HAL open science

Bivariate Cox model and copulas

Mohamed Achibi, Michel Broniatowski

▶ To cite this version:

Mohamed Achibi, Michel Broniatowski. Bivariate Cox model and copulas. 2009. hal-00430262v1

HAL Id: hal-00430262

https://hal.science/hal-00430262v1

Preprint submitted on 6 Nov 2009 (v1), last revised 29 Jun 2010 (v5)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

BIVARIATE COX MODEL AND COPULAS

MOHAMED ACHIBI AND MICHEL BRONIATOWSKI

Abstract. This paper introduces a new class of Cox models for dependent bivariate data. The impact of the covariate on the dependence of the variables is captured through the modification of their copula. Various classes of well known copulas are stable under the model (archimedean type and extreme value copulas), meaning that the role of the covariate acts in a simple and explicit way on the copula in the class; specific parametric classes are considered as well as approximation of Positive Quadrant Dependences through extreme value copulas which are stable under the action of the covariate.

1. Some model of dependence for duration data

Suppose each individual in a homogeneous population is subject to two failure times X and Y which are both observed. Assume also that X and Y are both absolutely continuous (a.c) non negative random variables (r.v's) with joint cumulative distribution function (cdf) $H(x,y) := \mathbb{P}(X \leq x, Y \leq y)$ and probability density function (pdf) $h(x,y)$. The corresponding survival distribution function (sdf) is $\overline{H}(x,y) := \mathbb{P}(X > x, Y > y)$, and the margins of H will be denoted by F and G, from which $\overline{H}(x,y) = 1 - F(x) - G(y) + H(x,y)$. The hazard rate of X, also known as the instantaneous or age specific failure rate is

$$\lambda_X(x) \equiv \lim_{dx \to 0} \frac{\mathbb{P}(x < X \leq x + dx \,|X > x)}{dx}$$

which may conveniently be written as

$$\lambda_X(x) = -\frac{d \log \overline{F}(x)}{dx}. \tag{1.1}$$

Regression models aim at modeling dependence upon explanatory variables. In the proportional hazard model the cause specific hazard functions satisfy

$$\begin{cases} \lambda_X^z(x) = \lambda_X^0(x)\Phi(z) \\ \lambda_Y^z(y) = \lambda_Y^0(y)\Psi(z) \end{cases} \tag{1.2}$$

Often the positive functions $\Phi(z)$ and $\Psi(z)$ are assumed to be parametric functions, with the standard $\Phi(z) = \exp(\alpha' z)$ and $\Psi(z) = \exp(\beta' z)$, $\alpha, \beta \in \mathbb{R}^d$. Investigators are rarely interested only in marginal behaviors as described in (1.2). This model ignores the dependence between X and Y and can only be of little interest for practical applications. In this paper we will consider models for dependence under explanatory variable. Let

$$\lambda_{Y|X=x}(y) \equiv \lim_{dy \to 0} \frac{\mathbb{P}(y < Y \leq y + dy \,|X = x, Y > y)}{dy} \tag{1.3}$$

and

$$\lambda_{Y|X>x}(y) \equiv \lim_{dy \to 0} \frac{\mathbb{P}(y < Y \leq y + dy \,|X > x, Y > y)}{dy}. \tag{1.4}$$

Date: November 05, 2009.

2000 *Mathematics Subject Classification.* 62H05;62N05.

Key words and phrases. Cox models, positive quadrant dependence, archimedean copula, extreme value copulas, asymmetric logistic copula, frailty models.

Modeling the multivariate dependency through (1.3) and (1.4) leads to different types of models, namely

$$\left\{ \begin{array}{c} \lambda_X^z(x) = \lambda_X^0(x)\Phi(z) \\ \lambda_{Y|X=x}^z(y) = \lambda_{Y|X=x}^0(y)\Psi(z) \end{array} \right. \tag{1.5}$$

or

$$\text{(M1)} \left\{ \begin{array}{c} \lambda_X^z(x) = \lambda_X^0(x)\Phi(z) \\ \lambda_{Y|X>x}^z(y) = \lambda_{Y|X>x}^0(y)\Psi(z) \end{array} \right. \tag{1.6}$$

Direct approaches based on regression type models cannot satisfy our purpose. Indeed consider for example a model defined through

$$\left\{ \begin{array}{l} X = r(z, U) \\ Y = s(z, V) \end{array} \right.$$

with $r(z, .)$ and $s(z, .)$ strictly increasing for all z. Then following Nelsen (2006), Theorem 2.4.3, (X, Y) has the same copula as (U, V) for all z, which implies that the covariate z plays no role in the dependency of X and Y. This fact motivates the present approach. We are interested mainly in positive dependency between the two variables. Both models (1.5) and (1.6) characterize the property that the failure of one component puts an extra load on the other component, as for example in studies involving a two organ system. Are such models mathematically valid ? That is, is there a sdf $\overline{H}^z$ for the r.v. (X, Y) and a baseline sdf $\overline{H}^0$ which are compatible with them ? Ignoring technicalities, this is indeed always the case for Model (1.5). Model (1.6) is valid under certain restrictions on the baseline hazard, as will be seen later. Model (1.5) has been widely studied, e.g. DeMasi et al. (1998). Model (1.6) has been introduce in Depire (2008). The usual paradigm in regression analysis is conditioning upon the observed value of the variable X. Note however that in the present setting, the covariate is in fact z. The main difficulty with the first model is that marginally it does not satisfy (1.2), the Cox paradigm, which trivially holds true for the model (1.6) with the setting $x = 0$. For statistical estimation both models have respective interest. The likelihood function in the first one can easily be written, and hence the parameters can be estimated by partial likelihood maximization. The second model allows a straightforward estimation of the parameters, which are defined independently by the dependence structure induced by the model, setting $x = 0$ in the second equation, and applying standard univariate estimation for the parameters of the functions Φ and Ψ. In this model it will be shown that the dependence features are captured through the copula of the bivariate baseline. Clearly model (1.6) leads to an easy description of the sdf of the r.v. (X, Y), while (1.5) is more adequate for handling the properties of the pdf. This difference motivates our interest in model (1.6). Since the sdf can easily be written as a function of the baseline and the covariate, it represents a natural model for the regression of basic dependence indices on the covariate. Let us now show the main results which we present in connection with model (1.6):

(1) The TP_2 class of sdf's is a subclass of the Positive Quadrant Dependence (PQD) sdf class and is stable under the model which is properly defined when the hazard baseline $\overline{H}^0$ is TP_2. This class appears quite naturally as the one under which the model is properly defined, and it is appropriate for the description of positive dependence between its margins.
(2) Since the TP_2 property of a multivariate sdf refers only to its copula, model (1.6) describes the changes of the baseline copula induced by the covariate. Also this implies that the model is valid independently of marginal distributions. Only the structure of dependence is involved in the domain of validity of the model.

(3) Two main classes of copulas are stable under the model namely: when the baseline bivariate copula is in such a class, so is the copula for all value of the covariate z. The class of extreme values copulas (evc) enjoys this property. The class of extended archimedean copulas is also stable under the model. This class results as a special by-product of a technique intended to produce asymmetric copulas due to Genest et al. (see Frees and Valdez, 2007). The so-called class of logistic asymmetric copulas (see Tawn, 1988), which is a simple extension of the Gumbel family of copulas, enjoys an important role in the present model. It is stable under the model and admits a simple parametrization. The covariate z acts in a simple adaptive way when the value of the covariate is changed. It is the only bivariate distribution in the class of frailty models which enjoys such properties in the model.

(4) We can show that to some extent, any PQD copula can be well approximated by an extreme value copula, which legitimates the restriction of the model to such class, which is also stable under the model. Inside this class the covariate z acts in an explicit and easy way.

This paper is organized as follows. In Section 2 we briefly recall the necessary background from bivariate dependence. Section 3 describes the model. In Section 4 we focus on the asymmetric Gumbel class of copulas, which is the natural parametric setting of our model; we also provide some connection with bivariate frailty models. At the opposite in Section 5 we show that general PQD dependence can be handled through the model, making use of some approximation results pertaining to extreme value copulas. All proofs are deferred to the Appendix.

2. Some useful facts in bivariate dependence

Let X and Y be two random variables (r.v) with joint sdf $\overline{H}$, with margins $\overline{F}$ and $\overline{G}$. All dependence properties of X and Y are captured through the *survival copula* C which is a cdf defined on $[0,1]\times[0,1]$ through

$$C(u,v) \;=\; \overline{H}(\overline{F}^{\leftarrow}(u),\overline{G}^{\leftarrow}(v))$$

where u and v belong to $[0,1]$ and where $\overline{F}^{\leftarrow}(t) := \sup\{x : \overline{F}(x)\geq t\}$. It is easily checked that C is indeed a copula. The definition of a copula is given in Nelsen (2006), definition 2.2.2. We will make use of the following definition and notation.

Definition 1. *A bivariate cdf H is min-infinitely divisible (min-id) if for all positive γ, $\overline{H}^{\gamma}$ is a sdf.*

Assume that H is min-id and let $\mathbb{V}=(X,Y)$ be a random vector with sdf $\overline{H}$. Then for all n in $\mathbb{N}$, $\overline{H}^{1/n}$ is a sdf. Further let (X_i,Y_i), $i=1,\ldots,n$, be n copies i.i.d with sdf $\overline{H}^{1/n}$. It holds

$$\mathbb{V}\stackrel{d}{=}(\min_i\; X_i,\min_i\; Y_i).$$

Definition 2. *X and Y are positively quadrant dependent (PQD) iff, for all (x,y) in $\mathbb{R}^2, \mathbb{P}(X>x,Y>y)\geq\mathbb{P}(X>x)\mathbb{P}(Y>y)$; in this case we also say that $\overline{H}$ is PQD.*

Definition 3. *A mapping ϕ from $\overline{\mathbb{R}}^2$ onto $\mathbb{R}$ is totally positive of order 2 (TP_2) if $\phi(x,y)\geq 0$ for all (x,y) in $\overline{\mathbb{R}}^2$ and $\begin{vmatrix}\phi(x_1,y_1) & \phi(x_1,y_2)\\ \phi(x_2,y_1) & \phi(x_2,y_2)\end{vmatrix} = \phi(x_1,y_1)\phi(x_2,y_2)-\phi(x_1,y_2)\phi(x_2,y_1)\geq 0$, for all $x_1<x_2$ and $y_1<y_2$.*

Remark 1. *When ϕ is $\mathcal{C}^2$, then ϕ is TP_2 iff*

$$\frac{\partial \phi}{\partial x}(x,y)\frac{\partial \phi}{\partial y}(x,y) \leq \frac{\partial^2 \phi}{\partial x \partial y}(x,y)\phi(x,y) \tag{2.7}$$

The proof is given in Resnick (1987), p.254.

We also recall the following results:

Theorem 1 (Joe (1997), Theorem 2.3)**.** *If $\overline{H}$ is a TP_2 sdf, then $\overline{H}$is PQD*

Theorem 2 (Joe (1997), Theorem 2.6)**.** *Let H be a cdf, then H is min-id iff $\overline{H}$ is TP_2*

The relation between properties of the s.d.f's and their copulas is captured in the following result:

Lemma 1. *Let $\overline{H}$ be a sdf with copula C. Then $\overline{H}$ is TP_2 iff C is TP_2.*

Proof. By Remark 1, $\overline{H}$ $TP_2 \Rightarrow \frac{\partial \overline{H}}{\partial x}(x,y)\frac{\partial \overline{H}}{\partial y}(x,y) \leq \frac{\partial^2 \overline{H}}{\partial x \partial y}(x,y) \times \overline{H}(x,y)$. Furthermore, $\overline{H}(x,y) = C(\overline{F}(x), \overline{G}(y))$. Therefore,

$$\begin{aligned}
\frac{\partial \overline{H}}{\partial x}(x,y) &= -\frac{\partial C}{\partial u}(u, \overline{G}(y))\Big|_{u=\overline{F}(x)} \times f(x) \\
\frac{\partial \overline{H}}{\partial y}(x,y) &= -\frac{\partial C}{\partial v}(\overline{F}(x), v)\Big|_{v=\overline{G}(y)} \times g(y) \\
\frac{\partial^2 \overline{H}}{\partial x \partial y}(x,y) &= \frac{\partial^2 C}{\partial u \partial v}(u,v)\Big|_{\substack{u=\overline{F}(x)\\ v=\overline{G}(y)}} \times f(x) \times g(y)
\end{aligned}$$

Hence,

$$\frac{\partial C}{\partial u}(u, \overline{G}(y))\Big|_{u=\overline{F}(x)} \times \frac{\partial C}{\partial v}(\overline{F}(x), v)\Big|_{v=\overline{G}(y)} \leq \frac{\partial^2 C}{\partial u \partial v}(u,v)\Big|_{\substack{u=\overline{F}(x)\\ v=\overline{G}(y)}} \times C(\overline{F}(x), \overline{G}(y))$$

■

Definition 4 (Archimedean copula)**.** *An Archimedean copula is a function C from $[0,1]^2$ to $[0,1]$ given by $C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v))$, where φ is a continuous strictly decreasing convex function from $[0,1]$ to $[0,\infty]$ such that $\varphi(1) = 0$, and where $\varphi^{[-1]}$ denotes the "pseudo-inverse" of φ:*

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t) & \text{for } t \text{ in } [0, \varphi(0)] \\ 0 & \text{for } t \geq \varphi(0) \end{cases}$$

When $\varphi(0) = \infty$, φ is said to be strict and $\varphi^{[-1]} \equiv \varphi^{-1}$.

The class of so called *extreme value copulas* (evc) is important in this model, although not related here with the theory of bivariate extremes; therefore we define an *extreme value copula* through the basic Pickands representation, without further reference to the theory of bivariate extremes.

Proposition 1 (The Pickands Theorem)**.** *C is an extreme value copula iff there exists a convex function A defined on $[0,1]$, which satisfies $A(0) = A(1) = 1$ and $\max(t, 1-t) \leq A(t) \leq 1$ such that*

$$C(u,v) = \exp\Big[\log(uv) A\Big(\frac{\log v}{\log uv}\Big)\Big] \tag{2.8}$$

The function A is referred to as the *dependence function* or *Pickands* function of the copula C.

3. Introducing covariates in dependence models

3.1. **Description of the model.** Not all baseline survival d.f's $\overline{H}^0$ defines a model, so that λ_X^z and $\lambda_{Y|X>x}^z$ are the marginal and conditional specific cause hazards for some bivariate sdf $\overline{H}^z$ with margins $\overline{F}^z$ and $\overline{G}^z$ under a given covariate z. We conclude from the first equation of (1.6) and (1.1) that $\overline{F}^z(x) = \left(\overline{F}^0(x)\right)^{\Phi(z)}$. By the second equation in (1.6), plugging $x = 0$, we get $\overline{G}^z(y) = \left(\overline{G}^0(y)\right)^{\Psi(z)}$. The model is defined when z holds if $\left(\overline{F}^0(x)\right)^{\Phi(z)} \left(\overline{H}^0_{Y|X>x}(y)\right)^{\Psi(z)}$ defines a sdf. Notice that

$$\overline{H}^z(x,y) = \left(\overline{H}^0(x,y)\right)^{\Psi(z)} \left(\overline{F}^0(x)\right)^{\Phi(z)-\Psi(z)} \tag{3.9}$$

which is indeed a sdf when $\Phi(z) \geq \Psi(z) > 0$ and $\left(\overline{H}^0(x,y)\right)^{\Psi(z)}$ is a sdf.

Also not all bivariate survival d.f's $\overline{H}^0$ are such that for all positive γ, $\left(\overline{H}^0\right)^{\gamma}$ is a sdf. Min-infinite divisibility of the baseline hazard seems to be a natural assumption here. Assume therefore that:

$$(\mathbf{H}) \qquad H^0 \text{ is min-infinitely divisible}$$

By Theorem 2 and Lemma 1, (**H**) holds iff $C_{\overline{H}^0}$ is TP_2. We have the following result, whose proof is postponed to the Appendix.

Proposition 2. *When (**H**) holds then $\overline{H}^z$ defined in (3.9) is a sdf for all z such that $\Phi(z) \geq \Psi(z) > 0$.*

Let us consider the case when $0 < \Phi(z) \leq \Psi(z)$. Analogously with (1.6) the model may then be written

$$(\mathrm{M2}) \begin{cases} \lambda_Y^z(y) = \lambda_Y^0(y)\Psi(z) \\ \lambda_{X|Y>y}^z(x) = \lambda_{X|Y>y}^0(x)\Phi(z) \end{cases} \tag{3.10}$$

permuting the role of X and Y. In a similar way to the above we have that

$$\overline{H}^z(x,y) = \left(\overline{H}^0(x,y)\right)^{\Phi(z)} \left(\overline{G}^0(y)\right)^{\Psi(z)-\Phi(z)}$$

is a proper sdf. To summarize the above arguments we state:

Let the model be defined by (1.6) if $\Phi(z) \geq \Psi(z)$ and by (3.10) if $\Phi(z) < \Psi(z)$. Call (**M**) the model defined in this way.

$$(\mathbf{M}) := (M1)\mathbb{1}_{\Phi(z)\geq\Psi(z)} + (M2)\mathbb{1}_{\Phi(z)<\Psi(z)}.$$

This model is well defined, even if $\Phi(z)$ and $\Psi(z)$ are not ordered uniformly on the covariate z (which can be multivariate); the functions Φ and Ψ can be easily estimated through the data, since they characterize the marginal Cox models in (**M**). Suppose that X and Y are fitted to the same scale under the baseline, namely $\overline{F}^0(t) = \overline{G}^0(t)$ for all t. Then $\Phi(z) \geq \Psi(z)$ implies $\overline{F}^z(t) \leq \overline{G}^z(t)$ for all t, stretching the fact that X becomes stochastically smaller than Y under the stress parameter z.

Identifiability of (**M**) holds; assume for example that $\Phi(z) \geq \Psi(z)$. Then $\Phi(z)$ and

$\Psi(z)$ are defined uniquely. Indeed, assume

$$\begin{aligned}\overline{H}^z(x,y) &= \left(\overline{H}^0(x,y)\right)^{\Psi(z)}\left(\overline{F}^0(x)\right)^{\Phi(z)-\Psi(z)}\\ &= \left(\overline{H}^0(x,y)\right)^{\Psi'(z)}\left(\overline{F}^0(x)\right)^{\Phi'(z)-\Psi'(z)},\end{aligned}$$

for all x, y. Then taking logarithms yields $\Phi(z) = \Phi'(z)$ and $\Psi(z) = \Psi'(z)$. When (**H**) holds then for all z , $\overline{H}^z$ is a sdf and

$$\begin{aligned}\overline{H}^z(x,y) &= \mathbb{1}_{\Phi(z)\geq\Psi(z)}\left(\overline{H}^0(x,y)\right)^{\Psi(z)}\left(\overline{F}^0(x)\right)^{\Phi(z)-\Psi(z)}\\ &\quad+\mathbb{1}_{\Phi(z)<\Psi(z)}\left(\overline{H}^0(x,y)\right)^{\Phi(z)}\left(\overline{G}^0(y)\right)^{\Psi(z)-\Phi(z)}.\end{aligned} \tag{3.11}$$

Min-infinite divisibility of the baseline will also make any H^z min-infinitely divisible, showing that this class is *stable* under (**M**). Indeed for any positive γ,

$$\begin{aligned}\left(\overline{H}^z(x,y)\right)^\gamma &= \mathbb{1}_{\Phi(z)\geq\Psi(z)}\left(\overline{H}^0(x,y)\right)^{\gamma\Psi(z)}\left(\overline{F}^0(x)\right)^{\gamma[\Phi(z)-\Psi(z)]}\\ &\quad+\mathbb{1}_{\Phi(z)<\Psi(z)}\left(\overline{H}^0(x,y)\right)^{\gamma\Phi(z)}\left(\overline{G}^0(y)\right)^{\gamma[\Psi(z)-\Phi(z)]},\end{aligned}$$

which still is a sdf. By Theorem 2 and Lemma 1 min-infinite divisibility is not a property of the cdf but of its copula. Formula (3.11) can be written for copulas through

$$\begin{aligned}C_{\overline{H}^z}(u,v) &= \mathbb{1}_{\Phi(z)\geq\Psi(z)}u^{\frac{\Phi(z)-\Psi(z)}{\Phi(z)}}C_{\overline{H}^0}\left(u^{\frac{1}{\Phi(z)}},v^{\frac{1}{\Psi(z)}}\right)^{\Psi(z)}\\ &\quad+\mathbb{1}_{\Phi(z)<\Psi(z)}v^{\frac{\Psi(z)-\Phi(z)}{\Psi(z)}}C_{\overline{H}^0}\left(u^{\frac{1}{\Phi(z)}},v^{\frac{1}{\Psi(z)}}\right)^{\Phi(z)}.\end{aligned} \tag{3.12}$$

Although (**H**) is in full accordance with univariate Cox models, it is only a sufficient condition for its existence. The following example illustrates this fact.

Example 1. *Let $C_{\overline{H}^0}(u,v) = uv\exp\{-\theta\log u\log v\}$, with $\theta\in(0;1]$. This is the Gumbel-Barnett family (see Nelsen, 2006, p.119). By (2.7) it is easy to check that $C_{\overline{H}^0}$ is not TP_2. Using (3.12) and assuming $\Phi(z)\geq\Psi(z)$ we obtain*

$$C_{\overline{H}^z}(u,v) = uv\exp\left\{-\frac{\theta}{\Phi(z)}\log u\log v\right\}$$

which still is a Gumbel-Barnett copula when $\frac{\theta}{\Phi(z)}$ belongs to $(0;1]$.

This example shows that (**H**) is indeed the only acceptable condition for existence. Otherwise the baseline hazard $\overline{H}^0$ defines a model only for specific values of the covariate. This motivates our interest in good classes of min-infinitely divisible copulas which we intend to regress on the covariate z.

3.2. **Stability properties of the model.** We introduce two classes of copulas which are stable under (**M**).

3.2.1. *Extended archimedean copulas.* Among all possible types of bivariate dependence which can be described through the present bivariate Cox model, there exists a class of copulas which contains the archimedean copulas and which enjoys peculiar stability properties.

From now on, denote

$$\alpha(z) = \min\left(\frac{\Psi(z)}{\Phi(z)},1\right) \tag{3.13}$$

and

$$\beta(z) = \min\left(\frac{\Phi(z)}{\Psi(z)}, 1\right). \tag{3.14}$$

Proposition 3. *When $\overline{H}^0$ has an archimedean copula $C_{\overline{H}^0}$ with generator φ_0 and (**H**) holds then, denoting $C_{\overline{H}^z}$ the copula of $\overline{H}^z$ and using* (3.12)*, we have*

$$C_{\overline{H}^z}(u,v) = u^{1-\alpha(z)}v^{1-\beta(z)}\varphi_z^{-1}\left(\varphi_z\left(u^{\alpha(z)}\right) + \varphi_z\left(v^{\beta(z)}\right)\right) \tag{3.15}$$

with

$$\varphi_z(t) = \varphi_0\left(t^{\frac{1}{\Phi(z)\alpha(z)}}\right) = \varphi_0\left(t^{\frac{1}{\Psi(z)\beta(z)}}\right). \tag{3.16}$$

More generally we have, denoting $\Pi(u,v) = uv$

Proposition 4. *When $\overline{H}^0$ has the copula*

$$C_{\overline{H}^0}(u,v) = \Pi(u^{1-\kappa}, v^{1-\eta})C_{\varphi_0}(u^\kappa, v^\eta),\ 0 \le \kappa, \eta \le 1 \tag{3.17}$$

*and (**H**) holds, then the copula of $\overline{H}^z$ is given by*

$$C_{\overline{H}^z}(u,v) = \Pi(u^{1-\alpha(z)\kappa}, v^{1-\beta(z)\eta})C_{\varphi_z}(u^{\alpha(z)\kappa}, v^{\beta(z)\eta}) \tag{3.18}$$

where C_φ denotes an archimedean copula with generator φ and φ_z is as in (3.16).

3.2.2. *Extreme values copulas.* We show that the class of evc's also enjoys stability properties, as seen in the present Section. Extreme values copula are TP_2 (Hürlimann, 2003). Among all classes of TP_2 copulas, the evc's enjoy good properties as seen in Proposition 9, since they can nicely approximate PQD copulas and are parametrized through a smooth function of only one variable, as developed in Section 5 hereunder.

Proposition 5. *When $\overline{H}^0$ has an evc $C_{\overline{H}^0}$ with Pickands function A then, denoting $C_{\overline{H}^z}$ the copula of $\overline{H}^z$ and using* (3.12)*, we have*

$$C_{\overline{H}^z}(u,v) = \exp\left[\log(uv)B^z\left(\frac{\log v}{\log uv}\right)\right] \tag{3.19}$$

with

(3.20)

$$B^z(s) = 1 - W(z)K(z) - sW(z)[1-K(z)] + W(z)[(1-s)K(z)+s]A\left(\frac{s}{K(z)(1-s)+s}\right)$$

where $K(z) = \frac{\Psi(z)}{\Phi(z)}$ and $W(z) = \min\left(\frac{1}{K(z)}, 1\right) = \beta(z)$.

Remark 2. *This basic result shows that $C_{\overline{H}^z}$ is an evc with Pickands function B^z. Proposition 5 shows that the class of evc's is stable under (**M**). Although the copula of $\overline{H}^0$ is an evc, this does not imply in any respect that its marginals should be extreme value sdf's.*

From (3.20) we deduce the *transition formula* which links $B^{z'}$ to B^z for two different values of the covariate. It holds

Proposition 6. *Under (**M**) let $\overline{H}^0$ has an evc. Then with the above notation, for all z, z',*

(3.21)

$$B^{z'}(s) = 1 - \frac{\alpha(z')}{\alpha(z)} + \left(\frac{\alpha(z')}{\alpha(z)} - \frac{\beta(z')}{\beta(z)}\right)s + \left[(1-s)\frac{\alpha(z')}{\alpha(z)} + s\frac{\beta(z')}{\beta(z)}\right]B^z\left(\frac{s\frac{\beta(z')}{\beta(z)}}{(1-s)\frac{\alpha(z')}{\alpha(z)} + s\frac{\beta(z')}{\beta(z)}}\right)$$

Proposition 6 proves that the transition from z to z' is independent of the baseline dependence function. Formulas (3.21) can be seen as a kind of expression of the proportional hazard property, which links two hazard rates independently on the baseline. When the covariate acts equally on X and Y, i.e. $\Phi(z) = \Psi(z)$ for all z, then $B^z(s) = A(s)$ for all values of s as seen in Proposition 5. Thus, the copula of $\overline{H}^z$ equals that of the baseline $\overline{H}^0$; the dependency structure of X and Y should not be altered through (**M**). Only the marginal distributions of X and Y in this case reflect the role of the covariate.

We propose some illustration. We use $\Phi(z) = e^{\alpha z}$, with $\alpha = 1.5$ and $\Psi(z) = e^{\beta z}$, with $\beta = 2$. Figures 3.1-3.4 illustrate formula (3.18). We represent the change of the density of $C_{\overline{H}^z}$ with z. In Figures 3.1 and 3.2 the archimedean copula is the Clayton copula whose generator is defined by $\varphi(t) = t^{-\theta} - 1$. We take $\theta = 3$. In Figure 3.1 we take $\kappa = \eta = 1$. In Figure 3.2 we take $\kappa = 0.6$ and $\eta = 0.01$. In those two figures the model tends rapidely to independence since the density of the copula tends to 1 as z increases (see also Figure 6.10). In Figures 3.3 and 3.4 the archimedean copula is the Gumbel copula whose generator is define by $\varphi(t) = (-\log(t))^\theta$. We take $\theta = 3$. In Figure 3.3 we take $\kappa = \eta = 1$. In Figure 3.4 we take $\kappa = 0.6$ and $\eta = 0.01$. For this baseline copula which is both archimedean and evc, the tendency to independence is much slower (see also Figure 6.11). Figure 3.5 illustrates the transition formula (3.20). The baseline copula is the Gumbel copula with $\theta = 3$. The Pickands function of the Gumbel copula is $A(t) = \left[t^\theta + (1-t)^\theta\right]^{\frac{1}{\theta}}$. The dependence functions are ordered wrt z. As z increases, the model tends to independent marginals.

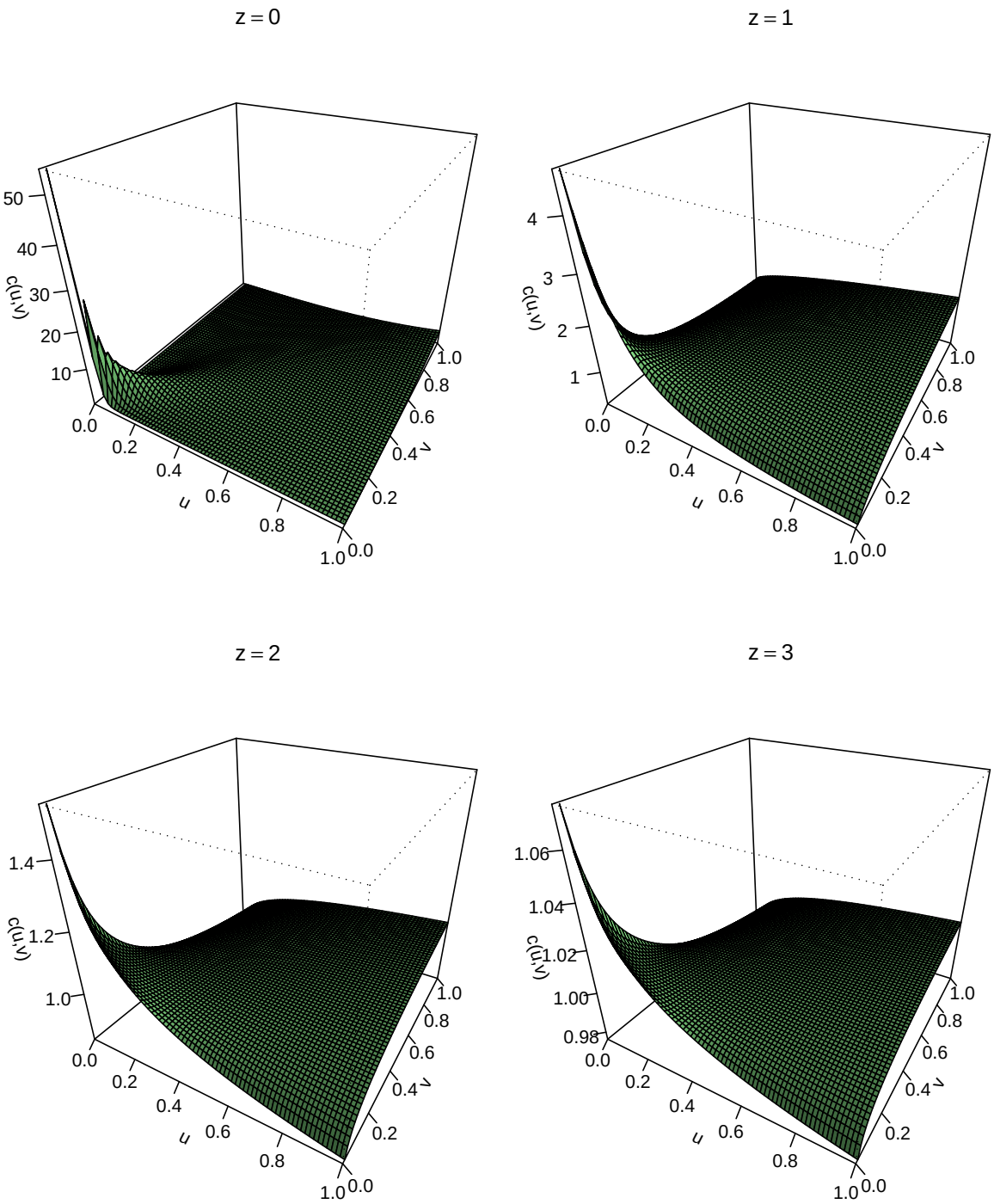

FIGURE 3.1. Illustration of (3.18) with the Clayton copula's density for the baseline ($z = 0$)

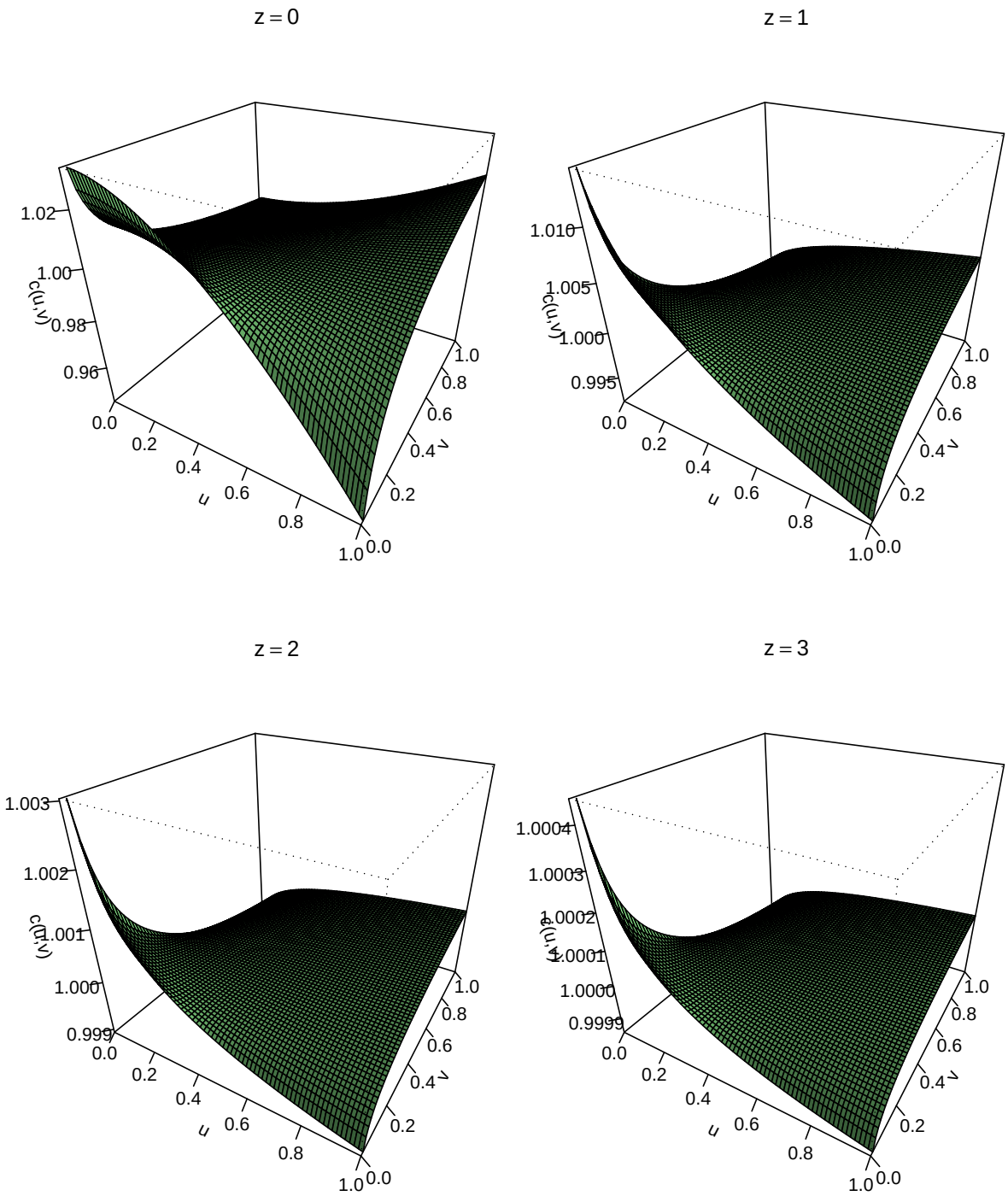

FIGURE 3.2. Illustration of (3.18) with the asymmetric Clayton copula's density for the baseline ($z = 0$)

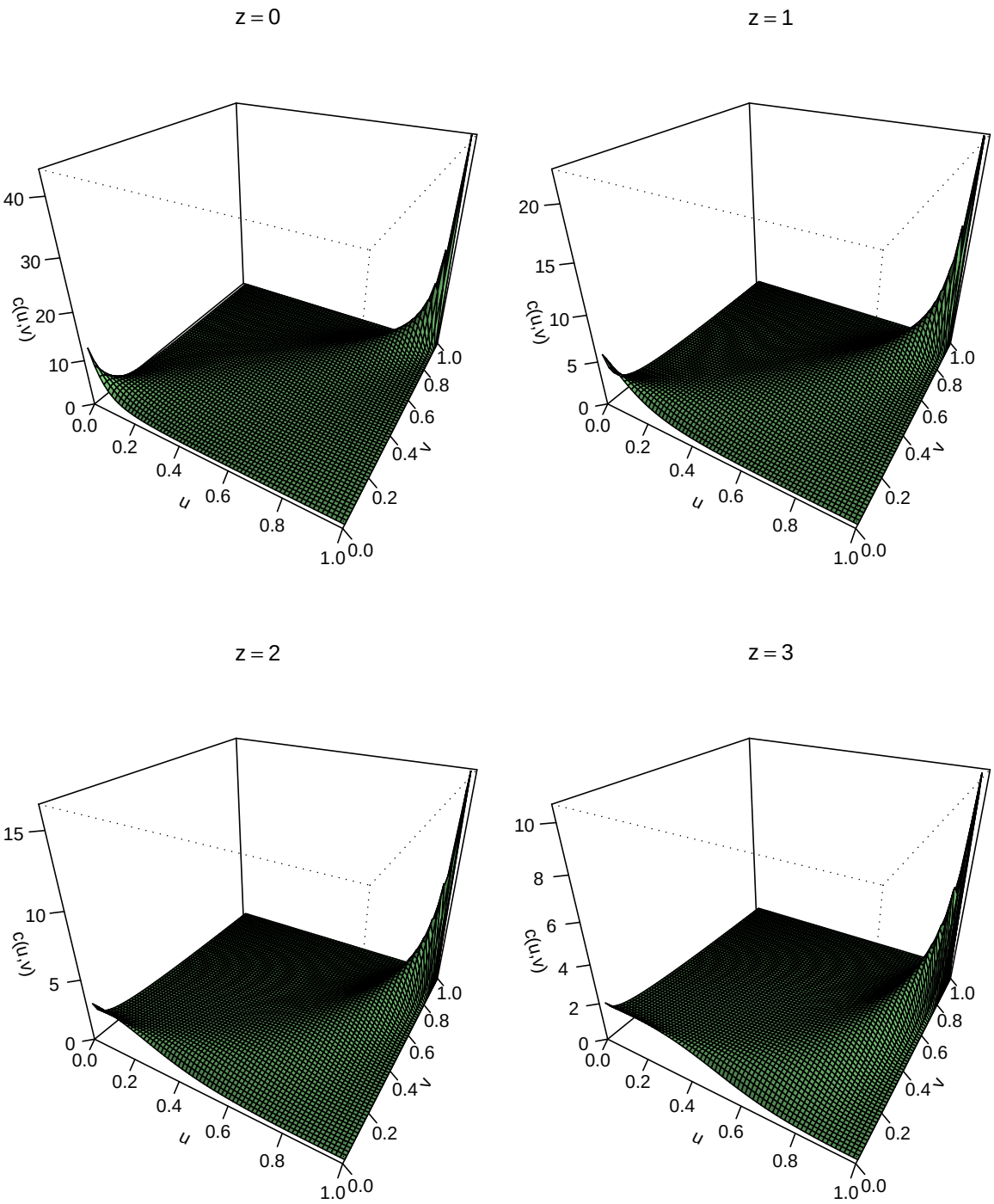

FIGURE 3.3. Illustration of (3.18) with the Gumbel copula's density for the baseline ($z = 0$)

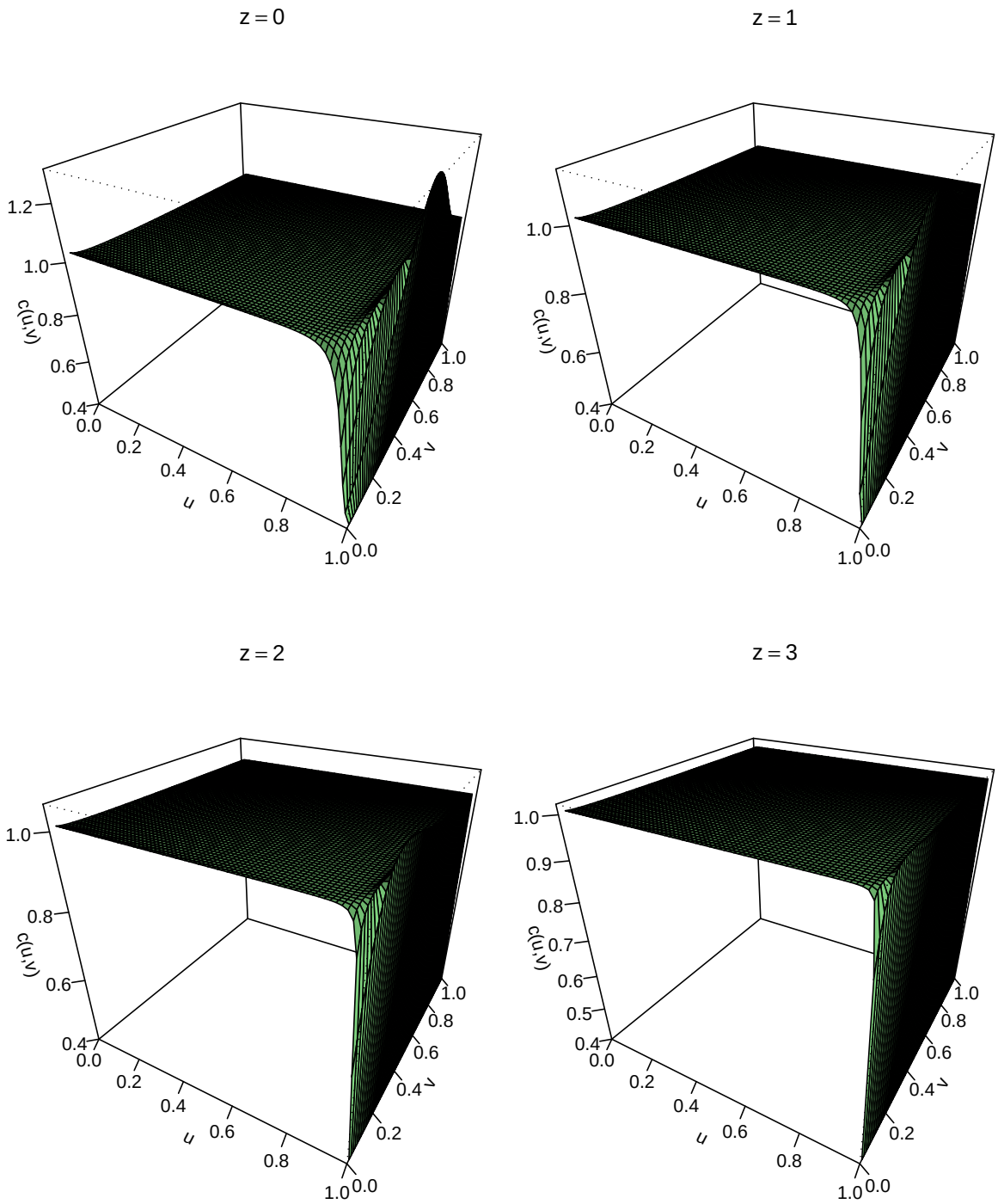

FIGURE 3.4. Illustration of (3.18) with the asymmetric Gumbel copula's density for the baseline ($z = 0$)

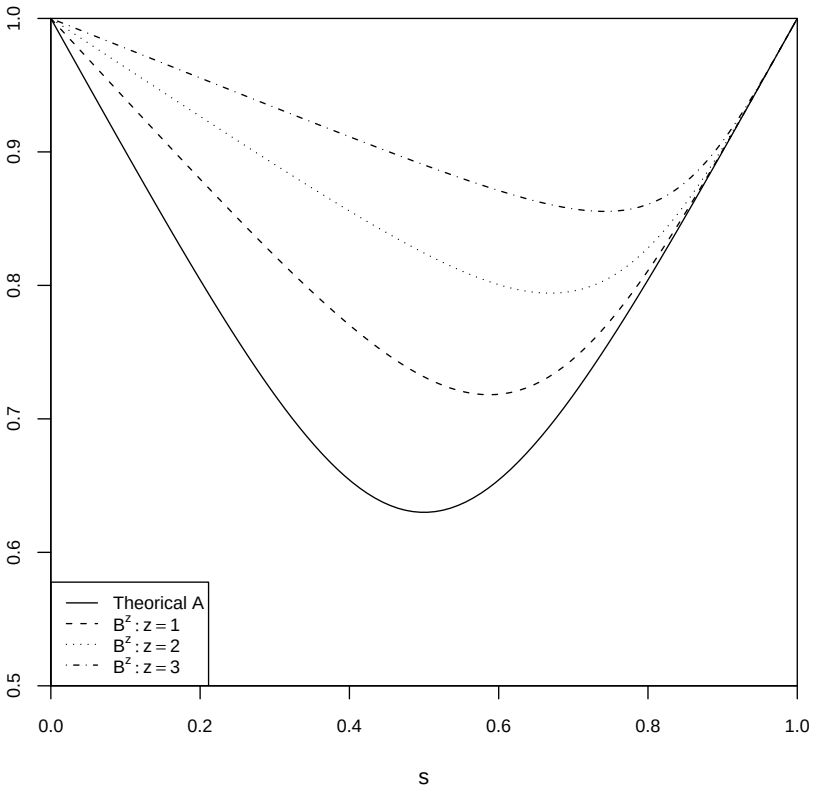

FIGURE 3.5. Illustration of formula (3.20)

4. Asymmetric logistic models of dependence

This section deals with specific parametric models for dependence which are stable under (**M**). We consider model (**M**) specialized in the case when the copula of $\overline{H}^0$ is a Gumbel copula. The margins of $\overline{H}^0$ can be any. It is a simple parametrized model of copulas, which is an evc on one hand, and which models frailty bivariate dependence, being hence an archimedean copula. Indeed it is the only copula satisfying jointly these two properties (see Nelsen (2006), Theorem 4.5.2; Genest and Rivest (1989), statement A).
The Gumbel copula writes

$$C(u,v) := \exp\left[-\left\{(-\log u)^\theta + (-\log v)^\theta\right\}^{1/\theta}\right]$$

with $\theta \geq 1$. The dependence function of this copula is

$$A(s) = \left[s^\theta + (1-s)^\theta\right]^{1/\theta}. \tag{4.22}$$

Assume that $\overline{H}^0$ has an evc with dependence function A. When the covariate z acts, the dependence function B^z defined through Proposition 5 determines the *asymmetric logistic* copula. This copula has three parameters $\alpha(z), \beta(z)$ and θ. Recall from (3.13), (3.14) and proposition 5 that

$$\alpha(z) = \min\left(\frac{\Psi(z)}{\Phi(z)}, 1\right)$$

and

$$\beta(z) = \min\left(\frac{\Phi(z)}{\Psi(z)}, 1\right).$$

It holds

$$B^z(s) = 1 - \alpha(z) + [\alpha(z) - \beta(z)]\, s + \left[\alpha(z)^\theta (1-s)^\theta + \beta(z)^\theta s^\theta\right]^{1/\theta}. \tag{4.23}$$

When $\alpha(z) = \beta(z)$ (which implies that they equal 1), i.e. when z acts equally on X and Y, $B^z(s) = A(s)$ for all s. The copula of $\overline{H}^z$ is

$$C_{\overline{H}^z}(u,v) = \Pi(u^{1-\alpha(z)}, v^{1-\beta(z)}) C(u^{\alpha(z)}, v^{\beta(z)}). \tag{4.24}$$

where $\Pi(u,v) := uv$ is the *product copula* (see Nelsen, 2006, p.11). The dependence function B^z is an asymmetric form of the Gumbel dependence function A defined in (4.22). This is the *asymmetric logistic model* in Tawn (1988) when the margins are standard exponential. As developped by Khoudraji (see Khoudraji, 1995, chap 4) and Genest et al. (see Frees and Valdez, 2007, Proposition 3), given two dependence functions A_1 and A_2 , two constants κ and η with $0 < \kappa, \eta < 1$ the function defined through

$$B(s) := (\kappa s + \eta \bar{s})\, A_1\left(\frac{\kappa s}{\kappa s + \eta \bar{s}}\right) + (\bar{\kappa} s + \bar{\eta}\bar{s})\, A_2\left(\frac{\bar{\kappa} s}{\bar{\kappa} s + \bar{\eta}\bar{s}}\right)$$

where $\bar{s}$ denotes $1 - s$, is the dependence function of the extreme value copula defined by $C_{A_1}(u^{1-\kappa}, v^{1-\eta}) C_{A_2}(u^\kappa, v^\eta)$. Genest et al. define this procedure as a technique to generate asymmetric copulas. The class of copulas defined in (4.24) has been introduced by Genest et al (see their Proposition 2 in Frees and Valdez (2007)).

We now analyze this class of copulas in terms of frailty models. The Gumbel copula is associated with a frailty model of order 1, namely

$$C(u,v) = \Lambda^{-1}(\Lambda(u) + \Lambda(v))$$

where $\Lambda^{-1}(s) = \int_0^\infty e^{-sw} dM_{1/\theta}(w) = e^{-s^{1/\theta}}$, $\theta > 1$, is the Laplace transform of the positive stable law $M_{1/\theta}$ on $\mathbb{R}^+$ with tail heaviness index $1/\theta$, location parameter 0, scale parameter 1 and skewness parameter 0 (see Ravishanker and Dey (2000) and the example 5 in Oakes (1989)). Denote W a positive random variable with cdf $M_{1/\theta}$. A bivariate sdf $\overline{H}^0$ with Gumbel copula C writes

$$\overline{H}^0(x,y) = \int_0^\infty \left\{\overline{F}(x)\overline{G}(y)\right\}^w dM_{1/\theta}(w),$$

for some sdf $\overline{F}$ and $\overline{G}$. Therefore $\overline{H}^0$ is a frailty bivariate sdf, with stable frailty measure and margins $\int_0^\infty \overline{F}(x)^w dM_{1/\theta}(w)$ and $\int_0^\infty \overline{G}(y)^w dM_{1/\theta}(w)$.
Let U_1 and U_2 be two independent r.v's, both independent of W. We assume that U_1 and U_2 have a positive stable law M^1 and M^2 with tail heaviness index $1/\theta$. The r.v U_1 (resp. U_2) has shape parameter $\frac{1}{\alpha(z)} - 1$ (resp. $\frac{1}{\beta(z)} - 1$). Both have location and skewness parameters 0. Define $S_i := U_i + W, i = 1, 2$ (see Drouet and Monbet, 2004). Denote $\varphi_i^{-1}(s)$ the Laplace transforms of the distribution of S_i. Denote further $\psi_i^{-1}(s)$ the Laplace transform of the distribution of U_i. Let M denote the probability measure of $(S_1\ , S_2)$. For arbitrary sdf $\overline{H}_1$ and $\overline{H}_2$ define the bivariate sdf

$$\overline{H}(x,y) := \int\int \left\{\overline{H}_1(x)\right\}^{s_1} \left\{\overline{H}_2(y)\right\}^{s_2} dM(s_1, s_2)$$

which we call a frailty model of order 2 since it implies a bivariate latent variable. Frailty models of order two have been considered in Marshall and Olkin (1988) (see their formula (2.2)). The marginals of $\overline{H}$ are $\overline{F}(x) = \int_0^\infty \left\{\overline{H}_1(x)\right\}^{s_1} dM_{S_1}(s_1)$ and $\overline{G}(y) =$

$\int_0^\infty \left\{\overline{H}_2(y)\right\}^{s_2} dM_{S_2}(s_2)$.
We prove that the copula of $\overline{H}$ is (4.24). Indeed

$$\overline{H}(x,y) = \int \left\{\overline{H}_1(x)\right\}^{u_1} dM^1(u_1) \int \left\{\overline{H}_2(y)\right\}^{u_2} dM^2(u_2) \int \left\{\overline{H}_1(x)\overline{H}_2(y)\right\}^{w} dM_{1/\theta}(w).$$

Introducing the Laplace transforms defined above and rewriting the marginals $\overline{F}(x) = \varphi_1^{-1}\left(-\log \overline{H}_1(x)\right)$ and $\overline{G}(y) = \varphi_2^{-1}\left(-\log \overline{H}_2(y)\right)$ we obtain the following expression for the copula of $\overline{H}$

$$C(u,v) = \psi_1^{-1}\left(\varphi_1(u)\right)\psi_2^{-1}\left(\varphi_2(v)\right)\Lambda^{-1}\left(\varphi_1(u)+\varphi_2(v)\right), \tag{4.25}$$

since $\overline{H}(x,y) = C(\overline{F}(x), \overline{G}(y))$. Substituting ψ_i and $\varphi_i, i = 1,2$ by their expressions in the above expression, noting that $\varphi_i^{-1} = \psi_i^{-1}\Lambda^{-1}$, (4.25) coincides with (4.24). We now prove that for an adequate choice of $\overline{H}_1$ and $\overline{H}_2$ the bivariate sdf $\overline{H}$ has same marginals as $\overline{H}^z$. Indeed let

$$\begin{aligned}
\overline{H}_1(x) &:= \exp\left[-\left(-\min\left(\Phi(z),\Psi(z)\right)\log \overline{F}^0(x)\right)^\theta\right] \\
\overline{H}_2(y) &:= \exp\left[-\left(-\min\left(\Phi(z),\Psi(z)\right)\log \overline{G}^0(y)\right)^\theta\right]
\end{aligned}$$

which yields $\overline{F}(t) = \overline{F}^z(t)$ and $\overline{G}(t) = \overline{G}^z(t)$ for all t. Therefore $\overline{H}$ and $\overline{H}^z$ coincide. We have proved

Proposition 7. *When $\overline{H}^0$ is a frailty bivariate sdf with Gumbel copula, then for all z, $\overline{H}^z$ is a frailty sdf of order 2 with asymmetric logistic copula given in (4.24).*

More generally we have:

Proposition 8. *The class of second order frailty models with asymmetric logistic copula is stable under (**M**).*

Proof. Let A denote the dependence function of an asymmetric logistic copula.

$$A(s) := 1-\kappa+(\kappa-\eta)s+\left[\kappa^\theta(1-s)^\theta+\eta^\theta s^\theta\right]^{1/\theta}.$$

By (3.20) it holds

$$B^z(s) = 1-\kappa'+(\kappa'-\eta')s+\left[\kappa'^\theta(1-s)^\theta+\eta'^\theta s^\theta\right]^{1/\theta}$$

with

$$\begin{aligned}
\kappa' &= \alpha(z)\kappa \\
\eta' &= \beta(z)\eta.
\end{aligned}$$

These new parameters are in $(0,1)$, as are κ and η. We have proved that the class of sdf with asymmetric logistic copula is stable under (**M**). Any sdf with such a copula is necessarily a frailty sdf of order 2. Indeed this follows from (4.25) which enables identifying the frailty measure M of $\overline{H}^z$ as the joint distribution of (S_1, S_2) as defined here above. ■

Remark 3. *It can be seen that the only sdf which are frailty of order 2 with evc are precisely the frailty models with asymmetric logistic copula $C(u,v) = \Pi(u^{1-\kappa}, v^{1-\eta})C_\theta(u^\kappa, v^\eta)$ with $0<\eta,\kappa<1$, and where C_θ is the standard Gumbel copula with parameter $\theta \geq 1$.*

5. Approximation of PQD copulas by EV copulas

Assume that $\overline{H}^0$ is PQD. $\overline{H}^0$ needs not define the model for all z, since (**H**) only implies PQD. In this section we propose to approximate any baseline PQD copula (hence sdf) by some related evc for which the model is properly defined. As seen hereabove in Section 3.2.2 this approximation scheme is stable through (**M**).

Denote $I := [0,1] \times [0,1]$. Let us parameterize the points in I in the following way. An evc is characterized through its dependence function A which takes equal values $A(s)$ when evaluated at $s = \frac{\log v}{\log uv}$. For fixed s in $(0,1]$ defined the *arc* $\mathfrak{s}$ through

$$\mathfrak{s} := \{(u,v) = (v^{\frac{1}{s}-1}, v) : \ v \in [0,1]\}.$$

This arc is the set of points (u,v) on which any dependence function A is constant and equals $A(s)$. The family of the arcs $\mathfrak{s}$ covers I.

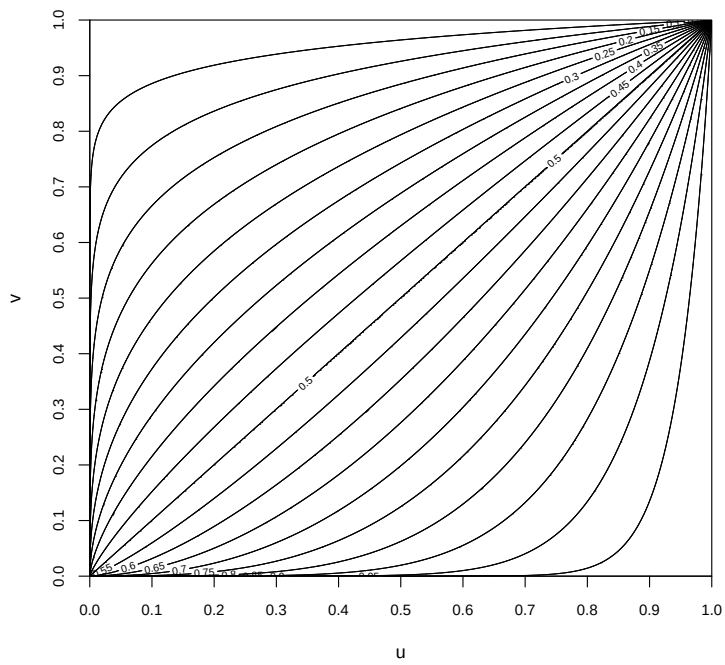

Figure 5.6. Parametrization of I^2

For any copula C define on $\mathfrak{s}$ the function

$$\mathbb{A}(s,v) \equiv \frac{s}{\log v} \log C(v^{\frac{1}{s}-1}, v). \tag{5.26}$$

The function $\mathbb{A}$ mimics a dependence function A; indeed, when C is an evc, then $\mathbb{A}(s,v) \equiv A(s)$ as seen by direct substitution. Using $\mathbb{A}$, the copula C, analogously to (2.8), write $C(u,v) = \exp\left[\log(uv)\mathbb{A}\left(\frac{\log v}{\log uv}, v\right)\right]$. In order to construct a dependence function close to $\mathbb{A}$, average the values of $\mathbb{A}$ on $\mathfrak{s}$ through

$$\overline{\mathbb{A}}(s) \equiv \int_0^1 \mathbb{A}(s,v)dv. \tag{5.27}$$

Obviously other choices are possible which do not change in any way the results in the Proposition hereafter. Define now:

$$\widetilde{C}(v^{\frac{1}{s}-1}, v) = \exp\left[\frac{\log v}{s}\overline{\mathbb{A}}(s)\right] \tag{5.28}$$

which by a change of variables is also $\widetilde{C}(u,v) = \exp\left[\log(uv)\overline{\mathbb{A}}\left(\frac{\log v}{\log uv}\right)\right]$. For completeness define $\mathbb{A}(0,v) = 1$ and $\mathbb{A}(1,v) = 1$.

Proposition 9. *When C is PQD, then the copula $\widetilde{C}$ is an evc, and for all s in $[0,1]$,*

$$\sup_{(u,v)\in\mathfrak{s}} \left|C(u,v) - \widetilde{C}(u,v)\right| \leq 2e^{-1}\underset{\mathfrak{s}}{Osc}(\mathbb{A}) \tag{5.29}$$

where $Osc_{\mathfrak{s}}(\mathbb{A})$ denotes the oscillation of the function $\mathbb{A}$ on the arc $\mathfrak{s}$, namely

$$\underset{\mathfrak{s}}{Osc}(\mathbb{A}) := \sup_{s,v}\left|\mathbb{A}(s,v) - \overline{\mathbb{A}}(s)\right|.$$

The upper bound in (5.29) is indeed 0 when C is an evc. The graphs show that the bound can be large with respect to the true distance between the copulas as measured on the arcs.
We represent on theses graphs, Figure 5.7, the error between a PQD copula and its approximation (dotted line) and the upper bound of the former proposition (black line). The abscissa is s, which parametrizes the arcs in I, as describes above.
The first example is

$$C(u,v) = \left(1 + \left[(1/u-1)^\theta + (1/v-1)^\theta\right]^{1/\theta}\right)^{-1} \text{ with } \theta = 6; \tag{5.30}$$

and the second one is

$$C(u,v) = \exp\left(1 - \left[(1-\log u)^\theta + (1-\log v)^\theta - 1\right]^{1/\theta}\right) \text{ with } \theta = 2; \tag{5.31}$$

see Table 4.1, Examples (4.2.12) and (4.2.13) in Nelsen (2006).

A *PQD* copula may sometimes be approximated by an evc with good relative accuracy, as follows: let $A_v(s) := A(s,v)$ as defined in (5.27). For all v, $A_v(s)$ is a dependence function, hence generating an evc $\widetilde{C}$. Select $v^\star$ such that $\sup_{(s,v)\in\mathfrak{s}}\left|\frac{C(v^{\frac{1}{s}-1},v)-\widetilde{C}(v^{\frac{1}{s}-1},v)}{C(v^{\frac{1}{s}-1},v)}\right|$ is minimal among all such evc's approximations. Figure 5.8 illustrates this construction with the Clayton copula with $\theta = 3$. The minimal relative error is obtain when $A_v(s) = A_{0.2}(s)$. The propagation of this approximation through (**M**) can be illustrated in figure 5.9. Start at $z = 0$ with the Clayton copula ($\theta = 3$). The accuracy of the evc approximation is of order 13%. For $z \neq 0$, we calculate $C^z(u,v)$ by the result in section 3.2. We also compute $\widetilde{C}^z(u,v)$ through formula (3.19). The relative error between C^z and $\widetilde{C}^z$ is seen to stay bounded by 20% for reasonable values of z; indeed for example $z = 0.1$ induces $\Psi(z) = e^{2z} = 1.2$, which is meaningful in applications.

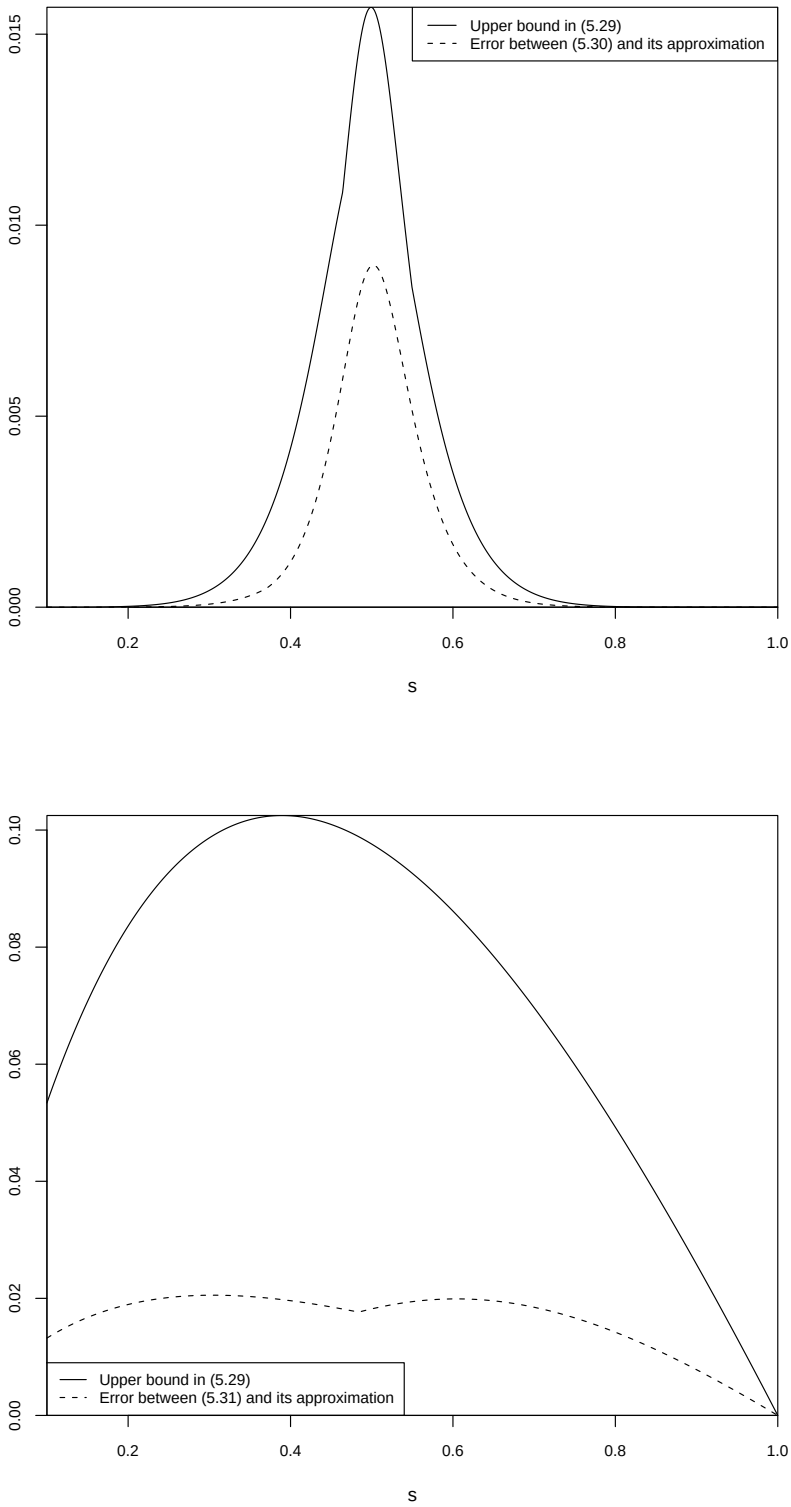

FIGURE 5.7. Illustration of (5.29)

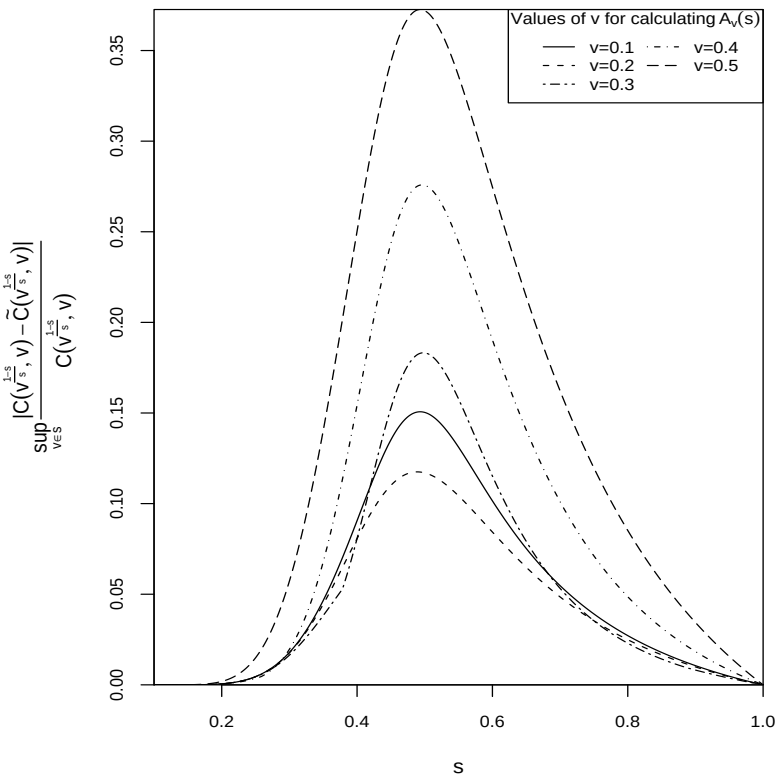

FIGURE 5.8. Choice of $v^\star$

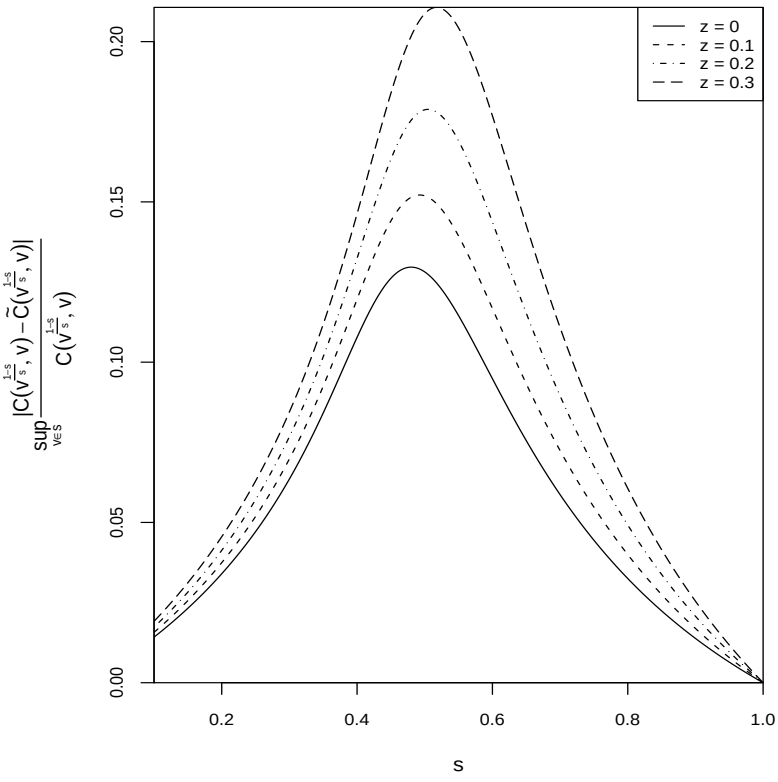

FIGURE 5.9. Propagation of the maximal relative error on $\mathfrak{s}$

6. Application: measure of dependence

We recall the expression of Spearman's rho: $\rho_C = 12 \int_0^1 \int_0^1 C(u,v)dudv - 3$

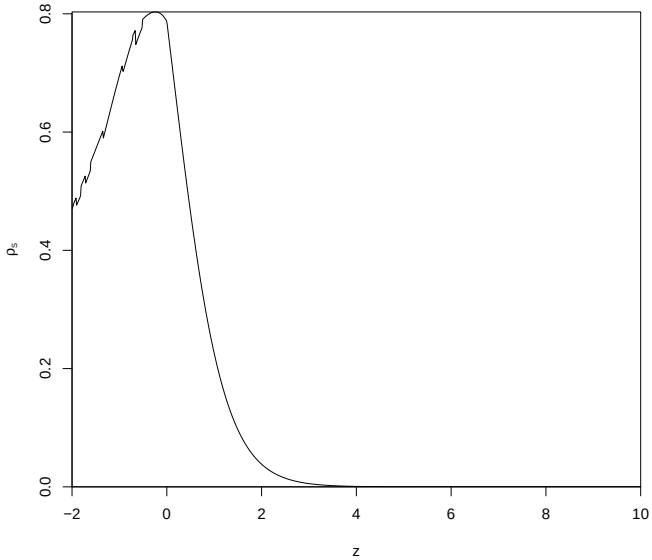

Figure 6.10. Change of Spearman's rho for Clayton copula ($\kappa = \eta = 1$ in (3.18))

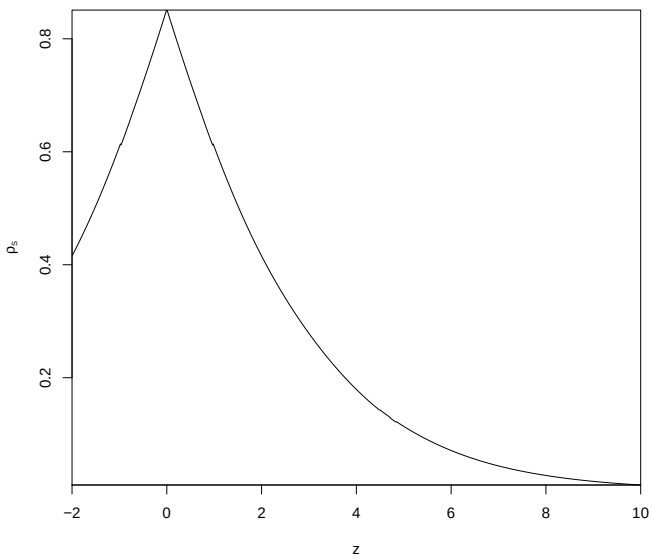

Figure 6.11. Change of Spearman's rho for Gumbel copula in (4.24)

Appendix A. Proof of Proposition 2

Trivially we show that $\lim_{x_j \to +\infty} \overline{H}^z(x_1, x_2) = 0$, $j = 1, 2$; and $\lim_{\substack{x_1 \to 0 \\ x_2 \to 0}} \overline{H}^z(x_1, x_2) = 1$

Now we will prove that $\overline{H}^z$ satisfies the rectangle inequality (Joe (1997), p.11) which we recall here: for all $(a_1, a_2), (b_1, b_2)$ with $a_1 < b_1$, $a_2 < b_2$,

$$\Delta H := \overline{H}^z(a_1, a_2) - \overline{H}^z(a_1, b_2) - \overline{H}^z(b_1, a_2) + \overline{H}^z(b_1, b_2) \geq 0.$$

$$\begin{aligned}\Delta H = &\left(\overline{F}^0(a_1)\right)^{\Phi(z)-\Psi(z)} \left[\left(\overline{H}^0(a_1, a_2)\right)^{\Psi(z)} - \left(\overline{H}^0(a_1, b_2)\right)^{\Psi(z)}\right] \\ &- \left(\overline{F}^0(b_1)\right)^{\Phi(z)-\Psi(z)} \left[\left(\overline{H}^0(b_1, a_2)\right)^{\Psi(z)} - \left(\overline{H}^0(b_1, b_2)\right)^{\Psi(z)}\right]\end{aligned}$$

We denote $r(t) = \left(\overline{H}^0(t, b_2)\right)^{\Psi(z)} - \left(\overline{H}^0(t, a_2)\right)^{\Psi(z)}$. If $\left(\overline{H}^0\right)^{\Psi(z)}$ is a sdf (which implies that $\left(\overline{H}^0\right)^{\Psi(z)}$ is a 2-increasing function), then by Lemma 2.1.3 in Nelsen (2006), the function r is nondecreasing. Therefore

$$\begin{aligned}\Delta H &= \left(\overline{F}^0(b_1)\right)^{\Phi(z)-\Psi(z)} r(b_1) - \left(\overline{F}^0(a_1)\right)^{\Phi(z)-\Psi(z)} r(a_1) \\ &= r(b_1)\left[\left(\overline{F}^0(b_1)\right)^{\Phi(z)-\Psi(z)} - \frac{r(a_1)}{r(b_1)}\left(\overline{F}^0(a_1)\right)^{\Phi(z)-\Psi(z)}\right].\end{aligned}$$

Under (**H**) and if $\Phi(z) \geq \Psi(z) > 0$, both $\left(\overline{H}^0\right)^{\Psi(z)}$ and $\left(\overline{F}^0\right)^{\Phi(z)-\Psi(z)}$ are sdf's. Use the fact that $r(b_1)$ is negative (which holds since $\left(\overline{H}^0\right)^{\Psi(z)}$ is a decreasing function of its second argument) to obtain

$$r(b_1)\left[\left(\overline{F}^0(b_1)\right)^{\Phi(z)-\Psi(z)} - \frac{r(a_1)}{r(b_1)}\left(\overline{F}^0(a_1)\right)^{\Phi(z)-\Psi(z)}\right] \geq 0.$$

Note that if (**H**) does not hold, $\overline{H}^z$ is still a sdf when $\Phi(z) \geq \Psi(z) \geq 1$. ■

Appendix B. Proof of Proposition 3

Write $C_{\overline{H}^0}(u, v) = \varphi_0^{-1}(\varphi_0(u) + \varphi_0(v))$. Using (3.12) some calculus yields (3.15). We now prove that $\varphi_z(t) = \varphi_0\left(t^{\frac{1}{\min(\Phi(z),\Psi(z))}}\right)$ is also a generator for all values of t in $[0, 1]$.

(1) $\varphi_z(1) = \varphi_0(1) = 0$
(2) $\varphi_z(t)$ is strictly decreasing in t, since $t^{\frac{1}{\min(\Phi(z),\Psi(z))}}$ is increasing in t and $\varphi_0(t)$ is strictly decreasing.
(3) φ_z is convex, since $C_{\overline{H}^0}$ is TP_2. Indeed, the property of TP_2 for an archimedean copula is equivalent to $\varphi_0'(t) + t\varphi_0''(t) \geq 0$. And we have,

$$\begin{aligned}\varphi_z''(t) &= m(m-1)t^{m-2}\varphi_0'(t^m) + (mt^{m-1})^2\varphi_0''(t^m) \\ &\geq m^2 t^{m-2}\left[\varphi_0'(t^m) + t^m\varphi_0''(t^m)\right] \geq 0\end{aligned}$$

where $m = \frac{1}{\min(\Phi(z),\Psi(z))}$. ■

Appendix C. Proof of Proposition 5

If $K(z) < 1$ then

$$\begin{aligned}
\hat{C}_{H^z}(u,v) &= \exp\left[\frac{\Phi(z)-\Psi(z)}{\Phi(z)}\ln u\right]\exp\left[\left(\frac{\ln u}{\Phi(z)}+\frac{\ln v}{\Psi(z)}\right)A\left(\frac{\frac{\ln v}{\Psi(z)}}{\frac{\ln u}{\Phi(z)}+\frac{\ln v}{\Psi(z)}}\right)\right]^{\Psi(z)} \\
&= \exp\left[\frac{\Phi(z)-\Psi(z)}{\Phi(z)}\ln u+\left(\frac{\Psi(z)}{\Phi(z)}\ln u+\ln v\right)A\left(\frac{\ln v}{\frac{\Psi(z)}{\Phi(z)}\ln u+\ln v}\right)\right] \\
&\underset{(u=v^{\frac{1}{s}-1})}{=} \exp\left[\left(\frac{1}{s}-1\right)\frac{\Phi(z)-\Psi(z)}{\Phi(z)}\ln v+\left[\frac{\Psi(z)}{\Phi(z)}\left(\frac{1}{s}-1\right)+1\right]A\left(\frac{1}{\left(\frac{1}{s}-1\right)\frac{\Psi(z)}{\Phi(z)}+1}\right)\ln v\right] \\
&= \exp\left[\frac{\ln v}{s}\left\{\left[(1-s)\frac{\Phi(z)-\Psi(z)}{\Phi(z)}\right]+\left[(1-s)\frac{\Psi(z)}{\Phi(z)}+s\right]A\left(\frac{s}{(1-s)\frac{\Psi(z)}{\Phi(z)}+s}\right)\right\}\right] \\
&= \exp\left[\ln(uv)B_1^z\left(\frac{\ln v}{\ln uv}\right)\right]
\end{aligned}$$

with

$$B_1^z(s) = (1-s)\frac{\Phi(z)-\Psi(z)}{\Phi(z)}+\left[(1-s)\frac{\Psi(z)}{\Phi(z)}+s\right]A\left(\frac{s}{(1-s)\frac{\Psi(z)}{\Phi(z)}+s}\right) \tag{C.32}$$

Hence

$$\begin{aligned}
B_1^z(s) &= (1-s)[1-K(z)]+[(1-s)K(z)+s]\,A\left(\frac{s}{(1-s)K(z)+s}\right) \\
&= 1-K(z)-s[1-K(z)]+[(1-s)K(z)+s]\,A\left(\frac{s}{(1-s)K(z)+s}\right)
\end{aligned}$$

If $K(z) > 1$

$$\begin{aligned}
\hat{C}_{H^z}(u,v) &= \exp\left[\frac{\Psi(z)-\Phi(z)}{\Psi(z)}\ln v\right]\exp\left[\left(\frac{\ln u}{\Phi(z)}+\frac{\ln v}{\Psi(z)}\right)A\left(\frac{\frac{\ln v}{\Psi(z)}}{\frac{\ln u}{\Phi(z)}+\frac{\ln v}{\Psi(z)}}\right)\right]^{\Phi(z)} \\
&= \exp\left[\frac{\Psi(z)-\Phi(z)}{\Psi(z)}\ln v+\left(\ln u+\frac{\Phi(z)}{\Psi(z)}\ln v\right)A\left(\frac{\ln v}{\frac{\Psi(z)}{\Phi(z)}\ln u+\ln v}\right)\right] \\
&\underset{(u=v^{\frac{1}{s}-1})}{=} \exp\left[\frac{\Psi(z)-\Phi(z)}{\Psi(z)}\ln v+\left[\left(\frac{1}{s}-1\right)+\frac{\Phi(z)}{\Psi(z)}\right]A\left(\frac{1}{\left(\frac{1}{s}-1\right)\frac{\Psi(z)}{\Phi(z)}+1}\right)\ln v\right] \\
&= \exp\left[\frac{\ln v}{s}\left\{\left(s\frac{\Psi(z)-\Phi(z)}{\Psi(z)}\right)+\left[(1-s)+s\frac{\Phi(z)}{\Psi(z)}\right]A\left(\frac{s}{(1-s)\frac{\Psi(z)}{\Phi(z)}+s}\right)\right\}\right] \\
&= \exp\left[\ln(uv)B_2^z\left(\frac{\ln v}{\ln uv}\right)\right]
\end{aligned}$$

with

$$B_2^z(s) = s\frac{\Psi(z)-\Phi(z)}{\Psi(z)}+\left[(1-s)+s\frac{\Phi(z)}{\Psi(z)}\right]A\left(\frac{s}{(1-s)\frac{\Psi(z)}{\Phi(z)}+s}\right) \tag{C.33}$$

Hence

$$B_2^z(s) = s\left(1-\frac{1}{K(z)}\right)+\left[\frac{s}{K(z)}+1-s\right]A\left(\frac{s}{(1-s)K(z)+s}\right)$$
$$= -\frac{s}{K(z)}[1-K(z)]+\frac{1}{K(z)}\left[(1-s)K(z)+s\right]A\left(\frac{s}{(1-s)K(z)+s}\right)$$

If $K(z)=1$

$$\hat{C}_{H^z}(u,v) = \exp\left[\ln(uv)B_3^z\left(\frac{\ln v}{\ln uv}\right)\right]$$

with

$$B_3^z(s) = A(s) \tag{C.34}$$

We have proved that whatever $K(z)$

$$B^z(s) = 1-\min(K(z),1)-s\min\left(\frac{1}{K(z)},1\right)[1-K(z)]$$
$$+\min\left(\frac{1}{K(z)},1\right)\left[(1-s)K(z)+s\right]A\left(\frac{s}{(1-s)K(z)+s}\right)$$
$$= 1-W(z)K(z)-sW(z)[1-K(z)]+W(z)\left[(1-s)K(z)+s\right]A\left(\frac{s}{(1-s)K(z)+s}\right)$$

We prove that B^z is a dependence function. It holds

$$\begin{aligned} B^z(0) &= 1-W(z)K(z)+W(z)K(z)A(0) \\ &= 1, \text{ since } A(0)=1 \end{aligned}$$

and

$$\begin{aligned} B^z(1) &= 1-W(z)K(z)-W(z)[1-K(z)]+W(z)A(1) \\ &= 1-W(z)+W(z)A(1) \\ &= 1, \text{ since } A(1)=1. \end{aligned}$$

We prove the upper and lower bounds for B^z.

Upper bound. Using $A(s)\le 1$ for all s in$[0,1]$

$$B^z(s) \le 1-W(z)K(z)-sW(z)[1-K(z)]+W(z)[(1-s)K(z)+s] = 1.$$

Lower bound. Using $A(s)\ge \max(s,1-s)$

$$B^z(s) \ge 1-W(z)K(z)-sW(z)[1-K(z)]+W(z)\max\left[s,(1-s)K(z)\right].$$

We prove that the RHS in the above display is larger than both s and $1-s$.
Since $\max\left[s,(1-s)K(z)\right]\ge s$,

$$\begin{aligned} RHS &\ge 1-W(z)K(z)-sW(z)[1-K(z)]+sW(z) \\ &= 1-(1-s)\min(K(z),1)\ge s. \end{aligned}$$

Since $\max\left[s,(1-s)K(z)\right]\ge (1-s)K(z)$,

$$\begin{aligned} RHS &\ge 1-W(z)K(z)-sW(z)[1-K(z)]+(1-s)W(z)K(z) \\ &= 1-sW(z)\ge 1-s \end{aligned}$$

as sought. It remains to prove that $B^z(s)$ is a convex function. Some calculus yields

$$\frac{\partial^2 B^z}{\partial s^2}(s) = \frac{W(z)K^2(z)}{[(1-s)K(z)+s]^3}\frac{\partial^2}{\partial t^2}A(t)\Big|_{t=\frac{s}{(1-s)K(z)+s}} \ge 0$$

as sought. ■

Appendix D. Proof of Proposition 6

Write

$$B^{z'}(s) = 1-W(z')K(z')-sW(z')[1-K(z')]+W(z')[(1-s)K(z')+s]A\left(\frac{s}{K(z')(1-s)+s}\right).$$

In the above display it holds

$$A\left(\frac{s}{K(z')(1-s)+s}\right) = A\left(\frac{\frac{s}{(1-s)\frac{K(z')}{K(z)}+s}}{K(z)\left(1-\frac{s}{(1-s)\frac{K(z')}{K(z)}+s}\right)+\frac{s}{(1-s)\frac{K(z')}{K(z)}+s}}\right).$$

The RHS in this latter expression can be written as a function of $B^z\left(\frac{s}{(1-s)\frac{K(z')}{K(z)}+s}\right)$. Some calculus yields

(D.35)

$$B^{z'}(s) = 1 - W(z')K(z') - sW(z')[K(z) - K(z')] - [1 - W(z)K(z)]\frac{W(z')}{W(z)}\left[(1-s)\frac{K(z')}{K(z)} + s\right]$$
$$+ \frac{W(z')}{W(z)}\left[(1-s)\frac{K(z')}{K(z)} + s\right] B^z\left(\frac{s}{(1-s)\frac{K(z')}{K(z)} + s}\right)$$

which is (3.21). ■

Appendix E. Proof of Proposition 9

We first prove that $\widetilde{C}$ is an evc. For this, we state that for all v in $[0,1]$,

(1) $\max(s, 1-s) \leq \mathbb{A}(s,v) \leq 1$
(2) $\lim_{s\to 0} \mathbb{A}(s,v) = \lim_{s\to 1} \mathbb{A}(s,v) = 1$
(3) the function $s \mapsto \mathbb{A}(s,v)$ is convex.

This will entail that $\overline{\mathbb{A}}$ defined in (5.27) is a dependence function. By Pickands Theorem (Proposition 1) it will follow that $\widetilde{C}$ is an evc.

(1) Proof of the upper bound. Since C is PQD, it holds

$$C(u,v) \geq \Pi(u,v) = uv.$$

Hence $C(v^{\frac{1}{s}-1}, v) \geq v^{\frac{1}{s}}$ which is to say that

$$\mathbb{A}(s,v) = \frac{s}{\log v}\log C(v^{\frac{1}{s}-1}, v) \leq 1.$$

The lower bound. Use the upper Frechet bound for copulas $C(u,v) \leq \min(u,v)$ to get

$$\log C(v^{\frac{1}{s}-1}, v) \leq \log\left[\min(v^{\frac{1}{s}-1}, v)\right] = \min\left[\left(\frac{1}{s}-1\right)\log v, \log v\right]$$

which yields

$$\frac{s}{\log v}\log \hat{C}(v^{\frac{1}{s}-1}, v) \geq \frac{1}{\log v}\min[(1-s)\log v, s\log v] = \max[s, 1-s].$$

(2) This holds true by the definition of $\mathbb{A}$.

(3) Let s_1, s_2 in $[0,1]$ and $0 \leq \lambda \leq 1$.

$$\begin{aligned}
\mathbb{A}(\lambda s_1 + (1-\lambda)s_2, v) &= \frac{\lambda s_1 + (1-\lambda)s_2}{\log v} \log C\Big(v^{\frac{1}{\lambda s_1 + (1-\lambda)s_2}-1}, v\Big) \\
&= \frac{\lambda s_1}{\log v} \log C\left(\exp\left[\left(\frac{1}{\lambda s_1 + (1-\lambda)s_2} - 1\right)\log v\right], v\right) \\
&+ \frac{(1-\lambda)s_2}{\log v} \log C\left(\exp\left[\left(\frac{1}{\lambda s_1 + (1-\lambda)s_2} - 1\right)\log v\right], v\right) \\
&=: \frac{\lambda s_1}{\log v} A + \frac{(1-\lambda)s_2}{\log v} B.
\end{aligned}$$

It holds

$$\begin{aligned}
\frac{\lambda s_1}{\log v} A &\leq \frac{\lambda s_1}{\log v} \log C\left(\exp\left[\left(\frac{1}{\lambda s_1} - 1\right)\log v\right], v\right) \\
&= \lambda \mathbb{A}(s_1, v) \frac{\log C\Big(v^{\frac{1}{\lambda s_1}-1}, v\Big)}{\log C\Big(v^{\frac{1}{s_1}-1}, v\Big)} \\
&\leq \lambda \mathbb{A}(s_1, v).
\end{aligned}$$

Use the same argument for $\frac{(1-\lambda)s_2}{\log v} B$ in order to conclude.

We now prove that $s \mapsto \overline{\mathbb{A}}(s)$ is also a dependence function, which proves that $\widetilde{C}$ is an evc.
Indeed,

$$\max(s, 1-s) \leq \overline{\mathbb{A}}(s) = \int_0^1 \mathbb{A}(s,v)dv \leq 1.$$

Furthermore $\overline{\mathbb{A}}(0) = \overline{\mathbb{A}}(1) = 1$ and $\overline{\mathbb{A}}$ is convex, using the convexity of $s \mapsto \mathbb{A}(s,v)$ for all v; this proves the claim.

Let $\alpha := \sup_{(u,v)\in\mathfrak{s}} \left| \exp\left[\log(uv)\mathbb{A}\left(\frac{\log v}{\log uv}, v\right)\right] - \exp\left[\log(uv)\overline{\mathbb{A}}\left(\frac{\log v}{\log uv}\right)\right] \right|$. It holds

$$\begin{aligned}
\alpha &= \sup_{(u,v)\in\mathfrak{s}} \left| \exp\left[\log(uv)\mathbb{A}\left(\frac{\log v}{\log uv}, v\right)\right] - \exp\left[\log(uv)\overline{\mathbb{A}}\left(\frac{\log v}{\log uv}\right)\right] \right| \\
&= \sup_{s,v} \left| \exp\left[\frac{\log v}{s}\mathbb{A}(s,v)\right] - \exp\left[\frac{\log v}{s}\overline{\mathbb{A}}(s)\right] \right|
\end{aligned}$$

Use the fact that for all a, b, $\left|e^a - e^b\right| \leq \max\left(e^a, e^b\right)|a-b|$ to obtain

$$\begin{aligned}
\alpha &\leq \sup_{s,v} \left[\left| \frac{\log v}{s}\left[\mathbb{A}(s,v) - \overline{\mathbb{A}}(s)\right] \right| \max\left(\exp\left[\frac{\log v}{s}\mathbb{A}(s,v)\right], \exp\left[\frac{\log v}{s}\overline{\mathbb{A}}(s)\right] \right) \right] \\
&\leq \sup_{s,v} \left[\left| \frac{\log v}{s}\left[\mathbb{A}(s,v) - \overline{\mathbb{A}}(s)\right] \right| \exp\left[\frac{\log v}{s} \min\left(\mathbb{A}(s,v), \overline{\mathbb{A}}(s)\right)\right] \right] \\
&\leq \sup_{s,v} \left| \mathbb{A}(s,v) - \overline{\mathbb{A}}(s) \right| \sup_{s,v} \left(\left| \frac{\log v}{s} \right| \exp\left[\frac{\log v}{s} \min\left(\mathbb{A}(s,v), \overline{\mathbb{A}}(s)\right)\right] \right).
\end{aligned}$$

Note that $\sup_{s,v} \left| \mathbb{A}(s,v) - \overline{\mathbb{A}}(s) \right| = \underset{\mathfrak{s}}{\mathrm{Osc}}(\mathbb{A})$. Since $\mathbb{A}(s,v) \geq \max(s, 1-s)$ and $\overline{\mathbb{A}}(s) \geq \max(s, 1-s)$

$$\min[\mathbb{A}(s,v), \overline{\mathbb{A}}(s)] \geq \max(s, 1-s) \geq \frac{1}{2}.$$

Hence

$$\frac{\log v}{s}\min\Big(\mathbb{A}(s,v),\overline{\mathbb{A}}(s)\Big)\leq\frac{\log v}{2s}$$

which in turn yields

$$\sup_{s,v}\left(\left|\frac{\log v}{s}\right|\exp\left[\frac{\log v}{s}\min\Big(\mathbb{A}(s,v),\overline{\mathbb{A}}(s)\Big)\right]\right)\leq\sup_{s,v}\left|\frac{\log v}{s}\right|e^{\frac{\log v}{2s}}=\sup_{x\geq 0}xe^{-\frac{x}{2}}=2e^{-1}.$$

■

References

DeMasi, R. A., B. Qaqish, and P. K. Sen (1998). A family of bivariate failure time distributions with proportional crude and conditional hazards. *Comm. Statist. Theory Methods 27*(2), 365–384.

Depire, A. (2008). *Modélisation Markovien - Modèles de régression de copules et valeurs extrêmes - Application aux systèmes d'aide à la conduite.* Ph. D. thesis, Université de Reims Champagne-Ardenne.

Drouet, D. and V. Monbet (2004). Construction de copules asymétriques avec dépendance extrême. Application à des données océanométéorologiques. *Actes XXXIVe Journées de Statistique, Montpellier*.

Frees, E. W. and E. A. Valdez (2007). Understanding relationships using copulas', north american actuarial journal, 2 (1), january, 1-25. *International Library Of Critical Writings In Economics 211*, 401.

Genest, C. and L. Rivest (1989, August). A characterization of Gumbel's family of extreme value distributions. *Statistics & Probability Letters 8*(3), 207–211.

Hürlimann, W. (2003). Hutchinson-Lai's conjecture for bivariate extreme value copulas. *Statistics & Probability Letters 61*(2), 191–198.

Joe, H. (1997). *Multivariate models and dependence concepts.* Boca Raton ; London ; New York: Chapman & Hall/CRC.

Khoudraji, A. (1995). *Contributions à l'études des copules et à la modélisation de valeurs extrêmes bivariées.* Ph. D. thesis, Université Laval, Québec, Canada.

Marshall, A. W. and I. Olkin (1988). Families of multivariate distributions. *Journal of the American Statistical Association*, 834–841.

Nelsen, R. B. (2006). *An introduction to copulas* (2nd ed.). New York ; Berlin ; Heidelberg: Springer.

Oakes, D. (1989). Bivariate survival models induced by frailties. *Journal of the American Statistical Association*, 487–493.

Ravishanker, N. and D. K. Dey (2000). Multivariate survival models with a mixture of positive stable frailties. *Methodology and Computing in Applied Probability 2*, 293–308.

Resnick, S. I. (1987). *Extreme values, regular variation, and point processes*, Volume 4 of *Applied Probability. A Series of the Applied Probability Trust.* New York: Springer-Verlag.

Tawn, J. A. (1988). Bivariate extreme value theory: models and estimation. *Biometrika 75*(3), 397–415.

(Mohamed Achibi) Université Pierre et Marie Curie, LSTA-Paris 6, 175 Rue du Chevaleret, 75013 Paris, France
Snecma, Site de Villaroche, Rond-Point René Ravaud, 77550 Moissy Cramayel, France
E-mail address: mohamed.achibi@snecma.fr

(Michel Broniatowski) Université Pierre et Marie Curie, LSTA-Paris 6, 175 Rue du Chevaleret, 75013 Paris, France,
E-mail address: michel.broniatowski@upmc.fr