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# ON REPRESENTATIONS OF THE FEASIBLE SET IN CONVEX OPTIMIZATION

JEAN B. LASSERRE

ABSTRACT. We consider the convex optimization problem  $\min_{\mathbf{x}} \{f(\mathbf{x}) : g_j(\mathbf{x}) \leq 0, j = 1, \dots, m\}$  where  $f$  is convex, the feasible set  $\mathbf{K}$  is convex and Slater's condition holds, but the functions  $g_j$ 's are *not* necessarily convex. We show that for any representation of  $\mathbf{K}$  that satisfies a mild nondegeneracy assumption, every minimizer is a Karush-Kuhn-Tucker (KKT) point and conversely every KKT point is a minimizer. That is, the KKT optimality conditions are necessary and sufficient as in convex programming where one assumes that the  $g_j$ 's are convex. So in convex optimization, and as far as one is concerned with KKT points, what really matters is the geometry of  $\mathbf{K}$  and not so much its representation.

## 1. INTRODUCTION

Given differentiable functions  $f, g_j : \mathbb{R}^n \rightarrow \mathbb{R}, j = 1, \dots, m$ , consider the following convex optimization problem:

$$(1.1) \quad f^* := \inf_{\mathbf{x}} \{ f(\mathbf{x}) : \mathbf{x} \in \mathbf{K} \}$$

where  $f$  is convex and the feasible set  $\mathbf{K} \subset \mathbb{R}^n$  is convex and represented in the form:

$$(1.2) \quad \mathbf{K} = \{ \mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \leq 0, j = 1, \dots, m \}.$$

Convex optimization usually refers to minimizing a convex function over a convex set without precisizing its representation (see e.g. Ben-Tal and Nemirovsky [1, Definition 5.1.1] or Bertsekas et al. [3, Chapter 2]), and it is well-known that convexity of the function  $f$  and of the set  $\mathbf{K}$  imply that every local minimum is a global minimum. An elementary proof only uses the geometry of  $\mathbf{K}$ , not its representation by the defining functions  $g_j$ ; see e.g. Bertsekas et al. [3, Prop. 2.1.2].

The convex set  $\mathbf{K}$  may be represented by different choices of the (not necessarily convex) defining functions  $g_j, j = 1, \dots, m$ . For instance, the set

$$\mathbf{K} := \{ \mathbf{x} \in \mathbb{R}^2 : 1 - x_1x_2 \leq 0; \mathbf{x} \geq 0 \}$$

is convex but the function  $\mathbf{x} \mapsto 1 - x_1x_2$  is not convex on  $\mathbb{R}_+^2$ . Of course, depending on the choice of the defining functions ( $g_j$ ), several properties may or may not hold. In particular, the celebrated Karush-Kuhn-Tucker (KKT) optimality conditions *depend* on the representation of  $\mathbf{K}$ . Recall that  $\mathbf{x} \in \mathbf{K}$  is a KKT point if

$$(1.3) \quad \nabla f(\mathbf{x}) + \sum_{j=1}^m \lambda_j \nabla g_j(\mathbf{x}) = 0 \quad \text{and} \quad \lambda_j g_j(\mathbf{x}) = 0, j = 1, \dots, m,$$

for some nonnegative vector  $\lambda \in \mathbb{R}^m$ . (More precisely  $(\mathbf{x}, \lambda)$  is a KKT point.)

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*Convex programming* refers to the situation where  $f$  is convex *and* the defining functions  $g_j$  of  $\mathbf{K}$  are also convex. See for instance Ben-Tal and Nemirovsky [1, p. 335], Berkovitz [2, p. 179], Boyd and Vandenberghe [4, p. 7], Bertsekas et al. [3, §3.5.5], Nesterov and Nemirovskii [6, p. 217-218], and Hiriart-Urruty [5].

A crucial feature of convex programming is that when Slater's condition holds<sup>1</sup>, the KKT optimality conditions (1.3) are necessary and sufficient, which shows that a representation of the convex set  $\mathbf{K}$  with convex functions ( $g_j$ ) has some very attractive features.

The purpose of this note is to show that in fact, when  $\mathbf{K}$  is convex and as far as one is concerned with KKT points, what really matters is the geometry of  $\mathbf{K}$  and not so much its representation. Indeed, we show that if  $\mathbf{K}$  is convex and Slater's condition holds then the KKT optimality conditions (1.3) are also necessary and sufficient for *all* representations of  $\mathbf{K}$  that satisfy a mild nondegeneracy condition, no matter if the  $g_j$ 's are convex. So this attractive feature is not specific to representations of  $\mathbf{K}$  with convex functions.

That a KKT point is a local (hence global) minimizer follows easily from the convexity of  $\mathbf{K}$ . More delicate is the fact that any local (hence global) minimizer is a KKT point. Various *constraint qualifications* are usually required to hold at a minimizer, and when the  $g_j$ 's are convex the simple Slater's condition is enough. Here we show that Slater's condition is also sufficient for all representations of  $\mathbf{K}$  that satisfy a mild additional nondegeneracy assumption on the boundary of  $\mathbf{K}$ . Moreover under Slater's condition this mild nondegeneracy assumption is automatically satisfied if the  $g_j$ 's are convex.

## 2. MAIN RESULT

Let  $\mathbf{K} \subset \mathbb{R}^n$  be as in (1.2). We first start with the following non degeneracy assumption:

**Assumption 2.1** (nondegeneracy). *For every  $j = 1, \dots, m$ ,*

$$(2.1) \quad \nabla g_j(\mathbf{x}) \neq 0, \quad \text{whenever } \mathbf{x} \in \mathbf{K} \text{ and } g_j(\mathbf{x}) = 0.$$

Observe that under Slater's condition, (2.1) is automatically satisfied if  $g_j$  is convex. Indeed if  $g_j(\mathbf{x}) = 0$  and  $\nabla g_j(\mathbf{x}) = 0$  then by convexity 0 is the global minimum of  $g_j$  on  $\mathbb{R}^n$ . Hence there is no  $\mathbf{x}_0 \in \mathbf{K}$  with  $g_j(\mathbf{x}_0) < 0$ . We next state the following characterization of convexity.

**Lemma 2.2.** *With  $\mathbf{K} \subset \mathbb{R}^n$  as in (1.2), let Assumption 2.1 and Slater's condition both hold for  $\mathbf{K}$ . Then  $\mathbf{K}$  is convex if and only if for every  $j = 1, \dots, m$ :*

$$(2.2) \quad \langle \nabla g_j(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \leq 0, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbf{K} \text{ with } g_j(\mathbf{x}) = 0.$$

*Proof. Only if* part. Assume that  $\mathbf{K}$  is convex and  $\langle \nabla g_j(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle > 0$  for some  $j \in \{1, \dots, m\}$  and some  $\mathbf{x}, \mathbf{y} \in \mathbf{K}$  with  $g_j(\mathbf{x}) = 0$ . Then  $g_j(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) > 0$  for all sufficiently small  $t$ , in contradiction with  $\mathbf{x} + t(\mathbf{y} - \mathbf{x}) \in \mathbf{K}$  for all  $0 \leq t \leq 1$  (by convexity of  $\mathbf{K}$ ).

*If* part. By (2.2), at every point  $\mathbf{x}$  on the boundary of  $\mathbf{K}$ , there exists a supporting hyperplane for  $\mathbf{K}$ . As  $\mathbf{K}$  is closed with nonempty interior, by [8][Th. 1.3.3] the set  $\mathbf{K}$  is convex<sup>2</sup>.  $\square$

<sup>1</sup>Slater's condition holds for  $\mathbf{K}$  if for some  $\mathbf{x}_0 \in \mathbf{K}$ ,  $g_j(\mathbf{x}_0) < 0$  for every  $j = 1, \dots, m$ .

<sup>2</sup>The author wishes to thank Prof. L. Tuncel for providing him with the reference [8].

**Theorem 2.3.** *Consider the nonlinear programming problem (1.1) and let Assumption 2.1 and Slater's condition both hold. If  $f$  is convex then every minimizer is a KKT point and conversely, every KKT point is a minimizer.*

*Proof.* Let  $\mathbf{x}^* \in \mathbf{K}$  be a minimizer (hence a global minimizer) with  $f^* = f(\mathbf{x}^*)$ . We first prove that  $\mathbf{x}^*$  is a KKT point. The Fritz-John optimality conditions state that

$$\lambda_0 \nabla f(\mathbf{x}^*) + \sum_{j=1}^m \lambda_j \nabla g_j(\mathbf{x}^*) = 0; \quad \lambda_j g_j(\mathbf{x}^*) = 0, \quad j = 1, \dots, m,$$

for some non trivial nonnegative vector  $0 \neq \lambda \in \mathbb{R}^{m+1}$ . See e.g. Hiriart-Urruty [5, Th. page 77] or Polyak [7, Theor. 1, p. 271]. We next prove that  $\lambda_0 \neq 0$ . Suppose that  $\lambda_0 = 0$  and let  $J := \{j \in \{1, \dots, m\} : \lambda_j > 0\}$ . As  $\lambda \neq 0$  and  $\lambda_0 = 0$ , the set  $J$  is nonempty. Next, as  $g_j(\mathbf{x}_0) < 0$  for every  $j = 1, \dots, m$ , there is some  $\rho > 0$  such that  $B(\mathbf{x}_0, \rho) := \{\mathbf{z} \in \mathbb{R}^n : \|\mathbf{z} - \mathbf{x}_0\| < \rho\} \subset \mathbf{K}$  and  $g_j(\mathbf{z}) < 0$  for all  $\mathbf{z} \in B(\mathbf{x}_0, \rho)$  and all  $j \in J$ . Therefore we obtain

$$\sum_{j \in J} \lambda_j \langle \nabla g_j(\mathbf{x}^*), \mathbf{z} - \mathbf{x}^* \rangle = 0 \quad \forall \mathbf{z} \in B(\mathbf{x}_0, \rho),$$

which, by Lemma 2.2, implies that  $\langle \nabla g_j(\mathbf{x}^*), \mathbf{z} - \mathbf{x}^* \rangle = 0$  for every  $j \in J$  and every  $\mathbf{z} \in B(\mathbf{x}_0, \rho)$ . But this clearly implies that  $\nabla g_j(\mathbf{x}^*) = 0$  for every  $j \in J$ , in contradiction with Assumption 2.1. Hence  $\lambda_0 > 0$  and we may and will set  $\lambda_0 = 1$ , so that the KKT conditions hold at  $\mathbf{x}^*$ .

Conversely, let  $\mathbf{x} \in \mathbf{K}$  be an arbitrary KKT point, i.e.,  $\mathbf{x} \in \mathbf{K}$  satisfies

$$\nabla f(\mathbf{x}) + \sum_{j=1}^m \lambda_j \nabla g_j(\mathbf{x}) = 0; \quad \lambda_j g_j(\mathbf{x}) = 0, \quad j = 1, \dots, m,$$

for some nonnegative vector  $\lambda \in \mathbb{R}^m$ . Suppose that there exists  $\mathbf{y} \in \mathbf{K}$  with  $f(\mathbf{y}) < f(\mathbf{x})$ . Then we obtain the contradiction:

$$\begin{aligned} 0 &> f(\mathbf{y}) - f(\mathbf{x}) \\ &\geq \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \quad [\text{by convexity of } f] \\ &= - \sum_{j=1}^m \lambda_j \langle \nabla g_j(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \geq 0 \end{aligned}$$

where the last inequality follows from  $\lambda \geq 0$  and Lemma 2.2. Hence  $\mathbf{x}$  is a minimizer.  $\square$

Hence if  $\mathbf{K}$  is convex and both Assumption 2.1 and Slater's condition hold, there is a one-to-one correspondence between KKT points and minimizers. That is, the KKT optimality conditions are necessary and sufficient for all representations of  $\mathbf{K}$  that satisfy Slater's condition and Assumption 2.1.

However there is an important additional property when all the defining functions  $g_j$  are convex. Dual methods of the type

$$\sup_{\lambda \in \mathbb{R}_+^m} \left\{ \inf_{\mathbf{x}} f(\mathbf{x}) + \sum_{j=1}^m \lambda_j g_j(\mathbf{x}) \right\},$$

are well defined because  $\mathbf{x} \mapsto f(\mathbf{x}) + \sum_{j=1}^m \lambda_j g_j(\mathbf{x})$  is a convex function. In particular, the Lagrangian  $\mathbf{x} \mapsto L_f(\mathbf{x}) := f(\mathbf{x}) - f^* + \sum_{j=1}^m \lambda_j g_j(\mathbf{x})$ , defined from an

arbitrary KKT point  $(\mathbf{x}^*, \lambda) \in \mathbf{K} \times \mathbb{R}_+^m$ , is convex and nonnegative on  $\mathbb{R}^n$ , with  $\mathbf{x}^*$  being a global minimizer. If the  $g_j$ 's are not convex this is not true in general.

**Example 1.** Let  $n = 2$  and consider the problem

$$\mathbf{P} : f^* = \min \{ f(\mathbf{x}) : a - x_1x_2 \leq 0; \mathbf{Ax} \leq \mathbf{b}; \mathbf{x} \geq 0 \},$$

where  $a > 0$ ,  $\mathbf{A} \in \mathbb{R}^{m \times n}$ ,  $\mathbf{b} \in \mathbb{R}^m$ , and  $f$  is convex and differentiable. The set

$$\mathbf{K} := \{ \mathbf{x} \in \mathbb{R}^2 : a - x_1x_2 \leq 0; \mathbf{Ax} \leq \mathbf{b}; \mathbf{x} \geq 0 \}$$

is convex and it is straightforward to check that Assumption 2.1 holds. Therefore, by Theorem 2.3, if Slater's condition holds, every KKT point is a global minimizer. However, the Lagrangian

$$\mathbf{x} \mapsto f(\mathbf{x}) - f^* + \psi(a - x_1x_2) + \langle \lambda, \mathbf{Ax} - \mathbf{b} \rangle - \langle \mu, \mathbf{x} \rangle,$$

with nonnegative  $(\psi, \lambda, \mu) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n$ , may not be convex whenever  $\psi \neq 0$  (for instance if  $f$  is linear). On the other hand, notice that  $\mathbf{K}$  has the equivalent convex representation

$$\mathbf{K} := \left\{ \mathbf{x} \in \mathbb{R}^2 : \begin{bmatrix} x_1 & \sqrt{a} \\ \sqrt{a} & x_2 \end{bmatrix} \succeq 0; \mathbf{Ax} \leq \mathbf{b} \right\},$$

where for a real symmetric matrix  $\mathbf{B}$ , the notation  $\mathbf{B} \succeq 0$  stands for  $\mathbf{B}$  is positive semidefinite.

A topic of further investigation is concerned with computational efficiency. Can *efficient* algorithms be devised for some class of convex problems (1.1) where the defining functions  $g_j$  of  $\mathbf{K}$  are not necessarily convex?

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