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# The primitive idempotents of the $p$ -permutation ring

Serge Bouc and Jacques Thévenaz

**Abstract :** Let  $G$  be a finite group, let  $p$  be a prime number, and let  $K$  be a field of characteristic 0 and  $k$  be a field of characteristic  $p$ , both large enough. In this note we state explicit formulae for the primitive idempotents of  $K \otimes_{\mathbb{Z}} pp_k(G)$ , where  $pp_k(G)$  is the ring of  $p$ -permutation  $kG$ -modules.

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**Key words :**  $p$ -permutation, idempotent, trivial source.

## 1. Introduction

Let  $G$  be a finite group, let  $p$  be a prime number, and let  $K$  be a field of characteristic 0 and  $k$  be a field of characteristic  $p$ , both large enough. In this note we state explicit formulae for the primitive idempotents of  $K \otimes_{\mathbb{Z}} pp_k(G)$ , where  $pp_k(G)$  is the ring of  $p$ -permutation  $kG$ -modules (also called the trivial source ring).

To obtain these formulae, we first use induction and restriction to reduce to the case where  $G$  is cyclic modulo  $p$ , i.e.  $G$  has a normal Sylow  $p$ -subgroup with cyclic quotient. Then we solve the easy and well known case where  $G$  is a cyclic  $p'$ -group. Finally we conclude by using the natural ring homomorphism from the Burnside ring  $B(G)$  of  $G$  to  $pp_k(G)$ , and the classical formulae for the primitive idempotents of  $K \otimes_{\mathbb{Z}} B(G)$ .

Our formulae are an essential tool in [2], where Cartan matrices of Mackey algebras are considered, and some invariants of these matrices (determinant, rank) are explicitly computed.

## 2. $p$ -permutation modules

### 2.1. Notation.

- Throughout the paper,  $G$  will be a fixed finite group and  $p$  a fixed prime number. We consider a field  $k$  of characteristic  $p$  and we denote by  $kG$  the group algebra of  $G$  over  $k$ . We assume that  $k$  is large enough in the sense that it is a splitting field for every group algebra  $k(N_G(P)/P)$ , where  $P$  runs through the set of all  $p$ -subgroups of  $G$ .

- We let  $K$  be a field of characteristic 0 and we assume that  $K$  is large enough in the sense that it contains the values of all the Brauer characters of the groups  $N_G(P)/P$ , where  $P$  runs through the set of all  $p$ -subgroups of  $G$ .

We recall quickly how Brauer characters are defined. We let  $\bar{k}$  be an extension of  $k$  containing all the  $n$ -th roots of unity, where  $n$  is the  $p'$ -part of the exponent of  $G$ . We choose an isomorphism  $\theta : \mu_n(\bar{k}) \rightarrow \mu_n(\mathbb{C})$  from the group of  $n$ -th roots of unity in  $\bar{k}$  and the corresponding group in  $\mathbb{C}$ . If  $V$  is an  $r$ -dimensional  $kH$ -module for the group  $H = N_G(P)/P$  and if  $s$  is an element of the set  $H_{p'}$  of all  $p'$ -elements of  $H$ , the matrix of the action of  $s$  on  $V$  has eigenvalues  $(\lambda_1, \dots, \lambda_r)$  in the group  $\mu_n(\bar{k})$ . The *Brauer character*  $\phi_V$  of  $V$  is the central function defined on  $H_{p'}$ , with values in the field  $\mathbb{Q}[\mu_n(\mathbb{C})]$ , sending  $s$  to  $\sum_{i=1}^r \theta(\lambda_i)$ . The actual values of Brauer characters may lie in a subfield of  $\mathbb{Q}[\mu_n(\mathbb{C})]$  and we simply require that  $K$  contains all these values.

**2.2. Remark :** Let  $V$  be as above and let  $W$  be a  $t$ -dimensional  $kH$ -module. If  $s$  has eigenvalues  $(\mu_1, \dots, \mu_t)$  on  $W$ , its eigenvalues for the diagonal action of  $H$  on  $V \otimes_k W$  are  $(\lambda_i \mu_j)_{1 \leq i \leq r, 1 \leq j \leq t}$ . It follows that  $\phi_{V \otimes_k W}(s) = \sum_{i=1}^r \sum_{j=1}^t \theta(\lambda_i \mu_j) = \phi_V(s) \phi_W(s)$ .

- When  $H$  is a subgroup of  $G$ , and  $M$  is a  $kG$ -module, we denote by  $\text{Res}_H^G M$  the  $kH$ -module obtained by restricting the action of  $G$  to  $H$ . When  $L$  is a  $kH$ -module, we denote by  $\text{Ind}_H^G L$  the induced  $kG$ -module.

- When  $M$  is a  $kG$ -module, and  $P$  is a subgroup of  $G$ , the  $k$ -vector space of fixed points of  $P$  on  $M$  is denoted by  $M^P$ . When  $Q \leq P$  are subgroups of  $G$ , the *relative trace* is the map  $\text{tr}_Q^P : M^Q \rightarrow M^P$  defined by  $\text{tr}_Q^P(m) = \sum_{x \in [P/Q]} x \cdot m$ .

- When  $M$  is a  $kG$ -module, the *Brauer quotient* of  $M$  at  $P$  is the  $k$ -vector space

$$M[P] = M^P / \sum_{Q < P} \text{tr}_Q^P M^Q .$$

This  $k$ -vector space has a natural structure of  $k\overline{N}_G(P)$ -module, where as usual  $\overline{N}_G(P) = N_G(P)/P$ . It is equal to zero if  $P$  is not a  $p$ -group.

- If  $P$  is a normal  $p$ -subgroup of  $G$  and  $M$  is a  $k(G/P)$ -module, denote by  $\text{Inf}_{G/P}^G M$  the  $kG$ -module obtained from  $M$  by inflation to  $G$ . Then there is an isomorphism

$$(\text{Inf}_{G/P}^G M)[P] \cong M$$

of  $k(G/P)$ -modules.

- When  $G$  acts on a set  $X$  (on the left), and  $x, y \in X$ , we write  $x =_G y$  if  $x$  and  $y$  are in the same  $G$ -orbit. We denote by  $[G \backslash X]$  a set of representatives

of  $G$ -orbits on  $X$ , and by  $X^G$  the set of fixed points of  $G$  on  $X$ . For  $x \in X$ , we denote by  $G_x$  its stabilizer in  $G$ .

**2.3. Review of  $p$ -permutation modules.** We begin by recalling some definitions and basic results. We refer to [3], and to [1] Sections 3.11 and 5.5 for details :

**2.4. Definition.** A permutation  $kG$ -module is a  $kG$ -module admitting a  $G$ -invariant  $k$ -basis. A  $p$ -permutation  $kG$ -module  $M$  is a  $kG$ -module such that  $\text{Res}_S^G M$  is a permutation  $kS$ -module, where  $S$  is a Sylow  $p$ -subgroup of  $G$ .

The  $p$ -permutation  $kG$ -modules are also called *trivial source modules*, because the indecomposable ones coincide with the indecomposable modules having a trivial source (see [3] 0.4). Moreover, the  $p$ -permutation modules also coincide with the direct summands of permutation modules (see [1], Lemma 3.11.2).

**2.5. Proposition.**

1. If  $H$  is a subgroup of  $G$ , and  $M$  is a  $p$ -permutation  $kG$ -module, then the restriction  $\text{Res}_H^G M$  of  $M$  to  $H$  is a  $p$ -permutation  $kH$ -module.
2. If  $H$  is a subgroup of  $G$ , and  $L$  is a  $p$ -permutation  $kH$ -module, then the induced module  $\text{Ind}_H^G L$  is a  $p$ -permutation  $kG$ -module.
3. If  $N$  is a normal subgroup of  $G$ , and  $L$  is a  $p$ -permutation  $k(G/N)$ -module, the inflated module  $\text{Inf}_{G/N}^G L$  is a  $p$ -permutation  $kG$ -module.
4. If  $P$  is a  $p$ -group, and  $M$  is a permutation  $kP$ -module with  $P$ -invariant basis  $X$ , then the image of the set  $X^P$  in  $M[P]$  is a  $k$ -basis of  $M[P]$ .
5. If  $P$  is a  $p$ -subgroup of  $G$ , and  $M$  is a  $p$ -permutation  $kG$ -module, then the Brauer quotient  $M[P]$  is a  $p$ -permutation  $k\overline{N}_G(P)$ -module.
6. If  $M$  and  $N$  are  $p$ -permutation  $kG$ -modules, then their tensor product  $M \otimes_k N$  is again a  $p$ -permutation  $kG$ -module.

**Proof :** Assertions 1,2,3, and 6 are straightforward consequences of the same assertions for *permutation* modules. For Assertion 4, see [3] 1.1.(3). Assertion 5 follows easily from Assertion 4 (see also [3] 3.1).  $\square$

This leads to the following definition :

**2.6. Definition.** The  $p$ -permutation ring  $pp_k(G)$  is the Grothendieck group of the category of  $p$ -permutation  $kG$ -modules, with relations corresponding to direct sum decompositions, i.e.  $[M] + [N] = [M \oplus N]$ . The ring structure

on  $pp_k(G)$  is induced by the tensor product of modules over  $k$ . The identity element of  $pp_k(G)$  is the class of the trivial  $kG$ -module  $k$ .

As the Krull-Schmidt theorem holds for  $kG$ -modules, the additive group  $pp_k(G)$  is a free (abelian) group on the set of isomorphism classes of indecomposable  $p$ -permutation  $kG$ -modules. These modules have the following properties :

**2.7. Theorem.** [ [3] Theorem 3.2]

1. The vertices of an indecomposable  $p$ -permutation  $kG$ -module  $M$  are the maximal  $p$ -subgroups  $P$  of  $G$  such that  $M[P] \neq \{0\}$ .
2. An indecomposable  $p$ -permutation  $kG$ -module has vertex  $P$  if and only if  $M[P]$  is a non-zero projective  $k\overline{N}_G(P)$ -module.
3. The correspondence  $M \mapsto M[P]$  induces a bijection between the isomorphism classes of indecomposable  $p$ -permutation  $kG$ -modules with vertex  $P$  and the isomorphism classes of indecomposable projective  $k\overline{N}_G(P)$ -modules.

**2.8. Notation.** Let  $\mathcal{P}_{G,p}$  denote the set of pairs  $(P, E)$ , where  $P$  is a  $p$ -subgroup of  $G$ , and  $E$  is an indecomposable projective  $k\overline{N}_G(P)$ -module. The group  $G$  acts on  $\mathcal{P}_{G,p}$  by conjugation, and we denote by  $[\mathcal{P}_{G,p}]$  a set of representatives of  $G$ -orbits on  $\mathcal{P}_{G,p}$ .

For  $(P, E) \in \mathcal{P}_{G,p}$ , let  $M_{P,E}$  denote the (unique up to isomorphism) indecomposable  $p$ -permutation  $kG$ -module such that  $M_{P,E}[P] \cong E$ .

**2.9. Corollary.** The classes of the modules  $M_{P,E}$ , for  $(P, E) \in [\mathcal{P}_{G,p}]$  form a  $\mathbb{Z}$ -basis of  $pp_k(G)$ .

**2.10. Notation.** The operations  $\text{Res}_H^G$ ,  $\text{Ind}_H^G$ ,  $\text{Inf}_{G/N}^G$  extend linearly to maps between the corresponding  $p$ -permutations rings, denoted with the same symbol.

The maps  $\text{Res}_H^G$  and  $\text{Inf}_{G/N}^G$  are ring homomorphisms, whereas  $\text{Ind}_H^G$  is not in general. Similarly :

**2.11. Proposition.** Let  $P$  be a  $p$ -subgroup of  $G$ . Then the correspondence  $M \mapsto M[P]$  induces a ring homomorphism  $\text{Br}_P^G : pp_k(G) \rightarrow pp_k(\overline{N}_G(P))$ .

**Proof :** Let  $M$  and  $N$  be  $p$ -permutation  $kG$ -modules. The canonical bilinear map  $M \times N \rightarrow M \otimes_k N$  is  $G$ -equivariant, hence it induces a bilinear map  $\beta_P : M[P] \times N[P] \rightarrow (M \otimes_k N)[P]$  (see [3] 1.2), which is  $\overline{N}_G(P)$ -equivariant. Now if  $X$  is a  $P$ -invariant  $k$ -basis of  $M$ , and  $Y$  a  $P$ -invariant  $k$ -basis of  $N$ ,

then  $X \times Y$  is a  $P$ -invariant basis of  $M \otimes_k N$ . The images of the sets  $X^P, Y^P$ , and  $(X \times Y)^P$  are bases of  $M[P], N[P]$ , and  $(M \otimes_k N)[P]$ , respectively, and the restriction of  $\beta_P$  to these bases is the canonical bijection  $X^P \times Y^P \rightarrow (X \times Y)^P$ . It follows that  $\beta_P$  induces an isomorphism  $M[P] \otimes_k N[P] \rightarrow (M \otimes_k N)[P]$  of  $k\overline{N}_G(P)$ -modules. Proposition 2.11 follows.  $\square$

**2.12. Notation.** Let  $\mathcal{Q}_{G,p}$  denote the set of pairs  $(P, s)$ , where  $P$  is a  $p$ -subgroup of  $G$ , and  $s$  is a  $p'$ -element of  $\overline{N}_G(P)$ . The group  $G$  acts on  $\mathcal{Q}_{G,p}$ , and we denote by  $[\mathcal{Q}_{G,p}]$  a set of representatives of  $G$ -orbits on  $\mathcal{Q}_{G,p}$ .

If  $(P, s) \in \mathcal{Q}_{G,p}$ , we denote by  $N_G(P, s)$  the stabilizer of  $(P, s)$  in  $G$ , and by  $\langle Ps \rangle$  the subgroup of  $N_G(P)$  generated by  $Ps$  (i.e. the inverse image in  $N_G(P)$  of the cyclic group  $\langle s \rangle$  of  $\overline{N}_G(P)$ ).

**2.13. Remarks :**

- When  $H$  is a subgroup of  $G$ , there is a natural inclusion of  $\mathcal{Q}_{H,p}$  into  $\mathcal{Q}_{G,p}$ , as  $\overline{N}_H(P) \leq \overline{N}_G(P)$  for any  $p$ -subgroup  $P$  of  $H$ . We will consider  $\mathcal{Q}_{H,p}$  as a subset of  $\mathcal{Q}_{G,p}$ .
- When  $(P, s) \in \mathcal{Q}_{G,p}$ , the group  $N_G(P, s)$  is the set of elements  $g$  in  $N_G(P)$  whose image in  $\overline{N}_G(P)$  centralizes  $s$ . In other words, there is a short exact sequence of groups

$$(2.14) \quad \mathbf{1} \rightarrow P \rightarrow N_G(P, s) \rightarrow C_{\overline{N}_G(P)}(s) \rightarrow \mathbf{1} .$$

In particular  $N_G(P, s)$  is a subgroup of  $N_G(\langle Ps \rangle)$ .

**2.15. Notation.** Let  $(P, s) \in \mathcal{Q}_{G,p}$ . Let  $\tau_{P,s}^G$  denote the additive map from  $pp_k(G)$  to  $K$  sending the class of a  $p$ -permutation  $kG$ -module  $M$  to the value at  $s$  of the Brauer character of the  $\overline{N}_G(P)$ -module  $M[P]$ .

**2.16. Remarks :**

- It is clear that  $\tau_{P,s}^G(M)$  only depends on the restriction of  $M$  to the group  $\langle Ps \rangle$ . In other words

$$\tau_{P,s}^G = \tau_{P,s}^{\langle Ps \rangle} \circ \text{Res}_{\langle Ps \rangle}^G .$$

Furthermore, it is clear from the definition that

$$(2.17) \quad \tau_{P,s}^G = \tau_{\mathbf{1},s}^{\langle Ps \rangle / P} \circ \text{Br}_P^{\langle Ps \rangle} \circ \text{Res}_{\langle Ps \rangle}^G .$$

- It is easy to see that  $\tau_{P,s}^G$  only depends on the  $G$ -orbit of  $(P, s)$ , that is,  $\tau_{P^g, s^g}^G = \tau_{P,s}^G$  for every  $g \in G$ .

The following proposition is Corollary 5.5.5 in [1], but our construction of the species is slightly different (but equivalent, of course). For this reason, we sketch an independent proof :

### 2.18. Proposition.

1. The map  $\tau_{P,s}^G$  is a ring homomorphism  $pp_k(G) \rightarrow K$  and extends to a  $K$ -algebra homomorphism (a species)  $\tau_{P,s}^G : K \otimes_{\mathbb{Z}} pp_k(G) \rightarrow K$ .
2. The set  $\{\tau_{P,s}^G \mid (P, s) \in [\mathcal{Q}_{G,p}]\}$  is the set of all distinct species from  $K \otimes_{\mathbb{Z}} pp_k(G)$  to  $K$ . These species induce a  $K$ -algebra isomorphism

$$T = \prod_{(P,s) \in [\mathcal{Q}_{G,p}]} \tau_{P,s}^G : K \otimes_{\mathbb{Z}} pp_k(G) \rightarrow \prod_{(P,s) \in [\mathcal{Q}_{G,p}]} K .$$

**Proof :** By 2.17, to prove Assertion 1, it suffices to prove that  $\tau_{\mathbf{1},s}^{\langle Ps \rangle / P}$  is a ring homomorphism, since both  $\text{Res}_{\langle Ps \rangle}^G$  and  $\text{Br}_P^{\langle Ps \rangle}$  are ring homomorphisms. In other words, we can assume that  $P = \mathbf{1}$ . Now the value of  $\tau_{\mathbf{1},s}^G$  on the class of a  $kG$ -module  $M$  is the value  $\phi_M(s)$  of the Brauer character of  $M$  at  $s$ , so Assertion 1 follows from Remark 2.2.

For Assertion 2, it suffices to prove that  $T$  is an isomorphism. Since  $[\mathcal{P}_{G,p}]$  and  $[\mathcal{Q}_{G,p}]$  have the same cardinality, the matrix  $\mathcal{M}$  of  $T$  is a square matrix. Let  $(P, E) \in \mathcal{P}_{G,p}$ , and  $(Q, s) \in \mathcal{Q}_{G,p}$ . Then  $\tau_{Q,s}(M_{P,E})$  is equal to zero if  $Q$  is not contained in  $P$  up to  $G$ -conjugation, because in this case  $M_{P,E}(Q) = \{0\}$  by Theorem 2.7. It follows that  $\mathcal{M}$  is block triangular. As moreover  $M_{P,E}[P] \cong E$ , we have that  $\tau_{P,s}(M_{P,E}) = \phi_E(s)$ . This means that the diagonal block of  $\mathcal{M}$  corresponding to  $P$  is the matrix of Brauer characters of projective  $k\overline{N}_G(P)$ -modules, and these are linearly independent by Lemma 5.3.1 of [1]. It follows that all the diagonal blocks of  $\mathcal{M}$  are non singular, so  $\mathcal{M}$  is invertible, and  $T$  is an isomorphism.  $\square$

**2.19. Corollary.** *The algebra  $K \otimes_{\mathbb{Z}} pp_k(G)$  is a split semisimple commutative  $K$ -algebra. Its primitive idempotents  $F_{P,s}^G$  are indexed by  $[\mathcal{Q}_{G,p}]$ , and the idempotent  $F_{P,s}^G$  is characterized by*

$$\forall (R, u) \in \mathcal{Q}_{G,p}, \quad \tau_{R,u}^G(F_{P,s}^G) = \begin{cases} 1 & \text{if } (R, u) =_G (P, s) \\ 0 & \text{otherwise.} \end{cases}$$

## 3. Restriction and induction

**3.1. Proposition.** *Let  $H \leq G$ , and  $(P, s) \in \mathcal{Q}_{G,p}$ . Then*

$$\text{Res}_H^G F_{P,s}^G = \sum_{(Q,t)} F_{Q,t}^H ,$$

where  $(Q, t)$  runs through a set of representatives of  $H$ -conjugacy classes of  $G$ -conjugates of  $(P, s)$  contained in  $H$ .

**Proof :** Indeed, as  $\text{Res}_H^G$  is an algebra homomorphism, the element  $\text{Res}_H^G F_{P,s}^G$  is an idempotent of  $K \otimes_{\mathbb{Z}} \text{pp}_k(H)$ , hence it is equal to a sum of some distinct primitive idempotents  $F_{Q,t}^H$ . The idempotent  $F_{Q,t}^H$  appears in this decomposition if and only if  $\tau_{Q,t}^H(\text{Res}_H^G F_{P,s}^G) = 1$ . By Remark 2.16

$$\begin{aligned} \tau_{Q,t}^H(\text{Res}_H^G F_{P,s}^G) &= \tau_{Q,t}^{\langle Qt \rangle}(\text{Res}_{\langle Qt \rangle}^H \text{Res}_H^G F_{P,s}^G) \\ &= \tau_{Q,t}^{\langle Qt \rangle}(\text{Res}_{\langle Qt \rangle}^G F_{P,s}^G) \\ &= \tau_{Q,t}^G(F_{P,s}^G) . \end{aligned}$$

Now  $\tau_{Q,t}^G(F_{P,s}^G)$  is equal to 1 if and only if  $(Q, t)$  and  $(P, s)$  are  $G$ -conjugate. This completes the proof.  $\square$

**3.2. Proposition.** *Let  $H \leq G$ , and  $(Q, t) \in \mathcal{Q}_{H,p}$ . Then*

$$\text{Ind}_H^G F_{Q,t}^H = |N_G(Q, t) : N_H(Q, t)| F_{Q,t}^G .$$

**Proof :** Since  $K \otimes_{\mathbb{Z}} \text{pp}_k(G)$  is a split semisimple commutative  $K$ -algebra, any element  $X$  in  $K \otimes_{\mathbb{Z}} \text{pp}_k(G)$  can be written

$$(3.3) \quad X = \sum_{(P,s) \in [\mathcal{Q}_{G,p}]} \tau_{P,s}^G(X) F_{P,s}^G ,$$

and moreover for any  $(P, s) \in \mathcal{Q}_{G,p}$

$$\tau_{P,s}^G(X) F_{P,s}^G = X \cdot F_{P,s}^G .$$

Setting  $X = \text{Ind}_H^G F_{Q,t}^H$  in this equation gives

$$\begin{aligned} \tau_{P,s}^G(\text{Ind}_H^G F_{Q,t}^H) F_{P,s}^G &= (\text{Ind}_H^G F_{Q,t}^H) \cdot F_{P,s}^G \\ &= \text{Ind}_H^G(F_{Q,t}^H \cdot \text{Res}_H^G F_{P,s}^G) . \end{aligned}$$

By Proposition 3.1, the element  $\text{Res}_H^G F_{P,s}^G$  is equal to the sum of the distinct idempotents  $F_{P^y, s^y}^H$  associated to elements  $y$  of  $G$  such that  $\langle Ps \rangle^y \leq H$ . The product  $F_{Q,t}^H \cdot F_{P^y, s^y}^H$  is equal to zero, unless  $(Q, t)$  is  $H$ -conjugate to  $(P^y, s^y)$ , which implies that  $(Q, t)$  and  $(P, s)$  are  $G$ -conjugate. It follows that the only non zero term in the right hand side of Equation 3.3 is the term corresponding to  $(Q, t)$ . Hence

$$\text{Ind}_H^G F_{Q,t}^H = \tau_{Q,t}^G(\text{Ind}_H^G F_{Q,t}^H) F_{Q,t}^G .$$



Now by Remark 2.16 and the Mackey formula

$$\begin{aligned}\tau_{Q,t}^G(\text{Ind}_H^G F_{Q,t}^H) &= \tau_{Q,t}^{\langle Qt \rangle}(\text{Res}_{\langle Qt \rangle}^G \text{Ind}_H^G F_{Q,t}^H) \\ &= \tau_{Q,t}^{\langle Qt \rangle} \left( \sum_{x \in \langle Qt \rangle \backslash G/H} \text{Ind}_{\langle Qt \rangle \cap xH}^{\langle Qt \rangle} {}^x \text{Res}_{\langle Qt \rangle \cap xH}^H F_{Q,t}^H \right).\end{aligned}$$

By Proposition 3.1, the element  $\text{Res}_{\langle Qt \rangle \cap xH}^H F_{Q,t}^H$  is equal to the sum of the distinct idempotents  $F_{Q^y, ty}^{\langle Qt \rangle \cap xH}$  corresponding to elements  $y \in H$  such that  $\langle Qt \rangle^y \leq \langle Qt \rangle \cap xH$ . This implies  $\langle Qt \rangle^y = \langle Qt \rangle^x$ , i.e.  $y \in N_G(\langle Qt \rangle)x$ , thus  $x \in N_G(\langle Qt \rangle) \cdot H$ . This gives

$$\begin{aligned}\tau_{Q,t}^G(\text{Ind}_H^G F_{Q,t}^H) &= \tau_{Q,t}^{\langle Qt \rangle} \left( \sum_{\substack{x \in N_G(\langle Qt \rangle)H/H \\ y \in N_H(Q,t) \backslash N_G(\langle Qt \rangle)x}} {}^x F_{Q^y, ty}^{\langle Qt \rangle} \right) \\ &= \sum_{\substack{x \in N_G(\langle Qt \rangle)/N_H(\langle Qt \rangle) \\ y \in N_H(Q,t) \backslash N_G(\langle Qt \rangle)x}} \tau_{Q,t}^{\langle Qt \rangle} (F_{Q^{yx^{-1}}, tyx^{-1}}^{\langle Qt \rangle}) \\ &= \sum_{z \in N_H(Q,t) \backslash N_G(\langle Qt \rangle)} \tau_{Q,t}^{\langle Qt \rangle} (F_{Q^z, tz}^{\langle Qt \rangle}),\end{aligned}$$

where  $z = yx^{-1}$ . Finally  $\tau_{Q,t}^{\langle Qt \rangle} (F_{Q^z, tz}^{\langle Qt \rangle})$  is equal to 1 if  $(Q^z, tz)$  is conjugate to  $(Q, t)$  in  $\langle Qt \rangle$ , and to zero otherwise.

If  $u \in \langle Qt \rangle$  is such that  $(Q^z, tz)^u = (Q, t)$ , then  $zu \in N_G(Q, t)$ . But since  $[\langle Qt \rangle, t] \leq Q$ , we have  $\langle Qt \rangle \leq N_G(Q, t)$ , so  $u \in N_G(Q, t)$ , hence  $z \in N_G(Q, t)$ , and  $(Q^z, tz) = (Q, t)$ . It follows that

$$\tau_{Q,t}^G(\text{Ind}_H^G F_{Q,t}^H) = |N_G(Q, t) : N_H(Q, t)|,$$

which completes the proof of the proposition.  $\square$

**3.4. Corollary.** *Let  $(P, s) \in \mathcal{Q}_{G,p}$ . Then*

$$F_{P,s}^G = \frac{|s|}{|C_{\overline{N}_G(P)}(s)|} \text{Ind}_{\langle Ps \rangle}^G F_{P,s}^{\langle Ps \rangle}.$$

**Proof :** Apply Proposition 3.2 with  $(Q, t) = (P, s)$  and  $H = \langle Ps \rangle$ . Then  $N_H(Q, t) = \langle Ps \rangle$ , thus by Exact sequence 2.14

$$|N_G(Q, t) : N_H(Q, t)| = \frac{|P| |C_{\overline{N}_G(P)}(s)|}{|P| |s|} = \frac{|C_{\overline{N}_G(P)}(s)|}{|s|},$$

and the corollary follows.  $\square$

## 4. Idempotents

It follows from Corollary 3.4 that, in order to find formulae for the primitive idempotents  $F_{P,s}^G$  of  $K \otimes_{\mathbb{Z}} pp_k(G)$ , it suffices to find the formula expressing the idempotent  $F_{P,s}^{\langle Ps \rangle}$ . In other words, we can assume that  $G = \langle Ps \rangle$ , i.e. that  $G$  has a normal Sylow  $p$ -subgroup  $P$  with cyclic quotient generated by  $s$ .

**4.1. A morphism from the Burnside ring.** When  $G$  is an arbitrary finite group, there is an obvious ring homomorphism  $\mathcal{L}_G$  from the Burnside ring  $B(G)$  to  $pp_k(G)$ , induced by the *linearization* operation, sending a finite  $G$ -set  $X$  to the permutation module  $kX$ , which is obviously a  $p$ -permutation module. This morphism also commutes with restriction and induction : if  $H \leq G$ , then

$$(4.2) \quad \mathcal{L}_H \circ \text{Res}_H^G = \text{Res}_H^G \circ \mathcal{L}_G, \quad \mathcal{L}_G \circ \text{Ind}_H^G = \text{Ind}_H^G \circ \mathcal{L}_H.$$

Indeed, for any  $G$ -set  $X$ , the  $kH$ -modules  $k\text{Res}_H^G X$  and  $\text{Res}_H^G(kX)$  are isomorphic, and for any  $H$ -set  $Y$ , the  $kG$ -modules  $k\text{Ind}_H^G Y$  and  $\text{Ind}_H^G(kY)$  are isomorphic.

Similarly, when  $P$  is a  $p$ -subgroup of  $G$ , the ring homomorphism  $\Phi_P : B(G) \rightarrow B(\overline{N}_G(P))$  induced by the operation  $X \mapsto X^P$  on  $G$ -sets, is compatible with the Brauer morphism  $\text{Br}_P^G : pp_k(G) \rightarrow pp_k(\overline{N}_G(P))$  :

$$(4.3) \quad \mathcal{L}_{\overline{N}_G(P)} \circ \Phi_P = \text{Br}_P^G \circ \mathcal{L}_G.$$

This is because for any  $G$ -set  $X$ , the  $k\overline{N}_G(P)$ -modules  $k(X^P)$  and  $(kX)[P]$  are isomorphic.

The ring homomorphism  $\mathcal{L}_G$  extends linearly to a  $K$ -algebra homomorphism  $K \otimes_{\mathbb{Z}} B(G) \rightarrow K \otimes_{\mathbb{Z}} pp_k(G)$ , still denoted by  $\mathcal{L}_G$ . The algebra  $K \otimes_{\mathbb{Z}} B(G)$  is also a split semisimple commutative  $K$ -algebra. Its species are the  $K$ -algebra maps

$$K \otimes_{\mathbb{Z}} B(G) \rightarrow K, \quad X \mapsto |X^H|,$$

where  $H$  runs through the set of all subgroups of  $G$  up to conjugation. Here we denote by  $|X^H|$  the number of  $H$ -fixed points of a  $G$ -set  $X$  and this notation is then extended  $K$ -linearly to any  $X \in K \otimes_{\mathbb{Z}} B(G)$ . The primitive idempotents  $e_H^G$  of  $K \otimes_{\mathbb{Z}} B(G)$  are indexed by subgroups  $H$  of  $G$ , up to conjugation. They are given by the following formulae, found by Gluck ([4]) and later independently by Yoshida ([5]) :

$$(4.4) \quad e_H^G = \frac{1}{|N_G(H)|} \sum_{L \leq H} |L| \mu(L, H) G/L,$$

where  $\mu$  denotes the Möbius function of the poset of subgroups of  $G$ . The idempotent  $e_H^G$  is characterized by the fact that for any  $X \in K \otimes_{\mathbb{Z}} B(G)$

$$X \cdot e_H^G = |X^H| e_H^G .$$

**4.5. Remark :** Since  $|X^H|$  only depends on  $\text{Res}_H^G X$ , it follows in particular that  $X$  is a scalar multiple of the “top” idempotent  $e_G^G$  if and only if  $\text{Res}_H^G X = 0$  for any proper subgroup  $H$  of  $G$ . In particular, if  $N$  is a normal subgroup of  $G$ , then

$$(4.6) \quad (e_G^G)^N = e_{G/N}^{G/N} .$$

This is because for any proper subgroup  $H/N$  of  $G/N$

$$\text{Res}_{H/N}^{G/N} (e_G^G)^N = (\text{Res}_H^G e_G^G)^N = 0 .$$

So  $(e_G^G)^N$  is a scalar multiple of  $e_{G/N}^{G/N}$ . As it is also an idempotent, it is equal to 0 or  $e_{G/N}^{G/N}$ . Finally

$$|((e_G^G)^N)^{G/N}| = |(e_G^G)^G| = 1 ,$$

so  $(e_G^G)^N$  is non zero.

**4.7. The case of a cyclic  $p'$ -group.** Suppose that  $G$  is a cyclic  $p'$ -group, of order  $n$ , generated by an element  $s$ . In this case, there are exactly  $n$  group homomorphisms from  $G$  to the multiplicative group  $k^\times$  of  $k$ . For each of these group homomorphisms  $\varphi$ , let  $k_\varphi$  denote the  $kG$ -module  $k$  on which the generator  $s$  acts by multiplication by  $\varphi(s)$ . As  $G$  is a  $p'$ -group, this module is simple and projective. The (classes of the) modules  $k_\varphi$ , for  $\varphi \in \widehat{G} = \text{Hom}(G, k^\times)$ , form a basis of  $pp_k(G)$ .

Since moreover for  $\varphi, \psi \in \widehat{G}$ , the modules  $k_\varphi \otimes_k k_\psi$  and  $k_{\varphi\psi}$  are isomorphic, the algebra  $K \otimes_{\mathbb{Z}} pp_k(G)$  is isomorphic to the group algebra of the group  $\widehat{G}$ . This leads to the following classical formula :

**4.8. Lemma.** *Let  $G$  be a cyclic  $p'$ -group. Then for any  $t \in G$ ,*

$$F_{1,t}^G = \frac{1}{n} \sum_{\varphi \in \widehat{G}} \tilde{\varphi}(t^{-1}) k_\varphi ,$$

where  $\tilde{\varphi}$  is the Brauer character of  $k_\varphi$ .

**Proof :** Indeed for  $s, t \in G$

$$\tau_{\mathbf{1},t}^G\left(\frac{1}{n} \sum_{\varphi \in \widehat{G}} \tilde{\varphi}(s^{-1})k_\varphi\right) = \frac{1}{n} \sum_{\varphi \in \widehat{G}} \tilde{\varphi}(s^{-1})\tilde{\varphi}(t) = \delta_{s,t} ,$$

where  $\delta_{s,t}$  is the Kronecker symbol. □

**4.9. The case  $G = \langle Ps \rangle$ .** Suppose now more generally that  $G = \langle Ps \rangle$ , where  $P$  is a normal Sylow  $p$ -subgroup of  $G$  and  $s$  is a  $p'$ -element. In this case, by Proposition 3.1, the restriction of  $F_{P,s}^G$  to any proper subgroup of  $G$  is equal to zero. Moreover, since  $N_G(P, t) = G$  for any  $t \in G/P$ , the conjugacy class of the pair  $(P, t)$  reduces to  $\{(P, t)\}$ .

**4.10. Lemma.** *Suppose  $G = \langle Ps \rangle$ , and set  $E_G^G = \mathcal{L}_G(e_G^G)$ . Then*

$$E_G^G = \sum_{\langle t \rangle = \langle s \rangle} F_{P,t}^G .$$

**Proof :** By 4.2 and by Remark 4.5, the restriction of  $E_G^G$  to any proper subgroup of  $G$  is equal to zero. Let  $(Q, t) \in \mathcal{Q}_{G,p}$ , such that the group  $L = \langle Qt \rangle$  is a proper subgroup of  $G$ . By Proposition 3.2, there is a rational number  $r$  such that

$$F_{Q,t}^G = r \operatorname{Ind}_L^G F_{Q,t}^L .$$

It follows that

$$E_G^G \cdot F_{Q,t}^G = r \operatorname{Ind}_L^G ((\operatorname{Res}_L^G E_G^G) \cdot F_{Q,t}^L) = 0 .$$

Now  $E_G^G$  is an idempotent of  $K \otimes_{\mathbb{Z}} pp_k(G)$ , hence is it a sum of some of the primitive idempotents  $F_{Q,t}^G$  associated to pairs  $(Q, t)$  for which  $\langle Qt \rangle = G$ . This condition is equivalent to  $Q = P$  and  $\langle t \rangle = \langle s \rangle$ .

It remains to show that all these idempotents  $F_{P,t}^G$  appear in the decomposition of  $E_G^G$ , i.e. equivalently that  $\tau_{P,t}^G(E_G^G) = 1$  for any generator  $t$  of  $\langle s \rangle$ . Now by 4.6 and Remark 2.16

$$\tau_{P,t}^G(E_G^G) = \tau_{\mathbf{1},t}^{G/P}(\operatorname{Br}_P^G(E_G^G)) = \tau_{\mathbf{1},t}^{\langle s \rangle}(E_{\langle s \rangle}^{\langle s \rangle}) .$$

Now the value at  $t$  of the Brauer character of a permutation module  $kX$  is equal to the number of fixed points of  $t$  on  $X$ . By  $K$ -linearity, this gives

$$\tau_{\mathbf{1},t}^{\langle s \rangle}(E_{\langle s \rangle}^{\langle s \rangle}) = |(e_{\langle s \rangle}^{\langle s \rangle})^t| ,$$

and this is equal to 1 if  $t$  generates  $\langle s \rangle$ , and to 0 otherwise, as was to be shown. □

**4.11. Proposition.** *Let  $(P, s) \in \mathcal{Q}_{G,p}$ , and suppose that  $G = \langle Ps \rangle$ . Then*

$$F_{P,s}^G = E_G^G \cdot \text{Inf}_{G/P}^G F_{1,s}^{G/P} .$$

**Proof:** Set  $E_s = E_G^G \cdot \text{Inf}_{G/P}^G F_{1,s}^{G/P}$ . Then  $E_s$  is an idempotent of  $K \otimes_{\mathbb{Z}} \text{pp}_k(G)$ , as it is the product of two (commuting) idempotents. Let  $(Q, t) \in \mathcal{Q}_{G,p}$ . If  $\langle Qt \rangle \neq G$ , then  $\tau_{Q,t}^G(E_G^G) = 0$  by Lemma 4.10, thus  $\tau_{Q,t}^G(E_s) = 0$ . And if  $\langle Qt \rangle = G$ , then  $Q = P$  and  $\langle t \rangle = \langle s \rangle$ . In this case

$$\tau_{Q,t}^G(E_s) = \tau_{P,t}^G(E_G^G) \cdot \tau_{P,t}^G(\text{Inf}_{G/P}^G F_{1,s}^{G/P}) .$$

By Lemma 4.10, the first factor in the right hand side is equal to 1. The second factor is equal to

$$\begin{aligned} \tau_{P,t}^G(\text{Inf}_{G/P}^G F_{1,s}^{G/P}) &= \tau_{1,t}^{G/P} \text{Br}_P^G(\text{Inf}_{G/P}^G F_{1,s}^{G/P}) \\ &= \tau_{1,t}^{G/P}(F_{1,s}^{G/P}) = \delta_{t,s} , \end{aligned}$$

where  $\delta_{t,s}$  is the Kronecker symbol. Hence  $\tau_{P,t}^G(E_s) = \delta_{t,s}$ , and this completes the proof.  $\square$

**4.12. Theorem.** *Let  $G$  be a finite group, and let  $(P, s) \in \mathcal{Q}_{G,p}$ . Then the primitive idempotent  $F_{P,s}^G$  of the  $p$ -permutation algebra  $K \otimes_{\mathbb{Z}} \text{pp}_k(G)$  is given by the following formula :*

$$F_{P,s}^G = \frac{1}{|P||s||C_{\overline{N}_G(P)}(s)|} \sum_{\substack{\varphi \in \langle s \rangle \\ L \leq \langle Ps \rangle \\ PL = \langle Ps \rangle}} \tilde{\varphi}(s^{-1}) |L| \mu(L, \langle Ps \rangle) \text{Ind}_L^G k_{L,\varphi}^{\langle Ps \rangle} ,$$

where  $k_{L,\varphi}^{\langle Ps \rangle} = \text{Res}_L^{\langle Ps \rangle} \text{Inf}_{\langle s \rangle}^{\langle Ps \rangle} k_{\varphi}$ .

**Proof :** By Corollary 3.4, and Proposition 4.11

$$F_{P,s}^G = \frac{|s|}{|C_{\overline{N}_G(P)}(s)|} \text{Ind}_{\langle Ps \rangle}^G (E_{\langle Ps \rangle}^{\langle Ps \rangle} \cdot \text{Inf}_{\langle s \rangle}^{\langle Ps \rangle} F_{1,s}^{\langle s \rangle}) .$$

By Equation 4.4, this gives

$$F_{P,s}^G = \frac{|s|}{|C_{\overline{N}_G(P)}(s)|} \text{Ind}_{\langle Ps \rangle}^G \frac{1}{|P||s|} \sum_{L \leq \langle Ps \rangle} |L| \mu(L, \langle Ps \rangle) \text{Ind}_L^{\langle Ps \rangle} k \cdot \text{Inf}_{\langle s \rangle}^{\langle Ps \rangle} F_{1,s}^{\langle s \rangle} .$$

Moreover for each  $L \leq \langle Ps \rangle$

$$\begin{aligned} \text{Ind}_L^{\langle Ps \rangle} k \cdot \text{Inf}_{\langle s \rangle}^{\langle Ps \rangle} F_{1,s}^{\langle s \rangle} &\cong \text{Ind}_L^{\langle Ps \rangle} (\text{Res}_L^{\langle Ps \rangle} \text{Inf}_{\langle s \rangle}^{\langle Ps \rangle} F_{1,s}^{\langle s \rangle}) \\ &\cong \text{Ind}_L^{\langle Ps \rangle} \text{Inf}_{L/L \cap P}^L \text{Iso}_{LP/P}^{L/L \cap P} \text{Res}_{LP/P}^{\langle s \rangle} F_{1,s}^{\langle s \rangle} . \end{aligned}$$

Here we have used the fact that if  $L$  and  $P$  are subgroups of a group  $H$ , with  $P \trianglelefteq H$ , then there is an isomorphism of functors

$$\text{Res}_L^H \circ \text{Inf}_{H/P}^H \cong \text{Inf}_{L/L \cap P}^L \circ \text{Iso}_{LP/P}^{L/L \cap P} \circ \text{Res}_{LP/P}^{H/P} ,$$

which follows from the isomorphism of  $(L, H/P)$ -bisets

$$H \times_H (H/P) \cong {}_L(H/P)_{H/P} \cong (L/L \cap P) \times_{L/L \cap P} (LP/P) \times_{LP/P} (H/P) .$$

Now Proposition 3.1 implies that  $\text{Res}_{LP/P}^{\langle s \rangle} F_{1,s}^{\langle s \rangle} = 0$  if  $LP/P \neq \langle s \rangle$ , i.e. equivalently if  $PL \neq \langle Ps \rangle$ . It follows that

$$F_{P,s}^G = \frac{1}{|P| |C_{\overline{N}_G(P)}(s)|} \sum_{\substack{L \leq \langle Ps \rangle \\ PL = \langle Ps \rangle}} |L| \mu(L, \langle Ps \rangle) \text{Ind}_L^G (\text{Res}_L^{\langle Ps \rangle} \text{Inf}_{\langle s \rangle}^{\langle Ps \rangle} F_{1,s}^{\langle s \rangle}) .$$

By Lemma 4.8, this gives

$$F_{P,s}^G = \frac{1}{|P| |s| |C_{\overline{N}_G(P)}(s)|} \sum_{\substack{\varphi \in \widehat{\langle s \rangle} \\ L \leq \langle Ps \rangle \\ PL = \langle Ps \rangle}} \tilde{\varphi}(s^{-1}) |L| \mu(L, \langle Ps \rangle) \text{Ind}_L^G k_{L,\varphi}^{\langle Ps \rangle} ,$$

where  $k_{L,\varphi}^{\langle Ps \rangle} = \text{Res}_L^{\langle Ps \rangle} \text{Inf}_{\langle s \rangle}^{\langle Ps \rangle} k_\varphi$ , as was to be shown.  $\square$

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