$(1.5) \quad u_{tt}(t, x) + Lu(t, x) = 0, \quad L = (m^2 - \Delta)I$$

with $u = u(t, x) : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}^4$. Our first result is a formal virial identity for solutions of a general system of wave equations like (1.5), with L being any selfadjoint operator on $L^2(\mathbb{R}^n; \mathbb{C}^k)$. In the following, round brackets

$$(F, G) = \int_{\mathbb{R}^n} F \cdot \bar{G} dx$$

denote the inner product in $L^2(\mathbb{R}^n; \mathbb{C}^k)$, while $[S, T] = ST - TS$ is the commutator of operators.

Theorem 1.1 (Virial identity for the wave equation). *Given a function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$, define the quantity*

$$(1.6) \quad \Theta(t) = (\phi u_t, u_t) + \Re((2\phi L - L\phi)u, u).$$

Then any solution $u(t, x)$ of (1.5) satisfies the formal identities

$$(1.7) \quad \dot{\Theta}(t) = \Re([L, \phi]u, u_t)$$

$$(1.8) \quad \ddot{\Theta}(t) = -\frac{1}{2}([L, [L, \phi]]u, u).$$

We recall that the virial identity for a magnetic wave equation first appeared in [10, 11]; the abstract formulation given here can be applied to more general equations including magnetic Dirac systems. Thus, as an application of (1.7), (1.8) we obtain:

Theorem 1.2 (Virial identity for the Dirac equation). *Assume the operator \mathcal{H} defined by (1.3) satisfies assumption (A) and let $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a real valued function. Then any solution $u(t, x)$ of (1.4) satisfies the formal virial identity, where $DB = [\partial_j B_i]_{i,j=1,3}$ and $S = \frac{i}{4}\alpha \wedge \alpha$ is the spin operator:*

$$(1.9) \quad \begin{aligned} & 2 \int_{\mathbb{R}^3} \nabla_A u D^2 \phi \overline{\nabla_A u} - \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 \Delta^2 \phi + 2\Im \int_{\mathbb{R}^3} u \phi' B_\tau \cdot \overline{\nabla_A u} + 2 \int_{\mathbb{R}^3} [S \cdot (DB \nabla \phi) u] \cdot \overline{u} \\ & = -\frac{d}{dt} \Re \left(\int_{\mathbb{R}^3} u_t (2\nabla \phi \cdot \overline{\nabla_A u} + \overline{u} \Delta \phi) \right). \end{aligned}$$

Remark 1.1. In the following, we will always consider radial multipliers ϕ , in which case the last term at the left-hand side of (1.9) simplifies to

$$S \cdot DB \nabla \phi = \phi' S \cdot \partial_r B.$$

In order to deduce a smoothing estimate from the virial identity, it will be necessary to impose suitable smallness conditions on the components of B appearing in (1.9), for which no natural positivity assumption holds in general.

1.2. Local smoothing for the magnetic Dirac Equation. As a first application of the virial identity, we prove some smoothing estimates for the magnetic Dirac equation. For $f : \mathbb{R}^3 \rightarrow \mathbb{C}$, denote by

$$\|f\|_{L_r^p L^\infty(S_r)} := \left\| \sup_{|x|=r} |f| \right\|_{L_r^p} = \left(\int_0^{+\infty} \left(\sup_{|x|=r} |f| \right)^p dr \right)^{\frac{1}{p}}.$$

Moreover, by $\nabla_A^r u$ and $\nabla_A^\tau u$ we denote, respectively, the radial and tangential components of the covariant gradient $\nabla_A = \nabla - iA(t, x)$:

$$(1.10) \quad \nabla_A^r u = \frac{x}{|x|} \cdot \nabla_A u, \quad \nabla_A^\tau u = \nabla_A u - \frac{x}{|x|} \nabla_A^r u,$$

so that

$$|\nabla_A^r u|^2 + |\nabla_A^\tau u|^2 = |\nabla_A u|^2.$$

We can now state our main result.

Theorem 1.3. *Let \mathcal{H} satisfy the self-adjointness assumption (A). Let $B = \text{curl } A = B_1 + B_2$ with $B_2 \in L^\infty(\mathbb{R}^n)$ and introduce the quantities*

$$C_0 = \||x|^2 B_1\|_{L^\infty(\mathbb{R}^n)}, \quad C_1 = \||x|^{\frac{3}{2}} B_\tau\|_{L_r^2 L^\infty(S_r)}, \quad C_2 = \||x|^2 \partial_r B\|_{L_r^1 L^\infty(S_r)}.$$

We shall assume the smallness conditions

$$(1.11) \quad C_0 < \frac{1}{4}, \quad C_1^2 + 3C_2 + C_1 \sqrt{C_1^2 + 6C_2} \leq 1$$

and that the L^∞ part of B is absent in the massless case:

$$(1.12) \quad m = 0 \quad \implies \quad B_2 \equiv 0.$$

Then for all $f \in L^2$, the following estimate holds:

$$(1.13) \quad \int_{-\infty}^{+\infty} \sup_{R>0} \frac{1}{R} \int_{|x|\leq R} |e^{it\mathcal{H}} f|^2 dx dt \lesssim \|f\|_{L^2}^2.$$

Assume moreover that the second inequality in (1.11) is strict; then for any $f \in D(\mathcal{H})$ the following estimate is true:

$$(1.14) \quad \begin{aligned} & \int_{-\infty}^{+\infty} \sup_{R>0} \frac{1}{R} \int_{|x|\leq R} |\nabla_A e^{it\mathcal{H}} f|^2 dx dt + \|e^{it\mathcal{H}} f\|_{L_x^\infty L_t^2}^2 \\ & + \int_{-\infty}^{+\infty} \int_{\mathbb{R}^3} \frac{|\nabla_A^\tau e^{it\mathcal{H}} f|^2}{|x|} dx dt + \int_{-\infty}^{+\infty} \sup_{R>0} \frac{1}{R^2} \int_{|x|=R} |e^{it\mathcal{H}} f|^2 d\sigma dt \lesssim \|\mathcal{H}f\|_{L^2}^2. \end{aligned}$$

Remark 1.2. Notice that all the assumptions in the smoothing Theorem 1.3 are expressed in terms of the magnetic field B , and consequently the gauge invariance of the result is preserved.

Example 1.4. Explicit examples of magnetic fields satisfying assumption (1.11) are of the following form

$$(1.15) \quad \omega \left(\frac{x}{|x|} \right) \frac{x}{|x|} + \epsilon B(x),$$

where ω is a smooth function on the unit sphere, while ϵ is sufficiently small and $B : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ satisfies

$$|B_r(x)| \leq \frac{1}{|x|^{2-\delta} + |x|^{2+\delta}}, \quad |\partial_r B| \leq \frac{1}{|x|^{3-\delta} + |x|^{3+\delta}},$$

for some $\delta > 0$.

In the proof of Theorem 1.3, we shall use the following Hardy inequality for the magnetic Dirac operator, proved in the Appendix at the end of the paper. Compare with [9] for parallel results.

Proposition 1.5. *Let $B = \text{curl } A = B_1 + B_2$ and assume that*

$$(1.16) \quad C_0 = \| |x|^2 B_1 \|_{L^\infty(\mathbb{R}^n)} < \infty, \quad \| B_2 \|_{L^\infty(\mathbb{R}^n)} < \infty$$

Then, for any $f : \mathbb{R}^3 \rightarrow \mathbb{C}^4$ such that $\mathcal{H}f \in L^2$, and any $\epsilon < 1$, the following inequality holds:

$$(1.17) \quad m^2 \int |f|^2 + \left(\frac{1-\epsilon}{4} - C_0 \right) \int \frac{|f|^2}{|x|^2} + \epsilon \int |\nabla_A f|^2 \leq \left(1 + \frac{\|B_2\|_{L^\infty}}{m^2} \right) \int_{\mathbb{R}^3} |\mathcal{H}f|^2.$$

When $m = 0$, $B_2 \equiv 0$, the right hand side is to be interpreted simply as $\int |\mathcal{H}f|^2 dx$.

1.3. Strichartz Estimates. As a natural application of the local smoothing estimates, we now derive from them the Strichartz estimates for the perturbed Dirac equation. We recall that the solution $u(t, x) = e^{it\mathcal{D}} f$ of the free massless Dirac system with initial value $u(0, x) = f(x)$ satisfies

$$(1.18) \quad \| e^{it\mathcal{D}} f \|_{L^p \dot{H}_q^{\frac{1}{q} - \frac{1}{p} - \frac{1}{2}}} \lesssim \| f \|_{L^2},$$

for all wave admissible (p, q)

$$(1.19) \quad \frac{2}{p} + \frac{2}{q} = \frac{2}{2}, \quad 2 < p \leq \infty, \quad \infty > q \geq 2$$

while in the massive case $m \neq 0$ we have

$$(1.20) \quad \| e^{it(\mathcal{D}+m\beta)} f \|_{L^p \dot{H}_q^{\frac{1}{q} - \frac{1}{p} - \frac{1}{2}}} \lesssim \| f \|_{L^2},$$

for all Schrödinger admissible (p, q)

$$(1.21) \quad \frac{2}{p} + \frac{3}{q} = \frac{3}{2}, \quad 2 \leq p \leq \infty, \quad 6 \geq q \geq 2.$$

For a proof of these estimates see [8]. In the perturbed case we obtain exactly the same results:

Theorem 1.6. *Assume \mathcal{H} and \mathcal{D}_A satisfy (A). Moreover, assume that (1.11), (1.12) hold and that*

$$(1.22) \quad \sum_{j \in \mathbb{Z}} 2^j \sup_{|x| \sim 2^j} |A| < \infty.$$

Then the massless perturbed flow satisfies the Strichartz estimates

$$(1.23) \quad \|e^{it\mathcal{D}^A} f\|_{L^p \dot{H}_q^{\frac{1}{q} - \frac{1}{p} - \frac{1}{2}}} \lesssim \|f\|_{L^2}$$

for all wave admissible couple (p, q) , (in particular, $p \neq 2$), while in the massive case we have, for all Schrödinger admissible couple (p, q) ,

$$(1.24) \quad \|e^{it\mathcal{H}} f\|_{L^p \dot{H}_q^{\frac{1}{q} - \frac{1}{p} - \frac{1}{2}}} \lesssim \|f\|_{L^2} \quad (m \neq 0).$$

2. PROOF OF THE VIRIAL IDENTITIES

Proof of Theorem 1.1. The proof relies on a direct computation. By equation (1.5) we have

$$(2.1) \quad \frac{d}{dt}(\phi u_t, u_t) = -2\Re(\phi Lu, u_t),$$

and

$$(2.2) \quad \frac{d}{dt}\Re((2\phi L - L\phi)u, u) = \Re((\phi L + L\phi)u, u_t),$$

since ϕ and L are symmetric operators. Summing (2.1) and (2.2) we get (1.7). An additional differentiation gives

$$(2.3) \quad \frac{d}{dt}\Re([L, \phi]u, u_t) = \Re([L, \phi]u_t, u_t) - \Re([L, \phi]u, Lu).$$

Since $[L, \phi]$ is anti-symmetric, we have

$$(2.4) \quad \Re([L, \phi]u_t, u_t) = 0$$

and also

$$(2.5) \quad -\Re([L, \phi]u, Lu) = -\frac{1}{2} \{([L, \phi]u, Lu) + (Lu, [L, \phi]u)\} = -\frac{1}{2}([L, [L, \phi]]u, u).$$

Identities (2.3), (2.4) and (2.5) give (1.8). \square

Proof of Theorem 1.2. Let u be a solution to equation (1.4). Using the identity

$$0 = (i\partial_t - \mathcal{H})(i\partial_t + \mathcal{H})u = (-\partial_{tt} - \mathcal{H}^2)u,$$

we see that u solves a Cauchy problem of the form (1.5):

$$(2.6) \quad \begin{cases} u_{tt} + \mathcal{H}^2 u = 0 \\ u(0) = f \\ u_t(0) = i\mathcal{H}f \end{cases} \quad L = \mathcal{H}^2.$$

In order to apply (1.7), (1.8), we need to compute explicitly the commutators appearing in the formulas with the choice $L = \mathcal{H}^2$.

In the following we shall need the spin operator S , defined as the triplet of matrices

$$S = \frac{i}{4}\alpha \wedge \alpha = \frac{i}{4}(\alpha_2\alpha_3 - \alpha_3\alpha_2, \alpha_3\alpha_1 - \alpha_1\alpha_3, \alpha_1\alpha_2 - \alpha_2\alpha_1).$$

We also recall the formula

$$(2.7) \quad (\alpha \cdot F)(\alpha \cdot G) = F \cdot G + 2iS \cdot (F \wedge G)$$

which holds for any matrix-valued vector fields $F = (F^1, F^2, F^3)$, $G = (G^1, G^2, G^3)$, with $F^i, G^i \in \mathcal{M}_{4 \times 4}(\mathbb{C})$ (see [19] for an extensive list of algebraic identities connected to Dirac operators). Thus expanding the square \mathcal{H}^2 we have

$$(2.8) \quad \mathcal{H}^2 = \mathcal{H}_0^2 - \mathcal{H}_0(\alpha \cdot A) - (\alpha \cdot A)\mathcal{H}_0 + (\alpha \cdot A)(\alpha \cdot A),$$

where the unperturbed part is precisely

$$(2.9) \quad \mathcal{H}_0^2 = (m^2 - \Delta)I_4,$$

and I_4 denotes the identity matrix. Using (2.7) we compute

$$(2.10) \quad -\mathcal{H}_0(\alpha \cdot A) - (\alpha \cdot A)\mathcal{H}_0 + (\alpha \cdot A)(\alpha \cdot A) = i(\nabla \cdot A) + i(A \cdot \nabla) + |A|^2 - 2S \cdot (\nabla \wedge A + A \wedge \nabla).$$

Now observe that

$$(2.11) \quad \nabla \wedge A + A \wedge \nabla = \operatorname{curl} A = B$$

$$(2.12) \quad -\Delta + i(\nabla \cdot A) + i(A \cdot \nabla) + |A|^2 = (i\nabla - A)^2 = -\Delta_A.$$

In conclusion, by (2.8), (2.9), (2.10), (2.11), (2.12) we obtain

$$(2.13) \quad \mathcal{H}^2 = (m^2 - \Delta_A)I_4 - 2S \cdot B.$$

Analogously, in the massless case we have

$$(2.14) \quad \mathcal{D}_A^2 = -\Delta_A I_4 - 2S \cdot B.$$

Hence the commutator with ϕ reduces to

$$(2.15) \quad [\mathcal{H}^2, \phi] = [m^2, \phi] - [\Delta_A, \phi] - 2[S \cdot B, \phi] = -[\Delta_A, \phi].$$

Using the Leibnitz rule

$$\nabla_A(fg) = g\nabla_A f + f\nabla g,$$

we arrive at the explicit formula

$$(2.16) \quad [\mathcal{H}^2, \phi] = -[\Delta_A, \phi] = -2\nabla\phi \cdot \nabla_A - (\Delta\phi).$$

Recalling (1.6), (1.7), we obtain

$$(2.17) \quad \dot{\Theta}(t) = -\Re \left(\int_{\mathbb{R}^3} u_t (2\nabla\phi \cdot \overline{\nabla_A u} + \bar{u}\Delta\phi) \right)$$

We now compute the second commutator between \mathcal{H}^2 and ϕ . By (2.13), (2.16) we have

$$(2.18) \quad [\mathcal{H}^2, [\mathcal{H}^2, \phi]] = [\Delta_A, [\Delta_A, \phi]] + 2[S \cdot B, [\Delta_A, \phi]].$$

The term involving the magnetic Laplacian gives

$$(2.19) \quad \begin{aligned} (u, [\Delta_A, [\Delta_A, \phi]]u) &= 4 \int_{\mathbb{R}^n} \nabla_A u D^2 \phi \overline{\nabla_A u} - \int_{\mathbb{R}^n} |u|^2 \Delta^2 \phi \\ &\quad + 4\Im \int_{\mathbb{R}^n} u \phi' B_\tau \cdot \overline{\nabla_A u} \end{aligned}$$

(see formula (2.18) in [11] with $V \equiv 0$). By (2.16), the last term in (2.18) is equal to

$$[S \cdot B, [\Delta_A, \phi]] = 2[S \cdot B, \nabla\phi \cdot \nabla_A] = 2(S \cdot B \nabla\phi \cdot \nabla_A - \nabla\phi \cdot \nabla_A S \cdot B).$$

Both ϕ and the components of the field B are scalars; moreover, we have

$$[B, \nabla_A] = -DB,$$

where DB denotes the (matrix) gradient of the field B , and in conclusion

$$(2.20) \quad 2[S \cdot B, [\Delta_A, \phi]] = -4S \cdot (DB\nabla\phi).$$

Finally, identity (1.9) follows from (1.8), (2.17), (2.18), (2.19) and (2.20). \square

3. THE SMOOTHING ESTIMATES

The formal computations leading to the virial identity (1.9) make sense for sufficiently smooth solutions $u \in C(\mathbb{R}, H^{3/2})$ and the choice of multiplier ϕ we make below. Thanks to the density assumption (A), if we approximate data $f \in L^2$ (resp. $D(\mathcal{H})$) with $f_j \in C_c^\infty$, the corresponding solutions $u_j = e^{it\mathcal{H}}f_j$ will converge to the solution $u = e^{it\mathcal{H}}f$ in $C([-T, T]; L^2)$ (resp. $C([-T, T]; D(\mathcal{H}))$) for all $T > 0$.

We shall apply identity (1.9) to the solution $u = e^{it\mathcal{H}}f$ of the problem

$$(3.1) \quad \begin{cases} iu_t = \mathcal{H}u \\ u(0) = f \end{cases}$$

with an appropriate multiplier function ϕ .

3.1. Choice of the multiplier. Writing $r = |x|$, we define ϕ as follows (see [11])

$$\phi_0(x) = \int_0^{r=|x|} \phi'_0(s) ds,$$

where

$$\phi'_0 = \phi'_0(r) = \begin{cases} M + \frac{1}{3}r, & r \leq 1 \\ M + \frac{1}{2} - \frac{1}{6r^2}, & r > 1, \end{cases}$$

and M is a positive constant we will choose later. We have

$$\phi''_0(r) = \begin{cases} \frac{1}{3}, & r \leq 1 \\ \frac{1}{3r^3}, & r > 1 \end{cases}$$

while the bilaplacian is given by

$$\Delta^2 \phi_0(r) = -4\pi\delta_{x=0} - \delta_{|x|=1}.$$

Moreover, for any $R > 0$ we define

$$\phi_R(r) = R\phi_0\left(\frac{r}{R}\right),$$

so that by rescaling we have

$$(3.2) \quad \phi'_R(r) = \begin{cases} M + \frac{r}{3R}, & r \leq R \\ M + \frac{1}{2} - \frac{R^2}{6r^2}, & r > R \end{cases}$$

$$(3.3) \quad \phi''_R(r) = \begin{cases} \frac{1}{3R}, & r \leq R \\ \frac{1}{R} \cdot \frac{R^3}{3r^3}, & r > R \end{cases}$$

$$(3.4) \quad \Delta\phi_R(r) = \begin{cases} \frac{1}{R} + \frac{2M}{r}, & r \leq R \\ \frac{1+2M}{r}, & r > R \end{cases}$$

$$(3.5) \quad \Delta^2\phi_R(r) = -4\pi\delta_{x=0} - \frac{1}{R^2}\delta_{|x|=R}.$$

We notice that $\phi'_R, \phi''_R, \Delta\phi_R \geq 0$ and moreover

$$(3.6) \quad \sup_{r \geq 0} \phi'_R(r) \leq M + \frac{1}{2} \quad \sup_{r \geq 0} \phi''_R(r) \leq \frac{1}{3R}, \quad \Delta\phi_R(r) \leq \frac{1+2M}{r}.$$

3.2. Estimate of the RHS in (1.9). Consider the expression

$$\int_{\mathbb{R}^3} u_t(2\nabla\phi \cdot \overline{\nabla_A u} + u\Delta\phi) = (u_t, 2\nabla\phi \cdot \nabla_A u + \bar{u}\Delta\phi)_{L^2}$$

appearing at the right hand side in (1.9). Using the equation, we can replace u_t with

$$u_t = -i\mathcal{H}u = -im\beta u - i\mathcal{D}_A u.$$

By the selfadjointness of β , it is easy to check that

$$\Re[-im(\beta u, 2\nabla\phi \cdot \nabla_A u) - im(\beta u, \Delta\phi u)] = 0,$$

so that

$$\Re(u_t, 2\nabla\phi \cdot \nabla_A u + u\Delta\phi) = 2\Im(\mathcal{D}_A u, \nabla\phi \cdot \nabla_A u) + \Im(\mathcal{D}_A u, \Delta\phi u)$$

and by Young we obtain

$$(3.7) \quad \left| \Re \left(\int_{\mathbb{R}^3} u_t(2\nabla\phi \cdot \overline{\nabla_A u} + \bar{u}\Delta\phi) \right) \right| \leq \frac{3}{2} \|\mathcal{D}_A u\|_{L^2}^2 + \|\nabla\phi \cdot \nabla_A u\|_{L^2}^2 + \frac{1}{2} \|u\Delta\phi\|_{L^2}^2.$$

Recalling (3.6), and using Proposition 1.5 with the choice $\epsilon = 1 - 4C_0$, which is positive in virtue of the assumption $C_0 < 4^{-1}$, we have

$$(3.8) \quad \|\nabla\phi \cdot \nabla_A u\|_{L^2}^2 \leq \frac{1}{1 - 4C_0} \left(M + \frac{1}{2} \right) \|\mathcal{D}_A u\|_{L^2}^2.$$

The third term in (3.7), can be estimated using (3.6) and again the Hardy inequality (1.17):

$$(3.9) \quad \|u\Delta\phi\|_{L^2}^2 \leq \frac{4}{1 - 4C_0} (1 + 2M) \|\mathcal{D}_A u\|_{L^2}^2.$$

Summing up, by (3.7), (3.8) and (3.9) we conclude that

$$(3.10) \quad \left| \Re \left(\int_{\mathbb{R}^3} u_t(2\nabla\phi \cdot \overline{\nabla_A u} + \bar{u}\Delta\phi) \right) \right| \lesssim \|\mathcal{D}_A u\|_{L^2}^2,$$

for any $t \in \mathbb{R}$.

3.3. Estimate of the LHS in (1.9). We shall use the elementary identity

$$(3.11) \quad \nabla_A u D^2\phi \overline{\nabla_A u} = \frac{\phi'(r)}{r} |\nabla_A^\tau u|^2 + \phi''(r) |\nabla_A^r u|^2.$$

By (3.11), (3.2), (3.3) and (3.5), for the first two terms at the LHS of (1.9) we have

$$(3.12) \quad \begin{aligned} & 2 \int_{\mathbb{R}^3} \nabla_A u D^2\phi \overline{\nabla_A u} - \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 \Delta^2\phi \\ & \geq \frac{2}{3R} \int_{|x| \leq R} |\nabla_A u|^2 dx + 2M \int_{\mathbb{R}^3} \frac{|\nabla_A^\tau u|^2}{|x|} dx + 2\pi |u(t, 0)|^2 + \frac{1}{2R^2} \int_{|x|=R} |u|^2 d\sigma(x), \end{aligned}$$

for any $R > 0$, where $d\sigma$ denotes the surface measure on the sphere of radius R . For the perturbative term involving B_τ in (1.9), by the Hölder inequality and (3.6) we obtain

$$(3.13) \quad \begin{aligned} & 2\Im \int_{\mathbb{R}^3} u \phi'_R B_\tau \cdot \overline{\nabla_A u} \\ & \geq -(2M + 1) \left(\sup_{R>0} \frac{1}{R^2} \int_{|x|=R} |u|^2 d\sigma(x) \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \frac{|\nabla_A^\tau u|^2}{|x|} dx \right)^{\frac{1}{2}} \| |x|^{\frac{3}{2}} B_\tau \|_{L^2_r L^\infty(S_r)}. \end{aligned}$$

For the remaining term in (1.9), observe that the operator norm of the components of $S = (S^1, S^2, S^3)$ is

$$\|S^k\|_{C^4 \rightarrow C^4} = \frac{1}{2};$$

hence we can write

$$(3.14) \quad 2 \int_{\mathbb{R}^3} |u|^2 S \cdot [\nabla \phi_R DB] \\ \geq -3 \left(M + \frac{1}{2} \right) \left(\sup_{R>0} \frac{1}{R^2} \int_{|x|=R} |u|^2 d\sigma_R(x) \right) \| |x|^2 \partial_r B \|_{L^1_r L^\infty(S_r)},$$

since ϕ_R is radial. Now we introduce the norms

$$\|u\|_X^2 := \sup_{R>0} \frac{1}{R} \int_{|x|\leq R} |u|^2 dx \\ \|u\|_Y^2 := \sup_{R>0} \frac{1}{R^2} \int_{|x|=R} |u|^2 d\sigma_R(x).$$

Taking the supremum over $R > 0$ in (3.12) and summing with (3.13), (3.14), we obtain

$$(3.15) \quad 2 \int_{\mathbb{R}^3} \nabla_A u D^2 \phi_R \overline{\nabla_A u} - \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 \Delta^2 \phi_R \\ + 2\Im \int_{\mathbb{R}^3} u \phi'_R B_\tau \cdot \overline{\nabla_A u} + 2 \int_{\mathbb{R}^3} |u|^2 S \cdot [\nabla \phi_R DB] \\ \geq \frac{2}{3} \|\nabla_A u\|_X^2 + \left(\frac{1}{2} - 3 \left(M + \frac{1}{2} \right) \| |x|^2 \partial_r B \|_{L^1_r L^\infty(S_r)} \right) \|u\|_Y^2 \\ - (2M+1) \| |x|^{-\frac{1}{2}} \nabla_A^\tau u \|_{L^2} \| |x|^{\frac{3}{2}} B_\tau \|_{L^2_r L^\infty(S_r)} \|u\|_Y \\ + 2M \| |x|^{-\frac{1}{2}} \nabla_A^\tau u \|_{L^2}^2 + 2\pi |u(t, 0)|^2.$$

In order to deduce (1.13), (1.14), we need to ensure the positivity of the right-hand side of (3.15). Define p, q as

$$p = \| |x|^{-\frac{1}{2}} \nabla_A^\tau u \|_{L^2}, \quad q = \|u\|_Y,$$

while C_1, C_2 are defined in the statement of the Theorem. Then we are led to study the inequality

$$(3.16) \quad 2Mp^2 + \left(\frac{1}{2} - 3 \left(M + \frac{1}{2} \right) C_2 \right) q^2 - (2M+1)C_1 pq \geq 0$$

and it is immediate to check that (3.16) holds for all $p, q \geq 0$ and $M = C_1 / (2\sqrt{C_1^2 + 6C_2})$, provided C_1 and C_2 satisfy (1.11). Thus, dropping the corresponding nonnegative terms, we arrive at the estimate

$$(3.17) \quad 2 \int_{\mathbb{R}^3} \nabla_A u D^2 \phi_R \overline{\nabla_A u} - \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 \Delta^2 \phi_R \\ + 2\Im \int_{\mathbb{R}^3} u \phi'_R B_\tau \cdot \overline{\nabla_A u} + 2 \int_{\mathbb{R}^3} |u|^2 S \cdot [\nabla \phi_R DB] \\ \geq \frac{2}{3} \|\nabla_A u\|_X^2 \geq \frac{2}{3} \|\mathcal{D}_A u\|_X^2$$

where in the last step we used the pointwise inequality $|\mathcal{D}_A u| \leq |\nabla_A u|$. We now integrate in time the virial identity on $[-T, T]$, and using (3.17) and (3.10) we get

$$(3.18) \quad \int_{-T}^T \|\mathcal{D}_A u\|_X^2 dt \lesssim \|\mathcal{D}_A u(T)\|_{L^2}^2 + \|\mathcal{D}_A u(-T)\|_{L^2}^2.$$

Notice that all the above computations do not depend on the sign of the mass m .

Let us consider the range of D_A , from Proposition 1.5, we have that for $C_0 < 1/4$, $0 \notin \ker(D_A)$ so $\text{Ran}(D_A)$ is either L^2 if 0 is not in the essential spectrum of D_A or it is dense in L^2 . Now fix an arbitrary $g \in \text{Ran}(D_A)$, there exists $f \in D(D_A) = D(\mathcal{H})$ with $\mathcal{D}_A f = g$; we consider then the solution $u(t, x)$ of the problem

$$iu_t = -m\beta u + \mathcal{D}_A u, \quad u(0, x) = f$$

with opposite mass, and notice that u satisfies estimate (3.18). If we apply to this equation the operator \mathcal{D}_A we obtain, by the anticommutation rules,

$$i(\mathcal{D}_A u)_t = \beta m(\mathcal{D}_A u) + \mathcal{D}_A(\mathcal{D}_A u), \quad (\mathcal{D}_A u(0, x)) = \mathcal{D}_A f$$

or, in other words, the function $v = \mathcal{D}_A u$ solves the problem

$$iv_t = \mathcal{H}v, \quad v(0, x) = g \implies v = e^{it\mathcal{H}}g.$$

Hence (3.18) can be written

$$(3.19) \quad \int_{-T}^T \|v\|_X^2 dt \lesssim \|v(T)\|_{L^2}^2 + \|v(-T)\|_{L^2}^2 = 2\|g\|_{L^2}^2$$

and letting $T \rightarrow \infty$ we conclude that

$$\int_{-\infty}^{\infty} \|e^{it\mathcal{H}}g\|_X^2 dt \lesssim \|g\|_{L^2}^2$$

which is exactly (1.13) for $g \in \text{Ran}(D_A)$, which is dense in L^2 . So density arguments provide (1.13).

In order to prove (1.14), let us come back to (3.15). If we take the strict inequality in (3.16), which is equivalent to assume the strict inequality in (1.11), we have that

$$(3.20) \quad \begin{aligned} & 2 \int_{\mathbb{R}^3} \nabla_A u D^2 \phi_R \overline{\nabla_A u} - \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 \Delta^2 \phi_R \\ & + 2\Im \int_{\mathbb{R}^3} u \phi'_R B_\tau \cdot \overline{\nabla_A u} + 2 \int_{\mathbb{R}^3} |u|^2 S \cdot [\nabla \phi_R DB] \\ & \geq \frac{2}{3} \|\nabla_A u\|_X^2 + \epsilon \|u\|_Y^2 + \epsilon \| |x|^{-\frac{1}{2}} \nabla_A^\tau u \|_{L^2}^2 + 2\pi |u(t, 0)|^2. \end{aligned}$$

By this inequality and (3.10) we obtain, after an integration in time on $[-T, T]$,

$$\begin{aligned} \int_{-T}^T \left[\frac{2}{3} \|\nabla_A u\|_X^2 + \epsilon \|u\|_Y^2 + \epsilon \| |x|^{-\frac{1}{2}} \nabla_A^\tau u \|_{L^2}^2 + 2\pi |u(t, 0)|^2 \right] dt & \leq \\ & \leq \|\mathcal{D}_A u(T)\|_{L^2}^2 + \|\mathcal{D}_A u(-T)\|_{L^2}^2. \end{aligned}$$

The right hand side can be estimated using the obvious inequality

$$\|\mathcal{D}_A f\|_{L^2}^2 \leq \|\mathcal{H}f\|_{L^2}^2,$$

and the conservation of $\|\mathcal{H}u(t)\|_{L^2}$. In order to complete the proof of (1.14), it only remains to remark that the term $\|u\|_{L_x^\infty L_t^2}$ in the inequality is obtained by the term $\int_{-T}^T |u(t, 0)|^2 dt$ by translating in space the multiplier ϕ . Letting $T \rightarrow \infty$ we conclude the proof.

4. PROOF OF STRICHARTZ ESTIMATES

We pass to the proof of Theorem 1.6 for the massless case $\mathcal{H} = \mathcal{D}_A$. We rewrite $u = e^{it\mathcal{D}_A} f$ using the Duhamel formula:

$$(4.1) \quad u(t) = e^{it\mathcal{D}} f + \int_0^t e^{i(t-s)\mathcal{D}} \alpha \cdot Au(s) ds.$$

The term $e^{it\mathcal{D}}f$ is estimated directly via (1.18). For the Duhamel term, we follow the Keel-Tao method (see [14], [12]): by the Christ-Kiselev Lemma in [5], it is sufficient to estimate the untruncated integral

$$\int e^{i(t-s)\mathcal{D}}\alpha \cdot Au(s)ds = e^{it\mathcal{D}} \int e^{-is\mathcal{D}}\alpha \cdot Au(s)ds$$

since we are only interested in the non-endpoint case. Again by (1.18) we obtain

$$(4.2) \quad \left\| e^{it\mathcal{D}} \int_0^t e^{-is\mathcal{D}}\alpha \cdot Au(s)ds \right\|_{L^p \dot{H}_q^{\frac{1}{q} - \frac{1}{p} - \frac{1}{2}}} \lesssim \left\| \int e^{-is\mathcal{D}}\alpha \cdot Au(s)ds \right\|_{L^2}.$$

Now we use the dual form of the smoothing estimate (1.13), i.e.

$$(4.3) \quad \left\| \int e^{-is\mathcal{D}}\alpha \cdot Au(s)ds \right\|_{L^2} \leq \sum_{j \in \mathbb{Z}} 2^{\frac{j}{2}} \| |A| \cdot |u| \|_{L_t^2 L^2(|x| \sim 2^j)}.$$

Hence, by Hölder inequality, assumption (1.22) and estimate (1.13) we continue the estimate as follows

$$(4.4) \quad \sum_{j \in \mathbb{Z}} 2^{\frac{j}{2}} \| |A| \cdot |u| \|_{L_t^2 L^2(|x| \sim 2^j)} \leq \sum_{j \in \mathbb{Z}} 2^j \sup_{|x| \sim 2^j} |A| \cdot \sup_{j \in \mathbb{Z}} 2^{-\frac{j}{2}} \|u\|_{L_t^2 L^2(|x| \sim 2^j)} \lesssim \|f\|_{L^2},$$

and this concludes the proof of (1.23).

We pass now to the proof of (1.24) in the massive case. By mixing free Strichartz estimates with the dual of (1.13), for the Duhamel term we obtain

$$(4.5) \quad \left\| e^{it\mathcal{H}_0} \int_0^t e^{-is\mathcal{H}_0}\alpha \cdot Au(s)ds \right\|_{L^p \dot{H}_q^{\frac{1}{q} - \frac{1}{p} - \frac{1}{2}}} \lesssim \sum_{j \in \mathbb{Z}} 2^{\frac{j}{2}} \| |A| \cdot |u| \|_{L_t^2 L^2(|x| \sim 2^j)},$$

for any Schrödinger admissible couple (p, q) , with $p \geq 2$. The endpoint here can be recovered by using exactly the same technique as in [13], Lemma 3. The rest of the proof is completely analogous to the massless case.

APPENDIX A. MAGNETIC HARDY INEQUALITY FOR DIRAC

We now prove Proposition 1.5. Denote by (\cdot, \cdot) the inner product in $L^2(\mathbb{R}^3, \mathbb{C}^4)$, $\|\cdot\|$ the associated norm, and observe that, due to the formula (2.7), we have the relation

$$\begin{aligned} \|\mathcal{D}_A f\|^2 &= (\alpha \cdot \nabla_A f, \alpha \cdot \nabla_A f) = -((\alpha \cdot \nabla_A)(\alpha \cdot \nabla_A)f, f) \\ &= -(\nabla_A^2 f, f) - 2i(S \cdot (\nabla_A \wedge \nabla_A)f, f), \end{aligned}$$

where $S = \frac{i}{4}\alpha \wedge \alpha$ is the spin operator. Writing for brevity $\partial_j^A = \partial_j - iA^j$, we can compute explicitly

$$\nabla_A \wedge \nabla_A = ([\partial_2^A, \partial_3^A], [\partial_3^A, \partial_1^A], [\partial_1^A, \partial_2^A]) = iB,$$

where $B = \text{curl}A$. Hence by the previous relation we obtain

$$0 \leq \|\mathcal{D}_A f\|^2 = \|\nabla_A f\|^2 + 2(S \cdot Bf, f).$$

Notice that S is a triple of matrices of norm $\leq 1/2$, hence we can write

$$|2(S \cdot Bf, f)| \leq C_0 \| |x|^{-1} f \|^2$$

and this implies

$$(A.1) \quad \|\mathcal{D}_A f\|^2 \geq \|\nabla_A f\|^2 - C_0 \left\| \frac{f}{|x|} \right\|^2 - \|B_2\|_{L^\infty} \|f\|^2.$$

Now we recall the magnetic Hardy inequality

$$(A.2) \quad \frac{1}{4} \int \frac{|f|^2}{|x|^2} dx \leq \int |\nabla_A f|^2 dx,$$

which is proved in [11]. To complete the proof of (1.17), it is sufficient to write

$$\|\mathcal{H}f\|_{L^2}^2 = (\mathcal{H}^2 f, f) = m^2 \|f\|^2 + \|\mathcal{D}_A f\|^2$$

and use the preceding estimates.

The ϵ inequality is obtained using

$$(1 - \epsilon) \frac{1}{4} \int \frac{|f|^2}{|x|^2} dx + \epsilon \int |\nabla_A f|^2 dx \leq \int |\nabla_A f|^2 dx.$$

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