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ABSTRACT. We develop in this paper the first systematic treatment of the homogeneous Boltzmann equation for granular gases with non constant restitution coefficient, generalizing a large number of the results obtained recently for homogeneous granular gases with constant restitution coefficient to a broader class of physical restitution coefficients that depend on the collision impact velocity. Our analysis is carried along the following paths: first, we develop the L^1-theory which is based on the understanding of the moments of solution and leads as byproduct to the Haff's law and the so-called L^1-exponential tails theorem. Second, we investigate the L^p-theory for $1 < p < \infty$, proving in particular the propagation of L^p norms. Finally, we develop the L^∞-theory which produces the celebrated L^∞-exponential tails theorem as ultimate goal. In all the above steps, the study of the self-similar solutions to the Boltzmann equation plays a crucial role.

1. INTRODUCTION

1.1. **General setting.** Rapid granular flows can be successfully described by the Boltzmann equation conveniently modified to account for the energy dissipation due to the inelasticity of collisions. For such a description, one usually considers the collective dynamics of inelastic hard-spheres interacting through binary collisions [11, 25, 27]. The loss of mechanical energy due to collisions is characterized by the so-called normal restitution coefficient which quantifies the loss of relative normal velocity of a pair of colliding particles after the collision with respect to the impact velocity. Namely, if v and $v_\star$ denote the velocities of two particles before they collide, their respective velocities v' and $v'_\star$ after collisions are such that

$$(u' \cdot \widehat{n}) = -(u \cdot \widehat{n})\, e, \tag{1.1}$$

where the restitution coefficient e is such that $0 \leqslant e \leqslant 1$ and $\widehat{n} \in \mathbb{S}^2$ determines the impact direction, i.e. $\widehat{n}$ stands for the unit vector that points from the v-particle center to the $v_\star$-particle center at the instant of impact. Here above

$$u = v - v_\star, \qquad u' = v' - v'_\star,$$

denote respectively the relative velocity before and after collision. The major part of the investigation, at the physical as well as the mathematical levels, has been devoted to the particular case of a constant normal restitution. However, as described in the monograph

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[11], it appears that a more relevant description of granular gases should deal with a variable restitution coefficient $e(\cdot)$ depending on the impact velocity, i.e.

$$e := e(|u \cdot \widehat{n}|).$$

The most common model is the one corresponding to visco-elastic hard-spheres for which the restitution coefficient has been derived by SCHWAGER & PÖSCHEL [25]. For this peculiar model, $e(\cdot)$ admits the following representation as an infinite expansion series:

$$e(|u \cdot \widehat{n}|) = 1 + \sum_{k=1}^{\infty} (-1)^k a_k |u \cdot \widehat{n}|^{k/5}, \qquad u \in \mathbb{R}^3, \quad \widehat{n} \in \mathbb{S}^2 \tag{1.2}$$

where $a_k > 0$ for any $k \in \mathbb{N}$. We refer the reader to [11, 25] for the physical considerations leading to the above expression (see also the Appendix A for several properties of $e(\cdot)$ in the case of visco-elastic hard-spheres). This is the principal example we have in mind for most of the results in the paper, though, as we shall see, our approach will cover more general cases including the one of constant restitution coefficient.

In a kinetic framework, behavior of the granular flows is described, in the spatially situation we shall consider here, by the so-called velocity distribution $f(v,t)$ which represents the probability density of particles with velocity $v \in \mathbb{R}^3$ at time $t \geqslant 0$. The time-evolution of the one-particle distribution function $f(v,t)$, $v \in \mathbb{R}^3$, $t > 0$ satisfies the following

$$\partial_t f = \mathcal{Q}_e(f,f)(v,t), \qquad f(t=0,v) = f_0(v) \tag{1.3}$$

where $\mathcal{Q}_e(f,f)$ is the inelastic Boltzmann collision operator, expressing the effect of binary collisions of particles. The collision operator $\mathcal{Q}_e$ shares a common structure with the classical Boltzmann operator for elastic collision [13, 26] but is conveniently modified in order to take into account the inelastic character of the collision mechanism. In particular, $\mathcal{Q}_e$ depends in a very strong and explicit way on the restitution coefficient e. Of course, for $e \equiv 1$, one recovers the classical Boltzmann operator. We postpone to Section 2.1 the precise expression of $\mathcal{Q}_e$. Due to the dissipation of kinetic energy during collisions, in the absence of external forces, the granular temperature

$$\mathcal{E}(t) = \int_{\mathbb{R}^3} f(t,v)|v|^2 \, \mathrm{d}v$$

is continuously decreasing and is expected to go to zero as time goes to infinity, expressing the *cooling of the granular gases*. Determining the precise rate of decay to zero for the granular temperature is, among other things, one of the questions addressed in this paper. The asymptotic behavior for the granular temperature was first explained in [17] by HAFF at the beginning of the 80's for the case of constant restitution coefficient, thus, it has become standard to refer to this behavior simply as *Haff's law*.

Up to now, the mathematical investigation of the inelastic Boltzmann equation has been almost uniquely devoted to the case of a constant restitution coefficient. As well documented in the survey [27], the study of Boltzmann models for granular flows has been first restricted to the so-called inelastic Maxwell molecules where the collision rate is independent on the relative velocity [5, 6, 8, 12]. A more sophisticated model can be found in [8, Section 6.2] which deals with inelastic Maxwell molecules where the restitution coefficient depends on time through the temperature of the gas. Regarding the

convergence towards homogeneous cooling state in the case of Maxwell molecules we refer to the recent work by CARLEN, CARRILLO & CARVALHO [10]. The mathematical investigations of the more physically relevant case of hard-spheres interactions have been then initiated by GAMBA, PANFEROV & VILLANI [14] for diffusively heated gases. Since then, a systematic study of the hard-spheres case have been addressed in a series of papers by MISCHLER & MOUHOT, who, among other important results, provided the first rigorous proof of the Haff's law for hard-spheres interactions and constant normal restitution [19, 20]. Together with the work of BOBYLEV, GAMBA & PANFEROV [9], these two papers have been the principal inspiration source for the present work. Let us also mention that MISCHLER & MOUHOT also addressed the relevant problem of Homogeneous Cooling State, proving the existence of self-similar solutions and their stability with respect to the quasi-elastic limit [21, 22].

All the aforementioned works are dealing with the case of a constant normal restitution coefficient. From the mathematical viewpoint, the literature on granular gases with *variable restitution coefficient* is rather limited. In [19], the Cauchy problem for the homogeneous inelastic Boltzmann equation is studied in great detail and full generality including the class of restitution coefficients that we are dealing with in this paper. For the inhomogeneous inelastic Boltzmann equation the literature is yet more scarce, in this respect we mention the work by the first author [1] that treats the Cauchy problem in the case of near-vacuum data. It is worthwhile mentioning that the scarcity of results regarding existence of solutions for the inhomogeneous case is explained by the lack of entropy estimates for the inelastic Boltzmann equation, thus, powerful theories like the DiPerna-Lions *renormalized solutions* is no longer available. More complex behavior that involve boundaries, for instance clusters and Maxwell demons, are well beyond of the present techniques.

From the technical viewpoint we will implement a great deal of the machinery developed through the years for the theory of homogeneous granular gases with constant restitution coefficient (see [9, 21, 22, 24] and the recent contribution [3]) to a broader class of physical restitution coefficients depending on the collision impact velocity. Of course, additions, improvements and new ideas will be introduced as needed.

1.2. **Description of the results.** The present paper provides, to our knowledge, the first systematic study of the Boltzmann equation for granular gases with non constant restitution coefficient. A simple recipe to understand this work is the following scheme

L^1-theory	L^p-theory $(1 < p < \infty)$	L^∞-theory
Propagation of moments	Carleman representation	Pointwise estimates
Haff's law	Compactness properties of $\mathcal{Q}_e^+$	Uniform boundedness
L^1-tails theorem	Propagation of L^p-norms	L^∞-tails theorem

Thus, in Sections 2, 3 and 4 we develop the L^1-theory which has as byproducts the Haff's law and the L^1-tails theorem. Sections 4 and 5 develop all the integrability estimates needed for the propagation of the L^p-norms for $1 < p < \infty$, which is treated in Section 6. Finally in Section 7 the L^∞-theory is developed to obtain the L^∞-tails theorem. As the reader will notice, this division is natural since the techniques involved in each of them will vary considerably. For the L^1-theory the basic tool is the so-called Povzner lemma, which was developed in full grace in [9], as opposed to the compactness arguments necessary for the L^p-theory. The latter has been first developed by P. L. LIONS in the 90's [18] and fully exploited then by MOUHOT & VILLANI [24]. Meanwhile, the L^∞-theory is based in several technical observations, and a comparison principle introduced recently by GAMBA, PANFEROV & VILLANI [15]. Certainly, we will have to work hard to adapt all these arguments to our present situation that proved to be rewarding at the end. One of the main tools in our analysis will be the study of self-similar solutions to the Boltzmann equation (1.3). Precisely, we shall repeatedly look for solutions to (1.3) of the form

$$f(t,v) = V(t)^3 g(\tau(t), V(t)v)$$

where $\tau(\cdot)$ and $V(\cdot)$ are time scaling functions to be specified later. The rescaled solution $g(\tau, w)$ turns out to be a solution of an evolution problem of the type:

$$\partial_\tau g(\tau,w) + \xi(\tau)\nabla_w \cdot (wg(\tau,w)) = \mathcal{Q}_{\widetilde{e}(\tau)}(g,g)$$

for some $\xi(\tau)$ depending on the time scale τ and $\mathcal{Q}_{\widetilde{e}(\tau)}(g,g)$ is a collision operator associated to a time-dependent restitution coefficient $\widetilde{e}(\tau)$ (see Section 2.4 for details). The most notable difference with respect to the case of a constant restitution coefficient is that the rescaled collision operator depends on the (rescaled) time τ, leading to a *non-autonomous problem* for g.

Let us explain in more details the results we obtain in this work. As we mentioned, the first part of the paper is devoted to the L^1-theory. The main result of this setting is the first rigorous proof of what we call *generalized Haff's law*. Precisely, the rate of cooling of the temperature $\mathcal{E}(t)$ of the solution to (1.3) is expected to be algebraic. From physical considerations and a careful dimensional analysis, HAFF [17] predicted that, for *constant restitution coefficient* the temperature $\mathcal{E}(t)$ of a granular gas should cool down at a quadratic rate:

$$\mathcal{E}(t) = O\left(\frac{1}{t^2}\right) \text{ as } t \to \infty.$$

Similar considerations lead SCHWAGER & PÖSCHEL [25] to the conclusion that, for the restitution coefficient associated to the visco-elastic hard-spheres (1.2), the decay should be slower than the above one, namely at an algebraic rate $t^{-5/3}$.

For the Boltzmann equation with constant restitution coefficient, Haff's law have been proved rigorously by MISCHLER & MOUHOT [21]. Their proof relies on the propagation of L^p norms for the time dependent self-similar solutions to (1.3) which implies non-concentration in these rescaled variables, i.e. the temperature in the rescaled variables is uniformly bounded from below whenever the initial datum satisfies some L^p bound with $p > 1$. Translating such an estimate in the original variables proves Haff's law.

We generalize their result by dealing with a general variable restitution coefficient $e(\cdot)$ satisfying reasonable assumptions, all of them being fulfilled by the physical model of visco-elastic hard-spheres (1.2). Actually, it turns out that the decay rate of $\mathcal{E}(t)$ depends heavily on the behavior of the restitution coefficient $e(|u\cdot\widehat{n}|)$ for small impact velocity. For instance, if there exist some constants $\alpha>0$ and $\gamma\geqslant 0$ such that

$$e(|u\cdot\widehat{n}|)\simeq 1-\alpha|u\cdot\widehat{n}|^{\gamma}\quad\text{ for }\quad|u\cdot\widehat{n}|\simeq 0$$

then we prove that

$$\mathcal{E}(t)=O\left(\frac{1}{t^{2/1+\gamma}}\right)\quad\text{ as }\quad t\to\infty.$$

We recover the results of [21] for $\gamma=0$ and the decay predicted in [25] for $\gamma=1/5$. Our approach is still based upon the study of self-similar solutions to (1.3). However, the proof does not need a compactness argument and is therefore transparent of L^p propagation ($p>1$). This is a major difference with [21] and provides in some sense a more direct approach since it is natural to expect that the decay of the energy should rely only on the L^1-theory of moments. This is indeed the case and we investigate directly the evolution of the temperature of the self-similar solutions to (1.3) and it will be clear, in Section 3.2, how some pertinent scaling provides a lower bound for this rescaled temperature. A crucial argument in our proof is the propagation of moments of any order for the solution f to (1.3) which can be proved following the, rather standard, approach developed by BOBYLEV, GAMBA, & PANFEROV [9] providing a sharp version of Povzner estimates. One of the novelties of our approach relies on the simple but interesting observation that moments of order $2p$ of f (with $p>1$) can all be controlled from above by the p-th power of the temperature $\mathcal{E}(t)$ (see Corollary 2.9). Finally, the L^1-theory ends up with Section 4, where the full power of the Povzner estimate is exploited to prove the propagation of exponential L^1-tails. The arguments here are rather standard and taken, with minor changes, from [9].

The second part of the paper begins in Section 5 which is the most technical of the document. In this section we present a full discussion of the regularity and integrability properties of the gain part of the collision operator $\mathcal{Q}^+_{B,e}$ associated to a general collision kernel $B(u,\sigma)=\Phi(|u|)b(\widehat{u}\cdot\sigma)$ satisfying Grad's cut-off assumption (see Section 2 for precise definition). All lemmas here will play an important role for both the L^p and L^∞ theories. This Section is divided in five subsections starting with a Carleman representation of the gain operator $\mathcal{Q}^+_{B,e}$. It is well-known that such a representation is essential for the study of regularizing properties of the gain operator $\mathcal{Q}^+_{B,e}$ when smooth assumptions are imposed on the kernel $B(u,\sigma)$. This has been studied for the classic (elastic) case in [18, 24, 28], and for the constant inelastic case in [20]. Our contribution here is to extend known results, in subsections 5.3 and 5.4, for the inelastic case with variable restitution coefficient. One of the technical difficulties relies on the fact that, since the estimates of Section 5 are aimed to be applied for the self-similar variables, we have to keep track of all the involved constants to make sure that they are *independent* of the restitution coefficient. This will allow us to overcome the technical problem of the time dependence of the gain operator in the self-similar variables. In subsection 5.2 we derive several convolution-like

estimates assuming minimal regularity of the angular kernel $b(\cdot)$. The techniques involved here are quite new (obtained from a combination of similar results of [3, 24]) and produce neat results. Finally, in subsection 5.5, we introduce two estimates involving exponential weights that will be essential for the proof of the L^∞-tails theorem. Notice that, in all Section 5, the reader will find an underlying structure in the estimates for the gain operator $\mathcal{Q}^+_{B,e}$, namely, they are composed of a big "good" part associated to the behavior of the angular kernel b in $(0,1)$ and a small "bad" part associated to the behavior of b in the end points $\{-1,1\}$ (see Preliminaries section for the angular kernel definition). It is well established that, having non concentration of energy at hand, the loss operator $\mathcal{Q}^-_e$ dominates each one of these parts (recall that the loss operator turns out to be independent of the restitution coefficient). This can close a good estimate for the full collision operator.

With the machinery of Section 5 at hand, the paper ends up with the propagation of L^p-norms $(1<p<\infty)$ for the self-similar solution in Section 6 and the propagation of L^∞-norm in Section 7. Surprisingly, the pointwise uniform propagation is not a direct consequence of the L^p-theory $(1<p<\infty)$ as the estimates for the L^p-theory degenerate in the limit $p\to\infty$. Thus, some extra work is needed to reach this limit (see Lemmas 7.1 and 7.2). For the last result of the paper, namely the study of L^∞-tails theorem, an additional ingredient comes into play, namely, a comparison principle for the Boltzmann equation proved in [15]. This result has been applied with success for the classical Boltzmann equation and here we apply it to the inelastic theory to prove the propagation of exponential bounds. As the reader progresses in the paper he will note that the program followed here is constructive, and the results of each Section depends on the previous ones. Thus, the optimality of the last result in Section 7, i.e. pointwise exponential bound propagation, is prescribed by the optimality of the L^1-exponential propagation of Section 4.

1.3. **Notations.** Let us introduce the notations we shall use in the sequel. Throughout the paper we shall use the notation $\langle\cdot\rangle=\sqrt{1+|\cdot|^2}$. We denote, for any $\eta\in\mathbb{R}$, the Banach space

$$L^1_\eta=\left\{f:\mathbb{R}^3\to\mathbb{R}\text{ measurable}\,;\quad \|f\|_{L^1_\eta}:=\int_{\mathbb{R}^3}|f(v)|\,\langle v\rangle^\eta\,\mathrm{d}v<+\infty\right\}.$$

More generally we define the weighted Lebesgue space $L^p_\eta(\mathbb{R}^3)$ $(p\in[1,+\infty),\,\eta\in\mathbb{R})$ by the norm

$$\|f\|_{L^p_\eta(\mathbb{R}^3)}=\left[\int_{\mathbb{R}^3}|f(v)|^p\,\langle v\rangle^{p\eta}\,\mathrm{d}v\right]^{1/p}.$$

The weighted Sobolev space $W^{k,p}_\eta(\mathbb{R}^3)$ $(p\in[1,+\infty),\,\eta\in\mathbb{R}$ and $k\in\mathbb{N})$ is defined by the norm

$$\|f\|_{W^{k,p}_\eta(\mathbb{R}^3)}=\left[\sum_{|s|\leqslant k}\|\partial^s_v f\|^p_{L^p_\eta}\right]^{1/p}$$

where ∂_v^s denotes the partial derivative associated with the multi-index $s \in \mathbb{N}^N$. In the particular case $p = 2$ we denote $H^k_\eta = W^{k,2}_\eta$. Moreover this definition can be extended to H^s_η for any $s \geqslant 0$ by using the Fourier transform $\mathcal{F}$.

2. Preliminaries: L^1-Theory

2.1. **The kinetic model.** We assume the granular particles to be perfectly smooth hard-spheres of mass $m = 1$ performing inelastic collisions. Recall that, as explained in Introduction, the inelasticity of the collision mechanism is characterized by a single parameter, namely the coefficient of normal restitution $0 \leqslant e \leqslant 1$ which we assume to be *non constant*. Precisely, let $(v, v_\star)$ denote the velocities of two particles before they collide. Their respective velocities after collisions v' and $v'_\star$ are given, in virtue of (1.1) and the conservation of momentum, by

$$v' = v - \frac{1+e}{2}\,(u\cdot\widehat{n})\widehat{n}, \qquad v'_\star = v_\star + \frac{1+e}{2}\,(u\cdot\widehat{n})\widehat{n}, \tag{2.1}$$

where the symbol u stands for the relative velocity $u = v - v_\star$ and $\widehat{n}$ is the impact direction. As explained in Introduction, from the physical viewpoint, a common approximation is to choose e as a suitable function of the impact velocity, i.e. $e := e(|u\cdot\widehat{n}|)$. The main assumptions on the function $e(\cdot)$ are listed in the following (see [1]):

Assumptions 2.1. *In all the paper, one assumes the following to hold:*

(1) The mapping $z \in \mathbb{R}_+ \mapsto e(z) \in (0,1]$ is absolutely continuous.
(2) The mapping $z \in \mathbb{R}^+ \to \vartheta(z) := z\,e(z)$ is strictly increasing.

Further assumptions on the function $e(\cdot)$ shall be needed later on. With the above assumption *(2)*, it is easy to check that the Jacobian of the transformation (2.1) is given by:

$$J := \left|\frac{\partial(v', v'_\star)}{\partial(v, v_\star)}\right| = |u\cdot\widehat{n}| + |u\cdot\widehat{n}|\frac{\mathrm{d}e}{\mathrm{d}z}(|u\cdot\widehat{n}|) = \frac{\mathrm{d}\vartheta}{\mathrm{d}z}(|u\cdot\widehat{n}|) > 0.$$

In practical situations, the restitution coefficient $e(\cdot)$ is usually chosen among the following three examples:

Example 2.2 (Constant restitution coefficient). *The most documented example in the literature is the one in which*

$$e(z) = e_0 \in (0,1] \qquad \textit{for any } z \geqslant 0.$$

Example 2.3 (Monotone decreasing). *A second example of interest is the one in which the restitution coefficient $e(\cdot)$ is a monotone decreasing function:*

$$e(z) = \frac{1}{1+az^\eta} \qquad \forall z \geqslant 0 \tag{2.2}$$

where $a > 0$, $\eta > 0$ are two given constants.

Example 2.4 (Viscoelastic hard-spheres). *The most physically relevant variable restitution coefficient is the one corresponding to the so-called viscoelastic hard-spheres* [11].

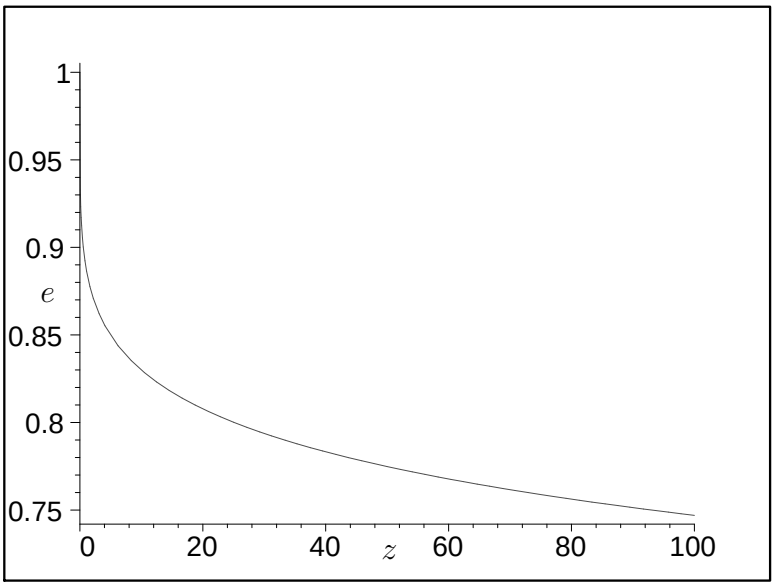

FIGURE 1. Restitution coefficient for viscoelastic hard-spheres given by Eq. (2.3) with $a = 0.12$

For such a model, the properties of the restitution coefficient have been derived in [11, 25] *and it can be shown that* $e(z)$ *is given by* (1.2) *or is defined implicitly by the following*

$$e(z) + az^{1/5}e(z)^{3/5} = 1 \tag{2.3}$$

where $a > 0$ *is a suitable positive constant depending on the material viscosity (see Figure 1).*

In the sequel, it shall be more convenient to deal with a second, equivalent, parametrization of the post-collisional velocities. Precisely, v and $v_\star$ being fixed, with $v \neq v_\star$, let $\widehat{u} = u/|u|$. Performing in (2.1) the change of unknown

$$\sigma = \widehat{u} - 2\,(\widehat{u}\cdot\widehat{n})\widehat{n} \in \mathbb{S}^2$$

this provides an alternative parametrization of the unit sphere $\mathbb{S}^2$ and, in this case, the impact velocity reads

$$|u\cdot\widehat{n}| = |u|\,|\widehat{u}\cdot\widehat{n}| = |u|\sqrt{\tfrac{1-\widehat{u}\cdot\sigma}{2}}.$$

Then, the post-collisional velocities $(v', v'_\star)$ are given by

$$v' = v - \beta\frac{u - |u|\sigma}{2}, \qquad v'_\star = v_\star + \beta\frac{u - |u|\sigma}{2} \tag{2.4}$$

where

$$\beta = \beta\left(|u|\sqrt{\tfrac{1-\widehat{u}\cdot\sigma}{2}}\right) = \frac{1+e}{2} \in \left(\tfrac{1}{2}, 1\right].$$

This representation allows us to give a precise definition of the Boltzmann collision operator in *weak form*. Given a collision kernel $B(u,\sigma)$ one defines the associated collision

operator $\mathcal{Q}_{B,e}$ through the weak formulation:

$$\int_{\mathbb{R}^3} \mathcal{Q}_{B,e}(f,g)(v)\psi(v)\,\mathrm{d}v = \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} f(v)g(v_\star)\mathcal{A}_{B,e}[\psi](v,v_\star)\;\mathrm{d}v_\star\,\mathrm{d}v \tag{2.5}$$

for any test function $\psi = \psi(v)$ where

$$\mathcal{A}_{B,e}[\psi](v,v_\star) = \int_{\mathbb{S}^2}\Big(\psi(v') + \psi(v'_\star) - \psi(v) - \psi(v_\star)\Big)B(u,\sigma)\,\mathrm{d}\sigma$$

with $v', v'_\star$ are defined in (2.4) and the collision kernel $B(u,\sigma)$ is given by

$$B(u,\sigma) = \Phi(|u|)b(\widehat{u}\cdot\sigma)$$

where $\Phi(\cdot)$ is a suitable nonnegative function known as *potential*, while the *angular kernel* $b(\cdot)$ is usually assumed to belong to $L^1(-1,1)$. For any fixed vector $\widehat{u}$, the angular kernel defines a measure on the sphere through the mapping $\sigma\in\mathbb{S}^2 \mapsto b(\widehat{u}\cdot\sigma)\in[0,\infty]$ and we will assume it to satisfy the renormalized *Grad's cut-off* assumption

$$\|b\|_{L^1(\mathbb{S}^2)} = 2\pi\,\|b\|_{L^1(-1,1)} = 1. \tag{2.6}$$

A particularly relevant model is the one of hard-spheres corresponding to $\Phi(|u|) = |u|$ and $b(\widehat{u}\cdot\sigma) = 1/4\pi$. We shall often in the sequel consider the ***generalized hard-spheres collision kernel*** for which $\Phi(|u|) = |u|$ and the angular kernel is non necessarily constant and satisfies (2.6). For the particular model of hard-spheres interactions, we shall simply denote the collision operator $\mathcal{Q}_{B,e}$ by $\mathcal{Q}_e$.

2.2. **On the Cauchy problem.** We consider the following homogeneous Boltzmann equation

$$\begin{cases} \partial_t f(t,v) &= \mathcal{Q}_{B,e}(f,f)(t,v) \qquad\qquad t>0,\ v\in\mathbb{R}^3 \\ f(0,v) &= f_0(v), \qquad\qquad v\in\mathbb{R}^3 \end{cases} \tag{2.7}$$

where the initial datum f_0 is a nonnegative velocity distribution such that

$$\int_{\mathbb{R}^3} f_0(v)\,\mathrm{d}v = 1, \quad \int_{\mathbb{R}^3} f_0(v)v\,\mathrm{d}v = 0 \quad \text{and} \quad \int_{\mathbb{R}^3} f_0(v)|v|^2\,\mathrm{d}v < \infty. \tag{2.8}$$

Notice that there is no loss of generality to assume the two first moments conditions in (2.8) since it is always possible to reduce to such a case by a scaling and translational argument. We shall say that a nonnegative $f = f(t,v)$ is a solution to (2.8) if $f\in\mathcal{C}([0,\infty), L^1_2(\mathbb{R}^3))$ and

$$\int_0^\infty \mathrm{d}t\int_{\mathbb{R}^3}\Big(f(t,v)\partial_t\phi(t,v) + \phi(t,v)\mathcal{Q}_{B,e}(f,f)(t,v)\Big)\,\mathrm{d}v = \int_{\mathbb{R}^3} f_0(v)\phi(0,v)\,\mathrm{d}v$$

holds for any compactly supported $\phi\in\mathcal{C}^1([0,\infty)\times\mathbb{R}^3)$. Under the Assumptions 2.1, it is not difficult to see that the assumptions **H1** and **H2** of [20] are fulfilled (actually, with the terminology of [20], we are dealing here with a non-coupled collision rate and, more precisely, with the so-called *generalized visco-elastic model*, see [20], p. 661). In particular, [20, Theorem 1.2] applies straightforwardly and allows us to state:

Theorem 2.5 (Mischler et *al.*). *For any nonnegative velocity distribution f_0 satisfying (2.8), there is a unique solution $f = f(t,v)$ to (2.7). Moreover,*

$$\int_{\mathbb{R}^3} f(t,v)\,\mathrm{d}v = 1, \qquad \int_{\mathbb{R}^3} f(t,v) v\,\mathrm{d}v = 0 \qquad \forall t \geqslant 0. \tag{2.9}$$

2.3. **Povzner-type inequalities.** We extend in this section the results of [9] and [21] to the case of variable restitution coefficient we are dealing with. We shall consider the general case of a collision operator $\mathcal{Q}_{B,e}$ associated to some general collision kernel

$$B(u,\sigma) = \Phi(|u|) b(\widehat{u}\cdot\sigma),$$

for some nonnegative integrable angular kernel $b(\cdot)$ satisfying the renormalized Grad's cut-off assumption (2.6). Let f be a given velocity distribution function with $f \geqslant 0$ satisfying

$$\int_{\mathbb{R}^3} f(v)\,\mathrm{d}v = 1, \qquad \int_{\mathbb{R}^3} f(v) v\,\mathrm{d}v = 0.$$

Let $\psi(v) = \Psi(|v|^2)$ be a given test-function with Ψ convex and nondecreasing. Then, Eq. (2.5) leads to

$$\int_{\mathbb{R}^3} \mathcal{Q}_{B,e}(f,f)(v)\psi(v)\,\mathrm{d}v = \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} f(v) f(v_\star) \mathcal{A}_{B,e}[\psi](v,v_\star)\ \mathrm{d}v_\star\,\mathrm{d}v$$

with

$$\mathcal{A}_{B,e}[\psi](v,v_\star) = \Phi(|u|)\left(A^+_{B,e}[\Psi](v,v_\star) - A^-_{B,e}[\Psi](v,v_\star)\right)$$

where

$$A^+_{B,e}[\Psi](v,v_\star) = \int_{\mathbb{S}^2} \left(\Psi(|v'|^2) + \Psi(|v'_\star|^2)\right) b(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma$$

while, using also (2.6),

$$A^-_{B,e}[\Psi](v,v_\star) = \int_{\mathbb{S}^2} \left(\psi(v) + \psi(v_\star)\right) b(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma = \left(\Psi(|v|^2) + \Psi(|v_\star|^2)\right).$$

Following [9], we define the velocity of the center of mass $U = \dfrac{v+v_\star}{2}$ so that

$$v' = U + \frac{|u|}{2}\omega, \qquad v'_\star = U - \frac{|u|}{2}\omega \qquad \text{with} \quad \omega = (1-\beta)\widehat{u} + \beta\sigma$$

where we recall that, for any vector $V \in \mathbb{R}^3$, we set $\widehat{V} = V/|V|$. Notice that, when e (or equivalently β) is constant, the strategy of [9] consists, roughly speaking, in performing a suitable change of unknown $\sigma \to \widehat{\omega}$ to compute $A^+_{B,e}[\psi]$. Since we are dealing with a variable β, we do not apply directly such a strategy here. Instead, notice that, since $|\omega| \leqslant 1$ and Ψ is increasing one has

$$\begin{aligned}\Psi(|v'|^2) + \Psi(|v'_\star|^2) &\leqslant \Psi\left(|U|^2 + \frac{|u|^2}{4} + |u||U|\widehat{U}\cdot\omega\right) + \Psi\left(|U|^2 + \frac{|u|^2}{4} - |u||U|\widehat{U}\cdot\omega\right)\\ &= \Psi\left(E\frac{1+\xi\,\widehat{U}\cdot\omega}{2}\right) + \Psi\left(E\frac{1-\xi\,\widehat{U}\cdot\omega}{2}\right)\end{aligned}$$

where we set $E := |v|^2 + |v_\star|^2 = 2|U|^2 + \frac{|u|^2}{2}$ and $\xi = 2|U|\,|u|/E$. Since $\Psi(\cdot)$ is convex, it is known that (see e.g. [9]) that, the mapping $\Psi_0(t) = \Psi(x+ty) + \Psi(x-ty)$ is even and nondecreasing for $t \geqslant 0$ and $x, y \in \mathbb{R}$. Therefore, since $\xi \leqslant 1$ one gets

$$\Psi(|v'|^2) + \Psi(|v'_\star|^2) \leqslant \Psi\left(E\frac{1+\widehat{U}\cdot\omega}{2}\right) + \Psi\left(E\frac{1-\widehat{U}\cdot\omega}{2}\right). \tag{2.10}$$

Assume that $\widehat{U}\cdot\sigma \geqslant 0$, then

$$\left|\widehat{U}\cdot\omega\right| = \left|(1-\beta)\widehat{U}\cdot\widehat{u} + \beta\widehat{U}\cdot\sigma\right| \leqslant (1-\beta) + \beta\widehat{U}\cdot\sigma,$$

therefore, using the fact that Ψ_0 is even and nondecreasing for $t \geqslant 0$, we conclude from (2.10) that

$$\Psi(|v'|^2) + \Psi(|v'_\star|^2) \leqslant \Psi\left(E\frac{2-\beta+\beta\widehat{U}\cdot\sigma}{2}\right) + \Psi\left(E\frac{\beta-\beta\widehat{U}\cdot\sigma}{2}\right).$$

When $\widehat{U}\cdot\sigma \leqslant 0$ a similar argument shows that

$$\Psi(|v'|^2) + \Psi(|v'_\star|^2) \leqslant \Psi\left(E\frac{2-\beta-\beta\widehat{U}\cdot\sigma}{2}\right) + \Psi\left(E\frac{\beta+\beta\widehat{U}\cdot\sigma}{2}\right).$$

Hence, setting $\tilde{b}(s) = b(s) + b(-s)$ and using these last two estimates with the change of variables $\sigma \to -\sigma$ we get

$$\begin{aligned} A^+_{B,e}[\Psi](v,v_\star) &\leqslant \int_{\{\widehat{U}\cdot\sigma\geqslant 0\}} \left[\Psi\left(E\frac{2-\beta+\beta\widehat{U}\cdot\sigma}{2}\right) + \Psi\left(E\frac{\beta-\beta\widehat{U}\cdot\sigma}{2}\right)\right]\tilde{b}(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma \\ &\leqslant \int_{\{\widehat{U}\cdot\sigma\geqslant 0\}} \left[\Psi\left(E\frac{3+\widehat{U}\cdot\sigma}{4}\right) + \Psi\left(E\frac{1-\widehat{U}\cdot\sigma}{4}\right)\right]\tilde{b}(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma. \end{aligned} \tag{2.11}$$

The second inequality can be shown writing

$$\frac{2-\beta+\beta\widehat{U}\cdot\sigma}{2} = \frac{1}{2} + \left(\frac{1}{2} - \frac{\beta}{2}\left(1-\widehat{U}\cdot\sigma\right)\right) \quad \text{and,}$$

$$\frac{\beta-\beta\widehat{U}\cdot\sigma}{2} = \frac{1}{2} - \left(\frac{1}{2} - \frac{\beta}{2}\left(1-\widehat{U}\cdot\sigma\right)\right).$$

The latter term in parenthesis is maximized when $\beta = 1/2$, thus the monotonicity of Ψ_0 implies the result.

Next, we particularize the previous estimates when $\Psi(x) = x^p$. This choice will lead to the study of the moments of solutions:

Lemma 2.6. *Let $q \geqslant 1$ be such that $b \in L^q(\mathbb{S}^2)$. Then, for any restitution coefficient $e(\cdot)$ satisfying Assumptions 2.1 and any real $p \geqslant 1$, there exists an explicit constant $\gamma_p > 0$*

such that

$$\Phi(|u|)^{-1}\mathcal{A}_{B,e}[|\cdot|^p](v,v_\star) \leqslant -(1-\gamma_p)\left(|v|^{2p}+|v_\star|^{2p}\right) + \gamma_p\left[\left(|v|^2+|v_\star|^2\right)^p - |v|^{2p} - |v_\star|^{2p}\right]. \quad (2.12)$$

This constant γ_p has the following properties:

(1) $\gamma_1 \leqslant 1$.
(2) *For $p \geqslant 1$ the map $p \mapsto \gamma_p$ is strictly decreasing. In particular, $\gamma_p < 1$ for $p > 1$.*
(3) $\gamma_p = O\left(1/p^{1/q'}\right)$ *for large p, where $1/q + 1/q' = 1$.*
(4) *For $q = 1$, one still has $\gamma_p \searrow 0$ as $p \to \infty$.*

Proof. Let $\Psi_p(x) = x^p$. From (2.11), one sees that

$$A^+_{B,e}[\Psi_p](v,v_\star) \leqslant \gamma_p\, E^p$$

where

$$\gamma_p = \sup_{\widehat{U},\widehat{u}} \int_{\widehat{U}\cdot\sigma\geqslant 0}\left[\Psi_p\left(\frac{3+\widehat{U}\cdot\sigma}{4}\right)+\Psi_p\left(\frac{1-\widehat{U}\cdot\sigma}{4}\right)\right]\tilde{b}(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma. \quad (2.13)$$

It is clear that the above inequality yields (2.12). Let us prove that γ_p satisfies the aforementioned conditions. First, we use Hölder inequality to obtain

$$\gamma_p \leqslant 4\pi\,\|b\|_{L^q(\mathbb{S}^2)}\left(\int_{-1}^1\left[\Psi_p\left(\frac{3+s}{4}\right)+\Psi_p\left(\frac{1-s}{4}\right)\right]^{q'}\mathrm{d}s\right)^{1/q'} < \frac{16\pi\,\|b\|_{L^q(\mathbb{S}^2)}}{(q'p+1)^{1/q'}}.$$

This proves that γ_p is finite and also proves item (3) for $q > 1$. For items (1) and (2) observe that the integral in the right-hand-side (2.11) is continuous in the vectors $\widehat{U},\ \widehat{u} \in \mathbb{S}^2$. This can be shown by changing the integral to polar coordinates. Thus, the supremum in these arguments is achieved. Therefore, there exist $\widehat{U}_0,\ \widehat{u}_0 \in \mathbb{S}^2$ (depending on the angular kernel b) such that

$$\gamma_p = \int_{\{\widehat{U}_0\cdot\sigma\geqslant 0\}}\left[\Psi_p\left(\frac{3+\widehat{U}_0\cdot\sigma}{4}\right)+\Psi_p\left(\frac{1-\widehat{U}_0\cdot\sigma}{4}\right)\right]\tilde{b}(\widehat{u}_0\cdot\sigma)\,\mathrm{d}\sigma.$$

A simple computation with this estimate shows that $\gamma_1 = \|b\|_{L^1(\mathbb{S}^2)} = 1$. Moreover, the integrand is a.e. strictly decreasing as p increases, this proves (2). Finally, let $p \to \infty$ in this expression and use Dominated convergence to conclude (4) for the case $q = 1$. □

Remark 2.7. *Notice that the above constant γ_p is independent of the variable restitution coefficient $e(\cdot)$.*

The above lemma is the analogous of [9, Corollary 1] for variable restitution coefficient $e(\cdot)$ and it proves that the subsequent results of [9] extend in a straightforward way to variable restitution coefficient. In particular, [9, Lemma 3] reads[1]:

[1]Notice that, though stated for hard-spheres interactions only, [9, Lemma 3] applies to our situation thanks to the above Lemma 2.6 and [9, Lemma 1].

Proposition 2.8. *Let f be a given velocity distribution function with $f \geqslant 0$ with*

$$\int_{\mathbb{R}^3} f(v)\,\mathrm{d}v = 1, \qquad \int_{\mathbb{R}^3} f(v) v\,\mathrm{d}v = 0.$$

For any $p \geqslant 1$, we set

$$m_p = \int_{\mathbb{R}^3} f(v)|v|^{2p}\,\mathrm{d}v.$$

Assume that the collision kernel $B(u,\sigma) = |u| b(\widehat{u}\cdot\sigma)$ is such that $b(\cdot)$ satisfies (2.6) with $b(\cdot) \in L^q(\mathbb{S}^2)$ for some $q \geqslant 1$. For any restitution coefficient $e(\cdot)$ satisfying Assumptions (2.1) and any real $p \geqslant 1$, one has

$$\int_{\mathbb{R}^3} \mathcal{Q}_{B,e}(f,f)(v)|v|^{2p}\,\mathrm{d}v \leqslant -(1-\gamma_p) m_{p+1/2} + \gamma_p\, S_p, \tag{2.14}$$

where,

$$S_p = \sum_{k=1}^{[\frac{p+1}{2}]} \binom{p}{k} \left(m_{k+1/2}\, m_{p-k} + m_k\, m_{p-k+1/2}\right),$$

$[\frac{p+1}{2}]$ denoting the integer part of $\frac{p+1}{2}$ and γ_p being the constant of Lemma 2.6.

As well-documented [20, 9], since S_p involves only moments of order $p-1/2$, the above estimate has important consequences on the propagation of moments for the solution to (2.7). We show in the following that actually the moments of such a solution can be controlled from above by the second moment. More precisely:

Corollary 2.9. *Let $B(u,\sigma) = |u| b(\widehat{u}\cdot\sigma)$ with $b(\cdot)$ satisfying (2.6) with $b(\cdot) \in L^q(\mathbb{S}^2)$ for some $q \geqslant 1$. Let f_0 be a nonnegative velocity distribution satisfying (2.8) and let $f(t,v)$ be the associated solution to (2.7). For any $t \geqslant 0$ and any $p \geqslant 1$ we define*

$$m_p(t) := \int_{\mathbb{R}^3} f(t,v)|v|^{2p}\,\mathrm{d}v \tag{2.15}$$

with the convention of notation $\mathcal{E}(t) = m_1(t)$.

If $m_p(0) < \infty$ then $\sup_{t\geqslant 0} m_p(t) < \infty$ and there exists a constant $K_p > 0$ such that

$$m_p(t) \leqslant K_p\, \mathcal{E}(t)^p \qquad \forall t \geqslant 0. \tag{2.16}$$

Proof. The first part of the corollary, namely

$$m_p(0) < \infty \Longrightarrow \sup_{t\geqslant 0} m_p(t) < \infty$$

is a classical consequence of (2.14) whose proof can be recovered from [9, 20]. Let us prove (2.16) holds for any real $p \geqslant 1$. First, one notes that, by a classical interpolation argument, it suffices to prove it for any p such that $2p \in \mathbb{N}$. Let us then prove the result by induction. It is clear that estimate (2.16) holds true for $p = 1$ with $K_1 = 1$. Let now $p > 1$, with $2p \in \mathbb{N}$ be fixed and assume that, for any integer $1 \leqslant j \leqslant p - 1/2$, there exists $K_j > 0$ such that $m_j(t) \leqslant K_j \mathcal{E}(t)^j$ holds for any $t \geqslant 0$. According to Proposition 2.8, one gets that

$$\frac{\mathrm{d}}{\mathrm{d}t} m_p(t) = \int_{\mathbb{R}^3} \mathcal{Q}_{B,e}(f,f)(t,v)|v|^{2p}\,\mathrm{d}v \leqslant -(1-\gamma_p) m_{p+1/2}(t) + \gamma_p\, S_p(t)$$

where

$$S_p(t) = \sum_{k=1}^{[\frac{p+1}{2}]} \left(\begin{array}{c} p \\ k \end{array} \right) \left(m_{k+1/2}(t)\, m_{p-k}(t) + m_k(t)\, m_{p-k+1/2}(t) \right), \quad t \geqslant 0.$$

Since $S_p(t)$ involves only moments of order less than $p-1/2$, our induction hypothesis implies that there exists a constant $C_p > 0$ such that

$$S_p(t) \leqslant C_p\, \mathcal{E}(t)^{p+1/2} \qquad \forall t \geqslant 0$$

where $C_p = \sum_{k=1}^{[\frac{p+1}{2}]} \left(\begin{array}{c} p \\ k \end{array} \right) \left(K_{k+1/2}\, K_{p-k} + K_k\, K_{p-k+1/2} \right)$. Moreover, since

$$m_{p+1/2}(t) \geqslant m_p^{1+1/2p}(t), \qquad \forall t \geqslant 0$$

according to Jensen's inequality (recall that we have here $m_0(t) = m_0(0) = 1$ for any t), we get that

$$\frac{\mathrm{d}}{\mathrm{d}t} m_p(t) \leqslant -(1-\gamma_p) m_p^{1+1/2p}(t) + \gamma_p\, C_p\, \mathcal{E}(t)^{p+1/2} \qquad \forall t \geqslant 0. \tag{2.17}$$

Let us choose then K_p such that

$$K_p > \max \left\{ \frac{m_p(0)}{(\mathcal{E}(0))^p},\ \left(\frac{C_p\, \gamma_p}{1-\gamma_p} \right)^{\frac{2p}{2p+1}} \right\},$$

then (2.16) holds. Indeed, we first notice that by continuity of both $m_p(t)$ and $\mathcal{E}(t)$ the estimate (2.16) holds at least for short time. Assume then there exists some time $t_\star$ such that $m_p(t_\star) = K_p \mathcal{E}(t_\star)^p$ then, from (2.17)

$$\frac{\mathrm{d}}{\mathrm{d}t} m_p(t_\star) \leqslant \left(\gamma_p\, C_p - (1-\gamma_p) K_p^{1+1/2p} \right) \mathcal{E}(t_\star)^{p+1/2} < 0$$

so that (2.16) still holds for subsequent times. □

2.4. **Self-similar variables.** As it was the case for constant restitution coefficient, it shall be often useful to deal with solutions of the Boltzmann equation in self-similar variables. Precisely, for a given collision kernel

$$B(u,\sigma) = \Phi(|u|) b(\widehat{u} \cdot \sigma)$$

with $b(\cdot)$ satisfying (2.6) and a given initial datum f_0 satisfying (2.8), let $f(t,v)$ be the solution to (2.7). We introduce a rescaled solution $g = g(\tau, w)$ such that

$$f(t,v) = V(t)^3 g(\tau(t), V(t) v) \tag{2.18}$$

where $\tau(\cdot)$ and $V(\cdot)$ are time scaling functions to be determined such that $\tau(0) = 0$ and $V(0) = 1$. Notice that, with this scaling one has

$$1 = \int_{\mathbb{R}^3} f(t,v)\, \mathrm{d}v = \int_{\mathbb{R}^3} g(\tau(t), w)\, \mathrm{d}w \qquad \forall t \geqslant 0$$

and $g(0,w) = f_0(w)$. A straightforward calculation shows that the function $g = g(\tau, w)$ satisfies the following:

$$V(t)^{-2}\mathcal{Q}_e(f,f)(t,v) = \dot\tau(t)V(t)\partial_\tau g(\tau,w) + \dot V(t)\nabla_w\cdot(wg(\tau,w))\Big|_{\substack{w=V(t)v\\ \tau=\tau(t)}} \tag{2.19}$$

where, here and in the sequel, we shall use the dot symbol for the derivative with respect to t. Moreover, the expression of the collision operator in self-similar variables is as follows:

$$\mathcal{Q}_{B,e}(f,f)\left(t,\frac{v}{V(t)}\right) = \mathcal{Q}_{B_\tau,\widetilde{e}_\tau}(g,g)(\tau(t),v)$$

where the rescaled collision kernel B_τ is given by

$$B_{\tau(t)}(u,\sigma) := V(t)\Phi\left(\frac{|u|}{V(t)}\right)b(\widehat{u}\cdot\sigma)$$

while the rescaled restitution coefficient $\widetilde{e}_\tau$ is defined by $\widetilde{e}_{\tau(t)}(z) := e\left(\frac{z}{V(t)}\right)$, $z>0$.

If the mapping $t\in\mathbb{R}^+ \longmapsto \tau(t)\in\mathbb{R}^+$ is one-to-one with inverse ζ, one can rewrite equation (2.19) in terms of τ only. Precisely, $g(\tau,w)$ is a solution to the following rescaled Boltzmann equation:

$$\lambda(\tau)\partial_\tau g(\tau,w) + \xi(\tau)\nabla_w\cdot(wg(\tau,w)) = \mathcal{Q}_{B_\tau,\widetilde{e}_\tau}(g,g)(\tau,w) \qquad \tau>0 \tag{2.20}$$

with $\lambda(\cdot) = \dot\tau(\zeta(\cdot))$, $\xi(\cdot) = \dot V(\zeta(\cdot))$,

$$B_\tau(u,\sigma) = V(\zeta(\tau))\Phi\left(\frac{|u|}{V(\zeta(\tau))}\right)b(\widehat{u}\cdot\sigma), \qquad \widetilde{e}_\tau(z) = e\left(\frac{z}{V(\zeta(\tau))}\right) \quad z>0.$$

Notice that, for generalized hard-spheres interactions (i.e. whenever $\Phi(|u|)=|u|$) one has $B_\tau = B$. For true hard-spheres interactions, i.e. $b(\cdot)=\frac{1}{4\pi}$ one simply denotes the rescaled collision operator by $\mathcal{Q}_{\widetilde{e}_\tau}$. It is very important to notice that the rescaled operator now depends on time, i.e. g is a solution to a non-autonomous problem. This is a major difference with respect to the case of constant restitution coefficient.

3. Free cooling of granular gases: generalized Haff's law

We prove in this section the so-called generalized *Haff's law for granular gases with variable restitution coefficient*. More precisely, we give the exact rate of decay of the temperature $\mathcal{E}(t)$ of the solution to Eq. (2.7). Notice that, in all this section, ***we are dealing only with the generalized hard-spheres collision kernel***:

$$B(u,\sigma) = |u|b(\widehat{u}\cdot\sigma)$$

where $b(\cdot)$ satisfies (2.6). Let f_0 be a nonnegative velocity distribution satisfying (2.8) and let $f(t,v)$ be the associated solution to the Cauchy problem (2.7). We denote its temperature $\mathcal{E}(t)$:

$$\mathcal{E}(t) = \int_{\mathbb{R}^3} f(t,v)|v|^2\,\mathrm{d}v.$$

Notice that the above Corollary 2.9 together with the last conditions of (2.8) implies that $\sup_{t\geqslant 0}\mathcal{E}(t)<\infty$. The evolution of $\mathcal{E}(t)$ is actually governed by the following:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t)=\int_{\mathbb{R}^3}\mathcal{Q}_{B,e}(f,f)(t,v)|v|^2\,\mathrm{d}v=\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3}f(t,v)f(t,v_\star)|u|\times$$
$$\times\int_{\mathbb{S}^2}\Big(|v'|^2+|v'_\star|^2-|v|^2-|v_\star|^2\Big)b(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma\;\mathrm{d}v_\star\,\mathrm{d}v$$

where we applied (2.5) with $\psi(v)=|v|^2$ and where $(v',v'_\star)$ are given by (2.4). One checks in a direct way that

$$|v'|^2+|v'_\star|^2-|v|^2-|v_\star|^2=-|u|^2\frac{1-\widehat{u}\cdot\sigma}{4}\left(1-e^2\left(|u|\sqrt{\frac{1-\widehat{u}\cdot\sigma}{2}}\right)\right)$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t)=-\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3}f(t,v)f(t,v_\star)|u|^3\,\mathrm{d}v\,\mathrm{d}v_\star$$
$$\int_{\mathbb{S}^2}\frac{1-\widehat{u}\cdot\sigma}{4}\left(1-e^2\left(|u|\sqrt{\frac{1-\widehat{u}\cdot\sigma}{2}}\right)\right)b(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma.$$

We compute this last integral over $\mathbb{S}^2$ (for fixed v and $v_\star$) using polar coordinates to get

$$|u|^3\int_{\mathbb{S}^2}\frac{1-\widehat{u}\cdot\sigma}{8}\left(1-e^2\left(|u|\sqrt{\frac{1-\widehat{u}\cdot\sigma}{2}}\right)\right)b(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma=$$
$$2\pi|u|^3\int_0^1\big(1-e^2(|u|y)\big)\,b(1-2y^2)y^3\,\mathrm{d}y=:\boldsymbol{\Psi}_e(|u|^2)$$

where we defined:

$$\boldsymbol{\Psi}_e(r):=2\pi r^{3/2}\int_0^1\big(1-e(\sqrt{r}z)^2\big)\,b\left(1-2z^2\right)z^3\,\mathrm{d}z,\qquad\forall r>0.\tag{3.1}$$

In other words, the evolution of the temperature $\mathcal{E}(t)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t)=-\int_{\mathbb{R}^3\times\mathbb{R}^3}f(t,v)f(t,v_\star)\boldsymbol{\Psi}_e(|u|^2)\,\mathrm{d}v\,\mathrm{d}v_\star,\qquad t\geqslant 0.$$

From now on, besides Assumptions 2.1, we assume that the restitution coefficient $e(\cdot)$ satisfies also the following:

Assumptions 3.1. *Assume that the mapping $z\mapsto e(z)\in(0,1]$ satisfies Assumptions 2.1 and*

(1) there exist $\alpha>0$ and $\gamma\geqslant 0$ such that

$$e(z)\simeq 1-\alpha\,z^\gamma\quad\text{for}\quad z\simeq 0$$

while $\liminf_{z\to\infty}e(z)=e_0<1$.

(2) $b(\cdot)\in L^q(\mathbb{S}^2)$ for some $q\geqslant 1$.

(3) the function $x>0\longmapsto\boldsymbol{\Psi}_e(x)$ defined in (3.1) is strictly increasing and convex over $(0,+\infty)$.

Remark 3.2. *For hard-spheres interactions, $b(\widehat{u}\cdot\sigma)=\frac{1}{4\pi}$ is constant and, setting $z=\sqrt{r}y$, it is possible to rewrite $\mathbf{\Psi}_e$ as*

$$\mathbf{\Psi}_e(r)=\frac{1}{2\sqrt{r}}\int_0^{\sqrt{r}}\left(1-e(y)^2\right)y^3\,\mathrm{d}y,\qquad\qquad r>0.$$

We prove in the Appendix that the above assumptions 3.1 are satisfied for the viscoelastic hard-spheres of Example 2.4 with $\gamma=1/5$. For constant restitution coefficient $e(z)=e_0$, Assumption 3.1 (1) is fulfilled with $\gamma=0$ and $\alpha=1-e_0$. Notice that, for hard-spheres collision kernel, Assumptions 3.1 (3) hold true if $e(\cdot)$ is continuously decreasing (see Lemma A. 1 in Appendix A).

3.1. **Upper bound for $\mathcal{E}(t)$.** We first prove the first part of Haff's law on the algebraic decay of $\mathcal{E}(t)$:

Proposition 3.3. *Let f_0 be a nonnegative velocity distribution satisfying (2.8) and let $f(t,v)$ be the associated solution to the Cauchy problem (2.7) where the variable restitution coefficient satisfies Assumptions 3.1. Then,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t)\leqslant-\mathbf{\Psi}_e(\mathcal{E}(t))\qquad\qquad\forall t\geqslant 0$$

where $\mathcal{E}(t)=\int_{\mathbb{R}^3}f(t,v)|v|^2\,\mathrm{d}v$. Moreover, there exist $C>0$ and $t_0>0$ such that

$$\mathcal{E}(t)\leqslant C\,(1+t)^{-\frac{2}{1+\gamma}}\qquad\qquad\forall t\geqslant t_0. \tag{3.2}$$

Proof. Recall that the evolution of the temperature is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t)=-\int_{\mathbb{R}^3\times\mathbb{R}^3}f(t,v)f(t,v_\star)\mathbf{\Psi}_e(|u|^2)\,\mathrm{d}v\,\mathrm{d}v_\star,\qquad t\geqslant 0, \tag{3.3}$$

where $u=v-v_\star$. Since $\mathbf{\Psi}_e(|\cdot|^2)$ is convex according to Assumption 3.1 *(2)* and $f(t,v_\star)\,\mathrm{d}v_\star$ is a probability measure over $\mathbb{R}^3$, Jensen's inequality implies

$$\int_{\mathbb{R}^3}f(t,v_\star)\mathbf{\Psi}_e(|u|^2)\,\mathrm{d}v_\star\geqslant\mathbf{\Psi}_e\left(\left|v-\int_{\mathbb{R}^3}v_\star f(t,v_\star)\,\mathrm{d}v_\star\right|^2\right)=\mathbf{\Psi}_e(|v|^2)$$

where we used (2.9). Applying again Jensen's inequality, we see that

$$\int_{\mathbb{R}^3}f(t,v)\mathbf{\Psi}_e(|v|^2)\,\mathrm{d}v\geqslant\mathbf{\Psi}_e\left(\int_{\mathbb{R}^3}f(t,v)|v|^2\,\mathrm{d}v\right),$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t)\leqslant-\mathbf{\Psi}_e(\mathcal{E}(t))\qquad\forall t\geqslant 0.$$

Since $\mathbf{\Psi}_e(\cdot)$ is strictly increasing with $\lim_{x\to0}\mathbf{\Psi}_e(x)=0$, this ensures that

$$\lim_{t\to\infty}\mathcal{E}(t)=0.$$

Moreover, according to Assumptions 3.1 *(1)*, it is clear from (3.1) that

$$\mathbf{\Psi}_e(x)\simeq C_\gamma x^{\frac{3+\gamma}{2}}\qquad\text{for}\qquad x\simeq 0$$

where C_γ is a positive constant, namely, $C_\gamma = 2\pi\alpha \int_0^1 y^{3+\gamma} b(1-2y^2)\,\mathrm{d}y < \infty$. Since $\mathcal{E}(t) \to 0$, there exists $t_0 > 0$ such that $\mathbf{\Psi}_e(\mathcal{E}(t)) \geqslant C_\gamma \mathcal{E}(t)^{\frac{3+\gamma}{2}}$ $\forall t \geqslant t_0$ which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \leqslant -C_\gamma \mathcal{E}(t)^{\frac{3+\gamma}{2}} \qquad \forall t \geqslant t_0.$$

This proves (3.2) □

Example 3.4. *In the case of constant restitution coefficient $e(z) = e_0 \in (0,1)$ for any $z \geqslant 0$, for hard-spheres interactions, one has*

$$\mathbf{\Psi}_e(x) = \frac{1-e_0^2}{8} x^{3/2}$$

and one recovers from (3.2) the decay of the temperature established from physical considerations (dimension analysis) in [17] and proved in [21], namely, $\mathcal{E}(t) \leqslant C(1+t)^{-2}$ for large t.

Example 3.5. *As already mentioned, for the restitution coefficient $e(\cdot)$ associated to viscoelastic hard-spheres (see Example 2.4), one has $\gamma = 1/5$ and the above estimate (3.2) leads a decay of the temperature faster than $(1+t)^{-5/3}$ which is the one obtained in [25] (see also [11]) from physical considerations and dimensional analysis.*

Remark 3.6. *Notice that, since $\mathcal{E}(t) \to 0$ as $t \to \infty$, it is possible to resume the arguments of [20, Prop. 5.1] to prove that the solution $f(t,v)$ to (2.7) converges to a Dirac mass as t goes to infinity, namely*

$$f(t,v) \underset{t\to\infty}{\longrightarrow} \delta_{v=0} \quad \textit{weakly} * \textit{in} \quad M^1(\mathbb{R}^3)$$

where $M^1(\mathbb{R}^3)$ denotes the space of normalized probability measures on $\mathbb{R}^3$.

3.2. **Lower bound for $\mathcal{E}(t)$: non-concentration on the self-similar variables.** Let us now prove that the above decay of the temperature is optimal under Assumptions 3.1. To do so, we argue as in [21] introducing self-similar variables (see Section 2.4). Precisely, for any solution $f(t,v)$ to (2.7) associated to an initial datum f_0 satisfying (2.8), we define the rescaled function $g = g(\tau, w)$ such that

$$f(t,v) = V(t)^3 g(\tau(t), V(t)v)$$

where $\tau(\cdot)$ and $V(\cdot)$ are time scaling functions to be determined such that $\tau(0) = 0$ and $V(0) = 1$. In such a case, g is a solution to (2.20) with $g(0,w) = f_0(w)$.

While the temperature $\mathcal{E}(t)$ of $f(t,v)$ is cooling down to zero (see Prop. 3.3), we identifies in this section suitable rescaled variables $\tau(\cdot)$ and $V(\cdot)$ for which the corresponding "temperature" of g is bounded away from zero. Precisely, for any $\tau > 0$, let

$$\mathbf{\Theta}(\tau) = \int_{\mathbb{R}^3} g(\tau, w)|w|^2\,\mathrm{d}w.$$

One sees from the rescaling (2.18) that

$$\mathcal{E}(t) = V(t)^{-2}\mathbf{\Theta}(\tau(t)) \qquad \forall t \geqslant 0. \tag{3.4}$$

Multiplying equation (2.19) by $|w|^2$ and integrating over $\mathbb{R}^3$ leads to

$$\dot\tau(t)V(t)\frac{\mathrm{d}\boldsymbol{\Theta}}{\mathrm{d}\tau}(\tau(t)) - 2\dot V(t)\boldsymbol{\Theta}(\tau(t)) = V(t)^{-2}\int_{\mathbb{R}^3}\mathcal{Q}_e(f,f)(t,w)|w|^2\,\mathrm{d}w$$
$$= -V(t)^{-2}\int_{\mathbb{R}^3\times\mathbb{R}^3} f(t,w)f(t,w_\star)\boldsymbol{\Psi}_e(|w-w_\star|^2)\,\mathrm{d}w\,\mathrm{d}w_\star$$

and using the re-scaling (2.18) again, we get

$$\dot\tau(t)V(t)\frac{\mathrm{d}\boldsymbol{\Theta}}{\mathrm{d}\tau}(\tau(t)) - 2\dot V(t)\boldsymbol{\Theta}(\tau(t))$$
$$= -V(t)^3\int_{\mathbb{R}^3\times\mathbb{R}^3} g(\tau(t),v)g(\tau(t),v_\star)\boldsymbol{\Psi}_e\left(|V(t)^{-1}u|^2\right)\,\mathrm{d}v\,\mathrm{d}v_\star \quad (3.5)$$

with $u = v - v_\star$. According to Assumption 3.1, we already noted that

$$\boldsymbol{\Psi}_e(x) \simeq C_\gamma x^{\frac{3+\gamma}{2}} \qquad \text{for } x \simeq 0$$

while, for large x, since $\liminf_{z\to\infty} e(z) = e_0 < 1$, it is clear that there exists $C_b > 0$ such that $\boldsymbol{\Psi}_e(x) \simeq C_b x^{3/2}$ for large x, namely $C_b = 2\pi(1-e_0^2)\int_0^1 b(1-2z^2)z^3\,\mathrm{d}z$. Thus, there exists a constant $C > 0$ such that

$$\boldsymbol{\Psi}_e(x) \leqslant Cx^{\frac{3+\gamma}{2}} \qquad \forall x > 0.$$

Consequently,

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} g(\tau(t),v)g(\tau(t),v_\star)\boldsymbol{\Psi}_e\left(|V(t)^{-1}u|^2\right)\,\mathrm{d}v\,\mathrm{d}v_\star$$
$$\leqslant CV(t)^{-(3+\gamma)}\int_{\mathbb{R}^3\times\mathbb{R}^3} g(\tau(t),v)g(\tau(t),v_\star)|u|^{3+\gamma}\,\mathrm{d}v\,\mathrm{d}v_\star$$
$$\leqslant 2^{3+\gamma}CV(t)^{-(3+\gamma)}\int_{\mathbb{R}^3} g(\tau(t),v)\,\mathrm{d}v\int_{\mathbb{R}^3} g(\tau(t),v_\star)|v_\star|^{3+\gamma}\,\mathrm{d}v_\star.$$

Since $\int_{\mathbb{R}^3} g(\tau(t),v)\,\mathrm{d}v = 1$,

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} g(\tau(t),v)g(\tau(t),v_\star)\boldsymbol{\Psi}_e\left(|V(t)^{-1}u|^2\right)\,\mathrm{d}v\,\mathrm{d}v_\star$$
$$\leqslant 2^{3+\gamma}CV(t)^{-(3+\gamma)}\int_{\mathbb{R}^3} g(\tau(t),v_\star)|v_\star|^{3+\gamma}\,\mathrm{d}v_\star.$$

It is easy to see, using Corollary 2.9 and, especially, Eq. (2.16) with $p = 3+\gamma$, that there exists $K > 0$ such that

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} g(\tau(t),v)g(\tau(t),v_\star)\boldsymbol{\Psi}_e\left(|V(t)^{-1}u|^2\right)\,\mathrm{d}v\,\mathrm{d}v_\star \leqslant KV(t)^{-(3+\gamma)}\boldsymbol{\Theta}(\tau(t))^{\frac{3+\gamma}{2}}$$

for any $t \geqslant 0$. This, together with (3.5), yields

$$\dot\tau(t)V(t)\frac{\mathrm{d}\boldsymbol{\Theta}}{\mathrm{d}\tau}(\tau(t)) - 2\dot V(t)\boldsymbol{\Theta}(\tau(t)) \geqslant -KV(t)^{-\gamma}\boldsymbol{\Theta}(\tau(t))^{\frac{3+\gamma}{2}}.$$

At this stage, one sees that choosing now $V(\cdot)$ and $\tau(\cdot)$ such that

$$\dot{V}(t) = \frac{1}{\gamma+1} V(t)^{-\gamma}, \qquad \dot{\tau}(t) V(t) = 1, \qquad \tau(0) = 0,\ V(0) = 1 \tag{3.6}$$

leads to the differential inequality

$$\frac{\mathrm{d}\mathbf{\Theta}}{\mathrm{d}\tau}(\tau(t)) \geqslant \left(\frac{2}{\gamma+1} \mathbf{\Theta}(\tau(t)) - K \mathbf{\Theta}(\tau(t))^{\frac{3+\gamma}{2}} \right) V(t)^{-\gamma}.$$

In particular, a simple application of the maximum principle implies that

$$\mathbf{\Theta}(\tau(t)) \geqslant \min \left\{ \mathbf{\Theta}(0),\ \left(\frac{2}{K(\gamma+1)} \right)^{2/(1+\gamma)} \right\}, \qquad \forall t \geqslant 0.$$

Turning back to the original variable f, we just proved the following:

Theorem 3.7. *For any initial distribution velocity $f_0 \geqslant 0$ satisfying the conditions given by (2.8), the solution $f(t,v)$ to the associated Boltzmann equation (2.7) satisfies the generalized Haff's law for variable restitution coefficient $e(\cdot)$ fulfilling Assumptions 3.1:*

$$c(1+t)^{-\frac{2}{1+\gamma}} \leqslant \mathcal{E}(t) \leqslant C(1+t)^{-\frac{2}{1+\gamma}}, \qquad t \geqslant 0 \tag{3.7}$$

where $\mathcal{E}(t) = \int_{\mathbb{R}^3} f(t,v)|v|^2 \,\mathrm{d}v$ and c, C are positive constants depending only on $e(\cdot)$ and $\mathcal{E}(0)$. More generally, the $p-$moment $m_p(t)$ defined in (2.15) satisfies

$$c_p(1+t)^{-\frac{2p}{1+\gamma}} \leqslant m_p(t) \leqslant C_p(1+t)^{-\frac{2p}{1+\gamma}}, \qquad t \geqslant 0,\ \ p \geqslant 1 \tag{3.8}$$

where the positive constants c_p, C_p depend on p, $m_p(0)$, $\mathcal{E}(0)$ and $e(\cdot)$.

Proof. We just proved that, for $g(\tau, w)$ given by (2.18) where the time scaling functions $\tau(\cdot)$ and $V(\cdot)$ are solutions to (3.6), there exists $c > 0$ such that $\mathbf{\Theta}(\tau(t)) \geqslant c$ for any $t > 0$. According to (3.4), this implies the following lower bound for the temperature $\mathcal{E}$:

$$\mathcal{E}(t) \geqslant \frac{c}{V(t)^2} \qquad \forall t \geqslant 0.$$

Since $\dot{V}(t) = \frac{1}{\gamma+1} V^{-\gamma}(t)$ with $V(0) = 1$, we get that $V(t) = (1+t)^{\frac{1}{\gamma+1}}$ for any $t \geqslant 0$ and obtain the desired lower bound, the upper bound being provided by Prop. 3.2. For general p-moments, the use of Jensen's inequality (for the lower bound) and Corollary 2.9 (for the upper bound) yield

$$c_p\, \mathcal{E}(t)^p \leqslant m_p(t) \leqslant C_p\, \mathcal{E}(t)^p$$

for some positive constants $c_p,\ C_p$. Then, (3.7) provides the conclusion. □

Example 3.8. *For constant restitution coefficient, $e(z) = e_0$ for any $z \geqslant 0$, since $\gamma = 0$, we recover, via a simpler argument, the classical Haff's law of [17] proved recently by Mischler and Mouhot [21]:*

$$c(1+t)^{-2} \leqslant \mathcal{E}(t) \leqslant C(1+t)^{-2}, \qquad t \geqslant 0.$$

Example 3.9. *For viscoelastic hard-spheres (see Example 2.4), as already said, $e(\cdot)$ fulfills Assumptions 3.1 with $\gamma = 1/5$ and Theorem 3.7 provides the first rigorous justification of the cooling rate conjectured in* [11, 25]*:*

$$c(1+t)^{-5/3} \leqslant \mathcal{E}(t) \leqslant C(1+t)^{-5/3}, \qquad t \geqslant 0.$$

Remark 3.10. *Theorem 3.7 shows that the decay of the temperature is governed, in some sense, by the behavior of the restitution coefficient $e(z)$ for small impact. The cooling of the gases is slower for increasing γ. However, if Assumptions 3.1 hold true, one sees that the cooling is still algebraic in time even for very large γ.*

From now, when dealing with $g = g(\tau, w)$ defined in (2.18), we shall always assume the time scaling functions $\tau(\cdot)$ and $V(\cdot)$ to satisfy (3.6), i.e.

$$V(t) = (1+t)^{\frac{1}{1+\gamma}}, \quad \tau(t) = \int_0^t \frac{\mathrm{d}s}{V(s)} = \frac{\gamma+1}{\gamma}\left((1+t)^{\frac{\gamma}{1+\gamma}} - 1\right), \qquad t \geqslant 0$$

where $\gamma > 0$ is the constant in Assumptions 3.1. In this case, from (2.20), $g(\tau, w)$ is a solution to the following Cauchy problem:

$$\left(\partial_\tau g + \xi(\tau)\nabla_w \cdot (wg)\right)(\tau, w) = \mathcal{Q}_{\widetilde{e}_\tau}(g, g)(\tau, w), \qquad g(0, w) = f_0(w), \tag{3.9}$$

where

$$\xi(\tau) = \frac{1}{\gamma\tau + (1+\gamma)}, \quad \text{and} \quad \widetilde{e}_\tau \,:\, z > 0 \longmapsto e\left(z\left(1 + \frac{\gamma}{\gamma+1}\tau\right)^{-1/\gamma}\right). \tag{3.10}$$

As already mentioned, in contrast to what happens for constant restitution coefficient, the "rescaled" collision operator $\mathcal{Q}_{\widetilde{e}_\tau}(g, g)$ is now depending on τ.

Remark 3.11. *If $\gamma = 0$ in Assumptions 3.1, then $V(t) = 1 + t$ while $\tau(t) = \ln(1+t)$. In such a case, $\xi(\tau) = 1$ is constant (see* [20]*).*

4. High-energy tails for the self-similar solution

We are interested in this Section in the study of the tail behavior of the solution $f(t, v)$ to the Boltzmann's equation (1.3). More precisely, we shall give an estimate of the high-energy tails of $f(t, v)$ through a time-dependent weighted integral bound of the solution to (1.3). Our approach is reminiscent to the work of Bobylev [7] recently improved in a series of paper [9, 21, 2]. Here again, in all this section, we shall deal with the generalized hard-spheres collision kernel:

$$B(u, \sigma) = |u| b(\widehat{u} \cdot \sigma)$$

where $b(\cdot)$ satisfies (2.6). To prove our result, it shall be convenient to deal, as we did in the previous section, with the self-similar solution $g = g(\tau, w)$ defined in (2.18).

For any $1 \leqslant p < \infty$ and any $\tau \geqslant 0$, we define:

$$\mathbf{m}_p(\tau) = \int_{\mathbb{R}^3} g(\tau, w)\, |w|^{2p}\, \mathrm{d}w.$$

Notice that (3.8) readily translates into

$$c_p \leqslant \mathbf{m}_p(\tau) \leqslant C_p \quad \text{for } \tau \geqslant 0 \tag{4.1}$$

where $c_p, C_p > 0$ are the constants in Theorem 3.7. One can prove the following Theorem, which generalizes [21, Proposition 3.1] to the case of a variable restitution coefficient.

Theorem 4.1 (**L^1-exponential tails Theorem**). *Let $B(u,\sigma) = |u| b(\widehat{u}\cdot\sigma)$ with $b(\cdot)$ satisfying (2.6) with $b(\cdot) \in L^q(\mathbb{S}^2)$ for some $q \geqslant 1$. Assume the variable restitution coefficient $e(\cdot)$ satisfy Assumptions 3.1. Let f_0 satisfying (2.8) and assume moreover that there exists $r_0 > 0$ such that*

$$\int_{\mathbb{R}^3} f_0(v) \exp\left(r_0|v|\right)\, \mathrm{d}v < \infty.$$

Let $g(\tau, w)$ be the self-similar solution defined by (2.18) where $f(t,v)$ is the solution to (2.7). Then, there exists some $r \leqslant r_0$ such that

$$\sup_{\tau\geqslant 0} \int_{\mathbb{R}^3} g(\tau, w) \exp\left(r|w|\right)\, \mathrm{d}w < \infty. \tag{4.2}$$

Consequently, the solution $f(t,v)$ satisfies

$$\sup_{t\geqslant 0} \int_{\mathbb{R}^3} f(t, v) \exp\left(rV(t)|v|\right)\, \mathrm{d}w < \infty. \tag{4.3}$$

Proof. The method of proof is by now rather standard and carefully documented in [9, 2] for a time-independent version. We sketch only the time-dependent proof which is divided in several steps:

Step 1. Note that formally

$$\int_{\mathbb{R}^3} g(\tau, w) \exp\left(r|w|^s\right)\, \mathrm{d}w = \sum_{k=0}^{\infty} \frac{r^k}{k!} \mathbf{m}_{sk/2}(\tau),$$

for any $r > 0$ and any $s > 0$. Hence, the summability of the integral is described by the behavior of the functions $\frac{\mathbf{m}_{sk/2}(\tau)}{k!}$. This motivates the introduction of the renormalized moments

$$z_p(\tau) := \frac{\mathbf{m}_p(\tau)}{\Gamma(ap+b)}, \quad \text{with } a = 2/s,$$

where $\Gamma(\cdot)$ denotes the Gamma function. We shall prove that the series converges for some $r < r_0$ and with $s = 1$ (i.e. $a = 2$). To do so, it is enough to prove that, for some $b < 1$ and $Q > 0$ large enough, one has $z_p(\tau) \leqslant Q^p$ for any $p \geqslant 1$ and any $\tau \geqslant 0$.

Step 2. Recall that, according to Lemma 2.6, the estimates proved in Povzner Lemma (Prop. 2.8) are independent of the restitution coefficient $e(\cdot)$. In particular, they hold for the time-dependent collision operator $\mathcal{Q}_{\widetilde{e}_\tau}$ providing bounds *which are uniform with respect* to τ. Specifically,

$$\int_{\mathbb{R}^3} \mathcal{Q}_{\widetilde{e}_\tau}(g,g)(\tau,w)|w|^{2p}\,\mathrm{d}w \leqslant -(1-\gamma_p)\mathbf{m}_{p+1/2}(\tau) + \gamma_p\, \mathcal{S}_p(\tau), \qquad \forall \tau \geqslant 0$$

where γ_p is the constant introduced in Lemma 2.6 and

$$\mathcal{S}_p(\tau) = \sum_{k=1}^{[\frac{p+1}{2}]} \left(\begin{array}{c} p \\ k \end{array} \right) \left(\mathbf{m}_{k+1/2}(\tau)\, \mathbf{m}_{p-k}(\tau) + \mathbf{m}_k(\tau)\, \mathbf{m}_{p-k+1/2}(\tau) \right).$$

Step 3. An important simplification, first observed in [9], consists in noticing that the term $\mathcal{S}_p$ satisfies

$$\mathcal{S}_p(\tau) \leqslant A\,\Gamma(ap+a/2+2b)\;\mathcal{Z}_p(\tau) \;\text{ for }\; a \geqslant 1,\; b>0,$$

where $A = A(a,b) > 0$ does not depend on p and

$$\mathcal{Z}_p(\tau) = \max_{1\leqslant k\leqslant k_p} \left\{ z_{k+1/2}(\tau)\; z_{p-k}(\tau), z_k(\tau)\; z_{p-k+1/2}(\tau) \right\}.$$

With such an estimate, the rather involved term $\mathcal{S}_p$ is more tractable.

Step 4. Using the above steps and the evolution problem (3.9) satisfied by the self-similar solution g, we check easily that

$$\frac{\mathrm{d}\mathbf{m}_p}{\mathrm{d}\tau}(\tau) + (1-\gamma_p)\mathbf{m}_{p+1/2}(\tau) \leqslant \gamma_p\,\Gamma(ap+a/2+2b)\mathcal{Z}_p(\tau) + 2p\,\xi(\tau)\mathbf{m}_p(\tau)$$

where we used the fact that $\int_{\mathbb{R}^3} |w|^{2p}\nabla_w\cdot(wg(\tau,w))\,\mathrm{d}w = -2p\,\mathbf{m}_p(\tau)$. Using the asymptotic formula

$$\lim_{p\to\infty}\frac{\Gamma(p+r)}{\Gamma(p+s)}p^{s-r} = 1,$$

and the fact that $\xi(\tau)\leqslant 1$, one concludes that there are constants $c_i > 0$ $(i=1,2)$ and $p_0 > 1$ large enough (recall that $\gamma_p \sim 1/p^{1/q'}$ for large p) so that

$$\frac{\mathrm{d}z_p}{\mathrm{d}\tau}(\tau) + c_1\,p^{a/2}z_p^{1+1/2p}(\tau) \leqslant c_2\,p^{a/2+b-1/q'}\;\mathcal{Z}_p(\tau) + 2p\;z_p(\tau) \qquad \forall\tau\geqslant 0,\; p\geqslant p_0,$$

where we also used the fact that $\mathbf{m}_{p+1/2}(\tau) \geqslant \mathbf{m}_p^{1+1/2p}(\tau)$ for any $\tau\geqslant 0$ thanks to Jensen's inequality.

Final step. We claim that if we choose $a=2$ and $0<b<1/q'$ it is possible to find $Q>0$ large enough so that $\mathbf{m}_p(\tau)\leqslant Q^p$. Indeed, let p_0 and $Q<\infty$ such that

$$\frac{c_2}{c_1}p_0^{b-1/q'} \leqslant \frac{1}{2},\;\text{ and }\; Q \geqslant \left\{ \max_{1\leqslant k\leqslant p_0}\sup_{\tau\geqslant 0} z_k(\tau), Q_0, \frac{16}{c_1^2}, 1 \right\},$$

here Q_0 is a constant such that $z_p(0)\leqslant Q_0^p$. This constant exists by the exponential integrability assumption on the initial datum. Moreover, since moments of g are uniformly propagated, the existence of such *finite* Q is guaranteed. Arguing now by induction and by standard comparison of ODE's, one proves as in [21, 9, 2] that $y_p(\tau) := Q^p$ satisfies for $p\geqslant p_0$

$$\frac{\mathrm{d}y_p}{\mathrm{d}\tau}(\tau) + c_1\,p^{a/2}y_p^{1+1/2p}(\tau) \geqslant c_2\,p^{a/2+b-1/q'}\;\mathcal{Z}_p(\tau) + 2p\;y_p(\tau), \quad y_p(0)\geqslant z_p(0)$$

therefore, $y_p(\tau)\geqslant z_p(\tau)\;\forall p\geqslant p_0$. Since this is trivially true for $p<p_0$ we obtain finally that

$$\mathbf{m}_p(\tau)\leqslant\Gamma(2p+b)Q^p, \qquad \forall p\geqslant 1, \tau\geqslant 0.$$

From Step 1, this is enough to prove the Theorem. □

Example 4.2. *For viscoelastic hard-spheres (see Example 2.4), as already said,* $V(t) = (1+t)^{5/3}$. *Therefore,*

$$\int_{\mathbb{R}^3} f_0(v)\exp\left(r_0|v|\right)\,\mathrm{d}v < \infty \Longrightarrow \sup_{t\geqslant 0}\int_{\mathbb{R}^3} f(t,v)\exp\left(r(1+t)^{5/6}|v|\right)\,\mathrm{d}v < \infty$$

for some $r < r_0$. In particular, using the terminology of [9], *$f(t,v)$ has an (uniform in time)* exponential tail of order 1.

5. Regularity properties of the collision operator

We extend now to variable restitution coefficient the regularity properties of the collision operator obtained originally in [18, 28], extended in [24] and generalized to cover the inelastic case in [21]. Of course, we only need to investigate the regularity properties of $\mathcal{Q}_e^+$ since the loss operator $\mathcal{Q}_e^-$ does not depend on the restitution coefficient. To do so, we shall need some basic estimates on the gain operator $\mathcal{Q}_{B,e}^+$ associated to a general collision kernel $B(u,\sigma)$ (see (2.5)).

5.1. **Carleman representation.** We establish here a technical representation of the gain term $\mathcal{Q}_{B,e}^+$ which is reminiscent of the classical Carleman representation in the elastic case (extended to the inelastic case for a constant restitution coefficient in [21]). Precisely, let $B(u,\sigma)$ be a general collision kernel of the form

$$B(u,\sigma) = \Phi(|u|)b(\widehat{u}\cdot\sigma)$$

where $\Phi(\cdot) \geqslant 0$ while $b(\cdot) \geqslant 0$ satisfies (2.6). For any $\psi = \psi(v)$, define the following linear operators:

$$\mathcal{S}_{\pm}(\psi)(u) = \int_{\mathbb{S}^2} \psi(u^{\pm}) b(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma, \qquad \forall u \in \mathbb{R}^3 \tag{5.1}$$

where we set

$$u^- = \beta\left(|u|\sqrt{\frac{1-\widehat{u}\cdot\sigma}{2}}\right)\ \frac{u-|u|\,\sigma}{2}, \quad \text{and} \quad u^+ = u - u^-.$$

Then, one has the following technical result

Lemma 5.1. *For any continuous functions ψ and φ, one has*

$$\int_{\mathbb{R}^3} \varphi(u)\mathcal{S}_-(\psi)(u)\Phi(|u|)\,\mathrm{d}u = \int_{\mathbb{R}^3} \psi(x)\Gamma_B(\varphi)(x)\,\mathrm{d}x$$

where the linear operator Γ_B is given by

$$\Gamma_B(\varphi)(x) = \int_{\omega^\perp} \mathcal{B}(z+\alpha(r)\omega,\alpha(r))\varphi(\alpha(r)\omega+z)\,\mathrm{d}\pi_z, \qquad x = r\omega,\ r\geqslant 0,\ \omega\in\mathbb{S}^2 \tag{5.2}$$

where $\mathrm{d}\pi_z$ is the Lebesgue measure in the hyperplane $\omega^\perp$ perpendicular to ω, $\alpha(\cdot)$ is the inverse of the mapping $s\mapsto s\beta(s)$ and

$$\mathcal{B}(z,\varrho) = \frac{8\Phi(|z|)}{|z|(\varrho\beta(\varrho))^2} b\left(1-2\frac{\varrho^2}{|z|^2}\right)\frac{\varrho}{1+\vartheta_z(\varrho)}, \qquad \varrho\geqslant 0, \quad z\in\mathbb{R}^3 \tag{5.3}$$

with $\vartheta(\cdot)$ defined in Assumption 2.1 (2) and $\vartheta_z(\cdot)$ denoting its derivative.

Proof. Up to divide $\mathcal{B}(z,\varrho)$ by $\Phi(|z|)$, there is no loss of generality in assuming for simplicity that $\Phi(|\cdot|)\equiv 1$. Let

$$I=\int_{\mathbb{R}^3}\varphi(u)\mathcal{S}_-(\psi)(u)\,\mathrm{d}u=\int_{\mathbb{R}^3}\varphi(u)\,\mathrm{d}u\int_{\mathbb{S}^2}\psi(u^-)b(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma.$$

For a fixed $u\in\mathbb{R}^3$, we perform the integration over $\mathbb{S}^2$ using the following formula

$$\int_{\mathbb{S}^2}F\left(\frac{u-|u|\sigma}{2}\right)\,\mathrm{d}\sigma=\frac{4}{|u|}\int_{\mathbb{R}^3}\delta(|x|^2-x\cdot u)F(x)\,\mathrm{d}x$$

which is valid for any given function F. Then,

$$I=4\int_{\mathbb{R}^3\times\mathbb{R}^3}\varphi(u)|u|^{-1}\delta(|x|^2-x\cdot u)\psi\big(x\beta(|x|)\big)b\left(1-2|x|^2/|u|^2\right)\,\mathrm{d}x\,\mathrm{d}u.$$

Setting now $u=z+x$ we get

$$I=4\int_{\mathbb{R}^3\times\mathbb{R}^3}\varphi(x+z)|x+z|^{-1}\delta(x\cdot z)\psi\big(x\beta(|x|)\big)b\left(1-2|x|^2/|x+z|^2\right)\,\mathrm{d}z\,\mathrm{d}x.$$

Finally, keeping x fixed, we remove the Dirac mass thanks to the identity

$$\int_{\mathbb{R}^3}F(z)\delta(x\cdot z)\,\mathrm{d}z=\frac{1}{|x|}\int_{x^\perp}F(z)\,\mathrm{d}\pi_z$$

which leads to

$$I=4\int_{\mathbb{R}^3}\psi\big(x\beta(|x|)\big)\frac{\mathrm{d}x}{|x|}\int_{x^\perp}\frac{\varphi(x+z)}{|x+z|}b\left(1-2|x|^2/|x+z|^2\right)\,\mathrm{d}\pi_z.$$

We now perform the x integral using polar coordinates $x=\varrho\omega$ and with the change of variables $r=\varrho\beta(\varrho)$ with inverse $\alpha(r)$. Notice that $\mathrm{d}r=\frac{1}{2}(1+\vartheta_z(\varrho))\,\mathrm{d}\varrho$ which yields

$$I=8\int_0^\infty\frac{\alpha(r)\,\mathrm{d}r}{1+\vartheta_z(\alpha(r))}\int_{\mathbb{S}^2}\psi(r\omega)\,\mathrm{d}\omega\int_{\omega^\perp}\frac{\varphi(z+\alpha(r)\omega)}{|z+\alpha(r)\omega|}b\left(1-2\frac{\alpha(r)^2}{|z+\alpha(r)\omega|^2}\right)\,\mathrm{d}\pi_z.$$

Turning back to cartesian coordinates $x=r\omega$, $\mathrm{d}x=r^2\,\mathrm{d}r\,\mathrm{d}\omega$, we get the desired expression $I=\int_{\mathbb{R}^3}\psi(x)\Gamma_B(\varphi)(x)\,\mathrm{d}x$ where Γ_B is given by (5.2). □

The above result leads to a Carleman-like expression for $\mathcal{Q}^+_{B,e}$:

Corollary 5.2 (Carleman representation). *Let $e(\cdot)$ satisfy Assumptions 2.1 and let*

$$B(u,\sigma)=\Phi(|u|)b(\widehat{u}\cdot\sigma)$$

satisfying (2.6). Then, for any velocity distributions f,g one has

$$\mathcal{Q}^+_{B,e}(f,g)(v)=\int_{\mathbb{R}^3}f(z)\left[(t_z\circ\Gamma_B\circ t_z)\,g\right](v)\,\mathrm{d}z$$

where $[t_v\psi](x)=\psi(v-x)$ for any $v,x\in\mathbb{R}^3$ and any test-function ψ.

Proof. The proof is a simple consequence of the above Lemma together with the following identity

$$\int_{\mathbb{R}^3\times\mathbb{R}^3}\mathcal{Q}^+_{B,e}(f,g)(v)\psi(v)\,\mathrm{d}v=\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3}f(v)g(v-u)\Phi(|u|)\mathcal{S}_-(t_v\psi)(u)\,\mathrm{d}v\,\mathrm{d}u\quad(5.4)$$

valid for any test-function ψ. □

5.2. **Convolution-like estimates for $\mathcal{Q}^+_{B,e}$.** For the so-called Variable Hard Spheres collision kernels (i.e. for $\Phi(|u|) = |u|^k$, $k \geqslant 0$), general convolution-like estimates are obtained in [3, Theorem 1] for non-constant restitution coefficient. However, such estimates are given in L^p_η with $\eta \geqslant 0$ and, for the applications we have in mind, we need to extend some of them to $\eta \leqslant 0$. This can be done easily using the method developed in [24] (see also [14]) and the estimates of [3] leading to the following theorem [2]:

Theorem 5.3. *Let $B(u,\sigma) = \Phi(|u|)b(\widehat{u}\cdot\sigma)$ where $b(\cdot)$ satisfies (2.6) and $\Phi(\cdot) \in L^\infty_{-k}$ for some $k \in \mathbb{R}$. Assume that $e(\cdot)$ fulfils Assumption 2.1. Then, for any $1 \leqslant p \leqslant \infty$ and any $\eta \in \mathbb{R}$, there exists $\mathbf{C}_{\eta,p,k}(B) > 0$ such that*

$$\left\|\mathcal{Q}^+_{B,e}(f,g)\right\|_{L^p_\eta} \leqslant \mathbf{C}_{\eta,p,k}(B)\,\|f\|_{L^1_{|\eta+k|+|\eta|}}\,\|g\|_{L^p_{\eta+k}}$$

where the constant $\mathbf{C}_{\eta,p,k}(B)$ is given by:

$$\mathbf{C}_{\eta,p,k}(B) = c_{k,\eta,p}\,\|\Phi\|_{L^\infty_{-k}}\,\gamma(\eta,p,b) \tag{5.5}$$

with a constant $c_{k,\eta,p} > 0$ depending only on k,η and p while

$$\gamma(\eta,p,b) = \int_{-1}^{1}\left(\frac{1-s}{2}\right)^{-\frac{3+\eta_+}{2p'}} b(s)\,\mathrm{d}s \tag{5.6}$$

where $1/p + 1/p' = 1$ and η_+ is the positive part of η. In the same way, there exists $\widetilde{\mathbf{C}}_{\eta,p,k}(B) > 0$ such that

$$\left\|\mathcal{Q}^+_{B,e}(f,g)\right\|_{L^p_\eta} \leqslant \widetilde{\mathbf{C}}_{\eta,p,k}(B)\,\|g\|_{L^1_{|\eta+k|+|\eta|}}\,\|f\|_{L^p_{\eta+k}}$$

where the constant $\widetilde{\mathbf{C}}_{\eta,p,k}(B)$ is given by:

$$\widetilde{\mathbf{C}}_{\eta,p,k}(B) = \widetilde{c}_{k,\eta,p}\,\|\Phi\|_{L^\infty_{-k}}\,\widetilde{\gamma}(\eta,p,b) \tag{5.7}$$

for some constant $\widetilde{c}_{k,\eta,p} > 0$ depending only on k,η and p while

$$\widetilde{\gamma}(\eta,p,b) = \int_{-1}^{1}\left(\frac{1+s}{2} + (1-\beta_0)^2\,\frac{1-s}{2}\right)^{-\frac{3+\eta_+}{2p'}} b(s)\,\mathrm{d}s \tag{5.8}$$

where $1/p + 1/p' = 1$ and $\beta_0 = \beta(0) = \frac{1+e(0)}{2}$.

Proof. Let $1 \leqslant p \leqslant \infty$ and $\eta \in \mathbb{R}$ be fixed and let $1/p' + 1/p = 1$. By duality,

$$\left\|\mathcal{Q}^+_{B,e}(f,g)\right\|_{L^p_\eta} = \sup\left\{\left|\int_{\mathbb{R}^3}\mathcal{Q}^+_{B,e}(f,g)(v)\psi(v)\,\mathrm{d}v\right| \;;\; \|\psi\|_{L^{p'}_{-\eta}} \leqslant 1\right\}.$$

As already mentioned (see (5.4))

$$\int_{\mathbb{R}^3}\mathcal{Q}^+_{B,e}(f,g)(v)\psi(v)\,\mathrm{d}v = \int_{\mathbb{R}^3\times\mathbb{R}^3} f(v)g(v-u)\mathcal{T}_-(t_v\psi)(u)\,\mathrm{d}v\,\mathrm{d}u$$

with

$$\mathcal{T}_-(\psi)(u) = \Phi(|u|)\mathcal{S}_-(\psi)(u), \qquad t_v\psi(x) = \psi(v-x)$$

[2] Notice that the constants $\gamma(\eta,p,b)$ and $\widetilde{\gamma}(\eta,p,b)$ given by (5.6) and (5.8) are of course not finite for any angular kernel b or parameters η, p. It is implicitly assumed that the Theorem applies for the range of parameters leading to finite constants (see also Remark 5.4).

where $\mathcal{S}_-$ has been defined in Section 5.1. With the notations of [3], one recognizes that $\mathcal{S}_-(h) = \mathcal{P}(h,1)$ for any h, so that, applying [3, Theorem 5] (with $q=\infty$ and $\alpha=-\eta$),

$$\|\mathcal{S}_-(h)\|_{L^{p'}_{-\eta}} \leqslant \gamma(\eta,p,b)\|h\|_{L^{p'}_{-\eta}}$$

with $\gamma(\eta,p,b)$ given by (5.6) (see [3, Eq. (2.15)], notice that, with respect to [3], we use the weight $\langle v\rangle^\eta$ instead of $|v|^\eta$ and this is the reason why we have to introduce η_+ in our definition of $\gamma(\eta,p,b)$). As a consequence,

$$\|\mathcal{T}(h)\|_{L^{p'}_{-\eta-k}} \leqslant \gamma(\eta,p,b)\|\Phi\|_{L^\infty_{-k}}\|h\|_{L^{p'}_{-\eta}}\,. \tag{5.9}$$

Now,

$$\begin{aligned}\left|\int_{\mathbb{R}^3}\mathcal{Q}^+_{B,e}(f,g)\psi\,\mathrm{d}v\right| &\leqslant \int_{\mathbb{R}^3}|f(v)|\,\mathrm{d}v\left(\int_{\mathbb{R}^3}|g(u)|\,[(t_v\circ\mathcal{T}\circ t_v)\psi]\,(u)\,\mathrm{d}u\right)\\ &\leqslant \|g\|_{L^p_{\eta+k}}\int_{\mathbb{R}^3}|f(v)|\,\|(t_v\circ\mathcal{T}\circ t_v)\psi\|_{L^{p'}_{-k-\eta}}\,\mathrm{d}v.\end{aligned}$$

Using the fact that, for any $s\in\mathbb{R}$, $\|t_v h\|_{L^{p'}_s}\leqslant 2^{|s|/2}\langle v\rangle^{|s|}\|h\|_{L^{p'}_s}$ for any v, we get

$$\begin{aligned}\left|\int_{\mathbb{R}^3}\mathcal{Q}^+_{B,e}(f,g)\psi\,\mathrm{d}v\right| &\leqslant 2^{|\eta+k|/2}\|g\|_{L^p_{\eta+k}}\int_{\mathbb{R}^3}|f(v)|\langle v\rangle^{|\eta+k|}\,\|(\mathcal{T}\circ t_v)\psi\|_{L^{p'}_{-k-\eta}}\,\mathrm{d}v\\ &\leqslant 2^{|\eta+k|/2}\gamma(\eta,p,b)\|\Phi\|_{L^\infty_{-k}}\|g\|_{L^p_{\eta+k}}\int_{\mathbb{R}^3}|f(v)|\langle v\rangle^{|\eta+k|}\,\|t_v\psi\|_{L^{p'}_{-\eta}}\,\mathrm{d}v\\ &\leqslant 2^{|\eta+k|+|\eta|/2}\gamma(\eta,p,b)\|\Phi\|_{L^\infty_{-k}}\|g\|_{L^p_{\eta+k}}\int_{\mathbb{R}^3}|f(v)|\langle v\rangle^{|\eta+k|+|\eta|}\,\|\psi\|_{L^{p'}_{-\eta}}\,\mathrm{d}v\end{aligned}$$

which proves the first part of the result. Now, to prove the second part, one notices that

$$\int_{\mathbb{R}^3}\mathcal{Q}^+_{B,e}(f,g)(v)\psi(v)\,\mathrm{d}v=\int_{\mathbb{R}^3\times\mathbb{R}^3}f(v-u)g(v)\mathcal{T}_+(t_v\psi)(u)\,\mathrm{d}v\,\mathrm{d}u$$

with $\mathcal{T}_+(\psi)(u)=\Phi(|u|)\mathcal{S}_+(\psi)(u)$ where $\mathcal{S}_+$ has been defined in (5.1). Using again the notations of [3], one has $\mathcal{S}_+(h)=\mathcal{P}(1,h)$ for any h, so that, applying [3, Theorem 5] (with now $p=\infty$ and $\alpha=-\eta$), we get that

$$\|\mathcal{S}_-(h)\|_{L^{p'}_{-\eta}}\leqslant\widetilde{\gamma}(\eta,p,b)\|h\|_{L^{p'}_{-\eta}}$$

where $\widetilde{\gamma}(\eta,p,b)$ given by (5.8). One concludes as above, exchanging the roles of f and g. □

Remark 5.4. *Clearly, the constants $\gamma(\eta,p,b)$ and $\widetilde{\gamma}(\eta,p,b)$ are not finite for any given η, p or b because of the possible singularity in $s=1$ or $s=-1$. However, if $p=1$ then $p'=\infty$ and both $\gamma(\eta,1,b)$ and $\widetilde{\gamma}(\eta,1,b)$ are finite for any integrable kernel $b(\cdot)$ and any $\eta\in\mathbb{R}$. Moreover, if one assumes additionally (as in [21]) that the angular kernel $b(\cdot)$ vanishes in the vicinity of $s=1$ then $\gamma(\eta,p,b)<\infty$ for any $1\leqslant p\leqslant\infty$ and any $\eta\in\mathbb{R}$. In the same way, if for instance $\beta_0=1$ and $b(\cdot)$ vanishes around $s=-1$, then $\widetilde{\gamma}(\eta,p,b)<\infty$ for any $1\leqslant p\leqslant\infty$ and any $\eta\in\mathbb{R}$.*

Remark 5.5. *Notice that the above constant* $\mathbf{C}_{\eta,p,k}(B)$ *given by* (5.5) *does not depend on the restitution coefficient* $e(\cdot)$ *while* $\widetilde{\mathbf{C}}_{\eta,p,k}$ *depends on the restitution coefficient* $e(z)$ *only through its value for* $z=0$*. Notice in particular, that, if* $e(0)=1$ *and* $b(s)=b(-s)$*, then* $\widetilde{\gamma}(\eta,p,b)=\gamma(\eta,p,b)$*. This is the case for the viscoelastic hard-spheres (see Example 2.4).*

For the special case, $p=\infty$, we can make the above a bit more precise, providing a pointwise estimate of the gain part:

Corollary 5.6. *Assume the variable restitution coefficient* $e(\cdot)$ *satisfy Assumptions 3.1. Assume also that the collision kernel is given by* $B(u,\sigma)=\Phi(|u|)b(\widehat{u}\cdot\sigma)$ *where* $b(\cdot)$ *satisfies* (2.6) *and* $\Phi(\cdot)\in L^\infty_{-k}$ *for some* $k\geqslant 0$*. Then, for any* $\varepsilon>0$*, the exists a positive constant* $C:=C(\|b\|_1,\varepsilon,k)$ *such that*

$$\mathcal{Q}^+_{B,e}(f,f)(v)\leqslant C\|f\|^2_{L^2_k}+\varepsilon\|f\|_{L^1_k}\|f\|_{L^\infty}\langle v\rangle^k,$$

for any nonnegative $f\in L^1_k\cap L^\infty(\mathbb{R}^3)$*.*

Proof. Due to the symmetry of the collision operator (recall that we look for an estimate of the *quadratic* operator $\mathcal{Q}^+_{B,e}(f,f)$) we may assume that b is supported in $[-1,0]$. Fix $\varepsilon>0$ and write

$$b(s)=b(s)\chi_{[-1+\delta,0]}+b(s)\chi_{[-1,-1+\delta)}:=b_1(s)+b_2(s),$$

with $\delta>0$ chosen later on. Then, we split

$$\mathcal{Q}^+_{B,e}(f,f)=\mathcal{Q}^+_{B_1,e}(f,f)+\mathcal{Q}^+_{B_2,e}(f,f)$$

where $B_i(u,\sigma)=\Phi(|u|)b_i(\widehat{u}\cdot\sigma)$ $i=1,2$. We estimate the first term using Young's inequality as obtained in [3, Theorem 1] to get

$$\left\|\mathcal{Q}^+_{B_1,e}(f,f)\right\|_{L^\infty}\leqslant c_k\|\Phi\|_{L^\infty_{-k}}C(B_1)\|f\|^2_{L^2_k}$$

for some numerical constant $c_k>0$ and where the constant $C(B_1)$ is given by (see [3, Eq. (3.3)]:

$$C(B_1):=\left(\int_{-1}^1\left(\frac{1-s}{2}\right)^{-3/2}b_1(s)\,\mathrm{d}s\right)^{1/2}\left(\int_{-1}^1\left(\frac{1+s}{2}\right)^{-3/2}b_1(s)\,\mathrm{d}s\right)^{1/2}.$$

Notice that $C(B_1)<\infty$ since b_1 is supported on $[-1+\delta,0]$. Moreover $C(B_1)$ is larger than the constant obtained in [3, Eq. (3.3)] (since we used the fact that $(1-\beta(0))^2\frac{1-s}{2}\geqslant 0$ for any $s\in(-1,1)$) but is independent of the restitution coefficient $e(\cdot)$. For the second term we use Theorem 5.3 with $\eta=-k$ to get

$$\begin{aligned}\mathcal{Q}^+_{B_2,e}(f,f)(v)&\leqslant\left\|\mathcal{Q}^+_{B_2,e}(f,f)\right\|_{L^\infty_{-k}}\langle v\rangle^k\\&\leqslant\mathbf{C}_{-k,\infty,k}(B_2)\|f\|_{L^1_k}\|f\|_{L^\infty}\langle v\rangle^k\end{aligned}$$

where the constant $\mathbf{C}_{-k,\infty,k}(B_2)$ is given by (5.5) and (5.6). In particular, since

$$\gamma(-k,\infty,b_2)=\int_{-1}^{-1+\delta}\left(\frac{1-s}{2}\right)^{-\frac{3}{2}}b(s)\,\mathrm{d}s$$

and $b(\cdot)$ is integrable in $[-1,0]$ due to the Grad's cut-off assumption (2.6), one sees that the parameter δ can be chosen small enough to get $\mathbf{C}_{-k,\infty,k}(B_2) \leqslant \varepsilon$. This achieves the proof. □

5.3. **Sobolev regularity for smooth collision kernel.** Let Φ and $b(\cdot)$ be smooth and compactly supported:

$$\Phi \in \mathcal{C}_0^\infty(\mathbb{R}^3 \setminus \{0\}), \qquad b \in \mathcal{C}_0^\infty(-1,1) \tag{5.10}$$

and let us denote by $\mathcal{Q}_{B,e}$ the associated collision operator defined by (2.5). Then, one has the technical result:

Lemma 5.7. *Assume that $e(\cdot)$ satisfies Assumptions 2.1 with $e(\cdot) \in \mathcal{C}^m(0,\infty)$ for some integer $m \in \mathbb{N}$. Then, under assumption (5.10) on the collision kernel, for any $0 \leqslant s \leqslant m$, there exists $C = C(s,B,e)$ such that*

$$\|\Gamma_B(f)\|_{H^{s+1}} \leqslant C(s,B,e)\ \|f\|_{H^s}\,, \qquad \forall f \in H^s$$

where Γ_B is the operator defined in Lemma 5.1 and the constant $C(s,B,e)$ depends only on s, on the collision kernel B and on the restitution coefficient $e(\cdot)$. More precisely, $C(s,B,e)$ depends on $e(\cdot)$ through the L^∞ norm of the derivatives $D^k e(\cdot)$ ($k = 1,\dots,m$) over some compact interval I bounded away from zero (that depends only on B).

We postpone the proof of this lemma and first prove the following:

Theorem 5.8. *Let $B(u,\sigma) = \Phi(|u|)b(\widehat{u}\cdot\sigma)$ satisfy (5.10) and let $e(\cdot)$ satisfy Assumption 2.1 with $e(\cdot) \in \mathcal{C}^m(0,\infty)$ for some integer $m \in \mathbb{N}$. Then, for any $0 \leqslant s \leqslant m$, one has*

$$\left\|\mathcal{Q}^+_{B,e}(f,g)\right\|_{H^{s+1}} \leqslant C(s,B,e)\,\|g\|_{H^s}\,\|f\|_{L^1}$$

where the constant $C(s,B,e)$ is the one obtained in Lemma 5.7.

Proof. Let $\mathcal{F}\left[\mathcal{Q}^+_{B,e}(f,g)\right](\xi)$ denote the Fourier transform of $\mathcal{Q}^+_{B,e}(f,g)$. According to Corollary 5.2,

$$\mathcal{F}\left[\mathcal{Q}^+_{B,e}(f,g)\right](\xi) = \int_{\mathbb{R}^3} f(v)\mathcal{F}\left[(t_v\circ\Gamma_B\circ t_v)\,g\right](\xi)\,\mathrm{d}v.$$

To simplify notation, set $G(v,\xi) = \mathcal{F}\left[(t_v\circ\Gamma_B\circ t_v)\,g\right](\xi)$. One has

$$\begin{aligned}
\left\|\mathcal{Q}^+_{B,e}(f,g)\right\|^2_{H^{s+1}} &= \int_{\mathbb{R}^3} \left|\mathcal{F}\left[\mathcal{Q}^+_{B,e}(f,g)\right](\xi)\right|^2 \langle\xi\rangle^{2(s+1)}\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^3} \langle\xi\rangle^{2(s+1)} \left|\int_{\mathbb{R}^3} f(v)G(v,\xi)\,\mathrm{d}v\right|^2 \mathrm{d}\xi \\
&\leqslant \|f\|_{L^1}\int_{\mathbb{R}^3\times\mathbb{R}^3} |f(v)|\,|G(v,\xi)|^2\langle\xi\rangle^{2(s+1)}\,\mathrm{d}\xi\,\mathrm{d}v.
\end{aligned} \tag{5.11}$$

Since $G(v,\xi) = \mathcal{F}\left[(t_v\circ\Gamma_B\circ t_v)\,g\right](\xi)$,

$$\int_{\mathbb{R}^3} |G(v,\xi)|^2\langle\xi\rangle^{2(s+1)}\,\mathrm{d}\xi = \|(t_v\circ\Gamma_B\circ t_v)\,g\|^2_{H^{s+1}} \leqslant C(s,B,e)^2\ \|g\|^2_{H^s}$$

where we used Lemma 5.7 and the fact that, for any v, t_v is an operator of norm one in any Sobolev space. Then, estimate (5.11) yields the desired estimate. □

We come now to the proof of Lemma 5.7.

Proof of Lemma 5.7. The proof of the regularity property of Γ_B can be obtained along the same lines as the one for elastic Boltzmann operator [24] (or, for the inelastic case with constant restitution coefficient [21]). More precisely, denote by $\widetilde{\Gamma_B}(f)(r,\omega) = \Gamma_B(f)(\alpha^{-1}(r),\omega) = \Gamma_B(f)(r\beta(r),\omega)$, i.e.

$$\widetilde{\Gamma_B}(f)(r,\omega) = \int_{\omega^\perp} \mathcal{B}(z+r\omega,r)\varphi(r\omega+z)\,\mathrm{d}\pi_z.$$

One notices that, from (5.10), there is $\delta > 0$ such that $b(x) = 0$ for $|x \pm 1| \leqslant \delta$ while $\{|z|\,;\, z \in \mathrm{Supp}(\Phi)\} \subset (a, M)$ for some positive constants $M > a > 0$. Then, by virtue of (5.3), for any fixed $r > 0$, $\omega \in \mathbb{S}^2$ and $z \in \omega^\perp$, one sees that $\mathcal{B}(z + r\omega, r) = 0$ if $|z|^2 > \frac{2-\delta}{\delta} r^2$ while, for $|z|^2 \leqslant \frac{2-\delta}{\delta} r^2$, one has $|z + r\omega|^2 \leqslant 2r^2/\delta$ so that $\mathcal{B}(z + r\omega, r) = 0$ if $r < \sqrt{\delta a^2/2}$. This means that

$$\mathcal{B}(z+r\omega,r) = 0 \qquad \forall r \notin I = \left(\sqrt{\delta a^2/2}, M\right),\ \omega \in \mathbb{S}^2 \text{ and any } z \perp \omega. \tag{5.12}$$

In particular, $\widetilde{\Gamma_B}(f)(r,\omega) = 0$ for any $r \notin I$ independently of f. Let

$$\mathcal{B}_0(z,\varrho) = \frac{1+\vartheta_z(\varrho)}{\varrho}\beta^2(\varrho)\mathcal{B}(z,\varrho) = \frac{\Phi(|z|)b\left(1-2\frac{\varrho^2}{|z|^2}\right)}{|z|\varrho^2}$$

and let $\widetilde{\Gamma_0}(f)$ be the associated operator, i.e.

$$\widetilde{\Gamma_0}(f)(r,\omega) = \int_{\omega^\perp} \mathcal{B}_0(z+r\omega,r)\varphi(r\omega+z)\,\mathrm{d}\pi_z.$$

Then, $\mathcal{B}_0$ does not depend on the restitution coefficient $e(\cdot)$ and $\widetilde{\Gamma_0}$ is exactly of the form of the operator T studied in [24, Theorem 3.1]. Therefore, arguing as in *op. cit.*, for any $s \geqslant 0$, there is an explicit constant $C_0 = C_0(s,\Phi,b)$ such that

$$\left\|\widetilde{\Gamma_0}(f)\right\|_{H^{s+1}} \leqslant C_0(s,\Phi,b)\ \|f\|_{H^s}, \qquad \forall f \in H^s.$$

Setting

$$G_e(\varrho) = \frac{\varrho}{(1+\vartheta_z(\varrho))\,\beta^2(\varrho)}\chi_I(\varrho)$$

where χ_I is the characteristic function of $I = \left(\sqrt{\delta a^2/2}, M\right)$ (see Eq. (5.12)), one sees that G_e is a $\mathcal{C}^m$ function over I whose derivatives $D^k G_e$ are bounded over I for any $k \leqslant m$ and

$$\widetilde{\Gamma_B}(f)(r,\omega) = G_e(r)\widetilde{\Gamma_0}(f)(r,\omega).$$

Therefore, for any $0 \leqslant s \leqslant m$, there is some constant $C = C(s,b,e)$ such that

$$\left\|\widetilde{\Gamma_B}(f)\right\|_{H^{s+1}} \leqslant C(s,B,e)\ \|f\|_{H^s}, \qquad \forall f \in H^s$$

where the constant $C(s,B,e)$ only depends on C_0 and $\max_{k=0,\dots,m}\|D^k G_e\|_{L^\infty(I)}$. Let us now explain how to deduce Lemma 5.7 from the above estimate. Assume first $s = k$ is

an integer. It is easy to check that, using polar coordinates,

$$\|\Gamma_B(f)\|_{H^k} = \sum_{|j|\leqslant k} \int_0^\infty F_j(\varrho)\varrho^2 \,\mathrm{d}\varrho \int_{\mathbb{S}^2} |\partial_v^j \widetilde{\Gamma_B}(f)(\varrho,\omega)|^2 \,\mathrm{d}\omega$$

where, for any $|j| \leqslant k$, the function $F_j(\varrho)$ can be written as

$$F_j(\varrho) = P_j(\vartheta^{(1)}(\varrho), \dots, \vartheta^{(j)}(\varrho))(1+\vartheta^{(1)}(\varrho))^{-n_j}$$

where $P_j(y_1,\dots,y_j)$ is a suitable polynomial, $n_j \in \mathbb{N}$ and $\vartheta^{(p)}$ denotes the p-th derivative of $\vartheta(\cdot)$. It is not difficult to see that, since $\vartheta \in \mathcal{C}^m(0,\infty)$ and I is a compact interval bounded away from zero, one has $\sup_{\varrho\in I} F_j(\varrho) = C_k < \infty$ for any $|j| \leqslant k$. Thus

$$\|\Gamma_B(f)\|_{H^k} \leqslant C_k \|\widetilde{\Gamma_B}(f)\|_{H^k} \tag{5.13}$$

where C_k is an explicit constant involving the L^∞ norm of the first k-th order derivatives of $\alpha(\cdot)$ on I. This proves that the conclusion of the Lemma 5.7 holds true for any integer $s \leqslant m$ and we deduce the general case by simple interpolation. □

Arguing exactly as in [24, Corollary 3.2] we translate the gain of regularity obtained in Theorem 5.8 into the following gain of integrability:

Corollary 5.9. *Let $B(u,\sigma) = \Phi(|u|)b(\widehat{u}\cdot\sigma)$ satisfy (5.10) and let $e(\cdot) \in \mathcal{C}^1(0,\infty)$ satisfy Assumption 2.1. Then, for any $1 < p < \infty$*

$$\left\|\mathcal{Q}^+_{B,e}(f,g)\right\|_{L^p} \leqslant C(p,B,e)\left(\|g\|_{L^q}\|f\|_{L^1} + \|g\|_{L^1}\|f\|_{L^q}\right)$$

where the constant $C(p,B,e)$ depends on $e(\cdot)$ and B through the constant $C(1,B,e)$ of Theorem 5.8 while $q < p$ is given by:

$$q = \begin{cases} \dfrac{5p}{3+2p} & \text{if} \quad p \in (1,6] \\ p/3 & \text{if} \quad p \in [6,\infty). \end{cases} \tag{5.14}$$

5.4. **Regularity and integrability for hard-spheres.** We consider in this section the case of a hard-spheres collision kernel

$$B(u,\sigma) = \frac{|u|}{4\pi}.$$

Of course, such a collision kernel does not enjoy the regularity properties assumed in the previous section. But, since the constant $\mathbf{C}_{\eta,p,q,k}(B)$ in Theorem 5.3 depends on $\|\Phi\|_{L^\infty_{-k}}$ and some (weighted) $L^1(\mathbb{S}^2)$-norm of $b(\cdot)$, we can adapt easily the method of [21] (see also [24]) which consists in splitting the collision kernel into a smooth part and a remainder part to get the following estimate:

Theorem 5.10. *Assume that $e(\cdot) \in \mathcal{C}^1(0,\infty)$ and satisfies Assumptions 2.1. For any $p \in [1,\infty)$ and $\delta > 0$, there exists a constant $C_\delta > 0$ depending only on $\delta > 0$ and the restitution coefficient $e(\cdot)$ such that*

$$\int_{\mathbb{R}^3} \mathcal{Q}^+_e(f,f)\, f^{p-1}\,\mathrm{d}v \leqslant C_\delta \|f\|_{L^1}^{1+p\theta}\|f\|_{L^p}^{p(1-\theta)} + \delta\,\|f\|_{L^1_2}\|f\|^p_{L^p_{1/p}}.$$

for some constant $\theta \in (0,1)$ depending only on p. Moreover, the constant C_δ depends on the restitution coefficient through the L^∞ norm of the derivatives $D^k e(\cdot)$ ($k=0,1$) over some compact interval of $(0,\infty)$ bounded away from zero (that depends only on B and δ).

Proof. The proof of this result follows exactly the same lines as the corresponding one in [21]. We recall it for the reader convenience. It is based on a suitable splitting of the collision operator $\mathcal{Q}_e^+$. Let $p \geqslant 1$ be fixed. Consider a smooth collision kernel of the form

$$B_{S_{m,n}}(|u|, \widehat{u}\cdot\sigma) = \Phi_{S_n}(|u|)\, b_{S_m}(\widehat{u}\cdot\sigma),$$

with Φ_{S_n} smooth and with compact support $\left[\frac{2}{n}, n\right]$, and b_{S_m} smooth and supported in $\left[-1+\frac{2}{m}, 1-\frac{2}{m}\right]$. We assume that

$$\lim_{m\to\infty} \|b_{S_m} - 1\|_{L^1(\mathbb{S}^2)} = 0 \quad \text{and} \quad \lim_{n\to\infty} \Phi_{S_n}(|u|) = |u| \qquad \forall u \in \mathbb{R}^3.$$

Set then

$$B_{SR_{m,n}}(|u|, \widehat{u}\cdot\sigma) = \Phi_{S_n}(|u|)\, b_{R_m}(\widehat{u}\cdot\sigma),$$

and

$$B_{RS_{m,n}}(|u|, \widehat{u}\cdot\sigma) = \Phi_{R_n}(|u|)\, b_{S_m}(\widehat{u}\cdot\sigma), \qquad B_{RR_{m,n}}(|u|, \widehat{u}\cdot\sigma) = \Phi_{R_n}(|u|)\, b_{R_m}(\widehat{u}\cdot\sigma)$$

where $\Phi_{R_n}(|u|) = |u| - \Phi_{S_n}(|u|)$ while $b_{R_m}(\widehat{u}\cdot\sigma) = 1 - b_{S_m}(\widehat{u}\cdot\sigma)$. With this in hands, one splits $\mathcal{Q}_e^+$ as

$$\mathcal{Q}_e^+ = \mathcal{Q}^+_{B_{S_{m,n}},e} + \mathcal{Q}^+_{B_{SR_{m,n}},e} + \mathcal{Q}^+_{B_{RS_{m,n}},e} + \mathcal{Q}^+_{B_{RR_{m,n}},e}$$

with obvious notations. Let m and n be fixed. Since $B_{S_{m,n}}(|u|, \widehat{u}\cdot\sigma)$ fulfils (5.10), one deduces from Corollary 5.9 that there is a constant $C(m,n)$ such that

$$\left\| \mathcal{Q}^+_{B_{S_{m,n}},e}(f,f) \right\|_{L^p} \leqslant C(m,n) \|f\|_{L^q}\, \|f\|_{L^1}$$

for some $q < p$ given by (5.14) which, by a simple application of Hölder's inequality, yields as in [21]:

$$\int_{\mathbb{R}^3} \mathcal{Q}^+_{B_{S_{m,n}},e}(f,f)\, f^{p-1}\, \mathrm{d}v \leqslant C(m,n)\, \|f\|_{L^q}\, \|f\|_{L^1}\, \|f\|_{L^p}^{p-1}. \tag{5.15}$$

Now, applying Theorem 5.3 with $k=1$

$$\left\| \mathcal{Q}^+_{B_{SR_{m,n}},e}(f,f) \right\|_{L^p_\eta} + \left\| \mathcal{Q}^+_{B_{RR_{m,n}},e}(f,f) \right\|_{L^p_\eta} \leqslant \varepsilon(m) \|f\|_{L^1_{|\eta+1|+|\eta|}}\, \|f\|_{L^p_{\eta+1}}$$

where $\varepsilon(m) \to 0$ as $m \to \infty$ since the constant in (5.5) depends continuously (through (5.6)) on the $L^1(\mathbb{S}^2)$-norm of b_{R_m} which is arbitrarily small as m grows. Using the above estimate with $\eta = -1/p'$, we get

$$\int_{\mathbb{R}^3} \left[\mathcal{Q}^+_{B_{SR_{m,n}},e}(f,f) + \mathcal{Q}^+_{B_{RR_{m,n}},e}(f,f) \right] f^{p-1}\, \mathrm{d}v \leqslant \varepsilon(m) \|f\|_{L^1_1}\, \|f\|^p_{L^p_{1/p}}. \tag{5.16}$$

It remains only to estimate $I := \displaystyle\int_{\mathbb{R}^3} \mathcal{Q}^+_{B_{RS_{m,n}},e}(f,f)\, f^{p-1}\, \mathrm{d}v$. As in [21], one notes that

$$\Phi_{R_n}(|v - v_\star|) \leqslant C n^{-1} \left(|v|^2 + |v_\star|^2 \right), \qquad \forall v, v_\star \in \mathbb{R}^3$$

where $C > 0$. Thus

$$I \leqslant Cn^{-1} \int_{\mathbb{R}^3\times\mathbb{R}^3} f(v)f(v_\star)\left(|v|^2+|v_\star|^2\right)\,\mathrm{d}v\,\mathrm{d}v_\star \int_{\mathbb{S}^2} f^{p-1}(v')b_{S_m}(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma.$$

Set

$$I_1 = \int_{\mathbb{R}^3\times\mathbb{R}^3} f(v)f(v_\star)|v|^2\,\mathrm{d}v\,\mathrm{d}v_\star \int_{\mathbb{S}^2} f^{p-1}(v')b_{S_m}(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma$$

and

$$I_2 = \int_{\mathbb{R}^3\times\mathbb{R}^3} f(v)f(v_\star)|v_\star|^2\,\mathrm{d}v\,\mathrm{d}v_\star \int_{\mathbb{S}^2} f^{p-1}(v')b_{S_m}(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma.$$

One sees that I_1 can be written as

$$I_1 = \int_{\mathbb{R}^3\times\mathbb{R}^3} \mathcal{Q}^+_{B_m,e}(F,f)(v)\psi(v)\,\mathrm{d}v$$

where

$$F(v) = |v|^2 f(v), \qquad \psi(v) = f^{p-1}(v) \in L^{p'}(\mathbb{R}^3)$$

and the collision kernel $B_m(|u|,\widehat{u}\cdot\sigma) = b_{S_m}(\widehat{u}\cdot\sigma)$. Applying Theorem 5.3 with $\eta = k = 0$ gives then

$$\begin{aligned} I_1 &\leqslant \left\|\mathcal{Q}^+_{B_m,e}(F,f)\right\|_{L^p}\,\|\psi\|_{L^{p'}} \leqslant \mathbf{C}_{0,p,0}(B_m)\,\|F\|_{L^1}\,\|f\|_{L^p}\,\|\psi\|_{L^{p'}} \\ &\leqslant \mathbf{C}_{0,p,0}(B_m)\,\|f\|_{L^1_2}\,\|f\|^p_{L^p} \end{aligned}$$

where $\mathbf{C}_{0,p,0}(B_m)$ is defined by (5.5). Now, with the same notations, we see that

$$I_2 = \int_{\mathbb{R}^3\times\mathbb{R}^3} \mathcal{Q}^+_{B_m,e}(f,F)(v)\psi(v)\,\mathrm{d}v$$

so that, applying now Theorem 5.3 with $\eta = 0$ and $k = -2$ (notice that $\Phi = 1 \in L^\infty_2$) yields

$$I_2 \leqslant \mathbf{C}_{0,p,-2}(B_m)\,\|f\|_{L^1_2}\,\|F\|_{L^p_{-2}}\,\|\psi\|_{L^{p'}} \leqslant \mathbf{C}_{0,p,-2}(B_m)\,\|f\|_{L^1_2}\,\|f\|^p_{L^p}.$$

Combining the two estimates for I_1 and I_2, there exists some constant $C = C(m)$ such that

$$I \leqslant \frac{C(m)}{n}\|f\|_{L^1_2}\,\|f\|^p_{L^p}.$$

This, with (5.15) and (5.16), gives

$$\begin{aligned} \int_{\mathbb{R}^3} \mathcal{Q}^+_e(f,f)\,f^{p-1}\,\mathrm{d}v \leqslant C(m,n)\,\|f\|_{L^q}\,\|f\|_{L^1}\,\|f\|^{p-1}_{L^p} + \\ + \varepsilon(m)\|f\|_{L^1_1}\,\|f\|^p_{L^p_{1/p}} + \frac{C(m)}{n}\|f\|_{L^1_2}\,\|f\|^p_{L^p} \end{aligned}$$

which, arguing as in [21], leads to the result choosing first m big enough then n big enough. □

Corollary 5.11. *Assume that $e(\cdot) \in \mathcal{C}^1(0,\infty)$ and satisfies Assumptions 2.1. For any $p \in [1,\infty)$ and $\delta > 0$, there exists a constant $C_\delta > 0$ depending only on $\delta > 0$ and the restitution coefficient $e(\cdot)$ such that*

$$\int_{\mathbb{R}^3} \mathcal{Q}_e^+(g,g)\, g^{p-1}\langle v\rangle^{\eta p}\,\mathrm{d}v \leqslant C_\delta \left\|g\right\|_{L^1_\eta}^{1+p\theta} \left\|g\right\|_{L^p_\eta}^{p(1-\theta)} + \delta \left\|g\right\|_{L^1_{2+\eta}} \left\|g\right\|_{L^p_{\eta+1/p}}^{p}, \qquad \forall \eta \geqslant 0$$

for some constant $\theta \in (0,1)$ depending only on p.

Proof. Let a nonnegative function g be fixed. For $\eta \geqslant 0$, set $f(v) = g(v)\langle v\rangle^\eta$. Noticing that $\langle v'\rangle^\eta \leqslant \langle v\rangle^\eta \langle v_\star\rangle^\eta$ for any $v, v_\star \in \mathbb{R}^3$, one checks easily using the weak formulation of $\mathcal{Q}_e^+$ that

$$\int_{\mathbb{R}^3} \mathcal{Q}_e^+(g,g)\, g^{p-1}\langle v\rangle^{\eta p}\,\mathrm{d}v = \int_{\mathbb{R}^3} \langle v\rangle^\eta \mathcal{Q}_e^+(g,g)\, f^{p-1}\,\mathrm{d}v \leqslant \int_{\mathbb{R}^3} \mathcal{Q}_e^+(f,f) f^{p-1}\,\mathrm{d}v$$

and we conclude with Theorem 5.10. □

Remark 5.12. *Notice that, for any $\delta > 0$, the above constant C_δ is exactly the one provided by Theorem 5.10.*

The (almost) explicit dependence of the above constant C_δ with respect to the restitution coefficient has some very important consequences in the study of the (time-dependent) collision operator $\mathcal{Q}_{\widetilde{e}_\tau}$ in self-similar variable. In particular, we can prove that, for such an operator, the corresponding constant $C_\delta = C_\delta(\tau)$ remains uniformly bounded with respect to $\tau > 0$. Precisely, one can state the following:

Corollary 5.13. *Assume that $e(\cdot) \in \mathcal{C}^1(0,\infty)$ and satisfies Assumptions 2.1. For any $\tau \geqslant 0$, let $\widetilde{e}_\tau$ be the restitution coefficient defined by (3.10) and let $\mathcal{Q}_{\widetilde{e}_\tau}(f,f)$ be the associated collision operator. For any $p \in [1,\infty)$ and $\delta > 0$, there exists a constant K_δ that does not depend on τ such that*

$$\int_{\mathbb{R}^3} \mathcal{Q}_{\widetilde{e}_\tau}(g,g)\, g^{p-1}\langle w\rangle^{\eta p}\,\mathrm{d}w \leqslant K_\delta \left\|g\right\|_{L^1_\eta}^{1+p\theta} \left\|g\right\|_{L^p_\eta}^{p(1-\theta)} + \delta \left\|g\right\|_{L^1_{2+\eta}} \left\|g\right\|_{L^p_{\eta+1/p}}^{p}, \qquad \forall \eta \geqslant 0$$

for some constant $\theta \in (0,1)$ depending only on p.

Proof. Clearly, from Corollary 5.11, for any $\tau \geqslant 0$, there exists $C_\delta(\tau)$ for which the above inequality holds and it suffices to prove that $K_\delta = \sup_{\tau\geqslant 0} C_\delta(\tau) < \infty$. Recall that, $C_\delta(\tau)$ depends on τ through the restitution coefficient $\widetilde{e}_\tau$ and, more precisely, $C_\delta(\tau)$ depends on the L^∞ norm of the derivatives $D^k\widetilde{e}_\tau(\cdot)$ ($k = 0,1$) over some compact interval of $(0,\infty)$ bounded away from zero (independent of τ). Now, for any $\tau \geqslant 0$, recall (see Section 4) that $\widetilde{e}_\tau(\cdot) = e\left(\frac{\cdot}{\lambda(\tau)}\right)$ where $\lambda(\tau) = \left(1 + \frac{\gamma}{\gamma+1}\tau\right)^{1/\gamma}$. Consequently, $D^k\widetilde{e}_t(\cdot) = \lambda^{-k}(\tau)(D^k e)\left(\frac{\cdot}{\lambda(\tau)}\right)$. Since $\lambda^{-1}(\tau)$ is continuous and goes to zero as τ goes to ∞, one sees that all the L^∞ norms of $D^k\widetilde{e}_t(\cdot)$ remain uniformly bounded with respect to τ. The same holds for $C_\delta(\tau)$ and the proof is achieved. □

5.5. **Pointwise exponential estimates of $\mathcal{Q}^+_{B,e}$.** We provide here some useful pointwise estimates of the gain part $\mathcal{Q}^+_{B,e}$ with exponential weights. Our results take benefit of the very recent *Young's inequality with exponential weights* obtained by R. ALONSO, E. CARNEIRO AND I. M. GAMBA [3]. These results are inspired by similar estimates for Maxwellians weights obtained for the elastic Boltzmann equation in [15] and apply to general collision kernels $B(u,\sigma)=\Phi(|u|)b(\widehat{u}\cdot\sigma)$ where the angular kernel $b(\cdot)$ and the variable restitution coefficient $e(\cdot)$ satisfy the following:

Assumptions 5.14. *In addition to Assumptions 2.1, the restitution coefficient $e(\cdot)$ is assumed to be non-increasing over $\mathbb{R}$ with $e(z)<1$ for $z>0$. Moreover, the collision kernel $B(u,\sigma)=\Phi(|u|)b(\hat{u}\cdot\sigma)$ where $\Phi(\cdot)\in L^\infty_{-k}$ for some $k\geqslant 0$ and the angular kernel satisfies the integral cut-off*

$$\ell_b:=\int_{-1}^1\left[\frac{1+s}{2}+(1-\beta(0))^2\frac{1-s}{2}\right]^{-3/2}b(s)\,\mathrm{d}s<\infty. \tag{5.17}$$

Remark 5.15. *It is clear that (5.17) implies in particular that $b(\cdot)$ satisfies Grad's cut-off assumption: $b(\cdot)\in L^1(\mathbb{S}^2)$. Note, however, that when $\beta(0)<1$, for instance in models with constant restitution coefficient, assumption (5.17) reduces to Grad's cut-off.*

For this kind of weights we have the following result taken from [3]:

Theorem 5.16. *Let $1\leqslant p,\,q,\,r\leqslant\infty$ with $1/p+1/q=1+1/r$. Assume that $B(u,\sigma)=\Phi(|u|)b(\hat{u}\cdot\sigma)$ and the restitution coefficient $e(\cdot)$ satisfy Assumptions 5.14. For any fixed $a>0$, define the exponential weight as*

$$\mathcal{M}_\lambda(v):=e^{-a|v|^\lambda}\qquad\lambda\geqslant 0,\;\;v\in\mathbb{R}^3.$$

Then, there is some constant $C>0$ such that

$$\left\|\mathcal{Q}^+_{B,e}(f,g)\,\mathcal{M}_\lambda^{-1}\right\|_{L^r(\mathbb{R}^3)}\leqslant C\,\|f\,\mathcal{M}_\lambda^{-1}\|_{L^p(\mathbb{R}^3)}\,\|g\,\mathcal{M}_\lambda^{-1}\|_{L^q(\mathbb{R}^3)}. \tag{5.18}$$

In the important case $(p,q,r)=(\infty,1,\infty)$ the (non-sharp) constant $C:=C(b,\beta)$ can be taken as $C=\kappa\ell_{b_0}$ for some $\kappa>0$ and ℓ_{b_0} defined in (5.17) with

$$b_0(s):=\left[1-\left(\frac{1+|\theta(s)|}{2}\right)^k\right]^{-1}b(s),$$

where

$$|\theta(s)|^2=(1-\beta(x))^2+\beta^2(x)+2\beta(x)(1-\beta(x))s\quad\text{with}\quad x=\sqrt{\tfrac{1-s}{2}}.$$

With this result at hand we have the following Proposition.

Proposition 5.17. *Let the collision kernel $B(\cdot,\cdot)$ and the restitution coefficient $e(\cdot)$ satisfy Assumptions 5.14 with moreover $e(0)=1$. Then for any $\varepsilon>0$, there exists $C(e,\varepsilon,\ell_b)>0$ such that*

$$\begin{aligned}\mathcal{Q}^+_{B,e}(f,g)(v)\,\mathcal{M}_\lambda^{-1}(v)\leqslant C(e,\varepsilon,\ell_b)\,\|f\,\mathcal{M}_\lambda^{-1}\|_{L^\infty(\mathbb{R}^3)}\,\|g\,\mathcal{M}_\lambda^{-1}\|_{L^1(\mathbb{R}^3)}\\ +\varepsilon\|f\,\mathcal{M}_\lambda^{-1}\|_{L^\infty(\mathbb{R}^3)}\,\|g\,\mathcal{M}_\lambda^{-1}\|_{L^1_k(\mathbb{R}^3)}\,\langle v\rangle^k\end{aligned} \tag{5.19}$$

holds for any $v\in\mathbb{R}^3$ and any nonnegative f,g.

Proof. Let f and g be two nonnegative distribution functions. As in the proof of Corollary 5.6 we break the angular kernel into two parts[3]:

$$b(s) = b(s)\chi_{[-1,1-\delta]} + b(s)\chi_{(1-\delta,1]} =: b_1(s) + b_2(s)$$

and let B_1 and B_2 denote the associated collision kernel: $B_i(u,\sigma) = \Phi(|u|)b_i(\hat{u}\cdot\sigma)$, $i = 1, 2$. Let us first estimate the collision operator associated to B_2. Using the dissipation of energy, one notices that

$$|v'|^\lambda = \left(|v'|^2\right)^{\lambda/2} \leqslant \left(|v|^2 + |v_\star|^2\right)^{\lambda/2} \leqslant |v|^\lambda + |v_\star|^\lambda,$$

so that

$$\mathcal{M}_\lambda^{-1}(v') \leqslant \mathcal{M}_\lambda^{-1}(v)\mathcal{M}_\lambda^{-1}(v_\star) \quad \forall v,\, v_\star \in \mathbb{R}^3.$$

Then, using the weak formulation of $\mathcal{Q}^+_{B_2,e}$ we conclude that

$$\begin{aligned}\mathcal{Q}^+_{B_2,e}(f,g)(v)\,\mathcal{M}_\lambda^{-1}(v) &\leqslant \mathcal{Q}^+_{B_2,e}(f\mathcal{M}_\lambda^{-1}, g\mathcal{M}_\lambda^{-1})(v)\\ &\leqslant \left\|\mathcal{Q}^+_{B_2,e}(f\mathcal{M}_\lambda^{-1}, g\mathcal{M}_\lambda^{-1})\right\|_{L^\infty_{-k}} \langle v\rangle^k .\end{aligned}$$

Therefore, according to Theorem 5.3,

$$\mathcal{Q}^+_{B_2,e}(f,g)(v)\,\mathcal{M}_\lambda^{-1}(v) \leqslant \widetilde{\mathbf{C}}_{-k,\infty,k}(B_2)\|f\,\mathcal{M}_\lambda^{-1}\|_{L^\infty(\mathbb{R}^3)}\,\|g\,\mathcal{M}_\lambda^{-1}\|_{L^1_k(\mathbb{R}^3)}\,\langle v\rangle^k$$

where the constant $\widetilde{\mathbf{C}}_{-k,\infty,k}(B_2)$ is given by (5.7). According to (5.8) and since $\beta(0) = 1$ and $b_2(\cdot)$ is integrable, for any fixed $\varepsilon > 0$, we can choose $\delta > 0$ small enough such that $\widetilde{\mathbf{C}}_{-k,\infty,k}(B_2) \leqslant \varepsilon$, i.e.

$$\mathcal{Q}^+_{B_2,e}(f,g)(v)\,\mathcal{M}_\lambda^{-1}(v) \leqslant \varepsilon\|f\,\mathcal{M}_\lambda^{-1}\|_{L^\infty(\mathbb{R}^3)}\,\|g\,\mathcal{M}_\lambda^{-1}\|_{L^1_k(\mathbb{R}^3)}\,\langle v\rangle^k .$$

Concerning the collision operator associated to B_1, using Theorem 5.16 with $(p,q,r) = (\infty,1,\infty)$ leads to the estimate

$$\mathcal{Q}^+_{B_1,e}(f,g)(v)\,\mathcal{M}_\lambda^{-1}(v) \leqslant C_e\|f\,\mathcal{M}_\lambda^{-1}\|_{L^\infty(\mathbb{R}^3)}\,\|g\,\mathcal{M}_\lambda^{-1}\|_{L^1(\mathbb{R}^3)}$$

for some constant $C_e > 0$ depending only on $b_1(\cdot)$, the restitution coefficient $e(\cdot)$ and ℓ_b. This concludes the proof. □

Remark 5.18. *Note that the integrand defining the constant C_e (see Theorem 5.16) has a singularity only at $s = 1$ because $|\theta(1)| = 1$ (when $\beta(0) = 1$). This singularity is avoided by the splitting technique used throughout this paper. On the contrary, the singularity at $s = -1$, when occurs, can not be avoided. This problem arises in the study of inelastic interactions because of the lack of symmetry that they induce, e.g. for the bilinear form of the inelastic collision operator, it is not correct to assume that b can be defined in half the domain. This is the main reason why we have to add the stronger (and undesired) integral cut-off hypothesis* (5.17).

[3]notice however that, since we are dealing with the bilinear operator $\mathcal{Q}_{B,e}(f,g)$ and not the symmetric $\mathcal{Q}_{B,e}(f,f)$, we cannot assume $b(\cdot)$ to be supported on $[-1,0]$ (see also Remark 5.18).

6. Propagation of Lebesgue norms: L^p-Theory $(1 < p < \infty)$.

In this section we are interested in proving the propagation of L^p-norms, with $1 \leqslant p < \infty$, for the self-similar profile and for the solution to the Boltzmann equation (2.7). We will use the main result of this section, and a couple of observations to prove the propagation of the L^∞-norm later in Section 7. Our strategy is based in the standard technique of energy estimates which will be carried out in the space of self-similar variables for optimality. As usual, we will assume that the restitution coefficient satisfies Assumptions 3.1 and, given a solution $f(t,v)$ to (2.7), the rescaled solution $g = g(\tau, w)$ is a solution to the Cauchy problem (3.9).

We adapt here the technique introduced in [24] for integrability propagation in the elastic case. Later in [21] such techniques have been applied to prove the propagation of L^p norms for inelastic interactions in the case of constant restitution coefficient. We extend their results to the case of a variable restitution coefficient satisfying Assumptions 3.1. One begins with the following Lemma which relies on the fact that the energy of the self-similar variable g is bounded from below:

Lemma 6.1. *Assume that $f_0 \in L^1_\eta$ for some $\eta > 2$. Then, there exists some positive constant $\nu_0 > 0$ such that*

$$\int_{\mathbb{R}^n} g(\tau, w_\star)|w - w_\star| \, \mathrm{d}w_\star \geqslant \max\{\nu_0, |w|\} \geqslant \frac{\nu_0}{2}\langle w\rangle, \qquad \forall w \in \mathbb{R}^3, \quad \tau > 0.$$

In particular,

$$\int_{\mathbb{R}^3} g^{p-1}\mathcal{Q}_e^-(g,g)\,\mathrm{d}w \geqslant \frac{\nu_0}{2}\int_{\mathbb{R}^3} g^p(\tau,w)(1+|w|^2)^{1/2}\,\mathrm{d}w = \frac{\nu_0}{2}\left\|g(\tau)\right\|^p_{L^p_{1/p}}.$$

Proof. Since $f_0 = g_0$, the propagation of p-moments in the rescaled profile g implies $\sup_{t\geqslant 0}\|g(\tau)\|_{L^1_\eta} < \infty$. Then, for any $R > 0$,

$$\int_{\{|w|\geqslant R\}} g(\tau,w)|w|^2\,\mathrm{d}w \leqslant \frac{1}{R^{\eta-2}}\sup_{\{\tau\geqslant 0\}}\|g(\tau)\|_{L^1_\eta}.$$

Recall that in Section 3 we proved that the energy $\mathbf{\Theta}(\tau)$ of g has a uniform lower bound $\mathbf{\Theta}_{\min} > 0$, therefore for sufficiently large R

$$\begin{aligned}\int_{\{|w|\leqslant R\}} g(\tau,w)|w|^2\,\mathrm{d}w &= \int_{\mathbb{R}^3} g(\tau,w)|w|^2\,\mathrm{d}w - \int_{\{|w|\geqslant R\}} g(\tau,w)|w|^2\,\mathrm{d}w\\ &\geqslant \mathbf{\Theta}_{\min} - \frac{1}{R^{\eta-2}}\sup_{\{\tau\geqslant 0\}}\|g(\tau)\|_{L^1_\eta} \geqslant \mathbf{\Theta}_{\min}/2 > 0.\end{aligned}$$

We conclude that,

$$\int_{\mathbb{R}^3} g(\tau,w)|w|\,\mathrm{d}w \geqslant \frac{1}{R}\int_{\{|w|\leqslant R\}} g(\tau,w)|w|^2\,\mathrm{d}w \geqslant \frac{\mathbf{\Theta}_{\min}}{2R} =: \nu_0 > 0.$$

Thus, using this observation and Jensen's inequality we get the conclusion. □

Then, one has the following

Theorem 6.2. *Assume the variable restitution coefficient $e(\cdot)$ satisfy Assumptions 3.1 for some* positive $\gamma > 0$. *Assume that $f_0 \in L^1_{2(1+\eta)} \cap L^p_\eta(\mathbb{R}^3)$ for some $1 \leqslant p < \infty$ and $\eta \geqslant 0$. Then the rescaled function g defined by (2.18) is such that*

$$\sup_{\tau \geqslant 0} \|g(\tau)\|_{L^p_\eta} < \infty.$$

As a consequence,

$$\sup_{t \geqslant 0} \left\{ V(t)^{3/p'} \|f(t)\|_{L^p} \right\} \leqslant \sup_{\tau \geqslant 0} \|g(\tau)\|_{L^p} < \infty$$

where $1/p + 1/p' = 1$.

Proof. Let $\eta \geqslant 0$ be fixed. Recall that (see Eq. (3.9)) the self-similar function $g(\tau, w)$ satisfies

$$\left(\partial_\tau g + \xi(\tau) \nabla_w \cdot (wg)\right)(\tau, w) = \mathcal{Q}_{\widetilde{e}_\tau}(g, g)(\tau, w), \qquad g(0, w) = f_0(w),$$

where $\xi(\tau)$ and $\widetilde{e}_\tau$ are defined in (3.10). Multiplying this equation by $g^{p-1}(\tau, w) \langle w \rangle^{\eta p}$ and integrating over $\mathbb{R}^3$ yields, after a few algebra,

$$\frac{1}{p} \frac{\mathrm{d} \|g(\tau)\|^p_{L^p_\eta}}{\mathrm{d}\tau} + 3 \left(1 - \frac{1}{p}\right) \xi(\tau) \|g\|^p_{L^p_\eta} = \int_{\mathbb{R}^3} \mathcal{Q}^+_{\widetilde{e}_\tau}(g, g) g^{p-1} \langle w \rangle^{\eta p} \, \mathrm{d}w -$$
$$\int_{\mathbb{R}^3} \mathcal{Q}^-(g, g) g^{p-1} \langle w \rangle^{\eta p} \, \mathrm{d}w + \eta \xi(\tau) \int_{\mathbb{R}^3} g^p(\tau, w) |w|^2 \langle w \rangle^{\eta p - 2} \, \mathrm{d}w.$$

Using Lemma 6.1 one has clearly

$$\int_{\mathbb{R}^3} \mathcal{Q}^-(g, g) g^{p-1} \langle w \rangle^{\eta p} \, \mathrm{d}w \geqslant \frac{\nu_0}{2} \|g(\tau)\|^p_{L^p_{\eta + 1/p}}.$$

Moreover, $C = \sup_{\tau \geqslant 0} \|g(\tau)\|_{L^1_{2+\eta}} > \infty$ by virtue of the propagation of moments (see (4.1)). Applying Corollary 5.13 with $\delta = \frac{\nu_0}{4C}$, one has

$$\frac{1}{p} \frac{\mathrm{d}}{\mathrm{d}\tau} \|g(\tau)\|^p_{L^p_\eta} + \frac{\nu_0}{4} \|g(\tau)\|^p_{L^p_{\eta + 1/p}} \leqslant K \|g(\tau)\|^{p(1-\theta)}_{L^p_\eta} + \xi(\tau) \left(\eta - \frac{3}{p'}\right) \|g(\tau)\|^p_{L^p_\eta} \quad \forall \tau > 0 \tag{6.1}$$

for some constant K independent of τ. Since $\gamma > 0$, the mapping $\xi(\tau)$ is nonincreasing with $\lim_{\tau \to \infty} \xi(\tau) = 0$ and it is not difficult to prove that (6.1) leads to the result. □

Remark 6.3. *Notice that, arguing as in* [21, Section 3.4], *the uniform L^p-norm (for $p > 1$) of the rescaled solution $g(\tau, w)$ provided by the above Theorem implies non-concentration of the rescaled temperature*

$$\mathbf{\Theta}(\tau) = \int_{\mathbb{R}^3} g(\tau, w) \, |w|^2 \, \mathrm{d}w.$$

Precisely, a simple use of Hölder's inequality shows that

$$\sup_{t \geqslant 0} \|g(\tau(t))\|_{L^p} \leqslant C_p \Longrightarrow \inf_{t \geqslant 0} \mathbf{\Theta}(\tau(t)) > \mathbf{\Theta}_{min} > 0.$$

Turning back to the original variable f, one gets that $\mathcal{E}(t) \geqslant \mathbf{\Theta}_{min} / V^2(t)$.

Remark 6.4. *If $e(\cdot)$ satisfies Assumptions 3.1 with $\gamma = 0$ (typically, if $e(\cdot)$ is a constant restitution coefficient), then $\xi(\tau) = 1$ for any $\tau > 0$ and the last part of the proof does not apply. However, one sees from (6.1) that the conclusion still holds if $\eta \leqslant 3/p'$. Moreover, for constant restitution coefficient, a version of Theorem 6.2 is given in* [21, Theorem 1.3].

7. Pointwise estimates: L^∞-Theory

In this last section we are interested in developing the L^∞-theory for solutions to Boltzmann equation (1.3). In early 80's, Arkeryd proved, inspired by the work of Carleman, that any solution to the elastic Boltzmann equation remain bounded as long as its initial datum is bounded [4]. His argument is rather clever and consists in proving a series of results that imply the L^∞ control of the gain part of the collision operator by the conserved quantities, the entropy, and the L^∞ norm of the initial datum. The Carleman representation plays a crucial role in his proof. We extend here his results to the inelastic case (with variable restitution coefficient). Of course, no entropy control is available here and we proceed in a different way, dealing with the self-similarity variable (see (2.18)).

7.1. L^∞-norm propagation.

The reader should notice that the results here are not a direct consequence of the L^p-theory, since the estimates involving L^p-norms for $p < \infty$ degenerate as $p \to \infty$. Unfortunately, this is an intrinsic feature of the methodology developed for such theory. The next two Lemmas overcome this problem.

Lemma 7.1. *Assume that the mapping $\tau \in [0,\infty) \longmapsto X(\tau, w)$ is absolutely continuous for almost every $w \in \mathbb{R}^3$ with $\|X(\tau)\|_{L^\infty} < \infty$ for any $\tau \in [0,\infty)$. If there exist positive constants $a, b > 0$ such that*

$$\frac{\mathrm{d}X}{\mathrm{d}\tau}(\tau, w) + aX(\tau, w)\langle w\rangle \leqslant b + \frac{a}{4}\|X(\tau)\|_{L^\infty}\langle w\rangle \quad \textit{for a.e. } w \in \mathbb{R}^3, \tag{7.1}$$

then,

$$\sup_{\tau\geqslant 0}\|X(\tau)\|_{L^\infty} \leqslant \max\left\{\|X_0\|_{L^\infty}, \frac{2b}{a}\right\}.$$

Proof. Let $C := \max\left\{\|X(0)\|_{L^\infty}, \frac{2b}{a}\right\}$. Assume that $\|X(\tau_0)\|_{L^\infty} > C$ for some $\tau_0 > 0$. There is no loss of generality in assuming that $\|X(\tau)\|_{L^\infty} \leqslant C$ for any $\tau \in (0, \tau_0)$. Then, there exists w_0 such that

$$X(\tau_0, w_0) > C_1 := \max\{\|X(\tau_0)\|_{L^\infty}/2, C\}, \qquad X(0, w_0) \leqslant C$$

and the mapping $\tau \longmapsto X(\tau, w_0)$ is absolutely continuous. Thus, there exists $\tau_\star \in [0, \tau_0)$ such that $X(\tau, w_0) \geqslant C_1$ for any $\tau \in [\tau_\star, \tau_0)$ and $X(\tau_\star, w_0) = C_1$. Then, since

$$\|X(\tau)\|_{L^\infty} \leqslant C < \|X(\tau_0)\|_{L^\infty} \leqslant 2C_1 \qquad \forall \tau \in (\tau_\star, \tau_0)$$

we see from (7.1) that

$$\frac{\mathrm{d}X}{\mathrm{d}\tau}(\tau, w_0) + \frac{a}{2}X(\tau, w_0)\langle w_0\rangle \leqslant b + \frac{a}{4}\langle w_0\rangle\big(\|X(\tau)\|_{L^\infty} - 2X(\tau, w_0)\big) \leqslant b \quad \forall \tau \in (\tau_\star, \tau_0).$$

In particular, $\frac{\mathrm{d}X}{\mathrm{d}\tau}(\tau, w_0) \leqslant b - C_1\, a/2 \leqslant 0$ for any $\tau \in (\tau_\star, \tau_0)$. Therefore,

$$C_1 < X(\tau_0, w_0) \leqslant X(\tau_\star, w_0) = C_1,$$

which is a contradiction. □

In order to use previous result, we need first to establish a non-uniform control on the L^∞-norm for the self-similar profile.

Lemma 7.2. *Let f_0 satisfying (2.8) with moreover $f_0 \in L^\infty(\mathbb{R}^3)$. Let $g(\tau,w)$ be the self-similar solution defined by (2.18) where $f(t,v)$ is the solution to (2.7). Then, there is a constant $\kappa > 0$ such that*

$$\|g(\tau)\|_{L^\infty} \leqslant \|f_0\|_{L^\infty} + \kappa\,\tau \qquad \forall \tau > 0.$$

Proof. Recall that the self-similar solution $g(\tau,w)$ satisfies (3.9). Split then the collision operator $\mathcal{Q}_{\widetilde{e}_\tau}(f,f)$ as in Corollary 5.6 (for fixed $\tau > 0$). Then, using the standard estimates of Theorem 5.3 and Lemma 7.1, for any $p \geqslant 2$

$$\begin{aligned}\int_{\mathbb{R}^3} g^{p-1} Q^+_{\widetilde{e}_\tau}(g,g)\,\mathrm{d}w &= \int_{\mathbb{R}^3} g^{p-1}\mathcal{Q}^+_{B_1,\widetilde{e}_\tau}(g,g)\,\mathrm{d}w + \int_{\mathbb{R}^3} g^{p-1}\mathcal{Q}^+_{B_2,\widetilde{e}_\tau}(g,g)\,\mathrm{d}w \\ &\leqslant C_1\,\|g(\tau)\|^2_{L^2_1}\,\|g(\tau)\|^{p-1}_{L^{p-1}} + \|g\|^{p-1}_{L^p_{1/p}}\left\|\mathcal{Q}^+_{B_2,\widetilde{e}_\tau}(g,g)\right\|_{L^p_{-1/p'}}.\end{aligned}$$

A simple interpolation inequality shows that

$$\|g(\tau)\|^{p-1}_{L^{p-1}} \leqslant \|g(\tau)\|^{\frac{1}{p-1}}_{L^1}\,\|g(\tau)\|^{p\frac{p-2}{p-1}}_{L^p} = \|g(\tau)\|^{p\frac{p-2}{p-1}}_{L^p}.$$

On the other hand, arguing as in the proof of Theorem 5.3 and choosing the mass of b_2 small enough, one obtains that, for any $\varepsilon > 0$, there is some constant $C > 0$ (independent of τ) such that

$$\int_{\mathbb{R}^3} g^{p-1}\mathcal{Q}^+_{\widetilde{e}_\tau}(g,g)\,\mathrm{d}w \leqslant C_1\,\|g(\tau)\|^2_{L^2_1}\,\|g(\tau)\|^{p\frac{p-2}{p-1}}_{L^p} + \varepsilon\,\|g(\tau)\|_{L^1_1}\,\|g(\tau)\|^p_{L^p_{1/p}}.$$

Note that our assumptions on f_0 imply $\|g_0\|_{L^2_1} < \infty$, hence by Theorem 6.2,

$$\sup_{\tau\geqslant 0}\|g(\tau)\|_{L^2_1} < \infty.$$

Using now the lower bound given in Lemma 6.1 and choosing

$$\varepsilon = K\left(\sup_{\tau\geqslant 0}\|g(\tau)\|_{L^1_1}\right)^{-1} > 0$$

we obtain

$$\int_{\mathbb{R}^3} g^{p-1}\mathcal{Q}_{\widetilde{e}_\tau}(g,g)(\tau,\omega)\,\mathrm{d}w \leqslant C_1\,\|g(\tau)\|^2_{L^2_1}\,\|g(\tau)\|^{p\frac{p-2}{p-1}}_{L^p} \leqslant C_2\,\|g(\tau)\|^{p\frac{p-2}{p-1}}_{L^p}.$$

Next, multiply by g^{p-1} with $p \geqslant 2$ and integrate in $\mathbb{R}^3$ the equation (3.9) satisfied by g. After few calculations one obtains

$$\frac{1}{p}\frac{\mathrm{d}X_p}{\mathrm{d}\tau}(\tau) \leqslant \int_{\mathbb{R}^3} g^{p-1}\mathcal{Q}_{\widetilde{e}_\tau}(g,g)(\tau,\omega)\,\mathrm{d}w \leqslant C_2 X_p^{\frac{p-2}{p-1}}(\tau),$$

where $X_p(\tau) := \|g(\tau)\|^p_{L^p}$. Therefore,

$$X_p(\tau) \leqslant \left(X_p(0)^{\frac{1}{p-1}} + \frac{C\,p}{p-1}\tau\right)^{p-1}$$

which translates to $\|g(\tau)\|_{L^p} \leqslant \left(\|g_0\|_{L^p}^{p'} + Cp'\tau\right)^{\frac{1}{p'}}$. Letting now $p \to \infty$, we get the conclusion. □

Finally we are ready to prove the L^∞ propagation.

Theorem 7.3. *Assume the variable restitution coefficient $e(\cdot)$ satisfy Assumptions 3.1. Let f_0 satisfying (2.8) with moreover $f_0 \in L^\infty(\mathbb{R}^3)$. Let $g(\tau, w)$ be the self-similar solution defined by (2.18) where $f(t, v)$ is the solution to (2.7). Then, there exists a positive constant $C > 0$ depending only on $\|f_0\|_{L^1_2}$ and $\|f_0\|_{L^\infty}$ such that*

$$\sup_{\tau \geqslant 0} \|g(\tau)\|_{L^\infty} \leqslant C.$$

Whence, $\|f(t)\|_{L^\infty} \leqslant C\, V(t)^3$ for any $t > 0$.

Proof. Recall that the equation for the self-similar profile can be written as

$$\begin{aligned}(\partial_\tau g + \xi(\tau)\nabla_w \cdot (wg))(\tau, \omega) &= (\partial_\tau g + \xi(\tau) w \cdot \nabla_w g + 3\xi(\tau) g)(\tau, \omega) \\ &= \mathcal{Q}_{\widetilde{e}_\tau}(g, g)(\tau, \omega).\end{aligned}$$

Defining the solution along the trajectories

$$g^{\#}(\tau, w) = g(\tau, \phi(\tau)w) \text{ with } \frac{\phi'(\tau)}{\phi(\tau)} = \xi(\tau),\ \phi(0) = V(0) = 1,$$

we can rewrite this equation as

$$\partial_\tau g^{\#}(\tau, w) + 3\xi(\tau) g^{\#}(\tau, w) = [\mathcal{Q}_{\widetilde{e}_\tau}(g, g)]^{\#}(\tau, w). \tag{7.2}$$

Using Corollary 5.6 and the lower bound for $\mathcal{Q}^-(g, g) = \mathcal{Q}^-_{\widetilde{e}_\tau}(g, g)$ given in Lemma 6.1 one sees that, for any $\varepsilon > 0$, there exists a constant $C > 0$ (independent of τ) such that

$$\begin{aligned}\partial_\tau g^{\#}(\tau, w) + 3\xi(\tau) g^{\#}(\tau, w) + \frac{\nu_0}{2} g^{\#}(\tau, w) \langle w \rangle^{\#} \\ \leqslant C \|g(\tau)\|_{L^2_1}^2 + \varepsilon \|g(\tau)\|_{L^1_1} \|g(\tau)\|_{L^\infty} \langle w \rangle^{\#}.\end{aligned} \tag{7.3}$$

Due to propagation of moments we can fix $\varepsilon := \frac{\nu_0}{8}\left(\sup_{\tau \geqslant 0} \|g(\tau)\|_{L^1_1}\right)^{-1}$. Notice that $\|g_0\|_{L^2_1}^2 \leqslant \|g_0\|_{L^\infty} \|g_0\|_{L^1_2} < \infty$, so that, according to Theorem 6.2,

$$\sup_{\tau \geqslant 0} \|g(\tau)\|_{L^2_1} < \infty.$$

Moreover, we can disregard the second term in the left hand side of (7.3) because is nonnegative. Thus, we conclude that, for $\tau > 0$

$$\partial_\tau g^{\#}(\tau, w) + \frac{\nu_0}{2} g^{\#}(\tau, w) \langle w \rangle^{\#} \leqslant C \sup_{\tau \geqslant 0} \|g(\tau)\|_{L^2_1}^2 + \frac{\nu_0}{8} \|g(\tau)\|_{L^\infty} \langle w \rangle^{\#}. \tag{7.4}$$

Moreover, according to Lemma 7.2 the norm $\|g(\tau)\|_{L^\infty}$ is finite for any $\tau > 0$. Notice that the mapping $\tau > 0 \longmapsto g^{\#}(\tau, w)$ is absolutely continuous for any fixed w as the

solution to the rescaled Boltzmann equation (3.9). Hence, using Lemma 7.1, inequality (7.4) implies that

$$\|g(\tau)\|_{L^\infty} = \left\|g^{\#}(\tau)\right\|_{L^\infty} \leqslant \max\left\{\|f_0\|_{L^\infty}\,,\ \frac{4C}{\nu_0}\sup_{\tau\geqslant 0}\|g(\tau)\|_{L^2_1}^2\right\} \tag{7.5}$$

which proves the Theorem. □

Remark 7.4. *Notice that the same reasoning applies readily to the elastic Boltzmann equation in an even easier way, providing a new proof of the results of* [4, 16]. *In particular, our new method shows that it is possible to relax the conditions on the collision kernel of* [4]. *Namely, while* [4] *applies to a* bounded $b(\cdot)$, *the above result applies to any integrable* $b(\cdot)$ *through the cutoff assumption* (2.6).

7.2. **Pointwise exponential bounds.** In GAMBA *et. al.* [15] the authors use a comparison principle that holds for weak solutions of the Boltzmann equation to prove that under the condition $f_0(v) \leqslant e^{-a_0|v|^2+c_0}$, the solution to the classical (elastic) Boltzmann equation remains pointwise uniformly controlled by a Maxwellian distribution. There are four main ingredients in the pointwise Maxwellian control proof for the elastic case:

- the L^∞-propagation,
- the L^1-Maxwellian propagation,
- the Young's inequality (with Maxwellian weights),
- the aforementioned comparison principle.

We have seen in Theorem 7.3 that the second point holds true for the inelastic Boltzmann equation with variable restitution coefficient (at least for the self-similar solution). Moreover, variants of the second and third points have been obtained in Sections 4 and 5.5 respectively. For such estimates, Maxwellian weights have to be replaced by only exponential weights (since the tail of self-similar solutions are of order only one, see Theorem 4.1). With all these results at hand, we are in position to prove the following result, inspired by [15]:

Theorem 7.5 (**L^∞-exponential tails Theorem**). *Assume that* $B(u,\sigma) = \Phi(|u|)b(\widehat{u}\cdot\sigma)$ *and the restitution coefficient* $e(\cdot)$ *satisfy Assumption 5.14 with* $k = 1$. *Let* f_0 *satisfying* (2.8) *and assume moreover that there are some constants* $a_0 > 0$ *and* $c_0 \in \mathbb{R}$ *such that*

$$f_0(v) \leqslant \exp\left(-a_0|v| + c_0\right) \qquad \text{for a. e. } v \in \mathbb{R}^3.$$

Let $g(\tau, w)$ *be the self-similar solution defined by* (2.18) *where* $f(t,v)$ *is the solution to* (2.7). *Then, there exist* $a > 0$ *and* $c \geqslant c_0$ *such that*

$$\sup_{\tau\geqslant 0} g(\tau, w) \leqslant \exp\left(-a|w| + c\right) \qquad \text{for a. e. } w \in \mathbb{R}^3.$$

Hence, $f(t,v) \leqslant V(t)^3 \exp\left(-a|V(t)v| + c\right)$ *for a. e.* $v \in \mathbb{R}^3$ *and any* $t \geqslant 0$.

Proof. Since $f_0(v) \leqslant \exp\left(-a_0|v| + c_0\right)$, we can apply the L^1-exponential propagation Theorem 4.1 which implies the existence of some $0 < a_1 < a_0$ such that

$$\sup_{\tau\geqslant 0}\int_{\mathbb{R}^3} g(\tau, w)\,\langle w\rangle \exp\left(a_1|w|\right)\,\mathrm{d}w < \infty,$$

or equivalently

$$\sup_{t\geqslant 0}\int_{\mathbb{R}^3} f(t,v)\,\langle V(t)v\rangle \exp\left(a_1|V(t)v|\right)\,\mathrm{d}v<\infty. \tag{7.6}$$

Moreover, since $f_0\in L^\infty$, we can apply Theorem 7.3 to conclude that

$$\sup_{\tau\geqslant 0}\|g(\tau)\|_{L^\infty}<\infty. \tag{7.7}$$

The next step it to use the comparison principle given in Appendix B, Theorem B.1. To this end define

$$\mathcal{M}(w):=\exp\left(-a|w|+c\right)$$

where $0<a<\min\{a_1,\frac{\nu_0}{8}\}$ is fixed (ν_0 being given in Lemma 6.1) and where the parameter $c\geqslant c_0\in\mathbb{R}$ will be chosen large enough in the sequel. Define then

$$\mathcal{M}_0(t,v):=V^3(t)\mathcal{M}(V(t)\,v)\qquad t\geqslant 0,\ v\in\mathbb{R}^3.$$

According to Proposition 5.17, for any fixed $\varepsilon>0$, there is some positive constant C_e such that

$$\mathcal{Q}_e^+(\mathcal{M}_0,f)(t,v)\leqslant \left\|f(t)\mathcal{M}_0^{-1}(t,v)\right\|_{L^1_1}\left(C_e+\varepsilon\,\langle v\rangle\right)\mathcal{M}_0(t,v).$$

Now, according to (7.6) and our choice of the parameter $a>0$, one has

$$\widetilde{C}:=\sup_{t\geqslant 0}\left\|f(t)\mathcal{M}_0^{-1}(t,v)\right\|_{L^1_1}<\infty$$

and

$$\mathcal{Q}_e^+(\mathcal{M}_0,f)(t,v)\leqslant \widetilde{C}\mathcal{M}(V(t)v)\left(C_e+\varepsilon\,\langle v\rangle\right)\qquad \forall t\geqslant 0,\ v\in\mathbb{R}^3.$$

Observe that $\widetilde{C}$ is independent of c. Then, moving to the self-similar variable one has

$$\mathcal{Q}^+_{\widetilde{e}_{\tau(t)}}(\mathcal{M},g)(\tau(t),V(t)v)=V^{-2}(t)\mathcal{Q}_e^+(\mathcal{M}_0,f)(t,v),$$

and since $V(t)\geqslant 1$,

$$\begin{aligned}\mathcal{Q}^+_{\widetilde{e}_{\tau(t)}}(\mathcal{M},g)(\tau(t),w)&\leqslant V^{-2}(t)\widetilde{C}\mathcal{M}(w)\left(C_e+\varepsilon\left\langle V^{-1}(t)w\right\rangle\right)\\&\leqslant \tilde{C}\mathcal{M}(w)\left(C_e+\epsilon\,\langle w\rangle\right)\end{aligned}$$

where $\widetilde{e}_\tau$ has been defined in Section 4. Fix now $\varepsilon=\nu_0/4\widetilde{C}$ and use Lemma 6.1 to conclude that

$$\mathcal{Q}_{\widetilde{e}_{\tau(t)}}(\mathcal{M},g)(\tau(t),w)\leqslant\left(C_0-\frac{\nu_0}{4}\langle w\rangle\right)\mathcal{M}(w),\qquad \forall t\geqslant 0,\ w\in\mathbb{R}^3.$$

Recall that equation (3.6) implies that $1\geqslant\dot{V}(t)\searrow 0$ as $t\to\infty$. Then, choosing any $R\geqslant\frac{C_0-3}{\nu_0/4-a}$ we conclude that for $t\geqslant 0$,

$$\dot{V}(t)(3-a|w|)\geqslant\left(C_0-\frac{\nu_0}{4}\langle w\rangle\right)\qquad\text{for any } |w|\geqslant R \text{ and any } t\geqslant 0.$$

In other words, for any $|w|\geqslant R$ and any $t\geqslant 0$:

$$\begin{aligned}\partial_\tau\mathcal{M}(w)+\dot{V}(t)\nabla_w\cdot(w\mathcal{M})(w)&=\dot{V}(t)(3-a|w|)\mathcal{M}(w)\\&\geqslant\left(C_0-\frac{\epsilon_0}{4}\langle w\rangle\right)\mathcal{M}(w)\geqslant\mathcal{Q}_{\widetilde{e}_{\tau(t)}}(\mathcal{M},g)(\tau(t),w)\end{aligned}\tag{7.8}$$

Then moving back to the original variables (t,v), inequality (7.8) reads

$$\begin{aligned}\partial_t \mathcal{M}_0(t,v) &= V^2(t)\dot{V}(t)(3-a|V(t)\ v|)\mathcal{M}(V(t)v)\\ &\geqslant V^2(t)\mathcal{Q}_{\widetilde{e}_{\tau(t)}}(\mathcal{M},g)(\tau(t),V(t)v) = \mathcal{Q}_e(\mathcal{M}_0,f)(t,v) \quad \text{for any } (t,v)\in U_R,\end{aligned}$$

where $U_R=\{(t,v)\in(0,\infty)\times\mathbb{R}^3\,;\,|V(t)\ v|\geqslant R\}$. Moreover, using Theorem 7.3 and (7.7) we have for sufficiently large c

$$\begin{aligned}\mathcal{M}_0(t,v) = V(t)^3\ \exp\left(-a|V(t)v|+c\right) &\geqslant V(t)^3\ \exp\left(-aR+c\right)\\ &\geqslant V(t)^3 \sup_{\tau\geqslant 0}\|g(\tau)\|_{L^\infty} \geqslant f(t,v)\ \text{ in }\ U_R^c.\end{aligned}$$

By choosing c large enough to satisfy also $f_0(v)\leqslant\mathcal{M}_0(0,v)$, we can conclude the proof using the comparison principle in Theorem B. 1 in the Appendix B. □

Appendix A: Viscoelastic hard-spheres

In this Appendix, we prove that our general Assumptions 3.1 are met by the restitution coefficient $e(\cdot)$ associated to the so-called viscoelastic hard-spheres as derived in [25] (see also [11, Chapter 4]). Before this, we state a more general result for general restitution coefficient and for hard-spheres collision kernel

$$B(u,\sigma)=\frac{|u|}{4\pi}\qquad \forall u\in\mathbb{R}^3,\ \sigma\in\mathbb{S}^2$$

for which we recall that $\mathbf{\Psi}_e$ as defined in (3.1) is given by:

$$\mathbf{\Psi}_e(x)=\frac{1}{2\sqrt{x}}\int_0^{\sqrt{x}}\left(1-e(z)^2\right)z^3\,\mathrm{d}z,\qquad\qquad x>0.$$

Lemma A. 1. *Assume that $e(\cdot)$ satisfies Assumption 2.1 and that the mapping $z\geqslant 0\mapsto e(z)$ is decreasing. Then, the associated function $\mathbf{\Psi}_e$ defined in (3.1) is strictly increasing and convex.*

Proof. Let us assume that $e_z(z)\leqslant 0$ for any $z\geqslant 0$ where $e_z(\cdot)$ denotes the derivative of $e(\cdot)$. Let $\Phi(x)=2\mathbf{\Psi}_e(x^2)$ for any $x>0$, i.e.

$$\Phi(x)=\frac{1}{x}\int_0^{x}\left(1-e^2(z)\right)z^3\,\mathrm{d}z,\qquad\qquad x>0.$$

It is easy to see that $\mathbf{\Psi}_e(\cdot)$ is convex if and only if $x\Phi_{xx}(x)-\Phi_x(x)\geqslant 0$ for any $x>0$ where Φ_{xx} and Φ_x denote the respectively the second and first derivative of Φ. A simple calculation shows that

$$x\Phi_{xx}(x)-\Phi_x(x)=-2x^3e_z(x)e(x)+\frac{3}{x^2}\int_0^x(1-e^2(z))z^3\,\mathrm{d}z,\qquad\forall x>0$$

and, since $e_z(x)\geqslant 0$ while $e(\cdot)\in(0,1]$, one gets that $x\Phi_{xx}(x)-\Phi_x(x)\geqslant 0$ for any $x>0$.

In the same way, since $e_z(\cdot)\leqslant 0$, the mapping $z\geqslant 0\mapsto(1-e^2(z))z^3$ is nondecreasing and one deduces easily that $\Phi_x(x)>0$ for any $x>0$. This obviously implies that $\mathbf{\Psi}_e(\cdot)$ is strictly increasing over $(0,+\infty)$. □

Now, for the visco-elastic hard-spheres as derived in [25], the restitution coefficient $e = e(z)$ is the solution of the equation

$$e(z) + \alpha z^{1/5} e(z)^{3/5} = 1 \tag{A.1}$$

where $\alpha > 0$ is a constant depending on the material viscosity (see precisely [11, Eq. (3.38), p. 40]). Notice there is also an alternative expression of the restitution coefficient $e(z)$ for viscoelastic hard-spheres as an infinite expansion series in power of $z^{1/5}$:

$$e(z) = 1 + \sum_{k=1}^{\infty} (-1)^k a_k z^{k/5}, \qquad z \geqslant 0$$

where $a_k > 0$ for any $k \in \mathbb{N}$. It has already proven, on the basis of (A.1) that Assumptions 2.1 are met (see [1, p. 1006]). Now, from Eq. (A.1), one sees that $\lim_{z\to 0^+} e(z) = 1$ and

$$e(z) \simeq 1 - \alpha z^{1/5} \qquad \text{for} \qquad z \simeq 0$$

which means that Assumption 3.1 *(1)* is met. Moreover, it is easy to prove from (A.1) that the restitution coefficient is continuously decreasing. Thus, according to Lemma A.1, the restitution coefficient $e(\cdot)$ associated to the visco-elastic hard-spheres satisfy Assumptions 3.1. Notice that our result covers more general models than the one of viscoelastic hard-spheres as illustrated by the following example:

Example A. 1. *For monotone decreasing restitution coefficient introduced in Example 2.3, Assumptions 3.1 are also met by virtue of the above Lemma. In such a case, according to* (2.2)*, the cooling of the temperature $\mathcal{E}(t)$ is as follows*

$$\mathcal{E}(t) = O\left((1+t)^{-\frac{2}{1+\eta}}\right) \qquad \textit{as} \qquad t \to \infty.$$

APPENDIX B: COMPARISON PRINCIPLE

We write here, for comfort, the GAMBA, PANFEROV & VILLANI comparison principle for the Boltzmann equation [15]. The proof of this result can be found in [15] for the elastic case. *Same* proof applies in the inelastic case. The result is stated in the context of the, very weak, *dissipative* solutions. In particular, spatially homogeneous solutions that we found in this work are dissipative solutions.

Theorem B. 1. *Let $f \in \mathcal{C}([0,T]; L^1(\mathbb{R}^3))$ be a dissipative solution of the Boltzmann equation and let g be a sufficiently regular function, such that $f_0 \leqslant g_0$ and*

$$\partial_t g - \mathcal{Q}_{B,e}(g, f) \geqslant 0 \ \text{ on } \ U$$

and $f \leqslant g$ on U^c, where U is a measurable subset of $[0,T] \times \mathbb{R}^3$. Then $f \leqslant g$ almost everywhere on $[0,T] \times \mathbb{R}^3$.

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